論文 / 著書情報
Article / Book Information

Title	Filter Bank Implementation of the Shift Operation in Orthonormal Wavelet Bases
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Citation	IEICE Trans. Fundamentals, Vol. E79-A, No. 3, pp. 291-296
Pub. date	1996, 3
URL	http://search.ieice.org/
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PAPER *Special Section of Selected Papers from the 8th Karuizawa Workshop on Circuits and Systems*

Filter Bank Implementation of the Shift Operation in Orthonormal Wavelet Bases

Achim GOTTSCHEBER†, ***Student Member and*** **Akinori NISHIHARA**†, ***Member***

SUMMARY The purpose of this paper is to provide a practical tool for performing a shift operation in orthonormal compactly supported wavelet bases. This translation τ of a discrete sequence, where τ is a real number, is suitable for filter bank implementations. The shift operation in this realization is neither related to the analysis filters nor to the synthesis filters of the filter bank. Simulations were done on the Daubechies wavelets with 12 coefficients and on complex valued wavelets. For the latter ones a real input sequence was used and split up into two subsequences in order to gain computational efficiency.

key words: *orthonormal wavelet bases, shift operations, filter banks with perfect reconstruction*

1. Introduction

Viewing the discrete wavelet transform theory through the octave filter bank, an important property, the computation of shift operations in orthonormal compactly supported wavelet bases, will be presented, which are e.g. useful in image processing to move pictures in the compressed form or for some methods which solve partial differential equations.

The particular advantage of dyadic wavelets is that at different times and at different frequencies, a different resolution can be obtained (octave spacing of the cutoff frequency). This quality is used in many engineering fields. So as well in the discipline of filter banks with perfect reconstruction and applications thereof. M-channel filter banks with perfect reconstruction process orthogonal signals which can also be described by using the wavelet transform (WT). Define the mother wavelet and scaling function as

$$\psi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_1(n)\phi(2t-n) \tag{1}$$

$$\phi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_0(n)\phi(2t-n). \tag{2}$$

And using the concept of multiresolution, where the signal space $L_2(\mathbf{R})$ and the nested signal subspaces (scales) are related as

$$V_{-\infty} \subset ...V_{-1} \subset V_0 \subset V_1 \subset ...V_{\infty} = L_2(\mathbf{R}), \tag{3}$$

we can define an orthogonal complement W_i for every subspace $V_i \subset V_{i+1}$ such that

$$V_{i+1} = V_i \bigoplus W_i, \tag{4}$$

$i \in \mathbf{Z}$, where the complementary space W_i is spanned by an orthonormal base

$$\psi_{i,k}(t) = 2^{\frac{i}{2}}\psi(2^i t - k), \tag{5}$$

$i, k \in \mathbf{Z}$, and the space V_i is spanned by an orthonormal base

$$\phi_{i,k}(t) = 2^{\frac{i}{2}}\phi(2^i t - k), \tag{6}$$

$i, k \in \mathbf{Z}$. We can then project a signal $x(t) \in L_2(\mathbf{R})$ into subspaces which then corresponds to a wavelet transform. The wavelet transform is defined as (general definition)

$$WT\{x(t), a, b\} = \frac{1}{\sqrt{|a|}} \int_{t=-\infty}^{\infty} x(t)\psi^*(\frac{t-b}{a})dt \tag{7}$$

where b (time shift) and a (scaling factor) are real valued continuous variables. The discrete choice, e.g. see [1] or [2], $a = 2^{-i}$, $b = 2^{-i}k$, $i, k \in \mathbf{Z}$ and $\psi(t) = h^*(-t)$ leads to

$$WT\{x(t), i, k\} = 2^{\frac{i}{2}} \int_{t=-\infty}^{\infty} x(t)h(k - 2^{-i}t)dt \tag{8}$$

Considering the two-channel discrete-time case ($M = 2$, finite number of subspaces, $V_0 = V_{-1} \bigoplus W_{-1}$), the wavelet coefficients, $x_q(n)$, can be expressed as (filtered and decimated signals)

$$x_q(n) = \sum_{m=-\infty}^{\infty} x(m)h_q(2n-m), \tag{9}$$

$q = 0, 1$. For the two-channel discrete-time case, the inverse wavelet transform follows as

$$x(m) = \sum_{q=0}^{1} \sum_{n=-\infty}^{\infty} x_q(n)g_q(m-2n). \tag{10}$$

The analysis filters h_q as well as the synthesis filters g_q are assumed to fulfill the perfect reconstruction property and particularly

$$\sum_m h_0(m) = \sqrt{2} \tag{11}$$

Manuscript received June 6, 1995.
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$$\sum_m h_0^2(m) = 1 \tag{12}$$

$$\sum_m h_1(m) = 0 \tag{13}$$

$$\sum_m h_1^2(m) = 1 \tag{14}$$

2. Shift Operation

2.1 Statement of the Problem

In [3] the nonstandard form of the shift operator for the wavelet decomposition of all circulant shifts of a vector is reported, and [4] shows an algorithm to approximate the shift operator for real shifts in the finest scale, where in both cases the shift operator is related to the analysis filters and is a recurrent algorithm, respectively. However, the shift operation presented in this paper is neither related to the analysis filters nor to the synthesis filters of the filter bank. In particular the shift operation in this realization is done directly on the wavelet coefficients.

A practical tool to perform a translation τ, $\tau \in \mathbf{R}$, more suitable for filter bank implementations of the wavelet transform is provided below. The shift problem, here shown for the two-channel case, may be described as follows: For filter banks with perfect reconstruction one can write by using the z-transform representation with the output signal $\hat{X}(z)$ and the input signal $X(z)$

$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}[G_0(z)H_0(z) + G_1(z)H_1(z)]X(z) \\ &\quad + \frac{1}{2}[G_0(z)H_0(-z) + G_1(z)H_1(-z)]X(-z) \\ &= F_0(z)X(z) + F_1(z)X(-z)\end{aligned} \tag{15}$$

where $F_0(z) = z^{-k}$ and $F_1(z) = 0$.
This can be written in matrix form as

$$\hat{X}(z) = \frac{1}{2}\left[\begin{array}{cc} G_0(z) & G_1(z) \end{array}\right] \left[\begin{array}{cc} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{array}\right] \left[\begin{array}{c} X(z) \\ X(-z) \end{array}\right]. \tag{16}$$

The intermediate process where the input signal $X(z)$ is split up in two channels, filtered by the analysis filters $H_0(z)$ and $H_1(z)$ and down sampled by the factor M $= 2$, follows for the subsignals $X_0(z)$ and $X_1(z)$ in matrix form as

$$\left[\begin{array}{c} X_0(z) \\ X_1(z) \end{array}\right] = \frac{1}{2}\left[\begin{array}{cc} H_0(z^{\frac{1}{2}}) & H_0(-z^{\frac{1}{2}}) \\ H_1(z^{\frac{1}{2}}) & H_1(-z^{\frac{1}{2}}) \end{array}\right] \left[\begin{array}{c} X(z^{\frac{1}{2}}) \\ X(-z^{\frac{1}{2}}) \end{array}\right]. \tag{17}$$

The output signal $\hat{X}(z)$ of the synthesis filters results from this intermediate process after upsampling by the factor L $= 2$ and filtering with $G_0(z)$ and $G_1(z)$ in

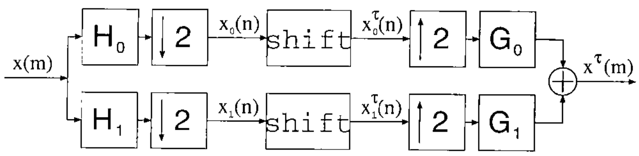

Fig. 1 Shift operation in orthonormal compactly supported wavelet bases, M=2.

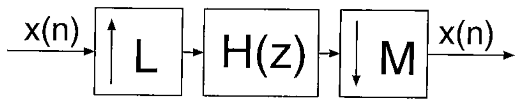

Fig. 2 L-fold expander followed by the interpolator filter $H(z)$ and a M-fold decimator (fractional sampling rate alteration).

$$\hat{X}(z) = \left[\begin{array}{cc} G_0(z) & G_1(z) \end{array}\right] \left[\begin{array}{c} X_0(z^2) \\ X_1(z^2) \end{array}\right] \tag{18}$$

Having the wavelet coefficients (subsignals) $x_0(n)$ and $x_1(n)$, the task is in finding the new coefficients (shifted subsignals) $x_0^\tau(n)$ and $x_1^\tau(n)$ such, that $x^\tau(m)$ is a shifted version of $x(m)$, where $x^\tau(m)$ is the τ shifted output signal and $x(m)$ the input signal. Figure 1 illustrates this in a pictorial way.

2.2 Shift Operation on the Wavelet Coefficients

The overall filter bank with perfect reconstruction has a constant group delay $F_0(z) = z^{-k}$. And it is possible to realize filter banks where the output signal is exactly the same as the input signal (theortically). However, our aim is to have an output signal which is a shifted version of the input signal. And we aim to perform it on the wavelet coefficients.

Since a rotation in the wavelet coefficients leads to an integer shift, the shift $0 \leqq \tau \leqq 1$ only needs to be considered. See also [5] page 261.

A signal $x(n)$ passed through an L-fold expander followed by the joined interpolator and decimator filter $H(z)$ and a M-fold decimator where L $=$ M, shown in Fig. 2, results in $y(n) = x(n)$.

Observing the fact that two sampling rate conversions (first upsampling by a factor L and second downsampling by the same factor M $=$ L) on the wavelet coefficients $x_0(n)$ and $x_1(n)$, but discarding by the downsampling process also the original values and therefore keeping other values, results in a set of shifted wavelet coefficients $x_0^\tau(n)$ and $x_1^\tau(n)$, we can imediately draw the conclusion that this yields finally to a shift of the sequence $x(m)$. Theoretically we can perform any desired shift by using this technique. In practice for some fractional shifts a dramatic increase of the sampling rate conversion circuit complexity follows. The solution to this problem is a continuously variable delay element (FIR-filter, $\tau \in \mathbf{R}$) described below, which has a reasonable complexity.

Let $x(nT_s)$ designate the sequence of the signal samples uniformly sampled with a sampling rate $1/T_s$ and n the sampling index of the signal samples. Also let $y(kT_i)$ designate the sequence of interpolated samples with an intermediate sampling rate $1/T_i$ where $T_i = T/K$. $1/T$ denotes the symbol rate, K a small integer and k the interpolator output index. The interpolator equation, for the k^{th} interpolator output sample, may then be expressed as

$$y(kT_i) = \sum_{n=M_1}^{M_2} x(nT_s)h_I(kT_i - nT_s). \tag{19}$$

M_1 and M_2 are limits determined by the finite response duration of a time-continuous, time-invariant, linear phase FIR filter $h_I(t)$. Since we aim also to have shifts $\tau \in \mathbf{R}$, the ratio T_i/T_s becomes irrational. Unlike conventional digital interpolation/decimation methods this method enables one to obtain an interpolated value $y(kT_i)$ which must not be related to a rational factor L/M of the sampling rate, where L and M are integer values. In order to synthesize a controllable delay, τ, in Eq. (19) a change of the indexing is required. The first control parameter n_k, the index for the k^{th} interpolant, is defined as†

$$n_k = \lceil \frac{kT_i}{T_s} \rceil \tag{20}$$

The second control parameter τ (delay), length of a fractional interval for the k^{th} interpolant ($0 \leqq \tau < 1$), is defined as

$$\tau = \frac{kT_i}{T_s} - n_k. \tag{21}$$

The filter index i, index on filter coefficients, is defined as

$$i = n_k - n. \tag{22}$$

Eq. (19) may be rewritten as

$$y(kT_i) = \sum_{i=I_1}^{I_2} x[(n_k - i)T_s]h_I[(i+\tau)T_s]. \tag{23}$$

The new limits I_1 and I_2 are large enough to include all nonzero coefficients of the finite impulse response of the filter; these modified limits are fixed numbers and do not depend upon n. The basepoint set length I, $I = I_2 - I_1 + 1$ and ranging from t_1 to t_2, indicates the required number of coefficients $h_I[(i+\tau)T_s]$ (taps) in the FIR filter. Interpolants $y(kT_i)$ are computed for times corresponding to

$$t = kT_i = (\tau + n_k)T_s. \tag{24}$$

For $t < t_2$ or $t \geqq t_1$ ($t_2 > t_1$) follows $h_I(t) = 0$. Thus the duration of the impulse response, T_0, is defined as

$$T_0 = t_2 - t_1. \tag{25}$$

It is straight forward that the inequality $IT_s \geqq T_0$ must be imposed for any method of implementing the interpolator in order that all non-zero portions of the continuous-time function are sampled correctly. A conventional interpolating polynomial, such as the Lagrange polynomial $I_N(t)$, leads to piecewise polynomials $L_i(t)$ (=Lagrange functions).

$$h_I(t) = \begin{cases} L_1(t) \\ L_2(t) \\ L_3(t) \\ . \\ . \\ . \end{cases} \tag{26}$$

The N^{th} degree polynomial $I_N(t)$, with the Lagrange functions $L_i(t)$, both given below, is fitted to the interpolant $y(kT_i)$ in such a way that the Lagrange functions $L_i(t)$ become identified with the coefficients of the interpolating filter $h_I(t)$.

$$I_N(t) = \sum_{i=0}^{N} L_i(t)x_i \tag{27}$$

and

$$L_i(t) = \prod_{j=0, j \neq i}^{N} \frac{t - t_i}{t_i - t_j}. \tag{28}$$

The reference [6] explains how to obtain Eq. (29) from Eq. (28) by substituting $t = (\tau + n_k)T_s$

$$L_i(\tau) = \prod_{j=I_1, j \neq i}^{I_2} \frac{\tau + j}{j - i}. \tag{29}$$

Replacing x_i with $x[(n_k - i)T_s]$ and $I_N(t)$ with $y(kT_i)$ yields to the interpolation function

$$y(kT_i) = \sum_{i=I_1}^{I_2} x[(n_k - i)T_s]L_i(\tau) \tag{30}$$

where each Lagrange function $L_i(\tau)$ of degree N is also defined as

$$L_i(\tau) = \sum_{j=0}^{N} b_j(i)\tau^j \tag{31}$$

and describes $h_I(t)$ in the i^{th} T_s interval. Only odd degree Lagrange functions [6] (N odd) should be employed in order to have a linear phase FIR interpolation filter with impulse response $h_I(t)$.

Using this technique to realize a continuously variable delay element, [7] and [8] show examples with an odd degree Lagrange function, which one of them we

† $\lceil x \rceil$ means: largest integer less or equal to the real number x.

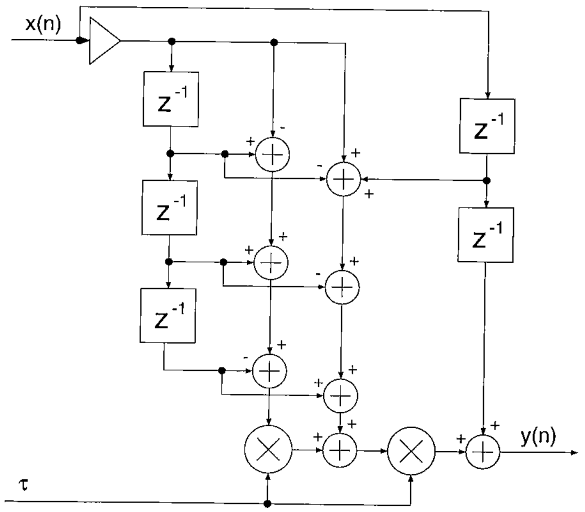

Fig. 3 FIR filter for shift operation on the wavelet coefficients.

Table 1 Filter coefficients $b_j(i)$ for the continuously controllable delay element, N =2, $I_1 = -2$, and $I_2 = 1$.

$i \downarrow$ $j \rightarrow$	0	1	2
−2	0	$-\alpha$	α
−1	0	$\alpha+1$	$-\alpha$
0	1	$\alpha-1$	$-\alpha$
1	0	$-\alpha$	α

adopt here for demonstrating a shift in orthonormal wavelet bases.

The coefficients $b_j(i)$ for the filter structure realizing a controllable delay, shown in Fig. 3, are given in Table 1 with $\alpha = 0.5$, $N = 2$, $I_1 = -2$ and $I_2 = 1$. Note, that having $\alpha \neq 0.5$ results in a dramatic increase of the delay element complexity. Using higher odd-degree Lagrange functions [9] yield to higer accuracy. Performance simulations are given in the next section.

3. Performance Simulations

Using the above given method we present in this section some simulation results of the shift operation in orthonrmal wavelet bases. Simulations were done by using the Daubechies wavelets with 12 coefficients (hence the order of the FIR filter obtained by spectral factorization of a linear phase half-band filter is 11) and on complex valued wavelets reported in [10], in order to test translations on the periodic function $x(m) = \cos(m)$ and the nonperiodic function $x(m) = \exp(-300\,m^2)$.

As both functions are real valued we used a method

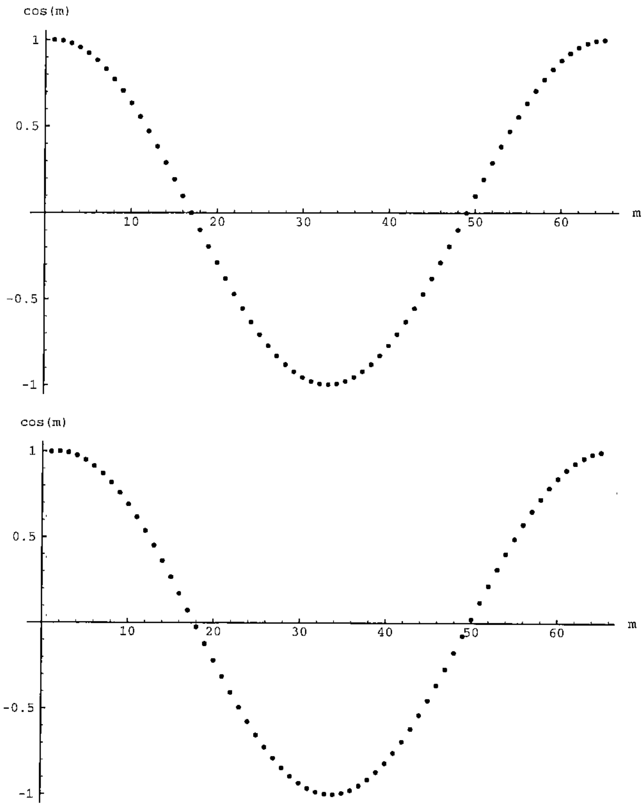

Fig. 4 Translations on the function $x(m) = \cos(m)$ for the shifts $\tau = 0$ and $\tau = 0.379$.

similar to the Fourier transform approach for the complex valued wavelets where $2N$ real samples are transformed with an N-sample complex transform. In other words, the imaginary part of the discrete transform was used to compute also real values which can be done by breaking the $2N$-point function $x(m)$, $m = 0, 1, 2, ..., 2N-1$, into two N-sample functions. Function $x(m)$ can be devided in half as follows:

$$a(m) = x(2m) \tag{32}$$

$$b(m) = x(2m+1) \tag{33}$$

with $m = 0, 1, 2, ..., N-1$.

Function $a(m)$ is equal to the even-numbered samples of $x(m)$, and $b(m)$ is equal to the odd-numbered samples of $x(m)$. Now, in order to use efficiently the wavelet transform, $a(m)$ and $b(m)$ are used to form the complex function $e(m)$.

$$e(m) = a(m) + jb(m) \tag{34}$$

with $m = 0, 1, 2, ..., N-1$.

The Figs. 4 and 5 show shifts for $\tau = 0$ and $\tau = 0.379$ for both test functions $x(m) = \cos(m)$ and $x(m) = \exp(-300\,m^2)$ on the Daubechies wavelets, respectively. We do not show here the plots for the complex wavelets as there can not be seen any difference within this graphical resolution to that for the Daubechies wavelets.

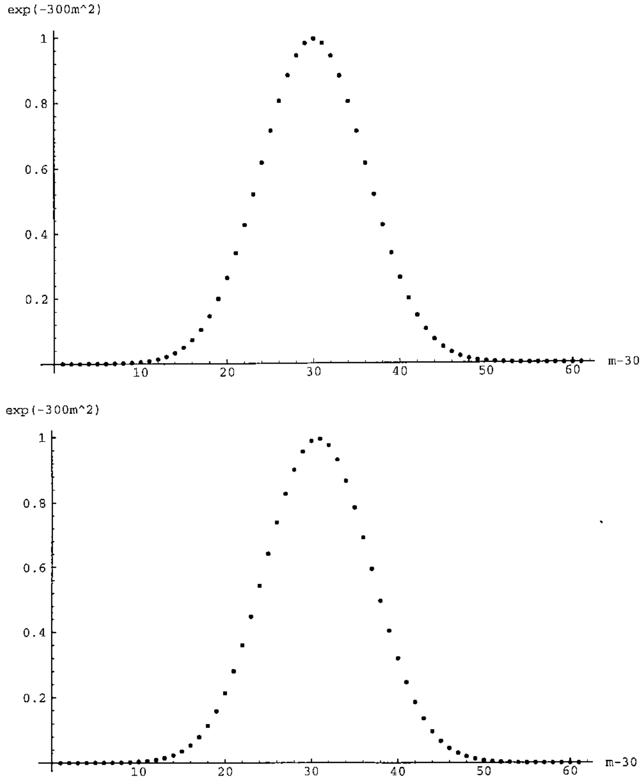

Fig. 5 Translations on the function $x(m) = \exp(-300\,m^2)$ for the shifts $\tau = 0$ and $\tau = 0.379$.

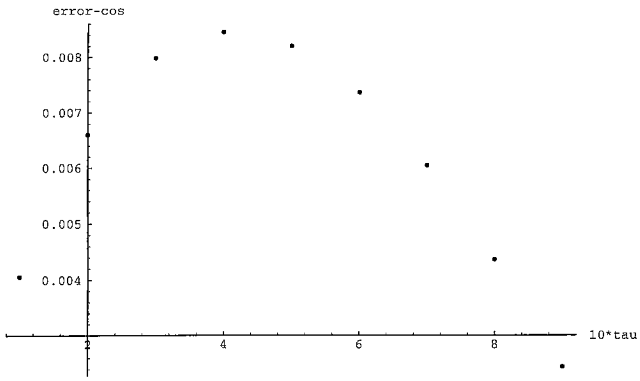

Fig. 6 Maximal absolute error, $x(m) = \cos(m)$.

Having used one particular delay element out of many possible ones, the accuracy of the delay element itself was tested by using the periodic function $x(m) = \cos(m)$ and the non-periodic function $x(m) = \exp(-300\,m^2)$. The maximal absolute error $e(\tau)$, where $e(\tau) = x(m) - x(m+\tau)$, is shown in Fig. 6 for the periodic input function (error-cos) with respect to 10τ and for the non-periodic input function (error-exp) with respect to 10τ in Fig. 7. For example, the maximal absolute error $e(\tau)$ in Fig. 6 for $\tau = 0.2$ is 0.0066.

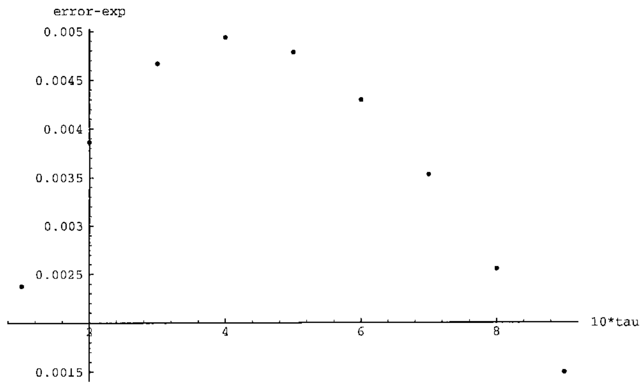

Fig. 7 Maximal absolute error, $x(m) = \exp(-300\,m^2)$.

4. Conclusions

A novel realization of the shift operation for a shift of $\tau \in \mathbf{R}$ in orthonormal wavelet bases suitable for filter bank implementations is proposed. Its performance is very simple by having a high accuracy at the same time. Higher accuracy can be achieved, if desired, by designing higher odd order Lagrange functions for the applied continuously variable delay element. Simulations were done on a real as well as on a complex filter bank with perfect reconstruction for the periodic function $x(m) = \cos(m)$ and for the nonperiodic function $x(m) = \exp(-300\,m^2)$. Additionally it was shown that one can split up a real input sequence into two subsequences in order to gain computational efficiency while using a complex filter bank and having a real input sequence.

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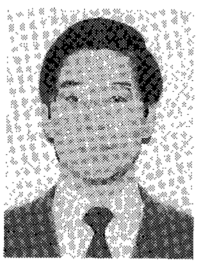

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