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PAPER

Design of 2-D Separable Denominator IIR Digital Filters in Spatial Domain

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SUMMARY A new design method for 2-D IIR digital filters, having a separable denominator, in the spatial domain is presented. The modified Gauss Method is applied in the iterating calculation of the filter coefficients. Also, the 1-D state space representation of the denominator is utilized in determining the impulse response of the designed IIR transfer function and its partial derivatives systematically while the numerator is expressed by a nonseparable polynomial. The error criterion function, which also includes the response outside the given region of support, is minimized in the least square sense. Convergence, together with the stability of the resulting filter, are guaranteed.

key words: *digital signal processing, optimization techniques*

1. Introduction

Several design methods for two-dimensional (2-D) IIR digital filters in the spatial domain have been presented [1]–[3]. The common design problem of all these methods is to estimate the filter coefficients, which are the design parameters, such that the designed filter provides a response closely fitting the given spatial domain response. In Ref. [1], the iterative prefilter design method exploits the estimate of denominator in the previous iteration to recalculate all designed parameters at the current iteration. However, the filter at any particular iteration can become unstable which results in severe numerical problems and the convergence conditions of the method are not known. For the Davidson-Fletcher-Powell optimization algorithm [2], a large number of iterations is necessary to converge and the stability of a resulting filter is not guaranteed. The method proposed in Ref. [3] requires splitting the iteration algorithm into 2 phases to find the stationary point of the equivalent minimization problem and the cost of calculation depends on the number of samples in the given region of support.

In this paper, a new design method for 2-D IIR digital filters, having a separable denominator, is developed. For a given response with finite region of support, the designed filter provides a response which minimizes the error criterion of entire spatial domain in the least square sense. This method can be considered as an extended version of the method proposed by Cadzow [4]. As in the former method [4], this developed method starts with estimating the error criterion function by a quadratic model, and then the modified Gauss method [4]–[6] is applied to the iterative minimization algorithm until the termination conditions are satisfied. With the separable denominator, the calculation of Hessian matrix in the algorithm is considerably simplified and also, the stability of the resulting filter can easily be obtained. The 1-D state space representation of the denominator is utilized in the calculation of the impulse response of the IIR transfer function and its partial derivatives. It is known that the modified Gauss method exhibits a quadratic-like convergence so the designed parameters can rapidly attain to a stationary point. The particular features of the proposed method are

- The entire response of the filter outside the given region of support, usually assumed to have zero value, is also minimized.
- The calculation of Hessian matrix is independent of the number of samples in the region of support so that the cost of calculation of the algorithm depends mainly on the order of the designed filter.

2. The Modified Gauss Method

The square error criterion in the spatial domain with quarter-plane causality is given by

$$f(\mathbf{x}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[h_o(m,n) - h(\mathbf{x},m,n)\right]^2, \tag{1}$$

where $h_o(m,n)$ is a sample of the given response with finite region of support, $h(\mathbf{x},m,n)$ is the impulse response of a 2-D IIR system and $\mathbf{x}$ is the parameter vector to be designed. Assume that

$$h_o(m,n) = 0 \quad , \quad (m,n) \notin R_e \tag{2}$$

where R_e is the region of support with $0 \leqq m \leqq M, 0 \leqq n \leqq N$.

Naturally, the problem of calculating $\mathbf{x}$ which minimizes Eq. (1) is highly nonlinear. For the unconstrained nonlinear minimization problem, it is known that the modified Gauss method can efficiently provide a satisfying solution. In applying this design method to the minimization algorithm, $f(\mathbf{x})$ is approximated up to second order in the Taylor series expansion and the

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stationary point is found by equating the gradient of $f(\mathbf{x})$ to zero. Then, $\mathbf{x}$ can be computed iteratively. The iterative relation is given by

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \rho\Delta\mathbf{x}^k , \tag{3}$$

where $\mathbf{x}^k$ and $\Delta\mathbf{x}^k$ are the parameter vector and the direction vector in the k th iteration, respectively , and ρ is a positive scalar. The value of ρ should be selected such that

$$f(\mathbf{x}^{k+1}) \; < \; f(\mathbf{x}^k) \tag{4}$$

is satisfied. $\Delta\mathbf{x}$ is given by

$$\Delta\mathbf{x} = -\mathbf{N}^{-1}\mathbf{q} \; . \tag{5}$$

The modified Hessian matrix $\mathbf{N}$ and the gradient vector $\mathbf{q}$ are defined, as in Ref. [5] and [6], by

$$\mathbf{N} \; = \; \left[\frac{\partial f(\mathbf{x})}{\partial x_u \partial x_v}\right]_{w\times w} , \tag{6}$$

for

$$\frac{\partial f(\mathbf{x})}{\partial x_u \partial x_v} \; \approx \; 2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\partial h(\mathbf{x},m,n)}{\partial x_u}\frac{\partial h(\mathbf{x},m,n)}{\partial x_v} \tag{7}$$

and

$$\mathbf{q} \; = \; \left[\frac{\partial f(\mathbf{x})}{\partial x_v}\right]_{w\times 1} , \tag{8}$$

for

$$\frac{\partial f(\mathbf{x})}{\partial x_v} = - \; 2\sum_{m=0}^{M}\sum_{n=0}^{N} h_o(m,n) \; \frac{\partial h(\mathbf{x},m,n)}{\partial x_v} \; + \; 2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h(\mathbf{x},m,n) \; \frac{\partial h(\mathbf{x},m,n)}{\partial x_v}, \tag{9}$$

where the dimension of $\mathbf{x}$ is $w \times 1$ and $x_u, x_v \in \mathbf{x}$.

It can be seen that the modified Hessian matrix corresponding to Eq. (6) and Eq. (7) is always positive definite. The algorithm iteratively computes $\mathbf{x}^k$, which converges to a stationary point in a quadratic-like manner, until the termination conditions are satisfied.

3. Computational Method

3.1 The Model of 2-D IIR Digital Filter

The model of a 2-D IIR transfer function used in the algorithm is

$$H(\mathbf{x},z_1,z_2) \; = \; \frac{z_1^{-1}z_2^{-1} \; C(\mathbf{c},z_1,z_2)}{A(\mathbf{a},z_1) \; B(\mathbf{b},z_2)} , \tag{10}$$

where $H(\mathbf{x},z_1,z_2)$ is the z transform of $h(\mathbf{x},m,n)$ and also,

$$A(\mathbf{a},z_1) \; = \; 1-\sum_{i=1}^{p} a_i z_1^{-i} \quad ,$$

$$B(\mathbf{b},z_2) \; = \; 1-\sum_{j=1}^{q} b_j z_2^{-j} \quad ,$$

$$C(\mathbf{c},z_1,z_2) \; = \; \sum_{i=0}^{p}\sum_{j=0}^{q} c_{ij} z_1^{-i} z_2^{-j} \quad ,$$

where

$\mathbf{a} = [a_1, a_2, \ldots, a_p]^T_{1\times p}$, $\mathbf{b} = [b_1, b_2, \ldots, b_q]^T_{1\times q}$,

$\mathbf{c} = [c_{00}, c_{10}, \ldots, c_{p0}, c_{01}, \ldots, c_{p1}, \ldots, c_{pq}]^T_{1\times(p+1)(q+1)}$,

and $a_i, b_j, c_{ij} \in \mathbf{x}$. The factor $z_1^{-1}z_2^{-1}$ is included in the numerator of $H(\mathbf{x},z_1,z_2)$ to simplify the impulse response function. As the region of support of a causal $h(\mathbf{x},m,n)$ is $1 \leqq m$ and $1 \leqq n$, all the samples of the response being approximated must be shifted accordingly such that

$$h_o'(m,n) \; = \; h_o(m-1,n-1) \; , \tag{11}$$

where $h_o'(m,n)$ is the new objective response to be approximated in Eq. (1). When the filter is implemented, $z_1^{-1}z_2^{-1}$ can be removed to obtain the desired response. $H(\mathbf{x},z_1,z_2)$ can be written in the following form;

$$\begin{aligned} H(\mathbf{x}, \; z_1,z_2) &= \frac{z_1^{-1}}{A(\mathbf{a},z_1)} \cdot \frac{z_2^{-1}}{B(\mathbf{b},z_2)} \cdot C(\mathbf{c},z_1,z_2) \\ &= H_a(\mathbf{a},z_1) \cdot H_b(\mathbf{b},z_2) \cdot C(\mathbf{c},z_1,z_2) \; . \end{aligned} \tag{12}$$

Then $h_a(m)$, the impulse response of $H_a(\mathbf{a},z_1)$ and $h_b(n)$, the impulse response of $H_b(\mathbf{b},z_2)$ can be obtained by the state space representation [7], as follows

$$h_a(m) = \begin{cases} [0 \; \cdots \; 1]\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ a_p & a_{p-1} & \cdots & a_1 \end{bmatrix}^{m-1}\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} & ; \; m \; = \; 1,2,\ldots \\ 0 & ; \; m \; \leqq \; 0 \end{cases} \tag{13}$$

$$h_b(n) = \begin{cases} [0 \; \cdots \; 1]\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ b_q & b_{q-1} & \cdots & b_1 \end{bmatrix}^{n-1}\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} & ; \; n \; = \; 1,2,\ldots \\ 0 & ; \; n \; \leqq \; 0 \end{cases} \tag{14}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant in each iteration. Hence, $h(\mathbf{x},m,n)$ can be given by the equation

$$h(\mathbf{x},m,n) \; = \; \sum_{i=0}^{p}\sum_{j=0}^{q} c_{ij} \; h_a(m-i) \; h_b(n-j) \; . \tag{15}$$

3.2 Calculation of the Direction Vector $\Delta\mathbf{x}$

3.2.1 The Partial Derivatives of $h(\mathbf{x}, m, n)$

To determine the direction vector $\Delta\mathbf{x}$ in each iteration, the elements of $\mathbf{N}$ and $\mathbf{q}$ have to be derived. In this section, the method to find the partial derivatives of $h(\mathbf{x}, m, n)$ is shown.

Consider the partial derivative of $H(\mathbf{x}, z_1, z_2)$ by x_u

$$\frac{\partial H(\mathbf{x}, z_1, z_2)}{\partial x_u} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\partial h(\mathbf{x}, m, n)}{\partial x_u} z_1^{-m} z_2^{-n} \; . \quad (16)$$

It is clear from Eq. (16) that $\partial h(\mathbf{x}, m, n)/\partial x_u$ can be calculated directly from the impluse response of $\partial H(\mathbf{x}, z_1, z_2)/\partial x_u$. The partial derivative of $H(\mathbf{x}, z_1, z_2)$ by each filter coefficient is given by;

$$\frac{\partial H(\mathbf{x}, z_1, z_2)}{\partial a_u} = z_1^{-u}\left(\frac{z_1^{-1}}{A^2(\mathbf{a}, z_1)}\right)\left(\frac{z_2^{-1}}{B(\mathbf{b}, z_2)}\right) C(\mathbf{c}, z_1, z_2) \;\; ; \;\; a_u \in \mathbf{a} \; , \quad (17)$$

$$\frac{\partial H(\mathbf{x}, z_1, z_2)}{\partial b_v} = z_2^{-v}\left(\frac{z_1^{-1}}{A(\mathbf{a}, z_1)}\right)\left(\frac{z_2^{-1}}{B^2(\mathbf{b}, z_2)}\right) C(\mathbf{c}, z_1, z_2) \;\; ; \;\; b_v \in \mathbf{b} \; , \quad (18)$$

$$\frac{\partial H(\mathbf{x}, z_1, z_2)}{\partial c_{uv}} = z_1^{-u} z_2^{-v}\left(\frac{z_1^{-1}}{A(\mathbf{a}, z_1)}\right)\left(\frac{z_2^{-1}}{B(\mathbf{b}, z_2)}\right) \;\; ; \;\; c_{uv} \in \mathbf{c} \; . \quad (19)$$

Define that

$$H_{\hat{a}}(\hat{\mathbf{a}}, z_1) = \frac{z_1^{-1}}{\hat{A}(\hat{\mathbf{a}}, z_1)} = \frac{z_1^{-1}}{A^2(\mathbf{a}, z_1)} \; ,$$

$$\text{where } \hat{A}(\hat{\mathbf{a}}, z_1) = 1 - \sum_{i=1}^{2p} \hat{a}_i z_1^{-i} = (1 - \sum_{i=1}^{p} a_i z_1^{-i})^2$$

and

$$H_{\hat{b}}(\hat{\mathbf{b}}, z_2) = \frac{z_2^{-1}}{\hat{B}(\hat{\mathbf{b}}, z_2)} = \frac{z_2^{-1}}{B^2(\mathbf{b}, z_2)} \; ,$$

$$\text{where } \hat{B}(\hat{\mathbf{b}}, z_2) = 1 - \sum_{j=1}^{2q} \hat{b}_j z_2^{-j} = (1 - \sum_{j=1}^{q} b_j z_2^{-j})^2 \; .$$

Here, $\hat{a}_i$ and $\hat{b}_j$ are the elements of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, respectively. Then $h_{\hat{a}}(m)$, the impulse response of $H_{\hat{a}}(\hat{\mathbf{a}}, z_1)$ and $h_{\hat{b}}(n)$, the impulse response of $H_{\hat{b}}(\hat{\mathbf{b}}, z_2)$ are given by

$$h_{\hat{a}}(m) = \begin{cases} [0 \cdots 1]\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ \hat{a}_{2p} & \hat{a}_{2p-1} & \cdots & \hat{a}_1 \end{bmatrix}^{m-1}\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} & ; \; m = 1, 2, \ldots \\ 0 & ; \; m \leqq 0 \end{cases} \quad (20)$$

and

$$h_{\hat{b}}(n) = \begin{cases} [0 \cdots 1]\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ \hat{b}_{2q} & \hat{b}_{2q-1} & \cdots & \hat{b}_1 \end{bmatrix}^{n-1}\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} & ; \; n = 1, 2, \ldots \\ 0 & ; \; n \leqq 0 \end{cases} \quad (21)$$

Thus, the partial derivative of $h(\mathbf{x}, m, n)$ by each filter coefficient is given as follows,

$$\frac{\partial h(\mathbf{x}, m, n)}{\partial a_u} = \sum_{i=0}^{p}\sum_{j=0}^{q} c_{ij}\; h_{\hat{a}}(m-i-u)\; h_b(n-j) \; , \quad (22)$$

$$\frac{\partial h(\mathbf{x}, m, n)}{\partial b_v} = \sum_{i=0}^{p}\sum_{j=0}^{q} c_{ij}\; h_a(m-i)\; h_{\hat{b}}(n-j-v) \; , \quad (23)$$

$$\frac{\partial h(\mathbf{x}, m, n)}{\partial c_{uv}} = h_a(m-u)\; h_b(n-v) \; . \quad (24)$$

It can readily be seen, from Eq. (17), that $\partial h(\mathbf{x}, m, n)/\partial a_{u+i}$ is the i th shifted version of $\partial h(\mathbf{x}, m, n)/\partial a_u$. The same property also holds in the cases of b_v and c_{uv}.

3.2.2 Calculation of $\mathbf{N}$

From Eq. (6) and Eq. (7), $\mathbf{N}$ can be expressed in a partitioned form as

$$\mathbf{N} = 2\begin{bmatrix} \mathbf{D}(a_u, a_v) & \vdots & \mathbf{D}(a_u, b_v) & \vdots & \mathbf{D}(a_u, c_{rs}) \\ \cdots\cdots & \vdots & \cdots\cdots & \vdots & \cdots\cdots \\ \mathbf{D}(b_u, a_v) & \vdots & \mathbf{D}(b_u, b_v) & \vdots & \mathbf{D}(b_u, c_{rs}) \\ \cdots\cdots & \vdots & \cdots\cdots & \vdots & \cdots\cdots \\ \mathbf{D}(c_{ij}, a_v) & \vdots & \mathbf{D}(c_{ij}, b_v) & \vdots & \mathbf{D}(c_{ij}, c_{rs}) \end{bmatrix} , \quad (25)$$

where $\mathbf{D}(x_u, x_v)$ is a sub-matrix defined as

$$\mathbf{D}(x_u, x_v) = \left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\partial h(\mathbf{x}, m, n)}{\partial x_u}\; \frac{\partial h(\mathbf{x}, m, n)}{\partial x_v}\right] . \quad (26)$$

The corresponding dimensions of the sub-matrices are as follows;

$\mathbf{D}(a_u, a_v) \; : \; p \times p$
$\mathbf{D}(b_u, b_v) \; : \; q \times q$
$\mathbf{D}(b_u, a_v) = [\mathbf{D}(a_u, b_v)]^T \; : \; q \times p$
$\mathbf{D}(c_{ij}, a_v) = [\mathbf{D}(a_u, c_{rs})]^T \; : \; (p+1)(q+1) \times p$
$\mathbf{D}(c_{ij}, b_v) = [\mathbf{D}(b_u, c_{ij})]^T \; : \; (p+1)(q+1) \times q$
$\mathbf{D}(c_{ij}, c_{rs}) \; : \; (p+1)(q+1) \times (p+1)(q+1)$

Clearly, $\mathbf{N}$ is a symmetric matrix. First, consider $\mathbf{D}(a_u, a_v)$. It can be seen, from Eq. (20) and Eq. (22), that the following shift property holds;

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{u+i}}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{v+i}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{u}}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{v}}, \quad (27)$$

where i is a positive integer. From Eq. (27), we obtain that $\mathbf{D}(a_u, a_v)$ is a symmetric Toeplitz matrix which all the values of $p \times p$ elements of $\mathbf{D}(a_u, a_v)$ can be determined from the p elements in the first column (or row). From Eq. (22), The value of an element of $\mathbf{D}(a_u, a_v)$ is given by

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{u}}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{v}} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{i=0}^{p}\sum_{j=0}^{q} c_{ij} h_{\hat{a}}(m-i-u)h_b(n-j)\right] \left[\sum_{r=0}^{p}\sum_{s=0}^{q} c_{rs} h_{\hat{a}}(m-r-v)h_b(n-s)\right] \\
&= \sum_{i,r=0}^{p}\sum_{j,s=0}^{q} c_{ij}c_{rs} \underbrace{\left[\sum_{m=0}^{\infty} h_{\hat{a}}(m-i-u)h_{\hat{a}}(m-r-v)\right]}_{t_1} \underbrace{\left[\sum_{n=0}^{\infty} h_b(n-j)h_b(n-s)\right]}_{t_2}. \quad (28)
\end{aligned}$$

In the right of Eq. (28), we see that t_2 depends on the value of j and s. It can be shown that, for example, when $(j, s) = (0, 1)$, the infinite sum provides the same value as when $(j, s) = (1, 2)$;

$$\begin{aligned}
(j,s)=(0,1) &: \sum_{n=0}^{\infty} h_b(n)h_b(n-1) = h_b(2)h_b(1) + h_b(3)h_b(2) + \ldots \\
(j,s)=(1,2) &: \sum_{n=0}^{\infty} h_b(n-1)h_b(n-2) = h_b(2)h_b(1) + h_b(3)h_b(2) + \ldots
\end{aligned}$$

From the example above, it can be deduced that the value of t_2 depends on the difference between j and s. The same situation also happens in the case of t_1; that is, for any fixed values of u and v, the value of t_1 depends on the difference between i and r. By making use of this result, the calculation of the elements of $\mathbf{D}(a_u, a_v)$ can be made compactly by putting together $c_{ij}c_{rs}$ which have the same values of $(i-r)$ and $(j-s)$. To determine the values in the first column of $\mathbf{D}(a_u, a_v)$, we suppose that $v = 1$ and then, it can be verified that the u th element in the first column of $\mathbf{D}(a_u, a_v)$ can be written as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{u}}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{1}} = \mathbf{V}_{bb}\mathbf{C}_1\mathbf{V}_{\hat{a}\hat{a}}(u). \quad (29)$$

The matrix $\mathbf{C}_1$ is defined as

$$\mathbf{C}_1 = \begin{bmatrix} \gamma_{(q)(-p)} & \gamma_{(q)(-p+1)} & \gamma_{(q)(p)} \\ \gamma_{(q-1)(-p)} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \gamma_{(-q+1)(p)} \\ \gamma_{(-q)(-p)} & \cdots & \gamma_{(-q)(p)} \end{bmatrix}, \quad (30)$$

where the dimension of $\mathbf{C}_1$ is $(2q+1)\times(2p+1)$. $\gamma_{(\delta_1)(\delta_2)}$ is a summation of $c_{ij}c_{rs}$ which have the same values of $\delta_1(=j-s)$ and $\delta_2(=i-r)$, defined as

$$\gamma_{(\delta_1)(\delta_2)} = \sum_{(j-s=\delta_1, i-r=\delta_2)} c_{ij}c_{rs},$$

and i, r, j, s are positive integers with $i, r \in [0, p]$ and $j, s \in [0, q]$. Let

$$S_{uv}(l) = \begin{cases} \sum_{m=0}^{\infty} h_u(m)\, h_v(m+l) & ; \quad l \geqq 0 \\ \sum_{m=0}^{\infty} h_u(m-l)\, h_v(m) & ; \quad l \leqq 0 \end{cases}$$

Then, we have

$$\begin{aligned}
\mathbf{V}_{\hat{a}\hat{a}}(u) = [&S_{\hat{a}\hat{a}}(u+p), S_{\hat{a}\hat{a}}(u+(p-1)), \\
&\ldots, S_{\hat{a}\hat{a}}(u-(p-1)), S_{\hat{a}\hat{a}}(u-p)\,]^T_{1\times(2p+1)}, \quad (31)
\end{aligned}$$

where $S_{\hat{a}\hat{a}}(l)$ corresponds to t_1 for $l = i - r$, and

$$\begin{aligned}
\mathbf{V}_{bb} = [&S_{bb}(q), \ldots, S_{bb}(1), S_{bb}(0), S_{bb}(-1), \\
&\ldots, S_{bb}(-q)]_{1\times(2q+1)}, \quad (32)
\end{aligned}$$

where $S_{bb}(l)$ corresponds to t_2 for $l = j - s$.

The method to find the value of $S_{\hat{a}\hat{a}}(n)$ or $S_{bb}(n)$ is discussed in Appendix. The matrix $\mathbf{D}(b_u, b_v)$ can be computed in completely the same way as $\mathbf{D}(a_u, a_v)$. Thus, the u th element of the first column of $\mathbf{D}(b_u, b_v)$ is given by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial b_{u}}\frac{\partial h(\mathbf{x},m,n)}{\partial b_{1}} = \mathbf{V}_{\hat{b}\hat{b}}(u)\mathbf{C}_1\mathbf{V}_{aa}. \quad (33)$$

where $\mathbf{V}_{\hat{b}\hat{b}}(u)$ and $\mathbf{V}_{aa}$ are defined similarly to $\mathbf{V}_{\hat{a}\hat{a}}(u)$ and $\mathbf{V}_{bb}$, respectively. Notice that $\mathbf{V}_{\hat{a}\hat{a}}(u)$ has a shift property that $\mathbf{V}_{\hat{a}\hat{a}}(u)$ with $1 < u \leqq p$ is the u th right-shifted version of $\mathbf{V}_{\hat{a}\hat{a}}(1)$. The same property also holds for $\mathbf{V}_{\hat{b}\hat{b}}(u)$.

In the case of $\mathbf{D}(b_u, a_v)$, the value of an element is given by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial b_{u}}\frac{\partial h(\mathbf{x},m,n)}{\partial a_{v}} = \sum_{i,r=0}^{p}\sum_{j,s=0}^{q} c_{ij}c_{rs} \quad (34)$$

$$\underbrace{\left[\sum_{m=0}^{\infty} h_{\hat{a}}(m-r-v)h_a(m-i)\right]}_{t_3}\underbrace{\left[\sum_{n=0}^{\infty} h_{\hat{b}}(n-j-u)h_b(n-s)\right]}_{t_4}.$$

Notice that t_3 does not change along a certain column of C_1, nor does t_4 along a certain row of C_1. Then, the element of $\mathbf{D}(b_u, a_v)$ in the u th row and v th column is

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial b_u}\frac{\partial h(\mathbf{x},m,n)}{\partial a_v}=\mathbf{V}_{\hat{b}b}(u)\mathbf{C}_1\mathbf{V}_{\hat{a}a}(v) \quad (35)$$

$\mathbf{V}_{\hat{a}a}(v)$ and $\mathbf{V}_{\hat{b}b}(u)$ are defined as,

$$\begin{aligned}\mathbf{V}_{\hat{a}a}(v) = [&S_{\hat{a}a}(v+p), S_{\hat{a}a}(v+(p-1)), \dots,\\ &S_{\hat{a}a}(v-(p-1)), S_{\hat{a}a}(v-p)\,]^T_{1\times(2p+1)}\,,\end{aligned} \quad (36)$$

where $S_{\hat{a}a}(l)$ corresponds to t_3 for $l = i - r$, and

$$\begin{aligned}\mathbf{V}_{\hat{b}b}(u) = \Big[&S_{\hat{b}b}(u+q), S_{\hat{b}b}(u+(q-1)), \dots,\\ &S_{\hat{b}b}(u-(q-1)), S_{\hat{b}b}(u-q)\,\Big]_{1\times(2q+1)}\,,\end{aligned} \quad (37)$$

where $S_{\hat{b}b}(l)$ corresponds to t_4 for $l = j - s$.

In the case of $\mathbf{D}(c_{ij}, a_v)$, the value of an element is given by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial c_{ij}}\frac{\partial h(\mathbf{x},m,n)}{\partial a_v}=\sum_{r=0}^{p}\sum_{s=0}^{q}c_{rs} \quad (38)$$
$$\left[\sum_{m=0}^{\infty}h_{\hat{a}}(m-r-v)h_a(m-i)\right]\left[\sum_{n=0}^{\infty}h_b(n-j)h_b(n-s)\right].$$

It can be verified that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial c_{ij}}\frac{\partial h(\mathbf{x},m,n)}{\partial a_v} = \mathbf{v}_{bb}(j)\ \mathbf{C}_2\ \mathbf{v}_{\hat{a}a}(i,v)\,, \quad (39)$$

where $\mathbf{C}_2$ is defined as

$$\mathbf{C}_2 = \begin{bmatrix} c_{p0} & c_{(p-1)0} & \cdots & c_{00}\\ c_{p1} & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & c_{0(q-1)}\\ c_{pq} & \cdots & c_{1q} & c_{0q}\end{bmatrix}_{q\times p}, \quad (40)$$

and

$$\mathbf{v}_{\hat{a}a}(i,v) = [S_{\hat{a}a}(p+v-i), \dots, S_{\hat{a}a}(v-i)\,]^T_{1\times(p+1)} \quad (41)$$
$$\mathbf{v}_{bb}(j) = [S_{bb}(j), \dots, S_{bb}(-(q-j))\,]_{1\times(q+1)}\,. \quad (42)$$

The shift property can also be applied in the calculation of the elements of $\mathbf{D}(c_{ij}, a_v)$. By considering the number of $\mathbf{v}_{\hat{a}a}(i,v)$ and $\mathbf{v}_{bb}(j)$ vectors, it can be seen that, for the $(p+1)(q+1)\times p$ matrix, there are $(q+1)\times 2p$ different elements needed to be calculated. The element of $\mathbf{D}(c_{ij}, b_v)$ can be determined in the same way as $\mathbf{D}(c_{ij}, a_v)$. Given that

$$\mathbf{v}_{aa}(i) = [S_{aa}(-(p-i)), \dots, S_{aa}(i)\,]^T_{1\times(p+1)}\,, \quad (43)$$
$$\mathbf{v}_{\hat{b}b}(j,v) = \Big[S_{\hat{b}b}(v-j), \dots, S_{\hat{b}b}(q+v-j)\,\Big]_{1\times(q+1)} \quad (44)$$

we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial c_{ij}}\frac{\partial h(\mathbf{x},m,n)}{\partial b_v} = \mathbf{v}_{\hat{b}b}(j,v)\ \mathbf{C}_2\ \mathbf{v}_{aa}(i)\,. \quad (45)$$

And $(p+1)\times 2q$ out of $(p+1)(q+1)\times q$ elements need to be computed. Notice that all the elements of $\mathbf{v}_{\hat{a}a}, \mathbf{v}_{bb}, \mathbf{v}_{aa}$ and $\mathbf{v}_{\hat{b}b}$ are the same as the ones in $\mathbf{V}_{\hat{a}a}, \mathbf{V}_{bb}, \mathbf{V}_{aa}$ and $\mathbf{V}_{\hat{b}b}$, which have already been calculated.

In the case of $\mathbf{D}(c_{ij}, c_{rs})$, the value of an element is given by

$$\sum_{m,n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial c_{ij}}\frac{\partial h(\mathbf{x},m,n)}{\partial c_{rs}} = \quad (46)$$
$$\left[\sum_{m=0}^{\infty}h_a(m-i)h_a(m-r)\right]\left[\sum_{n=0}^{\infty}h_b(n-j)h_b(n-s)\right].$$

The value of each term in the right of Eq. (46), corresponding to (i,r) or (j,s), has also been found in the calculation of $\mathbf{V}_{aa}$ or $\mathbf{V}_{bb}$. With the shift property, there are only $(p+1)\times(q+1)$ different elements needed to be computed.

Totally, with the model of IIR transfer function defined in Eq. (10), the $6pq + 4(p+q) + 1$ out of $[p+q+(p+1)(q+1)]^2$ elements of $\mathbf{N}$ need to be computed in each iteration.

3.2.3 Calculation of q

Next, consider the calculation of an element of $\mathbf{q}$ in Eq. (9). Since the given response is finite, the first term in the right of Eq. (9) can be found directly. For the last term, it can be shown that the infinite sum can be computed by using the result from $\mathbf{N}$. From Eq. (15) and Eq. (24), $h(\mathbf{x},m,n)$ can be written as

$$h(\mathbf{x},m,n) = \sum_{i=0}^{p}\sum_{j=0}^{q}c_{ij}\frac{\partial h(\mathbf{x},m,n)}{\partial c_{ij}}\,. \quad (47)$$

Substituting Eq. (47) into the last term of Eq. (9) and rearranging the term, we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}h(\mathbf{x},m,n)\frac{\partial h(\mathbf{x},m,n)}{\partial x_v} = \quad (48)$$
$$\sum_{i=0}^{p}\sum_{j=0}^{q}c_{ij}\underbrace{\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\partial h(\mathbf{x},m,n)}{\partial c_{ij}}\frac{\partial h(\mathbf{x},m,n)}{\partial x_v}\right]}_{t_5}.$$

Notice that t_5, corresponding to c_{ij} and x_v, has already been found in Eq. (39), Eq. (45) and Eq. (46). Define that

$$\mathbf{C}_3 = [c_{00}, \dots, c_{p0}, c_{01}, \dots, c_{p1}, \dots, c_{0q}, \dots, c_{pq}]. \quad (49)$$

Then, we obtain that the right of Eq. (48) equals the inner product of $\mathbf{C}_3$ and the corresponding column of $\mathbf{D}(c_{ij}, a_v), \mathbf{D}(c_{ij}, b_v)$ or $\mathbf{D}(c_{ij}, c_{rs})$. Hence, each element of $\mathbf{q}$ can readily be computed from Eq. (9).

3.2.4 Summary of The Calculation of $\Delta\mathbf{x}$

Now, we can summarize the calculation of the direction vector $\Delta\mathbf{x}$ as follows :

- Calculate $S_{aa}(l)$, $S_{bb}(l)$, $S_{\hat{a}\hat{a}}(l)$, $S_{\hat{b}\hat{b}}(l)$, $S_{\hat{a}a}(l)$, $S_{\hat{b}b}(l)$, also $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$.
- Substitute them into $\mathbf{V}_{aa}$, $\mathbf{V}_{bb}$, $\mathbf{V}_{\hat{a}\hat{a}}$, $\mathbf{V}_{\hat{b}\hat{b}}$, $\mathbf{V}_{\hat{a}a}$, $\mathbf{V}_{\hat{b}b}$, also $\mathbf{v}_{aa}$, $\mathbf{v}_{bb}$, $\mathbf{v}_{\hat{a}a}$, $\mathbf{v}_{\hat{b}b}$ and calculate all the submatrices of $\mathbf{N}$.
- Calculate $\mathbf{q}$, and then $\Delta\mathbf{x} = -\mathbf{N}^{-1}\mathbf{q}$.

It can be seen that the cost of calculation of the algorithm depends mainly on the calculation of $\mathbf{N}$, which is decided by the order of the IIR transfer function. Hence, the method provides a computational advantage, when the number of samples of the given response is large.

3.3 Initial Guess, Stabilization and Termination Conditions

The approximation of design parameters begins with guessing the initial values. Normally, the Shanks's method [2] is used to provide a good starting point. Then, the algorithm in Sect. 2 is operated iteratively in the calculation of $\mathbf{x}^{k+1}$ from $\mathbf{x}^k$. In each iteration, ρ has to be selected such that the denominator coefficients in $\mathbf{x}^{k+1}$ satisfy the stablility condition. Since the modified Gauss method always provides the descent direction and only the stable function can minimize Eq. (1), the value of ρ which stabilizes the transfer function and satisfies Eq. (3) and Eq. (4) should always be available. Some inventive methods in Ref. [6], [8] can also be applied in determining ρ.

The iteration is terminated when the error of the approximation becomes less than a preassigned value, or $\mathbf{x}$ reaches a stationary point. The normalized error criterion

$$\epsilon_2(\mathbf{x}) = \frac{\left[\sum_{(m,n)\in R_e} [\, h_o(m,n) - h(\mathbf{x},m,n)\,]^2 \right]^{\frac{1}{2}}}{\left[\sum_{(m,n)\in R_e} h_o^2(m,n) \right]^{\frac{1}{2}}}, \tag{50}$$

is used in measuring the quality of the approximation. The normalized maximum norm of error, used in determining the quality of approximation, is defined by

$$\epsilon_\infty(\mathbf{x}) = \frac{\max_{(m,n)\in R_e} |\, h_o(m,n) - h(\mathbf{x},m,n)\,|}{\max_{(m,n)\in R_e} |\, h_o(m,n)\,|}. \tag{51}$$

$\mathbf{x}$ is said to have converged to a stationary point when the square norm of $\Delta\mathbf{x}$ satisfies

$$(\Delta\mathbf{x})^T \cdot \Delta\mathbf{x} \leqq d \quad , \tag{52}$$

where d is a small positive scalar.

4. Illustrative Examples

Example 1: The given response is the first quadrant of a 23 × 23 circularly symmetric lowpass FIR filter with the cutoff frequency of 0.5π [rad] and the transition bandwidth of 0.224π [rad], designed by the linear programming technique [9]. The response was approximated by a 2-D IIR filter of order 4 × 4 in both the numerator and denominator. After the 12 × 12 samples in the first quadrant was designed, the final circularly symmetric response was constructed by using four quadrant filters in parallel [2].

The initial value of $\mathbf{x}$ was selected to be $a_i = b_j = -0.7$ $(1 \leqq i \leqq 4\ ,\ 1 \leqq j \leqq 4)$ and $c_{ij} = 0.01$ $(0 \leqq i \leqq 4\ ,\ 0 \leqq j \leqq 4)$. In each step, we simply selected the value of ρ to be 1, 1/2, 1/4, 1/8, ... respectively, until the error of Eq. (1) was improved. At the 15th iteration, d was found to be 5.738×10^{-7}, $\epsilon_2(\mathbf{x})$ was 0.050643 and $\epsilon_\infty(\mathbf{x})$ was 0.0270887. The resulting parameters are as follows,

$$\mathbf{a} = \mathbf{b} = \begin{array}{l} [\ 0.058787, \quad -0.506390, \\ \ \ 0.102317, \quad -0.060019\ ]^T \end{array} ,$$

$$\mathbf{c} = \begin{array}{rrr} [\ -0.071818, & -0.082540, & -0.032844, \\ -0.022323, & -0.001799, & -0.082534, \\ -0.082635, & -0.016656, & -0.015139, \\ 0.001455, & -0.032834, & -0.016648, \\ 0.017752, & 0.000911, & 0.000843, \\ -0.022319, & -0.015135, & 0.000909, \\ -0.011442, & -0.008080, & -0.001796, \\ 0.001453, & 0.000846, & -0.008081, \\ -0.001663\ \ ] & & \end{array}$$

The impulse response of the designed IIR filter is shown in Fig. 1 and the absolute error between impulse responses of FIR filter and the designed IIR filter ($|\ h_o(m,n) - h(m,n)\ |$, where $(m,n) \in R_e$) is shown in Fig. 2. The amplitude characteristics of the FIR filter and the designed IIR filter are shown in Fig. 3 and Fig. 4, respectively.

Example 2: The impluse response of the "Gaussian filter" given by,

$$f_{mn} = 0.256322\ e^{\,[-0.103203\{\ (m-4)\ ^2+(n-4)\ ^2\}]} ,$$

is approximated for $R_e = (0,0) \leqq (m,n) \leqq (10,10)$. The order of the 2-D IIR filter used in this approximation is 3 × 3 in both the numerator and denominator. This is the same specification as in Ref. [3]. In order to compare the approximation outside the region of support, we define another measuring function as

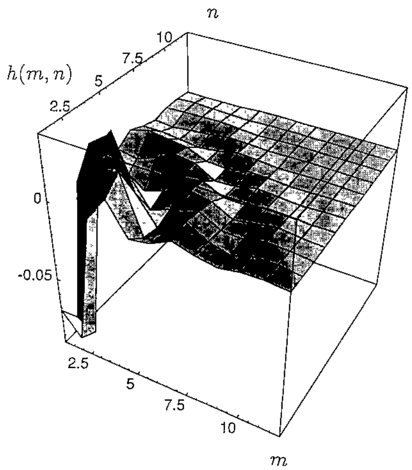

Fig. 1 The impulse response of the IIR lowpass filter.

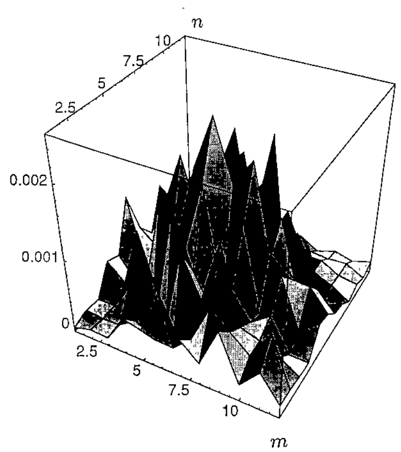

Fig. 2 The error between impulse responses of FIR prototype and IIR filter.

$$\epsilon_2'(\mathbf{x}) = \frac{\left[\sum_{(m,n)\in\hat{R}_e} [\, h_o(m,n) - h(\mathbf{x},m,n) \,]^2 \right]^{\frac{1}{2}}}{\left[\sum_{(m,n)\in\hat{R}_e} h_o^2(m,n) \right]^{\frac{1}{2}}}, \quad (53)$$

where $\hat{R}_e = \{(m,n) \mid (0,0) \leqq (m,n) \leqq (50,50)\, (m,n) \notin R_e\}$. The response of IIR function outside R_e and $\hat{R}_e$ is so small that it can be neglected in the comparison. And the negative ripple of $h(\mathbf{x},m,n)$ in R_e is also measured. The initial value of $\mathbf{x}$ and ρ value's selection were the same as in Example 1. At the 12th iteration, d was found to be 4.50498×10^{-8}. The resulting parameters are as follows,

$$\mathbf{a} = \mathbf{b} = [1.813177, -1.233216, 0.311953]^T,$$

$$\mathbf{c} = \begin{matrix} [\,0.00943040, & 0.00253772, & 0.00768342, \\ 0.00927025, & 0.00253772, & 0.00068174, \\ 0.00207115, & 0.00248973, & 0.00768342, \\ 0.00207115, & 0.00624516, & 0.00756444, \\ 0.00927025, & 0.00248973, & 0.00756444, \\ 0.00909975 \,] & & \end{matrix}$$

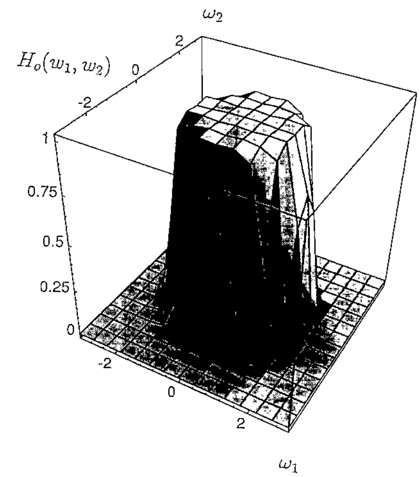

Fig. 3 The amplitude response of the FIR lowpass filter.

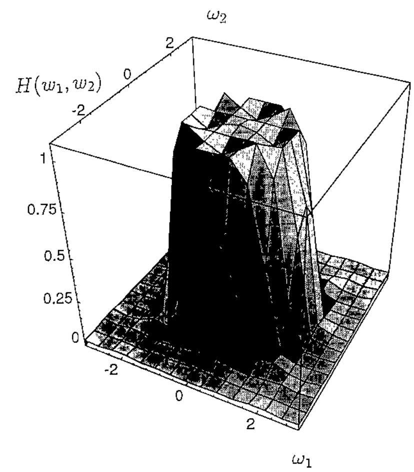

Fig. 4 The amplitude response of the IIR lowpass filter.

Table 1 Comparison of the error analysis.

	$\epsilon_2 \times 100$	ϵ_2'	$\epsilon_\infty \times 100$	Max. Negative Ripple in R_e
Design in [3]	1.5306	34.604	1.8403	always > 0
$h_1(m,n)$	2.9412	4.475	3.9057	all > 0
$h_2(m,n)$	1.4058	26.615	1.8622	all > 0

The error analysis of this IIR transfer function, named $h_1(m,n)$, together with those of the design in Ref. [3], are compared in the Table 1. The impulse response of $h_1(m,n)$ is shown in Fig. 5 and the absolute error between impulse responses of Gaussian filter and $h_1(m,n)$ ($|\, h_o(m,n) - h_1(m,n) \,|$, where $(m,n) \in R_e$) is shown in Fig. 6.

Next, the resulting parameters of Ref. [3] were used as the initial value of our design. Then, the same design procedure was repeated again, and the algorithm converged in the 2nd iteration. The resulting parameters are as follows,

$$\mathbf{a} = \mathbf{b} = [2.029145, -1.586161, 0.477145]^T,$$

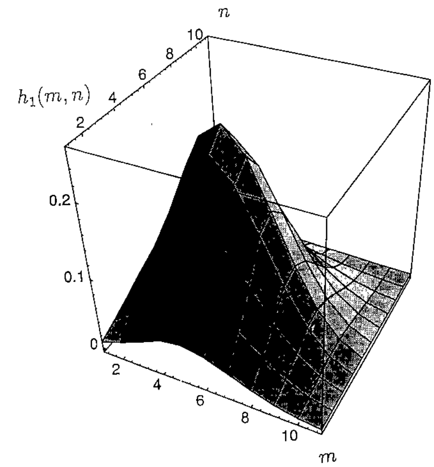

Fig. 5 The impulse response of $h_1(m,n)$.

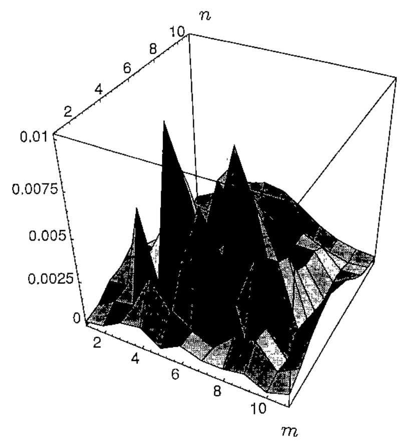

Fig. 6 The absolute error between impulse responses of Gaussian filter and $h_1(m,n)$.

$$\mathbf{c} = \begin{bmatrix} 0.00943001, & 0.00020926, & 0.00800834, \\ 0.00552796, & 0.00021596, & -0.00002314, \\ 0.00013811, & 0.00018366, & 0.00798094, \\ 0.00019669, & 0.00686095, & 0.00445545, \\ 0.00556815, & 0.00008934, & 0.00450664, \\ 0.00365848 & & \end{bmatrix}$$

The error analysis of this IIR function, named $h_2(m,n)$, is also compared in the Table 1. The impulse response of $h_2(m,n)$ is shown in Fig. 7 and the absolute error $| h_o(m,n) - h_2(m,n) |$, where $(m,n) \in R_e$, is shown in Fig. 8.

From the results of Example 2, it can be seen that a different initial value can lead to distinct values of resulting parameters. In the case of $h_1(m,n)$, although the total error in R_e is larger than that of the design in Ref. [3], the total error in $\hat{R}_e$ is much smaller. Since our design method has an ability to suppress the response outside the given region of support, the resulting IIR function thus yields a better response's performance in $\hat{R}_e$ in the case of the impulse response of Gaussian filter whose response tends to become smaller as (m,n)

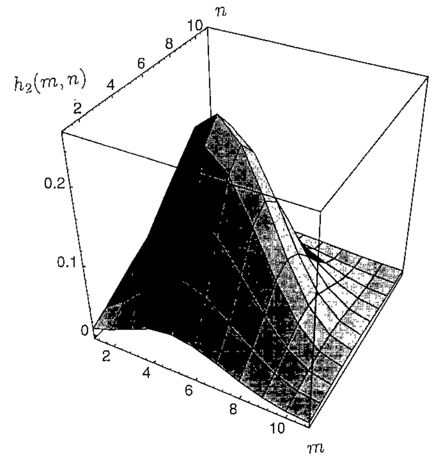

Fig. 7 The impulse response of $h_2(m,n)$.

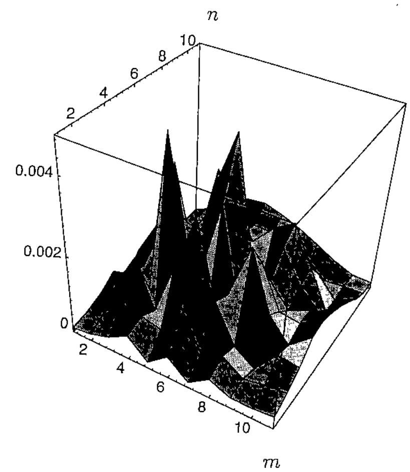

Fig. 8 The absolute error between impulse responses of Gaussian filter and $h_2(m,n)$.

increases. For $h_2(m,n)$, where the result from Ref. [3] was used as the initial value, we find that the total error in both regions can be reduced further. The basic difference in both designs is that, in our design, the error criterion Eq. (1), directly approximated by the Taylor's series up to second order, is utilized while, in Ref. [3], the equivalent criterion which only consists of a limited number of samples is optimized. Hence, the methods converge to different results. From Example 2, it can be said that our design can provide a well-suited result to the problem of approximating an impulse response with finite region of support if a suitable initial value is available.

5. Conclusion

The new design method for 2-D IIR digital filter, having a separable denominator, has been presented. It has been shown that a least square minimization of the error criterion in the entire spatial domain can be obtained. In favour of the characteristic of the method, the stable resulting filter is always available and the algorithm converges in a quadratic-like manner. The method can be well applied when the number of samples of the given response is large, since the cost of calculation depends mainly on the order of designed filter. The result from the examples shows that the method can provide a satisfying solution to the spatial domain approximation problem.

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Appendix

The infinite sum of impulse response functions like $h_a(n), h_{\hat{a}}(n), h_b(n)$ and $h_{\hat{b}}(n)$ can be determined directly by making partial fraction of denominators, and then finding the infinite sum of each separate term. Here, we introduce an equivalent way that uses the state space matrices in the calculation. The pulse transfer function matrix [7] of a linear time-invarient discrete-time system with zero initial conditions can be defined as

$$F(z) = \frac{Y(z)}{U(z)} = [\mathbf{C}(z\mathbf{I} - \mathbf{G})^{-1}\mathbf{H} + \mathbf{D}] , \quad \text{(A· 1)}$$

where $Y(z)$ and $U(z)$ are the output and input of system, respectively and all the state space matrices are defined as in Ref. [7]. In our case, $\mathbf{D}$ equals zero. Then, the impulse response of the system is given by

$$f(m) = \begin{cases} \mathbf{CG}^{m-1}\mathbf{H} & ; m = 1, 2, 3, \ldots \\ 0 & ; m \leqq 0 \end{cases} . \quad \text{(A· 2)}$$

Now, suppose that we want to determine

$$\begin{aligned} S &= \sum_{m=0}^{\infty} f(m)\, \hat{f}(m+l) \\ &= \sum_{m=0}^{\infty} \mathbf{H}^T(\mathbf{G}^T)^{m-1}\mathbf{C}^T\hat{\mathbf{C}}\, (\hat{\mathbf{G}})^{m+l-1}\, \hat{\mathbf{H}} , \quad \text{(A· 3)} \end{aligned}$$

where T is the transpose operation and l is the nonnegative integers. Define that

$$S = \mathbf{H}^T\mathbf{S}'\hat{\mathbf{H}} . \quad \text{(A· 4)}$$

Then, we obtain

$$\mathbf{S}' - \mathbf{G}^T\mathbf{S}'\hat{\mathbf{G}} = \mathbf{C}^T\hat{\mathbf{C}}\, (\hat{\mathbf{G}})^l , \quad \text{(A· 5)}$$

where f and $\hat{f}$ in each iteration are assumed to be stable responses. Equation (A· 5), which is known as the Liapunov equation, can be solved for $\mathbf{S}'$ by finding the solution of the linear equation system, and then S can be determined from Eq. (A· 4). To simplify the calculation, $\mathbf{G}$ and $\hat{\mathbf{G}}$ are transformed into diagonal forms or Jordan canonical forms.

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