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STABILITY OF A TRAVELING WAVE IN CURVATURE FLOWS FOR SPATIALLY NON-DECAYING INITIAL PERTURBATIONS

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Abstract. This paper is concerned with the long time behavior for the evolution of a curve governed by the curvature flow with constant driving force in two-dimensional space. Especially, the asymptotic stability of a traveling wave whose shape is a line is studied. We deal with moving curves represented by the entire graphs on the x-axis. By studying the Cauchy problem, the asymptotic stability of traveling waves with spatially decaying initial perturbations and the convergence rate are obtained. Moreover we establish the stability result where initial perturbations do not decay to zero but oscillate at infinity. In this case, we prove that one of the sufficient conditions for asymptotic stability is that a given perturbation is asymptotic to an almost periodic function in the sense of Bohr at infinity. Our results hold true with no assumptions on the smallness of given perturbations, and include the curve shortening flow problem as a special case.

1. **Introduction.** In this paper, we study the long time behavior of solutions to a Cauchy problem of the form

$$\frac{u_t}{\sqrt{1+u_x^2}} = \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} + k, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{2}$$

where $k \in \mathbb{R}$ is a given constant. Especially, we are concerned with the stability of a traveling wave whose shape is a line, that is, $u(x,t) = kt$.

Our motivation to study this problem comes from the theory of interfacial phenomena, which is a topic of considerable interest in applied mathematics. If two different chemical or physical states coexist, interfaces appear as a phase boundary between them, and the analysis of their motion is quite important.

Here we consider an interface $\Gamma(t)$ in $\mathbb{R}^2$ governed by a curvature flow with constant driving force $k \in \mathbb{R}$. Namely, we have

$$V = H + k, \tag{3}$$

where V and H are the normal velocity and the curvature of $\Gamma(t)$, respectively. This model appears in several fields. One of them is the study of dynamics of interfaces in an excitable media, for example, Belousov-Zhabotinsky reaction [3, 15]. Equation (3) also appears in the dynamics of interfaces in the Allen-Cahn equations. See [4]

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for instance. Moreover it appears in the reaction-diffusion systems of a competition type. See [9].

In this paper, we deal with the case where an interface $\Gamma(t)$ in $\mathbb{R}^2$ is represented by a graph on the x-axis; in other words, an initial curve $\Gamma(0)$ is given by a function $y = u_0(x), x \in \mathbb{R}$, and a moving curve $\Gamma(t), t > 0$ is expressed by $y = u(x,t), x \in \mathbb{R}, t > 0$. Under these assumptions, (3) is rewritten as the initial value problem (1) - (2).

A moving curve $\Gamma(t)$ is called a *traveling wave* of $V = H + k$ with the velocity $|\boldsymbol{v}|$, if $\Gamma(t)$ satisfies

$$\Gamma(t) = \Gamma(0) + \boldsymbol{v}t, \quad t > 0$$

for some vector $\boldsymbol{v} \in \mathbb{R}^2$. We assume that a traveling wave moves toward the y-axis without loss of generality. When $k \neq 0$, a stationary circle with radius $1/k$ is a traveling wave with $|\boldsymbol{v}| = 0$. Except for this circle and self-crossing ones, the traveling waves of $V = H + k\,(k \neq 0)$ are represented by a graphical form, after an appropriate rotation of the coordinates, and is one of the following two cases [3, 12]:

(i) *Traveling line*, that is, a line with velocity k, for example, $u(x,t) = kt$;

(ii) *V-shaped front*, which is convex and is asymptotic to the traveling lines as $|x| \to \infty$.

On the other hand, $k = 0$ means the so-called *curve shortening flow.* In this case, every line is a stationary wave, and there exists a traveling wave that is called the *Grim Reaper.* Furthermore the curve shortening flow has an extracting self-similar solution, which is convex and is asymptotic to the stationary lines as $|x| \to \infty$ as in [8, 11].

In this paper, letting k be an arbitrary constant, we show the stability of the traveling (or stationary) line $u(x,t) = kt$ for the Cauchy problem (1) - (2). We note that our results stated below include the case of $k = 0$, that is, the curve shortening flow case.

Now we state related works. Chou and Kwong [5] proved the existence of global solutions of (1) - (2) without the restriction of growth order. In [13], the authors showed that the V-shaped fronts are asymptotically stable with respect to spatially decaying perturbations. The proof of this result is given by constructing supersolutions and subsolutions. For the stability of extracting self-similar solutions in the curve shortening flow, see [8, 11].

To study the stability of a traveling line, we consider the function $\overline{u}(x,t) = u(x,t) - kt$. Since $\overline{u}_t = u_t - k$, we have a quasi-linear parabolic equation

$$\overline{u}_t = \frac{\overline{u}_{xx}}{1 + \overline{u}_x^2} + k(\sqrt{1 + \overline{u}_x^2} - 1) = (\arctan \overline{u}_x)_x + k(\sqrt{1 + \overline{u}_x^2} - 1).$$

Here we note that every constant, that is, $\overline{u}(x,t) \equiv (\text{const.})$, is a stationary solution. For the Cauchy problem of the heat equation, it is well known that every constant is a stationary solution and is asymptotically stable for spatially decaying perturbations. On the other hand, Collet and Eckmann [7] showed the counterexample for the asymptotic stability of constant solutions of the heat equation, where the bounded initial perturbation does not decay but oscillate slower and slower as $|x| \to \infty$. It is also known that the V-shaped front of the Cauchy problem (1) - (2) is not always asymptotically stable for similar perturbations as in [13]. Moreover Poláčik and Yanagida [16] studied related works on a supercritical semilinear diffusion equation.

This is the difficulty in considering asymptotic stability in the case where perturbations do not decay at infinity.

In what follows, $L^1(\mathbb{R})$, $L^\infty(\mathbb{R})$, $W^{2,\infty}(\mathbb{R})$, denote the Lebesgue or Sobolev spaces. For $\gamma \in (0,1)$, $C^\gamma(\mathbb{R})$ denotes the Hölder space, that is, the space of functions that are uniformly bounded and uniformly Hölder continuous with exponent γ on $\mathbb{R}$. $C^{2+\gamma}(\mathbb{R})$ means the space of functions with $u, u', u'' \in C^\gamma(\mathbb{R})$. For a domain $R_T = \mathbb{R} \times [0,T]$, $C^{\gamma,\gamma/2}(R_T)$ denotes the space of functions that are uniformly bounded and uniformly Hölder continuous with exponent γ and $\gamma/2$ with respect to x and t, respectively on R_T. $C^{2+\gamma,1+\gamma/2}(R_T)$ means the space of functions with $u, u_x, u_{xx}, u_t \in C^{\gamma,\gamma/2}(R_T)$.

First, we show the result on the asymptotic stability in the case where initial perturbations decay as $|x| \to \infty$.

Theorem 1. *Let $\gamma \in (0,1)$ be an arbitrary number. Suppose that $u_0 \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} u_0(x) = 0$. Then there exists a unique classical solution $u(x,t)$ to the Cauchy problem* (1) - (2) *up to $t = +\infty$. Moreover it holds that*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - kt| = 0.$$

Especially, if u_0 belongs to $C^{2+\gamma}(\mathbb{R}) \cap L^1(\mathbb{R})$, the solution $u(x,t)$ satisfies the estimate

$$\sup_{x\in\mathbb{R}} |u(x,t) - kt| \le C(1+t)^{-\frac{1}{2}}, \quad t > 0,$$

where C is a constant depending only on k and u_0.

The proof of Theorem 1 is given in Section 2. Next we show the result including the perturbations which do not decay at infinity. To this purpose we recall the definition of an *almost periodic function.*

Definition 1. A continuous function $f(x)$ for $x \in \mathbb{R}$ with values in $\mathbb{R}$ is called an *almost periodic function* in the sense of Bohr if, for every $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that, for every $p \in \mathbb{R}$, an interval $[\,p, p + l(\varepsilon)]$ contains at least one number q with

$$|f(x-q) - f(x)| < \varepsilon \quad \text{for all} \quad x \in \mathbb{R}. \tag{4}$$

For any almost periodic function f, there exists a *mean* $\mathcal{M}\{f\}$ defined by

$$\mathcal{M}\{f\} = \lim_{R\to\infty} \frac{1}{R} \int_s^{s+R} f(x)\,dx,$$

where the convergence is uniform with respect to $s \in \mathbb{R}$, and the limit is independent of s.

By this definition, every periodic function is an almost periodic function. Let $\mathcal{B}$ be the class of almost periodic functions. Here we mention some properties of the class $\mathcal{B}$. Each $f \in \mathcal{B}$ is bounded and uniformly continuous on $\mathbb{R}$. If $f, g \in \mathcal{B}$, then both $f(x) + g(x)$ and $f(x)g(x)$ belong to $\mathcal{B}$, where $\mathcal{M}\{f+g\} = \mathcal{M}\{f\} + \mathcal{M}\{g\}$ holds true. For a sequence $\{f_n\} \subset \mathcal{B}$, the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ belongs to $\mathcal{B}$ if the convergence is uniform on $\mathbb{R}$, where $\mathcal{M}\{f\} = \lim_{n\to\infty} \mathcal{M}\{f_n\}$ holds true. Note that a non-periodic function $f(x) = \sin x + \sin\sqrt{2}x$ belongs to $\mathcal{B}$ with $\mathcal{M}\{f\} = 0$. For further details, see [2, 6, 1] for instance.

Theorem 2. *Let $\gamma \in (0,1)$ be an arbitrary number. Assume that $u_0 \in C^{2+\gamma}(\mathbb{R})$ is an almost periodic function. Then there exists a unique classical solution $u(x,t)$ to the Cauchy problem* (1) - (2) *up to $t = +\infty$. Moreover it holds that*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - kt - \mu| = 0$$

for a constant μ. For each u_0, the constant μ is a nondecreasing function of $k \in \mathbb{R}$ with

$$\inf_{x\in\mathbb{R}} u_0(x) \le \mu \le \sup_{x\in\mathbb{R}} u_0(x).$$

Especially, $\mu = \mathcal{M}\{u_0\}$ holds true if $k = 0$.

Finally we show a result that gives uniform bounds for perturbations depending only on the asymptotic behavior of the given initial perturbation at infinity. This result extends a class of initial values for which the stability is determined.

Theorem 3. *Let $\gamma \in (0,1)$ be an arbitrary number. For some functions $\varphi_*(x)$ and $\varphi^*(x)$ that belong to $C^{2+\gamma}(\mathbb{R})$, let $u_*(x,t)$ and $u^*(x,t)$ be the solutions of* (1) - (2) *up to $t = +\infty$ with the initial values φ_* and φ^*, respectively. Assume that u_* and u^* satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u_*(x,t) - kt - \mu_*| = 0, \quad \lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u^*(x,t) - kt - \mu^*| = 0$$

for some constants μ_* and μ^*. Then for an initial value $u_0 \in C^{2+\gamma}(\mathbb{R})$ with

$$\lim_{x\to-\infty} |u_0(x) - \varphi_*(x)| = 0, \quad \lim_{x\to+\infty} |u_0(x) - \varphi^*(x)| = 0,$$

the solution $u(x,t)$ of (1) - (2) *satisfies*

$$\lim_{t\to\infty} \inf_{x\in\mathbb{R}} (u(x,t) - kt) = \min\{\mu_*, \mu^*\}, \quad \lim_{t\to\infty} \sup_{x\in\mathbb{R}} (u(x,t) - kt) = \max\{\mu_*, \mu^*\}.$$

By this theorem, we can extend Theorem 2 to the case where the initial value is asymptotic to an almost periodic function as $|x| \to \infty$.

Corollary 1. *Let $\gamma \in (0,1)$ be an arbitrary number. Assume that $f \in C^{2+\gamma}(\mathbb{R})$ is an almost periodic function. And assume that $g \in C^{2+\gamma}(\mathbb{R})$ satisfies $\lim_{|x|\to\infty} g(x) = 0$. Then there exists a unique classical solution $u(x,t)$ to the Cauchy problem* (1) - (2) *with the initial value $u_0 = f + g$ up to $t = +\infty$. Moreover it holds that*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - kt - \mu| = 0$$

for a constant μ that depends only on k and f, and is independent of g. Especially, $\mu = \mathcal{M}\{f\}$ holds true if $k = 0$.

This corollary states that one of the sufficient conditions for asymptotic stability of a traveling line is that a perturbation converges to an almost periodic function asymptotically as $|x| \to \infty$. The proofs of Theorem 2 and 3 are given in Section 3.

In Section 4, we give examples of perturbations for which the traveling lines lose the asymptotic stability. That is, if these perturbations are added on the traveling lines, they never recover to the original shape.

2. **Stability with spatially decaying perturbations.** As mentioned in the introduction, we consider the function $\overline{u}(x,t) = u(x,t) - kt$ instead of $u(x,t)$ in order to study the long time behavior of solutions to the Cauchy problem (1) - (2). Here we write $u(x,t)$ for $\overline{u}(x,t) = u(x,t) - kt$ for simplicity. Then we have

$$u_t = (\arctan u_x)_x + k(\sqrt{1+u_x^2} - 1), \quad x \in \mathbb{R},\ t > 0, \tag{5}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{6}$$

Hereafter, we shall investigate the long time behavior for this problem. We first show the global existence and some estimates for the solutions with initial value of class $C^{2+\gamma}(\mathbb{R})$. In what follows, we always assume that the initial value belongs to $C^{2+\gamma}(\mathbb{R})$ even if it is not mentioned specifically. Here $\gamma \in (0,1)$ is an arbitrary constant. The following proposition plays an important role in this paper.

Proposition 1. *Assume that $u_0 \in C^{2+\gamma}(\mathbb{R})$. Then there exists a classical solution $u(x,t)$ to the Cauchy problem (5) - (6) that belongs to $C^{2+\gamma,1+\gamma/2}(R_T)$, $R_T = \mathbb{R} \times [0,T]$ for any $T > 0$, and satisfies the following estimates:*

$$\sup_{x\in\mathbb{R},\,t>0} |u(x,t)| \ \le\ \|u_0\|_{L^\infty(\mathbb{R})}, \qquad \sup_{x\in\mathbb{R},\,t>0} |u_x(x,t)| \ \le\ \|u_0'\|_{L^\infty(\mathbb{R})},$$
$$\sup_{x\in\mathbb{R},\,t>0} |u_{xx}(x,t)| \ \le\ C_1, \qquad \sup_{x\in\mathbb{R},\,t>0} |u_t(x,t)| \ \le\ C_1.$$

Here C_1 is a constant depending only on k and $\|u_0\|_{W^{2,\infty}(\mathbb{R})}$. Moreover the comparison principle is valid, that is, if the solutions $u_1(x,t)$ and $u_2(x,t)$ to the problem (5) - (6) satisfy

$$u_1(x,0) \le u_2(x,0) \quad \textit{for} \quad x \in \mathbb{R},$$

then one has

$$u_1(x,t) \le u_2(x,t) \quad \textit{for} \quad x \in \mathbb{R},\ t > 0.$$

Thus the solution is unique for initial value of class $C^{2+\gamma}(\mathbb{R})$.

Proof. Let $M = \|u_0'\|_{L^\infty(\mathbb{R})}$. We first define a function $\eta(p)$ with

(i) $\eta(p)$ is of class $C^\infty(\mathbb{R})$ and satisfies

$$\eta(p) = \arctan p \quad \text{for} \quad |p| \le M,$$
$$|\eta(p)| \le |p| \quad \text{for} \quad p \in \mathbb{R};$$

(ii) $\eta'(p)$ is bounded uniformly for $p \in \mathbb{R}$ and satisfies $\inf_{p\in\mathbb{R}} \eta'(p) \ge \delta > 0$ for some constant δ;

(iii) $\eta''(p)$ is bounded uniformly for $p \in \mathbb{R}$.

Let any $T > 0$ be fixed. Now we consider

$$u_t = (\eta(u_x))_x + k(\sqrt{1+u_x^2} - 1), \quad x \in \mathbb{R},\ 0 < t < T, \tag{7}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{8}$$

By the definition of $\eta(p)$, we can apply the general theory for quasi-linear parabolic equations with principal part in divergence form, and obtain the classical solution $u(x,t) \in C^{2+\gamma,1+\gamma/2}(R_T)$ to the problem (7) - (8). See Chapter V of [14] for instance. If $u(x,t)$ satisfies

$$\sup_{x\in\mathbb{R},\,0<t\le T} |u_x(x,t)| \ \le\ M,$$

this solution is also the solution of (5) - (6) since $\eta(u_x) = \arctan u_x$ when $|u_x| \le M$. In order to obtain this estimate, we differentiate the equation with respect to x. If we set $v = u_x$, the function $v(x,t)$ satisfies

$$v_t = \eta'(v)v_{xx} + \left(\eta''(v)v_x + k\frac{v}{\sqrt{1+v^2}}\right)v_x, \quad x \in \mathbb{R},\ 0 < t < T, \tag{9}$$

$$v(x,0) = u_0'(x), \qquad x \in \mathbb{R}. \tag{10}$$

By the definition of $\eta(p)$, the equation (9) is uniformly parabolic and has coefficients that are continuous and bounded in R_T since u belongs to $C^{2+\gamma,1+\gamma/2}(R_T)$. Now we can apply the maximum principle. From the fact that $v \equiv \pm M$ are stationary solutions of (9) - (10), we obtain

$$\sup_{x\in\mathbb{R},\,0<t\le T} |v(x,t)| \;\le\; M.$$

Since $T > 0$ is arbitrary, we have the global solution to the problem (5) - (6), which has a uniform bound for $|u_x|$.

Next we prove the comparison principle. We define $w = u_1 - u_2$ for the solutions u_1, u_2 of (5) - (6) with the initial value $u_{10}, u_{20} \in C^{2+\gamma}(\mathbb{R})$, respectively. Then $w(x,t)$ belongs to $C^{2+\gamma,1+\gamma/2}(R_T)$ for any $T > 0$ and satisfies the equation

$$w_t = \alpha(x,t)w_{xx} + \alpha_x(x,t)w_x + k\,\beta(x,t)w_x, \tag{11}$$

where

$$\alpha(x,t) = \int_0^1 A'(\tau(u_1)_x + (1-\tau)(u_2)_x)\,d\tau,$$

$$\beta(x,t) = \int_0^1 B'(\tau(u_1)_x + (1-\tau)(u_2)_x)\,d\tau$$

and $A(p) = \arctan p$, $B(p) = \sqrt{1+p^2}$. The maximum principle is valid for (11) on R_T. Consequently, we have $w(x,t) \le 0$ if $w(x,0) \le 0$ for $x \in \mathbb{R}, 0 < t < T$. From this result, a uniform bound for $|u|$ also follows since $u \equiv \pm\|u_0\|_{L^\infty(\mathbb{R})}$ are stationary solutions of (5) - (6).

Finally we show uniform bounds for $|u_{xx}|$ and $|u_t|$. For any fixed $T > 0$, we define $u^h(x,t)$ as

$$u^h(x,t) = \frac{1}{h}\,(u(x,t+h) - u(x,t))$$

on $R_{T-h} = \mathbb{R} \times [0, T-h]$ for $h \in (0,T)$. Then $u^h(x,t)$ satisfies

$$u^h_t = \alpha(x,t,h)u^h_{xx} + \alpha_x(x,t,h)u^h_x + k\,\beta(x,t,h)u^h_x, \tag{12}$$

where we put

$$\alpha(x,t,h) = \int_0^1 A'(\rho(x,t,h,\tau))\,d\tau, \quad \beta(x,t,h) = \int_0^1 B'(\rho(x,t,h,\tau))\,d\tau$$

with $\rho(x,t,h,\tau) = \tau u_x(x,t+h) + (1-\tau)u_x(x,t)$. Recall that $A(p) = \arctan p$, $B(p) = \sqrt{1+p^2}$. Applying the maximum principle to (12), we have

$$\sup_{x\in\mathbb{R},\,0<t<T-h} |u^h(x,t)| \;\le\; \sup_{x\in\mathbb{R}} |u^h(x,0)|.$$

Passing to the limit as $h \to +0$, we obtain a uniform bound for $|u_t|$ as follows:

$$\sup_{x\in\mathbb{R},\,0<t\le T} |u_t(x,t)| \;\le\; \sup_{x\in\mathbb{R}} |u_t(x,0)| = \left\|(\arctan u_0')' + k\left(\sqrt{1+(u_0')^2} - 1\right)\right\|_{L^\infty(\mathbb{R})}.$$

By virtue of the uniform bounds for $|u_x|$ and $|u_t|$, a uniform bound for $|u_{xx}|$ also follows. Thus the proof of Proposition 1 is completed. $\square$

Now we give a proof of Theorem 1. It is based on the uniform bounds for $|u_{xx}|$ and $|u_x|$. We construct subsolutions and supersolutions.

Proof of Theorem 1. The existence of a solution globally in time follows from Proposition 1. It suffices to show the asymptotic stability of the solution $u(x,t)$ to the problem (5) - (6). First we show an upper estimate.

From Proposition 1 and the inequality $\sqrt{1+p^2}-1 \leq p^2$, we have

$$\begin{aligned} 0 = u_t - \frac{u_{xx}}{1+u_x^2} - k(\sqrt{1+u_x^2}-1) \ &\geq \ u_t - u_{xx} - \left(\frac{|u_{xx}|}{1+u_x^2} + |k|\right) u_x^2 \\ &\geq u_t - u_{xx} - c\, u_x^2 \end{aligned}$$

for some positive constant c. Then the solution $u(x,t)$ of (5) - (6) is a subsolution of

$$\widetilde{v}_t = \widetilde{v}_{xx} + c\widetilde{v}_x^2, \quad x \in \mathbb{R},\ t > 0, \tag{13}$$

$$\widetilde{v}(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{14}$$

Here we define the function $\widetilde{w}(x,t) = \exp(c\,\widetilde{v}(x,t))$. Then it is easily found that $\widetilde{w}$ satisfies the following Cauchy problem of the heat equation

$$\begin{aligned} &\widetilde{w}_t = \widetilde{w}_{xx}, && x \in \mathbb{R},\ t > 0, \\ &\widetilde{w}(x,0) = \exp(cu_0(x)), && x \in \mathbb{R}. \end{aligned}$$

Now we obtain an explicit expression of $\widetilde{v}(x,t)$ as

$$\widetilde{v}(x,t) = \frac{1}{c} \log \left(\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp(cu_0(y)) \exp\left(-\frac{(x-y)^2}{4t}\right) dy \right).$$

Note that $\exp(cu_0(y)) \geq \exp(-c\|u_0\|_{L^\infty(\mathbb{R})}) > 0$. We rewrite this expression as

$$\widetilde{v}(x,t) = \frac{1}{c} \log(1 + h(x,t)).$$

where

$$h(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} (\exp(cu_0(y)) - 1) \exp\left(-\frac{(x-y)^2}{4t}\right) dy.$$

Here if $u_0(x)$ decays to zero as $|x| \to \infty$, then $\exp(cu_0(x)) - 1$ also decays to zero as $|x| \to \infty$. Thus we have $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |h(x,t)| = 0$, and hence

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |\widetilde{v}(x,t)| = 0.$$

Moreover if u_0 belongs to $L^1(\mathbb{R})$, then $\exp(cu_0(y)) - 1$ also belongs to $L^1(\mathbb{R})$. Thus we obtain $\sup_{x\in\mathbb{R}} |h(x,t)| \leq C_2(1+t)^{-1/2}$ for some constant C_2 depending only on k and u_0. This implies

$$\sup_{x\in\mathbb{R}} |\widetilde{v}(x,t)| \ \leq \ C_3(1+t)^{-\frac{1}{2}}, \quad t > 0$$

for a constant C_3 depending only on k and u_0. This results for the problem (13) - (14) gives the upper estimate of $u(x,t)$ of (5) - (6). A lower estimate of u can be obtained in a similar and easier way. $\square$

3. **Stability with spatially non-decaying perturbations.** In this section we give proofs of Theorem 2 and Theorem 3. We first state some lemmas that play fundamental roles in the proof of Theorem 2.

Lemma 1. *Let $M > 0$, $m > 0$, $T > 0$, and $L > 0$ be given constants. Let $u(x,t)$ be the solution to the Cauchy problem* (5) - (6) *with*

$$\|u_0'\|_{L^\infty(\mathbb{R})} \le M, \quad \sup_{x\in\mathbb{R}} u_0(x) \le m, \tag{15}$$

$$u(a,t) \le 0 \ \ \text{and} \ \ u(b,t) \le 0 \quad \text{for} \quad 0 \le t \le T \tag{16}$$

for $a, b \in \mathbb{R}$ with $a < b$ and $b - a \le L$. Then there exists a positive constant λ depending only on M, m, T and L with

$$\max_{a\le x\le b} u(x,T) \ \le \ m - \lambda, \tag{17}$$

where λ depends continuously on $m \in (0, +\infty)$ for any fixed M, T and L.

Proof. Taking M large enough if necessary, we can assume $m/M < L/2$ without loss of generality. On the interval $[0, L]$ we consider a Dirichlet problem given by

$$\widehat{v}_t = (\arctan \widehat{v}_x)_x + k(\sqrt{1 + (\widehat{v}_x)^2} - 1), \quad 0 < x < L, \ 0 < t < T, \tag{18}$$

$$\widehat{v}(0,t) = 0, \ \ \widehat{v}(L,t) = 0, \qquad 0 \le t \le T, \tag{19}$$

$$\widehat{v}(x,0) = \varphi(x), \qquad 0 \le x \le L. \tag{20}$$

Here we take the initial value $\varphi(x) \in C^{2+\gamma}[0, L]$ with

$$\varphi(0) = 0, \ \ \varphi(L) = 0, \tag{21}$$

$$(\arctan \varphi'(x))' + k(\sqrt{1 + \varphi'(x)^2} - 1)|_{x=0,L} = 0, \tag{22}$$

$$\begin{array}{ll} Mx \le \varphi(x) \le m, & x \in [\,0, \ \frac{m}{M}\,], \\ \varphi(x) = m, & x \in [\,\frac{m}{M}, \ L - \frac{m}{M}\,], \\ M(L - x) \le \varphi(x) \le m, & x \in [\,L - \frac{m}{M}, \ L\,]. \end{array} \tag{23}$$

The conditions (21) - (22) are the compatibility condition of first order. Similarly as in the proof of Proposition 1, we find that the Dirichlet problem (18) - (20) has a unique classical solution $\widehat{v} \in C^{2+\gamma,1+\gamma}(\overline{Q}_T)$, $Q_T = (0, L) \times (0, T)$, and that the maximum principle holds true.

We define a constant λ as

$$\lambda = m - \max_{0\le x\le L} \widehat{v}(x,T).$$

Then the strong maximum principle for the Dirichlet problem (18) - (20) yields $\lambda > 0$.

Now we show the inequality (17). Here we may assume $a = 0$ without loss of generality. Indeed, if $a \ne 0$ we replace $\varphi(x)$ by $\varphi(x - a)$ and consider $\widehat{v}(x - a, t)$, which is the solution of (18) - (20) on $[a, a + L]$. Then we can deal with the case of $a \ne 0$ as in the case of $a = 0$.

In the case of $a = 0$ and $b = L$, we define $\widehat{w} = \widehat{v} - u$ on the interval $[0, L]$. By the definition of φ and the assumptions (15) and (16), we have

$$\widehat{w}(x,0) \ge 0 \quad \text{for} \quad 0 \le x \le L,$$

$$\widehat{w}(0,t) \ge 0 \ \ \text{and} \ \ \widehat{w}(L,t) \ge 0 \quad \text{for} \quad 0 \le t \le T.$$

Then we obtain $\widehat{w} \geq 0$ in $\overline{Q}_T$ as a consequence of the maximum principle. It follows that $\widehat{v} \geq u$ in $\overline{Q}_T$, and hence

$$m - \max_{0 \leq x \leq L} u(x,T) \;\geq\; m - \max_{0 \leq x \leq L} \widehat{v}(x,T) \;=\; \lambda.$$

Next we consider the case of $a = 0$ and $0 < b < L$. Since φ is nonnegative, $\widehat{v}(x,t)$ is also nonnegative in $\overline{Q}_T$. Thus we have the boundary condition $\widehat{v}(b,t) \geq 0 \geq u(b,t)$. Applying the maximum principle on the interval $[0,b]$, we have

$$m - \max_{0 \leq x \leq b} u(x,T) \;\geq\; m - \max_{0 \leq x \leq b} \widehat{v}(x,T) \;\geq\; \lambda.$$

Finally we show the continuous dependence of λ on $m \in (0,+\infty)$. To prove this fact, we consider the function $\varphi(x;m)$ defined for $x \in [0,L]$ and $m \in (0,+\infty)$ as follows:

(i) For each $m > 0$, $\varphi(x;m)$ belongs to $C^{2+\gamma}[0,L]$ and satisfies the conditions (21) - (23) as a function of x;

(ii) $\varphi(x;m)$ is continuous with respect to $m > 0$, that is,

$$\lim_{m' \to m} \max_{0 \leq x \leq L} |\varphi(x;m) - \varphi(x;m')| = 0$$

holds true if $m', m > 0$.

Let $\widehat{v}^{(m)}(x,t)$ be the solution to the Dirichlet problem (18) - (20) with the initial value $\varphi(x;m)$. We define a function $\lambda(m)$ as

$$\lambda(m) = m - \max_{0 \leq x \leq L} \widehat{v}^{(m)}(x,T).$$

Then $\lambda(m)$ is continuous function of $m > 0$ since $\widehat{v}^{(m)}(x,T)$ is continuous with respect to the initial value $\varphi(x;m)$ and $\varphi(x;m)$ is continuous with respect to m by the definition. Moreover $\lambda(m)$ satisfies (17) for each $m \in (0,+\infty)$. This completes the proof of Lemma 1. □

Next we show a simple lemma and provide an important property of the solution of (5) - (6) with an almost periodic function as an initial value. Roughly speaking, for each $t > 0$, such a solution has the same almost periodicity as that of the initial value.

Lemma 2. *Suppose that $u_0(x)$ is an almost periodic function that satisfies* (4) *as in Definition 1 with $l(\varepsilon)$. Let $u(x,t)$ be the solution of* (5) - (6). *Then, for every $p \in \mathbb{R}$, an interval $[p, p+l(\varepsilon)]$ contains at least one number q with*

$$|u(x-q,t) - u(x,t)| \;<\; \varepsilon \quad \textit{for} \quad x \in \mathbb{R},\ t > 0.$$

Proof. By the uniqueness of solutions as in Proposition 1, the solutions of (5) - (6) with the initial value $u_0(x) - \varepsilon$, $u_0(x-q)$ and $u_0(x) + \varepsilon$ are equivalent to $u(x,t) - \varepsilon$, $u(x-q,t)$ and $u(x,t) + \varepsilon$, respectively. Thus by the comparison principle, the inequality

$$u_0(x) - \varepsilon \;<\; u_0(x-q) \;<\; u_0(x) + \varepsilon \quad \text{for} \quad x \in \mathbb{R}$$

implies

$$u(x,t) - \varepsilon \;<\; u(x-q,t) \;<\; u(x,t) + \varepsilon \quad \text{for} \quad x \in \mathbb{R},\ t > 0.$$

This completes the proof. □

The two lemmas stated above yield the following result.

Proposition 2. *Suppose that $u_0(x)$ is an almost periodic function. Then there exists a constant μ such that the solution $u(x,t)$ of* (5) - (6) *satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t)-\mu| = 0. \tag{24}$$

In addition, for each u_0, the constant μ is a nondecreasing function of $k \in \mathbb{R}$ with

$$\inf_{x\in\mathbb{R}} u_0(x) \ \le\ \mu\ \le\ \sup_{x\in\mathbb{R}} u_0(x). \tag{25}$$

Proof. Since all constants are stationary solutions of (5) - (6), the functions $U^+(t)$ and $U^-(t)$ defined by

$$U^+(t) = \sup_{x\in\mathbb{R}} u(x,t), \quad U^-(t) = \inf_{x\in\mathbb{R}} u(x,t)$$

are nonincreasing and nondecreasing, respectively by virtue of the comparison principle. The constants $U^* = \lim_{t\to\infty} U^+(t)$ and $U_* = \lim_{t\to\infty} U^-(t)$ exist. Now we define $\mu = (U^*+U_*)/2$ and $\delta = (U^*-U_*)/2$. Then μ satisfies (25) because we have $U^-(0) \le U_* \le \mu \le U^* \le U^+(0)$ by the definition.

First we show $\delta = 0$. Then (24) follows. Since $u(x,t)-\mu$ also satisfies (5) - (6), we may assume $\mu = 0$ without loss of generality.

In what follows we derive a contradiction by assuming $\delta > 0$. Suppose that u_0 satisfies (4) as in Definition 1 with $l(\varepsilon)$. We define a constant $L = l(\delta/4)$ depending only on u_0 and δ.

Let $t_0 > 0$ and $x_0 \in \mathbb{R}$ be arbitrarily fixed. Then we have some points $a \in [x_0-L-1, x_0-1]$ and $b \in [x_0+1, x_0+L+1]$ with

$$u(a,t_0) < -\frac{\delta}{4} \quad \text{and} \quad u(b,t_0) < -\frac{\delta}{4}. \tag{26}$$

Indeed, by the definition of δ, we can take some point $x_* \in \mathbb{R}$ with $u(x_*,t_0) < -\delta/2$. By virtue of Lemma 2, the interval $[x_*-x_0+1, x_*-x_0+1+L]$ contains a number q with

$$|u(x_*-q,t_0)-u(x_*,t_0)| \ <\ \frac{\delta}{4}.$$

Setting $a = x_*-q$, we have $a \in [x_0-L-1, x_0-1]$ and

$$u(a,t_0) \ <\ u(x_*,t_0)+\frac{\delta}{4} \ <\ -\frac{\delta}{4},$$

which is the first inequality of (26). Similarly we can take a number $b \in [x_0+1, x_0+L+1]$ for the second inequality of (26).

Using the uniform bound for $|u_t|$ in Proposition 1, we obtain

$$u(a,t) \le 0 \quad \text{and} \quad u(b,t) \le 0 \quad \text{for} \quad t_0 \le t \le t_0+T$$

for some positive constant T depending only on u_0 and δ. We shall find a positive-valued function $\lambda(m)$ for $m > 0$ with

$$u(x_0,t_0+T) \ \le\ U^+(t_0) - \lambda\left(U^+(t_0)\right).$$

Using Lemma 1, we can choose $\lambda(\cdot)$ so that $\lambda(\cdot) \in C(0,+\infty)$ and that it depends only on $\|u_0'\|_{L^\infty(\mathbb{R})}$, T and $2(L+1)$. If $U^+(t_0) \ge \delta/2$, we get

$$u(x_0,t_0+T) \ \le\ U^+(t_0) - \lambda\left(U^+(t_0)\right) \ \le\ U^+(t_0) - \lambda_0.$$

Here a constant λ_0 is defined by

$$\lambda_0 = \min_{m\in[\delta/2,\, U^+(t_0)]} \lambda(m).$$

Note that λ_0 is well defined and is positive. Since $x_0 \in \mathbb{R}$ is arbitrary and λ_0 is independent of x_0, we obtain $U^+(t_0+T) \le U^+(t_0) - \lambda_0$.

If $U^+(t_0) - \lambda_0 \ge \delta/2$, the same argument can be carried out at $t = t_0 + T$. Namely, for any fixed $x_0 \in \mathbb{R}$, we have

$$u(x_0, t_0 + 2T) \;\le\; U^+(t_0) - \lambda_0 - \lambda\left(U^+(t_0) - \lambda_0\right) \;\le\; U^+(t_0) - 2\lambda_0.$$

Consequently, we find that $U^+(t_0 + nT) < \delta/2$ for some large n. It follows that $U^* < \delta/2$ because $U^+(t)$ is a nonincreasing. This contradicts the definition of δ. Thus we proved $\delta = 0$.

Next we show that μ is a nondecreasing function of $k \in \mathbb{R}$. For any fixed u_0, let $u_1(x,t)$ and $u_2(x,t)$ be the solutions to the problem (5) - (6) for $k = k_1$ and $k = k_2$, respectively. Then there exist constants μ_1 and μ_2 with

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u_j(x,t) - \mu_j| = 0, \quad j = 1, 2.$$

It suffice to show $\mu_1 \ge \mu_2$ if $k_1 \ge k_2$. If $k_1 \ge k_2$, we have

$$\begin{aligned} 0 &= (u_1)_t - (\arctan(u_1)_x)_x - k_1(\sqrt{1+(u_1)_x^2} - 1) \\ &\le (u_1)_t - (\arctan(u_1)_x)_x - k_2(\sqrt{1+(u_1)_x^2} - 1), \end{aligned}$$

which implies that u_1 is the supersolution to the problem (5) - (6) for $k = k_2$. Thus $u_1(x,t) \ge u_2(x,t)$ holds true, and hence $\mu_1 \ge \mu_2$ follows. This completes the proof. □

Remark 1. If $k = 0$, then $\mu = \mathcal{M}\{u_0\}$ holds true. Indeed, for any fixed $t > 0$, we have

$$\begin{aligned} \frac{1}{R}\left|\int_0^R (u(x,t) - u_0(x))dx\right| &= \frac{1}{R}\left|\int_0^R \left(\int_0^t u_t(x,s)\,ds\right) dx\right| \\ &= \frac{1}{R}\left|\int_0^R \left(\int_0^t (\arctan u_x)_x\,ds\right) dx\right| \\ &\le \frac{1}{R}\int_0^t \left|[\arctan u_x]_0^R\right| ds \;\le\; \frac{\pi t}{R}. \end{aligned}$$

Therefore we obtain $|\mathcal{M}\{u\}(t) - \mathcal{M}\{u_0\}| \le \lim_{R\to\infty} \pi t/R = 0$. One has

$$U^-(t) \;\le\; \mathcal{M}\{u\}(t) \equiv \mathcal{M}\{u_0\} \;\le\; U^+(t), \quad t > 0,$$

and hence $U_* = \mathcal{M}\{u_0\} = U^*$ in the limit $t \to \infty$.

Remark 2. Generically $\mu \ne \mathcal{M}\{u_0\}$ holds true if $k \ne 0$. Indeed, for $k > 0$ and $u_0(x) = \sin x$, we have

$$\begin{aligned} \mu = \lim_{t\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u(x,t)\,dx &= \frac{1}{2\pi}\int_0^{2\pi} u_0(x)\,dx + \frac{1}{2\pi}\int_0^\infty \left(\int_0^{2\pi} u_t(x,t)\,dx\right) dt \\ &= \frac{1}{2\pi}\int_0^\infty \left([\arctan u_x]_0^{2\pi} + \int_0^{2\pi} k(\sqrt{1+u_x^2} - 1)dx\right) dt \\ &= \frac{1}{2\pi}\int_0^\infty \left(\int_0^{2\pi} k\left(\sqrt{1+u_x^2} - 1\right) dx\right) dt \;>\; 0 \end{aligned}$$

by using the periodic boundary condition at $x = 0, 2\pi$. Thus μ differs from $\mathcal{M}\{u_0\} = 0$ in this case.

Proof of Theorem 2. Theorem 2 follows directly from Proposition 2 and Remark 1. □

Next we extend the stability results to the case where an initial value is asymptotic to some functions as $x \to \pm\infty$.

Lemma 3. *For some functions $\varphi_*(x)$ and $\varphi^*(x)$, let $\overline{u}_*(x,t)$ and $\overline{u}^*(x,t)$ be the solutions of* (5) - (6) *with the initial values φ_* and φ^*, respectively. Assume that $u_0(x)$ satisfies*

$$\lim_{x\to-\infty} |u_0(x) - \varphi_*(x)| = 0, \qquad \lim_{x\to+\infty} |u_0(x) - \varphi^*(x)| = 0.$$

Then the solution $u(x,t)$ of (5) - (6) *with the initial value u_0 satisfies*

$$\lim_{x\to-\infty} |u(x,t) - \overline{u}_*(x,t)| = 0, \qquad \lim_{x\to+\infty} |u(x,t) - \overline{u}^*(x,t)| = 0$$

for every fixed $t > 0$.

Proof. Let $T > 0$ be arbitrarily fixed. We first show $u(x,T) \to \overline{u}^*(x,T)$ as $x \to +\infty$. On the strip $R_T = \mathbb{R} \times [0,T]$, we consider the function $V = u - \overline{u}^*$. Then $V(x,t)$ is the solution of

$$\begin{aligned} &V_t = \alpha(x,t)\, V_{xx} + \alpha_x(x,t)\, V_x + k\,\beta(x,t)\, V_x, && x \in \mathbb{R},\ t > 0,\\ &V(x,0) = u_0(x) - \varphi^*(x), && x \in \mathbb{R}, \end{aligned}$$

where

$$\alpha(x,t) = \int_0^1 A'\left(\tau u_x + (1-\tau)\,(\overline{u}^*)_x\right) d\tau, \;\; \beta(x,t) = \int_0^1 B'\left(\tau u_x + (1-\tau)\,(\overline{u}^*)_x\right) d\tau$$

Recall that $A(p) = \arctan p$, $B(p) = \sqrt{1+p^2}$. From Proposition 1 the solutions u and $\overline{u}^*$ belong to $C^{2+\gamma,1+\gamma/2}(R_T)$. Thus the equation for V is uniformly parabolic, and the coefficients $\alpha(x,t), \alpha_x(x,t)$, and $\beta(x,t)$ belong to $C^{\gamma,\gamma/2}(R_T)$. Therefore we can express $V(x,t)$ as

$$V(x,t) = \int_{\mathbb{R}} Z(x,y,t)(u_0(y) - \varphi^*(y))\, dy, \tag{27}$$

where $Z(x,y,t)$ is the fundamental solution with

$$|Z(x,y,t)| \;\le\; C_4\, t^{-\frac{1}{2}} \exp\left(-C_5 \frac{(x-y)^2}{t}\right), \quad x, y \in \mathbb{R},\ 0 < t \le T \tag{28}$$

for some positive constants C_4 and C_5. This fact follows from the general theory for linear parabolic equations. See [10, 14] for instance.

From the estimate (28), the expression (27) implies that, for any $\varepsilon > 0$, there exists a constant $R > 0$ with

$$|V(x,T)| \;\le\; \varepsilon \quad \text{if} \quad x \ge R,$$

since $|u_0(x) - \varphi^*(x)| \to 0$ as $x \to +\infty$. Thus we have $V(x,T) \to 0$ as $x \to +\infty$, that is, $u(x,T) \to \overline{u}^*(x,T)$ as $x \to +\infty$.

Considering $u - \overline{u}_*$, we can obtain $u(x,T) \to \overline{u}_*(x,T)$ as $x \to -\infty$ in quite a similar manner. This completes the proof. □

Proof of Theorem 3. We only show the upper estimate since the lower estimate can be obtained similarly. It suffices to show that the solution $u(x,t)$ of (5) - (6) satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(x,t) = \mu^* \tag{29}$$

in the case of $\mu_* \le \mu^*$. Indeed, otherwise one can consider $u(-x,t)$ instead of $u(x,t)$.

Assume that $\mu_* \le \mu^*$ holds true. We denote $u_* - kt$ and $u^* - kt$ simply by u_* and u^*, respectively. Then $u_*(x,t)$ and $u^*(x,t)$ satisfy (5) - (6) with initial values φ_* and φ^*, respectively. For any $\varepsilon > 0$, we take a constant $T > 0$ with

$$\sup_{x\in\mathbb{R}} |u^*(x,T) - \mu^*| \ < \ \frac{\varepsilon}{4}, \quad \sup_{x\in\mathbb{R}} |u_*(x,T) - \mu_*| \ < \ \frac{\varepsilon}{4}.$$

From Lemma 3 there exists a large constant $R > 0$ with

$$\begin{aligned}
\sup_{x\ge R} |u(x,T) - \mu^*| \ &\le \ \sup_{x\ge R} |u(x,T) - u^*(x,T)| + \sup_{x\in\mathbb{R}} |u^*(x,T) - \mu^*| \ < \frac{\varepsilon}{2}, \\
\sup_{x\le -R} |u(x,T) - \mu_*| \ &\le \ \sup_{x\le -R} |u(x,T) - u_*(x,T)| + \sup_{x\in\mathbb{R}} |u_*(x,T) - \mu_*| \ < \frac{\varepsilon}{2}.
\end{aligned}$$

Since $\mu_* \le \mu^*$, we have

$$\begin{aligned}
u(x,T) \ &< \ \mu^* + \frac{\varepsilon}{2} \quad \text{if} \quad x \ge R, \\
u(x,T) \ &< \ \mu_* + \frac{\varepsilon}{2} \ \le \ \mu^* + \frac{\varepsilon}{2} \quad \text{if} \quad x \le -R.
\end{aligned}$$

Now we can define the function $\widehat{\varphi}(x)$ with

$$\begin{aligned}
u(x,T) \ \le \ \widehat{\varphi}(x), \quad x \in \mathbb{R}, \\
\lim_{|x|\to\infty} \widehat{\varphi}(x) = \mu^* + \frac{\varepsilon}{2}.
\end{aligned}$$

We consider the solution $\widehat{u}(x,t)$ of (5) - (6) with $\widehat{u}(x,T) = \widehat{\varphi}(x)$. By virtue of Theorem 1, we have $\widehat{u}(x,t) \to \mu^* + \varepsilon/2$ as $t \to \infty$. Since $u(x,t) \le \widehat{u}(x,t)$ for $t \ge T$, the comparison principle shows that $u(x,t) \le \widehat{u}(x,t) \le \mu^* + \varepsilon$ for $t \ge T_*$ for sufficiently large $T_* > 0$. This implies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(x,t) \ \le \ \mu^*.$$

Now it suffices to show that $\lim_{t\to\infty} \limsup_{x\to+\infty} |u(x,t) - \mu^*| = 0$ since this implies

$$\mu^* \ = \ \lim_{t\to\infty} \limsup_{x\to+\infty} u(x,t) \ \le \ \lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(x,t) \ \le \ \mu^*.$$

From Lemma 3, for any fixed $t > 0$, we have

$$\begin{aligned}
\limsup_{x\to+\infty} |u(x,t) - \mu^*| \ &\le \ \limsup_{x\to+\infty} |u(x,t) - u^*(x,t)| + \limsup_{x\to+\infty} |u^*(x,t) - \mu^*| \\
&= \ \limsup_{x\to+\infty} |u^*(x,t) - \mu^*|.
\end{aligned}$$

Then, by the assumption of Theorem 3, letting $t \to \infty$ yields

$$\begin{aligned}
\lim_{t\to\infty} \limsup_{x\to+\infty} |u(x,t) - \mu^*| \ &\le \ \lim_{t\to\infty} \limsup_{x\to+\infty} |u^*(x,t) - \mu^*| \\
&\le \ \lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u^*(x,t) - \mu^*| \ = \ 0.
\end{aligned}$$

This completes the proof of Theorem 3. □

Remark 3. Under the assumptions of Theorem 3, our proofs give some information about the behavior of solutions as $x \to \pm\infty$. From the proof stated just above, it is clear that the solution $u(x,t)$ of (1) - (2) satisfies

$$\lim_{t\to\infty} \limsup_{x\to-\infty} |u(x,t) - kt - \mu_*| = 0, \quad \lim_{t\to\infty} \limsup_{x\to+\infty} |u(x,t) - kt - \mu^*| = 0.$$

Especially if φ_*(or φ^*) is a constant, that is, $\varphi_* \equiv \mu_*$ ($\varphi^* \equiv \mu^*$), Lemma 3 implies that, for each fixed $t > 0$,

$$\lim_{x\to-\infty}(u(x,t)-kt) = \mu_* \quad \left(\lim_{x\to+\infty}(u(x,t)-kt) = \mu^*\right).$$

4. **Bounded initial perturbations which cause lack of asymptotic stability.** In this section, we discuss some examples and counter-examples for asymptotic stability of traveling lines. If a given initial value $u_0(x)$ is bounded in (5) - (6),

$$\inf_{x\in\mathbb{R}} u_0(x) \;\le\; u(x,t) \;\le\; \sup_{x\in\mathbb{R}} u_0(x)$$

holds true. This implies that a traveling line is always stable for bounded perturbations. The problem is the asymptotic stability for these perturbations.

Example 1. The solution $u(x,t)$ to the Cauchy problem (1) - (2) with the initial value $u_0(x) = \tanh x$ satisfies

$$\lim_{t\to\infty}\inf_{x\in\mathbb{R}}(u(x,t)-kt) = -1, \qquad \lim_{t\to\infty}\sup_{x\in\mathbb{R}}(u(x,t)-kt) = 1.$$

Moreover it holds that

$$\lim_{x\to-\infty}(u(x,t)-kt) = -1, \qquad \lim_{x\to+\infty}(u(x,t)-kt) = 1$$

for each fixed $t > 0$.

Though this example is intuitively clear, it is proved rigorously by our results. Namely, $\tanh x$ is asymptotic to ± 1 at infinity, and the solutions with the initial value $+1$ and -1 are given by $u^*(x,t) = kt+1$ and $u_*(x,t) = kt-1$, respectively. Thus Theorem 3 and Remark 3 give Example 1.

The next example shows difficulty of asymptotic stability for $k \neq 0$ compared with $k = 0$.

Example 2. Define the initial value $u_0(x) \in C^{2+\gamma}(\mathbb{R})$ to satisfy

$$\begin{aligned} u_0(x) &= 0 && \text{if} \quad x \in (-\infty,-1], \\ u_0(x) &= \sin x && \text{if} \quad x \in [1,+\infty). \end{aligned}$$

Then the solution $u(x,t)$ to the Cauchy problem (1) - (2) satisfies

$$\begin{aligned} &\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(x,t)| = 0 && \text{if} \quad k = 0, \\ &\lim_{t\to\infty}\inf_{x\in\mathbb{R}}(u(x,t)-kt) = 0, \quad \lim_{t\to\infty}\sup_{x\in\mathbb{R}}(u(x,t)-kt) = \mu && \text{if} \quad k > 0 \end{aligned}$$

for a positive constant μ. Especially, in the case of $k > 0$, it holds that

$$\lim_{x\to-\infty}(u(x,t)-kt) = 0$$

for each fixed $t > 0$, and that

$$\lim_{t\to\infty}\limsup_{x\to+\infty}|u(x,t)-kt-\mu| = 0.$$

This example follows from Theorem 3, Remark 2, and Remark 3. This example is due to the fact that a shift of the limiting traveling line occurs if $u_0(x) = \sin x$, while it does not occurs for $u_0 \equiv 0$. Arranging this example, we have the following one.

Example 3. Define the initial value $u_0(x) \in C^{2+\gamma}(\mathbb{R})$ to satisfy

$$\begin{aligned} u_0(x) &= \mu \quad &\text{if} \quad x \in (-\infty, -1], \\ u_0(x) &= \sin x \quad &\text{if} \quad x \in [1, +\infty). \end{aligned}$$

where μ is the constant defined as in Remark 2. Then the solution $u(x,t)$ to the Cauchy problem (1) - (2) satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - kt - \mu| = 0.$$

Example 2 and 3 show the peculiarity of our problem due to a shift of the limiting traveling line. Though the Cauchy problem of the linear equation $u_t = u_{xx} + k$ also has the traveling line $u(x,t) = kt$, this is not relevant to such phenomena.

Remark 4. For the traveling line $u(x,t) = kt$ of $u_t = u_{xx} + k$, a perturbation $w(x,t) = u(x,t) - kt$ satisfies the heat equation $w_t = w_{xx}$. Then same results as Theorem 1, 2, and 3 are obtained in a quite similar manner. Moreover, for any fixed $t > 0$, we have

$$\begin{aligned} \frac{1}{R}\left|\int_0^R (w(x,t) - u_0(x))dx\right| &= \frac{1}{R}\left|\int_0^R \left(\int_0^t w_t(x,s)\,ds\right)dx\right| \\ &= \frac{1}{R}\left|\int_0^R \left(\int_0^t w_{xx}(x,s)\,ds\right)dx\right| \\ &\le \frac{1}{R}\int_0^t \left|[w_x]_0^R\right| ds \;\le\; \frac{2t}{R}\|u_0'\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Therefore we obtain $\mathcal{M}\{w\}(t) \equiv \mathcal{M}\{u_0\}$ similarly as in Remark 1. Namely the Cauchy problem of the equation $u_t = u_{xx} + k$ has no relation to a shift of a limiting traveling line even if $k \neq 0$. Thus, for the initial value $u_0(x)$ as in Example 2, the solution $u(x,t)$ of $u_t = u_{xx} + k$ satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - kt| = 0$$

for any $k \in \mathbb{R}$.

Next we show other mechanisms of the curve evolution which produce some counter-examples for asymptotic stability. We prepare two lemmas, which evaluate the slowness needed for flattening out a curve.

Lemma 4. *Suppose that u_0 satisfies*

$$\begin{aligned} u_0(x) &\le 0 \quad &\textit{if} \quad x \in [a,b], \\ u_0(x) &\le 1 \quad &\textit{if} \quad x \in (-\infty, a) \cup (b, +\infty) \end{aligned}$$

for some constants $a, b \in \mathbb{R}$ with $a < b$. Then the solution $u(x,t)$ of (5) - (6) *satisfies*

$$u\left(\frac{a+b}{2}, t\right) \le 2\exp\left((1+|k|)\,t - \frac{b-a}{2}\right), \quad t > 0.$$

Proof. We can assume $a = -L$ and $b = L$ without loss of generality due to the shift invariance. We show $u(0,t) \le 2e^{(1+|k|)t-L}$. First we define a linear operator $\mathcal{L}$ as

$$\mathcal{L}[w] = w_t - \frac{1}{1+u_x^2}w_{xx} - k\,\frac{\sqrt{1+u_x^2}-1}{u_x}w_x = w_t - \alpha(x,t)w_{xx} - \beta(x,t)w_x.$$

Note that $\mathcal{L}[u]=0$. From Proposition 1 and the inequality $|\sqrt{1+p^2}-1|\le|p|$, we have

$$\frac{1}{1+\|u_0'\|^2_{L^\infty(\mathbb{R})}}\le\alpha(x,t)\le 1,\quad |\beta(x,t)|\le|k|. \tag{30}$$

Now we define $\widehat{W}(x,t)$ as

$$\widehat{W}(x,t)=e^{-x+(1+|k|)t-L}+e^{x+(1+|k|)t-L}.$$

Then we have $\mathcal{L}[\widehat{W}]\ge 0$. Indeed, from (30), we have

$$\begin{aligned}\mathcal{L}[\widehat{W}] &= \widehat{W}_t-\alpha(x,t)\widehat{W}_{xx}-\beta(x,t)\widehat{W}_x\\ &= ((1+|k|)-\alpha+\beta)\,e^{-x+(1+|k|)t-L}+((1+|k|)-\alpha-\beta)\,e^{x+(1+|k|)t-L}\\ &\ge ((1+|k|)-1-|k|)\,\widehat{W}\ \ge\ 0.\end{aligned}$$

Moreover, by the assumptions and the definition of $\widehat{W}$, we have $u_0(x)\le\widehat{W}(x,0)$. Thus the function $\widehat{V}(x,t)=\widehat{W}(x,t)-u(x,t)$ satisfies

$$\begin{aligned}&\mathcal{L}[\widehat{V}]\ =\ \widehat{V}_t-\alpha(x,t)\widehat{V}_{xx}-\beta(x,t)\widehat{V}_x\ \ge\ 0, && x\in\mathbb{R},\ t>0,\\ &\widehat{V}(x,0)\ge 0, && x\in\mathbb{R}.\end{aligned}$$

Then $\widehat{V}(x,t)\ge 0$ follows, and hence $u(x,t)\le\widehat{W}(x,t)$ holds true. Consequently, we obtain

$$u(0,t)\ \le\ 2\,e^{(1+|k|)t-L},\quad t>0.$$

This completes the proof. □

Lemma 5. *Suppose that u_0 satisfies*

$$\begin{aligned}u_0(x)\ &\ge\ 1\quad \textit{if}\quad x\in[a,b],\\ u_0(x)\ &\ge\ 0\quad \textit{if}\quad x\in(-\infty,a)\cup(b,+\infty)\end{aligned}$$

for some constants $a,b\in\mathbb{R}$ with $a<b$. Then the solution $u(x,t)$ of (5) - (6) satisfies

$$u\left(\frac{a+b}{2},t\right)\ \ge\ 1-2\exp\left((1+|k|)\,t-\frac{b-a}{2}\right),\quad t>0.$$

Proof. We define the function $\widehat{W}(x,t)$ as

$$\widehat{W}(x,t)=1-e^{-x+(1+|k|)t-L}-e^{x+(1+|k|)t-L}.$$

Then the proof of this lemma can be carried out similarly. □

Proposition 3. *Suppose that u_0 satisfies the following conditions (i), (ii).*

(*i*) $0\le u_0(x)\le 1$ *for* $x\in\mathbb{R}$*;*

(*ii*) *For any $L>0$, there exist constants $a_0,a_1\in\mathbb{R}$ with*

$$\begin{aligned}u_0(x)=0\quad &\textit{for}\quad x\in[a_0,\,a_0+L],\\ u_0(x)=1\quad &\textit{for}\quad x\in[a_1,\,a_1+L].\end{aligned}$$

Then the solution $u(x,t)$ to the Cauchy problem (5) - (6) satisfies, for each $t>0$,

$$\inf_{x\in\mathbb{R}}u(x,t)=0,\quad \sup_{x\in\mathbb{R}}u(x,t)=1. \tag{31}$$

Proof. We show $\inf_{x\in\mathbb{R}} u(x,t) = 0$. The comparison principle and the assumption $u_0(x) \geq 0$ imply $\inf_{x\in\mathbb{R}} u(x,t) \geq 0$.

Let $T > 0$ be arbitrarily fixed. For any $\varepsilon > 0$, we take sufficiently large constant L with

$$2\, e^{(1+|k|)T-\frac{L}{2}} \; < \; \varepsilon.$$

From the assumption, there exist a constant $a \in \mathbb{R}$ with $u_0(x) = 0$ for $x \in [a, a+L]$. Then lemma 4 shows

$$u\left(a + \frac{L}{2}, T\right) \; \leq \; 2\, e^{(1+|k|)T-\frac{L}{2}} \; < \; \varepsilon.$$

This implies $\inf_{x\in\mathbb{R}} u(x,T) = 0$. From Lemma 5, we obtain $\sup_{x\in\mathbb{R}} u(x,T) = 1$ similarly. □

Now we give concrete counter-examples. We exhibit two examples as corollaries of Proposition 3. The initial value of the first example does not decay but oscillate slower and slower as $|x| \to \infty$. This is an initial condition represented by $\sin(\log(|x|+1))$. On the other hand, the initial value of the second one is based on the *Bernoulli sequence*. This initial value may be called "random noise" type.

Example 4. Define a function $f(x)$ as

$$\begin{aligned} f(x) = 0 \quad &\text{if} \quad\quad (2n)^2 \leq |x| < (2n+1)^2, \\ f(x) = 1 \quad &\text{if} \quad (2n+1)^2 \leq |x| < (2n+2)^2, \end{aligned}$$

for $n = 0, 1, 2, \ldots$. Then the solution $u(x,t)$ of (5) - (6) with $u_0 = \eta * f \in C^{2+\gamma}(\mathbb{R})$ satisfies (31). Here $\eta*$ is *Friedrichs's mollifier*, that is, $\eta(x) \in C_0^\infty(\mathbb{R})$, $\eta(x) \geq 0$,$\|\eta\|_{L^1(\mathbb{R})} = 1$, and

$$(\eta * f)(x) = \int_{\mathbb{R}} \eta(x-y) f(y) dy.$$

Namely, the solution of (1) - (2) with $u_0 = \eta * f$ does not converge uniformly to the traveling line $u(x,t) = kt + \mu$ for any fixed μ.

Example 5. Define a function $f(x)$ as

$$f(x) \; = \; X_n, \quad n \leq x < n+1, \;\; n \in \mathbb{Z},$$

where $\{X_n\}_{n\in\mathbb{Z}}$ is the Bernoulli sequence, that is, X_n takes 0 or 1 for each $n \in \mathbb{N}$ randomly. Then the solution $u(x,t)$ of (5) - (6) with $u_0 = \eta * f$ satisfies (31). Namely, the solution of (1) - (2) with this initial value does not converge uniformly to the traveling line $u(x,t) = kt + \mu$ for any fixed μ.

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