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Explicit Formula for Predictive FIR Filters and Differentiators Using Hahn Orthogonal Polynomials

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SUMMARY An explicit expression for the impulse response coefficients of the predictive FIR digital filters is derived. The formula specifies a four-parameter family of smoothing FIR digital filters containing the Savitsky-Goaly filters, the Heinonen-Neuvo polynomial predictors, and the smoothing differentiators of arbitrary integer orders. The Hahn polynomials, which are orthogonal with respect to a discrete variable, are the main tool employed in the derivation of the formula. A recursive formula for the computation of the transfer function of the filters, which is the z-transform of a terminated sequence of polynomial ordinates, is also introduced. The formula can be used to design structures with low computational complexity for filters of any order.

key words: *Hahn polynomials, orthogonal polynomials of a discrete variable, polynomial signals, polynomial impulse response, predictive FIR filters, power moments, white noise, digital differentiators*

1. Introduction

Least-squares digital filters having the ability to pass the polynomial component of the input signal and suppress the power of its additive white noise component have a long multifaceted history. They have been studied by actuaries, mathematicians, analytical chemists, engineers and physicists both theoretically and experimentally. The smoothing effects of these filters on noisy experimental data were known to the actuaries of the late 19th century in connection with the problem of *graduation* of mortality tables. Various types of these filters, which were known to the early actuaries as adjustment formulas [1] or linear compounds [2], were intensively studied in the actuarial and mathematical literature of the first half of the 20th century. The first edition of an influential book [3] dealing with the subject and its mathematical background was published as early as 1924. Although many variations of the filters exist in the literature, they are most generally designed to possess the following features. Consider a causal discrete-time signal $x(n)$ of the form

$$x(n) = f(n) + e(n), \quad n \geq 0, \tag{1}$$

where $f(n)$ is a polynomial of degree M and $e(n)$ is a noise or error sequence. It is assumed that $e(n)$ is stationary with zero mean and known autocorrelation function. The filter provides a prediction of, or a smoothed value for, the input signal given by

$$y(n) = \sum_{i\geq 0} h_i\, x(n-i). \tag{2}$$

The coefficients h_i must be determined so that the following two groups of conditions are satisfied simultaneously. The first group of conditions deal with the processing of the polynomial component $f(n)$. In the general form, denoting the mth-order derivaive of $f(n)$ with respect to n by $f^{(m)}(n)$, and provided that $e(n) = 0$, it is required that h_i are so chosen that

$$y(n) = f^{(m)}(n-p) \tag{3}$$

for a fixed integer m and a real-valued delay parameter, or predication step, p. In many applications where the preservation of the polynomial component is a goal, the value $m = 0$ is adopted in (3). The second group of conditions are imposed on h_i in order to control the output standard deviation when the noise component $e(n)$ is present. These conditions vary in complexity from the simple classical case involving minimzation of the power of $e(n)$, which is assumed to be white noise, to the more complicated cases involving the minimization of the power of the higher order differences of $e(n)$ [4]. An ideal predictive filter suppressing the white noise component and delaying the polynomial component is show in Fig. 1.

In the engineering literature, the initial motivation behind the study of the filters was to extend the results of Zadeh and Ragazzini [5], who developed a continuous linear system for extracting the polynomial component from a mixture of signal and noise, to the discrete-time signals. For instance, Lees [7], Johnson [6] and Blum [8] studied and solved the problem to various degrees when h_i has a finite-length. Specifically, the results by Blum [8], being more general, are expressed in terms that are similar to the modern theory of discrete time signals and systems. Blum also designed IIR type filters. A notable result by Trench [9] formulates the problem in the z domain as opposed to the time-domain approach taken by Blum. It also allows the designer to formulate the problem for a wider variety of noise components.

An important signal processing application of these least-squares smoothers, popularized by the work of Savitzky and Golay [10], is the smoothing of noisy spectral

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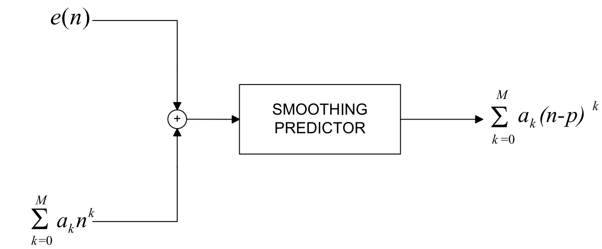
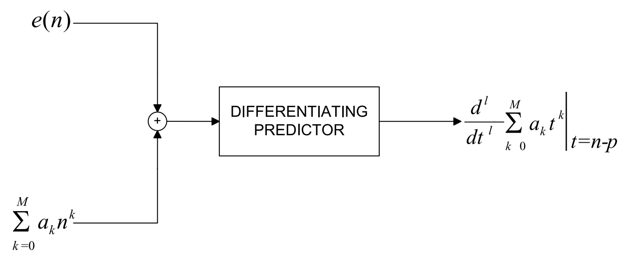


(a)



(b)

Fig. 1 Ideal predictive filters. The filter in (b) has the ability to differentiate the polynomial component.

data. Another rather recent incarnation of these filters can be found in a work of Heinonen and Neuvo [11], where a predictive digital filter was designed as a linear building block in a non-linear median filtering scheme. Like their classical predecessors, the Heinonen-Neuvo (H-N) filters operate on the input signal values within a finite-length sliding window and calculate a prediction value as the output. The prediction is formulated based on the requirement that the filter must be exact for the polynomial component $f(n)$, while yielding a minimized noise gain for the case where $e(n)$ is produced by a white noise source. The distinctive feature of the initial designs of the H-N filters is that, unlike most of their classical counterparts that predicted the true value of the sample located in the middle of the filtering window, they were designed to predict the value of the sample located one time index outside the filtering window. The H-N filters and their differentiator counterparts were later applied to porblems in control instrumentation [12], where they have been referred to as polynomial predictive filters. In their original form, the H-N filters were parametrized by the length of the filter and the degree of the polynomial component in the input signal. Nevertheless, the prediction or delay parameter p whose value is fixed in the Savitzky-Golay filters and the original H-N filters can be thought of as a third parameter. In the signal processing literature, the generalization of p to an arbitrary value was considered in [13], where a recursive structure for the realization of the filters with a low computational complexity was proposed. Henceforth, we shall refer to the both types of these filters as the H-N filters without making a distinction over the parameterization of the prediction parameter.

Focusing on the signal processing literature, closed-form formulas for the impulse response coefficients of the H-N filters can be found in [11] for polynomial components of degrees 1, 2, and 3. In [13], formulas for an arbitrary p were developed for polynomial models of degrees 1 and 2. In both cases, the formulas are given explicitly for h_i as polynomials in i whose coefficients depend on the degree of the polynomial model, the prediction paramter, and the overall length of the filter. In both cases, however, no general closed-form formula has been given. An exception in this regard, is a matrix formulation of the H-N filtering problem that results in a closed-form solution through the concept of the generalized inverse of a matrix [14]. This form of solution, requiring matrix inversion and matrix multiplication operations, can be viewed as a numerical solution to the problem.

It is the purpose of this paper to provide an explicit expression for the impulse response coefficients of the predictive smoothers, which include H-N filters, and their differentiator counterparts. Our results are given in the form of a concise formula for h_i as a polynomial in i, similar to those obtained in [11] and [13]. We use the orthogonal polynomials of a discrete variable to represent the solution. The use of the orthogonal polynomials of a discrete variable for the desgin of the classical Savitzky-Golay-type least-squares filters has some precedents in the engineering literature [8], [15], [16]. One can identify two forms of applications of these orthogonal polynomilas. The most common form of the application has been in connection with the representation and modeling of the input signal. For example, in [8] the author expresses the polynomial component $f(n)$ using the orthogonal polynomials of a discrete variable to derive an explicit expression for the impulse response of the filter. The other form of application of the orthogonal polynomials can be found in [15], where they were used in connection with the Hilbert space methods. In that approach, the polynomials are introduced as a tool in connection with the minimization process. Our approach, however, differs from these two existing categories. In this paper, the orthogonal polynomials are used to express the conditions for the polynomial processing requirements and, at the same time, to represent the impulse response coefficients h_i.

The other goal of this paper is to derive explicit rational expressions for the transfer function of the filters. Although such expressions are given in [13] for two examples with polynomial models of degrees 1 and 2, the general rational expression for the transfer function is not available in the literature. The rational form of the transfer function is of special interest for the recursive realization of all FIR filters having polynomial impulse responses in the form of structures of low computational complexity.

The organization of this paper is as follows. The design problem is stated in Sect. 2 for a polynomial predictor ($m = 0$) and a smoothing differentiator ($m \geq 1$). The frequency domain properties of the solution are also analyzed. Hahn polynomials and their orthogonality properties are reviewed in Sect. 3. An explicit solution is then derived in the form of a linear combination of the products of Hahn polynomials. The rational form of the transfer function is derived in Sect. 4. A recursive scheme for the computation of the rational transfer function is also developed. Conclusions are drawn in the final section.

2. Statement of Problem

After introducing the preliminaries and notations used in this paper, we provide a formulation of the design problem as that of the determination of a finite-length sequence having prescribed power moments. Both the basic smoothing problem and the more general smooth differentiation problem of an arbitrary order are considered here.

2.1 Statement of Basic Smoothing Problem

Consider a discrete-time signal expressed as

$$x(n) = \sum_{k=0}^{M} a_k\, n^k + e(n) \tag{4}$$

where $e(n)$ is a random, stationary, uncorrelated noise signal, or an error term, satisfying

$$E(e(n)) = 0, \quad E(e(n)e(n')) = \begin{cases} \sigma^2 & n = n' \\ 0 & n \neq n' \end{cases} \tag{5}$$

In other words, the signal $x(n)$ is modeled as a polynomial of finite degree $M \geq 0$, which constitutes the deterministic component of $x(n)$, plus a random component assumed to be due to a white noise source. The result of the convolution of $x(n)$ with an impulse response sequence h_i of lenght $N \geq 1$ is the desired smoothed output signal $y(n)$. The desired output can generally be the mth order derivative of the polynomial component evaluated at an arbitrary point as given by (3). Focusing on the case where $m = 0$ in (3), the problem is that of retaining the non-random polynomial component contained in the input signal without differentiation. This polynomial component is generated at the output with a delay of p samples. The numerical value of p, which can be an integer or a rational number, depends on the application at hand. This requirement is stated mathematically as

$$\sum_{i=0}^{N-1} h_i \sum_{k=0}^{M} a_k\,(n-i)^k = \sum_{k=0}^{M} a_k\,(n-p)^k. \tag{6}$$

The variance of the additive noise contained in the output is altered by the convolution and is given by

$$E\left(\left(\sum_{i=0}^{N-1} h_i\, e(n-i)\right)^2\right) = \sigma^2 \sum_{i=0}^{N-1} h_i^2. \tag{7}$$

To obtain a smooth output, it is required that h_i is so chosen that the noise gain, given by

$$\mathrm{NG} = \sum_{i=0}^{N-1} h_i^2, \tag{8}$$

is minimized.

Of the above two requirements, (6) and (8), on the impulse response coefficients, condition (6), which ensures the retention of the polynomial part, can be expressed in terms of the power moments of h_i. By noting that the ability of the filter to exactly delay any polynomial of degree M is equivalent to the property that all monomials x^k, $k = 0, 1, \ldots, M$, be exactly delayed in the same manner, one can obtain the equivalent conditions

$$\sum_{i=0}^{N-1} h_i(n-i)^k = (n-p)^k, \quad k = 0, 1, \ldots, M. \tag{9}$$

Application of the binomial expansion theorem to the left and right sides of (9) gives

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} n^j \sum_{i=0}^{N-1} i^{k-j} h_i = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} n^j p^{k-j}, \quad k = 0, 1, \ldots, M. \tag{10}$$

Since the relations (10) should hold as identities for all values of n, by comparing the same powers in the sums on the left and right, we arrive at the conditions

$$\sum_{i=0}^{N-1} i^j\, h_i = p^j \quad j = 0, 1, \ldots, M, \tag{11}$$

specifying the power moments of the impulse response coefficients from the 0th to the Mth order.

2.2 Frequency Domain Properties of Solution to Basic Smoothing Problem

To study the implications of imposing the moment conditions (11), we turn to the frequency domain. The frequency response of the filter is given by

$$H(\omega) = \sum_{i=0}^{N-1} h_i e^{-\mathbf{j}\, i\, \omega}, \tag{12}$$

where $\mathbf{j} = \sqrt{-1}$. On expanding the right side of (12) in the Taylor form about $\omega = 0$, and assuming that (11) holds, we obtain

$$\begin{aligned} H(\omega) &= \sum_{i=0}^{N-1} h_i \sum_{k=0}^{\infty} \frac{(-\mathbf{j}\, i\, \omega)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-\mathbf{j}\, \omega)^k}{k!} \sum_{i=0}^{N-1} i^k h_i \\ &= \sum_{k=0}^{M} \frac{(-\mathbf{j}\, \omega)^k}{k!} p^k + O(\omega^{M+1}), \end{aligned} \tag{13}$$

where $O(\omega^{M+1})$ contains all the terms with powers $(M+1)$ or higher. Consequently, we can write

$$H(\omega) = e^{-\mathbf{j}\, p\, \omega} + O_1(\omega^{M+1}), \tag{14}$$

which shows, since $O_1(\omega^{M+1})$ again contains all the terms with powers $(M+1)$ or higher, that the frequency response is a flat approximation to the ideal delay of p samples at $\omega = 0$. This confirms the well-known fact that the magnitude response of the filter has a flat shape around the zero frequency.

2.3 Shifted Moments

The p-shifted moments of a causal signal $x(n)$ are defined as

$$\mathcal{M}_p = \sum_n (n-p)^j x(n), \tag{15}$$

where the summation index n above as well as in all other sums where the range is not explicitly indicated runs over all integers. By applying $x(n)$ to a filter satisfying (9), we get an output $y(n)$ whose p-shifted moments are

$$\begin{aligned}\sum_n (n-p)^j y(n) &= \sum_n (n-p)^j \sum_{n'} x(n')h(n-n') \\ &= \sum_{n'} x(n') \sum_n (n-p)^j h(n-n') \\ &= \sum_{n'} n'^j x(n')\end{aligned} \tag{16}$$

This shows that the p-shifted moments of the output signal of a predictive filter is equal to the power moments of its input signal.

2.4 Smooth Differentiation Problem

We can recast the problem of exact integer-order differentiation of a degree M polynomial signal, which corresponds to the case where $m \geq 1$ in (3), as the equivalent problem of exact differentiation of the monomial x^k for $k = 0, 1, \ldots, M$. We start with the first-order differentiation and then generalize to the differentiation by an arbitrary higher integer order. The coeffecients h_i of an FIR system designed for the purpose of exact first-order differentiation of x^k must satisfy

$$\sum_{i=0}^{N-1} h_i (n-i)^k = \begin{cases} 0, & k = 0 \\ k(n-p)^{k-1}, & k = 1, \ldots, M. \end{cases} \tag{17}$$

In the expanded form, the conditions given by (17) can be expressed as the combination of the two equations

$$\sum_{i=0}^{N-1} h_i = 0, \tag{18}$$

$$\begin{aligned}&\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} n^j \sum_{i=0}^{N-1} i^{k-j} h_i \\ &\quad = k \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} n^j p^{k-1-j}, \quad k = 1, \ldots, M.\end{aligned} \tag{19}$$

Using (18) and the properties of the binomial coefficients, (19) can be written as

$$\begin{aligned}&\sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} n^j \sum_{i=0}^{N-1} i^{k-j} h_i \\ &\quad = -\sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} (k-j) n^j p^{k-1-j}, \quad k = 1, \ldots, M.\end{aligned} \tag{20}$$

After some simple algebraic manipulations, it can be shown that (20) is equivalent to

$$\sum_{i=0}^{N-1} i^j h_i = -jp^{j-1}, \quad j = 1, 2, \ldots, M. \tag{21}$$

The above set of power moment conditions together with (18) constitute the necessary and sufficient for a first-order FIR differentiator to be exact for polynomial signals of degree M.

In general, it can be shown, by mathematical induction, that the conditions on the impulse response coefficients of an exact mth order differentiator, $m \geq 1$, producing a delay of p samples for the Mth degree polynomial signals are given by

$$\begin{aligned}&\sum_{i=0}^{N-1} i^j h_i = 0 \quad j = 0, \ldots, m-1, \\ &\sum_{i=0}^{N-1} i^j h_i = (-1)^m j(j-1)\cdots(j-m+1)p^{j-m}, \\ &j = m, \ldots, M.\end{aligned} \tag{22}$$

We can use the falling factorial powers in order to unify the set of condition (22) with those given by (11) in a concise manner. The falling factorial power [17]

$$a^{\underline{b}} = a(a-1)\ldots(a-b+1) \tag{23}$$

possesses the property that for integers a and b, it vanishes when $0 \leq a < b$. It is also conventionally assumed that $a^{\underline{0}} = 1$. Using the falling factorial powers, the conclusion of this subsection is concisely stated as follows. For an FIR system, in order to be an exact mth order differentiator, $m = 0, 1, \ldots$, and to produce a delay of p samples on a polynomial input of order M, it is necessary and sufficient that

$$\sum_{i=0}^{N-1} i^j h_i = (-1)^m j^{\underline{m}} p^{j-m} \quad j = 0, \ldots, M. \tag{24}$$

The problem to be solved in this paper is that of the minimization of NG (8) under the linear side conditions given by (24). This result of this constrained minimization is the H-N-type predictive filters for $m = 0$ (and $p = N$ for example) and the predictive differentiators for $m \geq 1$.

2.5 Frequency Domain Properties of Solution to Smooth Differentiation Problem

The power moment conditions (24) have been derived in the time domain. They, however, have a direct frequency-domain interpretation. To see the relation between the two domains, we follow the steps taken in (13) and use the definition of the falling factorial powers to write

$$\begin{aligned} H(\omega) &= \sum_{k=0}^{\infty} \frac{(-\mathbf{j}\,\omega)^k}{k!} \sum_{i=0}^{N-1} i^k h_i \\ &= \sum_{k=0}^{M} \frac{(-\mathbf{j}\,\omega)^k}{k!} (-1)^m k^{\underline{m}}\, p^{k-m} + O_2(\omega^{M+1}) \\ &= (-\mathbf{j}\,\omega)^m \sum_{k=m}^{M} \frac{(-\mathbf{j}\,\omega)^{k-m}}{(k-m)!}\, p^{k-m} + O_2(\omega^{M+1}), \end{aligned} \tag{25}$$

where $O_2(\omega^{M+1})$ contains all the terms with powers $(M+1)$ or higher of ω. In the last step of the manipulations in (25), we have used the relation $k! = k^{\underline{m}}(k-m)!$. We can further write,

$$H(\omega) = (-\mathbf{j}\,\omega)^m\, e^{-\mathbf{j}\,\omega p} + O_3(\omega^{M+1}), \tag{26}$$

where $O_3(\omega^{M+1})$ contains all the terms having powers $(M+1)$ or higher of ω. This shows that the power moment conditions force the transfer function to approximate an mth order differentiation plus a delay of p samples in a flat manner about the zero frequency.

3. Solution by the Hahn Orthogonal Polynomials

As the first step toward obtaining the explicit solution, we express the side conditions derived in the preceding section using an orthonormal set of polynomials instead of simple monomial powers. The rationale behind this will become clear shortly. The orthonormal set of our choice is the Hahn polynomials. The Hahn polynomials may be defined in terms of the generalized hypergeometric series

$${}_3F_2(a_1, a_2, a_3;\ b_1, b_2;\ z) = \sum_{k=0}^{\infty} \frac{a_1^{\overline{k}}\, a_2^{\overline{k}}\, a_3^{\overline{k}}}{b_1^{\overline{k}}\, b_2^{\overline{k}}} \cdot \frac{z^k}{k!}, \tag{27}$$

where

$$a^{\overline{0}} = 1,\ a^{\overline{k}} = a(a+1)\cdots(a+k-1), \quad k \geq 1. \tag{28}$$

For a positive integer N and for real $\alpha > -1, \beta > -1$, the Hahn polynomials are defined by [18]

$$\begin{aligned} &Q_n(x;\ \alpha, \beta, N) \\ &\quad \triangleq {}_3F_2(-n, -x, n+\alpha+\beta+1;\ \alpha+1, -N+1;\ 1), \\ &n = 0, 1, \ldots, N-1. \end{aligned} \tag{29}$$

The right side of the above definition is a terminating hypergeomertic series and can be written in the form of a finite sum as

$$Q_n(x;\ \alpha, \beta, N) = \sum_{k=0}^{n} \frac{(-n)^{\overline{k}}\, (-x)^{\overline{k}}\, (n+\alpha+\beta+1)^{\overline{k}}}{(\alpha+1)^{\overline{k}}\, (-N+1)^{\overline{k}}\, k!}. \tag{30}$$

Hence, (29) defines a polynomial of degree at most n in x. For $n = 0$, we have $Q_0(x, \alpha, \beta, N) = 1$. A useful property of the Hahn polynomials is that, for a given n, they constitute a finite system of N polynomials orthogonal over the discrete values of x. Specifically, the orthogonality relation is given by [18]

$$\begin{aligned} &\sum_{x=0}^{N-1} Q_n(x;\ \alpha, \beta, N)\, Q_m(x;\ \alpha, \beta, N) \rho(x; \alpha, \beta, N) \\ &\quad = \frac{1}{\pi_n(\alpha, \beta, n)}\, \delta_{mn}, \end{aligned} \tag{31}$$

where δ_{mn} is the Kronecker symbol, the weight function is given by

$$\rho(x; \alpha, \beta, N) = \frac{\binom{\alpha+x}{x}\binom{\beta+N-1-x}{N-1-x}}{\binom{\alpha+\beta+N}{N-1}}, \tag{32}$$

and where

$$\begin{aligned} \pi_n(\alpha, \beta, N) &= \frac{\binom{N-1}{n}}{\binom{\alpha+\beta+N+n}{n}} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)} \\ &\quad \frac{\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\beta+n+1)\Gamma(n+1)} \frac{2n+\alpha+\beta+1}{\alpha+\beta+1}. \end{aligned} \tag{33}$$

In the orthogonality relation (31), we may select the parameters α and β in a way that $\rho(x; \alpha, \beta, N)$ is independent of x. This is achieved if we set $\alpha = \beta = 0$, resulting in

$$\sum_{x=0}^{N-1} Q_n(x; 0, 0, N)\, Q_m(x; 0, 0, N) = \frac{N}{\pi_n(0, 0, N)}\, \delta_{mn}, \tag{34}$$

As the second step toward obtaining the explicit solution, the power moment conditions (24) are expressed in terms of the Hahn polynomials. This is possible since for a given $M \leq N-1$, which is the degree of the polynomials passed without error, the Hahn polynomials provide an independant set of $M+1$ polynomials that may be used to express any given monomial of degree at most M. In other words, there exist coefficients c_l so that

$$x^j = \sum_{l=0}^{j} c_l\, Q_l(x, \alpha, \beta, N), \quad j = 0, 1, \ldots, M. \tag{35}$$

On applying this representation to (24), we obtain

$$\sum_{l=0}^{j} c_l \sum_{i=0}^{N-1} h_i\, Q_l(i,\alpha,\beta,N) = (-1)^m \sum_{l=0}^{j} c_l\, Q_l^{(m)}(p,\alpha,\beta,N), \quad j = 0,1,\ldots,M, \tag{36}$$

where $Q_l^{(m)}(p,\alpha,\beta,N)$ denots the mth derivative of $Q_l(p,\alpha,\beta,N)$ with respect to p. Thus, We can equivalently write

$$\sum_{i=0}^{N-1} h_i\, Q_j(i,\alpha,\beta,N) = (-1)^m Q_j^{(m)}(p,\alpha,\beta,N), \quad j = 0,1,\ldots,M. \tag{37}$$

As the last step toward obtaining the explicit solution, following [12], we employ the method of Lagrange multipliers. It can be shown that the mimization of NG is accomplished if the coefficients h_i are polynomials in i of degree M. At this point, we use the Hahn polynomials once again but this time in order to express the impulse response coefficients in the form

$$h_i = \sum_{l=0}^{M} \lambda_l\, Q_l(i,\alpha,\beta,N), \quad i = 0,1,\ldots,N-1. \tag{38}$$

The problem reduces to that of determining the $(M+1)$ unknowns λ_l. Substituting (38) into (37), we find that

$$\sum_{l=0}^{M} \lambda_l \sum_{i=0}^{N-1} Q_l(i,\alpha,\beta,N)\, Q_j(i,\alpha,\beta,N) = (-1)^m Q_j^{(m)}(p,\alpha,\beta,N), \quad j = 0,1,\ldots,M. \tag{39}$$

By (34), it follows that for $\alpha = \beta = 0$, we have

$$\lambda_l = \frac{(-1)^m}{N} \pi_l(0,0,N) Q_l^{(m)}(p,0,0,N), \quad l = 0,1,\ldots,M. \tag{40}$$

Hence, the impulse response coefficients of an FIR system of length N that produces the exact values of the mth derivative of a polynomial input of degree M, after a delay of p samples, and has a minimum white noise gain is given by

$$h_i = \sum_{l=0}^{M} \frac{(-1)^m}{N} \pi_l(0,0,N)\, Q_l(i,0,0,N) \times Q_l^{(m)}(p,0,0,N), \quad i = 0,1,\ldots,N-1. \tag{41}$$

Shown in Fig. 2(a) is the magnitude of the frequency response for a predictive smoother of length $N = 20$, producing a delay of $p = 9.5$ samples for the polynomial components of degree $M = 4$. Also, the four possible differentiators for the polynomials of degree 4, corresponding to $m = 1,2,3,4$ are shown in Fig. 2(b).

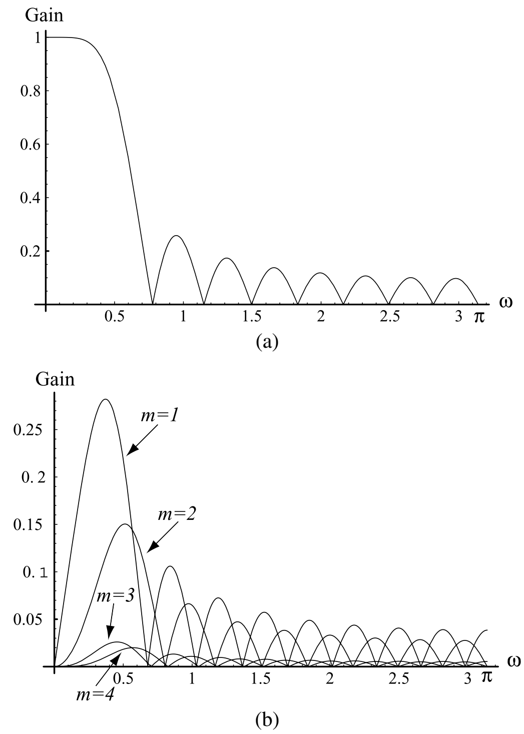


Fig. 2 Magnitude of the frequency response for (a) a predictive smoother $p = 9.5, M = 4, N = 20$, and four possible differentiators (b) $p = 9.5, M = 4, N = 20, m = 1,2,3,4$. The systems are all exact for polynomial components of degree four.

4. Development of Recursive Structures Using Rational Form of Transfer Functions

The results given in the preceding section indicate that the transfer function of the systems of our interest, like that of all other systems with a polynomial impulse response, is in fact a linear combination of the z-transforms of monomials n^k given by

$$\sum_{k=0}^{N-1} n^k z^{-n}. \tag{42}$$

Although we know that such polynomial impulse responses can be implemented recursively to reduce the computational complexity [13], [20], there is no systematic method for the derivation of the rational transfer function that is realized by such recursive structures. The purpose of this section is to introduce an explicit and systematic method for the derivation of the rational transfer functions for the realization of polynomial impulse responses.

A procedure for obtaining a closed-form formula for (42) when there are infinitely many terms is given in [21]. It was later remarked in [22] that this method involves the computation of the so-called Eulerian polynomials

$$A_m(z) = \sum_{j=0}^{m-1} \left\langle {m \atop j} \right\rangle z^{-j}, \quad m = 1,2,\ldots \qquad A_0(z) = 1 \tag{43}$$

where the coefficients $\left\langle {m \atop j} \right\rangle$, called the Eulerian numbers, can be computed using the recurrence [17]

$$\left\langle {m \atop j} \right\rangle = (j+1)\left\langle {m-1 \atop j} \right\rangle + (m-j)\left\langle {m-1 \atop j-1} \right\rangle. \tag{44}$$

The recurrence and its initial conditions result in a triangle of integers whose first few rows are of the form

$$\begin{array}{llllll} & \left\langle {m \atop 0} \right\rangle & \left\langle {m \atop 1} \right\rangle & \left\langle {m \atop 2} \right\rangle & \left\langle {m \atop 3} \right\rangle & \left\langle {m \atop 4} \right\rangle \\ m=0 & 1 & & & & \\ m=1 & 1 & 0 & & & \\ m=2 & 1 & 1 & 0 & & \\ m=3 & 1 & 4 & 1 & 0 & \\ m=4 & 1 & 11 & 11 & 1 & 0 \end{array} \tag{45}$$

Using the Eulerian polynomials, the infinite-length version of (42) is evaluated as

$$\sum_{n=0}^{\infty} n^k z^{-n} = \frac{z^{-1}A_k(z)}{(1-z^{-1})^{k+1}}. \tag{46}$$

The above formula, although very simple and effective, cannot be applied directly to our filters. We need to compute the z-transform of the truncated version of the above formula in an explicit manner. Such a formula exists and can be written as

$$\sum_{n=0}^{L} n^k z^{-n} = \frac{Q_{L+1}(z^{-1};k)}{(1-z^{-1})^{k+1}}, \tag{47}$$

where $Q_{L+1}(z^{-1};k)$ is a member of a family of polynomials whose coefficients depend on L and k. For a fixed L, the first few entries are

$$\begin{aligned} Q_{L+1}(z^{-1};0) &= 1 - z^{-L-1} \\ Q_{L+1}(z^{-1};1) &= z^{-1} - (L+1)z^{-L-1} + Lz^{-L-2} \\ Q_{L+1}(z^{-1};2) &= z^{-1} + z^{-2} - (L+1)^2 z^{-L-1} \\ &\quad +(2L^2+2L-1)z^{-L-2} - L^2 z^{-L-3} \end{aligned} \tag{48}$$

For a recursive evaluation of the numerator the recurrence [23],

$$\begin{aligned} Q_{L+1}(z^{-1};k+1) &= z^{-1}\Big((1-z^{-1})\frac{dQ_{L+1}(x;k)}{dx}\Big|_{x=z^{-1}} \\ &\quad +(k+1)Q_{L+1}(z^{-1};k)\Big), \\ &\quad k = 1,2,\ldots \end{aligned} \tag{49}$$

can be used. The formula is in fact the transfer function of the structures of [20] and [13] when they are tailored to generate a simple monomial.

5. Conclusion

The Hahn orthogonal polynomials have been used to express the conditions for exact polynomial processing and to express the impulse response coefficients h_i. The result is a concise and explicit formula for h_i expressed by the Hahn polynomials with variable is i and parameterized by the delay parameter, the order of differentiation, the degree of the polynomial component in the input signal, and the length of the filter. The resulting family of filter integrates the Savitzky-Golay filters as well as other existing predictive FIR smoothers and differentiators under a unified formula. Since the coefficients h_i are polynomials in i, the related problem of recursive implementation of a polynomial impulse response has been discussed and a closed-form expression for the related rational transfer functions has been introduced.

If it is desired to obtain the numerical values of the impulse response coefficients, one needs to evaluate Hahn polynomials and their derivatives at integer or possibly non-integer arguments. This task may be performed directly using the hypergeometric series given by (30). The series may be used in a direct manner to find the value of the Hahn polynomial and its derivative at a given argument. Another possibility is to use the recurrence relations of Weber and Erdélyi for Hahn polynomials [18], [19]. In that case, care must be taken with respect to the applicability of the recurrence relation for certain values of the arguments. The recurrence must also be adapted to a form suitable for the evaluation of the derivatives.

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