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Low-Sensitivity Second-Order Digital Filters
——Analysis and Design in Terms of Frequency Sensitivity——

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SUMMARY A new procedure to derive low-sensitivity second-order digital filters is presented. This procedure is simple compared to other existing methods. First, a denominator polynomial of a filter transfer function is characterized in terms of ω_0 and ω_b, which are defined as a center angular frequency and a 3 dB bandwidth of a bandpass filter, respectively. On the other hand, the denominator polynomial is also expressed as a general form including multiplier coefficients and some integer parameters corresponding to existent degrees of freedom. Then the sensitivities of $\cos\omega_0$ and $\tan\frac{\omega_b}{2}$ with respect to a multiplier coefficient, which are referred to as frequency sensitivities, are calculated. The integer parameters are determined so that the frequency sensitivities are reduced for small ω_0 and ω_b. This procedure yields several forms of transfer functions including those of well-known low-sensitivity circuits. One of them is new and the corresponding circuit structures are also given. In addition to its low sensitivity, the new filter has a useful feature; ω_0 and ω_b are determined independently by each of two multiplier coefficients.

1. Introduction

Low-sensitivity digital filters are of great importance in practical designs, for they can be realized with short word-length coefficients. If the number of multipliers in a low-sensitivity digital filter is restricted to the minimum, the filter can be implemented on a very small-size LSI chip.

The sensitivity problem has been studied by researchers, and many types of low-sensitivity structures have been proposed. Most of them are second-order blocks, since the cascade form implementation has the practical importance due to its simplicity and modularity. For the derivation of low-sensitivity narrow band second-order structures, Avenhaus used the density of possible pole locations as a sensitivity criterion[1], and Agarwal and Burrus shifted the z-plane coordinate such that the new origin is located at $z=1$[2]. The structural feature of these filters is that there exist some feedback loops containing delay element(s) and an integer multipler (when the integer is one, the actual multiplier is not used, though). The integer multipliers are considered to be extracted from the transfer function coefficients which are close to integers for narrow band filters. The author has generalized this extraction by introducing integer parameters which correspond to the existent degrees of freedom, and some low-sensitivity filters have been derived by determining the parameters appropriately. As the sensitivity measure, initially the amplitude sensitivity was used[3], and then, the pole sensitivity was introduced to avoid the frequency dependence[4]. These techniques, however, involve too complicated calculations to evaluate the sensitivity performance conveniently. In fact the exclusive search method was used in Ref. (4).

The author has also reported a simple and exact method to characterize the frequency response and the sensitivity performance of a second-order discrete-time filter in terms of the center angular frequency ω_0 and the 3 dB bandwidth ω_b[5]. The sensitivity analysis shows that the coefficient quantization effect is particularly severe when ω_0 and ω_b are small.

In this paper, this characterization is applied to the design of low-sensitivity filters. The point is that the sensitivities of $\cos\omega_0$ and $\tan\frac{\omega_b}{2}$ with respect to a multiplier coefficient, which are referred to as frequency sensitivities, are adopted as the measure. Since the frequency sensitivity has some similarity with the pole sensitivity, what is done in this paper is almost the same with that in Ref. (4), and it is impossible to get less sensitive filters than the optimum filters derived in Ref. (4). However, the following three points should be stressed. Firstly, the investigation here is much easier because of the tractable equations and it is expected that simple and useful filters are derived. Secondly, unlike the poles of a filter, the center frequency and the bandwidth or their variations can be directly measurable in laboratories. Finally, although the filters in this paper are designed so that the sensitivity is reduced for small ω_0 and ω_b, they can realize arbitrary specifications. On the other hand, each of many low-sensitivity filters derived in Ref. (4) can realize very limited specifications.

2. Transfer Characteristics and the Frequency Sensitivity

Consider a second-order bandpass digital filter whose transfer function is given by

$$H(z)=\frac{1-z^{-2}}{1+g_1 z^{-1}+g_2 z^{-2}}, \tag{1}$$

where g_1 and g_2 are expressed as functions of multiplier coefficients. Our main interest lies in the denominator of the transfer function, because the sensitivity of the nu-

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merator is usually much smaller than that of the denominator in the passband. The numerator may be any second-order polynomial, and the bandpass filter of Eq. (1) is used here for convenience in dealing the frequency response. The denominator polynomial derived in the following procedure can be combined with an arbitrary numerator polynomial. Then the sensitivity of the overall transfer function is still low when the passband lies in low frequency range. The corresponding circuit structures can be obtained by adding appropriate feedforward paths to the circuits in this paper.

The frequency response of Eq. (1) is obtained by substituting $z=e^{j\omega}$ (ω is normalized by a sampling frequency);

$$H(e^{j\omega})=\frac{j\,2\sin\omega}{(1+g_2)\cos\omega+g_1+j(1-g_2)\sin\omega}\,. \tag{2}$$

The normalized center angular frequency ω_0 and the 3 dB bandwidth ω_b are calculated to be

$$\cos\omega_0=\frac{-g_1}{1+g_2}\,, \tag{3a}$$

$$\tan\frac{\omega_b}{2}=\frac{1-g_2}{1+g_2}\,. \tag{3b}$$

Eqs. (3) can be rewritten as

$$g_1=\frac{-2\cos\omega_0}{1+\tan\frac{\omega_b}{2}}\,, \tag{4a}$$

$$g_2=\frac{1-\tan\frac{\omega_b}{2}}{1+\tan\frac{\omega_b}{2}}\,. \tag{4b}$$

Eqs. (4) lead to the following expression;

$$H(e^{j\omega})=\frac{j\left(1+\tan\frac{\omega_b}{2}\right)\sin\omega}{\cos\omega-\cos\omega_0+j\tan\frac{\omega_b}{2}\sin\omega}\,. \tag{5}$$

This expression for the frequency response is useful for its simplicity and validity for both complex conjugate poles and real poles filters. Forthermore, unlike the pole position, ω_0 and ω_b are directly measurable. Similar expressions can be derived for lowpass, highpass, allpass, and notch filters. The two parameters ω_0 and ω_b are also considered to characterize a denominator polynomial of a transfer function like ω_0 and Q in a continuous time filter. As a limit, when a sampling period tends to zero, i.e. ω_0 , ω_b , $\omega \ll 1$, Eq. (5) reduces to that of the continuous time filter.

The most popular and crucial sensitivity is the relative sensitivity of the amplitude response with respect to a multiplier coefficient x defined by

$$S_x^{|H|}=\frac{x}{|H|}\frac{\partial|H|}{\partial x}\,. \tag{6}$$

Using the parameters defined by Eq (3), Eq. (6) is decomposed as follows;

$$S_x^{|H|}=S_{\cos\omega_0}^{|H|}S_x^{\cos\omega_0}+S_{\tan(\omega_b/2)}^{|H|}S_x^{\tan(\omega_b/2)}\,, \tag{7}$$

where

$$S_{\cos\omega_0}^{|H|}=\frac{1}{|H|}\frac{\partial|H|}{\partial\cos\omega_0}\,,\quad S_{\tan(\omega_b/2)}^{|H|}=\frac{1}{|H|}\frac{\partial|H|}{\partial\tan\frac{\omega_b}{2}}\,, \tag{8a}$$

and

$$S_x^{\cos\omega_0}=x\frac{\partial\cos\omega_0}{\partial x}\,,\quad S_x^{\tan(\omega_b/2)}=x\frac{\partial\tan\frac{\omega_b}{2}}{\partial x}\,. \tag{8b}$$

Eqs. (8a) are determined only by the given transfer function and are independent of the circuit structure. This class of sensitivity is useful to give the measure of a transfer function and will clarify conditions where the sensitivity problem is particularly severe.

Substituting Eq. (5) into Eqs. (8a) gives

$$S_{\cos\omega_0}^{|H|}=\frac{\cos\omega-\cos\omega_0}{(\cos\omega-\cos\omega_0)^2+\tan^2\frac{\omega_b}{2}\sin^2\omega} \tag{9a}$$

$$S_{\tan(\omega_b/2)}^{|H|}=\frac{(\cos\omega-\cos\omega_0)^2-\tan\frac{\omega_b}{2}\sin^2\omega}{\left(1+\tan\frac{\omega_b}{2}\right)\left\{(\cos\omega-\cos\omega_0)^2+\tan^2\frac{\omega_b}{2}\sin^2\omega\right\}}\,. \tag{9b}$$

It is shown in Ref. (5) that Eqs. (9a) and (9b) have their absolute peak values at the bandeges and at the center frequency, respectively. The peak values increase when ω_0 and ω_b become small.

On the other hand, the frequency sensitivities defined by Eqs. (8b) are dependent both on the transfer function and on the circuit structure (how g_1 and g_2 are expressed in terms of multiplier coefficients). Therefore, the frequency sensitivities are useful measures to compare various types of circuits. In order to reduce the amplitude sensitivity of narrow band filters, the frequency sensitivities should be reduced for small ω_0 and ω_b .

3. Derivation of Low-Sensitivity Digital Filters

Since the digital filter whose transfer function is expressed by Eq. (1) has two degrees of freedom, it has a possibility to be realized using two multipliers. Let the names of these multipliers be a and b , which are also used to indicate multiplier coefficient values. In general, coefficients g_i are functions of multiplier coefficients. In order to realize the filter with a minimum number of multipliers (two multipliers in this case), the functions must be linear with respect to each multiplier. If, for example, g_1 contains the terms of a and a^2, at least two multipliers are necessary to realize them, and the total number of multipliers is not minimum. Now introducing arbitrary integer parameters m_{ij} , g_i's can be expressed by

$$g_1=m_{11}+m_{12}a+m_{13}b+m_{14}ab\,, \tag{10a}$$

$$g_2=m_{21}+m_{22}a+m_{23}b+m_{24}ab\,, \tag{10b}$$

which are linear functions with respect to each of a and

b. If m_{ij}'s are small integers, they may be realized without any actual multipliers and they don't cause roundoff errors. The parameters m_{ij} have close relations with the realized circuit structure. Since simple circuit structures are preferable, it is desired that there exist some restrictions on the values of m_{ij}'s, and if so, the corresponding circuit structure can be obtained using the technique in Ref. (3).

Eliminating b from Eqs. (10a) and (10b) gives the following second-order equation with respect to a;

$$(m_{14}m_{22}-m_{12}m_{24})a^2+\{m_{24}(g_1-m_{11})-m_{14}(g_2-m_{21})+m_{22}m_{13}-m_{12}m_{23}\}a+m_{23}(g_1-m_{11})-m_{13}(g_2-m_{21})=0. \tag{11}$$

Since a is a multiplier coefficient, it must be real. First, if the coefficient of a^2 in Eq. (11) is zero, i.e., if

$$m_{14}m_{22}=m_{12}m_{24}, \tag{12}$$

Eq. (11) has a real root for any g_1 and g_2, which means the filter can realize any given specifications. Now, the condition of Eq. (12) can be divided into the following two cases;

CASE 1; $m_{22}=m_{24}=0$

CASE 2; not both of m_{22} and m_{24} are equal to zero and Eq. (12) enables us to express $m_{12}=m_0 m_{22}$ and $m_{14}=m_0 m_{24}$, where m_0 is a new parameter.

Next, consider the case where Eq. (12) does not hold. The second-order equation (11) has its real roots if and only if its discriminant is not negative, i.e.

$$\{m_{24}(g_1-m_{11})-m_{14}(g_2-m_{21})+m_{22}m_{13}-m_{12}m_{23}\}^2-4(m_{14}m_{22}-m_{12}m_{24})\{m_{23}(g_1-m_{11})-m_{13}(g_2-m_{21})\}\geqq 0. \tag{13}$$

Eq. (13) holds for any g_1 and g_2 if both m_{13} and m_{23} are zero. Then a real nonzero root for a is always obtained. So the last case to be considered is

CASE 3; $m_{13}=m_{23}=0$.

For each case, the values of m_{ij}'s will be determined so that the frequency sensitivities with respect to a and b are reduced. It is noted in the preceding section that the sensitivity becomes very high when ω_0 and ω_b are small. This is equivalent to the well-known fact that the sensitivity problem is particularly severe when g_1 and g_2 are close to -2 and 1, respectively. In order to achieve low sensitivity for such narrow band filters, it is recomended in the literature[(2),(3),(4)] to let m_{11} be -2 and m_{21} be 1. Although these values can be obtained through the process of the sensitivity reduction, giving these values in advance makes the calculations much simpler.

1) CASE 1; $m_{22}=m_{24}=0$

In this case the transfer function coefficients have the forms of

$$g_1=-2+m_{12}a+m_{13}b+m_{14}ab, \tag{14a}$$

$$g_2=1+m_{23}b. \tag{14b}$$

The frequency sensitivities are calculated to be

$$S_a^{\cos\omega_0}=\cos\omega_0-1-(1+\frac{m_{13}}{m_{23}})\tan\frac{\omega_b}{2}, \tag{15a}$$

$$S_b^{\cos\omega_0}=\tan\frac{\omega_b}{2}(1+\tan\frac{\omega_b}{2})\frac{m_{12}m_{13}+2m_{14}+(m_{12}m_{23}-2m_{14})\cos\omega_0}{m_{12}m_{13}+(m_{12}m_{23}-2m_{14})\tan\frac{\omega_b}{2}}, \tag{15b}$$

$$S_a^{\tan(\omega_b/2)}=0, \tag{15c}$$

$$S_b^{\tan(\omega_b/2)}=\tan\frac{\omega_b}{2}(1+\tan\frac{\omega_b}{2}). \tag{15d}$$

Eqs. (15c) and (15d) are independent of m_{ij}'s and are already small. Eq. (15a) is also small unless m_{23} is zero. Therefore the sensitivity reduction for this case is done by reducing $S_b^{\cos\omega_0}$. Although Eq. (15b) is rather complicated, it vanishes if

$$m_{13}+m_{23}=0, \tag{16a}$$

and

$$m_{12}m_{23}-2m_{14}=0. \tag{16b}$$

Then Eqs. (15) reduce to

$$S_a^{\cos\omega_0}=\cos\omega_0-1, \tag{17a}$$

$$S_b^{\cos\omega_0}=0, \tag{17b}$$

$$S_a^{\tan(\omega_b/2)}=0, \tag{17c}$$

$$S_b^{\tan(\omega_b/2)}=\tan\frac{\omega_b}{2}(1+\tan\frac{\omega_b}{2}). \tag{17d}$$

Eq. (17b) shows that the center frequency is determined only by a, and Eq. (17c) shows that the bandwidth is determined only by b. This property is very useful for some applications such as variable filters or filter banks.

2) CASE 2; not both of m_{22} and m_{24} are zero

In this case m_{12} and m_{14} are considered to be dependent on m_{22} and m_{24} due to Eq. (12), and can be expressed as

$$m_{12}=m_0 m_{22}, \tag{18a}$$

$$m_{14}=m_0 m_{24}, \tag{18b}$$

where m_0 is a new parameter introduced to eliminate m_{12} and m_{14}. m_0 is not necessarily an integer, because it is a ratio of two integers. Then g_1 and g_2 are expressed by

$$g_1=-2+m_0(m_{22}+m_{24}b)a+m_{13}b, \tag{19a}$$

$$g_2=1+(m_{22}+m_{24}b)a+m_{23}b. \tag{19b}$$

From Eqs. (19) the frequency sensitivities are calculated as follows;

$$S_a^{\cos\omega_0}=\frac{(m_0+\cos\omega_0)\{m_{23}(1-\cos\omega_0)+(m_{13}+m_{23})\tan\frac{\omega_b}{2}\}}{m_{13}-m_0 m_{23}}, \tag{20a}$$

$$S_b^{\cos\omega_0} = -\left(1+\tan\frac{\omega_b}{2}\right)\left\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\right\} \cdot \frac{\{m_{22}(m_{13}+m_{23}\cos\omega_0)+2m_{24}(1-\cos\omega_0)\}}{m_{22}(m_{13}-m_0m_{23})(1+\tan\frac{\omega_b}{2}) + 2m_{24}\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\}}, \tag{20b}$$

$$S_a^{\tan(\omega_b/2)} = \left(1+\tan\frac{\omega_b}{2}\right)\frac{m_{23}(1-\cos\omega_0)+(m_{13}+m_{23})\tan\frac{\omega_b}{2}}{m_{13}-m_0m_{23}}, \tag{20c}$$

$$S_b^{\tan(\omega_b/2)} = \left(1+\tan\frac{\omega_b}{2}\right)\left\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\right\} \cdot \frac{\{-m_{22}m_{23}+(2m_{24}-m_{22}m_{23})\tan\frac{\omega_b}{2}\}}{m_{22}(m_{13}-m_0m_{23})(1+\tan\frac{\omega_b}{2}) + 2m_{24}\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\}}. \tag{20d}$$

Now it is desired that all of Eqs. (20) are reduced for small ω_0 and ω_b by determining m_{ij}'s appropriately. For small value of x, the trigonometric functions in Eqs. (20) can be approximated as;

$$\cos x \fallingdotseq 1-\frac{x^2}{2}, \tag{21a}$$

$$\tan x \fallingdotseq x. \tag{21b}$$

Eqs. (20) become rational functions by using these approximations, and can be reduced for small ω_0 and ω_b by eliminating low power terms of ω_0 and ω_b. The conditions to eliminate the constant and first-power terms of ω_0 or ω_b can be easily found as follows corresponding to each of Eqs. (20);

$$m_0=-1 \quad \text{or} \quad m_{13}+m_{23}=0 \tag{22a}$$

$$m_0=-1 \quad \text{or} \quad m_{13}+m_{23}=0 \quad \text{or} \quad m_{22}=0 \tag{22b}$$

$$m_{13}+m_{23}=0 \tag{22c}$$

$$m_0=-1 \quad \text{or} \quad m_{22}m_{23}=0 \tag{22d}$$

The above conditions cannot be made simultaneously. (If these conditions are given at the same time, Eqs. (19) become meaningless), and so the four subcases are considered separately.

2-1) CASE 2-1; $m_{13}+m_{23}=0$

Using the condition, Eqs. (20) reduce to

$$S_a^{\cos\omega_0} = -\frac{(m_0+\cos\omega_0)(1-\cos\omega_0)}{1+m_0}, \tag{23a}$$

$$S_b^{\cos\omega_0} = -\left(1+\tan\frac{\omega_b}{2}\right) \cdot \frac{\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\}(1-\cos\omega_0)(2m_{24}+m_{22}m_{13})}{m_{13}m_{22}(1+m_0)(1+\tan\frac{\omega_b}{2})+2m_{24}\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\}}, \tag{23b}$$

$$S_a^{\tan(\omega_b/2)} = -\left(1+\tan\frac{\omega_b}{2}\right)\frac{1-\cos\omega_0}{1+m_0}, \tag{23c}$$

$$S_b^{\tan(\omega_b/2)} = \left(1+\tan\frac{\omega_b}{2}\right) \cdot \frac{\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\}\{m_{22}m_{13}+(2m_{24}+m_{22}m_{13})\tan\frac{\omega_b}{2}\}}{m_{13}m_{22}(1+m_0)(1+\tan\frac{\omega_b}{2})+2m_{24}\{1-\cos\omega_0+(1+m_0)\tan\frac{\omega_b}{2}\}}. \tag{23d}$$

Among these four expressions, Eq. (23d) is not so small as others. A judicous choice to reduce it is to let m_{22} be zero. Since m_0 does not affect these expressions greatly, it may be made zero in order to simplify the circuit structure. Finally the frequency sensitivities for this case become

$$S_a^{\cos\omega_0} = -\cos\omega_0(1-\cos\omega_0), \tag{24a}$$

$$S_b^{\cos\omega_0} = -\left(1+\tan\frac{\omega_b}{2}\right)(1-\cos\omega_0), \tag{24b}$$

$$S_a^{\tan(\omega_b/2)} = -\left(1+\tan\frac{\omega_b}{2}\right)(1-\cos\omega_0), \tag{24c}$$

$$S_b^{\tan(\omega_b/2)} = \left(1+\tan\frac{\omega_b}{2}\right)\tan\frac{\omega_b}{2}. \tag{24d}$$

This form of transfer function is the same as that of the Circuit E of Avenhaus[1].

2-2) CASE 2-2; $m_0=-1$

Using the condition, Eqs. (20) reduce to

$$S_a^{\cos\omega_0} = -(1-\cos\omega_0)\frac{m_{23}(1-\cos\omega_0)+(m_{13}+m_{23})\tan\frac{\omega_b}{2}}{m_{13}+m_{23}}, \tag{25a}$$

$$S_b^{\cos\omega_0} = -\left(1+\tan\frac{\omega_b}{2}\right)(1-\cos\omega_0) \cdot \frac{m_{22}(m_{13}+m_{23}\cos\omega_0)+2m_{24}(1-\cos\omega_0)}{m_{22}(m_{13}+m_{23})(1+\tan\frac{\omega_b}{2})+2m_{24}(1-\cos\omega_0)}, \tag{25b}$$

$$S_a^{\tan(\omega_b/2)} = \left(1+\tan\frac{\omega_b}{2}\right) \cdot \frac{m_{23}(1-\cos\omega_0)+(m_{13}+m_{23})\tan\frac{\omega_b}{2}}{m_{13}+m_{23}}, \tag{25c}$$

$$S_b^{\tan(\omega_b/2)} = \left(1+\tan\frac{\omega_b}{2}\right)(1-\cos\omega_0) \cdot \frac{-m_{22}m_{23}+(2m_{24}-m_{22}m_{23})\tan\frac{\omega_b}{2}}{m_{22}(m_{13}+m_{23})(1+\tan\frac{\omega_b}{2})+2m_{24}(1-\cos\omega_0)}. \tag{25d}$$

Further reduction of the frequency sensitivities is dependent on given specifications, i.e. there are several degrees of freedom in choosing the values of m_{ij}'s, depending on the values of ω_0 and ω_b. Since the list of all these variations is not meaningful, the following two cases, which show good performance, are given here.

2-2-1) CASE 2-2-1; $m_{13}=m_{24}=0$

In this case the transfer function reduces to that of the realization 2 of Agarwal and Burrus[2].

2-2-2) CASE 2-2-2; $m_{23}=m_{24}=0$

The transfer function reduces to that of the digital

Table 1 Summary of frequency sensitivities.

CASE	$S_a^{\cos\omega_0}$	$S_b^{\cos\omega_0}$	$S_a^{\tan(\omega b/2)}$	$S_b^{\tan(\omega b/2)}$	$\sum\lvert S\rvert$
1 (New Biquad II)	$-(1-\cos\omega_0)$	0	0	$(1+\tan\frac{\omega b}{2})\tan\frac{\omega b}{2}$	$1-\cos\omega_0+\tan\frac{\omega b}{2}+\tan^2\frac{\omega b}{2}$
2-1 (Avenhaus)	$-(1-\cos\omega_0)\cos\omega_0$	$-(1-\cos\omega_0)(1+\tan\frac{\omega b}{2})$	$-(1+\tan\frac{\omega b}{2})(1-\cos\omega_0)$	$(1+\tan\frac{\omega b}{2})\tan\frac{\omega b}{2}$	$(1+\tan\frac{\omega b}{2})\{2(1-\cos\omega_0)+\tan\frac{\omega b}{2}\}+(1-\cos\omega_0)\cos\omega_0$
2-2-1 (Agarwal & Burrus)	$-(1-\cos\omega_0)(1-\cos\omega_0+\tan\frac{\omega b}{2})$	$-(1-\cos\omega_0)\cos\omega_0$	$(1+\tan\frac{\omega b}{2})(1-\cos\omega_0+\tan\frac{\omega b}{2})$	$-(1+\tan\frac{\omega b}{2})(1-\cos\omega_0)$	$(1+\tan\frac{\omega b}{2})\{3(1-\cos\omega_0)+\tan\frac{\omega b}{2}\}$
2-2-2 (Biquad I)	$-(1-\cos\omega_0)$	$-(1-\cos\omega_0)\tan\frac{\omega b}{2}$	0	$(1+\tan\frac{\omega b}{2})\tan\frac{\omega b}{2}$	$(1+\tan\frac{\omega b}{2})(1-\cos\omega_0+\tan\frac{\omega b}{2})$

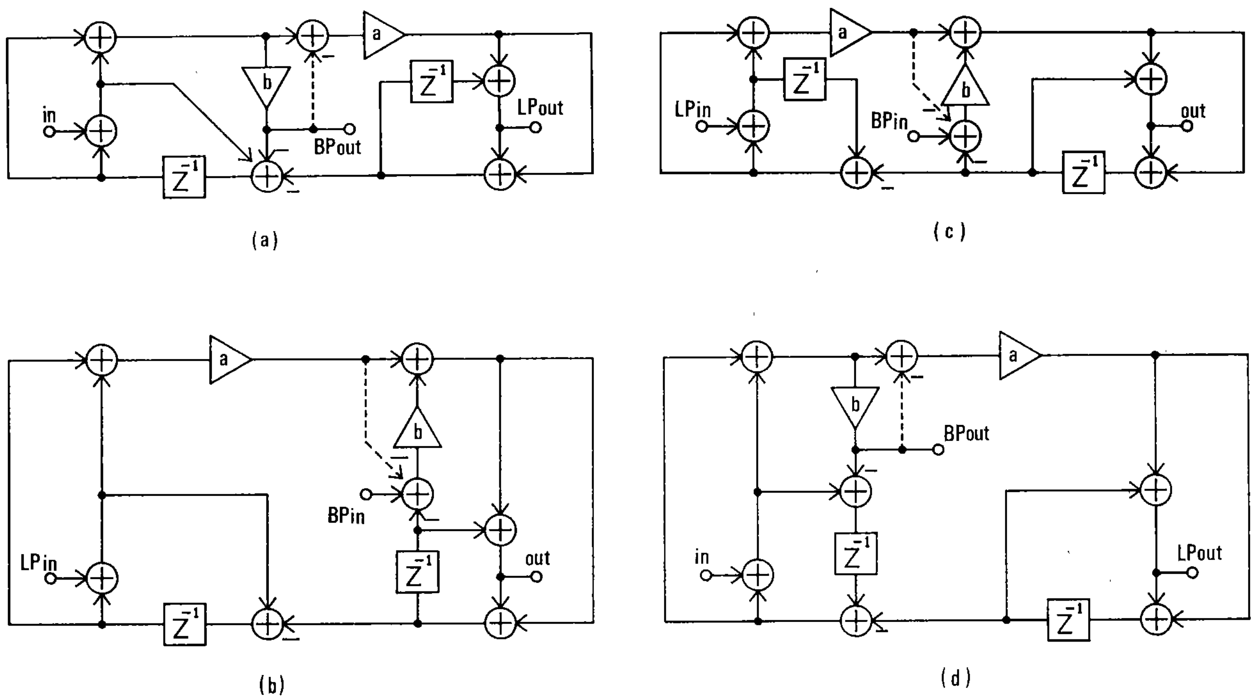

Fig. 1 Digital biquad circuits (I: without dotted path, II: with dotted path).

biquad[6].

2-3) CASE 2-3; $m_{22}=0$

No remarkable results are obtained except for the same transfer function as CASE 2-1.

2-4) CASE 2-4; $m_{23}=0$

No remarkable results are obtained except for the same transfer function as CASE 2-2-2.

3) CASE 3; $m_{13}=m_{23}=0$

Similar calculations are performed, but no remarkable results are obtained.

Now four types of transfer functions are derived. One of them is new and others are already known as low-sensitivity filters. Table 1 summarizes the frequency sensitivities for all cases. As a total measure, absolute sums of the frequency sensitivities are also listed in Table 1. Simple subtranction between the absolute sums shows that the four cases are ranked as follows:

1. CASE 1 (New)
2. CASE 2-2-2 (Biquad)
3. CASE 2-1 (Avenhaus)
4. CASE 2-2-1 (Agarwal and Burrus)

This order is independent of the values of ω_0 and ω_b. It should be noted, however, that this order is not always valid because sometimes appropriate weighting is necessary. Furthermore, since the amplitude sensitivity is evaluated by combining the frequency sensitivities with $S^{|H|}_{\cos\omega_0}$ and $S^{|H|}_{\tan(\omega_b/2)}$ which have the complicated frequency characteristics, the best form of transfer function may be dependent on particular applications. It can be shown that the derived filters of CASES 1 and 2-2-2 exhibit as low sensitivity as the best circuit in Ref. (4) when ω_0 and ω_b are small. Although the circuits in Ref. (4) have some restrictions in the realizable specifications (cf. Table 3 in Ref. (4)), all filters considered in this paper can realize any given specifications.

4. New Low-Sensitivity Digital Filter

It is shown that the CASE 1 filter has excellent sensitivity performance. The transfer function coefficients of the filter are written as follows;

$$g_1=-2+m_{12}a+m_{13}b-\frac{m_{12}m_{13}}{2}ab, \tag{26a}$$

$$g_2=1-m_{13}b. \tag{26b}$$

Substituting Eqs. (26) into Eqs. (3) leads to

$$\cos\omega_0=1-\frac{m_{12}a}{2}, \tag{27a}$$

$$\tan\frac{\omega_b}{2}=\frac{m_{13}b}{2-m_{13}b}, \tag{27b}$$

or

$$a=\frac{2(1-\cos\omega_0)}{m_{12}}, \tag{28a}$$

Table 2 Frequency sensitivities of variable filters.

Filter	$S_a^{\cos\omega_0}$	$S_b^{\tan(\omega_b/2)}$
proposed	$\cos\omega_0-1$	$\tan\frac{\omega_b}{2}(1+\tan\frac{\omega_b}{2})$
Nishihara[6]	$\cos\omega_0$	$\tan\frac{\omega_b}{2}(1+\tan\frac{\omega_b}{2})$
Nishimura et.al[7]	$\cos\omega_0$	$-\frac{1}{2}(1-\tan^2\frac{\omega_b}{2})$

$$b=\frac{2\tan\frac{\omega_b}{2}}{m_{13}(1+\tan\frac{\omega_b}{2})}. \tag{28b}$$

Eqs. (27) or Eqs. (28) clearly show that the center angular frequency and the bandwidth are determined independently by a and b, respectively. Two types of bandpass digital filters are known to have the same useful feature. They are the circuits shown in Fig. 10(b) in Ref. (6) and the filter in Ref. (7). Table 2 shows the comparative data for the nonzero components of the frequency sensitivities in these filters. For small values of ω_0 and ω_b, the proposed filter has the lowest sensitivity.

For a practical example of this new filter, let $m_{12}=4$ and $m_{13}=2$, and design a bandpass filter having unity gain at the center frequency ω_0. Then the transfer function is

$$H(z)=\frac{b(1-z^{-2})}{1+(-2+4a+2b-4ab)z^{-1}+(1-2b)z^{-2}}. \tag{29}$$

This transfer function is also obtained by replacing a by $a(1-b)$ in the transfer function of CASE 2-2-2 or the digital biquad circuits. Therefore, the circuit structures to realize Eq. (29) are easily obtained by modifying the circuits of Fig. 8 in Ref. (6) slightly, and are shown in Fig. 1. If the dotted paths in Fig. 1 are removed, the circuits reduce to the digial biquad circuits. There are four equivalent circuits, and Figs. 1(c) and (d) are the transposed circuits of (a) and (b), respectively.

Each of these circuits also provides a lowpass filter specified by

$$H(z)=\frac{a(1-b)(1+z^{-1})^2}{1+(-2+4a+2b-4ab)z^{-1}+(1-2b)z^{-2}}, \tag{30}$$

which has the unity DC gain. According to the similarity with the digital biquad circuits[6], the new circuit structures in Fig. 1 are now named digital biquad II circuits. Naturally, the circuits in Ref. (6) (or the ones without dotted paths in Fig. 1) are called biquad I circuits. In fact, in Fig. 1(a) or (d), if the signals of input, lowpass output, and bandpass output are summed up with appropriate weights, the filter can realize any second-order transfer function. Similarly, in Figs. 1(b) or (c), any second-order transfer function can be realized by feeding the input signal to the lowpass input, bandpass input, and output terminals with appropriate

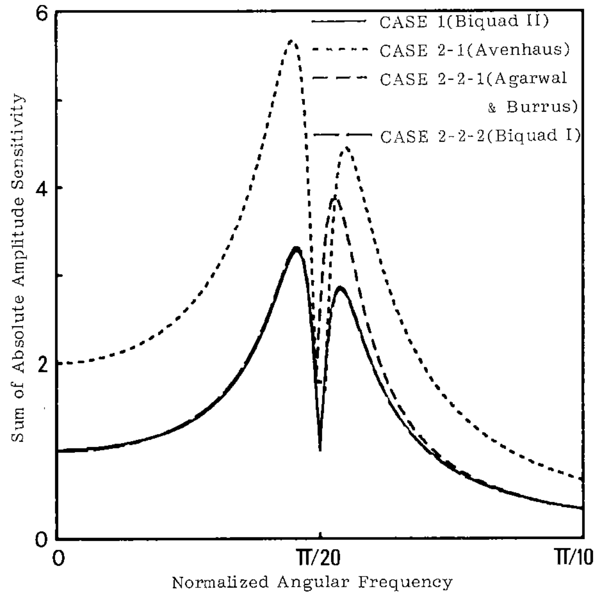

Fig. 2 Comparison of amplitude sensitivities.

weights.

The absolute sum of the amplitude sensitivity of this new filter with $\omega_0=\pi/20$ and $\omega_b=\pi/100$ is calculated using Eq. (7) and compared with the other types in Fig. 2. The values of the multiplier coefficients in this new biquad II filter are

$a = 0.006156,$

$b = 0.03047.$

The curves for the biquad II (CASE 1) and the biquad I (CASE 2-2-2) are almost the same, and the biquad II has a slightly lower peak. Both of the biquad circuits have much lower sensitivities than the other types. The amplitude sensitivity curves for the two filters, which are compared with the new filter in Table 2, are much worse and not drawn in Fig. 2.

5. Conclusion

The sensitivity problem for second-order digital filters is investigated from the viewpoint of the frequency sensitivity. The use of the frequency sensitivity enables us to develop the simple procedure to derive low-sensitivity transfer functions. Under condition that ω_0 and ω_b are small, this procedure yields the well-known low-sensitivity transfer functions as well as a new form. The simple circuit structures for the realization of the new form of transfer function are also given. In addition to the low-sensitivity property, the new circuits are practically useful, for their center frequency and bandwidth are determined independently by each of two multiplier coefficients. Roundoff noise and other finite wordlength effects of these filters still remain to be studied.

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