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TRAVELING FRONTS OF PYRAMIDAL SHAPES IN THE ALLEN–CAHN EQUATIONS*

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Abstract. This paper studies pyramidal traveling fronts in the Allen–Cahn equation or in the Nagumo equation. For the nonlinearity we are concerned mainly with the bistable reaction term with unbalanced energy density. Two-dimensional V-form waves and cylindrically symmetric waves in higher dimensions have been recently studied. Our aim in this paper is to construct truly three-dimensional traveling waves. For a pyramid that satisfies a condition, we construct a traveling front for which the contour line has a pyramidal shape. We also construct generalized pyramidal fronts and traveling waves of a hybrid type between pyramidal waves and planar V-form waves. We use the comparison principles and construct traveling fronts between supersolutions and subsolutions.

Key words. pyramidal traveling wave, Allen–Cahn equation, bistable

AMS subject classification. 35K57

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1. Introduction.

In this paper we consider the following equation:

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad \text{in } \mathbb{R}^3, \quad t > 0,$$
$$u\big|_{t=0} = u_0 \quad \text{in } \mathbb{R}^3.$$

Here a given function u_0 is bounded and of class C^1. The Laplacian Δ stands for $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. If the nonlinearity f is cubic, it is called the Allen–Cahn equation or the Nagumo equation. We study general nonlinear terms of a bistable type including cubic ones.

In the one-dimensional space, let $\Phi(x - kt)$ be a traveling wave that connects two stable equilibrium states ± 1 with speed k. By putting $\mu = x - kt$, Φ satisfies

$$-\Phi''(\mu) - k\Phi'(\mu) - f(\Phi(\mu)) = 0 \quad -\infty < \mu < \infty, \qquad \Phi(-\infty) = 1, \quad \Phi(\infty) = -1. \tag{1}$$

To fix the phase we set $\Phi(0) = 0$. Such one-dimensional traveling waves have been studied in many works. See Fife and McLeod [5], Aronson and Weinberger [1], Kanel' [10, 11], Chen [2], and Terman [18], for instance. We state equations for the unbalanced nonlinearity and the balanced one.

The unbalanced case is as follows. The following are the assumptions on f in this paper:

(A1) f is of class $C^1[-1, 1]$, with $f(\pm 1) = 0$ and $f'(\pm 1) < 0$.
(A2) $\int_{-1}^{1} f > 0$ holds true.
(A3) There exists $\Phi(\mu)$ that satisfies (1) for some $k \in \mathbb{R}$.

The assumption (A1) implies that f is bistable and that (A2) means that it is unbalanced. Note that (A2) implies $k > 0$. Under (A1), $(k, \Phi(\mu))$ is uniquely

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determined if it exists. For the proof of this uniqueness, see [5] or [2]. We show simple examples for f here.

Example 1. If f satisfies $f'(\beta) > 0$ and

$$f(s) > 0 \quad \text{for } \beta < s < 1,$$
$$f(s) < 0 \quad \text{for } -1 < s < \beta,$$

with some $\beta \in (-1, 1)$ in addition to (A1), then it is well known that (A3) is valid. See [5] or [2]. Especially, $f(u) = -(u+1)(u+a)(u-1)$ has a one-dimensional traveling wave $\Phi(\mu) = -\tanh(\mu/\sqrt{2})$ with speed $k = \sqrt{2}a$ for every $a \in [0, 1)$. This traveling wave is sometimes called the Huxley solution. See Nagumo, Yoshizawa, and Arimoto [14].

Example 2 (Fife and McLeod [5, Theorem 2.7]). Assume f satisfies (A1) and (A2). For $-1 < \lambda < 1$ assume that there exists (c_L, Φ_L) to

$$-\Phi_L''(\mu) - k\Phi_L'(\mu) - f(\Phi_L(\mu)) = 0 \quad -\infty < \mu < \infty,$$
$$\Phi_L(-\infty) = 1, \quad \Phi_L(\infty) = \lambda,$$

and there exists (c_R, Φ_R) to

$$-\Phi_R''(\mu) - k\Phi_R'(\mu) - f(\Phi_R(\mu)) = 0 \quad -\infty < \mu < \infty,$$
$$\Phi_R(-\infty) = \lambda, \quad \Phi_R(\infty) = -1.$$

If $c_L > c_R$, then (A3) holds true. If $c_L \le c_R$, there exists no solution to (1).

Example 3. For G with (B1) and (B2) below, we define

$$f(u) = -G'(u) + k\sqrt{2G(u)}$$

for $k > 0$. Then $\Phi_0(\mu)$ given by (3) is a solution to (1) for $k > 0$. If k is small enough, $f(u)$ satisfies (A1), (A2), and (A3).

We note that (A1) and (A2) do not always imply (A3), because we can construct such f with $c_L \le c_R$ in Example 2. If it exists, it is always monotone in μ as in Lemma 1. See [5, Lemma 2.1] for the proof of the monotony of one-dimensional fronts. We use this monotony and the comparison principles in this paper.

The balanced case is as follows:

$$\frac{\partial u}{\partial t} = \Delta u - G'(u) \quad \text{in } \mathbb{R}^3, \quad t > 0, \qquad u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^3. \tag{2}$$

The assumptions on G are as follows:

(B1) G is of class $C^2[-1, 1]$, with $G'(\pm 1) = 0$, $G''(\pm 1) > 0$.

(B2) $G(1) = 0$ and $G(s) > 0$ for $-1 < s < 1$.

Under (B1) and (B2), (2) has a standing wave solution $\Phi_0(x)$ to

$$-\Phi_0''(\mu) + G'(\Phi_0(x)) = 0 \quad -\infty < \mu < \infty,$$
$$\Phi_0(-\infty) = 1, \quad \Phi_0(\infty) = -1.$$

Φ_0 is given by

$$x = -\int_0^{\Phi_0} \frac{dv}{\sqrt{2G(v)}}. \tag{3}$$

The condition $G(1) = 0$ means that a potential density term G has minimizers with an equal depth. If G takes a negative value or zero in $(-1, 1)$, there exists no standing wave. Thus (B2) is the condition for the existence of a standing wave solution Φ_0. A typical balanced nonlinearity term is $-G'(u) = u - u^3$, with $G(u) = (1/4)(1 - u^2)^2$.

First we study traveling waves for the unbalanced nonlinearity. We adopt the moving coordinate of speed c toward the z-axis without loss of generality. We put $s = z - ct$ and $u(x, y, z, t) = w(x, y, s, t)$. We denote $w(x, y, s, t)$ by $w(x, y, z, t)$ for simplicity. Then we obtain

$$
\begin{aligned}
w_t - w_{xx} - w_{yy} - w_{zz} - cw_z - f(w) = 0 \quad &\text{in } \mathbb{R}^3, \quad t > 0, \\
w|_{t=0} = u_0 \quad &\text{in } \mathbb{R}^3.
\end{aligned}
\tag{4}
$$

Here w_t stands for $\partial w/\partial t$ and so on. We write the solution as $w(x, y, z, t; u_0)$. If v is a traveling wave with speed c, it satisfies

$$
\mathcal{L}[v] \stackrel{\text{def}}{=} -v_{xx} - v_{yy} - v_{zz} - cv_z - f(v) = 0 \quad \text{in } \mathbb{R}^3. \tag{5}
$$

We assume

$$
c > k
$$

throughout this paper. There exist many traveling waves in this situation, because k is the speed of a planar traveling wave, and the curvature effect often accelerates the speed.

In the two-dimensional plane there exists the following V-form wave.

THEOREM 1 (see [15]). *Under the assumptions* $c > k$, (A1), (A2), *and* (A3), *there exists* $v_*(x, y)$, *with*

$$
\begin{gathered}
-(v_*)_{xx} - (v_*)_{yy} - c(v_*)_y - f(v_*) = 0 \qquad \textit{for } (x, y) \in \mathbb{R}^2, \\
\lim_{R\to\infty} \sup_{x^2+y^2>R^2} \left| v_*(x, y) - \Phi\left(\frac{k}{c}\left(y - \frac{\sqrt{c^2 - k^2}}{k}|x|\right)\right)\right| = 0.
\end{gathered}
$$

Under these two equalities $v_*(x, y)$ *is uniquely determined.*

See also Hamel, Monneau, and Roquejoffre [8, 9] for V-form waves in the Allen–Cahn equation. Recently Haragus and Scheel [13] studied V-form waves in reaction-diffusion systems including the Allen–Cahn equation by using the bifurcation theory when the angle $\arctan(\sqrt{c^2 - k^2}/k)$ is small enough. Such a bifurcation technique is applicable to the cases where a one-dimensional traveling front loses its monotony.

For three- or higher-dimensional cases with cylindrical symmetry, Hamel, Monneau, and Roquejoffre [8, 9] studied conical traveling waves for unbalanced bistable nonlinearity. The proof is based on the results for bounded cylinders, and a passage to the limit gives the existence of a conical front in the whole domain.

Now we study three-dimensional traveling waves, and our aim is to search truly three-dimensional traveling waves that have pyramidal structures and are neither cylindrically symmetric nor reducible to two-dimensional traveling waves. For this purpose, we construct pyramidal traveling waves to (5). We apply the method of Ninomiya and Taniguchi [15, 16]. A supersolution for the V-form wave is constructed in [15] as follows. In the moving coordinate we put an almost flat planar front above the shape "V." This curve is almost flat, and then the real solution goes downwards with speed $c - k > 0$, since we are using a moving coordinate. This means that an almost flat stationary planar front is a supersolution. This method is based on the monotony of a one-dimensional traveling front and the comparison methods. The application of this method is restricted to equations for which the comparison principle holds true. In this paper, we put a mollified pyramid above a pyramid in $\mathbb{R}^3$ and construct a supersolution carefully, because a pyramidal wave is everywhere apart from a pyramid near the edges.

Let $n \geq 3$ be a given integer. We put

$$\tau \overset{\mathrm{def}}{=} \frac{\sqrt{c^2 - k^2}}{k} > 0. \tag{6}$$

Assume $(A_j, B_j) \in \mathbb{R}^2$ satisfies

$$A_j^2 + B_j^2 = 1 \qquad \text{for all } j = 1, \ldots, n \tag{7}$$

and

$$\begin{gathered} A_j B_{j+1} - A_{j+1} B_j > 0, \qquad 1 \leq j \leq n-1, \\ A_n B_1 - A_1 B_n > 0. \end{gathered} \tag{8}$$

We assume $(A_{j_1}, B_{j_1}) \neq (A_{j_2}, B_{j_2})$ if $j_1 \neq j_2$. Now $(-\tau A_j, -\tau B_j, 1)$ is the normal vector of a surface $\{z = \tau(A_j x + B_j y)\}$. We put

$$\begin{aligned} h_j(x,y) &\overset{\mathrm{def}}{=} \tau \left(A_j x + B_j y \right), \\ h(x,y) &\overset{\mathrm{def}}{=} \max_{1 \leq j \leq n} h_j(x,y) = \tau \max_{1 \leq j \leq n} \left(A_j x + B_j y \right). \end{aligned} \tag{9}$$

Then $z = h(x,y)$ represents a pyramid in $\mathbb{R}^3$. We set

$$\Omega_j = \{(x,y) \,|\, h(x,y) = h_j(x,y)\}$$

and obtain

$$\mathbb{R}^2 = \cup_{j=1}^n \Omega_j.$$

We locate $\Omega_1, \Omega_2, \ldots, \Omega_n$ counterclockwise as in Figure 1. To ensure this location we assumed (8). We set

$$E \overset{\mathrm{def}}{=} \cup_{j=1}^n \partial\Omega_j \subset \mathbb{R}^2.$$

Now the lateral surfaces of a pyramid are given by

$$S_j = \{(x,y,z) \in \mathbb{R}^3 \,|\, z = h_j(x,y), \quad (x,y) \in \Omega_j\}$$

for $j = 1, \ldots, n$. We put

$$\Gamma_j \overset{\mathrm{def}}{=} \begin{cases} S_j \cap S_{j+1} & \text{if } 1 \leq j \leq n-1, \\ S_n \cap S_1 & \text{if } j = n. \end{cases}$$

Then Γ_j represents an edge of a pyramid. Also

$$\Gamma \overset{\mathrm{def}}{=} \cup_{j=1}^n \Gamma_j$$

represents the set of all edges.

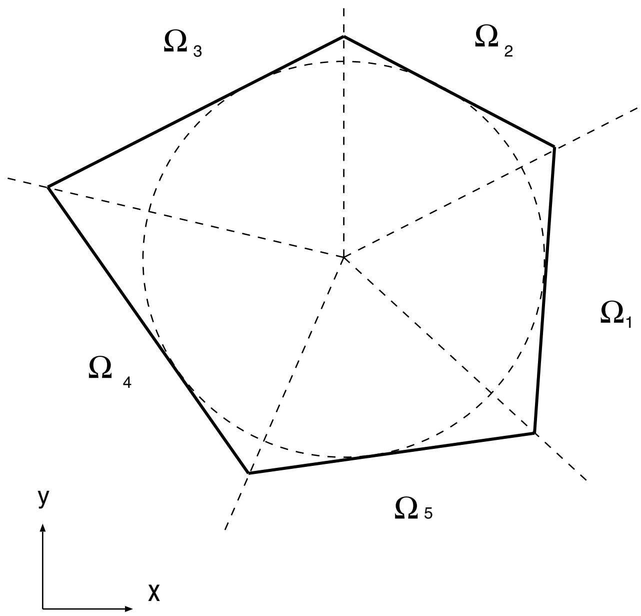

FIG. 1. *The decomposition of the x-y plane by Ω_j for $n = 5$.*

For every (A_j, B_j) with (7), (5) has a solution $\Phi\left((k/c)(z - h_j(x, y))\right)$, which is called a planar wave. Now we have

$$\Phi\left(\frac{k}{c}(z - h(x, y))\right) = \max_{1 \le j \le n} \Phi\left(\frac{k}{c}(z - h_j(x, y))\right) = \max_{1 \le j \le n} \Phi\left(\frac{k}{c}(z - a_j x - b_j y)\right).$$

This becomes a subsolution to (5). We define

$$D(\gamma) \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid \operatorname{dist}((x, y, z), \mathit{\Gamma}) > \gamma\} \tag{10}$$

for $\gamma > 0$. We will construct a supersolution that is larger than this subsolution and obtain a traveling wave between them.

The following theorem is the main assertion in this paper.

THEOREM 2. *Let $c > k$, and let $h(x, y)$ be given by* (9). *Under the assumptions* (A1), (A2), *and* (A3), *there exists a solution $V(x, y, z)$ to* (5) *with*

$$\Phi\left(\frac{k}{c}(z - h(x, y))\right) < V(x, y, z) < 1 \qquad \text{in } \mathbb{R}^3$$

and

$$\lim_{\gamma \to +\infty} \sup_{(x,y,z) \in D(\gamma)} \left| V(x, y, z) - \Phi\left(\frac{k}{c}(z - h(x, y))\right) \right| = 0, \tag{11}$$

$$V_z(x, y, z) < 0 \qquad \text{for all } (x, y, z) \in \mathbb{R}^3. \tag{12}$$

We state the proof of this theorem in section 3. A domain $D(\gamma)$ is a complement of a neighborhood of the edges. The property (11) implies that the geometric shape of V can be approximated by a combination of n planar waves except on a neighborhood of the edges (see Figure 2). We conjecture that the geometric shape of V

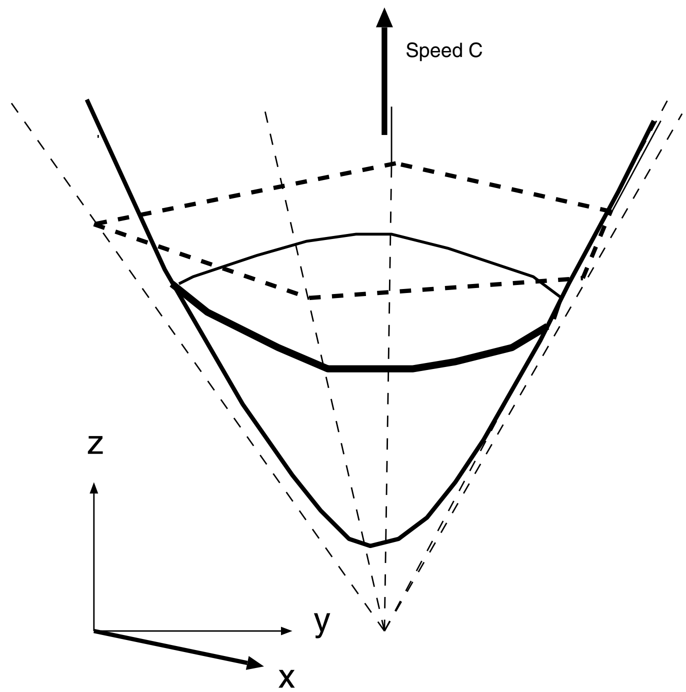

FIG. 2. *A pyramidal traveling wave.*

can be approximated by two-dimensional V-form waves on the edges and that V is a combination of planar waves and two-dimensional V-form waves. The uniqueness and the stability of V is yet to be proved.

Section 4 is devoted to applications of Theorem 2. A two-dimensional V-form wave in Theorem 1 immediately gives a three-dimensional wave $v_*(x, z)$. We call this wave a planar V-form wave. It is natural to search for a combination of a pyramidal wave and a planar V-form wave. In section 4 we study a traveling wave of a hybrid type between pyramidal waves and planar V-form waves as a special case of Theorem 2.

We studied pyramids whose lateral surfaces contain the origin in $\mathbb{R}^3$ in Theorem 2. We consider the case where the surfaces do not have a common point in section 5. Even in that case a combination of n planar waves gives three-dimensional traveling waves, and we construct generalized pyramidal traveling waves when the zero level sets of planar waves have no common point.

We study traveling waves for the balanced nonlinearity in section 6. For any given $c > 0$ we study

$$\mathcal{L}_0[v] \stackrel{\text{def}}{=} -v_{xx} - v_{yy} - v_{zz} - cv_z + G'(v) = 0 \quad \text{in } \mathbb{R}^3. \tag{13}$$

We call $-G'(u)$ in Example 3 a balanced nonlinearity. Cylindrically symmetric traveling waves for balanced nonlinearity have been studied in Chen et al [3] for two or higher dimensions. The limit of traveling waves for unbalanced nonlinearity terms when the difference of energy density goes to zero gives a traveling wave in (13). Pyramidal traveling waves for unbalanced nonlinear terms converge to traveling waves for a balanced nonlinearity term as the difference of the energy density goes to zero. Up to now the profile of the limit traveling wave is unknown and is yet to be studied.

The characterization and classification of all traveling waves for unbalanced and balanced nonlinearities will give interesting problems and are left for further studies.

2. Pyramids and mollified pyramids. In this section we make preparations. We state known results for one-dimensional traveling waves and construct mollified pyramids

LEMMA 1 (Fife and McLeod [5]). *Under the assumptions* (A1) *and* (A3), $\Phi(\mu)$ *as in* (1) *satisfies*

$$\Phi'(\mu) < 0 \qquad \text{for all } \mu \in \mathbb{R},$$
$$\max\{|\Phi'(\mu)|, |\Phi''(\mu)|, |\mu\Phi'(\mu)|\} \le K_0 \exp(-\kappa_0|\mu|).$$

Here K_0, κ_0 *are positive constants.*

There exists a positive constant δ_* $(0 < \delta_* < 1/4)$, with

$$-f'(s) > \kappa_1 \qquad \text{if } |s+1| < 2\delta_* \text{ or } |s-1| < 2\delta_*,$$

where

$$\kappa_1 \stackrel{\text{def}}{=} \frac{1}{2}\min\{-f'(-1), -f'(1)\} > 0.$$

We construct mollified pyramids. Let $\widetilde{\rho}(r) \in C^\infty[0,\infty)$ be a function with the following properties:

$$\widetilde{\rho}(r) > 0, \quad \widetilde{\rho}_r(r) \le 0 \qquad \text{for } r \ge 0,$$
$$\widetilde{\rho}(r) \equiv 1 \qquad \text{if } 0 \le r \le \tfrac{1}{2},$$
$$\widetilde{\rho}(r) = e^{-r} \qquad \text{if } r > 0 \text{ is large enough,}$$
$$2\pi \int_0^\infty r\widetilde{\rho}(r)dr = 1.$$

Then $\rho(x,y) \stackrel{\text{def}}{=} \widetilde{\rho}(\sqrt{x^2+y^2})$ belongs to $C^\infty(\mathbb{R}^2)$ and satisfies $\int_{\mathbb{R}^2} \rho = 1$. For a pyramid $z = h(x,y)$ we define a mollified pyramid $z = \varphi(x,y)$ as $\varphi(x,y) \stackrel{\text{def}}{=} \rho * h$, which means

$$\varphi(x,y) = \int_{\mathbb{R}^2} \rho(x-x', y-y')h(x',y')dx'dy' = \int_{\mathbb{R}^2} \rho(x',y')h(x-x',y-y')dx'dy'. \tag{14}$$

We set $(a_j, b_j) \stackrel{\text{def}}{=} \tau(A_j, B_j)$. Then $(a_j, b_j) \in \mathbb{R}^2$ satisfies

$$\frac{c}{\sqrt{1+a_j^2+b_j^2}} = k \qquad \text{for all } j = 1, \ldots, n. \tag{15}$$

We put

$$S(x,y) \stackrel{\text{def}}{=} \frac{c}{\sqrt{1+\varphi_x(x,y)^2+\varphi_y(x,y)^2}} - k. \tag{16}$$

Then we have the following lemma.

LEMMA 2. *Let* φ *and* S *be as in* (14) *and* (16), *respectively. Then one has*

$$\sup_{(x,y)\in\mathbb{R}^2} |D_x^{i_1} D_y^{i_2} \varphi(x,y)| < +\infty$$

for all integers $i_1 \geq 0$, $i_2 \geq 0$, *and*

$$\begin{aligned} & h(x,y) < \varphi(x,y) \leq h(x,y) + 2\pi\tau \int_0^\infty r^2 \widetilde{\rho}(r)\, dr \\ & |(\nabla\varphi)(x,y)| < \tau, \quad 0 < S(x,y) < c \end{aligned} \tag{17}$$

for all $(x,y) \in \mathbb{R}^2$.

Proof. Now ρ satisfies $|D_x^{i_1} D_y^{i_2} \rho(x,y)| \leq (\text{const}) e^{-\sqrt{x^2+y^2}}$ for large $\sqrt{x^2+y^2} > 0$. We get the first estimate from $D_x^{i_1} D_y^{i_2}(\rho * g_j) = (D_x^{i_1} D_y^{i_2} \rho) * g_j$. Note that $\rho * h_j = h_j$. Using $\rho > 0$, $h_j(x,y) \leq h(x,y)$, and $h_j(x,y) \not\equiv h(x,y)$, we have a strict inequality $h_j(x,y) < \varphi(x,y)$. Thus we get $h(x,y) = \max_{1 \leq j \leq n}(h_j(x,y)) < \varphi(x,y)$. Now

$$|h_j(x',y') - h_j(x,y)| \leq \tau\sqrt{(x'-x)^2 + (y'-y)^2}$$

gives

$$|h(x',y') - h(x,y)| \leq \tau\sqrt{(x'-x)^2 + (y'-y)^2}.$$

Thus we obtain

$$\varphi - h \leq \int_{\mathbb{R}^2} |h(x-x', y-y') - h(x,y)|\, \rho(x',y') dx'dy' \leq \tau \int_{\mathbb{R}^2} \sqrt{x^2+y^2} \rho(x,y) dxdy$$

and prove the first inequality. We have

$$(\nabla\varphi)(x) = \int_{\mathbb{R}^2} \rho(x',y')(\nabla h)(x-x', y-y')\, dx'dy'.$$

Here ∇h is a constant vector in each Ω_j, and at least two of these vectors are linearly independent. Thus we get a strict inequality

$$|(\nabla\varphi)(x)| < \int_{\mathbb{R}^2} \rho(x',y') |(\nabla h)(x-x', y-y')|\, dx'dy'.$$

The right-hand side equals

$$\int_{\mathbb{R}^2} \sqrt{a_j^2 + b_j^2}\, \rho(x',y')\, dx'dy' = \tau.$$

Clearly $S < c$ is valid, and $S > 0$ follows from $|\nabla\varphi| < \tau$. This completes the proof. □

The following proposition plays a key role in this paper.

Proposition 1. *For every integer* $i_1 \geq 0$, $i_2 \geq 0$, *with* $2 \leq i_1 + i_2 \leq 3$,

$$\sup_{(x,y)\in\mathbb{R}^2} \frac{|(D_x^{i_1} D_y^{i_2} \varphi)(x,y)|}{S(x,y)} < +\infty$$

holds true.

The proof of this proposition is given at the end of this section.

We study the difference of a mollified pyramid and the original pyramid, that is, $\varphi(x,y) - h(x,y)$. We put

$$\widetilde{\varphi}_j(x,y) \stackrel{\text{def}}{=} \varphi(x,y) - h_j(x,y) = \varphi(x,y) - a_j x - b_j y. \tag{18}$$

Then we have $\varphi(x,y) - h(x,y) = \widetilde{\varphi}_j(x,y)$ in Ω_j. It suffices to study $\widetilde{\varphi}_j(x,y)$ in each Ω_j for studying $\varphi(x,y) - h(x,y)$ in $\mathbb{R}^2$. To do this we study here the simplest case. For

$$q(x,y) \stackrel{\text{def}}{=} \max\{x,0\} = \begin{cases} -x & x < 0, \\ 0 & x \geq 0, \end{cases}$$

we define

$$\begin{aligned} P(x) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \rho(x',y') q(x-x', y-y')\, dx'dy' \\ &= -\int_{-\infty}^{\infty} \left(\int_x^{\infty} \rho(x',y')(x-x')\, dx' \right) dy' > 0. \end{aligned} \tag{19}$$

This $P(x)$ is a mollified function for $q(x,y)$. We use it to estimate $\varphi(x,y) - h(x,y)$ because it stands for the influence of a lateral surface when we construct a mollified pyramid from the original pyramid. Then we have

$$\begin{aligned} P'(x) &= -\int_{-\infty}^{\infty} \left(\int_x^{\infty} \rho(x',y)\, dx' \right) dy < 0, \\ P''(x) &= \int_{-\infty}^{\infty} \rho(x,y)\, dy = \int_{-\infty}^{\infty} \widetilde{\rho}(\sqrt{x^2+y^2})\, dy > 0, \\ P^{(3)}(x) &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+y^2}} \widetilde{\rho}_r(\sqrt{x^2+y^2})\, dy \leq 0. \end{aligned}$$

Especially we have

$$P''(x) = \int_{-\infty}^{\infty} e^{-\sqrt{x^2+y^2}}\, dy, \quad P^{(3)}(x) = -\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+y^2}} e^{-\sqrt{x^2+y^2}}\, dy,$$

if $x > 0$ is large enough. Now we have the following lemma.

Lemma 3. *Let $P(x)$ be as in* (19). *Then it satisfies*

$$\lim_{x\to\infty} \frac{P(x)}{\sqrt{2\pi x} e^{-x}} = 1$$

and

$$\lim_{x\to\infty} \frac{|P^{(i)}(x)|}{P(x)} = 1, \quad 0 < \inf_{x\geq 1} \frac{|P^{(i)}(x)|}{P(x)} \leq \sup_{x \geq 1} \frac{|P^{(i)}(x)|}{P(x)} < +\infty$$

for all i with $1 \leq i \leq 3$.

Proof. For $x > 0$ we use $y = \sqrt{s^2 + 2sx}$ and obtain

$$2\int_0^{\infty} e^{-\sqrt{x^2+y^2}}\, dy = 2e^{-x} \int_0^{\infty} e^{-s} \frac{s+x}{\sqrt{s^2+2sx}}\, ds = \sqrt{2x} e^{-x} Q(x).$$

Here

$$Q(x) \stackrel{\text{def}}{=} \int_0^{\infty} \frac{1}{\sqrt{s}} e^{-s} \left(1 + \frac{s}{x}\right) \left(1 + \frac{s}{2x}\right)^{-\frac{1}{2}}\, ds.$$

By Lebesgue's convergence theorem we have

$$\lim_{x\to\infty} Q(x) = \int_0^{\infty} \frac{1}{\sqrt{s}} e^{-s}\, ds = \sqrt{\pi}.$$

Thus we have

$$P''(x) = \sqrt{2\pi x}e^{-x}(1 + o(1)) \qquad \text{as } x \to \infty. \tag{20}$$

Similarly we get

$$\begin{aligned} &2\int_0^\infty \frac{x}{\sqrt{x^2+y^2}} e^{-\sqrt{x^2+y^2}}\, dy = \sqrt{2x}e^{-x}\int_0^\infty e^{-s}\sqrt{\frac{2x}{s^2+2sx}}\, ds \\ &= \sqrt{2x}e^{-x}\int_0^\infty e^{-s}\frac{1}{\sqrt{s}}\left(1+\frac{s}{2x}\right)^{-\frac{1}{2}} ds = \sqrt{2\pi x}e^{-x}\left(1+o(1)\right) \qquad \text{as } x \to \infty. \end{aligned}$$

Thus we obtain

$$\lim_{x\to\infty} \frac{-P^{(3)}(x)}{P''(x)} = 1.$$

Now the Cauchy mean value theorem gives

$$\frac{P''(x)}{P'(x)} = \frac{P^{(3)}(x')}{P''(x')}$$

for some $x' > x$. This yields

$$\lim_{x\to\infty} \frac{P''(x)}{-P'(x)} = 1.$$

Similarly we have

$$\lim_{x\to\infty} \frac{-P'(x)}{P(x)} = \lim_{x\to\infty} \frac{P''(x)}{-P'(x)} = 1.$$

Thus we obtain

$$\lim_{x\to\infty} \frac{-P'(x)}{P(x)} = \lim_{x\to\infty} \frac{P''(x)}{-P'(x)} = \lim_{x\to\infty} \frac{-P^{(3)}(x)}{P''(x)} = 1.$$

This completes the proof. $\square$

Now we come back to study

$$\widetilde{\varphi}_j(x,y) = \varphi(x,y) - h_j(x,y) = \left(\rho * (h - h_j)\right)(x,y)$$

in Ω_j. Hereafter we assume $(x,y) \in \Omega_j$. We write

$$\boldsymbol{a}_j = (a_j, b_j) \qquad (1 \le j \le n).$$

Then we get

$$0 < \tau^2 - |\nabla\varphi|^2 = -2\boldsymbol{a}_j \cdot \nabla\widetilde{\varphi}_j - |\nabla\widetilde{\varphi}_j|^2. \tag{21}$$

We have

$$h(x,y) - h_j(x,y) = \begin{cases} (a_{j+1}-a_j)x + (b_{j+1}-b_j)y & \text{in } \Omega_{j+1}, \\ (a_{j-1}-a_j)x + (b_{j-1}-b_j)y & \text{in } \Omega_{j-1}. \end{cases}$$

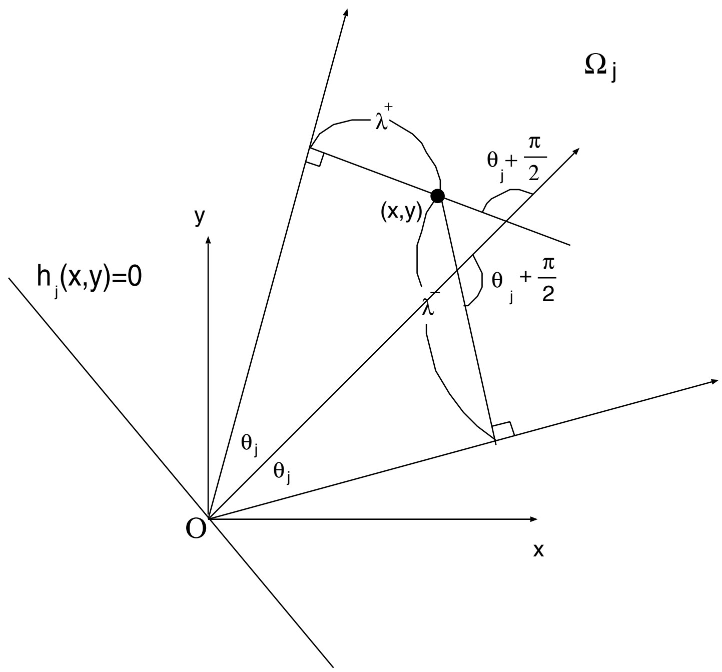

FIG. 3. *The definition of* $\lambda^{\pm}$.

Now

$$m_j^+ \stackrel{\text{def}}{=} \sqrt{(a_{j+1}-a_j)^2+(b_{j+1}-b_j)^2}, \quad m_j^- \stackrel{\text{def}}{=} \sqrt{(a_{j-1}-a_j)^2+(b_{j-1}-b_j)^2} \tag{22}$$

give the gradients of the adjacent surfaces S_{j+1} and S_{j-1} from a surface S_j, respectively. Let the angle of Ω_j be denoted by $2\theta_j$, with $\theta_j \in (0, \pi/2)$ for $j = 1, \dots, n$ as in Figure 3. For $(x, y) \in \Omega_j$, let λ^+ and λ^- be the lengths of the perpendiculars onto $\partial\Omega_j$. We have

$$\lambda^+ = \frac{(a_j - a_{j+1})x + (b_j - b_{j+1})y}{m_j^+}, \quad \lambda^- = \frac{(a_j - a_{j-1})x + (b_j - b_{j-1})y}{m_j^-}. \tag{23}$$

We study $\widetilde{\varphi}_j(x, y)$ and its derivatives in Ω_j when $\sqrt{x^2+y^2}$ is large enough. The number of the nearest latent surfaces for $(x, y) \in \Omega_j$ is at most two. This fact suggests that $\widetilde{\varphi}_j(x, y)$ can be approximated by $m^+P(\lambda^+) + m^-P(\lambda^-)$ in Ω_j if $\sqrt{x^2+y^2} \to \infty$ up to the derivatives. We have

$$\widetilde{\varphi}_j = \rho * (\max\{h_{j+1} - h_j, 0\}) + \rho * (\max\{h_{j-1} - h_j, 0\}) + \rho * g_j,$$

where

$$g_j \stackrel{\text{def}}{=} h - h_j - \max\{h_{j+1} - h_j, 0\} - \max\{h_{j-1} - h_j, 0\}.$$

Using P, we write the first and the second terms as

$$(\rho * (\max\{h_{j+1} - h_j, 0\}))\,(x, y) = m_j^+ P(\lambda^+),$$
$$(\rho * (\max\{h_{j-1} - h_j, 0\}))\,(x, y) = m_j^- P(\lambda^-),$$

respectively. We estimate the third term. We have

$$g_j \equiv 0 \quad \text{on } \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$$

and

$$h_j(x,y) \geq 0 \qquad \text{for all } (x,y) \in \Omega_j.$$

The distance between (x,y) and a line $\{(x,y) \mid h_j(x,y) = 0\}$ is $(1/\tau)h_j(x,y)$. The gradients of the planes h_j $(1 \leq j \leq n)$ are at most τ. We put $\Lambda^+ = \mathrm{dist}((x,y), \Omega_{j+2})$ and $\Lambda^- = \mathrm{dist}((x,y), \Omega_{j-2})$. We have $0 < \sin\theta_j < 1$ and $\min\{\lambda^+, \lambda^-\} \leq (h_j(x,y)/\tau)\sin\theta_j$. There exists $\gamma_0 > 1$ such that we have

$$\gamma_0 \min\{\lambda^+, \lambda^-\} \leq \min\left\{\Lambda^+, \Lambda^-, \frac{1}{\tau}h_j(x,y)\right\}.$$

The following lemma is useful to estimate $\rho * g_j$ and $\varphi(x,y) - h(x,y)$.

Lemma 4. *For every j $(1 \leq j \leq n)$, one has*

$$\varphi(x,y) - h(x,y) = m_j^+ P(\lambda^+) + m_j^- P(\lambda^-) + (\rho * g_j)(x,y) \qquad \text{for all } (x,y) \in \Omega_j,$$

where $m_j^\pm$, $\lambda^\pm$ are given by (22) *and* (23), *respectively. For all nonnegative integers i_1, i_2, with $0 \leq i_1 + i_2 \leq 3$, one has*

$$\left|D_x^{i_1} D_y^{i_2} (\rho * g_j)(x,y)\right| \leq K\left(\gamma_0 \min\{\lambda^+, \lambda^-\}\right)^{\frac{1}{2}} \exp\left(-\gamma_0 \min\{\lambda^+, \lambda^-\}\right)$$

for $(x,y) \in \Omega_j$ and $x^2 + y^2 \geq 1$. Here $K > 0$ and $\gamma_0 > 1$ are constants independent of j, i_1, and i_2. In particular one has

$$\lim_{\lambda\to\infty} \sup\left\{S(x,y) \mid (x,y) \in \mathbb{R}^2, \mathrm{dist}((x,y), E) \geq \lambda\right\} = 0,$$
$$\lim_{\lambda\to\infty} \sup\left\{\varphi(x,y) - h(x,y) \mid (x,y) \in \mathbb{R}^2, \mathrm{dist}((x,y), E) \geq \lambda\right\} = 0.$$

Proof. We already obtained the first equality. We decompose g_j as

$$g_j = g_j\chi_{\{h_{j+2}-h_{j+1}>0\}} + g_j\chi_{\{h_{j-2}-h_{j-1}>0\}} + g_j\chi_{\{h_{j+2}-h_{j+1}\leq 0, h_{j-2}-h_{j-1}\leq 0\}},$$

where $\chi_{\{h_{j+2}-h_{j+1}>0\}}$ is the characteristic function of $\{h_{j+2} - h_{j+1} > 0\}$ and so on. For all nonnegative integers i_1, i_2, with $0 \leq i_1 + i_2 \leq 3$, we take $\gamma_1 > 0$ so large to get $|D_x^{i_1} D_y^{i_2} \rho(x,y)| \leq \gamma_1 \rho(x,y)$ for all $(x,y) \in \mathbb{R}^2$. Then applying Lemma 3 we obtain

$$|(D_x^{i_1} D_y^{i_2} \rho) * g_j(x,y)| \leq 6\tau\gamma_1 P(\Lambda^+) + 6\tau\gamma_1 P(\Lambda^-) + 6\tau\gamma_1 P\left(-\frac{1}{\tau}h_j(x,y)\right)$$

for all $(x,y) \in \mathbb{R}^2$. Using $D_x^{i_1} D_y^{i_2}(\rho * g_j) = (D_x^{i_1} D_y^{i_2} \rho) * g_j$, we get the desired inequality. The last two equalities follow from this inequality. ∎

We have

$$\widetilde{\varphi}_j(x,y) = m_j^+ P(\lambda^+) + m_j^- P(\lambda^-) + (\rho * g_j)(x,y).$$

Using Lemma 4 we obtain

$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{\widetilde{\varphi}_j(x,y)}{m_j^+ P(\lambda^+) + m_j^- P(\lambda^-)} = 1,$$
$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{\boldsymbol{a}_j \cdot (\nabla\widetilde{\varphi}_j)(x,y)}{-\tau m_j^+ P'(\lambda^+)\cos(\theta_j + \frac{\pi}{2}) - \tau m_j^- P'(\lambda^-)\cos(\theta_j + \frac{\pi}{2})} = 1.$$

For all integers $i_1 \geq 0$, $i_2 \geq 0$, with $2 \leq i_1 + i_2 \leq 3$, we can estimate $|D_x^{i_1} D_y^{i_2} \widetilde{\varphi}_j(x,y)|$ by

$$\left|P''(\lambda^+)\right| + \left|P^{(3)}(\lambda^+)\right| + \left|P''(\lambda^-)\right| + \left|P^{(3)}(\lambda^-)\right|.$$

From Lemma 4 there exists a constant $M > 0$, with

$$|D_x^{i_1} D_y^{i_2} \widetilde{\varphi}_j(x,y)| \leq M\left(P(\lambda^+) + P(\lambda^-)\right) \qquad \text{in } \Omega_j \tag{24}$$

for every j $(1 \leq j \leq n)$ and all integers $i_1 \geq 0$, $i_2 \geq 0$, with $0 \leq i_1 + i_2 \leq 3$.

The definition of $S(x,y)$ and (21) give

$$\frac{k^3}{2c^2}\left(-2\boldsymbol{a}_j \cdot \nabla\widetilde{\varphi}_j - |\nabla\widetilde{\varphi}_j|^2\right) < S(x,y) < \frac{k^2}{c+k}\left(-2\boldsymbol{a}_j \cdot \nabla\widetilde{\varphi}_j - |\nabla\widetilde{\varphi}_j|^2\right). \tag{25}$$

Lemma 5. *For any given* $\omega > 0$

$$0 < \inf\left\{S(x,y) \mid \operatorname{dist}((x,y),E) \leq \omega\right\}$$

holds true.

Proof. It suffices to prove the lemma assuming $(x,y) \in \Omega_j$ and $\operatorname{dist}((x,y),\partial\Omega_j) \leq \omega$. We have

$$\begin{aligned}
&-2(\boldsymbol{a}_j, \nabla(m_j^+ P(\lambda^+))) - \left|\nabla(m_j^+ P(\lambda^+))\right|^2 \\
&= -P'(\lambda^+)\left(2(\boldsymbol{a}_j, \boldsymbol{a}_j - \boldsymbol{a}_{j+1}) + P'(\lambda^+)|\boldsymbol{a}_{j+1} - \boldsymbol{a}_j|^2\right) \\
&\geq -P'(\lambda^+)\left(2(\boldsymbol{a}_j, \boldsymbol{a}_j - \boldsymbol{a}_{j+1}) - \frac{1}{2}|\boldsymbol{a}_{j+1} - \boldsymbol{a}_j|^2\right) = -P'(\lambda^+)\left(|\boldsymbol{a}_j|^2 - (\boldsymbol{a}_j, \boldsymbol{a}_{j+1})\right) > 0.
\end{aligned}$$

As $\sqrt{x^2+y^2} \to \infty$, we can assume λ^+ remains finite and $\lambda^- \to \infty$ without loss of generality. Then the inequality stated above implies

$$\lim_{r\to\infty} \inf\left\{S(x,y) \,|\, (x,y) \in \Omega_j, \operatorname{dist}((x,y),\partial\Omega_j) \leq \omega, x^2+y^2 \geq r^2\right\} > 0.$$

This completes the proof. $\square$

Now we prove the following lemma.

Lemma 6. *There exists positive constants* ν_1, ν_2 *so that*

$$0 < \nu_1 \leq \frac{\varphi(x,y) - h(x,y)}{S(x,y)} \leq \nu_2$$

holds true for $(x,y) \in \mathbb{R}^2$.

Proof. We note that $(\varphi(x,y) - h(x,y))/S(x,y)$ is a positive function in $\mathbb{R}^2$. Without loss of generality, we assume (x,y) lies in Ω_j. Due to Lemma 5 it suffices to prove that it remains no less than a positive constant as $\sqrt{x^2+y^2} \to \infty$ under the condition $|\nabla\widetilde{\varphi}_j| \to 0$. We have

$$\limsup_{\sqrt{x^2+y^2}\to\infty} \left|\frac{\widetilde{\varphi}_j(x,y)}{-\boldsymbol{a}_j \cdot (\nabla\widetilde{\varphi}_j)(x,y)}\right| = \frac{1}{\tau\sin\theta_j} \limsup_{\sqrt{x^2+y^2}\to\infty} \left|\frac{m_j^+ P(\lambda^+) + m_j^- P(\lambda^-)}{-m_j^+ P'(\lambda^+) - m_j^- P'(\lambda^-)}\right|.$$

The right-hand side takes a positive bounded value. Using Lemma 5, (25), and this fact, we complete the proof. $\square$

Proof of Proposition 1. Without loss of generality we can assume $(x, y) \in \Omega_j$ for some j. By Lemma 5 it suffices to prove

$$\sup_{(x,y)\in\Omega_j} \frac{|(D_x^{i_1} D_y^{i_2} \widetilde{\varphi}_j)(x,y)|}{S(x,y)} < +\infty \tag{26}$$

for each $i_1 \geq 0$, $i_2 \geq 0$, with $2 \leq i_1 + i_2 \leq 3$, under the condition $|\nabla \widetilde{\varphi}_j| \to 0$. From (24) we obtain

$$\lim_{\sqrt{x^2+y^2}\to\infty} \left| \frac{(D_x^{i_1} D_y^{i_2} \widetilde{\varphi}_j)(x,y)}{-\boldsymbol{a}_j \cdot (\nabla \widetilde{\varphi}_j)(x,y)} \right| \leq \frac{M'}{\tau \sin\theta_j} \lim_{\sqrt{x^2+y^2}\to\infty} \left| \frac{m_j^+ P(\lambda^+) + m_j^- P(\lambda^-)}{-m_j^+ P'(\lambda^+) - m_j^- P'(\lambda^-)} \right|.$$

Here $M' > 0$ is a constant. The right-hand side is bounded. Using this estimate, (25), and (21), we obtain (26). This completes the proof. $\square$

3. Proof of Theorem 2. In this section we prove Theorem 2 by constructing a supersolution and a subsolution and by finding a pyramidal traveling wave between them.

For $\alpha \in (0, 1)$ we consider the graph of

$$z = \frac{1}{\alpha} \varphi(\alpha x, \alpha y). \tag{27}$$

Later we will choose α to be small enough. We use this function as a mollified pyramid. We note that

$$\frac{1}{\alpha} h(\alpha x, \alpha y) = h(x, y).$$

We use a rescaled coordinate (ξ, η, ζ) as

$$\xi = \alpha x, \quad \eta = \alpha y, \quad \zeta = \alpha z$$

and write (27) as $\zeta = \varphi(\xi, \eta)$.

For $(x_0, y_0) \in \mathbb{R}^2$, the tangent plane of (27) at $(x_0, y_0, \alpha^{-1}\varphi(\alpha x_0, \alpha y_0))$ is expressed by

$$-\varphi_\xi(\xi_0, \eta_0)(x - x_0) - \varphi_\eta(\xi_0, \eta_0)(y - y_0) + z - \frac{1}{\alpha}\varphi(\xi_0, \eta_0) = 0,$$

where $\xi_0 = \alpha x_0$, $\eta_0 = \alpha y_0$. The length of the perpendicular from (x_0, y_0, z_0) onto the tangent plane is

$$\frac{\left| z_0 - \frac{1}{\alpha}\varphi(\xi_0, \eta_0) \right|}{\sqrt{1 + \varphi_\xi(\xi_0, \eta_0)^2 + \varphi_\eta(\xi_0, \eta_0)^2}}.$$

We define

$$\widehat{\mu} \stackrel{\text{def}}{=} \frac{z - \frac{1}{\alpha}\varphi(\alpha x, \alpha y)}{\sqrt{1 + \varphi_\xi(\alpha x, \alpha y)^2 + \varphi_\eta(\alpha x, \alpha y)^2}} = \frac{1}{\alpha} \frac{\zeta - \varphi(\xi, \eta)}{\sqrt{1 + \varphi_\xi(\xi, \eta)^2 + \varphi_\eta(\xi, \eta)^2}}. \tag{28}$$

Then we have

$$\widehat{\mu}_z = \frac{1}{\sqrt{1 + \varphi_\xi^2 + \varphi_\eta^2}}, \qquad \widehat{\mu}_{zz} = 0.$$

Also we get

$$\widehat{\mu}_x = -\frac{\varphi_\xi}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}} + \alpha\widehat{\mu}F_1(\xi,\eta), \qquad \widehat{\mu}_{xx} = \alpha G_{11}(\xi,\eta) + \alpha^2\widehat{\mu}H_{11}(\xi,\eta),$$

where

$$\begin{aligned}
F_1(\xi,\eta) &\stackrel{\text{def}}{=} \sqrt{1+\varphi_\xi^2+\varphi_\eta^2}\left(\frac{1}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\xi, \\
G_{11}(\xi,\eta) &\stackrel{\text{def}}{=} -\left(\frac{\varphi_\xi}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\xi - \varphi_\xi\left(\frac{1}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\xi \\
&= \frac{(-1+\varphi_\xi^2-\varphi_\eta^2)\varphi_{\xi\xi} + (2\varphi_\xi^2+2\varphi_\xi\varphi_\eta)\varphi_{\xi\eta}}{(1+\varphi_\xi^2+\varphi_\eta^2)^{\frac{3}{2}}}, \\
H_{11}(\xi,\eta) &\stackrel{\text{def}}{=} (F_1(\xi,\eta))_\xi + F_1(\xi,\eta)^2.
\end{aligned}$$

Similarly we obtain

$$\widehat{\mu}_{xy} = \alpha G_{12}(\xi,\eta) + \alpha^2\widehat{\mu}H_{12}(\xi,\eta),$$

where

$$\begin{aligned}
G_{12}(\xi,\eta) &\stackrel{\text{def}}{=} -\left(\frac{\varphi_\xi}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\eta - \varphi_\eta\left(\frac{1}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\xi, \\
H_{12}(\xi,\eta) &\stackrel{\text{def}}{=} (F_1(\xi,\eta))_\eta + F_1(\xi,\eta)F_2(\xi,\eta).
\end{aligned}$$

We get

$$\widehat{\mu}_y = -\frac{\varphi_\eta}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}} + \alpha\widehat{\mu}F_2(\xi,\eta), \qquad \widehat{\mu}_{yy} = \alpha G_{22}(\xi,\eta) + \alpha^2\widehat{\mu}H_{22}(\xi,\eta),$$

where

$$\begin{aligned}
F_2(\xi,\eta) &\stackrel{\text{def}}{=} \sqrt{1+\varphi_\xi^2+\varphi_\eta^2}\left(\frac{1}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\eta, \\
G_{22}(\xi,\eta) &\stackrel{\text{def}}{=} -\left(\frac{\varphi_\eta}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\eta - \varphi_\eta\left(\frac{1}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_\eta, \\
H_{22}(\xi,\eta) &\stackrel{\text{def}}{=} (F_2(\xi,\eta))_\eta + F_2(\xi,\eta)^2.
\end{aligned}$$

We define

$$U(x,y,z) = \Phi(\widehat{\mu}) + \sigma(x,y), \tag{29}$$

where $\widehat{\mu}$ is as in (28) and

$$\sigma(x,y) \stackrel{\text{def}}{=} \varepsilon S(\alpha x, \alpha y).$$

Here we will fix $\varepsilon > 0$ later. We have

$$U_z = \frac{1}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\Phi'(\widehat{\mu}), \quad U_{zz} = \frac{1}{1+\varphi_\xi^2+\varphi_\eta^2}\Phi''(\widehat{\mu}),$$

and

$$U_{xx}+U_{yy} = \Phi^{'}(\widehat{\mu})(\widehat{\mu}_{xx}+\widehat{\mu}_{yy}) + \Phi^{''}(\widehat{\mu})(\widehat{\mu}_x^2+\widehat{\mu}_y^2) + \sigma_{xx}+\sigma_{yy}.$$

Thus we get

$$\begin{aligned} U_{xx}+U_{yy} &= \alpha\Phi^{'}(\widehat{\mu})(G_{11}(\xi,\eta)+G_{22}(\xi,\eta)) + \alpha^2\widehat{\mu}\Phi^{'}(\widehat{\mu})(H_{11}(\xi,\eta)+H_{22}(\xi,\eta)) \\ &\quad + \Phi^{''}(\widehat{\mu})\frac{\varphi_\xi^2+\varphi_\eta^2}{1+\varphi_\xi^2+\varphi_\eta^2} - 2\alpha\widehat{\mu}\Phi^{''}(\widehat{\mu})\frac{\varphi_\xi(\xi,\eta)F_1(\xi,\eta)+\varphi_\eta(\xi,\eta)F_2(\xi,\eta)}{\sqrt{1+\varphi_\xi(\xi,\eta)^2+\varphi_\eta(\xi,\eta)^2}} \\ &\quad + \alpha^2\widehat{\mu}^2\Phi^{''}(\widehat{\mu})(F_1(\xi,\eta)^2+F_2(\xi,\eta)^2) + \sigma_{xx}+\sigma_{yy}. \end{aligned}$$

We calculate $\mathcal{L}[U]$ as

$$\begin{aligned} \mathcal{L}[U] &= -\Phi^{''}(\widehat{\mu}) - \frac{c}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\Phi^{'}(\widehat{\mu}) - f(\Phi+\sigma) \\ &\quad - \alpha\Phi^{'}(\widehat{\mu})(G_{11}(\xi,\eta)+G_{22}(\xi,\eta)) - \alpha^2\widehat{\mu}\Phi^{'}(\widehat{\mu})(H_{11}(\xi,\eta)+H_{22}(\xi,\eta)) \\ &\quad + 2\alpha\widehat{\mu}\Phi^{''}(\widehat{\mu})\frac{\varphi_\xi(\xi,\eta)F_1(\xi,\eta)+\varphi_\eta(\xi,\eta)F_2(\xi,\eta)}{\sqrt{1+\varphi_\xi(\xi,\eta)^2+\varphi_\eta(\xi,\eta)^2}} \\ &\quad - \alpha^2\widehat{\mu}^2\Phi^{''}(\widehat{\mu})(F_1(\xi,\eta)^2+F_2(\xi,\eta)^2) - \varepsilon\alpha^2(S_{\xi\xi}+S_{\eta\eta}). \end{aligned}$$

We have

$$S_{\xi\xi}(\xi,\eta)+S_{\eta\eta}(\xi,\eta) = \left(\frac{c}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_{\xi\xi} + \left(\frac{c}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right)_{\eta\eta}$$

and define

$$\begin{aligned} R(\xi,\eta,\mu;\varepsilon,\alpha) &\overset{\text{def}}{=} -\Phi^{'}(\mu)(G_{11}(\xi,\eta)+G_{22}(\xi,\eta)) - \alpha\mu\Phi^{'}(\mu)(H_{11}(\xi,\eta)+H_{22}(\xi,\eta)) \\ &\quad + 2\mu\Phi^{''}(\mu)\frac{\varphi_\xi(\xi,\eta)F_1(\xi,\eta)+\varphi_\eta(\xi,\eta)F_2(\xi,\eta)}{\sqrt{1+\varphi_\xi(\xi,\eta)^2+\varphi_\eta(\xi,\eta)^2}} \\ &\quad - \alpha\mu^2\Phi^{''}(\mu)(F_1(\xi,\eta)^2+F_2(\xi,\eta)^2) - \varepsilon\alpha\left(S_{\xi\xi}(\xi,\eta)+S_{\eta\eta}(\xi,\eta)\right). \end{aligned}$$

Thus we get

$$\mathcal{L}[U] = -\Phi^{''}(\widehat{\mu}) - \frac{c}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\Phi^{'}(\widehat{\mu}) - f(\Phi+\sigma) + \alpha R(\xi,\eta,\widehat{\mu};\varepsilon,\alpha).$$

Using $-\Phi^{''}(\mu) - k\Phi^{'}(\mu) - f(\Phi) = 0$, we obtain

$$\mathcal{L}[U] = -\Phi^{'}(\widehat{\mu})S(\xi,\eta) - \sigma\int_0^1 f^{'}(\Phi(\widehat{\mu})+s\sigma)ds + \alpha R(\xi,\eta,\widehat{\mu};\varepsilon,\alpha).$$

We estimate $|R(\xi,\eta,\mu;\varepsilon,\alpha)|$ using

$$|R(\xi,\eta,\mu;\varepsilon,\alpha)| \le \max\left\{|\Phi^{'}(\mu)|, |\mu\Phi^{'}(\mu)|, |\mu\Phi^{''}(\mu)|, |\mu^2\Phi^{''}(\mu)|\right\}$$
$$\times \big(|G_{11}(\xi,\eta)| + |G_{22}(\xi,\eta)| + |H_{11}(\xi,\eta)| + |H_{22}(\xi,\eta)| + 2|F_1(\xi,\eta)+F_2(\xi,\eta)|$$
$$+|F_1(\xi,\eta)|^2 + |F_2(\xi,\eta)|^2 + |S_{\xi\xi}(\xi,\eta)| + |S_{\eta\eta}(\xi,\eta)|\big)$$

if $0<\alpha<1$. The first term $|G_{11}(\xi,\eta)|$ includes the second derivatives of φ as in the definition of G_{11}. Other terms also include the second or third derivatives of φ. Using Lemmas 1 and 2, we estimate all terms and obtain

$$|G_{11}(\xi,\eta)| + |G_{22}(\xi,\eta)| + |H_{11}(\xi,\eta)| + |H_{22}(\xi,\eta)| + 2|F_1(\xi,\eta)+F_2(\xi,\eta)|$$
$$+|F_1(\xi,\eta)|^2 + |F_2(\xi,\eta)|^2 + |S_{\xi\xi}(\xi,\eta)| + |S_{\eta\eta}(\xi,\eta)|$$
$$\le A' \sum_{2\le i_1+i_2\le 3} |(D_\xi^{i_1} D_\eta^{i_2}\varphi)(\xi,\eta)| \qquad \text{for all } (\xi,\eta)\in\mathbb{R}^2$$

with a constant A'. Using Proposition 1 we find a constant A so that

$$\frac{|R(\xi,\eta,\mu;\varepsilon,\alpha)|}{S(\xi,\eta)} < A$$

holds true for all $(\xi,\eta)\in\mathbb{R}^2$, $\mu\in\mathbb{R}$, $\varepsilon\in(0,1)$, and $\alpha\in(0,1)$. Constants A' and A depend only on f and c. We continue to calculate $\mathcal{L}[U]$ as

$$\mathcal{L}[U] = S(\xi,\eta)\left(-\Phi^{'}(\widehat{\mu}) - \varepsilon\int_0^1 f^{'}(\Phi(\widehat{\mu})+s\sigma)ds + \alpha\frac{R(\xi,\eta,\widehat{\mu};\varepsilon,\alpha)}{S(\xi,\eta)}\right).$$

Thus we get

$$\mathcal{L}[U] \ge S(\xi,\eta)\left(-\Phi^{'}(\widehat{\mu}) - \varepsilon\int_0^1 f^{'}(\Phi(\widehat{\mu})+s\sigma)ds - \alpha A\right). \tag{30}$$

Now we choose ε and α as was mentioned before. We take ε small enough to get

$$0<\varepsilon<\min\left\{\frac{1}{2}, \frac{\delta_*}{c}, \frac{2K_0}{c}, \frac{\min_{-1+\delta_*\le\Phi(p)\le 1-\delta_*}(-\Phi^{'}(p))}{4\max_{|s|\le 1+\delta_*}|f^{'}(s)|}\right\}. \tag{31}$$

Then we choose α small enough to get

$$0<\alpha<\min\left\{\frac{1}{2}, \frac{\varepsilon\kappa_1}{2A}, \frac{\min_{-1+\delta_*\le\Phi(p)\le 1-\delta_*}(-\Phi^{'}(p))}{4A}, \frac{k\kappa_0\nu_1}{\log\left(\frac{2K_0}{c\varepsilon}\right)}\right\}. \tag{32}$$

Now we show that U is a supersolution and is larger than the maximum of planar solutions.

LEMMA 7. *Assume ε and α satisfy* (31) *and* (32)*, respectively. Let U be as in* (29)*. Then*

$$\mathcal{L}[U]>0 \qquad \text{in } \mathbb{R}^3$$

holds true. Moreover

$$\Phi\left(\frac{k}{c}(z-h(x,y))\right) < U(x,y,z) \qquad \text{in } \mathbb{R}^3$$

holds true.

Proof. If $\Phi(\widehat{\mu}) < -1+\delta_*$ or $\Phi(\widehat{\mu}) > 1-\delta_*$, we have $|s\varepsilon S| \le s\varepsilon c \le \delta_*$ for $0 \le s \le 1$ in view of Lemma 2. We get $\Phi(\widehat{\mu}) + s\varepsilon S < -1 + 2\delta_*$ or $\Phi(\widehat{\mu}) + s\varepsilon S > 1 - 2\delta_*$. Combining $-\Phi'(\widehat{\mu}) > 0$ and (30), we obtain

$$\mathcal{L}[U] \ge S(\xi,\eta)\,(\varepsilon\kappa_1 - \alpha A) > 0.$$

If $-1+\delta_* \le \Phi(\widehat{\mu}) \le 1-\delta_*$, then we have

$$\mathcal{L}[U] \ge S(\xi,\eta)\left(\min_{-1+\delta_*\le\Phi(p)\le 1-\delta_*}(-\Phi'(p)) - \varepsilon \max_{|s|\le 1+\delta_*}|f'(s)| - \alpha A\right) > 0.$$

In both cases we proved that U is a supersolution.

We use a similar argument as in [15] to prove the latter statement. It suffices to prove

$$\Phi\left(\frac{k}{c}(z - a_j x - b_j y)\right) < U(x,y,z) \tag{33}$$

for fixed j. Temporarily we denote a_j, b_j simply by a, b to prove (33). If

$$\widehat{\mu} \le \frac{k}{c}(z - ax - by),$$

we get

$$U(x,y,z) > \Phi(\widehat{\mu}) \ge \Phi\left(\frac{k}{c}(z - ax - by)\right).$$

Thus it suffices to prove (33) by assuming

$$\widehat{\mu} > \frac{k}{c}(z - ax - by).$$

Substituting the definition of $\widehat{\mu}$ into this inequality, we obtain

$$\frac{z - ax - by + \left(ax + by - \frac{1}{\alpha}\varphi(\xi,\eta)\right)}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}} > \frac{k}{c}(z - ax - by),$$

which is equivalent to

$$\left(\frac{c}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}} - k\right)(z - ax - by) \ge \frac{c}{\alpha}\,\frac{\varphi(\xi,\eta) - a\xi - b\eta}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}.$$

Combining this inequality with the definition of $S(\xi,\eta)$, we get

$$z - ax - by \ge \frac{c\nu_1}{\alpha\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}} \ge \frac{k\nu_1}{\alpha}. \tag{34}$$

Using $\alpha(ax+by) = a\xi + b\eta \le \varphi(\xi,\eta)$, we obtain

$$\begin{aligned}
U(x,y,z) &- \Phi\left(\frac{k}{c}(z-ax-by)\right) \\
&\ge \Phi\left(\frac{z-ax-by}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}}\right) - \Phi\left(\frac{k}{c}(z-ax-by)\right) + \varepsilon S(\xi,\eta) \\
= \frac{(z-ax-by)S(\xi,\eta)}{c} &\int_0^1 \Phi'\left(\left(\frac{\theta}{\sqrt{1+\varphi_\xi^2+\varphi_\eta^2}} + \frac{k}{c}(1-\theta)\right)(z-ax-by)\right) d\theta \\
&+ \varepsilon S(\xi,\eta) \\
&\ge S(\xi,\eta)\left(\varepsilon - \frac{1}{c}\sup_{|\mu|\ge\frac{k\nu_1}{\alpha}}\left|\mu\Phi'\left(\frac{k}{c}\mu\right)\right|\right).
\end{aligned}$$

By virtue of Lemma 1 and (32) we have

$$\frac{1}{c}\sup_{|\mu|\ge\frac{k\nu_1}{\alpha}}\left|\mu\Phi'\left(\frac{k}{c}\mu\right)\right| < \frac{\varepsilon}{2}$$

and obtain

$$U(x,y,z) - \Phi\left(\frac{k}{c}(z-ax-by)\right) > \frac{\varepsilon}{2}S(\xi,\eta) > 0,$$

which yields (33). This completes the proof. $\square$

Thus U is a supersolution to (5). Now we prove the main assertion.

Proof of Theorem 2. We put

$$\underline{v}(x,y,z) = \Phi\left(\frac{k}{c}(z-h(x,y))\right) \tag{35}$$

and consider solutions of (4) given by $w(x,y,z,t;\underline{v})$ and $w(x,y,z,t;U)$. Since U is a supersolution and $\underline{v}$ is a subsolution, we have

$$\underline{v} \le w(x,y,z,t;\underline{v}) \le w(x,y,z,t;U) \le U$$

for $(x,y,z)\in\mathbb{R}^3$ and $t\ge 0$ by using [17, Theorem 3.4]. Then

$$V(x,y,z) \stackrel{\text{def}}{=} \lim_{t\to\infty} w(x,y,z,t;\underline{v}) \tag{36}$$

exists in $L^\infty(\mathbb{R}^3)$, with

$$\underline{v}(x,y,z) < V(x,y,z) < U(x,y,z) \qquad \text{in } \mathbb{R}^3.$$

This $V(x,y,z)$ is a solution of (5). See Sattinger [17, Theorem 3.6] for detailed arguments. Now we have

$$\underline{v}(x,y,z) < V(x,y,z) < \Phi(\widehat{\mu}) + \varepsilon S.$$

Now we prove (11). Let ε be arbitrarily given. Let U be as in (29). It suffices to prove

$$\sup_{(x,y,z)\in D(\gamma)} \left(U(x,y,z) - \Phi\left(\frac{k}{c}(z-h(x,y))\right)\right) < 2\varepsilon \tag{37}$$

if $\gamma > 0$ is large enough. Assume the contrary. Then there exists (γ_n) such that we have

$$\lim_{n\to\infty} \gamma_n = \infty, \quad (x_n, y_n, z_n) \in D(\gamma_n), \tag{38}$$

and

$$\left|\Phi(\widehat{\mu}_n) - \Phi\left(\frac{k}{c}(z_n - h(x_n, y_n))\right)\right| \geq \varepsilon. \tag{39}$$

Here we put $\xi_n = \alpha x_n$, $\eta_n = \alpha y_n$, $\zeta_n = \alpha z_n$, and

$$\widehat{\mu}_n = \frac{1}{\alpha}\frac{\zeta_n - \varphi(\xi_n, \eta_n)}{\sqrt{1+\varphi_\xi(\xi_n,\eta_n)^2 + \varphi_\eta(\xi_n,\eta_n)^2}} = \frac{z_n - h(x_n,y_n) - \frac{1}{\alpha}(\varphi(\xi_n,\eta_n) - h(\xi_n,\eta_n))}{\sqrt{1+\varphi_\xi(\xi_n,\eta_n)^2 + \varphi_\eta(\xi_n,\eta_n)^2}}.$$

If we have $\lim_{n\to\infty} \operatorname{dist}((\xi_n,\eta_n), E) = \infty$, then we obtain $\lim_{n\to\infty} |\varphi(\xi_n,\eta_n) - h(\xi_n,\eta_n)| = 0$ and $\lim_{n\to\infty} S(\xi_n,\eta_n) = 0$ by applying Lemma 4. Recall $E \overset{\text{def}}{=} \cup_{j=1}^n \partial\Omega_j \subset \mathbb{R}^2$. Then we get

$$\lim_{n\to\infty} \left|\widehat{\mu}_n - \frac{k}{c}(z_n - h(x_n,y_n))\right| = 0.$$

This contradicts (39). If $\operatorname{dist}((\xi_n,\eta_n), E)$ remains finite uniformly in n, then (38) implies that $\lim_{n\to\infty}(z_n - h(x_n,y_n)) = \pm\infty$ and $\lim_{n\to\infty} \widehat{\mu}_n = \pm\infty$, respectively. This contradicts (39). This completes the proof of Theorem 2. $\square$

4. Application of Theorem 2. In this section we state applications of Theorem 2. Traveling waves in Theorem 2 have a contour line of a pyramidal shape if the normal vectors of lateral surfaces are linearly independent. What is the shape of traveling waves in Theorem 2 if lateral surfaces are linearly dependent? In this section we show an example of such a traveling wave.

LEMMA 8. *Let $h(x,y)$ be given by (9) with (7) and (8). Assume that $h(-x,y) = h(x,y)$ and that at least one A_j is positive. For any fixed y, assume that $h(x,y)$ is nondecreasing for $x > 0$. Then V in Theorem 2 satisfies*

$$V(-x,y,z) = V(x,y,z) \qquad \text{in } \mathbb{R}^3,$$
$$V_x(x,y,z) > 0 \qquad \text{for } x > 0.$$

The same statement holds for y.

Proof. We have $\underline{v}(-x,y,z) = \underline{v}(x,y,z)$ and thus $w(-x,y,z,t;\underline{v}) = w(x,y,z,t;\underline{v})$. Then V given by (36) satisfies $V(-x,y,z) = V(x,y,z)$. We have $(\underline{v})_x(x,y,z) \geq 0$ for $x > 0$. Now $w_x(x,y,z,t;\underline{v})$ satisfies the derivative of (4) by x in $\{(x,y,z) \in \mathbb{R}^3 \mid x > 0\}$ with the Neumann boundary condition $w_x(x,y,z,t;\underline{v}) = 0$ on $\{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$. Then the comparison principle gives $w_x \geq 0$ and thus $V_x \geq 0$ for $x > 0$. From Theorem 2, $V_x \not\equiv 0$, and thus we get $V_x > 0$. $\square$

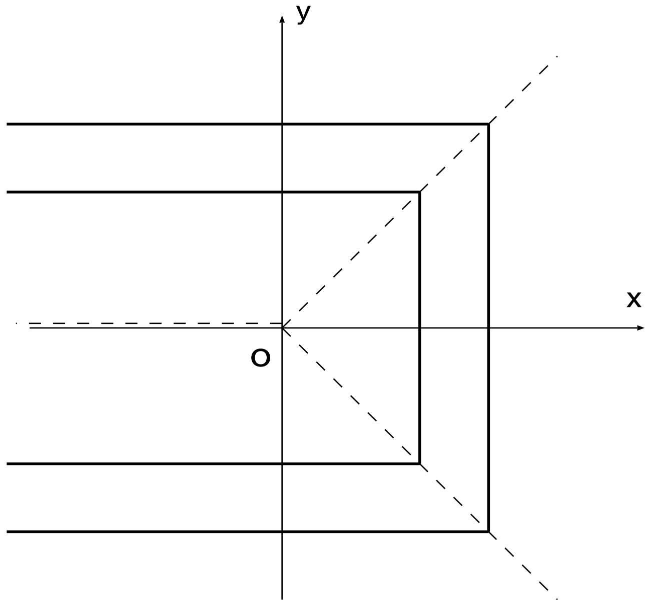

Fig. 4. *The contour lines of* $\tau \max\{x, |y|\}$.

We consider

$$h_1(x, y) = \tau y, \quad h_2(x, y) = -\tau y,$$

and thus $h(x, y) = \tau|y|$. Theorem 2 and its proof are applicable to this case. Then $V(x, y, z)$ as in Theorem 2 equals $v_*(y, z)$, where v_* is as in Theorem 1. The uniqueness follows from that of Theorem 1 in this case. We call this a planar V-form wave.

As an application of Theorem 2 we consider the following example:

$$h_1(x, y) = \tau x, \quad h_2(x, y) = \tau y, \quad h_3(x, y) = -\tau y,$$

and thus

$$h(x, y) = \max_{1 \le j \le 3} h_j(x, y) = \tau \max\{x, |y|\}. \tag{40}$$

See Figure 4. The edge lines are given by

$$\begin{aligned}
\Gamma_1 &= \{(x, y, z) \,|\, x = y = z, z \ge 0\}, \\
\Gamma_2 &= \{(x, y, z) \,|\, x = -y = z, z \ge 0\}, \\
\Gamma_3 &= \{(x, 0, 0) \,|\, x \le 0\}.
\end{aligned}$$

We have $\Gamma = \cup_{j=1}^{3} \Gamma_j$ and $D(\gamma)$ as in (10).

Proposition 2. *Assume* $c > k$, (A1), (A2), *and* (A3). *Let* $V_1(x, y, z)$ *be a solution of* (5) *as in Theorem* 2 *for* (40). *Then* $V_1(x, y, z)$ *satisfies* $V_1(x, -y, z) = V_1(x, y, z)$ *and*

$$\begin{aligned}
0 \le V_1(x, 0, 0) &\qquad \textit{for all } x \le 0, \\
(V_1)_z(x, y, z) < 0, \quad (V_1)_x(x, y, z) > 0 &\qquad \textit{in } \mathbb{R}^3, \\
(V_1)_y(x, y, z) > 0 &\qquad \textit{if } (x, y, z) \in \mathbb{R}^3,\ y > 0.
\end{aligned}$$

Proof. We put $v_1^-(x,y,z) \stackrel{\text{def}}{=} \Phi\left((k/c)(z-\tau\max\{x,|y|\})\right)$. It suffices to prove $(V_1)_x > 0$. We have $(v_1^-)_x \geq 0$ in $\mathbb{R}^3$. The comparison principle yields

$$w_x(x,y,z,t;v_1^-) \geq 0, \quad (V_1)_x \geq 0 \qquad \text{in } \mathbb{R}^3.$$

The maximum principle gives $(V_1)_x > 0$. $(V_1)_y > 0$ follows from Lemma 8 for $y > 0$. This completes the proof. □

From Theorem 2, $V_1(x,y,z)$ satisfies

$$\lim_{\gamma\to\infty} \sup_{(x,y,z)\in D(\gamma)} \left| V_1(x,y,z) - \Phi\left(\frac{k}{c}(z-\tau\max\{x,|y|\})\right)\right| = 0.$$

If $x < 0$ and $|x|$ is large enough, V_1 has a profile of the planar V-form wave. If $x > 0$ is large, V_1 has a profile of a pyramidal wave. Thus V_1 is a hybrid of them.

5. Generalized pyramidal traveling waves. The lateral surfaces of a pyramid have a common point. As a combination of planar traveling waves associated with the surfaces, we construct a pyramidal traveling wave in Theorem 2. How about if the surfaces have no common point? In this section we treat planes that have no common point and construct a generalized pyramidal traveling wave from a combination of planar traveling waves.

We introduce the following example:

$$\begin{aligned} h_1(x,y) &= \tau x, & h_2(x,y) &= \tau y, \\ h_3(x,y) &= -\tau x, & h_4(x,y) &= -\tau y. \end{aligned}$$

Then we have

$$h(x,y) = \tau\max\{|x|,|y|\}. \tag{41}$$

Let V_2 be a solution as in Theorem 2 for (41). Then Lemma 8 gives

$$\begin{aligned} &(V_2)_x(x,y,z) > 0 && \text{for } x > 0, \\ &(V_2)_y(x,y,z) > 0 && \text{for } y > 0. \end{aligned}$$

Let $U_2(x,y,z)$ be a supersolution as in Lemma 7 for (41). For any given $a \geq 0$, we define

$$\begin{aligned} \widetilde{h}_1(x,y) &= \tau(x-a), & \widetilde{h}_2(x,y) &= \tau y, \\ \widetilde{h}_3(x,y) &= -\tau(x+a), & \widetilde{h}_4(x,y) &= -\tau y, \end{aligned}$$

and

$$\widetilde{h}(x,y;a) \stackrel{\text{def}}{=} \max_{1\leq j\leq 4} \widetilde{h}_j(x,y;a) = \tau\max\{|y|,|x|-a\}. \tag{42}$$

The edges of a pyramid $z = \widetilde{h}(x,y;a)$ are given by

$$\begin{aligned} \widetilde{\Gamma}_1 &= \{(x,y,z) \mid z = \tau(x-a), x-a = y, z \geq 0\}, \\ \widetilde{\Gamma}_2 &= \{(x,y,z) \mid z = \tau y, y = -x-a, z \geq 0\}, \\ \widetilde{\Gamma}_3 &= \{(x,y,z) \mid z = -\tau(x+a), x+a = y, z \geq 0\}, \\ \widetilde{\Gamma}_4 &= \{(x,y,z) \mid z = -\tau y, -y = x-a, z \geq 0\}. \end{aligned}$$

We put $\widetilde{\Gamma} = \cup_{j=1}^{4} \widetilde{\Gamma}_j$ and

$$\widetilde{D}(\gamma) \stackrel{\text{def}}{=} \left\{ (x, y, z) \in \mathbb{R}^3 \mid \operatorname{dist}((x, y, z), \widetilde{\Gamma}) > \gamma \right\}.$$

We set

$$v_2^-(x, y, z) \stackrel{\text{def}}{=} \Phi\left(\frac{k}{c}(z - \widetilde{h}(x, y; a))\right) = \max_{1 \le j \le 4} \Phi\left(\frac{k}{c}(z - \widetilde{h}_j(x, y; a))\right).$$

Let $w(x, y, z, t; v_2^-)$ be the solution of (4) with an initial condition $w|_{t=0} = v_2^-$. From the comparison principle we obtain

$$v_2^-(x, y, z) < w(x, y, z, t; v_2^-) < U_2(x - x_0, y, z) \tag{43}$$

for any x_0 with $|x_0| \le a$. Thus we get

$$v_2^-(x, y, z) < w(x, y, z, t; v_2^-) \le \inf_{-a \le x_0 \le a} U_2(x - x_0, y, z).$$

Then we get the limit function

$$\widetilde{V}(x, y, z) \stackrel{\text{def}}{=} \lim_{t \to \infty} w(x, y, z, t; v_2^-) \qquad \text{in } C^2_{\text{loc}}(\mathbb{R}^3).$$

This satisfies (5). See Sattinger [17] for the general arguments. For every $x_0 \in [-a, a]$ we have

$$\widetilde{h}(x, y; a) \le \tau \max\{|y|, |x - x_0|\}$$

and thus

$$v_2^-(x, y, z) \le \Phi\left(\frac{k}{c}(z - \tau \max\{|y|, |x - x_0|\})\right).$$

We consider each side as an initial function of (4) and send $t \to \infty$. Then we get

$$\Phi\left(\frac{k}{c}(z - \tau \max\{|y|, |x| - a\})\right) < \widetilde{V}(x, y, z) < V_2(x - x_0, y, z).$$

The strict inequality follows from the strong maximum principle. See Figure 5.

Theorem 3. *Assume $c > k$,* (A1), (A2), *and* (A3). *Let V_2 be the solution of* (5) *in Theorem* 2 *for $h(x, y) = \tau \max\{|x|, |y|\}$. There exists a solution $\widetilde{V}(x, y, z)$ to* (5) *with*

$$\Phi\left(\frac{k}{c}(z - \tau \max\{|y|, |x| - a\})\right) < \widetilde{V}(x, y, z) < \inf_{-a \le x_0 \le a} V_2(x - x_0, y, z)$$

and

$$(\widetilde{V})_z(x, y, z) < 0 \qquad \text{in } \mathbb{R}^3.$$

$\widetilde{V}$ satisfies $\widetilde{V}(-x, y, z) = \widetilde{V}(x, y, z)$, $\widetilde{V}(x, -y, z) = \widetilde{V}(x, y, z)$, and

$$(\widetilde{V})_x(x, y, z) > 0 \qquad \text{for } x > 0,$$
$$(\widetilde{V})_y(x, y, z) > 0 \qquad \text{for } y > 0.$$

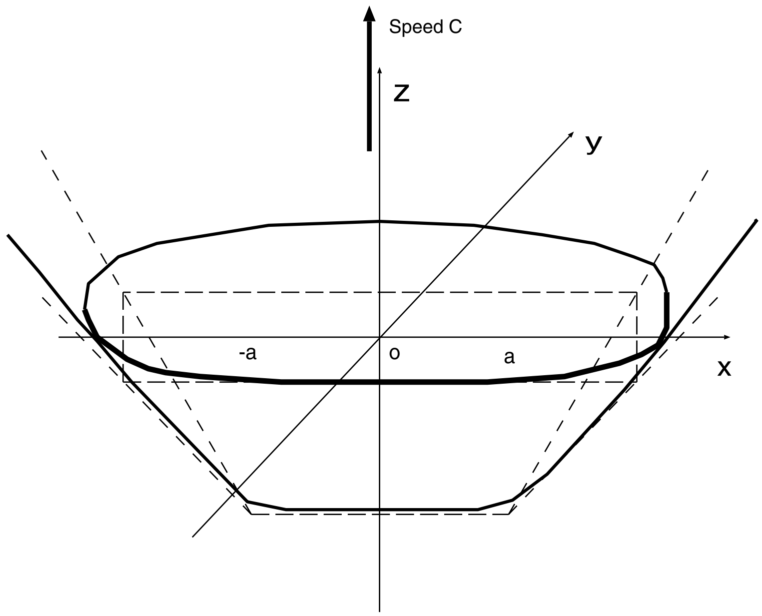

FIG. 5. *A generalized pyramidal traveling wave.*

Moreover

$$\lim_{\gamma\to\infty} \sup_{(x,y,z)\in D(\gamma)} \left| \widetilde{V}(x,y,z) - \Phi\left(\frac{k}{c}(z - \widetilde{h}(x,y;a))\right)\right| = 0$$

holds true.

Proof. Since $(v_2^-)_z \le 0$, we get $w_z(x,y,z,t;v_2^-) \le 0$ and also get $(\widetilde{V})_z < 0$. Lemma 8 and the proof are applicable to $\widetilde{h}(x,y;a)$. Thus we get $(\widetilde{V})_x > 0$ for $x > 0$ and $(\widetilde{V})_y > 0$ for $y > 0$. The asymptotic property of $\widetilde{V}(x,y,z)$ follows from that of V_2 in Theorem 2. ∎

This $\widetilde{V}(x,y,z)$ is a generalized pyramidal traveling wave. The method of this section might be applicable to a general case. The classification of all generalized pyramidal waves will give interesting problems.

6. Traveling fronts for balanced bistable nonlinearity. In this section we study traveling waves for balanced nonlinearity. Recently Chen et al. [3] constructed two-dimensional traveling waves and n-dimensional cylindrically symmetric traveling waves for balanced nonlinearity. They constructed such traveling waves as the limit of traveling waves for an unbalanced nonlinearity term when the difference of the energy density goes to zero.

Now we construct traveling waves for balanced nonlinearity by taking the limit of pyramidal traveling waves for unbalanced nonlinearity terms when the difference of the energy density goes to zero.

We consider (2) with a balanced nonlinear term $-G'(u)$. Let $c > 0$ be arbitrarily fixed. We study (13) in section 1. We define

$$(44) \qquad \mathcal{L}_\delta[v] \stackrel{\text{def}}{=} -v_{xx} - v_{yy} - v_{zz} - cv_z - f_\delta(v) = 0 \quad \text{in } \mathbb{R}^3$$

for any δ with $0 < \delta < 1$, where

$$f_\delta(v) \stackrel{\text{def}}{=} -G'(v) + \delta c\sqrt{2G(v)}.$$

Putting $k = \delta c$, we see that $\Phi_0(\mu)$ given by (3) satisfies (1). Let $V_\delta(x,y,z)$ be a solution of (44) as in Theorem 2 for

$$h_\delta(x,y) = \frac{\sqrt{1-\delta^2}}{\delta}\max\{|x|,|y|\}.$$

We fix $\lambda_1 \in (-1,1)$, with $G'(\lambda_1) < 0$. Let $z_1(\delta)$ be defined by

$$V_\delta(0,0,z_1(\delta)) = \lambda_1. \tag{45}$$

We construct a solution of (13) as the limit of $V_\delta(x,y,z+z_1(\delta))$.

Proposition 3. *Assume* (B1) *and* (B2). *Let $c > 0$ be arbitrarily fixed. Let $V_\delta(x,y,z)$ be a solution of* (44) *as in Theorem* 2 *for $h_\delta(x,y) = (\sqrt{1-\delta^2}/\delta)\max\{|x|,|y|\}$. There exists $1 > \delta_1 > \delta_2 > \cdots > \delta_i > \cdots \to 0$ so that one has*

$$\lim_{i\to\infty} V_{\delta_i}(x,y,z+z_1(\delta_i)) = V_*(x,y,z) \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^3).$$

This solution V_* satisfies $V_*(0,0,0) = \lambda_1$ and

$$\mathcal{L}_0[V_*] = 0, \quad (V_*)_z < 0 \qquad \text{in } \mathbb{R}^3.$$

Proof. We denote $V_{\delta_i}(x,y,z+z_1(\delta_i))$ simply by $v_i(x,y,z)$. Let $B(N)$ be a closed ball defined by

$$B(N) \stackrel{\text{def}}{=} \left\{(x,y,z) \mid \sqrt{x^2+y^2+z^2} \le N\right\}$$

for $N \in \mathbb{N}$. For any fixed N, $v_i(x,y,z)$ satisfies

$$\mathcal{L}[v_i] = 0, \quad -1 < v_i < 1 \qquad \text{in } B(N).$$

For any $p > 1$, (v_i) is bounded in $L^p(B(N))$. The Schauder interior estimates [6, Theorem 9.11] imply that

$$\sup_i \|v_i\|_{W^{2,p}(B(N))} < \infty.$$

We take p so large as to get $1 - 3/p > \beta > 0$. Then $W^{2,p}(B(N))$ is compactly embedded in $C^{1,\beta}(\overline{B(N)})$. By taking a subsequence (v_i) converges in $C^{1,\beta}(\overline{B(N)})$ as $i \to \infty$. Applying the Schauder interior estimates [6, Corollary 6.3] again, we find that (v_i) converges in $C^{2,\beta}(\overline{B(N)})$. By the diagonal argument we find a subsequence (v_i) that converges in $C^{2,\beta}_{\mathrm{loc}}(\mathbb{R}^3)$. Let V_* be the limit function. Then it satisfies (13). Since $(v_i)_z < 0$ in $\mathbb{R}^3$, we have $(V_*)_z \le 0$ in $\mathbb{R}^3$. From Lemma 8 we have $(v_i)_{xx}(0,0,0) \ge 0$ and $(v_i)_{yy}(0,0,0) \ge 0$ and thus $(V_*)_{xx}(0,0,0) \ge 0$ and $(V_*)_{yy}(0,0,0) \ge 0$. If $(V_*)_z \equiv 0$, we obtain a contradiction by $G'(\lambda_1) < 0$ and $\mathcal{L}_0[V_*] = 0$ at the origin. By the strong maximum principle, we get $(V_*)_z < 0$ in $\mathbb{R}^3$. $\square$

This V_* might inherit pyramidal structures, or it might not. This problem is yet to be studied. If we replace $h_\delta(x,y)$ by $(\sqrt{1-\delta^2}/\delta)\max_{1\le j\le n}(A_jx + B_jy)$ with (7) and (8), we get the associated limit traveling waves from the argument stated above and also find interesting open problems. The classification and the stability of all traveling waves for unbalanced and balanced nonlinearity have a wide variety of unknown problems and are left for further studies.

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REFERENCES

[1] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics*, in Partial Differential Equations and Related Topics, Lecture Notes in Math. 446, J. A. Goldstein, ed., Springer, New York, 1975., pp. 5–49.

[2] X. Chen, *Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations*, Adv. Differential Equations, 2 (1997), pp. 125–160.

[3] X. Chen, J-S. Guo, F. Hamel, H. Ninomiya, and J-M Roquejoffre, *Traveling waves with paraboloid like interfaces for balanced bistable dynamics*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), pp. 369–393.

[4] P. C. Fife, *Dynamics of internal layers and diffusive interfaces*, in CBMS-NSF Regional Conf. Ser. in Appl. Math. 53, SIAM, Philadelphia, 1988.

[5] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Ration. Mech. Anal., 65 (1977), pp. 335–361.

[6] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[7] F. Hamel, R. Monneau, and J.-M. Roquejoffre, *Stability of travelling waves in a model for conical flames in two space dimensions*, Ann. Sci. École Norm. Sup. (4), 37 (2004), pp. 469–506.

[8] F. Hamel, R. Monneau and J.-M. Roquejoffre, *Existence and qualitative properties of multidimensional conical bistable fronts*, Discrete Contin. Dyn. Syst., 13 (2005), pp. 1069–1096.

[9] F. Hamel, R. Monneau, and J.-M. Roquejoffre, *Asymptotic properties and classification of bistable fronts with Lipschitz level sets*, Discrete Contin. Dyn. Syst., 14 (2006), pp. 75–92.

[10] Y. I. Kanel', *Certain problems on equations in the theory of burning*, Sov. Math. Dokl., 2 (1961), pp. 48–51.

[11] Y. I. Kanel', *Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory*, Mat. Sb., 59 (1962), pp. 245–288.

[12] F. Hamel and N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in* $\mathbb{R}^N$, Arch. Ration. Mech. Anal., 157 (2001), pp. 91–163.

[13] M. Haragus and A. Scheel, *Corner defects in almost planar interface propagation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 23 (2006), pp. 283–329.

[14] J. Nagumo, S. Yoshizawa, and S. Arimoto, *Bistable transmission lines*, IEEE Trans. Circuit Theory, CT-12, No. 3, 1965, pp. 400–412.

[15] H. Ninomiya and M. Taniguchi, *Existence and global stability of traveling curved fronts in the Allen-Cahn equations*, J. Differential Equations, 213 (2005), pp. 204–233.

[16] H. Ninomiya and M. Taniguchi, *Global stability of traveling curved fronts in the Allen-Cahn equations*, Discrete Contin. Dyn. Syst., 15 (2006), pp. 819–832.

[17] D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J., 21 (1972), pp. 979–1000.

[18] D. Terman, *Directed graphs and traveling waves*, Trans. Amer. Math. Soc., 289 (1985), pp. 809–847.