PAPER

# A Design Method of 2-D Maximally Flat Diamond-Shaped Half-Band FIR Filters

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**SUMMARY** This paper proposes a new technique for designing 2-D diamond-shaped half-band FIR filters with maximally flat amplitude property. First, a 2-D half-band FIR filter is defined and several interesting properties of this filter are explained. Then a design method using these properties is explained and some design examples are shown. Finally, we discuss the advantages of our method by comparing with the previously proposed method.

## 1. Introduction

Recent advances of very large scale integrated (VLSI) circuits have allowed the realtime operations of two-dimensional (2-D) digital filters. There has been a great deal of interest in designing 2-D digital filters and their applications to 2-D digital signal processing.

The 2-D FIR filters shown in Figs. 2(a) and (b) are of special importance for practical applications. The filter of Fig. 2(b) is known as a 90-degree fan filter, which is used to process geoseismic data, for example. The filter of Fig. 2(a) (Diamond-shaped filter) can be used as a downsampling filter for quincuncially sampled 2-D data[1]. Another application is an interlace-to-noninterlace scanning converter of television image signals[2].

Several methods have been proposed to design the transfer functions of these filters. For example, an optimal design method using $l_p$ norm[3] and a design method by McClellan transform[4] are well-known. These methods are mainly used for designing a highly selective 2-D filter with passband ripples. On the other hand, a maximally flat 2-D filter is often preferred for image signal processing because it usually has less ringing in the step response compared with filters with passband ripples. Tonge has proposed a design method of the maximally flat filter of Fig. 2(a) by multiplying together two maximally flat 1-D filters and rotating the frequency response through 45 degrees[1]. Because of the rotation of the frequency response, the impulse response of the designed filter has, however, a diamond-shaped region of support, which results in a drawback that the implementation of such a filter requires a large number of delays.

Manuscript received September 29, 1989.

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As shown later, diamond-shaped filters and 90-degree fan filters are typical 2-D half-band FIR filters which are the simple extension of 1-D half-band filters into 2-D case. This paper proposes a new design method of diamond-shaped filters and 90-degree fan filters which has square region of the impulse response. The proposing method is very simple because it makes use of some inherent properties of 2-D octagonally symmetric half-band FIR filters. The design examples shown later and the comparison with Tonge's method indicate the validity and the effectiveness of our methods.

## 2. Definition of 2-D Half-Band FIR Filters

### 2.1 Preliminaries

The transfer function of a 2-D zero-phase FIR filter (odd-tap) is given as follows:

$$H(z_1, z_2)=\sum_{m=(M-1)/2}^{(M-1)/2}\sum_{n=-(N-1)/2}^{(N-1)/2} h(m, n)z_1^{-m}z_2^{-n} \qquad (1)$$

$$h(m, n)=h(-m, -n), \qquad (2)$$

where $M$, $N$ are the filter length in the spatial domain and they are restricted to be odd here. The frequency response is evaluated by substituting

$$z_1=e^{j\omega_1},\ z_2=e^{j\omega_2}, \qquad (3)$$

where $\omega_1$, $\omega_2$ are the normalized frequencies. The frequency response of the filter given by Eqs. (1) and (2) is symmetric with respect to the origin in the frequency plane. Furthermore the condition

$$h(m, n)=h(-m, n)=h(m, -n)=h(-m, -n) \qquad (4)$$

makes the frequency response of Eq. (1) symmetric with respect to the $\omega_1$ and $\omega_2$ axes (quadrantal symmetry[5]). From Eqs. (1) and (4), the frequency response is calculated as

$$H(e^{j\omega_1}, e^{j\omega_2})=h(0, 0)+\sum_{m=0}^{(M-1)/2} 2h(m, 0)\cos(m\omega_1)$$
$$+\sum_{n=0}^{(N-1)/2} 2h(0, n)\cos(n\omega_2)$$
$$+\sum_{m=1}^{(M-1)/2}\sum_{n=1}^{(N-1)/2} 4h(m, n)\cos(m\omega_1). \qquad (5)$$

From Eq. ( 5 ), it is clear that the frequency response of a 2-D zero-phase FIR filter is purely real. This makes the following discussion very simple.

### 2.2 Definition of 2-D Half-Band FIR Filters

2-D half-band FIR filters are defined for 2-D zero-phase FIR filters with quadrantal symmetry as follows ;
[Definition] A quadrantally symmetric 2-D zero-phase FIR filter is said to be a 2-D half-band FIR filter if and only if

$$H(e^{j\omega_1}, e^{j\omega_2})+H(e^{j(\pi-\omega_1)}, e^{j(\pi-\omega_2)})=1 \qquad (6)$$

is satisfied for arbitrary $\omega_1$ and $\omega_2$.
It should be noted that if either $M$ or $N$ is even, the transfer function never satisfies Eq. ( 6 ). From this reason, we cannot define the half-band filter for even-tap filters.

### 2.3 Properties of 2-D Half-Band FIR Filters

From the above definition, the following properties of a 2-D half-band FIR filter can be derived.
( i ) The impulse response satisfies

$$h(m, n)=\begin{cases}0.5 & (m=n=0)\\ 0 & (m+n=\text{even},\ m\neq 0,\ n\neq 0).\end{cases} \qquad (7)$$

( ii ) The frequency response is symmetric about $(\omega_1, \omega_2, H)=(\pi/2, \pi/2, 0.5)$ in the space $0\leqq\omega_1, \omega_2\leqq\pi$.
The derivation of the property ( i ) is shown in Appendix A. This property makes it very simple to design and implement a 2-D half-band FIR filter because about 50 percent of the impulse response coefficients are zero. Figure 1 indicates the impulse response of a 2-D half-band FIR filter for the case $M=N=11$. On the other hand the property ( ii ) is trivial from the definition. Figure 2 shows examples of the frequency response of 2-D half-band FIR filters. Figures 2( a ) and ( b ) indicate that 2-D half-band FIR filters include both a diamond-shaped filter and a 90-degree fan filter as a special case. Figure 2( c ) is just an example for more general type of 2-D half-band FIR filters.

Since diamond shaped filters are symmetric about $\omega_1=\omega_2$, the transfer function satisfies

$$H(e^{j\omega_1}, e^{j\omega_2})=H(e^{j\omega_2}, e^{j\omega_1}) \qquad (8)$$

under the condition

$$M=N. \qquad (9)$$

From Eqs. ( 5 ), ( 8 ), and ( 9 ), the relation

$$h(m, n)=h(n, m) \quad \left(-\frac{N-1}{2}\leqq m,\ n\leqq\frac{N-1}{2}\right) \qquad (10)$$

can be easily derived for a diamond-shaped filter. Eqs. ( 4 ) and (10) indicate that the octagonal symmetry[(5)] holds for diamond-shaped filters. On the other hand, it is observed from Fig. 2 that the 90-degree fan filter is obtained by shifting the frequency response of the diamond-shaped filter in the direction of $\omega_2$ by $\pi$.
Thus, the 90-degree fan filter satisfies

$$h(m, n)=-h(n, m) \quad \left(-\frac{N-1}{2}\leqq m,\ n\leqq\frac{N-1}{2}\right). \qquad (11)$$

As the diamond-shaped filter and the 90-degree fan filter

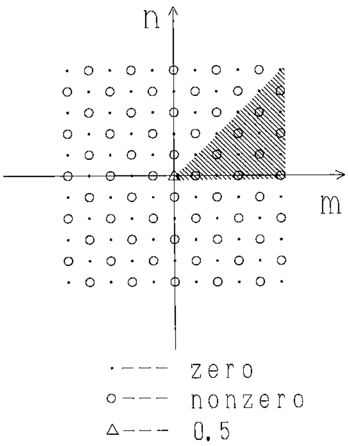
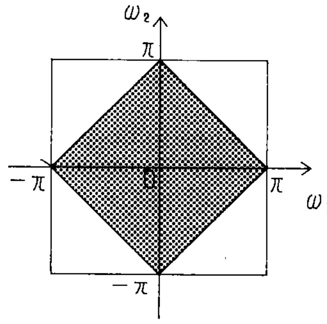
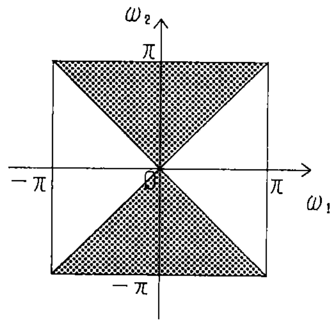


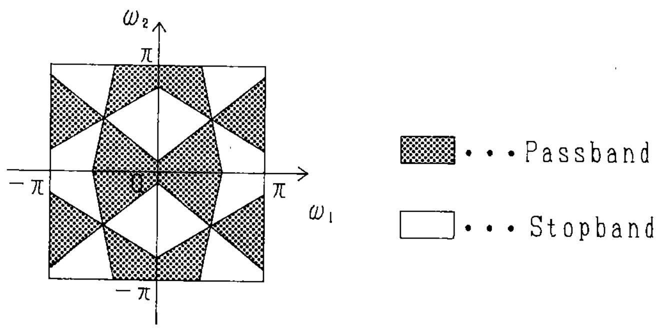

Fig. 1 Impulse response of a 2-D half-band FIR filter for $M=N=11$.




( a ) Diamond-shaped filter.




( b ) 90-degree fan filter.




( c ) General type.

Fig. 2 Examples of 2-D half-band FIR filters.

can be easily transformed from each other by using Eqs. (10) and (11), we restrict the following discussion to the design of diamond-shaped half-band FIR filters.

## 3. Design Method

### 3.1 Number of Impulse Response Coefficients

2-D FIR filters are designed by determining MN parameters (i.e., $h(m, n)(-(M-1)/2 \leqq m \leqq (M-1)/2, -(N-1)/2 \leqq n \leqq (N-1)/2)$) so as to satisfy the specifications. In the case of designing diamond-shaped filters, the number of parameters, however, reduces drastically because the octagonal symmetry and the property of Eq. (7) hold. Thus the design is completed by determining the nonzero impulse response coefficients (i.e., $h(m, n)(m+n=\text{odd})$) in the shaded wedge of Fig. 1. For the filter of length $N$, the number of the nonzero coefficients in this area is given as

$$k_N=\left[\frac{P+1}{2}\right]\left[\frac{P+2}{2}\right], \tag{12}$$

where $[x]$ denotes the maximum integer not exceeding $x$, and $P$ is given by

$$P=(N-1)/2. \tag{13}$$

The derivation of Eqs. (12) and (13) is shown in Appendix B.

### 3.2 Determination of the Impulse Response

Our problem is to design a maximally flat diamond-shaped half-band filter. This problem is equivalent to finding $k_N$ impulse response coefficients which makes the frequency response maximally flat.

By applying the property of the impulse response (i.e., Eq. (7)) and the octagonal symmetry (i.e., Eqs. (4), (9), and (10)) to Eq. (5), the frequency response of the diamond-shaped filter $H_D(e^{j\omega_1}, e^{j\omega_2})$ is obtained as

$$\begin{aligned} H_D(e^{j\omega_1}, e^{j\omega_2}) &= 0.5+\sum_{m=0}^{[(P+1)/2]} 2h(2m-1, 0)\{\cos(2m-1)\omega_1 \\ &\quad +\cos(2m-1)\omega_2\}+\sum_{m=1}^{[(P+1)/2]}\sum_{n=1}^{[P/2]} 4h(2m-1, 2n) \\ &\quad \cdot\{\cos(2m-1)\omega_1\cdot\cos(2n)\omega_2 \\ &\quad +\cos(2n)\omega_1\cdot\cos(2m-1)\omega_2\}. \end{aligned} \tag{14}$$

The following two conditions are required in order to design a maximally flat filter according to the definition of the maximally flatness.

(a) The amplitude equals 1 at $\omega_1=\omega_2=0$.

(b) The derivatives from the 1st to the $L$-th order of the amplitude are all zero at $\omega_1=\omega_2=0$.

The condition (a) adjusts the gain level of the transfer function in order that the attenuation at $\omega_1=\omega_2=\pi$ becomes infinite. On the other hand, the condition (b) specifies the maximally flat property. The integer $L$ is determined from the degree of freedom for the design (i.e., the number of the nonzero coefficients $k_N$) as shown later.

Next we examine the higher order derivatives of Eq. (14) which are functions of two variables. We use the following abbreviation for simplicity.

$$H_D(e^{j\omega_1}, e^{j\omega_2})=H_D(\omega_1, \omega_2) \tag{15}$$

Since Eq. (14) is a sum of cosine functions of $\omega_1$ and $\omega_2$, it has an arbitrary order of partial differential with respect to $\omega_1$ and $\omega_2$. Thus an arbitrary order of total differential of Eq. (14) exists, which is written as

$$d^r H_D(\omega_1, \omega_2)=\left(\frac{\partial}{\partial\omega_1}d\omega_1+\frac{\partial}{\partial\omega_2}d\omega_2\right)^r H_D, \tag{16}$$

where $r$ is a positive integer. From Eq. (16), the $r$-th direction differential coefficient in the direction $\theta$, indicated in Fig. 3, is given by

$$\frac{d^r H_D}{d\omega^r}=\left(\frac{\partial}{\partial\omega_1}\cos\theta+\frac{\partial}{\partial\omega_2}\sin\theta\right)^r H_D, \tag{17}$$

where

$$\omega_1=\omega\cos\theta,\ \ \omega_2=\omega\sin\theta. \tag{18}$$

The condition (b) requires

$$\left.\frac{d^r H_D}{d\omega^r}\right|_{\omega_1=\omega_2=0}=0 \quad (r=1, 2, \cdots, L) \tag{19}$$

in an arbitrary direction $\theta$. Equation (19) holds if and only if

$$\left.\frac{\partial^r H_D}{\partial\omega_1^i\partial\omega_2^j}\right|_{\omega_1=\omega_2=0}=0 \tag{20}$$

$$(i, j\geqq 0,\ i+j=r) \quad (r=1, 2, \cdots, L),$$

because $\theta$ is arbitrary. Since Eq. (14) is a function of cosine, Eq. (20) holds automatically when at least one of the integers $r$, $i$, $j$ is odd. Furthermore,

$$\frac{\partial^r H_D}{\partial\omega_1^i\partial\omega_2^j}=\frac{\partial^r H_D}{\partial\omega_1^j\partial\omega_2^i} \tag{21}$$

holds for arbitrary $i$ and $j$ because Eq. (14) is a symmet-

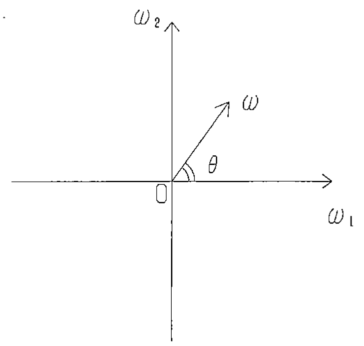



Fig. 3 Direction of the differential coefficient.

ric expression with respect to $\omega_1$ and $\omega_2$. Thus Eq. (20) can be simplified as

$$\left.\frac{\partial^r H_D}{\partial\omega_1^{2i}\partial\omega_2^{2j}}\right|_{\omega_1=\omega_2=0}=0 \tag{22}$$

$$(i \geqq j \geqq 0,\ 2i+2j=r) \quad (r=2, 4, \cdots, L)$$

On the other hand, the condition (a) requires

$$H_D(\omega_1, \omega_2)|_{\omega_1=\omega_2=0}=1. \tag{23}$$

Since the 0-th order derivative of a function is considered to represent the function itself, Eq. (23) is a special case of Eq. (22), for $r=0$, except for the constant in the right-hand side. Thus the condition (a) can be included in the condition (b) as a special case. Therefore, Eqs. (22) and (23) are written together as

$$\left.\frac{\partial^r H_D}{\partial\omega_1^{2i}\partial\omega_2^{2j}}\right|_{\omega_1=\omega_2=0}=\begin{cases}1 & (r=0)\\ 0 & (r\neq 0)\end{cases} \tag{24}$$

$$(i \geqq j \geqq 0,\ 2i+2j=r) \quad (r=0, 2, 4, \cdots, L).$$

For example, Eq. (24) represents the nine equations for $L=8$. These equations are

$$H_{D00}^{0}=1, \tag{25}$$

$$H_{D20}^{2}=0, \tag{26}$$

$$H_{D40}^{4}=0, \tag{27}$$

$$H_{D22}^{4}=0, \tag{28}$$

$$H_{D60}^{6}=0, \tag{29}$$

$$H_{D42}^{6}=0, \tag{30}$$

$$H_{D80}^{8}=0, \tag{31}$$

$$H_{D62}^{8}=0, \tag{32}$$

$$H_{D44}^{8}=0, \tag{33}$$

where

$$H_{Dij}^{r}=\left.\frac{\partial^r H_D}{\partial\omega_1^{i}\partial\omega_2^{j}}\right|_{\omega_1=\omega_2=0}. \tag{34}$$

The number of the equations that Eq. (24) represents for arbitrary $L$ is given by

$$c_L=\left[\frac{L+4}{4}\right]\left[\frac{L+6}{4}\right]. \tag{35}$$

The derivation of Eq. (35) is shown in Appendix C.

Substitution of Eq. (14) into Eq. (24) leads to

$$H_{D00}^{0}=0.5+\sum_{m=0}^{[(P/2]}4h(2m-1, 0)$$

$$+\sum_{m=1}^{[(P+1)2]}\sum_{n=1}^{[P/2]}8h(2m-1, 2n)=1 \tag{36}$$

$$H_{Dr0}^{r}=(-1)^{r/2}\sum_{m=0}^{[(P+1)/2]}2h(2m-1, 0)\cdot(2m-1)^r$$

$$+(-1)^{r/2}\sum_{m=1}^{[(P+1)/2]}\sum_{n=1}^{[P/2]}4h(2m-1, 2n)$$

$$\cdot((2m-1)^r+(2n)^r)=0 \quad (r=\text{Even}) \tag{37}$$

$$H_{D(2i)(2j)}^{r}=(-1)^{r/2}\sum_{m=1}^{[(P+1)/2]}\sum_{n=1}^{[P/2]}4h(2m-1, 2n)$$

$$\cdot((2m-1)^{2i}(2n)^{2j}+(2n)^{2i}(2m-1)^{2j}$$

$$=0 \quad (r=\text{Even},\ 2i+2j=r). \tag{38}$$

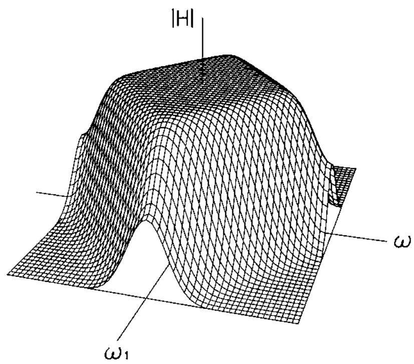
It is observed that Eqs. (36), (37), and (38) consist of linear combinations of the nonzero coefficients $h(m, n)$ and they are linearly independent.

The comparison of Eq. (12) with (35) indicates that, if we set

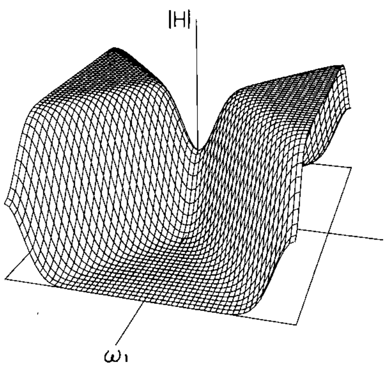
$$L=N-3, \tag{39}$$

the number of the nonzero coefficients coincides with the number of the equations given by Eq. (24) (i. e., $k_N=c_L$). Thus we can obtain a simultaneous equation whose solutions are the nonzero coefficients of a maximally flat diamond-shaped filter. By solving this simultaneous equation, a maximally flat diamond-shaped filter can be designed.

It should be noted that our method forces the designing filter to have the flatness of $(N-3)$-th order (i. e., the property that the derivatives of the amplitude from the 1st to the $(N-3)$-th order are all zero) at the origin in any direction. Moreover, the $(N-2)$-th order derivative is zero because $N-2$ is odd as mentioned before. Thus, it is concluded that the degree of flatness of a maximally




(a) Diamond-shaped filter.




(b) 90-degree fan filter.

Fig. 4 Perspective plots of the designed filter.

flat diamond-shaped filter is $N-2$.

## 4. Design Examples

In this section, a diamond-shaped filter and a 90-degree fan filter with $23\times 23$ taps are designed as examples.

From Eq. (12), these filters have 36 nonzero coefficients. They are determined by numerically solving the simultaneous equation. Figure 4 (a) shows the amplitude characteristics of the designed diamond-shaped filter.

A 90-degree fan filter can be easily obtained from the diamond-shaped filter by using Eqs. (10) and (11). Figure 4(b) shows the 90-degree fan filter transformed from Fig. 4(a).

It is observed from the perspective plots that the designed filters have flat amplitude characteristics in both passband and stopband. These examples indicate the validity of our approach.

## 5. Comparison with Tonge's Method

### 5.1 Brief Review of Tonge's Method

In order to compare our design method with Tonge's' we first review his method briefly. He designed the diamond-shaped filter by the following procedure[1].

(1) Design a 1-D maximally flat half-band filter[6].

(2) Design a 2-D separable filter by multiplying together two orthogonal nonzero impulse responses of 1-D filters in (1).

(3) Rotate the frequency response of the filter in (2) through 45 degrees.

(4) Adjust the gain level of the rotated filter in (3).

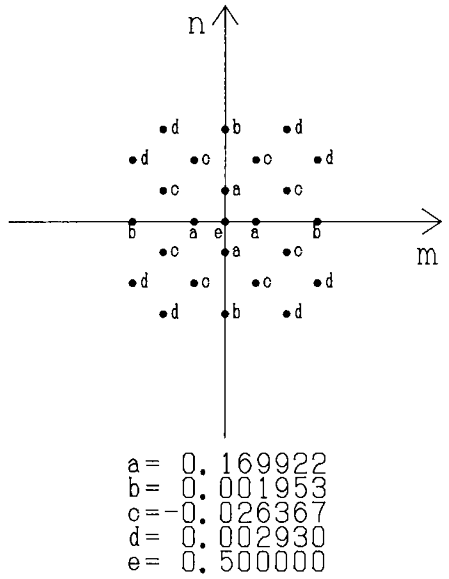
The designed filter has the same degree of flatness in all directions at the origin. The degree of flatness is $2M+1$ for the filter length $4M+3$, where $M$ is an arbitrary nonnegative integer.

| M | | 0 | 1 | 2 | 3 | 4 | … | 10 | … | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| Flatness | | 1 | 3 | 5 | 7 | 9 | … | 21 | … | 41 |
| Tonge's method | N | 3 | 7 | 11 | 15 | 19 | … | 43 | … | 83 |
| | $K_I$ | 1 | 3 | 6 | 10 | 15 | … | 66 | … | 231 |
| | $K_T$ | 5 | 17 | 37 | 65 | 101 | … | 485 | … | 1765 |
| Proposed method | N | 3 | 5 | 7 | 9 | 11 | … | 23 | … | 43 |
| | $K_I$ | 1 | 2 | 4 | 6 | 9 | … | 36 | … | 121 |
| | $K_T$ | 5 | 13 | 25 | 41 | 61 | … | 265 | … | 925 |

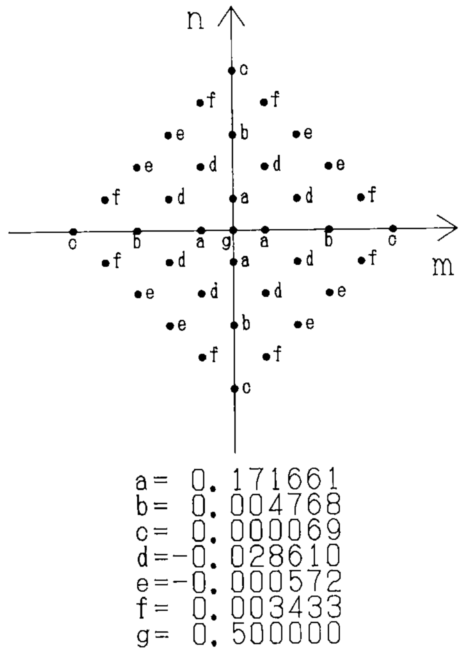
N ; Number of taps
$K_I$ ; Number of independent nonzero impulse response
$K_T$ ; Total number of nonzero impulse response

### 5.2 Comparison of the Two Methods

In this section, we show the comparison of the designed diamond-shaped filters by the proposing method and Tonge's method. Filters designed by these two methods are compared with respect to the following values required to realize the same degree of flatness; the number of taps $N$, the number of independent nonzero impulse responses $K_I$, and the total number of nonzero impulse responses $K_T$.




(a) Proposed method.




(b) Tonge's method.

Fig. 5 Impulse responses of the designed filters.

For the filter with the degree of flatness $2M+1$ ($M$: arbitrary nonnegative integer), these values are given as follows;
For the proposing method

$$N=4M+3 \tag{40}$$

$$K_I=\frac{(M+1)(M+2)}{2} \tag{41}$$

$$K_T=4(M+1)^2+1, \tag{42}$$

and for Tonge's method

$$N=2M+3 \tag{43}$$

$$K_I=\left[\frac{M+2}{2}\right]\left[\frac{M+3}{2}\right] \tag{44}$$

$$K_T=2(M+1)(M+2)+1. \tag{45}$$

These values are summarized in Table 1 for the specific degree of flatness. As an example, the designed impulse responses are shown in Fig. 5 for $M=2$.

Table 1 shows that in order to realize the same degree of flatness, Tonge's method requires approximately twice the number of taps and multipliers for large $M$ as compared with the proposing method. Therefore, by using the proposing method, the amount of hardware required is reduced to about half of that used by Tonge's method. This fact shows clearly the advantage of the proposing method.

## 6. Conclusion

A new design method for 2-D maximally flat diamond-shaped half-band FIR filters was proposed. Our method is very simple and straightforward because it uses some inherent properties of 2-D half-band FIR filters. The designed filter has the same flatness at the origin in any direction on the frequency plane. A 90-degree fan filter can be also designed by transforming the transfer function of a diamond-shaped filter. Two design examples and the comparison with Tonge's method show the validity and usefulness of our method.

The design of higher order filters, however, requires a large amount of numerical calculations in solving the simultaneous equation. This problem could be solved if explicit solutions were found which directly represent the impulse response coefficients for an arbitrary filter length. This is left for future work.

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## Appendix A: Derivation of the Property (i)

The property (i) (i.e., Eq. (7)) can be derived by substituting Eq. (5) into Eq. (6). This substitution leads to

$$\begin{aligned}&2h(0,0)+\sum_{m=0}^{(M-1)/2}2h(m,0)\{\cos(m\omega_1)\\&\quad+\cos(m(\pi-\omega_1))\}\\&\quad+\sum_{n=0}^{(N-1)/2}2h(0,n)\{\cos(n\omega_2)+\cos(n(\pi-\omega_2))\}\\&\quad+\sum_{m=1}^{(M-1)/2}\sum_{n=1}^{(N-1)/2}4h(m,n)\{\cos(m\omega_1)\cos(n\omega_2)\\&\quad+\cos(m(\pi-\omega_1)\cos(n(\pi-\omega_2))\}=1.\end{aligned} \tag{A・1}$$

Using the relation

$$\cos(m(\pi-\omega))=(-1)^m\cdot\cos(m\omega), \tag{A・2}$$

we have

$$\begin{aligned}&2h(0,0)+\sum_{m=0}^{(M-1)/2}2h(m,0)\cos(m\omega_1)\{1+(-1)^m\}\\&\quad+\sum_{n=0}^{(N-1)/2}2h(0,n)\cos(n\omega_2)\{1+(-1)^n\}\\&\quad+\sum_{m=1}^{(M-1)/2}\sum_{n=1}^{(N-1)/2}4h(m,n)\cos(m\omega_1)\cos(n\omega_2)\\&\quad\cdot\{1+(-1)^{m+n}\}=1.\end{aligned} \tag{A・3}$$

The definition of 2-D half-band FIR filters states that Eq. (A・3) holds for arbitrary $\omega_1$ and $\omega_2$. Thus,

$$\left.\begin{aligned}&2h(0,0) &&=1\\&2h(m,0)\{1+(-1)^m\} &&=0\\&2h(0,n)\{1+(-1)^n\} &&=0\\&4h(m,n)\{1+(-1)^{m+n}\}&&=0\end{aligned}\right\} \tag{A・4}$$

must be satisfied. Then Eq. (7) is derived from Eq. (A・4). Conversely, the sufficiency can be shown easily by using Eqs. (A・1)-(A・4) in reverse order.

## Appendix B: Derivation of Eqs. (12) and (13)

Let $k'(t)$ denotes the number of the nonzero coefficients which lie on the vertical line $m=t$ in the wedge for an arbitrary filter length. From Fig. 1, it is easily observed that

$$k'(t)=\left[\frac{t+1}{2}\right]. \tag{A·5}$$

Since $N=3$ is the smallest size for a diamond-shaped filter, Eq. (12) should be proved for an arbitrary odd integer $N \geqq 3$. This is done by the mathematical induction as follows;
(1) From Fig. 1, it is clear that Eq. (12) holds for $N=3$.
(2) Now we assume that Eq. (12) holds for $N=K$, i. e.,

$$k_K=\left[\frac{P+1}{2}\right]\left[\frac{P+2}{2}\right], \tag{A·6}$$

where $P$ is given by Eq. (13) for $N=K$. Then, for $N=K+2$,

$$k_{K+3}=k_K+k'(P+1). \tag{A·7}$$

By Eq. (A·5),

$$k_{K+3}=\left[\frac{P+1}{2}\right]\left[\frac{P+2}{2}\right]+\left[\frac{P+2}{2}\right]$$
$$=\left[\frac{P+2}{2}\right]\left[\frac{P+3}{2}\right]. \tag{A·8}$$

(3) Thus, Eq. (12) holds for an arbitrary odd integer $N \geqq 3$ because Eq. (13) gives $P+1$ for $N=K+2$.

**Appendix C: Derivation of Eq. (35)**

Let $c'(r)$ denote the number of equations represented by Eq. (24) for a fixed even integer $r$. By using the notation of Eq. (34), Eq. (24) gives the following equations for the case that $r/2$ is even,

$$\begin{aligned} H^r_{Dr0} &=1, \\ H^r_{D(r-2)2} &=0, \\ H^r_{D(r-4)4} &=0, \\ &\vdots \\ H^r_{D(r/2)(r/2)} &=0, \end{aligned} \tag{A·9}$$

For the case that $r/2$ is odd,

$$\begin{aligned} H^r_{Dr0} &=1, \\ H^r_{D(r-2)2} &=0, \\ H^r_{D(r-4)4} &=0, \\ &\vdots \\ H^r_{D(r/2+1)(r/2-1)} &=0. \end{aligned} \tag{A·10}$$

From Eqs. (A·9) and (A·10), $c'(r)$ is given by

$$c'(r)=\left[\frac{r+4}{4}\right]. \tag{A·11}$$

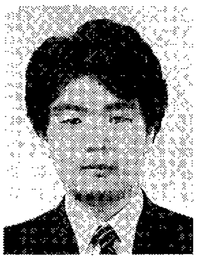
By using Eq. (A·11), Eq. (35) is also proved by the mathematical induction as follows;
(1) Clearly, Eq. (35) holds for $L=0$.
(2) Assume that Eq. (35) holds for $L=K$. For $L=K+2$,

$$c_{K+2}=c_K+c'(K+2). \tag{A·12}$$

By Eq. (A·9),

$$c_{K+2}=\left[\frac{K+4}{4}\right]\left[\frac{K+6}{4}\right]+\left[\frac{K+6}{4}\right]$$
$$=\left[\frac{K+6}{4}\right]\left[\frac{K+8}{4}\right]. \tag{A·13}$$

(3) Then Eq. (35) is true for an arbitrary even integer.



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