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PAPER

A Design Method of 2-D Diamond-Shaped Half-Band IIR Filters

Toshiyuki YOSHIDA†, Akinori NISHIHARA† and Nobuo FUJII†, Members

SUMMARY It is well known that both 1-D and 2-D half-band (HB) filters are efficient filters suitable for sampling rate converters by a factor of 2. Two kinds of HB filters have been already proposed, i.e., HB FIR filters and HB IIR filters, and their design methods have been well established. HB FIR filters show symmetric amplitude characteristics, whereas HB IIR filters do not. This paper proposes a design method of new 2-D IIR digital filters whose amplitude characteristics are similar to that of 2-D HB FIR filters. We call such filters 2-D HB IIR filters here in this paper. First a design method of 1-D HB IIR filters is explained. Then a simple design method of 2-D HB IIR filters is shown which uses 1-D HB IIR filters as prototype filters. Two design examples of 2-D HB IIR filters indicate the validity of our approach. The attenuation of the proposed filter is approximately the same as that of the filter designed by Ansari. It is also shown that the proposed filter is able to realized the complete zero-phase characteristics. As a result, it is concluded that the filter is suitable for 2-D decimators by a factor of 2.

1. Introduction

The VLSI technologies nowadays have enabled the real-time operation of not only 1-D but also 2-D filters. It is, however, still important to develop a design method of efficient digital filters which require less amount of calculation during the operation.

A 1-D half-band (HB) FIR filter is widely known as an efficient filter since it has a desirable property that about 50 percent of its coefficients are zero. Because of this property and its symmetric frequency response, a 1-D HB FIR filter can be suitably used in sampling rate converters, i.e., interpolators and decimators, by a factor of 2[1]. By extending 1-D HB FIR filters, 2-D HB FIR filters are obtained which inherit some useful properties of 1-D HB FIR filters. A diamond-shaped filter is a typical 2-D HB FIR filter suitable for an interpolator or a decimator of quincuncial sampling lattice[2]. The authors have presented several design methods of the diamond-shaped HB FIR filters[3],[4]. Another important class of 2-D HB FIR filters is that of 90-degree fan filters which have long been used in processing geoseismic data[5]. It is well known that these two filters are easily transformed from one to another by applying lowpass-highpass transformation with respect to one of two frequencies[3],[4].

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On the other hand, a class of 1-D IIR filters suited for sampling rate conversions by a factor of 2 is also proposed, referred to as HB IIR filters[6]-[10]. In Ref. (6) Ansari designed elliptic HB IIR filters as efficient interpolators and decimators, whereas in Refs. (7)-(9) the same filters are derived from a different viewpoint, e.g. design of IIR QMF systems. In Ref. (10) an iterative design method without any knowledge of elliptic functions is proposed. Furthermore Ansari proposed a design of 90-degree fan filters using 1-D HB IIR filters as prototypes[11]. Thus by using the above mentioned transformation, a diamond-shaped IIR filter can be also obtained from the 90-degree fan filters. A diamond-shaped IIR filter belongs to 2-D HB IIR filters and is also useful for 2-D sampling rate converters.

A remarkable property of HB IIR filters is that the passband ripple is much smaller than the stopband ripple[6],[10],[11]. This property is, however, of no advantage for decimators because the decimation process causes unavoidable aliasing that introduces distortions in the baseband signal even if the passband ripple of the decimator is small. This fact shows that the characteristics of HB FIR filters are adequate to decimators because the passband and stopband ripples are equal although cut-off characteristics are inferior to that of HB IIR filters. The above discussion motivates a study of 1-D and 2-D IIR filters which exhibit the symmetric amplitude characteristics similar to that of 1-D and 2-D HB FIR filters, respectively.

This paper proposes a design method of a class of IIR filters whose amplitude characteristics is symmetric as HB FIR filters and whose cut-off characteristics are as good as that of IIR filters. In this paper we call such filters HB IIR filters. In Sect. 2, we first discuss the definition of 1-D HB IIR filters by investigating some properties of 1-D HB FIR filters and then explain a design method of 1-D HB IIR filters as a preparation for the extension to 2-D case. In Sect. 3, we extend the definition of 1-D HB IIR filters to 2-D filters and show a very simple design method of diamond-shaped HB IIR filters using 1-D HB IIR filters as prototype filters. In Sect. 4 some design examples are given and the advantages of these filters are discussed.

2. Definition and Design Method of 1-D HB IIR Filters

2.1 Definition of 1-D HB IIR Filters

In Ref. (12) an HB FIR filter is defined as a zero-phase FIR filter $H(z)$ which satisfies

$$H(e^{j\omega})+H(e^{j(\pi-\omega)})=1. \tag{1}$$

The frequency response of N-tap HB FIR filter is written as

$$H(e^{j\omega})=0.5+\sum_{k=1}^{(N+1)/4} 2h(2k-1)\cos(2k-1)\omega. \tag{2}$$

Figure 1 shows the frequency response of an HB FIR filter. It indicates that the frequency response is symmetric with respect to $(\omega, |H|)=(\pi/2, 1/2)$.

Using a transformation

$$x=\cos\omega \tag{3}$$

and Tchebyshev polynomials, $H(e^{j\omega})$ can be transformed into a polynomial of variable x. Let $F'(x)$ be

$$F'(x)=H(e^{j\omega})|_{\cos\omega=x}. \tag{4}$$

By setting

$$F(x)=2F'(x)-1, \tag{5}$$

$F(x)$ becomes a polynomial in x approximating the function $\text{sgn}(x)$ in the interval $[-1, 1]$, as shown in Fig. 1.

Conversely, consider a rational function $F(x)$ which aproximates the function $\text{sgn}(x)$ in the interval $[-1, 1]$:

$$F(x)=\frac{F_N(x)}{F_D(x)}, \tag{6}$$

where $F_N(x)$ and $F_D(x)$ are polynomials in x. Using Eqs. (5) and (3), we get a function of ω which corresponds to the frequency response of an IIR lowpass filter symmetric with respect to $(\omega, |H|)=(\pi/2, 1/2)$. The transfer function of the IIR lowpass filter is obtained by using

$$x=\cos\omega=\frac{z+z^{-1}}{2}, \tag{7}$$

and setting

$$H(z)=F'(x)|_{x=\frac{z+z^{-1}}{2}}. \tag{8}$$

Eq. (8) is, however, unstable because substitution of z^{-1} into z never changes Eq. (8) (e.g., the poles off the unit circle occur in reciprocal pairs). We can obtain the stable transfer function by folding the poles outside the unit circle into the unit circle because such operation does not affect the amplitude response.

According to the above discussion, we define a 1-D HB IIR filter as follows. Among all 1-D IIR filters, an N-th order 1-D IIR filter $H(z)$ is said to have real frequency response if $H^*(z)$ can be found which satisfies

$$|H^*(e^{j\omega})|=|H(e^{j\omega})| \tag{9}$$

$$H^*(e^{j\omega})=\frac{\sum_{i=0}^{N/2} b_i\cos(i^{w})}{\sum_{i=0}^{N/2} a_i\cos(i\omega)}, \tag{10}$$

where a_i and b_i are real constants and N is an even integer. A 1-D HB IIR filter is defined for a 1-D IIR filter with real frequency response as follows:

Definition 1: An N-th order 1-D IIR filter with real frequency response is said to have half-band property if its real frequency response $H^*(e^{j\omega})$ satisfies

$$H^*(e^{j\omega})+H^*(e^{j(\pi-\omega)})=1. \tag{11}$$

It should be noted that the order N must be even, otherwise any transfer functions never satisfy Eq. (11). From Definition 1 it is found that a filter satisfying definition 1 is symmetric with respect to $(\omega, |H^*|)=(\pi/2, 1/2)$ and exhibits amplitude characteristics similar to that of 1-D HB FIR filters.

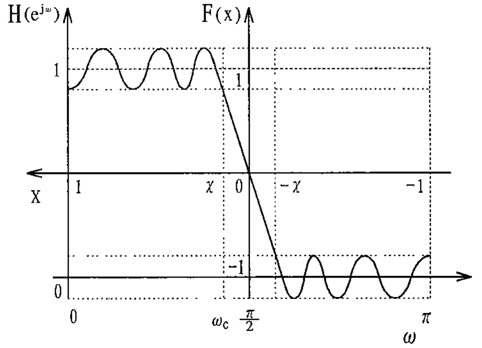

Fig. 1 Example of a 1-D HB FIR filter and the corresponding $F(x)$.

2.2 Design Method of 1-D HB IIR Filters

In order to design an N-th order 1-D HB IIR lowpass filter, we first obtain a rational function $F(x)$ of the order $N/2$ approximating $\text{sgn}(x)$ in the interval $[-1, 1]$. It is obvious that a maximally flat approximation of $\text{sgn}(x)$ leads to a maximally flat lowpass filter and an equiripple approximation to an equiripple lowpass filter.

It is clear that the function

$$F(x)=\frac{\text{Odd}[(1+x)^{N/2}]}{\text{Ev}[(1+x)^{N/2}]} \tag{12}$$

is the maximally flat rational approximation of $\text{sgn}(x)$

because $F(x)\pm 1$ has $N/2$ zeros at $x=\mp 1$, where Odd $[\cdot]$ and Ev$[\cdot]$ represent the odd and the even part of a polynomial, respectively. Eq. (12) can be factored as

$$F(x)=\frac{Nx}{2}\prod_{i=1}^{N/4}\frac{1+\dfrac{x^2}{\tan^2\dfrac{2i}{N}\pi}}{1+\dfrac{x^2}{\tan^2\dfrac{2i-1}{N}\pi}} \tag{13}$$

for $N/2$ is even and

$$F(x)=\frac{Nx}{2}\prod_{i=1}^{(N-2)/4}\frac{1+\dfrac{x^2}{\tan^2\left(\dfrac{i}{N}+\dfrac{1-(-1)^i}{4}\right)\pi}}{1+\dfrac{x^2}{\tan^2\left(\dfrac{i}{N}+\dfrac{1+(-1)^i}{4}\right)\pi}} \tag{14}$$

for $N/2$ is odd. Eqs. (13) and (14) can be proved by the same method shown in the appendix of Ref. (5), but the rigorous proof is omitted here.

On the other hand, an equiripple approximation leads to[13]

$$F(x)=\frac{2\lambda}{1+\lambda}\frac{x}{M\chi}\cdot\prod_{i=1}^{[N/4]}\frac{1+\dfrac{x^2}{\chi^2\mathrm{tn}^2\left(\dfrac{4i}{N}K(\chi'),\chi'\right)}}{1+\dfrac{x^2}{\chi^2\mathrm{tn}^2\left(\dfrac{4i-2}{N}K(\chi'),\chi'\right)}}, \tag{15}$$

where $[N/4]$ represents the integer not exceeding $N/4$. The constant χ is obtained from the specified cutoff frequency ω_c of the designing filter as

$$\chi=\cos\omega_c. \tag{16}$$

$K(\chi)$ represents the complete elliptic integral of the first kind for the modulus χ and χ' is the complementary modulus of χ. λ and λ' are the constants which satisfy

$$K(\lambda)=\frac{K(\chi)}{M} \tag{17}$$

and

$$K(\lambda')=\frac{2K(\chi')}{NM}, \tag{18}$$

respectively, where M is a constant. M, λ, λ' and the function tn can be calculated easily by using elliptic theta functions. The ripple δ is given by

$$\delta=\frac{1-\lambda}{1+\lambda}. \tag{19}$$

In Eqs. (13)-(15), the degree of the numerator is $(N/2-1)$ and that of the denominator is $N/2$ for even $N/2$, while for odd $N/2$ the degree of the numerator is $N/2$ and that of the denominator is $(N/2-1)$ because the relation

$$\frac{1}{\mathrm{tn}(K(\chi'),\chi')}=\frac{1}{\tan\left(\dfrac{\pi}{2}\right)}=0 \tag{20}$$

holds.

From Eqs. (13)-(15), $F(x)$ can be written as

$$F(x)=\frac{F_N(x)}{F_D(x)}=cx\frac{\prod_{i=1}^{(N-4)/4}\left(1+\dfrac{x^2}{n_i^2}\right)}{\prod_{i=1}^{N/4}\left(1+\dfrac{x^2}{d_i^2}\right)} \tag{21}$$

for even $N/2$ and

$$F(x)=\frac{F_N(x)}{F_D(x)}=cx\frac{\prod_{i=1}^{(N-2)/4}\left(1+\dfrac{x^2}{n_i^2}\right)}{\prod_{i=1}^{(N-2)/4}\left(1+\dfrac{x^2}{d_i^2}\right)} \tag{22}$$

for odd $N/2$, where c, n_i and d_i are the constants determined from Eqs. (13)-(15). Substitution of Eq. (7) into Eqs. (21) and (22) leads to

$$\begin{aligned}F(x)|_{(z+z^{-1})/2}&=\frac{2cz(z^2+1)\prod_{i=1}^{(N-4)/4}\dfrac{z^4+(2+4n_i^2)z^2+1}{n_i^2}}{\prod_{i=1}^{N/4}\dfrac{z^4+(2+4d_i^2)z^2+1}{d_i^2}}\\&\equiv\frac{2zG_N(z^2)}{G_D(z^2)}\end{aligned} \tag{23}$$

for even $N/2$ and

$$\begin{aligned}F(x)|_{(z+z^{-1})/2}&=\frac{c(z^2+1)\prod_{i=1}^{(N-2)/4}\dfrac{z^4+(2+4n_i^2)z^2+1}{n_i^2}}{2z\prod_{i=1}^{(N-2)/4}\dfrac{z^4+(2+4d_i^2)z^2+1}{d_i^2}}\\&\equiv\frac{G_N(z^2)}{2zG_D(z^2)}\end{aligned} \tag{24}$$

for odd $N/2$, respectively. From Eqs. (23) and (24), we can write

$$F(x)|_{(z+z^{-1})/2}=\frac{Z(z)\,G_N(z^2)}{G_D(z^2)}, \tag{25}$$

where

$$Z(z)=\begin{cases}2z & (N/2:\ \text{Even})\\(2z)^{-1} & (N/2:\ \text{Odd})\end{cases} \tag{26}$$

and $G_N(z^2)$ and $G_D(z^2)$ are even mirror-image polynomials.

If we set

$$H^*(z)=\left.\frac{F(x)+1}{2}\right|_{x=(z+z^{-1})/2}$$

$$= \frac{Z(z)\, G_N(z^2) + G_D(z^2)}{2G_D(z^2)}, \tag{27}$$

$H^*(z)$ satisfies the definition of 1-D HB IIR filters. The transfer function $H^*(z)$ is, however, unstable because the poles occur in reciprocal pairs as shown before. Folding all poles outside the unit circle into the unit circle can convert $H^*(z)$ into a stable transfer function $H(z)$, which changes the original denominator $G_D(z^2)$ in Eqs. (23) and (24) into a new denominator

$$G'_D(z^2) = \begin{cases} \prod_{i=1}^{N/4} \frac{\{(\sqrt{d_i^2+1}+d_i)z^2+(\sqrt{d_i^2+1}-d_i)\}^2}{d_i^2} & (N/2:\ \text{Even}) \\ \prod_{i=1}^{(N-2)/4} \frac{\{(\sqrt{d_i^2+1}+d_i)z^2+(\sqrt{d_i^2+1}-d_i)\}^2}{d_i^2} & (N/2:\ \text{Odd}). \end{cases} \tag{28}$$

By this modification, a stable 1-D HB IIR filter $H(z)$ can be written as

$$H(z) = \frac{Z(z)\, G_N(z^2) + G_D(z^2)}{2G'_D(z^2)} \tag{29}$$

The design procedures are summarized as follows. First we determine the order of the filter N and the cutoff frequency ω_c when designing an equiripple filter. Then we calculate $F(x)$ from Eqs. (13)-(15) to get coefficients c, d_i and n_i. By substituting these coefficients into Eq. (29), the final stable transfer function of the 1-D HB IIR filter is obtained.

2.3 Design Examples and Discussion

In order to show an example, a 1-D HB IIR filter is designed. The designed filter is 6-th order equiripple lowpass filter with cutoff frequency $\omega_c = 1.3823$[rad/s]. The transfer function is calculated as

$$H(z) = \frac{\left\{\begin{array}{c} 1.5403+4.7155z+8.7014z^2+10.431z^3 \\ +8.7014z^4+4.7155z^5+1.5403z^6 \end{array}\right\}}{z(5.97434+18.8621z^2+14.8877z^4)} \tag{30}$$

and its amplitude characteristics is given in Fig. 2, which indicates that the amplitude characteristics is symmetric about $(\omega, |H|) = (\pi/2, 1/2)$.

To clarify the difference between the proposed filters and generalized HB filters by Ansari[6], we compare these two filters in this section. Ansari's HB IIR filter $H_A(z)$ is defined as a lowpass filter with transfer function of the form given by[6].

$$H_A(z) = \frac{1}{2} + \frac{z}{2} T(z^2), \tag{31}$$

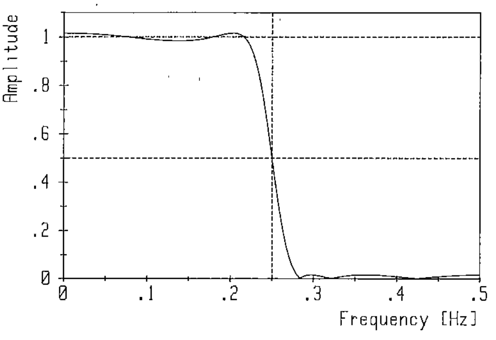

Fig. 2 Amplitude characteristics of the designed 1-D HB IIR filter.

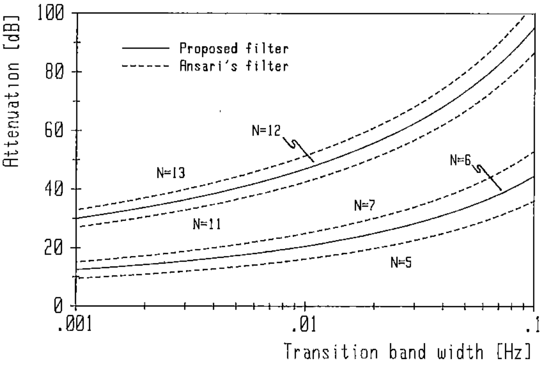

Fig. 3 Relation between the transition band width and the attenuation.

where the desired passband amplitude is assumed to be 1.0 and $T(z^2)$ is a unit magnitude allpass function written as

$$T(z^2) = \prod_{i=1}^{L} \frac{a_i z^2 + 1}{z^2 + a_i}. \tag{32}$$

The passband ripple δ_p and the stopband ripple δ_s are related by

$$\delta_p \approx \frac{1}{2}\delta_s^2, \tag{33}$$

which shows that the passband ripple is much smaller than the stopband ripple, as shown in Sect. 1.

Since $H_A(z)$ exhibits the power-complementary property[(7)-(9)]:

$$|H_A(e^{j\omega})|^2 + |H_A(e^{j(\pi-\omega)})|^2 = 1, \tag{34}$$

a comparison of Eq. (34) with Eq. (11) shows that $H_A^2(z)$ satisfies the definition 1 and that N-th order $H(z)$ can be obtained by squaring $N/2$-th order $H_A(z)$. An equiripple filter $H(z)$ designed by this method is, however, different from Eq. (29) because all the transmission zeros become double zeros and the attenuation of the resulting filter is 6 dB less than that given by Eq. (29).

In order to use the proposed filters as decimators, it is required that the stopband attenuation should be sufficiently large. Figure 3 indicates the relation between the transition band width and the stopband attenuation for both filters with specific orders, i.e., the proposed filters with 6 and 12-th order and Ansari's filters with 5, 7, 11 and 13-th order, where the transition band width means the difference of the stopband edge and the passband edge in Hz. Figure 3 shows that similar stopband attenuation can be obtained for both filters although they cannot be compared directly for the same order because the order of the proposed filters and Ansari's filters must be even and odd, respectively. Therefore it is concluded that both filters have approximately the same performance in the stopband when used as decimators. In the passband, however, the proposed filter has a drawback that twice the signal distortion occurs as compared with Ansari's filter when used for decimators, because the proposed filter has symmetric amplitude characteristics and satisfies.

$$\delta_p = \delta_s. \tag{35}$$

As mentioned before, the poles of the proposed filter $H^*(z)$ occur in reciprocal pairs. It is easily confirmed from Eqs. (23) and (24) that the denominator of $H^*(z)$ is written as

$$G_D(z^2) = G_i(z^2) \cdot G_o(z^2) \quad (|G_i(e^{2j\omega})| = |G_o(e^{2j\omega})|), \tag{36}$$

where all the poles of $G_i(z^2)$ and $G_o(z^2)$ lie inside and outside of the unit circle, respectively. Since the filter $1/G_i(z^2)$ is a causal stable while $1/G_o(z^2)$ is a noncausal stable filter, the denominator $G_D(z^2)$ can be realized as a zero-phase IIR filter[14] if the input sequence is a finite-duration sequence. In 1-D signal processing such filter can be seldom used because very few finite input sequences are encountered. In 2-D signal processing, however, such filter becomes especially important because many finite extent signals such as stationary pictures exist and the linearity of the phase is a necessary condition for 2-D filtering.

3. Definition and Design Method of 2-D HB IIR Filters

3.1 Definition of 2-D HB IIR filters

Corresponding to a 1-D IIR filter with real frequency response, we first define a 2-D IIR filter with real frequency response as follows. A 2-D IIR filter $H(z_1, z_2)$ is said to have real frequency response if $H^*(z_1, z_2)$ can be found which satisfies

$$|H^*(e^{j\omega_1}, e^{j\omega_2})| = |H(e^{j\omega_1}, e^{j\omega_2})| \tag{37}$$

$$H^*(e^{j\omega_1}, e^{j\omega_2}) = \frac{\sum_i \sum_j b_{ij} \cos(i\omega_1 + j\omega_2)}{\sum_i \sum_j a_{ij} \cos(i\omega_1 + j\omega_2)}, \tag{38}$$

where a_{ij} and b_{ij} are real constants. A 2-D HB IIR filter is defined for a 2-D IIR filter with real frequency response as follows:

Definition 2: An $M \times N$-th order 2-D IIR filter with real frequency response is said to have half-band property if its real frequency response $H^*(e^{j\omega_1}, e^{j\omega_2})$ satisfies

$$H^*(e^{j\omega_1}, e^{j\omega_2}) + H^*(e^{j(\pi-\omega_1)}, e^{j(\pi-\omega_2)}) = 1. \tag{39}$$

Again the order M and N must be even. A filter satisfying Definition 2 is symmetric with respect to $(\omega_1, \omega_2, |H|) = (\pi/2, \pi/2, 0.5)$ and its amplitude characteristics is similar to that of a 2-D HB FIR filter. The above definition is, however, more general than the definition of 2-D HB FIR filters given in Refs. (3) and (4) because the quadrantal symmetry is not assumed.

3.2 Design Method of 2-D Diamond-Shaped HB IIR Filters

The design of a diamond-shaped filter is performed by multiplying together two orthogonal 1-D filters and rotating the result by 45°. Thus we first assume $M = N$. Figure 4 shows a rotation of 2-D separable filter by 45°, where (ω_1', ω_2') and (ω_1, ω_2) are original and rotated coordinates, respectively. A relation

$$\begin{cases} \omega_1' = \dfrac{\omega_2 + \omega_1}{2} \\ \omega_2' = \dfrac{\omega_2 - \omega_1}{2} \end{cases} \tag{40}$$

holds between these frequencies and in the corresponding z-domain this relation is written as

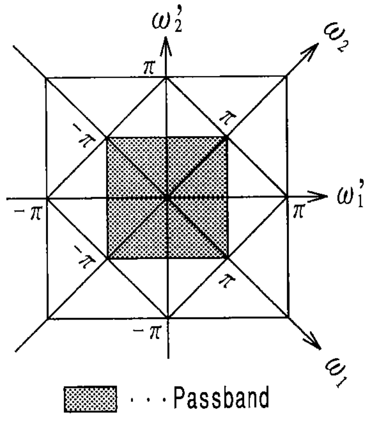

Fig. 4 Rotation of a 2-D separable lowpass filter.

$$\begin{cases} z_1' = \sqrt{z_2 z_1} \\ z_2' = \sqrt{z_2/z_1}. \end{cases} \quad (41)$$

To get the separable filter to be rotated, we begin with a consideration of $N/2$-th order rational function $F(x)$ in Sect. 2. If we multiply $F((z_1'+z_1'^{-1})/2)$ and $F((z_2'+z_2'^{-1})/2)$ together, we get a 2-D separable filter

$$H_2(z_1', z_2') \equiv kF\left(\frac{z_1'+z_1'^{-1}}{2}\right)F\left(\frac{z_2'+z_2'^{-1}}{2}\right) = \frac{kZ(z_1')Z(z_2')G_N(z_1'^2)G_N(z_2'^2)}{G_D(z_1'^2)G_D(z_2'^2)}, \quad (42)$$

where

$$k = \begin{cases} 1 & \text{(in maximally flat approximation)} \\ (1+\delta^2)^{-1} & \text{(in equiripple approximation)} \end{cases} \quad (43)$$

and δ is the ripple width given by Eq. (19). The coefficient k adjusts the 2-D ripple to oscillate between $\pm 2\delta k$ around 1 in the equiripple approximation. Substitution of Eq. (41) into Eq. (42) rotates the frequency response by 45°, which leads to

$$H_1(z_1, z_2) \equiv H_2(\sqrt{z_2 z_1}, \sqrt{z_2/z_1}) = \frac{N_1(z_1, z_2)}{D_1(z_1, z_2)}, \quad (44)$$

where

$$N_1 = kZ(\sqrt{z_2 z_1})Z(\sqrt{z_2/z_1})G_N(z_2 z_1)G_N(z_2/z_1) \quad (45)$$

$$D_1 = G_D(z_2 z_1)G_D(z_2/z_1). \quad (46)$$

For example, H_1 can be calculated from the $F(x)$ used in Eq. (30) as

$$H_1(z_1, z_2) = \frac{\boldsymbol{z}_{N2}^t \boldsymbol{A} \boldsymbol{z}_{N1}}{\boldsymbol{z}_{D2}^t \boldsymbol{B} \boldsymbol{z}_{D1}}, \quad (47)$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 2.37 & 0 & 0 & 0 \\ 0 & 0 & 13.4 & 0 & 13.4 & 0 & 0 \\ 0 & 13.4 & 0 & 75.6 & 0 & 13.4 & 0 \\ 2.37 & 0 & 75.6 & 0 & 75.6 & 0 & 2.37 \\ 0 & 13.4 & 0 & 75.6 & 0 & 13.4 & 0 \\ 0 & 0 & 13.4 & 0 & 13.4 & 0 & 0 \\ 0 & 0 & 0 & 2.37 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{z}_{Ni}[1 \ z_i \ z_i^2 \ z_i^3 \ z_i^4 \ z_i^5 \ z_i^6]^t \quad (i=1, 2)$$

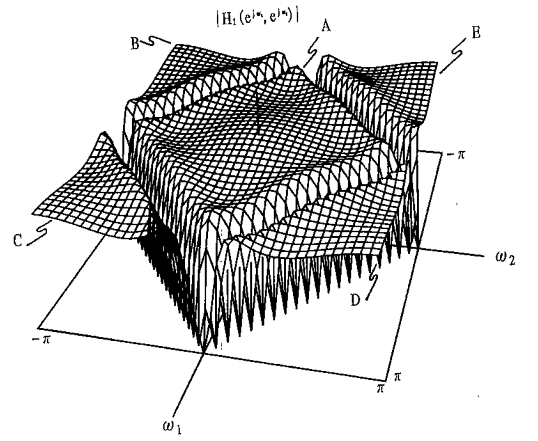

Fig. 5 Frequency response of the transfer function Eq. (47).

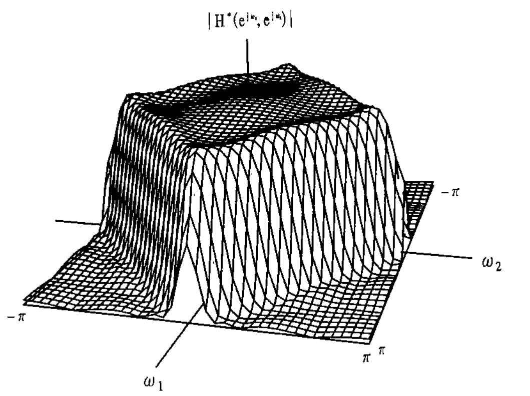

Fig. 6 Frequency response of the transfer function Eq. (48).

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 22.2 & 0 & 0 \\ 0 & 49.2 & 0 & 49.2 & 0 \\ 22.2 & 0 & 108 & 0 & 22.2 \\ 0 & 49.2 & 0 & 49.2 & 0 \\ 0 & 0 & 22.2 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{z}_{Di} = [1 \ z_i \ z_i^2 \ z_i^3 \ z_i^4]^t \quad (i=1, 2).$$

Figure 5 shows the frequency response of the transfer function Eq. (47), where the stopband never be formed. We can convert the bands B, C, D, E in Fig. 5 into stopbands by setting

$$H^*(z_1, z_2) = \frac{H_1(z_1, z_2)+1}{2} = \frac{\left\{ \begin{array}{c} kZ(\sqrt{z_2 z_1})Z(\sqrt{z_2/z_1})G_N(z_2 z_1)G_N(z_2/z_1) \\ +G_D(z_2 z_1)G_D(z_2/z_1) \end{array} \right\}}{2G_D(z_2 z_1)G_D(z_2/z_1)}. \quad (48)$$

The frequency response of Eq. (48) is given in Fig. 6, which shows that the frequency response is symmetric about $(\omega_1, \omega_2, |H^*|) = (\pi/2, \pi/2, 0.5)$.

Now consider a multiplication of two orthogonal 1-D HB IIR filters $H^*(z)$ given in Eq. (27), which is written as

$$
\begin{aligned}
H'(z_1', z_2') &\equiv H^*(z_1') H^*(z_2') \\
&= \frac{\left\{\begin{array}{c} Z(z_1') Z(z_2') G_N(z_1'^2) G_N(z_2'^2) \\ + G_D(z_1'^2) G_D(z_2'^2) \\ + Z(z_1') G_N(z_1'^2) G_D(z_2'^2) \\ + Z(z_2') G_N(z_2'^2) G_D(z_1'^2) \end{array}\right\}}{G_D(z_1'^2) G_D(z_2'^2)}.
\end{aligned}
\tag{49}
$$

Substitution of Eq. (41) rotates the frequency response of Eq. (49) :

$$
\begin{aligned}
&H'(z_1, z_2) \\
&= \frac{\left\{\begin{array}{c} Z(\sqrt{z_2 z_1}) Z(\sqrt{z_2/z_1}) G_N(z_2 z_1) G_N(z_2/z_1) \\ + G_D(z_2 z_1) G_D(z_2/z_1) \\ + Z(\sqrt{z_2 z_1}) G_N(z_2 z_1) G_D(z_2/z_1) \\ + Z(\sqrt{z_2/z_1}) G_N(z_2/z_1) G_D(z_2 z_1) \end{array}\right\}}{G_D(z_2 z_1) G_D(z_2/z_1)}.
\end{aligned}
\tag{50}
$$

The third and fourth terms in the numerator contain half-delays because $Z(z)$ is written as Eq. (26). It is found that by eliminating the third and fourth terms Eq. (50) coincides with Eq. (48) except for the coefficient 2 in the denominator and k in the numerator. Using this property we can obtain a simple design method of 2-D diamond-shaped HB IIR filters. The design method can be stated as follows :

(1) Specify the order of the filter N and the cutoff frequency ω_c when designing an equiripple filter.

(2) Design a 1-D HB IIR filter $H^*(z)$ with the specification of (1).

(3) Multiply together $H^*(z_1')$ and $H^*(z_2')$ to get $H'(z_1', z_2')$.

(4) Rotate the filter $H'(z_1', z_2')$ by substituting Eq. (41) and eliminate half-delays to get

$$
\begin{aligned}
&H^*(z_1, z_2) \\
&= \frac{\left\{\begin{array}{c} Z(\sqrt{z_2 z_1}) Z(\sqrt{z_2/z_1}) G_N(z_2 z_1) G_N(z_2/z_1) \\ + G_D(z_2 z_1) G_D(z_2/z_1) \end{array}\right\}}{G_D(z_2 z_1) G_D(z_2/z_1)}.
\end{aligned}
$$

(5) Multiply the denominator by 2 and the first term of the numerator by k determined by Eq. (43).

In the step 5, when the ripple width δ is very small, the corresponding coefficient k becomes close to 1. Thus in this case, it is considered that the multiplication by k can be omitted for practical use. Note also that to get a causal stable 2-D HB IIR filter we have to use causal stable filter $H(z)$ in Eq. (29) instead of $H^*(z)$ in Eq. (27).

4. Design Examples and Discussion of Application

4. 1 Design Examples

Example 1 is a 6×6-th order maximally flat diamond-shaped filter. The transfer function is calculated as

$$
H(z_1, z_2) = \frac{\boldsymbol{z}_{N2}^t \boldsymbol{A}_1 \boldsymbol{z}_{N1}}{\boldsymbol{z}_{D2}^t \boldsymbol{B}_1 \boldsymbol{z}_{D1}},
\tag{51}
$$

where

$$
\boldsymbol{A}_1 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 15 & 0 & 15 & 0 & 0 \\
0 & 15 & 120 & 225 & 120 & 15 & 0 \\
1 & 36 & 225 & 400 & 225 & 36 & 1 \\
0 & 15 & 120 & 225 & 120 & 15 & 0 \\
0 & 0 & 15 & 0 & 15 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

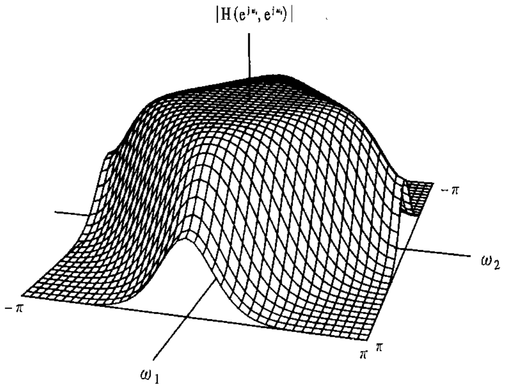

Fig. 7 Amplitude characteristics of the filter in Example 1.

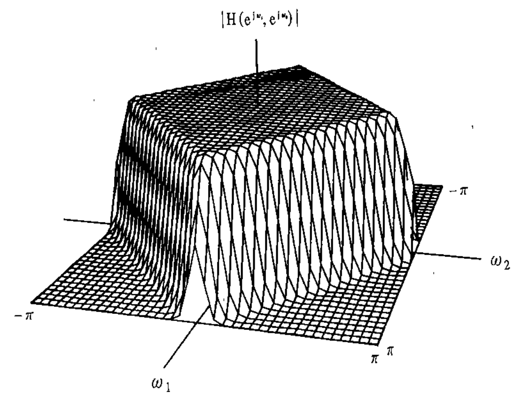

Fig. 8 Amplitude characteristics of the filter in Example 2.

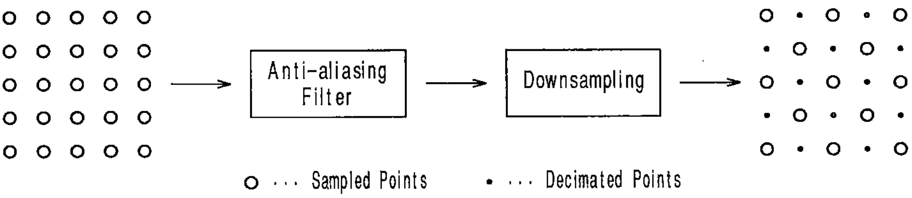

Fig. 9 Diagram of a quincuncial sampling system.

$$z_{Ni}=\begin{bmatrix}1 & z_i & z_i^2 & z_i^3 & z_i^4 & z_i^5 & z_i^6\end{bmatrix}^t \quad (i=1,2)$$

$$B_1=\begin{bmatrix}0 & 0 & 72 & 0 & 0\\ 0 & 48 & 0 & 432 & 0\\ 8 & 0 & 288 & 0 & 648\\ 0 & 48 & 0 & 432 & 0\\ 0 & 0 & 72 & 0 & 0\end{bmatrix}$$

$$z_{Di}=[1 \ z_i \ z_i^2 \ z_i^3 \ z_i^4]^t \quad (i=1,2).$$

The amplitude characteristics is shown in Fig. 7.

Example 2 is an 8×8-th order equiripple diamond-shaped filter with cutoff frequency $\omega_c=1.3823$[rad/s] ($f_c=0.22$[Hz]). The amplitude characteristics is shown in Fig. 8 while the transfer function is omitted here.

4. 2 Application of Diamond-Shaped HB IIR Filters

The amplitude characteristics of a diamond-shaped HB IIR filter $H(z_1, z_2)$ is approximately written as

$$H(e^{j\omega_1}, e^{j\omega_2})\begin{cases}\approx 1 & (|\omega_1|+|\omega_2|<\pi)\\ =0.5 & (|\omega_1|+|\omega_2|=\pi),\\ \approx 0 & (|\omega_1|+|\omega_2|>\pi)\end{cases} \qquad (52)$$

which is the same as that of a 2-D HB FIR filter. Such a kind of filters can be efficiently used as decimators for a quincuncial sampling system. Figure 9 shows the diagram of the system which converts a rectangularly sampled signal into a quincuncially sampled one. Here a 2-D rectangularly sampled discrete signal is first filtered in order to avoid aliasing and then downsampled to get a quincuncial sampling pattern whose sampling density is half of the density of the original signal[2]. Although various shapes of the anti-aliasing filter are known, a diamond shape is one of the feasible shapes[2]. Therefore a 2-D diamond-shaped HB IIR filter can be used as an anti-aliasing filter of this system because it has desirable amplitude characteristics shown in Eq. (52).

As mentioned in Sect. 1, a diamond-shaped filter can be obtained from the 90-degree fan filter designed by Ansari[11]. Since both the proposed filters and Ansari's filters use corresponding 1-D filters as prototypes, it is concluded from Fig. 3 that the stopband attenuation of these filters are approximately same although the passband ripple of the proposed filter is much larger than that of Ansari's.

The advantage of the proposed filter is that the complete zero-phase characteristics can be realized for a finite extent input sequence[2] because substitution of Eq. (36) into the denominator of Eq. (48) leads to

$$\begin{aligned}&G_D(z_2z_1)\,G_D(z_2/z_1)\\ &=G_i(z_2z_1)\,G_i(z_2/z_1)\,G_o(z_2z_1)\,G_o(z_2/z_1),\end{aligned} \qquad (53)$$

where

$$\begin{aligned}&|G_i(e^{j(\omega_2+\omega_1)})|\,|G_i(e^{j(\omega_2-\omega_1)})|\\ &=|G_o(e^{j(\omega_2+\omega_1)})|\,|G_o(e^{j(\omega_2-\omega_1)})|.\end{aligned} \qquad (54)$$

Another advantage of using a 2-D HB IIR filter as decimators lies in its denominator. From Eq. (48) and Eq. (51) in the example 1, it is found that the nonzero coefficients in the denominator form a quincuncial pattern. Thus the sampling points to be decimated after filtering do not have to be calculated because they never be needed to calculate other points. From this property it is concluded that the proposed 2-D HB IIR filter corresponds to a special extension of the filter designed by Martinez and Parks[15] to 2-D filters.

5. Conclusion

In this paper a design method of 2-D diamond-shaped HB IIR filters has been proposed. We first considered a definition and a design method of 1-D HB IIR filters. Then we extended the definition of 1-D HB IIR filters to 2-D case and explained a simple design method of 2-D diamond-shaped HB IIR filters using the corresponding 1-D HB IIR filters as prototypes. The proposed filter exhibits approximately the same stopband attenuation as the Ansari's filter. It is also shown that both 1-D and 2-D HB IIR filters proposed in this paper are able to realize the complete zero-phase characteristics if the input sequences are finite duration or extent. In order to realize the proposed filter, further consideration such as the direction of recursion will be required.

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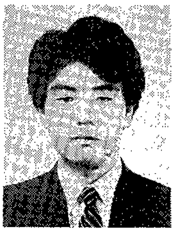

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