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PAPER

Analytic Constructions of Periodic and Non-periodic Complementary Sequences

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SUMMARY An analytic approach for the generation of non-periodic and periodic complementary sequences is advanced for lengths that are powers of two. The periodic complementary sequences can be obtained using symmetric or anti-symmetric extensions. The properties of their autocorrelation functions are studied. The non-periodic complementary sequences are the intersection between anti-symmetric and symmetric periodic sequences. These non-periodic and periodic complementary sequences are identified to be special cases of non-periodic and periodic (or cyclic) orthogonal wavelet transforms. This relationship leads to the novel approach.

key words: *correlation, discrete Fourier transforms, orthogonal functions, sequences, transforms, wavelet transforms*

1. Introduction

There is a wealth of literature on the theory and design of pseudo-random (or pseudo-noise) sequences for communications with different properties of their autocorrelation and cross-correlation functions (ACF and CCF) [1]–[5], [12]–[27].

Perfect-reconstruction (PR) filter banks have been intensely studied over the last twenty years. Orthogonal filter banks provide orthogonal bases for the Hilbert space of square-summable sequences [6]. Furthermore provided that the filters satisfy constraints additional to PR, regular (or smooth) continuous-time functions (scaling functions and wavelets) can be obtained, which are orthogonal bases for the space of square-integrable functions [6].

The main purpose of this paper is firstly to demonstrate the relationship between wavelet transform theory and the theory of complementary sequences, and secondly to develop novel formulae for the analytic construction of complementary sequences using wavelet (or filter bank) theory. We shall consider one important class of sequences, namely complementary sequences. These sequences were recently found to be efficient in a new modulation for wireless communications, called spread-signature CDMA [11]. The connection between two-channel orthogonal FIR filter banks and *aperiodic* complementary sequences was observed by several researchers [8], [10] and is not novel. *Periodic* complementary sequences were advanced in [27]. It is shown here that they can be approached using the cyclic wavelet transform. This allows us to develop systematic algorithms for their generation. These two new sets of orthogonal sequences are generalizations of the Golay sequences in the sense that the Golay sequences are members of both of these sets. The novel approach allows to derive explicit formulas for the systematic generation of Golay sequences, when the length is a power of two. Previously these sequences could only be generated using computer searches.

The paper is organized as follows. In Sect. 2 we review filter bank theory and complementary sequences. Section 3 is devoted to orthogonal periodic symmetric codes, and Sect. 4—to anti-symmetric codes. Section 5 is devoted to explicit formulas for Golay complementary pairs.

2. Two-Channel Orthogonal FIR Filter Banks and Aperiodic Complementary Sequences

Two-channel orthogonal FIR filter banks are the most fundamental and widely used class of filter banks [6], [7]. They consist of two parts (Fig. 1): an analysis part of two filters $H_0(z)$ and $H_1(z)$, each followed by downsampling, and a synthesis part, consisting of upsampling in each channel followed by two filters $G_0(z)$ and $G_1(z)$. It is easily shown that the output signal, $\hat{X}(z)$ is given by

$$\hat{X}(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]X(z) + \frac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]X(-z) \quad (1)$$

In perfect-reconstruction (PR) filter banks we have $\hat{X}(z) = X(z)$ and therefore

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2 \quad (2)$$

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \quad (3)$$

The aperiodic auto-correlation function (ACF) of the impulse responses $h_0[n]$ and $h_1[n]$ are half-band functions:

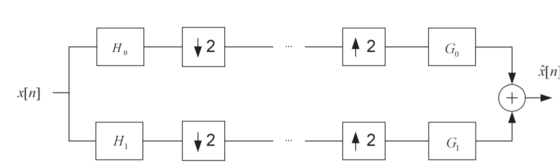

Fig. 1 A two-channel filter bank.

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$$\langle h_0[n], h_0[n+2k]\rangle = \delta_k \tag{4}$$
$$\langle h_1[n], h_1[n+2k]\rangle = \delta_k, \tag{5}$$

while the cross-correlation is identically zero

$$\langle h_0[n], h_1[n+2k]\rangle = 0 \tag{6}$$

The synthesis filters are completely determined from the analysis filters:

$$G_0(z) = H_1(-z) = z^{-N}\tilde{H}_0(z) \tag{7}$$
$$G_1(z) = -H_0(-z) = z^{-N}\tilde{H}_1(z) \tag{8}$$

where the $\tilde{\ }$ operation means transposition, conjugation of the coefficients and replacing z by z^{-1}. In the time-domain $h_1[n]$ is related to $h_0[n]$ according to

$$h_1[n] = -h_0[N-n](-1)^{-n+1} \tag{9}$$

where N is the order of the filters and is necessarily odd. Any two sequences $h_0[n]$ and $h_1[n]$ with the auto-correlation and cross-correlation properties in (4), (5) and (6) define an orthogonal wavelet transform and the two sequences are an orthogonal basis for the Hilbert space of square-summable sequences. Provided that $H_0(z)$ is regular, the impulse response of the iteration $\prod_{l=0}^{i-1} H_0(z^{2^l})$ converges to a continuous-time function called scaling function and the impulse response of the iteration $\left[\prod_{l=0}^{i-2} H_0(z^{2^l})\right] H_1(z^{2^{i-1}})$ converges to a continuous-time function called a wavelet.

The theory of Golay-Rudin-Shapiro (or complementary) sequences dates back to 1949 [2]. By definition a complementary series consists of two finite sequences of 1's and −1's such that the sum of autocorrelation functions of the two sequences is constant. These complementary sequences have been rediscovered several times. They have challenging properties from a theoretical perspective, and since the coefficients are binary, have obvious computational advantages in practical implementations. Thus, two sequences of length l,

$$A = (a_0, a_1, \ldots a_l), \tag{10}$$
$$B = (b_0, b_1, \ldots b_l), \tag{11}$$

where each entry equals 1 or −1, form a pair of Golay complementary sequences if they satisfy the $l-1$ conditions

$$\sum_{i=0}^{l-j-1} (a_i a_{i+j} + b_i b_{i+j}) = 0 \tag{12}$$

for $j = 1, \cdots l-1$. Using polynomial notation the two sequences A and B are complementary if and only if

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) = 2l \tag{13}$$

If A and B are complementary then the following operations produce complementary sequences of the same length: (1) Negating A and/or B; (2) Reversing A and/or B; (3) Negating the polyphase components of A and B. There are formulas to produce longer complementary pairs, starting from shorter ones [1], for example, if A and B are complementary of length l, $C(z) = A(z) + z^{-l}B(z)$ and $D(z) = A(z) - z^{-l}B(z)$ are also complementary of length $2l$. This construction is iterative, which is different from the explicit approach advanced here. Using the formulas introduced in this paper for a given length all complementary pairs can be generated when the length is a power of two, which implicitly takes into account all properties of the complementary pairs mentioned above.

Complementary sequences have found various applications in CDMA wireless communication systems [11] and data communications systems [19]. Note that there is a close relationship between PR filter banks and Golay-Rudin-Shapiro systems, which has not been recognized before:

Theorem 1 (Cooklev '95) The Golay-Rudin-Shapiro (GRS) polynomial pairs are polyphase components of a lowpass filter in an orthogonal maximally-decimated two-channel FIR filter bank.

Proof: Suppose we are given a filter $H(z)$ of length $2l-1$ with coefficients which are only +1 and −1 satisfying $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = const = 4$. It can be proven that the polyphase components of $H(z)$ satisfy (13), i.e. they form a GRS polynomial pair:

$$\begin{aligned} 4l &= \left[H_0(z^2) + z^{-1}H_1(z^2)\right]\left[H_0(z^{-2}) + zH_1(z^{-2})\right] \\ &\quad + \left[H_0(z^2) - z^{-1}H_1(z^2)\right]\left[H_0(z^{-2}) - zH_1(z^{-2})\right] \\ &= 2\left[H_0(z^2)H_0(z^{-2}) + H_1(z^2)H_1(z^{-2})\right] \end{aligned} \tag{14}$$

Therefore the polyphase components of every power-complementary filter $H(z)$ are a GRS pair. Now it is straightforward to establish that the filter with polyphase components equal to a GRS pair is power-complementary. Q.E.D.

Years before the advent of wavelet transforms it was recognized that these GRS pairs are the polyphase components of E-sequences, which provide orthonormal bases for the Hilbert space [5]. The E-sequences having zero values of the autocorrelation function in even shifts have correlation properties close to optimal. It has apparently escaped evidence the fact that these E-sequences have the same properties as the product $P(z)$ in filter bank theory. Note that while there are PR FIR filter banks of every even length, the requirement the length of the Golay sequences to be even is not sufficient. J. Byrnes in [10] realized that GRS sequences are related to filter banks, but he did not state exactly that they are the polyphase components. The above theorem was proven for the first time in [8]. It seems that the first lowpass filter with more than 2 coefficients for FIR perfect-reconstruction filter banks have been designed by Golay as early as 1949. Note that the restriction the coefficients to be binary (1 and −1) constrains the zeros of the filter $H(z)$ and as a result the filter bank is non-regular, i.e. the impulse response of the iteration $\prod_{l=0}^{i-1} H(z^{2^l})$ does not converge to a continuous-time function.

Following Golay's work, mathematical properties, computer searches and existence problems for certain lengths were further investigated by various researchers.

Different applications have required different generalizations of the original concept of Golay to be made [1]. For their research into surface acoustic wave (SAW) devices Tseng and Liu studied complementary sets of sequences [18]. Welti advanced sequences of vectors which could be successfully used in pulsed radar for range detection [15]. Complex-valued complementary sequences were considered by Frank; they have become known as Frank codes and have applications in the area of radar pulse compression. Subcomplementary and supercomplementary sequences are two relatively new extensions of GRS sequences. The fact that using a GRS pair we can build an orthogonal filter $H(z)$ which forms a basis for square-summable sequences was observed in [5]. It is, however, clear that the set of all possible extensions of GRS sequences is isomorphic to the set of all possible filter banks. Just as all filter banks have useful properties, by using the filter bank framework new sequences can be obtained that have useful properties.

3. Orthogonal Periodic Symmetric Codes

It must be noted at this point, that the theory of filter banks is usually developed assuming linear (or aperiodic) convolutions. However, when filter banks are used in data compression to avoid the increase in the number of samples (which would have compromized the compression performance) periodic (or cyclic) convolution is used. The corresponding wavelet transforms are called periodic (or cyclic). In this paper we use specifically cyclic wavelet transforms to design cyclic extensions of complementary sequences. Here we consider the problem of the design of orthogonal system $\{s_0, s_1, \cdots s_{M-1}\}$. It is convenient and simple to assume that all orthogonal signals s_i are generated by cyclic shifts of $s_0 = (a_0\ a_1 \cdots a_{N-1})$ and that the sequence s_i is periodic with period $N : a_{N+i} = a_i$. To simplify the signal processing operations it is desirable to deal with binary symbols, i.e. ± 1. It is clear that the maximum size of this cyclic code, that is the maximum number of different codewords, is equal to N. If the code is of maximum size, then the sequences s_i will be generated by single cyclic shifts and $M = N$. This problem is similar with filter bank theory. The codewords play the role of impulse responses of digital filters in a filter bank. If orthogonality is imposed orthogonal cyclic codes of maximum size do not exist. Following the wavelet transform approach, however, orthogonal periodic codes can be constructed with size equal to $N/2$. The properties of sequences depend on their autocorrelation functions (ACFs). Since we assumed periodic sequences it is convenient to use the periodic autocorrelation function (PACF)

$$r[n] = \sum_{i=0}^{N-1} a[i]a[\langle i+n\rangle_N] \quad a[i] \in 1, -1 \tag{15}$$

where $a[N+i] = a[i]$ and $\langle . \rangle$ is the modulo notation.

Theorem 2 The system of codewords formed by double cyclic shifts of the sequence $s_0 = (a_0, a_1, \cdots, a_{N-1})$ with length N is orthogonal iff

$$r[2n] = 0, \quad n = 1, 2, \cdots N/4, \tag{16}$$

and its size is $N/2$.

Proof: Clearly $s_{2n} = (a_{2n}, a_{2n+1}, \cdots, a_{2n+N-1})$ is formed by double cyclic shifts of the sequence s_0. If we assume that s_{2n} is orthogonal to s_0, then

$$a_0 \cdot a_{2n} + a_1 \cdot a_{2n+1} + a_2 \cdot a_{2n+2} + \cdots + a_{N-1} \cdot a_{2n+N-1} = 0. \tag{17}$$

It is seen that the left side of (17) is equal to the periodic autocorrelation function in (15), i.e. $r[2n] = 0$. Now, if it is assumed that $r[2n] = 0$, then orthogonality follows from (17). Q.E.D.

Using the discrete Fourier transform (DFT) it can be written that

$$\begin{aligned} R[k] &= \sum_{n=0}^{N-1} r[n] W_N^{nk} \\ &= \sum_{n=0}^{N-1}\sum_{i=0}^{N-1} a[i]a[\langle i+n\rangle_N] W_N^{nk} \quad \langle i+n\rangle_N = l \\ &= \sum_{i=0}^{N-1} a[i] \sum_{l=0}^{N-1} a[l] W_N^{(l-i)k} \\ &= A[k]A[-k] = |A[k]|^2 \end{aligned} \tag{18}$$

In our notation

$$W_N = e^{-j2\pi/N}. \tag{19}$$

A fundamental property of the DFT is that it assumes periodicity in both time- and frequency-domains. Note that the DFT of the PACF is non-negative, which corresponds to the condition that the frequency response of the product filter in filter banks be non-negative. A polyphase decomposition can be applied on the PACF

$$R(z) = R_0(z^2) + z^{-1} R_1(z^2), \tag{20}$$

which in the DFT domain corresponds to

$$R[k] = R_0[2k] + W_N^{-k} R_1[2k] \tag{21}$$

where

$$R_0[2k] = \sum_{i=0}^{N/2-1} r[2i] W_N^{i2k} \tag{22}$$

$$R_1[2k] = \sum_{i=0}^{N/2-1} r[2i+1] W_N^{i2k} \tag{23}$$

Note that $R_0[2k]$ and $R_1[2k]$, as well as the similarly defined $A_0[2k]$ and $A_1[2k]$, are not DFTs themselves. Since all even-indexed coefficients $r[2i]$ are equal to zero, with the exception of $r[0]$ we get

$$R_0[2k] = r[0] = N \tag{24}$$

Taking (21) into consideration we get

$$R[k] = \left(A_0[2k] + W_N^k A_1[2k]\right)\left(A_0[-2k] + W_N^k A_1[-2k]\right)$$

$$= A_0[2k]A_0[-2k] + A_1[2k]A_1[-2k] + W_N^k A_1[2k]A_0[-2k] + W_N^{-k} A_0[2k]A_1[-2k] \quad (25)$$

Note that $W_N^k A_1[2k]A_0[-2k]$ and $W_N^{-k} A_0[2k]A_0[-2k]$ are complex conjugates of each other. The conclusion is that

$$R_0\,[2k] = |A_0[2k]|^2 + |A_1[2k]|^2 \quad (26)$$

$$W_N^k R_1[2k] = 2\mathrm{Re}\left\{W_N^k A_1[2k]A_0[-2k]\right\} \quad (27)$$

The necessary and sufficient condition for orthogonality of the codewords is

$$|A_0[2k]|^2 + |A_1[2k]|^2 = N \quad (28)$$

The problem is how to find all orthogonal filters with binary coefficients? The conditions of orthogonality are invariant under the following operations:

- Sign inversion, i.e. if $A(z)$ is a codeword, then $-A(z)$ is also a codeword.
- Inversion of the order $a_i \leftarrow a_{N-1-i}$, i.e. if $A(z)$ is a codeword, then $z^{-N}\tilde{A}(z)$ is also a codeword.
- Cyclic shifts

3.1 The Structure of Codewords

The polynomial representation of the codeword s_0 is given by $A(z)$, which can be decomposed as

$$A(z) = A_0(z^2) + z^{-1}A_1(z^2) \quad (29)$$

In the same way, as it was done before it can be established that these polyphase components are complementary sequences, which are periodic, however. (The non-periodic complementary sequences are the GRS sequences) An interesting question is whether these complementary sequences are themselves codewords.

The PCF of $(a_i, a_{i+2}, \cdots a_{i+N-2})$, $i = 0, 1$ are

$$r_i[n] = \sum_{k=0}^{N/2-1} a[2k+i]a[2(k+n)+i] \quad (30)$$

where the indices must be evaluated (mod N). Therefore

$$R_i[k] = \sum_{n=0}^{N/2-1} r_i[n]W_{N/2}^{nk} = A_i[k]A_i[-k] = |A_i[k]|^2 \quad i = 0, 1 \quad (31)$$

From (30) it follows that

$$\begin{aligned} r_0[n] + r_1[n] &= \sum_k a[2k]a[2k+2n] \\ &\quad + \sum_k a[2k+1]a[2k+2n+1] \\ &= r[2n] \quad n = 1, \cdots N/4 \end{aligned} \quad (32)$$

Since we know that $r[2n] = 0$, $n \neq 0$, the necessary and sufficient conditions for orthogonality are

1. Each of these complementary sequences are themselves codewords, i.e. $r_i[2n] = 0$ and

$$r_0[2n-1] = -r_1[2n-1] \quad n = 1, 2, \cdots N/8 \quad (33)$$

2. The complementary sequences are not codewords, i.e. the condition $r_i[2n] = 0$ fails for at least one n; then

$$r_0[n] = -r_1[n], \quad n = 1, 2, \cdots N/8 \quad (34)$$

Let G_N be the set of all codewords with length N. This set is a union of two sets: the set G_N^1 of codewords the polyphase components of which are themselves codewords of length $N/2$ and the set G_N^2 of codewords the polyphase components of which are not codewords.

Theorem 3 A periodic cyclic code with length $N = 2^k$ exists for all values of k greater than 2.

Proof: First, it will be shown that a periodic cyclic code exists for k= 2. It is recognized that filter banks whose polyphase components are GRS polynomials with length $N/2$ belong to the set G_N. Therefore $(1, 1, 1, -1)$ is a codeword. By cyclic shifts and sign inversions we can get 7 other codewords, or the total size of the set G_4 is 8. Now, suppose that $A_0(W_{N/2}^k) \in G_{N/2}$ which has the polyphase decomposition

$$A_0(W_N^{2k}) = A_{00}(W_N^{4k}) + W_N^{2k}A_{01}(W_N^{4k}) \quad (35)$$

Since $A_0(W_{N/2}^k) \in G_{N/2}$ the condition of orthogonality

$$\left|A_{00}(W_N^{4k})\right|^2 + \left|A_{01}(W_N^{4k})\right|^2 = N/2 \quad (36)$$

holds. Let us define

$$A_1(W_N^{2k}) = \pm W_N^{kl}\left[A_{00}(W_N^{4k}) - W_N^{2k}A_{01}(W_N^{4k})\right] \quad (37)$$

The polyphase components of A_1 also satisfy (35) and thus A_1 is also a codeword, $A_1 \in G_{N/2}$. The sequence $\tilde{A}_1$ is also a codeword. Finally A_0 and A_1 can be shown to be polyphase components of a codeword with length N by taking into account (28) and (35)

$$\left|A_0(W_N^{2k})\right|^2 + \left|A_1(W_N^{2k})\right|^2 = N \quad (38)$$

Therefore $A(z) = A_0(z^2)+z^{-1}A_1(z^2)$ is a codeword belonging to the set G_N. Q.E.D.

From this construction it is obvious that the polyphase components of A_1 are not independent of those of A_0. For even values of l and apart from a shift, the first polyphase components of A_0 and A_1 coincide, and the second polyphase components are opposites. From (37) we understand that apart from shifting the sequence as a consequence of the term W_N^l A_1 is constructed by interleaving the two $N/4$-length sequences and changing the sign of the second one. This operation is the same as the one in property 10 of Ref. [3] for aperiodic sequences.

3.2 Construction of Codewords

It is convenient to introduce four DFTs:

$$B(W_4^k) = 1 - W_4^k - W_4^{2k} - W_4^{3k} \tag{39}$$
$$C(W_4^k) = 1 + W_4^k - W_4^{2k} - W_4^{3k} \tag{40}$$
$$D(W_4^k) = 1 + W_4^k + W_4^{2k} + W_4^{3k} \tag{41}$$
$$E(W_4^k) = 1 - W_4^k + W_4^{2k} - W_4^{3k} \tag{42}$$

Using these four DFTs we shall try to construct codewords satisfying the orthogonality condition (28).

3.2.1 Codewords with Length N=4

For N=4, we can see $\left|B(W_4^k)\right|^2 = 4$, so codewords for N=4 can be constructed by cyclic shifts and sign inversions of B:

$$A[k] = A(W_4^k) = \pm W_4^{kl} B(W_4^k) \quad l \in \{0, 1, 2, 3\}. \tag{43}$$

The capacity of this code is $Q_4 = 8$.

3.2.2 Codewords with Length N=8

For N= 8 the codewords can be constructed starting from codewords of length 4:

$$\begin{aligned} A[k] = A(W_8^k) &= \pm W_8^{kl} B\left(W_8^{2k}\right)\left(1 \pm W_8^{k(2m+1)}\right) \\ & m \in \{0, 1, 2, 3\} \;\; l \in \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned} \tag{44}$$

In this case the total number of such sequences is $4 \cdot 4 \cdot 2 \cdot 2 = 2^6$. For example when l= 0 and m= 0 we get the codeword $(1, 1, -1, -1, -1, -1, -1, -1)$. Table 1 lists all 64 sequences.

3.2.3 Codewords with Length N=16

For N= 16 we know that all codewords from G_8 are first polyphase components of codewords from G_{16}. The second polyphase components can be found from (36) and (37)

$$\begin{aligned} A_{16,1}(W_{16}^k) &= \pm W_{16}^{kl} B(W_{16}^{4k}) \left[\left(1 \pm W_{16}^{2k(2m+1)}\right)\right. \\ &\quad \left. + W_{16}^{2p+1}\left(1 - \pm W_{16}^{2k(2m+1)}\right)\right], \\ & m, p \in \{0, 1, 2, 3\} \\ & l \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}. \end{aligned} \tag{45}$$

By counting the number of free variables we get for the capacity of this code $G_{16}^1 = 4 \cdot 16 \cdot 8 \cdot 2 = 2^{10}$. But there are more orthogonal sets of sequences in G_{16}. The set G_8 does not exhaust all complementary pairs. Note that

$$2\left|C(W_4^k)\right|^2 + \left|D(W_4^k)\right|^2 + \left|E(W_4^k)\right|^2 = 16 \quad \forall k \in \{0, 1, 2, 3\} \tag{46}$$

Therefore we can construct more codewords in the set G_{16} as follows

$$\begin{aligned} A_{16,2}(W_{16}^k) &= W_{16}^{lk}\left[C(W_{16}^{4k}) \pm W_{16}^{2k} E(W_{16}^{4k})\right. \\ &\quad \left. \pm W_{16}^{(2p+1)k}\left(C(W_{16}^{4k}) \pm W_{16}^{2k} D(W_{16}^{4k})\right)\right] \\ & p \in \{0, 1, 2, 3\} \quad l \in \{0, 1, 2, 3, 4, 5, 6, 7\} \end{aligned} \tag{47}$$

This can be verified to be codeword by checking the PCF:

$$\begin{aligned} \left|A_{16,2}(W_{16}^k)\right|^2 &= \left|C \pm W_{16}^{2k} E\right|^2 + \left|C \pm W_{16}^{2k} D\right|^2 \\ &\quad + 2\mathrm{Re}\left\{W_{16}^{k(2p+1)}(C \pm W_{16}^{2k} D)(C \pm W_{16}^{2k} E)^*\right\} \\ &= 16 + 2\mathrm{Re}\left\{W_{16}^{(2m+1)k}\,|C|^2\right\} \\ &= 16 + 8\mathrm{Re}\left\{W_{16}^{(2p+1)k}(1 - W_{16}^{8k})\right\} \end{aligned} \tag{48}$$

Table 1 Periodic symmetric complementary sequences with length N=8.

1	-1	1	1	-1	1	-1	1	-1
2	-1	-1	1	1	-1	1	-1	1
3	1	-1	-1	1	1	-1	1	-1
4	-1	1	-1	-1	1	1	-1	1
5	1	-1	1	-1	-1	1	1	-1
6	-1	1	-1	1	-1	-1	1	1
7	1	-1	1	-1	1	-1	-1	1
8	1	1	-1	1	-1	1	-1	-1
9	1	-1	-1	1	-1	1	-1	1
10	1	1	-1	-1	1	-1	1	-1
11	-1	1	1	-1	-1	1	-1	1
12	1	-1	1	1	-1	-1	1	-1
13	-1	1	-1	1	1	-1	-1	1
14	1	-1	1	-1	1	1	-1	-1
15	-1	1	-1	1	-1	1	1	-1
16	-1	-1	1	-1	1	-1	1	1
17	-1	-1	1	1	1	1	1	1
18	1	-1	-1	1	1	1	1	1
19	1	1	-1	-1	1	1	1	1
20	1	1	1	-1	-1	1	1	1
21	1	1	1	1	-1	-1	1	1
22	1	1	1	1	1	-1	-1	1
23	1	1	1	1	1	1	-1	-1
24	-1	1	1	1	1	1	1	-1
25	1	1	-1	-1	-1	-1	-1	-1
26	-1	1	1	-1	-1	-1	-1	-1
27	-1	-1	1	1	-1	-1	-1	-1
28	-1	-1	-1	1	1	-1	-1	-1
29	-1	-1	-1	-1	1	1	-1	-1
30	-1	-1	-1	-1	-1	1	1	-1
31	-1	-1	-1	-1	-1	-1	1	1
32	1	-1	-1	-1	-1	-1	-1	1
33	-1	-1	1	1	1	-1	1	-1
34	-1	-1	-1	1	1	1	-1	1
35	1	-1	-1	-1	1	1	1	-1
36	-1	1	-1	-1	-1	1	1	1
37	1	-1	1	-1	-1	-1	1	1
38	1	1	-1	1	-1	-1	-1	1
39	1	1	1	-1	1	-1	-1	-1
40	-1	1	1	1	-1	1	-1	-1
41	1	1	-1	-1	-1	1	-1	1
42	1	1	1	-1	-1	-1	1	-1
43	-1	1	1	1	-1	-1	-1	1
44	1	-1	1	1	1	-1	-1	-1
45	-1	1	-1	1	1	1	-1	-1
46	-1	-1	1	-1	1	1	1	-1
47	-1	-1	-1	1	-1	1	1	1
48	1	-1	-1	-1	1	-1	1	1
49	-1	1	1	-1	1	1	1	1
50	1	-1	1	1	-1	1	1	1
51	1	1	-1	1	1	-1	1	1
52	1	1	1	-1	1	1	-1	1
53	1	1	1	1	-1	1	1	-1
54	-1	1	1	1	1	-1	1	1
55	1	-1	1	1	1	1	-1	1
56	1	1	-1	1	1	1	1	-1
57	1	-1	-1	1	-1	-1	-1	-1
58	-1	1	-1	-1	1	-1	-1	-1
59	-1	-1	1	-1	-1	1	-1	-1
60	-1	-1	-1	1	-1	-1	1	-1
61	-1	-1	-1	-1	1	-1	-1	1
62	1	-1	-1	-1	-1	1	-1	-1
63	-1	1	-1	-1	-1	-1	1	-1
64	-1	-1	1	-1	-1	-1	-1	1

where $C = C(W_4^k) \quad D = D(W_4^k)$. This means that the PCF is half-band, or

$$r[2n] = 0 \tag{49}$$

$$r[n+1] \in \{0, \pm 4\} \tag{50}$$

Taking into consideration the number of sequences in (47) it is figured out that there are 2^9 such sequences and the total capacity of G_{16} is $Q_{16} = 2^{10} + 2^9 = 3 \cdot 2^9$. Equations (45) and (47) represent an explicit construction of codewords with N= 16. For example when $l = m = p = 0$ from (45) we get

$$A_{16,1}(W_{16}^k) = B\left[(1 + W_{16}^2) + W_{16}^1(1 - W_{16}^2)\right] \tag{51}$$

There are eight codewords generated by $A_{16,1}$:

$$111-1-1-1-11-1-1-11-1-1-11 \tag{52}$$

$$1-1-1-1-11-1-1-11-1-1-1111 \tag{53}$$

$$-1-1-11-1-1-11-1-1-11111-1 \tag{54}$$

$$-11-1-1-11-1-1-11111-1-1-1 \tag{55}$$

$$-1-1-11-1-1-11111-1-1-1-11 \tag{56}$$

$$-11-1-1-11111-1-1-1-11-1-1 \tag{57}$$

$$-1-1-11111-1-1-1-11-1-1-11 \tag{58}$$

$$-11111-1-1-1-11-1-1-11-1-1 \tag{59}$$

The analytical formulas get too complicated for higher values of N.

4. Orthogonal Anti-symmetric Periodic Codes

In this section again orthogonal codes are constructed using wavelet-based approach. The orthogonal sets that are obtained offer high capacities and simple signal processing operations. The orthogonal set of codewords is $\{s_0\ s_1 \cdots s_{M-1}\}$ where $s_i = (a[2i], a[2i+1], \cdots, a[2i+N-1])$, and $a[N+i] = -a[i]$, $a[2N+i] = a[i]$. It also assumed that $a[i]$ can take only two values: 1, or -1. The number of codewords is $M = N/2$ each having length N. Since periodicity is assumed the properties of the periodic autocorrelation function

$$r[n] = \sum_{i=0}^{N-1} a[i]a[i+n] \tag{60}$$

which has a period equal to $2N$, are very important. The periodic autocorrelation function has the following properties:

1. $r[0] = \sum_{i=0}^{N-1} a^2[i] = N \qquad (61)$
2. $r[\pm N] = \sum_{i=0}^{N-1} a[i]a[i \pm N] = -\sum_{i=0}^{N-1} a^2[i] = -N \qquad (62)$
3. $r[n] = -\sum_{i=0}^{N-1} a[i]a[i \pm N + n] = -r[n \pm N] = r[-n] \qquad (63)$
4. $r[N/2] = 0$. This property can be established by

$$\begin{aligned} r[N/2] &= \sum_{i=0}^{N-1} a[i]a[N/2+i] \\ &= a[0]a[N/2] + a[1]a[N/2+1] + \cdots \\ &\quad + a[N/2-1]a[N-1] \\ &\quad - a[N/2]a[0] - a[N/2+1]a[1] - \cdots \\ &\quad - a[N-1]a[N/2-1] = 0 \end{aligned} \tag{64}$$

5. $r[2n] = 0 \pmod 4$, $r[2n+1] = 2 \pmod 4$. These properties are not trivial and need a proof. It is convenient to use the transform $b[i] = (1 - a[i])/2$. Then the PAF becomes

$$\begin{aligned} r[n] &= \sum_{i=0}^{N-1} (1 - 2b[i])(1 - 2b[i+n]) \\ &= \sum_{i=0}^{N-1} (1 - 2b[i] - 2b[i+n] + 4b[i]b[i+n]) \\ &= N - 2\left[\sum_{i=0}^{N-1} (b[i] + b[i+n])\right] + 4\sum_{i=0}^{N-1} b[i]b[i+n] \end{aligned} \tag{65}$$

But considering the anti-symmetry $a[N+i] = -a[i]$ we have

$$1 = b[N+i] + b[i] \tag{66}$$

and therefore

$$\sum_{i=0}^{N-1} (b[i] + b[i+n]) = 2\sum_{i=0}^{N-1} b[i] + n - 2\sum_{i=0}^{n-1} b[i]. \tag{67}$$

Finally

$$r[n] = N - 4\sum_{i=0}^{N-1} b[i] - 2n + 4\sum_{i=0}^{n-1} b[i] + 4\sum_{i=0}^{N-1} b[i]b[i+n] \tag{68}$$

Since N is a power of 2, the properties are easily established. Again the necessary and sufficient condition to have orthogonality is that the autocorrelation is half-band:

$$r[2n] = 0,\ n = 1, 2, \cdots N/4 - 1 \tag{69}$$

which also means that

$$R_0[2k] = |A_0[2k]|^2 + |A_1[2k]|^2 = N. \tag{70}$$

Now, for the polyphase components, the PCF of $(a_i, a_{i+2}, \cdots a_{i+N-2})$, $i = 0, 1$ are

$$r_i[n] = \sum_{k=0}^{N/2-1} a[2k+2]a[2(k+n)+i];\ a[N+i] = -a[i], \tag{71}$$

$$R_i[k] = \sum_{n=0}^{N/2-1} r_i[n]W_{N/2}^{nk} = A_i[k]A_i[-k] = |A_i[k]|^2,\ i = 0, 1 \tag{72}$$

where $r[2n] = r_0[n] + r_1[n]$ and therefore, a necessary and sufficient condition for orthogonality is that

$$r_0[n] = -r_1[n] \quad n = 1, 2, \cdots N/4 - 1 \tag{73}$$

The vectors for which the above condition is fulfilled are called complementary. The complementary property is invariant under the following transformations:

1. If $A_0(z)$ and $A_1(z)$ are complementary, then $\tilde{A}_0(z)$ and $\tilde{A}_1(z)$ will also be complementary.
2. The complementary property is invariant under cyclic shifts.

The GRS sequences are contained entirely in the new class of sequences, i.e. they are a subset of it. This implies, of course, that the number of the new sequences exceeds the number of GRS sequences for the same length.

The codewords of length N are obtained as $s_{2i} = (a[2i], a[2i+1], \cdots, a[2i+N-1])$. One sequence generates two codes s_{2i} and s_{2i+1} with capacities $M = N/2$ each having $N/2$ codewords. There are two cases:

1. The polyphase components (i.e. the complementary vectors) of a codeword are themselves codewords:

$$r_0[2n] = r_1[2n] = 0 \quad r_0[2n+1] = -r_1[2n+1] \tag{74}$$

2. The polyphase components of a codeword are not codewords themselves:

$$r_i[2n] \neq 0 \tag{75}$$

but still then $r_0[n] = -r_1[n]$ continues to hold.
It is clear that in the first case an orthogonal antisymmetric periodic code with length N and volume $N/2$ can be constructed iteratively, starting from codewords with length $N/2$. Suppose we have a codeword with length $N/2$, which is also a complementary vector, $A_0(z) \in G_{N/2}$. It must be the first polyphase component of a codeword in the set G_N, but it can be further decomposed using the polyphase decomposition

$$A_0[2k] = A_{00}[4k] + W^1_{N/2} A_{01}[4k] \tag{76}$$

The second polyphase component can be constructed in two ways. The first is

$$A_1[2k] = W^{2l}\left(A_{00}(4k) - W^{2k} A_{01}(4k)\right) \tag{77}$$

and the second

$$A_1[2k] = W^{-2l}\left(A_{00}(4k) - \tilde{W}^{2k} A_{01}(4k)\right) \tag{78}$$

It is obvious that $A_1 \in G_{N/2}$, since A_1 has polyphase components which have equal magnitudes as the polyphase components of A_0. Then, a codeword can be constructed, of which A_0 and A_1 are the first and second polyphase components, correspondingly:

$$|A_0|^2 + |A_1|^2 = 2\left[|A_{00}|^2 + |A_{01}|^2\right] = N \tag{79}$$

4.1 Construction of Codewords

From the properties of the autocorrelation function, discussed in the beginning of this section, it follows that all combinations of four digits that can take the values of $+1$ and -1 are codewords and therefore the volume of the code is $Q_4 = 2^4$.

4.1.1 Codewords with Length N=8

For N=8, we need eight elements to construct the codewords

$$P_1 = 1 + W_8^1 + W_8^2 + W_8^3 \tag{80}$$
$$P_2 = 1 + W_8^1 - W_8^2 - W_8^3 \tag{81}$$
$$P_3 = 1 + W_8^1 + W_8^2 - W_8^3 \tag{82}$$
$$P_4 = 1 - W_8^1 - W_8^2 - W_8^3 \tag{83}$$
$$Q_1 = 1 - W_8^1 + W_8^2 - W_8^3 \tag{84}$$
$$Q_2 = 1 - W_8^1 + W_8^2 + W_8^3 \tag{85}$$
$$Q_3 = 1 + W_8^1 - W_8^2 + W_8^3 \tag{86}$$
$$Q_4 = 1 - W_8^1 - W_8^2 + W_8^3 \tag{87}$$

The eight basic elements and inversions can exhaust all combinations of four digits, i.e. all codewords in the set G_4. Note that

$$|P_1|^2 = |P_2|^2 = |P_3|^2 = |P_4|^2 = 4 + 2\sqrt{2} \tag{88}$$
$$|Q_1|^2 = |Q_2|^2 = |Q_3|^2 = |Q_4|^2 = 4 - 2\sqrt{2} \tag{89}$$

From (70), when N= 8, the necessary condition for orthogonality is $|A_0|^2 + |A_1|^2 = 8$. From (88) and (89) we have $|P_1|^2 + |Q_1|^2 = |P_1|^2 + |Q_2|^2 = |P_1|^2 + |Q_3|^2 = |P_1|^2 + |Q_4|^2 = 8$ and P_1 can be replaced by P_2, P_3, P_4. Therefore, we can construct the codewords by

$$\begin{aligned} &A_8 = \pm W_{16}^l\left(M \pm W_{16}^1 N\right) \; M \in \{P_1, P_2, P_3, P_4\} \\ &N \in \{Q_1, Q_2, Q_3, Q_4\} \;\; l \in \{0, 1\} \end{aligned} \tag{90}$$

The total numbers of different codewords will be $2 \cdot 2 \cdot 2^2 \cdot 2 \cdot 2^2 = 2^7$. They are listed in Table 2.

4.1.2 Codewords with Length N=16

For N=16 the value of $\left|P_i + W_{16}^2 Q_i\right|^2$ is not constant for $i = 1, 2, 3, 4$. The following DFTs are defined:

$$S_{4(i-1)+j} = P_i + W_{32}^2 Q_j \tag{91}$$
$$T_{4(i-1)+j} = P_i - W_{32}^2 Q_j \tag{92}$$
$$U_{4(i-1)+j} = Q_i + W_{32}^2 P_j \tag{93}$$
$$V_{4(i-1)+j} = Q_i - W_{32}^2 P_j \tag{94}$$

for $i \in \{1, 2, 3, 4\}$, $j \in \{1, 2, 3, 4\}$. The squared absolute values of these DFTs take only four values, and therefore we separate them in four groups:
Group1

$$\begin{aligned} |S_1|^2 &= |S_2|^2 = |S_7|^2 = |S_8|^2 = |S_{13}|^2 = |T_{10}|^2 = |T_{12}|^2 \\ &= |T_{15}|^2 = |U_3|^2 = |U_7|^2 = |U_9|^2 = |U_{12}|^2 \\ &= |U_{16}|^2 = |V_1|^2 = |V_6|^2 = |V_{14}|^2 = 10.1648 \end{aligned} \tag{95}$$

Group2

$$\begin{aligned} |S_{10}|^2 &= |S_{12}|^2 = |S_{15}|^2 = |T_1|^2 = |T_2|^2 = |T_7|^2 = |T_8|^2 \\ &= |T_{13}|^2 = |U_1|^2 = |U_6|^2 = |U_{14}|^2 = |V_3|^2 = |V_7|^2 \end{aligned}$$

Table 2 Anti-symmetric periodic complementary sequences with length N=8.

1	-1	-1	-1	-1	-1	1	-1	-1	65	-1	1	-1	-1	1	1	1	-1
2	-1	-1	-1	1	-1	-1	1	1	66	-1	1	-1	1	-1	-1	-1	1
3	-1	-1	1	1	1	1	1	-1	67	-1	1	-1	1	1	-1	-1	-1
4	-1	-1	-1	-1	-1	1	1	-1	68	-1	1	1	-1	-1	-1	1	-1
5	-1	-1	-1	1	1	-1	1	1	69	-1	1	1	-1	1	1	1	-1
6	1	-1	1	1	1	1	1	-1	70	-1	1	-1	1	1	1	-1	-1
7	-1	-1	-1	-1	1	-1	-1	-1	71	-1	1	1	1	-1	-1	1	-1
8	-1	-1	1	-1	-1	-1	1	1	72	-1	1	-1	1	1	1	-1	1
9	-1	-1	-1	-1	1	-1	-1	1	73	-1	1	1	1	-1	1	1	-1
10	-1	-1	1	-1	-1	1	1	1	74	-1	1	1	-1	-1	-1	-1	-1
11	-1	-1	-1	-1	1	1	-1	1	75	1	-1	-1	-1	-1	-1	1	-1
12	-1	-1	1	1	-1	1	1	1	76	1	-1	-1	-1	1	-1	1	-1
13	-1	-1	-1	-1	1	1	1	-1	77	-1	1	1	-1	1	-1	-1	-1
14	-1	-1	1	1	1	-1	1	1	78	1	-1	1	-1	-1	-1	1	-1
15	-1	-1	-1	1	-1	-1	-1	-1	79	1	-1	1	-1	1	1	1	-1
16	-1	1	-1	-1	-1	-1	1	1	80	1	-1	1	1	1	-1	1	-1
17	-1	-1	-1	1	-1	-1	-1	1	81	-1	1	1	1	-1	-1	-1	-1
18	-1	1	-1	-1	-1	1	1	1	82	1	1	-1	-1	-1	-1	1	-1
19	-1	-1	-1	1	-1	-1	1	-1	83	1	1	-1	-1	1	-1	1	-1
20	-1	1	-1	-1	1	-1	1	1	84	-1	1	1	1	-1	1	-1	-1
21	-1	1	-1	-1	1	1	1	1	85	1	1	-1	1	-1	-1	1	-1
22	-1	-1	-1	1	-1	1	-1	-1	86	-1	1	1	1	-1	1	-1	1
23	-1	1	-1	1	-1	-1	1	1	87	1	1	-1	1	-1	1	1	-1
24	-1	-1	-1	1	-1	1	1	-1	88	1	1	-1	1	1	-1	1	-1
25	-1	1	-1	1	1	-1	1	1	89	1	1	-1	1	1	1	1	-1
26	-1	-1	-1	1	1	-1	1	-1	90	-1	1	1	1	1	-1	-1	-1
27	-1	1	1	-1	1	-1	1	1	91	1	1	1	-1	-1	-1	1	-1
28	-1	1	1	-1	1	1	1	1	92	1	1	1	-1	1	1	1	-1
29	-1	-1	-1	1	1	1	-1	1	93	-1	1	1	1	1	1	-1	-1
30	-1	1	1	1	-1	1	1	1	94	1	1	1	1	-1	-1	1	-1
31	-1	-1	-1	1	1	1	1	-1	95	-1	1	1	1	1	1	-1	1
32	-1	1	1	1	1	-1	1	1	96	1	1	1	1	-1	1	1	-1
33	-1	-1	1	-1	-1	-1	-1	-1	97	1	-1	-1	-1	-1	1	-1	-1
34	1	-1	-1	-1	-1	-1	1	1	98	1	-1	-1	-1	1	-1	-1	-1
35	-1	-1	1	-1	-1	-1	-1	1	99	1	-1	-1	-1	1	-1	-1	1
36	1	-1	-1	-1	-1	1	1	1	100	1	-1	-1	-1	1	1	-1	1
37	-1	-1	1	-1	-1	-1	1	-1	101	1	-1	-1	1	-1	-1	-1	-1
38	1	-1	-1	-1	1	-1	1	1	102	1	-1	-1	1	-1	-1	-1	1
39	1	-1	-1	-1	1	1	1	1	103	1	-1	-1	1	-1	1	-1	-1
40	-1	-1	1	-1	-1	1	-1	1	104	1	-1	-1	1	1	1	-1	1
41	1	-1	-1	1	-1	1	1	1	105	1	-1	1	-1	-1	1	-1	-1
42	1	-1	-1	1	1	1	1	1	106	1	-1	1	-1	1	1	-1	-1
43	-1	-1	1	-1	1	-1	-1	-1	107	1	-1	1	1	-1	-1	-1	1
44	1	-1	1	-1	-1	-1	1	1	108	1	-1	1	1	1	-1	-1	1
45	-1	-1	1	-1	1	-1	-1	1	109	1	-1	1	1	-1	-1	-1	-1
46	1	-1	1	-1	-1	1	1	1	110	1	1	-1	-1	-1	-1	-1	1
47	-1	-1	1	-1	1	1	-1	1	111	1	1	-1	-1	-1	1	-1	1
48	1	-1	1	1	-1	1	1	1	112	1	-1	1	1	-1	1	-1	-1
49	-1	-1	1	-1	1	1	1	-1	113	1	1	-1	1	-1	-1	-1	1
50	1	-1	1	1	1	-1	1	1	114	1	1	-1	1	1	1	-1	1
51	-1	-1	1	1	-1	1	-1	1	115	1	-1	1	1	1	-1	-1	-1
52	1	1	-1	1	-1	1	1	1	116	1	1	1	-1	-1	-1	-1	1
53	1	1	-1	1	1	1	1	1	117	1	1	1	-1	-1	1	-1	1
54	-1	-1	1	1	1	-1	1	-1	118	1	1	1	-1	1	-1	-1	1
55	1	1	1	-1	1	-1	1	1	119	1	1	1	-1	1	1	-1	1
56	1	1	1	-1	1	1	1	1	120	1	-1	1	1	1	1	-1	-1
57	-1	-1	1	1	1	1	-1	1	121	1	1	1	1	-1	-1	-1	1
58	1	1	1	1	-1	1	1	1	122	1	1	1	1	1	-1	-1	1
59	1	1	1	1	1	-1	1	1	123	1	1	-1	-1	-1	1	-1	-1
60	-1	1	-1	-1	-1	-1	-1	1	124	1	1	-1	-1	1	-1	-1	-1
61	-1	1	-1	-1	-1	1	-1	-1	125	1	1	-1	1	1	-1	-1	-1
62	-1	1	-1	-1	-1	1	-1	1	126	1	1	-1	1	1	1	-1	-1
63	-1	1	-1	-1	-1	1	1	-1	127	1	1	1	-1	-1	1	-1	-1
64	-1	1	-1	-1	1	-1	-1	-1	128	1	1	1	-1	1	1	-1	-1

$$= |V_9|^2 = |V_{12}|^2 = |V_{16}|^2 = 5.8352 \quad (96)$$

Group3

$$\begin{aligned} |S_3|^2 &= |S_5|^2 = |S_9|^2 = |S_{11}|^2 = |S_{16}|^2 = |T_4|^2 = |T_6|^2 \\ &= |T_{14}|^2 = |U_2|^2 = |U_4|^2 = |U_5|^2 = |U_{10}|^2 \\ &= |U_{11}|^2 = |V_8|^2 = |V_{13}|^2 = |V_{15}|^2 = 13.2263 \quad (97) \end{aligned}$$

Group4

$$\begin{aligned} |S_4|^2 &= |S_6|^2 = |S_{14}|^2 = |T_2|^2 = |T_5|^2 = |T_9|^2 = |T_{11}|^2 \\ &= |T_{16}|^2 = |U_8|^2 = |U_{13}|^2 = |U_{15}|^2 = |V_2|^2 = |V_4|^2 \\ &= |V_5|^2 = |V_{10}|^2 = |V_{11}|^2 = 2.7737 \quad (98) \end{aligned}$$

From (70) we know that the sufficient condition is $|A_0|^2 + |A_1|^2 = 16$. So the codewords can be constructed by.

$$A_{16,1} = A(W_{32}) = \pm W_{32}^l (X \pm W_{32}^1 Y) \quad l \in \{0, 1\} \quad (99)$$

$$X \in \{S_1, S_2, S_7, S_8, S_{13}, T_{10}, T_{12}, T_{15}, U_3, U_7, U_9, U_{12}, U_{16}, V_1, V_6, V_{14}\}$$

$$Y \in \{S_{10}, S_{12}, S_{15}, T_1, T_2, T_7, T_8, T_{13}, U_1, U_6, U_{14}, V_3, V_7, V_9, V_{12}, V_{16}\}$$

$$A_{16,2} = A(W_{32}) = \pm W_{32}^l (Z \pm W_{32}^1 J) \quad l \in \{0, 1\} \quad (100)$$

$$Z \in \{S_3, S_5, S_9, S_{11}, S_{16}, T_4, T_6, T_{14}, U_2, U_4, U_5, U_{10}, U_{11}, V_8, V_{13}, V_{15}\}$$

$$J \in \{S_4, S_6, S_{14}, T_2, T_5, T_9, T_{11}, T_{16}, U_8, U_{13}, U_{15}, V_2, V_4, V_5, V_{10}, V_{11}\}$$

The total number of codewords of $A_{16,1}$ is $2{\cdot}2{\cdot}2^4{\cdot}2{\cdot}2^4 = 2^{11}$ and the total number of codewords of $A_{16,2}$ is $2{\cdot}2{\cdot}2^4{\cdot}2{\cdot}2^4 = 2^{11}$. This makes the total number of codewords with length 16 equal to 2^{12}.

5. Explicit Formulas for GRS Pairs

The problem of generation of all Golay-Rudin-Shapiro sequences is of considerable importance. For example, in the context of wireless communications, every user is assigned a different sequence, and then it is necessary to generate all sequences of a given length. The non-periodic correction function is

$$r[m] \sum_{i=0}^{N-1-m} a[i]a[i+m] \quad (101)$$

In the previous two sections two periodic extensions of the sequence $(a[0]a[1]\cdots a[N+i])$ were considered: symmetric, where $a[N+i] = a[i]$, and anti-symmetric, where $a[N+i] = -a[i]$. It is convenient to denote the periodic autocorrelation functions by $r_s[m]$ and $r_a[m]$ for the symmetric and anti-symmetric cases, respectively. Note that the relationship among the non-periodic autocorrelation function $r[m]$ and the two periodic autocorrelation functions $r_s[m]$ and $r_a[m]$ is

$$r[m] = r_s[m] + r_a[m] \quad (102)$$

$$r_s[m] = r[m] + r[N-m] \quad (103)$$

$$r_a[m] = r[m] - r[N-m] \quad (104)$$

Theorem 4 The necessary and sufficient condition the non-periodic autocorrelation function to be half-band,

$$r[2m] = 0 \quad m = 1, 2, \cdots N/2, \quad (105)$$

is that

$$r_s[2m] = r_a[2m] = 0. \quad (106)$$

The proof can immediately be obtained using (102).
Corollary 1 The set of Golay sequences is the intersection of the sets of codewords belong to the orthogonal symmetric and antisymmetric cyclic codes. In other words the Golay sequences are simultaneously codewords of two codes.

5.1 Construction of Complementary Sequences

For N=4, it is easily verified that the total capacity is 8.

5.1.1 Complementary Sequences with Length N=8

When N=8 the codewords in the symmetric code are described by the formula

$$A[k] = A(W_8^k) = \pm W_8^{kl} B(W_8^{2k})(1 \pm W_8^{k(2m+1)})$$
$$m \in \{0, 1, 2, 3\}\, l \in \{0, 1, 2, 3, 4, 5, 6, 7\} \quad (107)$$

The DFTs of the codewords in the anti-symmetric case are

$$A_8 = \pm W_{16}^l \left(M \pm W_{16}^1 N\right) \quad M \in \{P_1, P_2, P_3, P_4\}$$
$$N \in \{Q_1, Q_2, Q_3, Q_4\} \quad l \in \{\,0,1\,\}. \quad (108)$$

In the anti-symmetric code P_3, P_4, Q_2, Q_3 can be the cyclic shifts and inversion of $B(W_4^1)$. The GRS sequences for N=8 are the intersection of (107) and (108) and can be constructed as

$$A_8 = \pm W_{16}^l \left(M \pm W_{16}^1 N\right) \quad M \in \{P_3, P_4\}$$
$$N \in \{Q_2, Q_3\} \quad l \in \{\,0,1\,\}. \quad (109)$$

The volume of the set is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$, which coincides with the known previously obtained estimate.

5.1.2 Codewords with Length N=16

For N=16, the symmetric codewords can be written in the following form:

$$\begin{aligned} A_{16,1}(W_{16}^k) = \pm W_{16}^{kl} B(W_{16}^{4k}) \Big[&\left(1 \pm W_{16}^{2k(2m+1)}\right) \\ &+ W_{16}^{2p+1}\left(1 - \pm W_{16}^{2k(2m+1)}\right)\Big] \quad (110) \end{aligned}$$

$$m, p \in \{0, 1, 2, 3\}$$
$$l \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$

$$\begin{aligned} A_{16,2}(W_{16}^k) = W_{16}^{lk} \Big[&C(W_{16}^{4k}) \pm W_{16}^{2k} E(W_{16}^{4k}) \\ &\pm W_{16}^{(2p+1)k}\left(C(W_{16}^{4k}) \pm W_{16}^{2k} D(W_{16}^{4k})\right)\Big] \quad (111) \end{aligned}$$

$$p \in \{0, 1, 2, 3\} \quad l \in \{0, 1, 2, 3, 4, 5, 6, 7\}$$

In the anti-symmetric case codewords can be constructed by choosing $T_{10}, T_{15}, U_7, U_{12}$ from Group 1, $S_{10}, S_{15}, V_7, V_{12}$ from Group 2, $S_{11}, T_{14}, U_{11}, V_8$ from Group 3, and $S_{14}, T_{11}, U_8, V_{11}$ from Group 4. Therefore

$$A_{16,1} = A(W_{32}) = \pm W_{32}^{l}(X \pm W_{32}^{1} Y) \quad l \in \{0, 1\} \tag{112}$$
$$X \in \{T_{10}, T_{15}, U_7, U_{12}\} \quad Y \in \{S_{10}, S_{15}, V_7, V_{12}\}$$
$$A_{16,2} = A(W_{32}) = \pm W_{32}^{l}(Z \pm W_{32}^{1} J) \quad l \in \{0, 1\} \tag{113}$$
$$Z \in \{S_{11}, T_{14}, U_{11}, V_8\} \quad J \in \{S_{14}, T_{11}, U_8, V_{11}\}$$

For the symmetric case, from (111) the codewords are constructed by the cyclic shifts of $C(W_{16}^{4k}), D(W_{16}^{4k}), E(W_{16}^{4k})$. Therefore in the anti-symmetic case, P_2, Q_4 corresponds to cyclic shifts of $C(W_{16}^{4k})$, P_1 corresponds to $D(W_{16}^{4k})$, and Q_1 corresponds to $E(W_{16}^{4k})$. The combination of P_1, Q_4 and P_2, Q_1 can be used to construct the Golay sequences with length N=16. Therefore, referring to Eqs. (95)–(98), S_5, T_4, U_2, V_{13} can be chosen in Group 3, and S_4, T_5, U_{13}, V_2 can be chosen in Group 4, in the following way:

$$A_{16,3} = A(W_{32}) = \pm W_{32}^{l}(X \pm W_{32}^{1} Y) \quad l \in \{ 0,1 \} \tag{114}$$
$$X \in \{S_5, U_2\} \quad Y \in \{S_4, U_{13}\}$$
$$A_{16,4} = A(W_{32}) = \pm W_{32}^{l}(Z \pm W_{32}^{1} J) \quad l \in \{ 0,1 \} \tag{115}$$
$$Z \in \{T_4, V_{13}\} \quad J \in \{T_5, V_2\}$$

From (112) the number of codewords can be generated is $2 \cdot 2 \cdot 2^2 \cdot 2^2 = 2^6$, and from (113) the number of codewords can be generated is $2 \cdot 2 \cdot 2^2 \cdot 2^2 = 2^6$. From (114) the number of codewords can be generated is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$, and from (115) the number of codewords can generated is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$. So total volume of the set for E sequences with length 16 will be $2^6 + 2^6 + 2^5 + 2^5 = 2^7 + 2^6 = 192$, which coincides with the previously obtained estimate.

6. Conclusions

We have developed novel analytic solutions for periodic and aperiodic complementary sequences. These analytic solutions allow the generation of all periodic and non-periodic complementary sequences when the length is a power of two. Previously this could only be accomplished with computer searches. The approach that has been proposed here can be used, in principle, for other complementary sequences, like ternary complementary sequences, advanced in [21].

The novelty in this work from the point of view of wavelet transforms is that non-regular wavelets are constructed here. Note that filter banks have always been designed so that in addition to perfect reconstruction the filters have "good" frequency responses; e.g. $H_0(z)$ has always been required to be a good low pass filter and $H_1(z)$ is to be a good high pass filter [7]. This requirement is equivalent to requiring that the filter bank offer energy concentration and perform well in applications which require energy concentration like compression. Complementary sequences correspond to filter banks where the filters $H_0(z)$ and $H_1(z)$ are not "good" filters in this traditional sense, i.e. these filters have pseudo-random frequency responses. These filters offer energy spreading as opposed to energy compaction. This property is not desirable in data compression, but very desirable in CDMA wireless communication systems [11].

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