Title	Eigensignals of Downsamplers in Time and Transform Domains
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Citation	IEICE Trans. Fundamentals, Vol. E90-A, No. 9, pp. 1904-1912
Pub. date	2007, 9
URL	http://search.ieice.org/
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PAPER

Eigensignals of Downsamplers in Time and Transform Domains

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SUMMARY As a fundamental building block of multirate systems, the downsampler, also known as the decimator, is a periodically time-varying linear system. An eigensignal of the downsampler is defined to be an input signal which appears at the output unaltered or scaled by a non-zero coefficient. In this paper, the eigensignals are studied and characterized in the time and z domains. The time-domain characterization is carried out using number theoretic principles, while the one-sided z-transform and Lambert-form series are used for the transform-domain characterization. Examples of non-trivial eigensignals are provided. These include the special classes of multiplicative and completely multiplicative eigensignals. Moreover, the locus of poles of eigensignals with rational z transforms are identified.

key words: *decimation, downsampling, eigensignal, filterbanks, Lambert series, Möbius function, multirate systems, quasi-polynomials, time-varying systems*

1. Introduction

In the context of digital signal processing, an eigensignal for a discrete-time system is defined to be an input signal which appears at the output of the system unaltered, or scaled by a non-zero coefficient. It is well-known that linear time-invariant systems have exponential eigensignals and simple proofs of this fact can be found in the textbook literature (see for example [1]). Some non-linear filtering operations are also known to possess eigensignals. For instance, the signals which are invariant under median filtering are called root signals [2]. The root signals can be thought of as eigensignals having a scaling factor equal to unity. As another example, a class of nonlinear, non-homogeneous, time-varying systems that have exponential eigensignals were studied in [3]. Surprisingly, the non-trivial eigensignals of a simple linear periodically time-varying system like a decimator are not well studied.

The purpose of this paper is to fill this existing gap by identifying and characterizing the eigensignals of a downsampler. This task is carried out in the time and z domains, in Sects. 2 and 3, respectively, by providing rigorous characterizations. The time-domain approach, which is based on number theoretic principles, provides a general characterization of the eigensignals that is valid for both the prime or composite decimation rates M. On the other hand, the basic z-domain treatment, which is based on applying the time-domain characterization to the one-sided z-transform, results in a general z-domain representation of the eigensignals and gives rise to an expression in the form of a linear combination of fundamental eigensignals. An alternative z-domain approach is also taken by expressing the z-transform as a Lambert-type series. It results in a characterization in the form of linear constraints on the coefficients of the Lambert-form representation of the z-transform. It is shown that it is possible to obtain an explicit expression for the coefficients of the Lambert form for the case of prime decimation rates. In Sect. 4, special classes of eigensignals for prime and composite decimation rates are identified. These are multiplicative and completely multiplicative eigensignals and are closely related to a class of functions arising in number theory called the multiplicative arithmetical functions [4]. Finally, the study of an interesting class of eigensignals with a rational z transform is the subject of Sect. 5. In particular, we provide a necessary condition for rational eigensignals by showing that all of their poles must lie on the unit circle. Concluding remarks are given in Sect. 6, where examples of potential application of the eigensignal are also discussed.

2. Treatment in Time Domain

The time-domain response of an M-to-1 downsampler to a causal input signal $x(n)$ is given by the relation [5]

$$y(n) = x(Mn), \qquad n \geq 0. \tag{1}$$

Since downsampling is an operation in which parts of the input signal are discarded, we intuitively expect that if $x(n)$ contains a sufficient degree of redundancy, then its downsampled version could be a signal identical to it. Such signals, invariant under downsampling, may be called eigenfunctions, or more appropriately, eigensignals, for the downsampler. This notion can be favorably broadened by allowing the output signal to be proportional to $x(n)$ by a non-zero scaling constant.

Definition 1: The signal $i_{c,M}(n)$ is an eigensignal for an M-to-1 downsampler, with the scaling constant $c \neq 0$, if and

Manuscript received August 31, 2006.
Manuscript revised February 16, 2007.
Final manuscript received May 21, 2007.

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DOI: 10.1093/ietfec/e90–a.9.1904

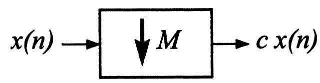

Fig. 1 The M-to-1 downsampler and an eigensignal.

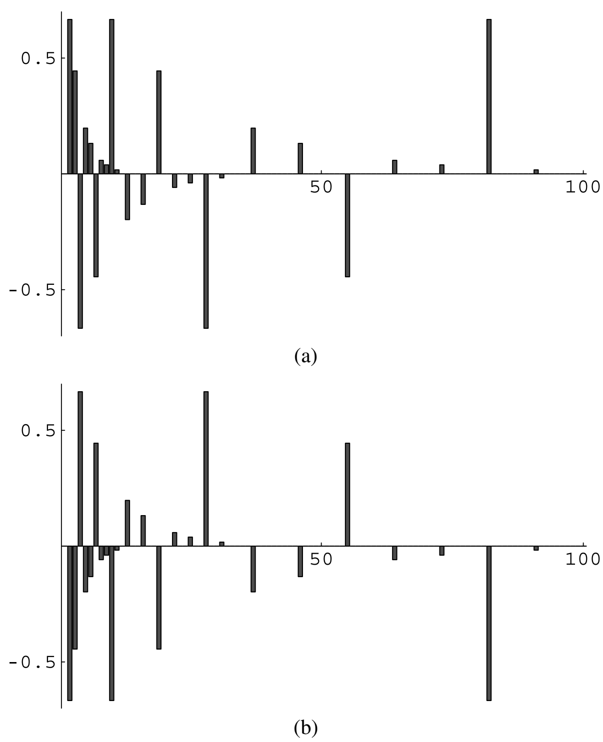

Fig. 2 Time plot of an eigensignal for $M = 3$, $c = -1$. The non-zero samples of the input (a) and output (b) are shown by bars and the horizontal axis represents the discrete time index n.

only if

$$i_{c,M}(Mn) = c\, i_{c,M}(n), \qquad n \geq 0. \tag{2}$$

In the above definition, the signal and the scaling constant may take on real or complex values. Figure 1 depicts a block-diagram representation of this concept. A trivial example of an eigensignal, which is common to all downsamplers, is the unit-step signal $i_{1,M}(n) = 1$, $n \geq 0$, where $c = 1$ and $M \geq 2$ is an integer. Another trivial example, common to all downsamplers, is the ramp signal given by $i_{M,M}(n) = n$, $n \geq 0$, where $c = M$ and $M \geq 2$ is an integer. A non-trivial example is given in Fig. 2 where an eigensignal for a 3-to-1 decimator ($M = 3$) with $c = -1$ is shown.

2.1 Characterization

An eigensignal $i_{c,M}(n)$ may be regarded as a solution to the functional equation given by (2). To provide a general solution, we distinguish two separate cases for the value of the time index n: the value at the origin and at the indices that are multiples of M. At the origin, substituting $n = 0$ into (2), we obtain the equation

$$i_{c,M}(0) = c\, i_{c,M}(0),$$

whose general solution is given by

$$i_{c,M}(0) = \delta_{c,1}\, x_0, \tag{3}$$

where x_0 is an arbitrary number and the Kronecker delta $\delta_{c,1}$ is defined by

$$\delta_{c,1} = \begin{cases} 1, & c = 1; \\ 0, & c \neq 1. \end{cases} \tag{4}$$

The above expression for the solution at the origin restricts the zeroth sample of the eigensignal to take on the value zero whenever $c \neq 1$. However, when $c = 1$, the eigensignal may freely take on an arbitrary value at the origin, including zero.

Iteration can be used for the treatment of the non-zero-valued time indices that are multiples of M. Specifically, the value of the eigensignal at $n = M^q r$ is given by

$$i_{c,M}(M^q n) = c^q i_{c,M}(n), \quad n > 0,\ q \geq 0. \tag{5}$$

Two fundamental properties of the eigensignals may be inferred from (3) and (5).

Property 1: The simplest nonzero eigensignal is the signal $i_{1,M} = \{x_0, 0, 0, \cdots\}$, where $x_0 \neq 0$.

Property 2: If the eigensignal has at least one non-zero-valued sample, other than the origin, at $n = n_0$, then it must have infinitely many non-zero-valued samples at the time indices $n = M^{q_0} n_0$ $(q_0, n_0 \geq 1)$.

Let M^q be the largest integer power of M contained in n. Then, *any* integer time index n can always be written uniquely in the form

$$n = M^q\, r, \tag{6}$$

where $q \geq 0$, and r is a positive integer satisfying

$$r \bmod M \neq 0 \quad \text{and} \quad r \geq 1.$$

Note that the above representation is not restrictive in any sense and for the time indices n that do not contain powers of M, we simply have $q = 0$ in (6). Consequently, the functional equation (2) can be rewritten as

$$\begin{aligned} &i_{c,M}(M^q r) = c^q\, i_{c,M}(r), \\ &r \geq 1, \quad q \geq 0, \quad r \bmod M \neq 0. \end{aligned} \tag{7}$$

The redundant samples of the eigensignal correspond to those values of n for which $q \geq 1$ in (6) and (7). These samples cannot be chosen independently. However, the remaining samples, for which $q = 0$, may be chosen freely. The above development leads to the following proposition.

Proposition 1: The signal $i_{c,M}(n)$ is an eigensignal if and only if the conditions (3) and (7) are satisfied simultaneously.

Proof. Sufficiency of the conditions can be verified by direct substitution into (2). Necessity of the conditions is a direct consequence of the derivation process.

2.2 Construction of Eigensignals by Multiplication

The time-domain treatment of the eigensignals gives a simple method for constructing non-trivial eigensignals. Specifically, it is possible to create a new eigensignal by modulating an eigensignal $i_{c_0,M}(n)$ using another eigensignal $i_{c_1,M}(n)$. The modulated signal is obtained by multiplication in the time domain where two eigensignals $i_{c_0,M}(n)$ and $i_{c_1,M}(n)$ are used to give

$$y(n) = i_{c_0,M}(n) i_{c_1,M}(n). \tag{8}$$

To show that $y(n)$ is an eigensignal, we use Proposition 1 and verify the validity of the conditions given by (3) and (7). For the value at the origin, we have

$$y(0) = \delta_{c_0,1}\,\delta_{c_1,1}\,x_0\,x_1. \tag{9}$$

From the definition of the Kronecker delta, the left side of (9) does not change if the right side is multiplied by $\delta_{c_0c_1,1}$, where

$$\delta_{c_0c_1,1} = \begin{cases} 1, & c_0c_1 = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$y(0) = \delta_{c_0c_1,1}\,\delta_{c_0,1}\,\delta_{c_1,1}\,x_0\,x_1 \tag{10}$$

showing the validity of (3) for $y(n)$ if the scaling factor is taken to be $c_0\,c_1$ and the value at the origin to be $\delta_{c_0,1}\,\delta_{c_1,1}\,x_0\,x_1$. On the other hand, at $n = M^q r$ we have

$$\begin{aligned} y(M^q r) &= (c_0\,c_1)^q\, i_{c_0,M}(r)\, i_{c_1,M}(r) \\ &= (c_0\,c_1)^q\, y(r), \qquad r \bmod M \neq 0, \end{aligned} \tag{11}$$

which is also consistent with the scaling factor c_0c_1 and confirms the validity of (7).

As an example, let us consider the use of the ramp signal for the purpose of modulation. Starting from a given eigensignal $i_{c_1,M}(n)$, a related eigensignal of the form $n\,i_{c_1,M}(n)$ can be created having the scaling factor Mc_1. Repetition of this process is a simple method to construct more complicated eigensignals.

3. Characterization in z Domain

In the z-domain, $I_{c,M}(z)$ is an eigensignal of an M-to-1 downsampler if and only if

$$I_{c,M}(z)\big\downarrow_M = c\, I_{c,M}(z), \tag{12}$$

where the decimation operator, signified by a downward arrow on the left, should be interpreted according to the relation

$$I_{c,M}(z)\big\downarrow_M = \frac{1}{M}\sum_{k=0}^{M-1} I_{c,M}(z^{\frac{1}{M}} W_M^k). \tag{13}$$

The symbol W_M stands for an Mth root of unity and is adopted to be equal to $\exp(-j\frac{2\pi}{M})$. The presence of a fractional power of z in the definition (12) may be conceived as an unfavorable attribute. The issue can be alleviated by noting that any two discrete-time signals are identical if and only if their upsampled versions, obtained by regular zero insertion, are identical as well. Thus, by applying 1-to-M upsampling to the both sides of (12), we obtain

$$I_{c,M}(z)\big\downarrow_M\big\uparrow_M = c\, I(z^M), \tag{14}$$

where the upward arrow on the left signifies the operation of 1-to-M upsampling. Conversely, applying M-to-1 downsampling to the both sides of (14), and noting that for any signal $X(z)$, $X(z)\uparrow_M\downarrow_M = X(z)$, we obtain (12). Hence, a z-domain definition of an eigensignal, equivalent to Definition 1 and free from the fractional powers of z, is as follows.

Definition 2: The signal $I_{c,M}(z)$ is an eigensignal of an M-to-1 downsampler with the scaling constant c if and only if

$$\sum_{k=0}^{M-1} I_{c,M}(zW_M^k) = c\,M\,I_{c,M}(z^M). \tag{15}$$

Note that any linear combination of an arbitrary number of eigensignals $I_{c,M}^{(i)}(z)$, $i = 1, 2, \ldots$, having identical scaling constants and rate-change factors, is also an eigensignal. In the following, we consider two different z-domain approaches to the problem of eigensignal characterization.

3.1 Power-Series Approach

The first characterization is based on the one-sided z-transform in the power form

$$I_{c,M}(z) = \sum_{n\geq 0} i_{c,M}(n)\, z^{-n}.$$

From (6), and Proposition 1, it follows that for an eigensignal we have

$$I_{c,M}(z) = \delta_{c,1}\,x_0 + \sum_{q\geq 0}\ \sum_{\substack{r\geq 1 \\ r \bmod M\neq 0}} c^q\, i_{c,M}(r) z^{-M^q r},$$

which can be rewritten in a more convenient form, after interchanging the order of summations, as

$$I_{c,M}(z) = \delta_{c,1}\,x_0 + \sum_{\substack{r\geq 1 \\ r \bmod M\neq 0}} i_{c,M}(r)\Phi_r(z;c), \tag{16}$$

where

$$\Phi_r(z;c) = \sum_{q\geq 0} c^q z^{-M^q r}, \qquad r \bmod M \neq 0, \tag{17}$$

is called the rth *fundamental eigensignal* for the M-to-1 downsampler. The term fundamental eigensignal is coined based on the assertion that $\Phi_r(z;c)$ is the “simplest” non-trivial infinite-length eigensignal. An important formal property of $\Phi_r(z;c)$ is that

$$\sum_{\substack{r\geq 1\\ r \bmod M\neq 0}} \Phi_r(z;1) = \frac{z^{-1}}{1-z^{-1}}. \tag{18}$$

This is the partitioning property of the fundamental eigensignals which holds only for $c = 1$. The above development results in the following proposition.

Proposition 2: $I_{c,M}(z)$ represents the z transform of an eigensignal if and only if it can be expressed as a linear combination of $\Phi_r(z;c)$ where M must not be a divisor of r.

Proof. The sufficiency of the condition given by (16) can be verified by calculating the inverse z-transform and applying the conditions specified in Proposition 1. The necessity of the conditions is a direct consequence of the derivation process.

3.2 Construction of Eigensignals by Differentiation

A simple method for constructing new eigensignals from a given eigensignal $I_{c,M}(z)$ with a closed-form z transform is to apply the operator $-z\frac{d}{dz}$. In particular, the following Proposition holds.

Proposition 3: Given the eigensignal $I_{c,M}(z)$, the signal $-z\frac{d}{dz}I_{c,M}(z)$ is also an eigensignal for an M-to-1 downsampler with the scaling factor cM.

To prove the validity of Proposition 3, we apply the operator to the fundamental eigensignals. We thus write

$$-z\frac{d}{dz}\Phi_r(z;c) = r\sum_{q\geq 0}(cM)^q z^{-M^q r}. \tag{19}$$

It follows that

$$-z\frac{d}{dz}I_{c,M}(z) = \sum_{\substack{r\geq 1\\ r \bmod M\neq 0}} r i_{c,M}(r)\,\Phi_r(z;cM). \tag{20}$$

The right side of (20) is still consistent with the general form of an eigensignal given by (16) by assuming a scaling factor equal to cM and a zero-valued constant term. It is not difficult to verify that the application of $-z\frac{d}{dz}$ to the z transform of an eigensignal is identical to modulation with the ramp signal in the time domain. Nevertheless, if the z transform is given explicitly, it is more convenient to apply this z- domain operator.

3.3 Lambert-Form Characterization

The fact that the unit-step signal is an eigensignal for any downsampler inspires us to examine an alternative characterization of the eigensignals $I_{c,M}(z)$ by considering their formal representation as Lambert-form series. A Lambert series is an infinite series of the form

$$L(Z) = \sum_{n\geq 1} a_n \frac{Z^n}{1-Z^n},$$

which was introduced by J.H. Lambert in 1771 [6]. We employ the familiar z^{-1} to represent the indeterminate Z. Note that in the above form, the formal expansion of the Lambert series in terms of the powers of z^{-1} does not yield a constant term. Therefore, we augment the series by adding a constant term to it, and write

$$L(z) = a_0 + \sum_{n\geq 1} a_n \frac{z^{-n}}{1-z^{-n}}. \tag{21}$$

The interested reader should consult [6] for convergence issues. However, for the particular problem at hand, we only need the formal properties of Lambert series and convergence is not of interest.

Formally, the z-transform of a signal can be expressed in either the power form or the Lambert form. The Lambert form (21) is a representation based on the linear combination of the delayed unit-step function and its upsampled versions. Since the two representations are formally equivalent, we may write

$$\sum_{n\geq 0} x(n) z^{-n} = x(0) + \sum_{n\geq 1} a_n \frac{z^{-n}}{1-z^{-n}}. \tag{22}$$

The above relation only indicates that the same positive powers of z^{-1} on the both sides of (22) are equal once the right side is formally expanded in z^{-1}. By expanding the right side above, we obtain

$$x(n) = \sum_{i|n} a_i, \qquad n \geq 1, \tag{23}$$

where the sum is taken over all positive divisors of n. We can easily solve for a_n in terms of $x(i)$ using the well-known Möbius inversion formula [7]. This yields

$$a_n = \sum_{i|n} \mu\left(\frac{n}{i}\right) x(i), \qquad n \geq 1, \tag{24}$$

where $\mu(.)$ is the Möbius function. The Möbius function takes on values from the set $\{0, 1, -1\}$ depending on the number of repetitive prime factors of its argument. Specifically, we have

$$\mu(n) = \begin{cases} 1, & n = 1, \\ 0, & n \text{ has one or more repeated} \\ & \text{prime factors,} \\ (-1)^k, & n \text{ is the product of } k \text{ distinct} \\ & \text{prime factors.} \end{cases}$$

The relations (23) and (24) enable us to switch between the power and Lambert forms of a z-transform. For another signal processing application of the Möbius function and associated inversion formula see [8].

The advantage of dealing with the Lambert form is that the action of a downsampler on the unit-step signal and its upsampled versions is very amenable to mathematical treatment. Specifically, we show that

$$\left(a_n \frac{z^{-n}}{1-z^{-n}}\right)\Big\downarrow_M = a_n \frac{z^{-\frac{n}{\gcd(n,M)}}}{1-z^{-\frac{n}{\gcd(n,M)}}}. \tag{25}$$

The validity of this relation follows from

$$\begin{aligned}
\left(\frac{z^{-n}}{1-z^{-n}}\right)\Big\downarrow_M &= \left(\sum_{i\geq 1} z^{-ni}\right)\Bigg\downarrow_M = \sum_{\substack{i\geq 1\\ M|ni}} z^{-\frac{ni}{M}}\\
&= \sum_{\substack{i\geq 1\\ \frac{M}{\gcd(n,M)}|\frac{n}{\gcd(n,M)}i}} z^{-\frac{ni}{M}}\\
&= \sum_{\substack{i\geq 1\\ \frac{M}{\gcd(n,M)}|i}} z^{-\frac{ni}{M}}\\
&= \sum_{\substack{i=\frac{M}{\gcd(n,M)}i'\\ i'\geq 1}} z^{-\frac{ni}{M}}\\
&= \sum_{i'\geq 1} z^{-\frac{n}{\gcd(n,M)}i'}\\
&= \frac{z^{-\frac{n}{\gcd(n,M)}}}{1-z^{-\frac{n}{\gcd(n,M)}}}.
\end{aligned} \tag{26}$$

Consequently, for $I_{c,M}(z)$ to be an eigensignal, it is necessary and sufficient that the relationship

$$\begin{aligned}
&i_{c,M}(0) + \sum_{n\geq 1} a_n \frac{z^{-\frac{n}{\gcd(n,M)}}}{1-z^{-\frac{n}{\gcd(n,M)}}}\\
&= c\, i_{c,M}(0) + c\sum_{n\geq 1} a_n \frac{z^{-n}}{1-z^{-n}}
\end{aligned}$$

holds formally. Since the zeroth sample, $i_{c,M}(0)$, has already been characterized in (3), the Lambert form of the eigensignal is thus expressed as

$$I_{c,M}(z) = \delta_{c,1}\, x_0 + \sum_{n\geq 1} a_n \frac{z^{-n}}{1-z^{-n}}, \tag{27}$$

where the coefficients a_n should satisfy the condition

$$c\, a_n = \sum_{\substack{i\geq n\\ \frac{i}{\gcd(i,M)}=n}} a_i, \qquad n \geq 1. \tag{28}$$

Compared to (7), which is a multiplicative condition, (28) is an additive condition, i.e., it is expressed as a linear combination of the Lambert coefficients. The above result is expressed by the following proposition.

Proposition 4: The signal $I_{c,M}(n)$ is an eigensignal if and only if (3) holds and the coefficients a_n of its Lambert form satisfy (28).

Example

As an application of the Lambert-form characterization, let us design an eigensignal by finding a solution to the linear system (28) under the assumption that

$$a_j = 0, \qquad \gcd(j, M) > 1.$$

This restriction reduces (28) to a linear system of equations of the form

$$c\, a_n = a_n, \qquad \gcd(n, M) = 1. \tag{29}$$

To obtain a non-trivial solution to the above system, we should set $c = 1$. The Lambert-form representation of the eigensignal in this example then becomes

$$I_{1,M}(z) = \sum_{\substack{n\geq 1\\ \gcd(n,M)=1}} a_n \frac{z^{-n}}{1-z^{-n}}. \tag{30}$$

This particular eigensignal has the property that each unit-step signal involved in the combination is an eigensignal itself.

3.4 Explicit Lambert-Form Formula for Prime M

We restrict our attention to prime decimation rates and derive a solution to the characterizing equations (28). For a prime decimation rate, two distinct types of equations are identified in the system specified by (28) and given by

$$\begin{aligned}
c\, a_n &= a_n + a_{Mn}, &\quad \gcd(n, M) = 1,\\
c\, a_n &= a_{Mn}, &\quad \gcd(n, M) = M.
\end{aligned} \tag{31}$$

Let us write $n = M^q r$, where q and r are determined as described earlier in Sect. 2. A solution to (31) then becomes

$$a_n = \begin{cases} c^{q-1}(c-1)a_r & q\geq 1\\ a_r & q = 0 \end{cases} \quad r \bmod M \neq 0. \tag{32}$$

This means that only those coefficients that are not multiples of M may be chosen in an arbitrary manner. The Lambert form of the eigensignal is thus given by

$$\begin{aligned}
I_{c,M}(z) &= \delta_{c,1}\, x_0 + \sum_{\substack{r\geq 1\\ r \bmod M\neq 0}} a_r \frac{z^{-r}}{1-z^{-r}}\\
&\quad + \sum_{q\geq 1} \sum_{\substack{r\geq 1\\ r \bmod M\neq 0}} c^{q-1}(c-1)a_r \frac{z^{-M^q r}}{1-z^{-M^q r}}.
\end{aligned} \tag{33}$$

This is a general formula for the Lambert form of eigensignals for the case where the decimation rate is a prime number. Also note that (33) is consistent with the result obtained in the example given in the preceding section. Specifically, by setting $c = 1$, the constant term and the two-fold summation vanish in (33), and the resulting expression is equivalent to (30).

4. Multiplicative Eigensignals

This section is devoted to the study of two classes of eigensignals that have deep connections with the multiplicative number theoretic or arithmetic functions [9].

4.1 Prime Decimation Rates

An arithmetical function, also known as a number-theoretic function, is a function whose domain is the set of positive integers [4]. A special class of arithmetical functions, known as the multiplicative arithmetical functions, is defined by the relation [4], [7]

$$f(mn) = f(m)f(n) \qquad \text{whenever} \quad \gcd(m, n) = 1. \tag{34}$$

The value of a multiplicative arithmetical function at $n = 1$ is restricted to

$$f(1) = 1. \tag{35}$$

A well-known example of such functions is the Möbius function [7].

When M is prime, the factors in (6) satisfy the relation

$$\gcd(M^q, r) = 1. \tag{36}$$

Thus, a special class of eigensignals emerges by letting $i_{c,M}(n)$ be a multiplicative arithmetical function $f(n)$ satisfying an additional condition of the form

$$f(M^q) = c^q, \qquad q = 0, 1, \cdots . \tag{37}$$

This requirement is clearly consistent with (35). Moreover, in this case, the value of c is determined by setting $q = 1$, i.e.,

$$c = f(M). \tag{38}$$

Consequently, for a prime decimation rate, any multiplicative arithmetical function $f(n)$ satisfying (37) is an eigensignal with the scaling factor $f(M)$. Such eigensignals are called *multiplicative eigensignals*.

Definition 3: For a prime M, a multiplicative eigensignal is defined to be a signal satisfying

$$\begin{aligned} &i_{c,M}(0) = \delta_{c,1}\, x_0 \\ &i_{c,M}(M^q) = c^q, \quad q \geq 0 \\ &i_{c,M}(mn) = i_{c,M}(m)\, i_{c,M}(n), \qquad \gcd(m, n) = 1 \end{aligned} \tag{39}$$

From the above definition, it follows that a multiplicative eigensignal can be designed by specifying its samples at the time indices which are powers of primes while imposing the restriction that its value at the qth power of M be equal to c^q.

4.2 Composite Decimation Rates

If M is a composite number, condition (36) does not hold for all values of q and r and an ordinary multiplicative function cannot be used to create an eigensignal. However, it is possible to use a more restrictive type of multiplicative functions called completely multiplicative arithmetical functions. An arithmetical function $f(n)$ which is not identically zero is completely multiplicative if [4]

$$\begin{aligned} &f(1) = 1, \\ &f(mn) = f(m)f(n), \qquad \forall\, m, n. \end{aligned} \tag{40}$$

By assuming that $i_{c,M}(n)$ is a completely multiplicative arithmetical function, the right side of (7) becomes

$$i_{c,M}(M^q r) = (i_{c,M}(M))^q\, i_{c,M}(r).$$

It then follows that $i_{c,M}(n)$ is an eigensignal with the scaling factor

$$c = i_{c,M}(M). \tag{41}$$

Definition 4: A completely multiplicative eigensignal is defined to be a signal satisfying

$$\begin{aligned} &i_{c,M}(0) = \delta_{c,1}\, x_0 \\ &i_{c,M}(1) = 1 \\ &i_{c,M}(mn) = i_{c,M}(m)\, i_{c,M}(n) \end{aligned} \tag{42}$$

To design a completely multiplicative eigensignal, we only need to specify its values at prime time indices. These values may be freely defined with no restriction. The values of the signal at composite time indices are specified by the multiplicative property. In short, for composite or prime decimation rates M, any completely multiplicative arithmetical function is an eigensignal with the scaling factor $c = i_{c,M}(M)$.

An example of a completely multiplicative eigensignal is the power signal $x(n) = n^k$. Another example is Liouville's function defined according to

$$x(n) = (-1)^{\alpha_1 + \cdots + \alpha_t}, \tag{43}$$

where integers $\alpha_1, \ldots, \alpha_t$ are the exponents in the unique prime factorization of n, i. e.,

$$n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}.$$

A notable property of completely multiplicative eigensignals is that they are valid as eigensignals for all decimation rates $M \geq 2$. The parameter whose value varies with the adopted decimation rate is the scaling factor c. For instance, for the completely multiplicative eigensignal of Fig. 3, which is Liouvilles's function, $c = \pm 1$ depending on the value of M. The converse of this proposition is also true. That is, if $i_{c,M}(n)$ is an eigensignal for all decimation rates $M \geq 2$, and $i_{c,M}(1) = 1$, then $i_{c,M}(n)$ is a completely multiplicative arithmetical function. Finally, note that the so-called Bell transform [10] has been applied by mathematician to study and represent multiplicative and completely multiplicative arithmetical functions.

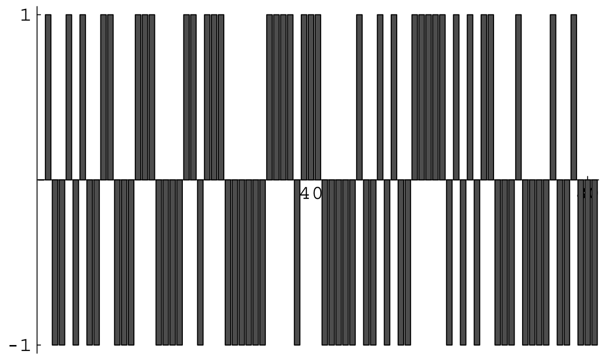

Fig. 3 A completely multiplicative eigensignal satisfying the definition given by (43).

5. Rational Eigensignals

An example of eigensignals with rational z-transforms is the family of signals

$$\underbrace{\left(-z^{-1}\frac{d}{dz}\right)\cdots\left(-z^{-1}\frac{d}{dz}\right)}_{p\text{ times}}\frac{1}{1-z^{-1}}$$

derived from the unit-step signal by the p-fold application of the operator introduced in Sect. 3.2. They are eigensignals common to all M-to-1 downsamplers and have rational transfer functions. In Sect. 3, where the Lambert series was used to study the eigensignals in the z- domain, we also dealt with eigensignals with rational z-transforms. In the following, we introduce a result showing that the locus of the poles of a rational eigensignal is tightly restricted, and as a result, it is not possible to construct rational eigensignals whose poles are located at arbitrary points of the complex plane [11].

Proposition 5: The poles of the rational eigensignal $I_{c,M}(z) = \frac{P(z)}{Q(z)}$ lie on the unit circle.

The validity of Proposition 5 can be proved by invoking Definition 2 and writing

$$\sum_{k=0}^{M-1}\frac{P(zW_M^k)}{Q(zW_M^k)} = cM\frac{P(z^M)}{Q(z^M)}. \tag{44}$$

It follows that

$$\frac{\sum_{k=0}^{M-1}P(zW_M^k)\prod_{l\neq k}Q(zW_M^l)}{\prod_{k=0}^{M-1}Q(zW_M^k)} = cM\frac{P(z^M)}{Q(z^M)}. \tag{45}$$

Therefore,

$$\prod_{k=0}^{M-1}Q(zW_M^k) = Q(z^M). \tag{46}$$

Let r_l, $l = 0, \cdots, N-1$, be the roots of $Q(z)$ and assume that the constant coefficient of $Q(z)$ is unity. We can rewrite (46) as

$$\prod_{k=0}^{M-1}\prod_{l=0}^{N-1}(1 - r_l W_M^k\, z^{-1}) = \prod_{l=0}^{N-1}(1 - r_l\, z^{-M}). \tag{47}$$

Changing the order of products on the left side of (47), we may write

$$\prod_{l=0}^{N-1}(1 - r_l^M\, z^{-M}) = \prod_{l=0}^{N-1}(1 - r_l\, z^{-M}). \tag{48}$$

Now, let us pick the pole $r = r_0$. Obviously, we must have $r_0^M = r_{k_1}$, for some k_1, in order that (48) is satisfied. We then have either $r_0 = 1$, which is a real root of unity, or, since there are a finite number of poles, after performing i iterations of the forms

$$r_0^M = r_{k_1}, \qquad r_{k_1}^M = r_{k_2}, \quad \ldots, \quad r_{k_{i-1}}^M = r_{k_i},$$

we must return to the original index. This means that $r_0^{M^i} = r_0$, which is again equivalent to the fact that r_0 is a real or complex root of unity.

The restriction on the locus of the poles of rational eigensignals has an interesting implication in the time domain. The inverse z transform of an eigensignal whose poles lie on the unit circle is a quasi-polynomial [12]. A quasi-polynomial eigensignal may be expressed as a function of the form

$$i_{c,M}(n) = g_d(n)n^d + g_{d-1}(n)n^{d-1} + \cdots + g_0(n), \tag{49}$$

where each $g_i(n)$, $i = 0, 1, \ldots, d$ is a periodic function with an integer period. Equivalently, $i_{c,M}(n)$ is a quasi-polynomial eigensignal if there exists an integer $N_0 > 0$ (called the quasi-period of the eigensignal) and polynomials $p_0(n), p_1(n), \ldots, p_{N_0-1}(n)$ such that

$$i_{c,M}(n) = p_i(n), \qquad \text{where } i = n \bmod N_0. \tag{50}$$

This shows that eigensignals with rational z-transforms only exist within the class of quasi-polynomial signals.

6. Conclusion

It has been shown that the eigensignals of a downsampler are not limited to the trivial unit-impulse or unit-step signals. Eigensignals have been studied and characterized in the time and z domains. In the time domain, the characterization problem has been formulated through a functional equation. The solutions to the functional equation have been completely characterized using number theoretic techniques. In the z domain, the characterization has led to a representation of the eigensignals as a linear combination of what we have called the fundamental eigensignals. A Lambert-form characterization has been carried out as well. An explicit z-domain representation of the eigensignals for prime decimation rates has been derived in the Lambert form. Some special multiplicative eigensignal have also been studied. Finally, it has been stated that the locus of the poles for the eigensignals with rational z-transforms is the unit circle.

In a recent paper [13]†, the authors have successfully used the unit-step signal to characterize perfect reconstruction conditions for nonuniform filterbanks. The more general class of the eigensignals derived in this paper may find similar applications in the theory of filterbanks. Another possible application is the development of signal decomposition scenarios based on the eigensignals. An open question that has not been treated in this paper is the frequency-domain characterization of the eigensignals. The concepts and signals developed in this paper may find industrial applications in testing and verification of decimators in hardware or software-based equipment in digital signal processing.

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†See [14] for the correction of a paragraph deleted in the post-production phase of the publication of [13].

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