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GLOBAL STABILITY OF TRAVELING CURVED FRONTS IN THE ALLEN-CAHN EQUATIONS

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Abstract. This paper is concerned with the global stability of a traveling curved front in the Allen-Cahn equation. The existence of such a front is recently proved by constructing supersolutions and subsolutions. In this paper, we introduce a method to construct new subsolutions and prove the asymptotic stability of traveling curved fronts globally in space.

1. **Introduction.** Front propagation phenomena are often observed in various fields (e.g. [16]). We study a traveling curved front with V-formed contour lines in $\mathbb{R}^2$ in bistable media as one of the simplest non-planar multidimensional traveling waves. Such a front propagates due to the curvature effect and the difference of energy densities between two states.

In this paper we consider a solution $u(\cdot,\cdot,t) \in L^\infty(\mathbb{R}^2)$ of the Allen-Cahn equation:

$$\begin{aligned} u_t &= \Delta u + f(u) && (x,y) \in \mathbb{R}^2,\ t>0, \\ u(x,y,0) &= u_0(x,y) && (x,y) \in \mathbb{R}^2. \end{aligned} \tag{1.1}$$

Here the nonlinear term f is of bistable type and a given initial function u_0 is bounded and of class C^1.

We consider the case where the constant states -1 and 1 are asymptotically stable and where the region of the state 1 is getting larger and larger and finally it covers the whole space. When the state 1 propagates, we can observe the characteristic profiles. In the one-dimensional space, one of the typical solutions is a traveling wave solution which always keeps its shape. Setting $u(x,t) = \Phi(\eta)$ and $\eta = x - kt$, we have

$$\begin{aligned} -\Phi^{''}(\eta) - k\Phi^{'}(\eta) - f(\Phi) &= 0, \qquad \Phi^{'}(\eta) < 0 \qquad (-\infty < \eta < \infty), \\ \Phi(-\infty) &= 1, \qquad \Phi(+\infty) = -1. \end{aligned} \tag{1.2}$$

The following is the standing assumptions on f.

(A1) f is of class C^1 with $f(1) = 0$, $f(-1) = 0$, $f'(1) < 0$ and $f'(-1) < 0$.

(A2) $\displaystyle\int_{-1}^{1} f(s)\,ds > 0$.

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(A3) $f(s) < 0$ and $f'(s) < 0$ for $s > 1$. $f(s) > 0$ and $f'(s) < 0$ for $s < -1$.
(A4) There exist $k > 0$ and a function $\Phi(\eta)$ satisfying (1.2).

In (1.2), the speed k is uniquely determined and the traveling wave Φ is unique up to phase shift under the assumptions (A1), (A2) and (A3) as in [5] and [3].

Example 1. If f satisfies

$$\begin{aligned} f(s) &> 0 \qquad \text{for } s \in (\beta, 1), \\ f(s) &< 0 \qquad \text{for } s \in (-1, \beta), \\ f'(\beta) &> 0 \end{aligned}$$

for some $\beta \in (-1, 1)$ in addition to (A1)-(A3), then it satisfies (A4).

A typical example of f is

$$f(u) = -(u+1)(u+a)(u-1), \tag{1.3}$$

where $a \in (0, 1)$ is a given number. For the nonlinearity (1.3), one has $k = \sqrt{2}a$ and $\Phi(\eta) = -\tanh\left(\eta/\sqrt{2}\right)$.

There are many works on one-dimensional traveling waves. One can see [5], [1], [10] and [11].

Example 2. Another example is the following nonlinear term

$$f(u) = -u\left(u^2 - \frac{1}{4}\right)(u^2 - 1) + a(1 - u^2)\sqrt{8u^2 + 1},$$

where $0 < a < 1/4$. This f satisfies $f(\pm 1) = 0$, and it has three zero points in $(-1, 1)$ if $|a|$ is small enough. To solve (1.2) it suffices to solve the following two relations

$$\begin{aligned} \Phi''(\eta) &= \Phi\left(\Phi^2 - \frac{1}{4}\right)(\Phi^2 - 1) \\ -k\Phi'(\eta) &= a(1 - \Phi^2)\sqrt{8\Phi^2 + 1} \end{aligned} \qquad -\infty < \eta < \infty.$$

From the first relation Φ is a unique solution of

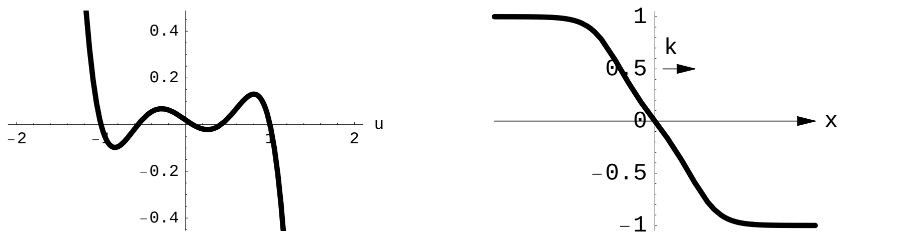
$$\int_0^{\Phi(\eta)} \frac{dv}{(1 - v^2)\sqrt{8v^2 + 1}} = -\frac{\sqrt{6}}{12}\eta$$

if one assumes $\Phi(0) = 0$. This $\Phi(\eta)$ satisfies the second relation if $k = 2\sqrt{6}a$. Terman [19] studied the existence of traveling waves when the nonlinear function has multiple zeros.

For multi-dimensional traveling wave solutions under the bistable condition (A1), Fife [4] first studied traveling curved fronts as a singular limit problem by putting a small coefficient to the diffusion term. The existence and stability of traveling curved fronts in (1.1) are recently studied by [14], [8] and [9]. For the existence and the stability of two-dimensional traveling curved fronts, see [14]. For the existence and properties two-dimensional traveling curved fronts and higher dimensional conical waves, see [8] and [9].

Related works in the interface equation are studied in [13] and [12]. See [13] and the references therein for the detail.

For nonlinearity appearing in combustion problems, the stability of the traveling curved fronts is studied in [7] for exponentially decaying perturbations that belongs to $[-1, 1]$.



The graph of f Profile of the corresponding traveling wave

FIGURE 1. The profiles of f and a traveling wave in Example 2 for $a = 1/50$.

For the bistable nonlinearity including the Allen-Cahn equations, one can expect a wider class of initial perturbations which leads to the asymptotic stability of traveling curved fronts, since the two rest states ($+1$ and -1) are asymptotically stable. This is the main issue of this paper.

We study traveling wave solutions. We assume that the solutions travel towards y-direction without loss of generality. We put

$$u(x, y, t) = w(x, y - ct, t), \qquad z = y - ct.$$

Then we obtain

$$\begin{aligned} w_t - w_{xx} - w_{zz} - cw_z - f(w) &= 0, \qquad (x, z) \in \mathbb{R}^2,\ t > 0. \\ w|_{t=0} &= u_0 \qquad \text{in } \mathbb{R}^2. \end{aligned} \tag{1.4}$$

We denote the solution of (1.4) with $w(x, z, 0; u_0) = u_0(x, z)$ by $w(x, z, t; u_0)$.

The traveling wave v with speed c in y-direction becomes a stationary solution of (1.4), namely, v satisfies

$$\mathcal{L}[v] = 0 \tag{1.5}$$

where

$$\mathcal{L}[v] \stackrel{\text{def}}{=} -v_{xx} - v_{zz} - cv_z - f(v).$$

Hereafter the speed c is an arbitrary constant that is bigger than k. We see that $\Phi(k(z \pm m_* x)/c)$ satisfy (1.5), where

$$m_* \stackrel{\text{def}}{=} \frac{\sqrt{c^2 - k^2}}{k} > 0.$$

They are so called *planar traveling fronts.*

Since the maximum of subsolutions is also a subsolution, it turns out that

$$\begin{aligned} v^-(x, z) &\stackrel{\text{def}}{=} \max\left\{ \Phi\left(\frac{k}{c}(z - m_* x)\right), \Phi\left(\frac{k}{c}(z + m_* x)\right) \right\} \\ &= \Phi\left(\frac{k}{c}(z - m_*|x|)\right) \end{aligned}$$

is a subsolution of (1.4). This $v^-(x, z)$ is strictly monotone decreasing in z.

If v^- is given as an initial condition, it converges to a traveling curved front. The existence of such traveling waves is shown in [14].

Theorem 1.1 ([14, Theorems 1.2 and 1.3])**.** *There exists a traveling wave solution* $u(x,y,t) = v_*(x, y-ct)$ *of* (1.1) *with*

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v_*(x,z) - v^-(x,z)| = 0, \tag{1.6}$$

$$v^-(x,z) < v_*(x,z).$$

This traveling curved front v^* *also satisfies the following property: for* $0 < \delta < 1$, *there exists a positive constant* γ *such that*

$$(v_*)_z \quad \geq \quad \gamma > 0 \quad \textit{if } -1+\delta \leq v_* \leq 1-\delta.$$

Moreover, if $u_0(x,y)$ *satisfies*

$$\lim_{R\to\infty} \sup_{x^2+y^2>R^2} |u_0(x,y) - v^-(x,y)| = 0, \tag{1.7}$$

$$v^-(x,y) \leq u_0(x,y), \tag{1.8}$$

then the solution $u(x,y,t;u_0)$ *of* (1.1) *satisfies*

$$\lim_{t\to\infty} \sup_{(x,y)\in\mathbb{R}^2} |u(x,y,t;u_0) - v_*(x,y-ct)| = 0.$$

If $v(x,z)$ is a solution of (1.5) and satisfies

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v(x,z) - v^-(x,z)| = 0, \tag{1.9}$$

then $v(x, y-ct)$ is called a *traveling curved front*. Actually the contours of the traveling curved front are seen as in Figure 2. Theorem 1.1 shows that the traveling curved front with the speed c exists for each $c > k$. Moreover, it says the convergence of the solutions satisfying (1.7) and (1.8) to the traveling curved front. The assumption (1.7) is essential to the speed of the traveling curved front. We can expect that the solution starting from v^- converges to v_* because the solution for $t > 0$ must satisfy (1.8). Due to the continuity of the initial functions we can also expect that the solution of (1.1) converges to v_* under the assumption (1.7) even if an initial state close to v^- is less than v^-. Actually we will show that the theorem holds true without the assumption (1.8). For this purpose we use supersolutions and subsolutions. In [14] a supersolution is constructed so that it is larger than any given initial state u_0 with (1.7). Now we need to introduce a subsolution that is smaller than any given u_0 with (1.7).

We construct such a subsolution in Section 2 and prove the following theorem.

Theorem 1.2 (global stability)**.** *If* $u_0(x,y)$ *satisfies* (1.7), *then the solution* $u(x,y,t;u_0)$ *of* (1.1) *satisfies*

$$\lim_{t\to\infty} \sup_{(x,y)\in\mathbb{R}^2} |u(x,y,t;u_0) - v_*(x,y-ct)| = 0.$$

This theorem implies that v_* is the unique traveling curved front.

2. **Construction of subsolutions.**

We introduce several constants for preparation. There exists a positive constant δ_1 $(0 < \delta_1 < 1/4)$ with

$$-f'(s) > \kappa_1 \qquad \text{if } |s+1| < 2\delta_1 \text{ or } |s-1| < 2\delta_1,$$

where

$$\kappa_1 \stackrel{\text{def}}{=} \frac{1}{2} \min\{-f'(-1), -f'(1)\} > 0.$$

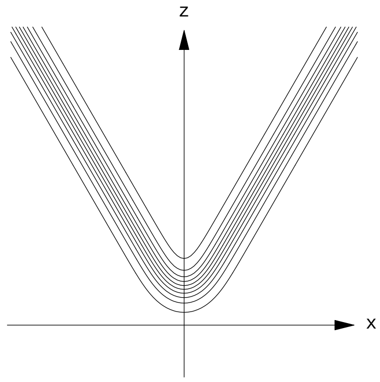


FIGURE 2. The contours of the numerical traveling curved front with the nonlinear function f as in (1.3), $a = 1/2$ and $m_* = \sqrt{3}$.

Since $\Phi(\mu)$ is monotone decreasing in μ, we can define positive constants A and B by

$$\Phi(-A) = 1 - \frac{\delta_1}{2}, \qquad \Phi(B) = -1 + \frac{\delta_1}{2},$$

respectively. Then

$$-1 + \frac{\delta_1}{2} < \Phi(\mu) < 1 - \frac{\delta_1}{2}$$

is equivalent to $-A < \mu < B$.

There exist positive constants γ_1 and K_1 with

$$\max\{|\Phi'(\mu)|, |\Phi''(\mu)|\} \le K_1 \operatorname{sech}(\gamma_1|\mu|) \tag{2.10}$$

for any $\mu \in \mathbb{R}$.

Take

$$\psi(\zeta) \stackrel{\text{def}}{=} \frac{1}{m_*\gamma_2} \log\Big(1 + \exp(\gamma_2\zeta)\Big). \tag{2.11}$$

The right-hand side equals

$$\frac{1}{2m_*} \int_{-\infty}^{\zeta} \left(1 + \tanh\left(\frac{1}{2}\gamma_2\zeta'\right)\right) d\zeta'.$$

There exist some constants $K_i > 0$ $(i = 2, 3, 4)$ and $\gamma_2 > 0$ so that $\psi(\zeta)$ satisfies

$$\max\left\{\left|\psi(\zeta) - \frac{\zeta}{m_*}\right|, \left|\psi'(\zeta) - \frac{1}{m_*}\right|\right\} \le K_2 \operatorname{sech}(\gamma_2\zeta) \qquad \text{for } \zeta \ge 0, \tag{2.12}$$

$$\max\{|\psi(\zeta)|, |\psi'(\zeta)|\} \le K_2 \operatorname{sech}(\gamma_2\zeta) \qquad \text{for } \zeta \le 0, \tag{2.13}$$

$$\max\{|\psi''(\zeta)|, |\psi'''(\zeta)|\} \le K_2 \operatorname{sech}(\gamma_2\zeta) \qquad \text{for } \zeta \in \mathbb{R}, \tag{2.14}$$

$$k - \frac{c\psi'(\zeta)}{\sqrt{1 + \psi'(\zeta)^2}} \ge K_3 \min\{1, \exp(-\gamma_2\zeta)\} \qquad \text{for } \zeta \in \mathbb{R}, \tag{2.15}$$

$$0 \le \frac{c}{\sqrt{1 + \psi'(\zeta)^2}} - km_* \le K_4 \min\{1, \exp(-\gamma_2\zeta)\} \qquad \text{for } \zeta \in \mathbb{R}. \tag{2.16}$$

Then we construct a subsolution to (1.5) as follows.

Theorem 2.1 (subsolution)**.** *There exist a positive constant ε_0 and a positive function $\alpha_0(\varepsilon)$ so that, for $0<\varepsilon<\varepsilon_0$ and $0<\alpha<\alpha_0(\varepsilon)$,*

$$v_1(x,z;\varepsilon,\alpha) \overset{\text{def}}{=} \Phi\left(\frac{\psi(\alpha z)-\alpha x}{\alpha\sqrt{1+\psi'(\alpha z)^2}}\right) - \varepsilon\,\mathrm{sech}(\gamma_2\alpha z) \tag{2.17}$$

is a subsolution to (1.5)*. Moreover, there exists a positive constant γ_3 with*

$$(v_1)_x \geq \gamma_3 \quad \textit{if } -1+\delta_1 \leq v_1 \leq 1-\delta_1. \tag{2.18}$$

Proof. We set

$$\begin{aligned} \xi &\overset{\text{def}}{=} \alpha x, \qquad \zeta \overset{\text{def}}{=} \alpha z,\\ \mu &\overset{\text{def}}{=} \frac{1}{\sqrt{1+\psi'(\alpha z)^2}}\left(\frac{1}{\alpha}\psi(\alpha z)-x\right),\\ \sigma(\zeta) &\overset{\text{def}}{=} \varepsilon\,\mathrm{sech}(\gamma_2\zeta). \end{aligned} \tag{2.19}$$

Then we have $v_1=\Phi(\mu)-\sigma(\zeta)$. The chain rule gives

$$\mu_x = -\frac{1}{\sqrt{1+\psi'(\zeta)^2}}, \qquad \mu_{xx}=0, \tag{2.20}$$

$$\mu_z = \frac{\psi'(\zeta)}{\sqrt{1+\psi'(\zeta)^2}} - \frac{\alpha\psi'(\zeta)\psi''(\zeta)}{1+\psi'(\zeta)^2}\mu, \tag{2.21}$$

$$\mu_{zz} = \frac{\alpha\psi''(\zeta)(1-\psi'(\zeta)^2)}{(1+\psi'(\zeta)^2)^{\frac{3}{2}}} + \frac{\alpha^2 g(\zeta)}{(1+\psi'(\zeta)^2)^2}\mu, \tag{2.22}$$

where

$$g(\zeta) = 3\psi'(\zeta)^2\psi''(\zeta)^2 - \left(1+\psi'(\zeta)^2\right)\left(\psi''(\zeta)^2+\psi'(\zeta)\psi'''(\zeta)\right).$$

Using these equalities and (1.2), we get

$$\begin{aligned} \mathcal{L}[v_1] &= -\frac{\Phi''(\mu)}{1+\psi'(\zeta)^2} - (\Phi'(\mu)\mu_z)_z - c\Phi'(\mu)\mu_z - f(\Phi(\mu)-\sigma(\zeta))\\ &\quad +\alpha^2\sigma''(\zeta)+c\alpha\sigma'(\zeta)+\Phi''(\mu)+k\Phi'(\mu)+f(\Phi) \end{aligned}$$

and thus

$$\begin{aligned} \mathcal{L}[v_1] &= \left(1-\frac{1}{1+\psi'(\zeta)^2}-\mu_z^2\right)\Phi''(\mu) - \mu_{zz}\Phi'(\mu) + (k-c\mu_z)\,\Phi'(\mu)\\ &\quad +f(\Phi(\mu))-f(\Phi(\mu)-\sigma(\zeta))+\alpha^2\sigma''(\zeta)+c\alpha\sigma'(\zeta)\\ &= I_1+I_2+I_3, \end{aligned}$$

$$\begin{aligned} I_1 &\overset{\text{def}}{=} \left(1-\frac{1}{1+\psi'(\zeta)^2}-\mu_z^2\right)\Phi''(\mu) + \left(-\mu_{zz}+\frac{c\alpha\psi'(\zeta)\psi''(\zeta)}{1+\psi'(\zeta)^2}\mu\right)\Phi'(\mu),\\ I_2 &\overset{\text{def}}{=} \left(k-\frac{c\psi'(\zeta)}{\sqrt{1+\psi'(\zeta)^2}}\right)\Phi'(\mu),\\ I_3 &\overset{\text{def}}{=} f(\Phi(\mu))-f(\Phi(\mu)-\sigma(\zeta))+\alpha^2\sigma''(\zeta)+c\alpha\sigma'(\zeta). \end{aligned}$$

By (2.21) and (2.22), we have

$$\begin{aligned} I_1 &= \alpha\left(-\left(\frac{\psi'(\zeta)\psi''(\zeta)}{1+\psi'(\zeta)^2}\right)^2\alpha\mu^2+\frac{2\psi'(\zeta)^2\psi''(\zeta)}{(1+\psi'(\zeta)^2)^{\frac{3}{2}}}\mu\right)\Phi''(\mu) \\ &\quad +\alpha\left(\frac{\psi''(\zeta)^2+\psi'(\zeta)\psi'''(\zeta)}{1+\psi'(\zeta)^2}\alpha\mu-\frac{3\psi'(\zeta)^2\psi''(\zeta)^2}{(1+\psi'(\zeta)^2)^2}\alpha\mu\right. \\ &\quad \left.-\frac{(1-\psi'(\zeta)^2)\psi''(\zeta)}{(1+\psi'(\zeta)^2)^{\frac{3}{2}}}+\frac{c\psi'(\zeta)\psi''(\zeta)}{1+\psi'(\zeta)^2}\mu\right)\Phi'(\mu) \end{aligned}$$

From (2.10)–(2.16), we have

$$|I_1|\le K_5\alpha\,\mathrm{sech}(\gamma_2\zeta)$$

with positive constant K_5 that is independent of ε and α. Now we assume

$$0<\varepsilon<\varepsilon_0\le\frac{\delta_1}{2}. \tag{2.23}$$

We have

$$I_2 \le -K_3|\Phi'(\mu)|\min\{1,\exp(-\gamma_2\zeta)\}. \tag{2.24}$$

Here we have

$$I_3=-\sigma(\zeta)\int_0^1(-f'(\Phi(\mu)-s\sigma(\zeta)))\,ds+\alpha\sigma''(\zeta)+c\alpha\sigma'(\zeta).$$

If α is small enough to satisfy

$$0<\alpha<\min\left\{1,\frac{\kappa_1}{2\gamma_2}\inf_{\zeta\in\mathbb{R}}\frac{\mathrm{sech}\,\zeta}{\gamma_2|\mathrm{sech}''\,\zeta|+|c||\mathrm{sech}'\,\zeta|}\right\}, \tag{2.25}$$

we get

$$|I_3| \le K_6\varepsilon\,\mathrm{sech}(\gamma_2\zeta) \qquad \text{if } -A<\mu<B, \tag{2.26}$$

$$I_3 \le -\frac{1}{2}\kappa_1\sigma(\zeta) \qquad \text{if } \mu\le -A \text{ or } \mu\ge B. \tag{2.27}$$

If $\mu\le -A$ or $\mu\ge B$, we have

$$\mathcal{L}[v_1]\le-\frac{1}{2}\kappa_1\sigma(\zeta)+K_5\alpha\,\mathrm{sech}(\gamma_2\zeta)<0$$

assuming

$$\alpha<\frac{\kappa_1}{2K_5}\varepsilon. \tag{2.28}$$

If $-A\le\mu\le B$, we have

$$\mathcal{L}[v_1]\le-K_3|\Phi'(\mu)|\min\{1,\exp(-\gamma_2\zeta)\}+(K_5\alpha+K_6\varepsilon)\,\mathrm{sech}(\gamma_2\zeta).$$

The right-hand side is negative if

$$K_5\alpha+K_6\varepsilon\le K_3\min_{-A\le\mu\le B}|\Phi'(\mu)|\inf_{\zeta\in\mathbb{R}}\frac{\min\{1,\exp(-\zeta)\}}{\mathrm{sech}\,\zeta}. \tag{2.29}$$

Thus we obtain

$$\mathcal{L}[v_1]<0 \qquad \text{in } \mathbb{R}^2$$

under (2.23), (2.25), (2.28) and (2.29).

Finally we prove (2.18). We have

$$(v_1)_x=\frac{-\Phi'(\mu)}{\sqrt{1+\psi'(\zeta)^2}}>0.$$

Since

$$\sup_{\zeta\in\mathbb{R}} |\psi'(\zeta)| \leq \frac{1}{m_*},$$

it suffices to show that $|\mu|$ is bounded if $-1+\delta_1 \leq v_1 \leq 1-\delta_1$. Recalling that $v_1 = \Phi(\mu) - \varepsilon \operatorname{sech}(\gamma_2\zeta)$ and $0 < \varepsilon < \delta_1/2$, we obtain

$$-1 + \frac{\delta_1}{2} \leq \Phi(\mu) \leq 1 - \frac{\delta_1}{2}$$

and thus $-A \leq \mu \leq B$ from $-1+\delta_1 \leq v_1 \leq 1-\delta_1$. Putting

$$\gamma_3 \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+(1/m_*)^2}} \min_{-A\leq\mu\leq B} |\Phi'(\mu)|,$$

we obtain (2.18). ∎

This lemma implies that

$$v_2(x,z) \stackrel{\text{def}}{=} v_1(-x,z) \tag{2.30}$$

is also subsolution of (1.5) with

$$-(v_2)_x \geq \gamma_3 \quad \text{if } -1+\delta_1 \leq v_2 \leq 1-\delta_1.$$

Lemma 2.2. *Let $w_j(x,z,t)$ $(j=1,2)$ be defined by*

$$\begin{aligned} w_1(x,z,t) &\stackrel{\text{def}}{=} v_1(x-\rho\delta(1-e^{-\beta t}),z) - \delta e^{-\beta t}, \\ w_2(x,z,t) &\stackrel{\text{def}}{=} v_2(x+\rho\delta(1-e^{-\beta t}),z) - \delta e^{-\beta t}. \end{aligned}$$

Then there exists a large positive constant ρ such that, for any $\delta \in (0,\delta_1/2]$, w_1 and w_2 are also subsolutions of (1.4).

Proof. First we show for w_1. Theorem 2.1 and simple calculation imply that

$$\begin{aligned} (w_1)_t + \mathcal{L}[w_1] &= -(v_1)_x\rho\delta\beta e^{-\beta t} + \delta\beta e^{-\beta t} - (v_1)_{xx} - (v_1)_{zz} \\ &\quad -c(v_1)_z - f(v_1 - \delta e^{-\beta t}) \\ &\leq -\delta e^{-\beta t}\left(\rho\beta(v_1)_x - \beta - \int_0^1 f'(v_1 - s\delta e^{-\beta t})ds\right). \end{aligned}$$

We get

$$\rho\beta(v_1)_x - \beta - \int_0^1 f'(v_1 - s\delta e^{-\beta t})\,ds \quad \geq \quad \beta\left(\gamma_3\rho - 1 - \frac{K_7}{\beta}\right)$$

when $-1+\delta_1 \leq v_1 \leq 1-\delta_1$ where

$$K_7 \stackrel{\text{def}}{=} \sup_{|s|\leq 2} |f'(s)|. \tag{2.31}$$

For $v_1 < -1+\delta_1$ or $v_1 > 1-\delta_1$, we have an estimate

$$\rho\beta(v_1)_x - \beta - \int_0^1 f'(v_1 \pm s\delta e^{-\beta t})ds \quad \geq \quad \kappa_1 - \beta.$$

We choose a small positive constant $\beta > 0$ and a large positive constant ρ satisfying

$$0 < \beta < \kappa_1, \quad \rho > \frac{\beta + K_7}{\beta\gamma_3}. \tag{2.32}$$

This immediately implies

$$(w_1)_t + \mathcal{L}[w_1] \leq 0.$$

Thus w_1 is a subsolution.

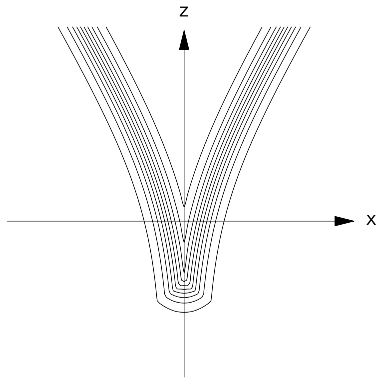


FIGURE 3. The graph of v_4 and its contours.

Since $v_2(x,z) = v_1(-x,z)$, we have

$$\begin{aligned}\mathcal{L}[w_2] &= (v_2)_x \rho\delta\beta e^{-\beta t} + \delta\beta e^{-\beta t} - (v_2)_{xx} - (v_2)_{zz} - c(v_2)_z - f(v_2 - \delta e^{-\beta t}) \\ &\le -\delta e^{-\beta t}\left(-\rho\beta(v_2)_x - \beta - \int_0^1 f'(v_2 - s\delta e^{-\beta t})ds\right).\end{aligned}$$

Similarly we can see that w_2 is also a subsolution. □

3. **Proof of Theorem 1.2.** For simplicity, we denote $w(x,z,t;u_0)$ by $w(x,z,t)$ hereafter. Define

$$\begin{aligned} v_3(x,z) &\stackrel{\text{def}}{=} v_*(x, z+M), \\ v_4(x,z) &\stackrel{\text{def}}{=} \max\{v_1(x,z;\varepsilon,\alpha), v_2(x,z;\varepsilon,\alpha), v_3(x,z)\}, \end{aligned} \tag{3.33}$$

where M is a positive constant specified later. Recall that w_1 and w_2 are defined in Lemma 2.2. We also set

$$\begin{aligned} w_3(x,z,t) &\stackrel{\text{def}}{=} v_3(x, z+\rho\delta(1-e^{-\beta t})) - \delta e^{-\beta t}, \\ w_4(x,z,t) &\stackrel{\text{def}}{=} \max\{w_1(x,z,t), w_2(x,z,t), w_3(x,z,t)\}. \end{aligned}$$

Note that $w_4(x,z,0) = v_4(x,z) - \delta$.

Lemma 3.1. *For $0<\varepsilon<\varepsilon_0$, $0<\alpha<\alpha_0(\varepsilon)$ and $M>0$, the limit of $w(x,z,t;v_4)$ as $t\to\infty$ exists in $L^\infty(\mathbb{R}^2)$ and the limit function*

$$v_{**}(x,z) \stackrel{\text{def}}{=} \lim_{t\to\infty} w(x,z,t;v_4)$$

satisfies

$$\begin{aligned} \mathcal{L}[v_{**}] &= 0, \\ v_{**} &\le v_*, \\ \lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v_{**} - v^-(x,z)| &= 0. \end{aligned}$$

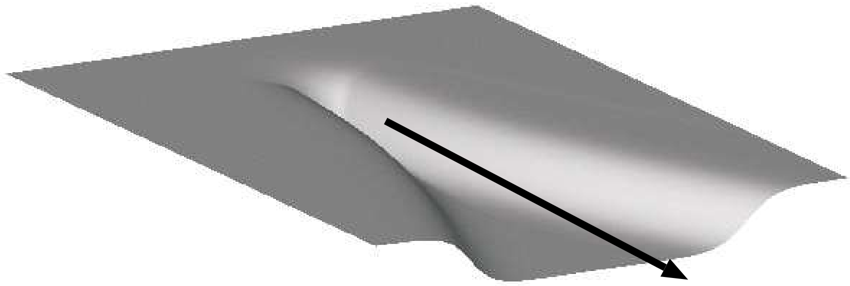


FIGURE 4. A bird's-eye view of v_4.

Proof. From (2.17), (2.30), (3.33) and the properties of ψ, we obtain

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v_1(x,z) - v^-(x,z)| = 0. \tag{3.34}$$

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v_4(x,z) - v^-(x,z)| = 0. \tag{3.35}$$

We note that

$$v_4 \le \max\{v_4, v^-\}.$$

Since v_4 is a subsolution of (1.5), we have

$$v_4 \le w(x,z,t;v_4) \le w(x,z,t;\max\{v_4,v^-\})$$

and thus

$$v_4 \le w(x,z,t;v_4) \le w(x,z,t;\max\{v_4,v^-\}). \tag{3.36}$$

From Theorem 1.1 we have

$$\lim_{t\to\infty} \sup_{(x,z)\in\mathbb{R}^2} |w(x,z,t;\max\{v_4,v^-\}) - v_*(x,z)| = 0.$$

Since v_4 is a subsolution with (3.36), $w(x,z,t;v_4)$ is monotone increasing in t and the limiting function v_{**} exists with

$$\mathcal{L}[v_{**}] = 0, \tag{3.37}$$

$$v_4 \le v_{**} \le v_*. \tag{3.38}$$

See Sattinger [18] for instance. Thus

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v_{**}(x,z) - v^-(x,z)| = 0$$

follows from (3.38), (3.35) and

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v_*(x,z) - v^-(x,z)| = 0.$$

∎

Lemma 3.2. *For any initial function u_0 satisfying* (1.7) *and any positive constant δ, there exist positive constants α, ε, T and M satisfying*

$$v_4(x,z) - \delta \le w(x,z,T) \qquad \textit{for } (x,z) \in \mathbb{R}^2.$$

Proof. We define $W_i(t)$ $(i = 1, 2)$ by

$$W_i'(t) = f(W_i(t)),$$

with initial conditions

$$\begin{aligned} W_1(0) &= \inf_{(x,z)\in\mathbb{R}^2} u_0(x,z), \\ W_2(0) &= \sup_{(x,z)\in\mathbb{R}^2} u_0(x,z). \end{aligned}$$

Then we have

$$W_1(t) \le w(x,z,t) \le W_2(t) \quad \text{for } (x,z) \in \mathbb{R}^2,\ t > 0$$

and thus

$$-1 \le \liminf_{t\to\infty} w(x,z,t) \le \limsup_{t\to\infty} w(x,z,t) \le 1 \qquad \text{for } (x,z) \in \mathbb{R}^2.$$

Namely, for any $\delta > 0$, there exists $T > 0$ with

$$-1 - \frac{\delta}{2} < w(x,z,t) \qquad \text{for } (x,z) \in \mathbb{R}^2,\ t \ge T. \tag{3.39}$$

By Lemma 4.5 in [14], we have

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |w(x,z,T) - v^-(x,z)| = 0. \tag{3.40}$$

By (1.6), (3.39) and (3.40) we can choose a large constant M with

$$v_3(x,z) - \delta = v_*(x, z+M) - \delta < w(x,z,T) \qquad \text{for } (x,z) \in \mathbb{R}^2.$$

By (3.40) there exists a positive constant R with

$$v^-(x,z) - \frac{\delta}{2} \le w(x,z,T) \qquad \text{for } x^2 + z^2 \ge R^2.$$

Using (3.34) we have

$$v_1(x,z;\varepsilon,\alpha) - \delta \le v^-(x,z) - \frac{\delta}{2} \le w(x,z,T) \qquad \text{for } x^2 + z^2 \ge R^2.$$

For $x^2 + z^2 \le R^2$, there exists a small positive constant α with

$$\begin{aligned} \frac{1}{\sqrt{1+\psi'(\alpha z)^2}} \left(\frac{1}{\alpha}\psi(\alpha z) - x\right) &\ge \frac{1}{\sqrt{1+\max_{|z|\le R}\psi'(\alpha z)^2}} \left(\frac{1}{\alpha}\min_{|z|\le R}\psi(\alpha z) - x\right) \\ &\ge \Phi^{-1}\left(-1 + \frac{\delta}{2}\right). \end{aligned}$$

Thus we have

$$v_1(x,z) - \delta \le w(x,z,T) \qquad \text{for } (x,z) \in \mathbb{R}^2. \tag{3.41}$$

Similarly we also have (3.41) with replacing v_1 by v_2. ∎

We use $\varepsilon > 0$ and $\alpha > 0$ as in Lemma 3.2. Hereafter we denote $v_j(x,z;\varepsilon,\alpha)$ simply by $v_j(x,z)$ for $j = 1, 2$ respectively.

Proposition 3.3. *The following holds true:*

$$v_{**}(x,z) = v_*(x,z)$$

for all x and z.

Proof. By the definition of v_{**} and Lemma 3.1, we have

$$v_*(x, z+M) \le v_{**}(x,z) \le v_*(x,z) \qquad \text{in } \mathbb{R}^2.$$

Define

$$\Lambda \stackrel{\text{def}}{=} \inf\{\lambda \ : \ v_*(x, z+\lambda) \le v_{**}(x,z)\}$$

Thus we have $0 \le \Lambda \le M$. We show $\Lambda = 0$ by contradiction. Assume $\Lambda > 0$. By (1.7), $v_*(\cdot, \cdot + \Lambda) \not\equiv v_{**}(\cdot,\cdot)$. By the strong maximum principle, we have

$$v_*(x, z+\Lambda) < v_{**}(x,z) \qquad \text{in } \mathbb{R}^2. \tag{3.42}$$

As in [14, (5.4)], we can take R_* so large that

$$2\rho \sup_{\left|z - m_*|x|\right| > R_* - \rho\delta_1} |(v_*)_z(x, z+\Lambda)| < 1.$$

By (3.42) and $\max\{v_1(x,z), v_2(x,z)\} \le v_{**}(x,z) \le v_*(x,z)$, there is a small positive constant $h(< \delta_1/2)$ with

$$v_*(x, z+\Lambda - 2\rho h) < v_{**}(x,z) \qquad \text{in } D, \tag{3.43}$$

where

$$D \stackrel{\text{def}}{=} \left\{(x,z) \ : \ \left|z - m_*|x|\right| \le R_*\right\}.$$

In $\mathbb{R}^2 \backslash D$, we have

$$v_*(x, z+\Lambda-2\rho h) - v_*(x, z+\Lambda) = -2\rho h \int_0^1 (v_*)_z(x, z+\Lambda - 2s\rho h) ds \le h.$$

This implies

$$v_*(x, z+\Lambda - 2\rho h) - h \le v_*(x, z+\Lambda) \qquad \text{in } \mathbb{R}^2\backslash D. \tag{3.44}$$

Combining (3.42) – (3.44), we have

$$v_*(x, z+\Lambda-2\rho h) - h \le v_{**}(x,z) \qquad \text{in } \mathbb{R}^2.$$

Now [14, Lemma 4.4] says that $v_*(x, z+\Lambda - 2\rho h + \rho h(1-e^{-\beta t})) - he^{-\beta t}$ is a subsolution of (1.4). Thus, for any $t > 0$,

$$v_*(x, z+\Lambda - 2\rho h + \rho h(1-e^{-\beta t})) - he^{-\beta t} \le v_{**}(x,z) \qquad \text{in } \mathbb{R}^2.$$

Letting t tend to infinity, we have

$$v_*(x, z+\Lambda - \rho h) \le v_{**}(x,z) \qquad \text{in } \mathbb{R}^2.$$

This contradict the definition of Λ. Thus we have $\Lambda = 0$ and $v_* \equiv v_{**}$. ∎

Proof of Theorem 1.2. We denote $w(x,z,t)$ by $w(x,z,t;u_0)$ for simplicity. Note that

$$\min\{u_0, v^-\} \le u_0 \le \max\{u_0, v^-\}.$$

Theorem 1.1 says that, for any given $\delta > 0$, there exists $T > 0$ with

$$\sup_{(x,z)\in\mathbb{R}^2} |w(x,z,t;\max\{u_0,v^-\}) - v_*(x,z)| < \delta$$

for $t > T$. Using this inequality and the comparison principle, we have

$$w(x,z,t) < w(x,z,t;\max\{u_0,v^-\}) < v_*(x,z) + \delta$$

for all $(x,z) \in \mathbb{R}^2$, $t > T$. Thus we get

$$\limsup_{t\to\infty} w(x,z,t) \le v_*(x,z)$$

for $(x,z) \in \mathbb{R}^2$. Then it suffices to show that

$$\liminf_{t\to\infty} w(x,z,t;\min\{u_0,v^-\}) = v_*(x,z)$$

for $(x,z) \in \mathbb{R}^2$ to prove the theorem. We use $\min\{u_0,v^-\}$ instead of u_0. Hereafter we assume $u_0 \le v^-$ without loss of generality. Lemma 3.2 immediately shows

$$w_4(x,z,t) \le w(x,z,T+t) \le v_*(x,z) \qquad \text{for } (x,z) \in \mathbb{R}^2$$

and then

$$w(x,z,s;w^t) \le w(x,z,T+t+s) \le v_*(x,z) \qquad \text{for } (x,z) \in \mathbb{R}^2$$

where

$$w^t(x,z) \stackrel{\text{def}}{=} w_4(x,z,t).$$

By this inequality and Lemma 2.2, [14, Lemma 4.1] and by letting t tend to infinity, we obtain

$$w(x,z,s;v_5^\delta) \le \liminf_{t\to\infty} w(x,z,T+t+s) \le \limsup_{t\to\infty} w(x,z,T+t+s) \le v_*(x,z)$$

for $(x,z) \in \mathbb{R}^2$, where

$$v_5^\delta(x,z) \stackrel{\text{def}}{=} \max\{v_1(x-\rho\delta,z), v_2(x+\rho\delta,z), v_*(x,z+M+\rho\delta)\}.$$

Since δ can be chosen arbitrarily small, we have

$$w(x,z,s;v_5^0) \le \liminf_{t\to\infty} w(x,z,T+t) \le \limsup_{t\to\infty} w(x,z,T+t) \le v_*(x,z)$$

for $(x,z) \in \mathbb{R}^2$. Thus letting $s \to \infty$ again, we have

$$v_{**}(x,z) \le \liminf_{t\to\infty} w(x,z,t) \le \limsup_{t\to\infty} w(x,z,t) \le v_*(x,z).$$

Proposition 3.3 implies that $w(x,z,t)$ converges to $v_*(x,z)$ as $t \to \infty$. □

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