

論文 / 著書情報
Article / Book Information

Title	Semismall perturbations, semi-intrinsic ultracontractivity, and integral representations of nonnegative solutions for parabolic equations
Author	Pedro J. Mendez-Hernandez, Minoru Murata
Journal/Book name	Journal of Functional Analysis, Vol. 257, , pp. 1799–1827
Issue date	2009, 7
URL	http://www.journals.elsevier.com/journal-of-functional-analysis/
DOI	http://dx.doi.org/10.1016/j.jfa.2009.05.028
Note	このファイルは著者（最終）版です。 This file is author (final) version

Semismall perturbations,
semi-intrinsic ultracontractivity,
and integral representations of nonnegative
solutions for parabolic equations

Pedro J. Mendez-Hernandez
Escuela de Matemática, Universidad de Costa Rica
San José, Costa Rica
e-mail: pedro.mendez@ucr.ac.cr

Minoru Murata
Department of Mathematics, Tokyo Institute of Technology,
Oh-okayama, Meguro-ku, Tokyo, 152-8551 Japan
e-mail: minoru3@math.titech.ac.jp

The address for manuscript correspondence:

Minoru Murata
Department of Mathematics, Tokyo Institute of Technology,
Oh-okayama, Meguro-ku, Tokyo, 152-8551 Japan
e-mail: minoru3@math.titech.ac.jp
telephone: 81-3-5734-2210
fax: 81-3-5734-2738

Abstract

We consider nonnegative solutions of a parabolic equation in a cylinder $D \times I$, where D is a noncompact domain of a Riemannian manifold and $I = (0, T)$ with $0 < T \leq \infty$ or $I = (-\infty, 0)$. Under the assumption [SSP] (i.e., the constant function 1 is a semismall perturbation of the associated elliptic operator on D), we establish an integral representation theorem of nonnegative solutions: In the case $I = (0, T)$, any nonnegative solution is represented uniquely by an integral on $(D \times \{0\}) \cup (\partial_M D \times [0, T))$, where $\partial_M D$ is the Martin boundary of D for the elliptic operator; and in the case $I = (-\infty, 0)$, any nonnegative solution is represented uniquely by the sum of an integral on $\partial_M D \times (-\infty, 0)$ and a constant multiple of a particular solution. We also show that [SSP] implies the condition [SIU] (i.e., the associated heat kernel is semi-intrinsically ultracontractive).

Key Words and Phrases: semismall perturbation, semi-intrinsic ultracontractivity, parabolic equation, nonnegative solution, integral representation, Martin boundary, elliptic equation

1 Introduction

This paper is a continuation of [34]. It is concerned with integral representations of nonnegative solutions to parabolic equations and perturbation theory for elliptic operators.

We consider nonnegative solutions of a parabolic equation

$$(\partial_t + L)u = 0 \quad \text{in} \quad D \times I, \quad (1.1)$$

where $\partial_t = \partial/\partial t$, L is a second order elliptic operator on a noncompact domain D of a Riemannian manifold M , and I is a time interval: $I = (0, T)$ with $0 < T \leq \infty$ or $I = (-\infty, 0)$.

During the last few decades, much attention has been paid to the structure of all nonnegative solutions to a parabolic equation, perturbation theory for elliptic operators, and their relations. (See [1], [2], [4], [5], [6], [11], [14], [17], [19], [20], [22], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [36], [37], [38], [40], [41], [42].) Among others, Murata [34] has established integral representation theorems of nonnegative solutions to the equation (1.1) under the condition [IU] (i.e., intrinsic ultracontractivity) on the minimal fundamental solution $p(x, y, t)$ for (1.1). Furthermore, he has shown that [IU] implies [SP] (i.e., the constant function 1 is a small perturbation of L on D). It is known ([30]) that [SP] implies [SSP] (i.e., 1 is a semismall perturbation of L on D).

In this paper, we show that [SSP] implies [SIU] (i.e., semi-intrinsic ultracontractivity) and give integral representation theorems of nonnegative solutions to (1.1) under the condition [SSP]. We consider that [SSP] is one of the weakest possible condition for getting "explicit" integral representation theorems.

Now, in order to state our main results, we fix notations and recall several notions and facts. Let M be a connected separable n -dimensional smooth manifold with Riemannian metric of class C^0 . Denote by ν the Riemannian measure on M . $T_x M$ and TM denote the tangent space to M at $x \in M$ and the tangent bundle, respectively. We denote by $\text{End}(T_x M)$ and $\text{End}(TM)$ the set of endomorphisms in $T_x M$ and the corresponding bundle, respectively. The inner product on TM is denoted by $\langle X, Y \rangle$, where $X, Y \in TM$; and $|X| = \langle X, X \rangle^{1/2}$. The divergence and gradient with respect to the metric on M are denoted by div and ∇ , respectively. Let D be a noncompact domain of M . Let L be an elliptic differential operator on D of the form

$$Lu = -m^{-1} \text{div}(mA \nabla u) + Vu, \quad (1.2)$$

where m is a positive measurable function on D such that m and m^{-1} are bounded on any compact subset of D , A is a symmetric measurable section on D of $\text{End}(TM)$, and V is a real-valued measurable function on D such that

$$V \in L^p_{\text{loc}}(D, m d\nu) \quad \text{for some } p > \max(\frac{n}{2}, 1).$$

Here $L^p_{\text{loc}}(D, m d\nu)$ is the set of real-valued functions on D locally p -th integrable with respect to $m d\nu$. We assume that L is locally uniformly elliptic on D , i.e., for any compact set K in D there exists a positive constant λ such that

$$\lambda |\xi|^2 \leq \langle A_x \xi, \xi \rangle \leq \lambda^{-1} |\xi|^2, \quad x \in K, \quad (x, \xi) \in TM.$$

We assume that the quadratic form Q on $C_0^\infty(D)$ defined by

$$Q[u] = \int_D (\langle A \nabla u, \nabla u \rangle + V u^2) m d\nu$$

is bounded from below, and put

$$\lambda_0 = \inf \left\{ Q[u]; u \in C_0^\infty(D), \int_D u^2 m d\nu = 1 \right\}.$$

Then, for any $a < \lambda_0$, $(L - a, D)$ is subcritical, i.e., there exists the (minimal positive) Green function of $L - a$ on D . We denote by L_D the selfadjoint operator in $L^2(D; m d\nu)$ associated with the closure of Q . The minimal fundamental solution for (1.1) is denoted by $p(x, y, t)$, which is equal to the integral kernel of the semigroup e^{-tL_D} on $L^2(D, m d\nu)$.

Let us recall several notions related to [SSP].

[IU] λ_0 is an eigenvalue of L_D ; and there exists, for any $t > 0$, a constant $C_t > 0$ such that

$$p(x, y, t) \leq C_t \phi_0(x) \phi_0(y), \quad x, y \in D,$$

where ϕ_0 is the normalized positive eigenfunction for λ_0 .

This notion was introduced by Davies-Simon [13], and investigated extensively because of its important consequences (see [7], [8], [9], [10], [12], [23], [24], [31], [34], [42], and references therein). It looks, on the surface, not related to perturbation theory. But it has turned out ([34]) that [IU] implies the following condition [SP] for any $a < \lambda_0$.

[**SP**] The constant function 1 is a small perturbation of $L - a$ on D , i.e., for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D \setminus K} G(x, z) G(z, y) m(z) d\nu(z) \leq \varepsilon G(x, y), \quad x, y \in D \setminus K,$$

where G is the Green function of $L - a$ on D .

This condition is a special case of the notion introduced by Pinchover [37]. Recall that [SP] implies the following condition [SSP] (see [30]).

[**SSP**] The constant function 1 is a semismall perturbation of $L - a$ on D , i.e., with x^0 being a fixed reference point in D , for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D \setminus K} G(x^0, z) G(z, y) m(z) d\nu(z) \leq \varepsilon G(x^0, y), \quad y \in D \setminus K.$$

This condition [SSP] implies that L_D admits a complete orthonormal base of eigenfunctions $\{\phi_j\}_{j=0}^\infty$ with eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ repeated according to multiplicity; furthermore, for any $j = 1, 2, \dots$, the function ϕ_j/ϕ_0 has a continuous extension $[\phi_j/\phi_0]$ up to the Martin boundary $\partial_M D$ of D for $L - a$ (see Theorem 6.3 of [38]).

We show in this paper that [SSP] also implies the following condition [SIU].

[**SIU**] λ_0 is an eigenvalue of L_D ; and there exist, for any $t > 0$ and compact subset K of D , positive constants A and B such that

$$A \phi_0(x) \phi_0(y) \leq p(x, y, t) \leq B \phi_0(x) \phi_0(y), \quad x \in K, \quad y \in D.$$

This notion was introduced by Bañuelos-Davis [9], where they called it one half IU. Here we should recall that [IU] implies that for any $t > 0$ there exists a constant $c_t > 0$ such that

$$c_t \phi_0(x) \phi_0(y) \leq p(x, y, t), \quad x, y \in D.$$

We see that the same argument as in the proof of Theorem 3.1 in [25] (or the argument in the proof of Theorem 1.2 below) shows that [SIU] implies the following condition [NUP] (i.e., non-uniqueness for the positive Cauchy problem).

[NUP] The Cauchy problem

$$(\partial_t + L)u = 0 \quad \text{in } D \times (0, T), \quad u(x, 0) = 0 \quad \text{on } D \quad (1.3)$$

admits a nonnegative solution u which is not identically zero.

We say that [UP] holds for (1.3) when any nonnegative solution of (1.3) is identically zero. We note that [UP] implies that the constant function 1 is a "big" perturbation of $L - a$ on D in some sense (see Theorem 2.1 of [32]).

Fix $a < \lambda_0$, and suppose that [SSP] holds. Let $D^* = D \cup \partial_M D$ be the Martin compactification of D for $L - a$, which is a compact metric space. Denote by $\partial_m D$ the minimal Martin boundary of D for $L - a$, which is a Borel subset of the Martin boundary $\partial_M D$ of D for $L - a$. Here, we note that $\partial_M D$ and $\partial_m D$ are independent of a in the following sense: if [SSP] holds, then for any $b < \lambda_0$ there is a homeomorphism Φ from the Martin compactification of D for $L - a$ onto that for $L - b$ such that $\Phi|_D = \text{identity}$, and Φ maps the Martin boundary and minimal Martin boundary of D for $L - a$ onto those for $L - b$, respectively (see Theorem 1.4 of [30]).

Now, we are ready to state our main results. In the following theorems we assume that [SSP] holds for some fixed $a < \lambda_0$.

Theorem 1.1 The condition [SSP] implies [SIU].

Theorem 1.2 Assume [SSP]. Then, for any $\xi \in \partial_M D$ there exists the limit

$$\lim_{D \ni y \rightarrow \xi} \frac{p(x, y, t)}{\phi_0(y)} \equiv q(x, \xi, t), \quad x \in D, \quad t \in \mathbf{R}. \quad (1.4)$$

Here, as functions of (x, t) , $\{p(x, y, t)/\phi_0(y)\}_y$ converges to $q(x, \xi, t)$ as $y \rightarrow \xi$ uniformly on any compact subset of $D \times \mathbf{R}$. Furthermore, $q(x, \xi, t)$ is a continuous function on $D \times \partial_M D \times \mathbf{R}$ such that

$$q > 0 \quad \text{on } D \times \partial_M D \times (0, \infty), \quad (1.5)$$

$$q = 0 \quad \text{on } D \times \partial_M D \times (-\infty, 0], \quad (1.6)$$

$$(\partial_t + L)q(\cdot, \xi, \cdot) = 0 \quad \text{on } D \times \mathbf{R}. \quad (1.7)$$

Theorem 1.3 Assume [SSP]. Consider the equation (1.1) for $I = (0, T)$ with $0 < T \leq \infty$. Then, for any nonnegative solution u of (1.1) there exists

a unique pair of Borel measures μ on D and λ on $\partial_M D \times [0, T)$ such that λ is supported by the set $\partial_m D \times [0, T)$, and

$$u(x, t) = \int_D p(x, y, t) d\mu(y) + \int_{\partial_M D \times [0, t)} q(x, \xi, t - s) d\lambda(\xi, s) \quad (1.8)$$

for any $(x, t) \in D \times I$.

Conversely, for any Borel measures μ on D and λ on $\partial_M D \times [0, T)$ such that λ is supported by $\partial_m D \times [0, T)$ and

$$\int_D p(x^0, y, t) d\mu(y) < \infty, \quad 0 < t < T, \quad (1.9)$$

$$\int_{\partial_M D \times [0, t)} q(x^0, \xi, t - s) d\lambda(\xi, s) < \infty, \quad 0 < t < T, \quad (1.10)$$

where x^0 is a fixed point in D , the right hand side of (1.8) is a nonnegative solution of (1.1) for $I = (0, T)$ with $0 < T \leq \infty$.

The proof of this theorem will be given in Sections 4 and 5. It is based upon the abstract integral representation theorem established in [34], without assuming [IU], via a parabolic Martin representation theorem and Choquet's theorem (see [18], [21], [35]). Its key step is to identify the parabolic Martin boundary.

This theorem is an improvement of Theorem 1.2 of [34]; where the condition [IU], which is more stringent than [SSP], is assumed. It is also an answer to a problem raised in Remark 4.13 of [34]. Note that (1.8) gives explicit integral representations of nonnegative solutions to (1.1) provided that the Martin boundary $\partial_M D$ of D for $L - a$ is determined explicitly. We consider that [SSP] is one of the weakest possible condition for getting such explicit integral representations.

Let us recall that when [UP] holds for (1.3), the structure of all nonnegative solutions to (1.1) for $I = (0, T)$ is extremely simple. Namely, the following theorem holds (see [5]).

Fact AT Assume [UP]. Then, for any nonnegative solution u of (1.1) with $I = (0, T)$, there exists a unique Borel measure μ on D such that

$$u(x, t) = \int_D p(x, y, t) d\mu(y), \quad (x, t) \in D \times I. \quad (1.11)$$

Conversely, for any Borel measure μ on D satisfying (1.9), the right hand side of (1.11) is a nonnegative solution of (1.1) with $I = (0, T)$.

It is quite interesting that when [UP] holds, the elliptic Martin boundary disappears in the parabolic representation theorem; while it enters in many cases of [NUP].

Finally, we state an integral representation theorem for the case $I = (-\infty, 0)$.

Theorem 1.4 Assume [SSP]. Consider the equation (1.1) for $I = (-\infty, 0)$. Then, for any nonnegative solution u of (1.1) there exists a unique pair of a nonnegative constant α and a Borel measure λ on $\partial_M D \times (-\infty, 0)$ supported by the set $\partial_m D \times (-\infty, 0)$ such that

$$u(x, t) = \alpha e^{-\lambda_0 t} \phi_0(x) + \int_{\partial_M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s) \quad (1.12)$$

for any $(x, t) \in D \times (-\infty, 0)$.

Conversely, for any nonnegative constant α and a Borel measure λ on $\partial_M D \times (-\infty, 0)$ such that it is supported by $\partial_m D \times (-\infty, 0)$ and

$$\int_{\partial_M D \times (-\infty, t)} q(x^0, \xi, t - s) d\lambda(\xi, s) < \infty, \quad -\infty < t < 0, \quad (1.13)$$

the right hand side of (1.12) is a nonnegative solution of (1.1).

This theorem is an improvement of Theorem 6.1 of [34], where [IU] is assumed instead of [SSP].

Here, in order to illustrate a scope of Theorems 1.3 and 1.4, we give a simple example. Further examples will be given in Section 7.

Example 1.5 Let D be a domain in $\mathbf{R}^2$ with finite area. Then, by Theorem 6.1 of [33], the constant function 1 is a small perturbation of $L = -\Delta$ on D . Thus Theorems 1.3 and 1.4 hold true for the heat equation

$$(\partial_t - \Delta)u = 0 \quad \text{in } D \times I.$$

Note that there exist many bounded planar domains for which the heat semigroup is not intrinsically ultracontractive (see Example 1 of [13] and Section 4 of [9]). Thus, the last assertion of this example is new for such domains.

The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1.1, and Theorem 1.2 is proved in Section 3. Sections 4 and 5 are devoted to the proof of Theorem 1.3. In Section 4 we show it in the case of $I = (0, \infty)$. In Section 5 we show it in the case of $I = (0, T)$ with $0 < T < \infty$ by making use of results to be given in Section 4. Theorem 1.4 is proved in Section 6. Finally we shall give two more concrete examples in Section 7 with emphasis on sharpness of concrete sufficient conditions of [SSP].

2 [SSP] implies [SIU]

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 We may and shall assume that $a = 0 < \lambda_0$. Let G be the Green function of L on D . For any $t > 0$, put

$$G_t(x, y) = \int_t^\infty p(x, y, s) ds,$$

$$G^t(x, y) = \int_0^t p(x, y, s) ds.$$

Then $G = G_t + G^t$. Let us show that for any $t > 0$ and any compact subset K of D there exists a constant $A > 0$ such that

$$A \phi_0(x) \phi_0(y) \leq p(x, y, t), \quad x \in K, \quad y \in D. \quad (2.1)$$

Fix a compact subset K . We may assume that $x^0 \in K$. Let $K_1 \subset D$ be a compact neighborhood of K . Then the same argument as in the proof of Theorem 1.5 of [30] shows that

$$C^{-1} G(x^0, z) \leq \phi_0(z) \leq C G(x^0, z), \quad z \in D \setminus K_1, \quad (2.2)$$

for some constant $C > 0$. Fix $t > 0$, and put

$$\epsilon_t = \frac{1}{2\lambda_0} (1 - e^{-t\lambda_0}).$$

By [SSP] and (2.2), there exists a compact subset $K_2 \supset K_1$ such that

$$\int_{D \setminus K_2} \phi_0(z) G(z, y) d\mu(z) \leq \epsilon_t \phi_0(y), \quad y \in D \setminus K_2, \quad (2.3)$$

where $d\mu(z) = m(z) d\nu(z)$. Since

$$\frac{\phi_0(y)}{\lambda_0} = \int_D G(y, z) \phi_0(z) d\mu(z),$$

and $G(y, z) = G(z, y)$, (2.3) yields

$$\begin{aligned} \frac{\phi_0(y)}{\lambda_0} &\leq \int_{K_2} G_t(z, y) \phi_0(z) d\mu(z) + \int_{K_2} G^t(z, y) \phi_0(z) d\mu(z) \\ &\quad + \epsilon_t \phi_0(y) \end{aligned} \quad (2.4)$$

for any $y \in D \setminus K_2$. By Fubini's theorem,

$$\begin{aligned} \int_D G_t(z, y) \phi_0(z) d\mu(z) &= \int_t^\infty ds \int_D p(z, y, s) \phi_0(z) d\mu(z) \\ &= \int_t^\infty e^{-\lambda_0 s} \phi_0(y) ds \\ &= \frac{1}{\lambda_0} e^{-\lambda_0 t} \phi_0(y). \end{aligned}$$

Thus

$$\int_{K_2} G_t(z, y) \phi_0(z) d\mu(z) \leq \frac{1}{\lambda_0} e^{-\lambda_0 t} \phi_0(y).$$

This together with (2.4) implies

$$\epsilon_t \phi_0(y) \leq \int_{K_2} G^t(z, y) \phi_0(z) d\mu(z). \quad (2.5)$$

Choose a compact subset K_3 whose interior includes K_2 . By the parabolic Harnack inequality, there exists a constant C_1 depending on t, K_2, K_3 such that

$$p(z, y, s) \leq C_1 p(x, y, 2t),$$

for any $x, z \in K_2, y \in D \setminus K_3$, and $0 < s \leq t$. We have

$$\begin{aligned} G^t(z, y) &= \int_0^t p(z, y, s) ds \\ &\leq C_1 t p(x^0, y, 2t), \quad z \in K_2, y \in D \setminus K_3. \end{aligned} \quad (2.6)$$

Thus

$$\int_{K_2} G^t(z, y) \phi_0(z) d\mu(z) \leq \left[C_1 t \int_{K_2} \phi_0(z) dz \right] p(x^0, y, 2t).$$

This together with (2.5) implies

$$\phi_0(y) \leq C_2 p(x^0, y, 2t), \quad y \in D \setminus K_3, \quad (2.7)$$

where

$$C_2 = \frac{1}{\epsilon_t} C_1 t \int_{K_2} \phi_0(z) d\mu(z).$$

By the parabolic Harnack inequality,

$$p(x^0, y, 2t) \leq C p(x, y, 3t), \quad x \in K, y \in D,$$

for some constant $C > 0$. This together with (2.7) yields the desired inequality (2.1). It remains to show that for any $t > 0$ and a compact subset K of D there exists a constant B such that

$$p(x, y, t) \leq B \phi_0(x) \phi_0(y), \quad x \in K, y \in D. \quad (2.8)$$

Fix a compact subset K . We may assume that $x^0 \in K$. Let $K_1 \subset D$ be a compact neighborhood of K . By the parabolic Harnack inequality there exists a constant $c > 0$ such that

$$c p(x^0, y, t) \leq p(z, y, 2t), \quad z \in K_1, y \in D.$$

Thus, for any $y \in D$,

$$\begin{aligned} e^{-2t\lambda_0} \phi_0(y) &= \int_D \phi_0(z) p(z, y, 2t) d\mu(z) \\ &\geq \int_{K_1} \phi_0(z) p(z, y, 2t) d\mu(z) \\ &\geq c \left[\int_{K_1} \phi_0(z) d\mu(z) \right] p(x^0, y, t). \end{aligned}$$

This implies (2.8), since

$$C p(x^0, y, t) \geq p(x, y, t/2), \quad x \in K, y \in D,$$

for some constant $C > 0$. (We should note that in proving (2.8) we have only used the consequence of [SSP] that ϕ_0 is a positive eigenfunction.) $\square$

Remark 2.1 It is an open problem whether [SIU] implies [SSP] or not. Furthermore, the problem whether [SSP] implies [SP] or not in the case $n > 1$ is still open.

3 Parabolic Martin kernels

In this section we prove Theorem 1.2. Throughout the present section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$. Let G be the Green function of L on D . For any $0 < \delta < t$, put

$$G_\delta^t(x, y) = \int_\delta^t p(x, y, s) ds. \quad (3.1)$$

We denote by $\partial_M D$ the Martin boundary of D for L . In order to prove Theorem 1.2, we need two lemmas.

Lemma 3.1 Let $\xi \in \partial_M D$. Suppose that a sequence $\{y_n\}_{n=1}^\infty \subset D$ converges to ξ , and there exists the limit

$$\lim_{n \rightarrow \infty} \frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} = w(z, t), \quad z \in D. \quad (3.2)$$

Then

$$\lim_{n \rightarrow \infty} \int_D G(x, z) \frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} d\mu(z) = \int_D G(x, z) w(z, t) d\mu(z) \quad (3.3)$$

for any $x \in D$, where $d\mu(z) = m(z)d\nu(z)$.

Proof Fix $x \in D$. Let $K_1 \subset D$ be a compact neighborhood of x . By [SSP], there exists a constant $C > 0$ such that

$$C^{-1} \phi_0(y) \leq G(x, y) \leq C \phi_0(y), \quad y \in D \setminus K_1. \quad (3.4)$$

Let $\epsilon > 0$. Then there exists a compact subset $K \supset K_1$ such that

$$\int_{D \setminus K} G(x, z) \frac{G(z, y)}{G(x, y)} d\mu(z) < \frac{\epsilon}{3C}, \quad y \in D \setminus K.$$

Thus, for n sufficiently large,

$$\begin{aligned} \int_{D \setminus K} G(x, z) \left[\frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} \right] d\mu(z) &\leq \int_{D \setminus K} G(x, z) \left[\frac{C G(z, y_n)}{G(x, y_n)} \right] d\mu(z) \\ &< \frac{\epsilon}{3}. \end{aligned}$$

By Fatou's lemma,

$$\int_{D \setminus K} G(x, z) w(z, t) d\mu(z) \leq \frac{\epsilon}{3}.$$

By Theorem 1.1, there exist constants A_1 and A_2 such that

$$A_1 \phi_0(x) \phi_0(y) \leq p(x, y, \delta) \leq A_2 \phi_0(x) \phi_0(y), \quad x \in K, \quad y \in D.$$

Then, for any $t > \delta$, the semigroup property yields

$$A_1 e^{-\lambda_0(t-\delta)} \phi_0(x) \phi_0(y) \leq p(x, y, t) \leq A_2 e^{-\lambda_0(t-\delta)} \phi_0(x) \phi_0(y) \quad (3.5)$$

for any $x \in K, y \in D$. Thus there exists a constant $B > 0$ such that for any n

$$\frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} \leq B \phi_0(z), \quad z \in K.$$

Then Lebesgue's dominated convergence theorem yields

$$\lim_{n \rightarrow \infty} \int_K G(x, z) \left[\frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} \right] d\mu(z) = \int_K G(x, z) w(z, t) d\mu(z).$$

Therefore, for n sufficiently large,

$$\left| \int_D G(x, z) \left[\frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} \right] d\mu(z) - \int_D G(x, z) w(z, t) d\mu(z) \right| < \epsilon.$$

This shows (3.3). $\square$

By Lemma 6.1 of [38], it follows from [SSP] that there exists the limit

$$\lim_{D \ni y \rightarrow \xi} \frac{G(y, z)}{\phi_0(y)} = h(\xi, z), \quad (\xi, z) \in \partial_M D \times D, \quad (3.6)$$

and h is a positive continuous function on $\partial_M D \times D$. From this we show the following lemma.

Lemma 3.2 Under the same assumptions as in Lemma 3.1, one has

$$\begin{aligned} \int_D h(\xi, z) G_\delta^t(z, x) d\mu(z) &= \lim_{n \rightarrow \infty} \int_D \frac{G(y_n, z)}{\phi_0(y_n)} G_\delta^t(z, x) d\mu(z) \\ &= \int_D G(x, z) w(z, t) d\mu(z) \end{aligned} \quad (3.7)$$

for any $x \in D$.

Proof Fix $x \in D$. Let $K_1 \subset D$ be a compact neighborhood of x . By Theorem 1.1, (3.4) and (3.5), there exists a constant $C_1 > 0$ such that

$$C_1 G(z, x) \leq G_\delta^t(z, x) \leq G(z, x), \quad z \in D \setminus K_1.$$

Let $\epsilon > 0$. By [SSP], there exists a compact subset $K \supset K_1$ such that

$$\int_{D \setminus K} \left[\frac{G(y_n, z)}{\phi_0(y_n)} \right] G_\delta^t(z, x) d\mu(z) < \frac{\epsilon}{3}, \quad (3.8)$$

for n sufficiently large. By Fatou's lemma,

$$\int_{D \setminus K} h(\xi, z) G_\delta^t(z, x) d\mu(z) \leq \frac{\epsilon}{3}. \quad (3.9)$$

On the other hand, for any sufficiently large n

$$\left[\frac{G(y_n, z)}{\phi_0(y_n)} \right] G_\delta^t(z, x) \leq C_2, \quad z \in K,$$

where C_2 is a positive constant. By Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_K \frac{G(y_n, z)}{\phi_0(y_n)} G_\delta^t(z, x) d\mu(z) = \int_K h(\xi, z) G_\delta^t(z, x) d\mu(z). \quad (3.10)$$

Combining (3.8), (3.9) and (3.10), we get the first equality. It remains to show the second equality of (3.7). By Fubini's theorem and the symmetry

$$p(x, y, t) = p(y, x, t),$$

we have

$$\begin{aligned} \int_D G(y_n, z) G_\delta^t(z, x) d\mu(z) &= \int_0^\infty dr \int_\delta^t ds p(y_n, x, r + s) \\ &= \int_D G(x, z) G_\delta^t(z, y_n) d\mu(z). \end{aligned}$$

This together with Lemma 3.1 implies the second equality. $\square$

Proof of Theorem 1.2 Let $\{y_j\}_{j=1}^\infty \subset D$ be any sequence converging to $\xi \in \partial_M D$. Put

$$u_j(x, t) = \frac{p(x, y_j, t)}{\phi_0(y_j)} \quad \text{for } t > 0, \quad u_j(x, t) = 0 \quad \text{for } t \leq 0. \quad (3.11)$$

Since [SIU] holds, it follows from the parabolic Harnack inequality and local a priori estimates for nonnegative solutions to parabolic equations (see [6] and [16]) that there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ such that u_{j_k} converges, as $k \rightarrow \infty$, uniformly on any compact subset of $D \times \mathbf{R}$ to a solution u of the equation

$$(\partial_t + L) u = 0 \quad \text{in } D \times \mathbf{R}$$

satisfying $u > 0$ on $D \times (0, \infty)$ and $u = 0$ on $D \times (-\infty, 0]$. Thus, in order to prove Theorem 1.2, it suffices to show that the limit function u is independent of $\{y_{j_k}\}_{k=1}^\infty$ and uniquely determined by ξ . Let $\{y_j\}_{n=1}^\infty$ and $\{y'_j\}_{n=1}^\infty$ be two sequences in D converging to ξ . Define u_j by (3.11), and u'_j by (3.11) with y_j replaced by y'_j . Suppose that $\{u_j\}_{j=1}^\infty$ and $\{u'_j\}_{j=1}^\infty$ converge to u and u' , respectively. For any $t > \delta > 0$, put

$$w(z, t) = \int_\delta^t u(z, s) ds, \quad w'(z, t) = \int_\delta^t u'(z, s) ds.$$

Then we have

$$\lim_{n \rightarrow \infty} \frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} = w(z, t), \quad \lim_{n \rightarrow \infty} \frac{G_\delta^t(z, y'_n)}{\phi_0(y'_n)} = w'(z, t).$$

By Lemma 3.2,

$$\begin{aligned} \int_D G(x, z) w(z, t) d\mu(z) &= \int_D h(\xi, z) G_\delta^t(z, x) d\mu(z) \\ &= \int_D G(x, z) w'(z, t) d\mu(z). \end{aligned}$$

Thus $w(x, t) = w'(x, t)$, which implies $u(x, t) = u'(x, t)$. This completes the proof of Theorem 1.2. $\square$

4 Integral representations; the case $I = (0, \infty)$

In this section we prove Theorem 1.3 in the case $T = \infty$.

We first state an abstract integral representation theorem which holds without [SSP]. For $x \in D$ and $r > 0$, we denote by $B(x, r)$ the geodesic ball in the Riemannian manifold M with center x and radius r . Let x^0 be a reference point in D . Choose a nonnegative continuous function a on D such

that $a(x) = 1$ on $B(x^0, r^0)$ and $a(x) = 0$ outside $B(x^0, 2r^0)$ for some $r^0 > 0$ with $B(x^0, 3r^0) \Subset D$. Choose a nonnegative continuous function b on $\mathbf{R}$ such that $0 < b(t) < e^{\gamma t}$ on $(1, \infty)$ for some $\gamma < \lambda_0$, and $b(t) = 0$ on $(-\infty, 1]$. Denote by β the measure defined by $d\beta(x, t) = a(x)b(t)m(x) d\nu(x)dt$. For any nonnegative measurable function u on $Q = D \times (0, \infty)$, we write

$$\beta(u) = \iint_Q u(x, t) d\beta(x, t).$$

Denote by $P(Q)$ the set of all nonnegative solutions of (1.1) with $I = (0, \infty)$, and put

$$P_\beta(Q) = \{u \in P(Q); \beta(u) < \infty\}.$$

Note that for any $u \in P(Q)$ there exists a function b as above such that $\beta(u) < \infty$; thus $P(Q) = \bigcup_\beta P_\beta(Q)$. Furthermore, the parabolic Harnack inequality shows that if $\beta(u) = 0$, then $u = 0$. Now, let us define the β -Martin boundary $\partial_M^\beta Q$ of Q with respect to $\partial_t + L$ along the line given in [21] and [18]. Put

$$\begin{aligned} p(x, t; y, s) &= p(x, y, t - s), \quad t > s, \quad x, y \in D, \\ p(x, t; y, s) &= 0, \quad t \leq s, \quad x, y \in D. \end{aligned}$$

Define the β -Martin kernel K_β by

$$K_\beta(x, t; y, s) = \frac{p(x, t; y, s)}{\beta(p(\cdot; y, s))}, \quad (x, t), (y, s) \in Q,$$

where $\beta(p(\cdot; y, s)) = \iint_Q p(z, r; y, s) d\beta(z, r)$. Note that $\beta(p(\cdot; y, s)) < \infty$ for any $(y, s) \in Q$, since $0 < b(t) < e^{\gamma t}$ on $(1, \infty)$ for some $\gamma < \lambda_0$. Let $\{D_j\}_{j=1}^\infty$ be an exhaustion of D such that each D_j is a domain with smooth boundary, $D_j \Subset D_{j+1} \Subset D$, $\bigcup_{j=1}^\infty D_j = D$, and $B(x^0, 3r^0) \Subset D_1$. Put $Q_j = D_j \times (1/j, j)$. For $Y = (y, s)$, $Z = (z, r) \in Q$, let

$$\delta_\beta(Y, Z) = \sum_{j=1}^\infty 2^{-j} \sup_{X \in Q_j} \frac{|K_\beta(X; Y) - K_\beta(X; Z)|}{1 + |K_\beta(X; Y) - K_\beta(X; Z)|}.$$

Then we see that δ_β is a metric on Q , and the topology on Q induced by δ_β is equivalent to the original topology of Q . Denote by $Q^{\beta*}$ the completion of Q with respect to the metric δ_β . Put $\partial_M^\beta Q = Q^{\beta*} \setminus Q$. A sequence $\{Y^k\}_{k=1}^\infty$ in Q

is called a fundamental sequence if $\{Y^k\}_{k=1}^\infty$ has no point of accumulation in Q and $\{K_\beta(\cdot; Y^k)\}_{k=1}^\infty$ converges uniformly on any compact subset of Q to a nonnegative solution of (1.1) with $I = (0, \infty)$. By the local a priori estimates for solutions of (1.1), for any $\Xi \in \partial_M^\beta Q$ there exist a unique nonnegative solution $K_\beta(\cdot; \Xi)$ of (1.1) and a fundamental sequence $\{Y^k\}_{k=1}^\infty$ in Q such that

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\infty} 2^{-j} \sup_{X \in Q_j} \frac{|K_\beta(X; Y^k) - K_\beta(X; \Xi)|}{1 + |K_\beta(X; Y^k) - K_\beta(X; \Xi)|} = 0.$$

Thus the metric δ_β is canonically extended to $Q^{\beta*}$. Furthermore, $Q^{\beta*}$ becomes a compact metric space, since by the parabolic Harnack inequality, any sequence $\{Y^k\}_{k=1}^\infty$ with no point of accumulation in Q has a fundamental subsequence. We call $K_\beta(\cdot; \Xi)$, $\partial_M^\beta Q$ and $Q^{\beta*}$ the β -Martin kernel, β -Martin boundary and β -Martin compactification for $(Q, \partial_t + L)$, respectively. Note that $\beta(K_\beta(\cdot; \Xi)) \leq 1$ by Fatou's lemma; and so $K_\beta(\cdot; \Xi) \in P_\beta(Q)$. A nonnegative solution $u \in P_\beta(Q)$ is said to be minimal if for any nonnegative solution $v \leq u$ there exists a nonnegative constant C such that $v = Cu$. Put

$$\partial_m^\beta Q = \left\{ \Xi \in \partial_M^\beta Q; K_\beta(\cdot; \Xi) \text{ is minimal and } \beta(K_\beta(\cdot; \Xi)) = 1 \right\},$$

which we call the minimal β -Martin boundary for $(Q, \partial_t + L)$.

Observe that $D \times [0, \infty)$ is embedded into $Q^{\beta*}$, and $D \times \{0\} \subset \partial_M^\beta Q$. Indeed, with $y \in D$ fixed, for any sequence $\{Y^k\}_{k=1}^\infty$ in Q with $\lim_{k \rightarrow \infty} Y^k = (y, 0)$ we have $\lim_{k \rightarrow \infty} K_\beta(x, t; Y^k) = p(x, t; y, 0) / \beta(p(\cdot; y, 0))$; furthermore, $K_\beta(\cdot; y, 0) \neq K_\beta(\cdot; z, 0)$ if $y \neq z$. We also note that any sequence $\{Y^k = (y^k, s^k)\}_{k=1}^\infty$ in Q with $\lim_{k \rightarrow \infty} s^k = \infty$ is a fundamental sequence, since $\lim_{k \rightarrow \infty} K_\beta(\cdot; Y^k) = 0$. We denote by ϖ the point in $\partial_M^\beta Q$ corresponding to the Martin kernel which is identically zero: $K_\beta(\cdot; \varpi) = 0$. Put

$$\mathcal{L}_m^\beta Q = \partial_m^\beta Q \setminus (D \times \{0\} \cup \{\varpi\}).$$

We obtain the following abstract integral representation theorem in the same way as in the proof of Theorem 2.1 and Lemma 2.2 of [34].

Theorem 4.1 For any $u \in P_\beta(Q)$, there exists a unique pair of finite Borel measures κ on D and λ on $\partial_M^\beta Q \setminus (D \times \{0\})$ such that λ is supported by the set $\mathcal{L}_m^\beta Q$,

$$u(x, t) = \int_D \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))} d\kappa(y) + \int_{\mathcal{L}_m^\beta Q} K_\beta(x, t; \Xi) d\lambda(\Xi) \quad (4.1)$$

for any $(x, t) \in Q$, and

$$\beta(u) = \kappa(D) + \lambda(\mathcal{L}_m^\beta Q). \quad (4.2)$$

Furthermore, the function

$$v(x, t) = u(x, t) - \int_D \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))} d\kappa(y)$$

is a nonnegative solution of the equation

$$(\partial_t + L)v = 0 \quad \text{in } D \times \mathbf{R}$$

such that $v = 0$ on $D \times (-\infty, 0]$.

Conversely, for any finite Borel measures κ on D and λ on $\partial_M^\beta Q \setminus (D \times \{0\})$ such that λ is supported by the set $\mathcal{L}_m^\beta Q$, the right hand side of (4.1) belongs to $P_\beta(Q)$.

We put

$$P_\beta^0(Q) = \left\{ v \in P_\beta(Q); \lim_{t \downarrow 0} v(x, t) = 0 \text{ on } D \right\}.$$

We show Theorem 1.3 on the basis of Theorem 4.1. To this end it suffices to show (1.8) for $u \in P_\beta^0(Q)$. The key step in the proof is to identify $\mathcal{L}_m^\beta Q$. Under the condition [SSP], we shall show that $\mathcal{L}_m^\beta Q = \partial_m D \times [0, \infty)$. In the remainder of this section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$.

Lemma 4.2 For any domains U and W with $U \Subset W \Subset D$, there exist positive constants C and α such that

$$p(x, y, t) \leq C f(t) \phi_0(x) \phi_0(y), \quad x \in U, \ y \in D \setminus W, \ t > 0, \quad (4.3)$$

where $f(t) = e^{-\alpha/t}$ for $0 < t < 1$, and $f(t) = e^{-\lambda_0 t}$ for $t \geq 1$. Furthermore,

$$q(x, \xi, t) \leq C f(t) \phi_0(x), \quad x \in U, \ \xi \in \partial_M D, \ t > 0, \quad (4.4)$$

$$G(x, y) \leq C \phi_0(x) \phi_0(y), \quad x \in U, \ y \in D \setminus W, \quad (4.5)$$

where G is the Green function of L on D .

This lemma is shown in the same way as Lemmas 4.2 and 4.4 of [34].

Let $K(x, \xi)$ be the Martin kernel for L on D with reference point $x^0 \in D$, i.e., $K(x^0, \xi) = 1$, $\xi \in \partial_M D$. The following lemma gives a relation between K and q .

Lemma 4.3 For any $\xi \in \partial_M D$,

$$\lim_{D \ni y \rightarrow \xi} \frac{G(x, y)}{\phi_0(y)} = \int_0^\infty q(x, \xi, t) dt, \quad x \in D, \quad (4.6)$$

$$K(x, \xi) = \frac{\int_0^\infty q(x, \xi, t) dt}{\int_0^\infty q(x^0, \xi, t) dt}, \quad x \in D. \quad (4.7)$$

This lemma is shown in the same way as Lemma 4.5 of [34]

Lemma 4.4 Let $\xi, \eta \in \partial_M D$, $0 \leq s, r < \infty$ and $C > 0$. If

$$q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad (x, t) \in Q,$$

then $\xi = \eta$, $s = r$ and $C = 1$.

Proof Since $q(x, \xi, \tau) > 0$ for $\tau > 0$ and $q(x, \xi, \tau) = 0$ for $\tau \leq 0$, we obtain that $s = r$. Thus $q(x, \xi, \tau) = q(x, \eta, \tau)$. This together with (4.7) implies that $K(\cdot, \xi) = K(\cdot, \eta)$ on D . Hence $\xi = \eta$, and so $C = 1$. $\square$

Now, let β be a measure on $Q = D \times (0, \infty)$ as described in the beginning of this section: $d\beta(x, t) = a(x)b(t)m(x) d\nu(x) dt$. The following proposition determines the β -Martin boundary $\partial_M^\beta Q$, β -Martin compactification $Q^{\beta*}$, and β -Martin kernel K_β for $(\partial_t + L, Q)$. Recall that $p(x, t; y, s) = p(x, y, t - s)$ and $K_\beta(\cdot; y, s) = p(\cdot; y, s)/\beta(p(\cdot; y, s))$. We write

$$q(x, t; \xi, s) = q(x, \xi, t - s)$$

for $\xi \in \partial_M D$ and $0 \leq s < \infty$.

Proposition 4.5 (i) The β -Martin boundary $\partial_M^\beta Q$ of Q for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial_M D \times [0, \infty)$ and the one point set $\{\varpi\}$:

$$\partial_M^\beta Q = D \times \{0\} \cup \partial_M D \times [0, \infty) \cup \{\varpi\}. \quad (4.8)$$

In particular, $\partial_M^\beta Q$ does not depend on β .

(ii) The β -Martin compactification $Q^{\beta*}$ of Q for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0, \infty)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of ϖ is given by the family $\{\varpi\} \cup D^* \times (N, \infty)$, $N > 1$. In particular, $Q^{\beta*}$ does not depend on β .

(iii) The β -Martin kernel K_β is given as follows: For $(x, t) \in Q$,

$$K_\beta(x, t; y, 0) = \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))}, \quad (y, 0) \in D \times \{0\}, \quad (4.9)$$

$$K_\beta(x, t; \xi, s) = \frac{q(x, t; \xi, s)}{\beta(q(\cdot; \xi, s))}, \quad (\xi, s) \in \partial_M D \times [0, \infty), \quad (4.10)$$

and $K_\beta(x, t; \varpi) = 0$.

This proposition is shown in the same way as Proposition 4.8 of [34].

Lemma 4.6 Let $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, \infty)$. Then there exists a finite Borel measure γ on $\partial_m D$ supported by $\partial_m D$ such that

$$q(\cdot; \xi, s) = \int_{\partial_m D} q(\cdot; \eta, s) d\gamma(\eta). \quad (4.11)$$

Proof For reader's convenience, we give a sketch of the proof for the case $s = 0$. (For details, see the proof of Lemma 4.10 of [34].) By the elliptic Martin representation theorem, there exists a unique finite Borel measure μ on $\partial_m D$ supported by $\partial_m D$ such that

$$K(x, \xi) = \int_{\partial_m D} K(x, \eta) d\mu(\eta).$$

This together with (4.7) implies

$$\int_0^\infty q(x, \xi, t) dt = \int_{\partial_m D} \left(\int_0^\infty q(x, \eta, t) dt \right) d\gamma(\eta), \quad (4.12)$$

where $d\gamma(\eta) = [H(x^0, \xi)/H(x^0, \eta)] d\mu(\eta)$ with

$$H(x, \eta) = \int_0^\infty q(x, \eta, t) dt.$$

For $\alpha > 0$, denote by G_α the Green function of $L + \alpha$ on D . By the resolvent equation and [SSP], we then have

$$\begin{aligned} & \int_0^\infty e^{-\alpha t} q(x, \eta, t) dt \\ &= \int_0^\infty q(x, \eta, t) dt - \alpha \int_D G_\alpha(x, z) \left(\int_0^\infty q(z, \eta, t) dt \right) m(z) d\nu(z), \end{aligned} \quad (4.13)$$

for any $\eta \in \partial_M D$. By combining (4.12) and (4.13), we get

$$\int_0^\infty e^{-\alpha t} \left(\int_{\partial_m D} q(x, \eta, t) d\gamma(\eta) \right) dt = \int_0^\infty e^{-\alpha t} q(x, \xi, t) dt.$$

Thus the Laplace transforms of $q(x, \xi, t)$ and $\int_{\partial_m D} q(x, \eta, t) d\gamma(\eta)$ coincide; and so (4.11) holds. $\square$

Lemma 4.7 Let $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, \infty)$. Then $q(\cdot; \xi, s)$ is not minimal.

Proof For reader's convenience, we give a proof. We have (4.11). Suppose that $q(\cdot; \xi, s)$ is minimal. Then, along the line given in the proof of Lemma 12.12 of [15], we obtain from (4.11) that the support of γ consists of a single point. Thus, for some $\eta \in \partial_m D$ and constant C

$$q(\cdot; \xi, s) = Cq(\cdot; \eta, s).$$

Hence, by Lemma 4.4, $\xi = \eta$; which is a contradiction. $\square$

Lemma 4.8 Let $(\xi, s) \in \partial_m D \times (0, \infty)$. Then $q(\cdot; \xi, s)$ is minimal if and only if $q(\cdot; \xi, 0)$ is minimal.

Proof Assume that $q(\cdot; \xi, 0)$ is minimal. Suppose that a nonnegative solution u of (1.1) satisfies $u(\cdot) \leq q(\cdot; \xi, s)$ on Q . Put $v(x, t) = u(x, t + s)$. Then $v(\cdot) \leq q(\cdot; \xi, 0)$. Thus $v(\cdot) = Cq(\cdot; \xi, 0)$ for some constant C . Hence $u(x, t) = Cq(x, t; \xi, s)$ for $t > s$, and $u(x, t) = 0 = Cq(x, t; \xi, s)$ for $t \leq s$. This shows that $q(\cdot; \xi, s)$ is minimal. Next, assume that $q(\cdot; \xi, s)$ is minimal. Suppose that a nonnegative solution u of (1.1) satisfies $u(\cdot) \leq q(\cdot; \xi, 0)$ on Q . Put $v(x, t) = u(x, t - s)$ for $t > s$, and $v(x, t) = 0$ for $0 < t \leq s$. Then $v(\cdot) \leq q(\cdot; \xi, s)$. Thus $v(\cdot) = Cq(\cdot; \xi, s)$ for some constant C . Hence $u(x, t) = Cq(x, t; \xi, 0)$. This shows that $q(\cdot; \xi, 0)$ is minimal. $\square$

By Theorem 4.1 and Lemmas 4.7 and 4.8, we have the following proposition.

Proposition 4.9 There exists a Borel subset R of $\partial_M D$ such that

$$R \subset \partial_m D, \quad \mathcal{L}_m^\beta Q = R \times [0, \infty),$$

for any $u \in P_\beta^0(Q)$ there exists a unique Borel measure λ on $\partial_M D \times [0, \infty)$ which is supported by $R \times [0, \infty)$ and satisfies

$$u(x, t) = \int_{R \times [0, \infty)} q(x, \xi, t - s) d\lambda(\xi, s) \quad (x, t) \in Q. \quad (4.14)$$

Lemma 4.10 Let $(\xi, s) \in \partial_m D \times [0, \infty)$. Then $q(\cdot; \xi, s)$ is minimal.

Proof Suppose that $q(\cdot; \xi, 0)$ is not minimal. Then $\xi \notin R$ and

$$q(x, \xi, t) = \int_{R \times [0, \infty)} q(x, \eta, t - s) d\lambda(\eta, s)$$

for some Borel measure λ . We have

$$\begin{aligned} K(x, \xi) \int_0^\infty q(x^0, \xi, t) dt &= \int_0^\infty q(x, \xi, t) dt \\ &= \int_{R \times [0, \infty)} d\lambda(\eta, s) K(x, \eta) \int_0^\infty q(x^0, \eta, t) dt. \end{aligned}$$

Thus

$$K(x, \xi) = \int_R K(x, \eta) d\Lambda(\eta)$$

for some Borel measure Λ . But $\xi \in \partial_m D \setminus R$ and $R \subset \partial_m D$. This contradicts the uniqueness of a representing measure in the elliptic Martin representation theorem. Hence $q(\cdot; \xi, 0)$ is minimal; which together with Lemma 4.8 shows Lemma 4.10. $\square$

Completion of the proof of Theorem 1.3 in the case $I = (0, \infty)$ By Lemma 4.10, $R = \partial_m D$ and

$$\mathcal{L}_m^\beta Q = \partial_m D \times [0, \infty).$$

Thus Proposition 4.9 shows Theorem 1.3. $\square$

5 Proof of Theorem 1.3; the case $0 < T < \infty$

In this section we prove Theorem 1.3 in the case $0 < T < \infty$ by making use of the results in Section 4. To this end, the following proposition plays a crucial role.

Proposition 5.1 Let $\xi \in \partial_M D$ and $0 \leq s < r < \infty$. Then

$$\int_D p(x, y, t - r) q(y, r; \xi, s) d\mu(y) = q(x, t; \xi, s), \quad x \in D, \quad t > r, \quad (5.1)$$

where $d\mu(y) = m(y) d\nu(y)$

Proof We first show (5.1) for $\xi \in \partial_m D$. Define $u(x, t)$ by

$$\begin{aligned} u(x, t) &= q(x, t; \xi, s), & 0 < t \leq r, \\ u(x, t) &= \int_D p(x, y, t - r) q(y, r; \xi, s) d\mu(y), & r < t < \infty. \end{aligned} \quad (5.2)$$

(We call u the minimal extension of q from $t = r$.) Then we see that u is a nonnegative solution of $(\partial_t + L)u = 0$ in $D \times (0, \infty)$ such that $u(\cdot) \leq q(\cdot; \xi, s)$ on $D \times (0, \infty)$. By Lemma 4.10, $u(\cdot) = Cq(\cdot; \xi, s)$ for some constant C . But $u(x, t) = q(x, t; \xi, s)$ for $0 < t \leq r$. Thus $C = 1$, and so $u(\cdot) = q(\cdot; \xi, s)$.

Next, let $\xi \notin \partial_m D$. By Lemma 4.6, there exists a finite Borel measure γ on $\partial_M D$ supported by $\partial_m D$ such that

$$q(\cdot; \xi, s) = \int_{\partial_m D} q(\cdot; \eta, s) d\gamma(\eta). \quad (5.3)$$

Thus

$$\begin{aligned} & \int_D p(x, y, t - r) q(y, r; \xi, s) d\mu(y) \\ &= \int_{\partial_m D} d\gamma(\eta) \int_D p(x, y, t - r) q(y, r; \eta, s) d\mu(y) \\ &= \int_{\partial_m D} q(x, t; \eta, s) d\gamma(\eta) \\ &= q(x, t; \xi, s). \end{aligned}$$

This proves (5.1). □

Lemma 5.2 Let $\xi, \eta \in \partial_M D$, $0 \leq s, r < T$ and $C > 0$. If

$$q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad x \in D, \quad 0 < t < T, \quad (5.4)$$

then $\xi = \eta$, $s = r$ and $C = 1$.

Proof Choose u such that $\max(r, s) < u < T$, and construct minimal extensions of both sides of (5.4) from $t = u$. Then, by (5.1) we have

$$q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad x \in D, \quad 0 < t < \infty.$$

By Lemma 4.4, this implies that $\xi = \eta$, $s = r$ and $C = 1$. $\square$

Now, let β be a measure on $Q = D \times (0, T)$ defined by

$$d\beta(x, t) = a(x)b(t)m(x) d\nu(x)dt.$$

Here $a(x)$ is a nonnegative continuous function on D as described in the beginning of Section 4, and $b(t)$ is a nonnegative continuous function on $\mathbf{R}$ such that $b(t) > 0$ on $(T/2, T)$ and $b(t) = 0$ on $\mathbf{R} \setminus (T/2, T)$. Let $K_\beta(\cdot; \Xi)$, $\partial_M^\beta Q$, $\partial_m^\beta Q$, and $Q^{\beta*}$ be the β -Martin kernel, β -Martin boundary, minimal β -Martin boundary, and β -Martin compactification for $(Q, \partial_t + L)$ with $Q = D \times (0, T)$, respectively. The following proposition is an analogue of Proposition 4.5, and is shown in the same way.

Proposition 5.3 (i) The β -Martin boundary $\partial_M^\beta Q$ of Q for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial_M D \times [0, T)$ and the one point set $\{\varpi\}$:

$$\partial_M^\beta Q = D \times \{0\} \cup \partial_M D \times [0, T) \cup \{\varpi\}. \quad (5.5)$$

In particular, $\partial_M^\beta Q$ does not depend on β .

(ii) The β -Martin compactification $Q^{\beta*}$ of Q for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0, T)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of ϖ is given by the family $\{\varpi\} \cup D^* \times (T - \varepsilon, T)$, $0 < \varepsilon < T/2$. In particular, $Q^{\beta*}$ does not depend on β .

(iii) The β -Martin kernel K_β is given as follows: For $(x, t) \in Q$,

$$K_\beta(x, t; y, 0) = \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))}, \quad (y, 0) \in D \times \{0\}, \quad (5.6)$$

$$K_\beta(x, t; \xi, s) = \frac{q(x, t; \xi, s)}{\beta(q(\cdot; \xi, s))}, \quad (\xi, s) \in \partial_M D \times [0, T), \quad (5.7)$$

and $K_\beta(x, t; \varpi) = 0$.

Lemma 5.4 Let $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, T)$. Then $q(\cdot; \xi, s)$ is not minimal.

Proof Suppose that $q(\cdot; \xi, s)$ is minimal. Then we obtain from (5.3) that

$$q(x, \xi, t - s) = Cq(x, \eta, t - s), \quad x \in D, \quad 0 < t < T,$$

for some $\eta \in \partial_m D$ and $C > 0$. By Lemma 5.2, this is a contradiction. $\square$

Lemma 5.5 Let $(\xi, s) \in \partial_m D \times [0, T)$. Then $q(\cdot; \xi, s)$ is minimal.

Proof Let u be a nonnegative solution of $(\partial_t + L)u = 0$ in Q such that $u(\cdot) \leq q(\cdot; \xi, s)$ in Q . For $r \in (s, T)$, let u_r be the minimal extension of u from $t = r$. By Proposition 5.1,

$$u_r(x, t) \leq q(x, t; \xi, s), \quad x \in D, \quad t > 0.$$

By Lemma 4.10, there exists a constant C_r such that $u_r(x, t) = C_r q(x, t; \xi, s)$ for $t > 0$. But $u_r(x, t) = u(x, t)$ for $0 < t < r$. Thus C_r is independent of r ; and so $u(\cdot) = Cq(\cdot; \xi, s)$ in Q for some constant C . $\square$

Completion of the proof of Theorem 1.3 in the case $0 < T < \infty$
Put

$$\mathcal{L}_m^\beta Q = \partial_m^\beta Q \setminus (D \times \{0\} \cup \{\varpi\}).$$

By Proposition 5.3, Lemmas 5.4 and 5.5, we get

$$\mathcal{L}_m^\beta Q = \partial_m D \times [0, T).$$

Thus, Theorem 2.1 of [34] which is an analogue of Theorem 4.1 completes the proof. $\square$

6 Integral representations; the case $I = (-\infty, 0)$

In this section we prove Theorem 1.4. We begin with the following proposition, which can be shown in the same way as in the proof of Theorem 1 of [9] (see also [39]).

Proposition 6.1 Assume [SIU]. Then

$$\lim_{t \rightarrow \infty} \frac{e^{\lambda_0 t} p(x, y, t)}{\phi_0(x) \phi_0(y)} = 1 \quad \text{uniformly in } (x, y) \in K \times D \quad (6.1)$$

for any compact subset K of D .

In the rest of this section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$. By Theorem 1.1, we have the following corollary of Proposition 6.1.

Corollary 6.2 Assume [SSP]. Then, for any compact subset K of D and $N > 1$,

$$\lim_{s \rightarrow -\infty} \frac{p(x, y, t - s)}{e^{\lambda_0 s} \phi_0(y)} = e^{-\lambda_0 t} \phi_0(x) \quad \text{uniformly in } (x, y, t) \in K \times D \times (-N, 0).$$

Lemma 6.3 The solution $e^{-\lambda_0 t} \phi_0(x)$ is minimal.

Proof Suppose that $e^{-\lambda_0 t} \phi_0(x)$ is not minimal. Then, in view of Corollary 6.2, the same argument as in the proof of Theorem 1.3 shows that for any nonnegative solution u of the equation

$$(\partial_t + L)u = 0 \quad \text{in } Q = D \times (-\infty, 0)$$

there exists a unique Borel measure λ on $\partial_M D \times (-\infty, 0)$ supported by the set $\partial_m D \times (-\infty, 0)$ such that

$$u(x, t) = \int_{\partial_M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s), \quad (x, t) \in Q.$$

Thus

$$e^{-\lambda_0 t} \phi_0(x) = \int_{\partial_M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s), \quad (x, t) \in Q, \quad (6.2)$$

for such a measure λ . Now, fix x . It follows from Theorems 1.1 and 1.2 that for any $\delta > 0$ there exists a positive constant C_δ such that

$$C_\delta^{-1} \leq \frac{q(x, \xi, \tau)}{e^{-\lambda_0 \tau} \phi_0(x)} \leq C_\delta, \quad \tau \geq \delta, \quad \xi \in \partial_M D. \quad (6.3)$$

By (4.4),

$$q(x, \xi, \tau) \leq C e^{-\alpha/\tau} \phi_0(x), \quad \xi \in \partial_M D, \quad 0 < \tau < 1, \quad (6.4)$$

for some positive constants α and C . By (6.2) and (6.3),

$$e^{\lambda_0} \phi_0(x) \geq \int_{\partial_M D \times (-\infty, -2)} C_1^{-1} e^{-\lambda_0(-1-s)} d\lambda(\xi, s).$$

Thus

$$\int_{\partial_M D \times (-\infty, -2)} e^{\lambda_0 s} d\lambda(\xi, s) \leq C_1 \phi_0(x). \quad (6.5)$$

For $t < -2$ and $0 < \delta < 1$, we have

$$\phi_0(x) = \int_{\partial_M D \times \{(-\infty, t-\delta] \cup (t-\delta, t)\}} e^{\lambda_0(t-s)} q(x, \xi, t-s) e^{\lambda_0 s} d\lambda(\xi, s). \quad (6.6)$$

In view of (6.4) and (6.5), we choose δ so small that the integral on $\partial_M D \times (t-\delta, t)$ of the right hand side of (6.6) is smaller than $\phi_0(x)/3$. Then, in view of (6.3) and (6.5), we choose $t < -2$ with $|t|$ being so large that the integral on $\partial_M D \times (-\infty, t-\delta]$ of the right hand side of (6.6) is smaller than $\phi_0(x)/3$. This is a contradiction. $\square$

Completion of the proof of Theorem 1.4 By virtue of Corollary 6.2 and Lemma 6.3, the same argument as in the proof of Theorem 1.3 shows Theorem 1.4. $\square$

7 Examples

In this section we give two examples in order to illustrate a scope of Theorem 1.3. Throughout this section L_0 is a uniformly elliptic operator on $\mathbf{R}^n$ of the form

$$L_0 u = - \sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j u),$$

where $a(x) = [a_{ij}(x)]_{i,j=1}^n$ is a symmetric matrix-valued measurable function on $\mathbf{R}^n$ satisfying, for some $\Lambda > 0$,

$$\Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad x, \xi \in \mathbf{R}^n.$$

7.1. Let $V(x)$ be a measurable function in $L_{\text{loc}}^\infty(\mathbf{R}^n)$, and $L = L_0 + V(x)$ on $D = \mathbf{R}^n$.

Theorem 7.1 Suppose that there exist a positive constant $c < 1$ and a positive continuous increasing function ρ on $[0, \infty)$ such that

$$c [\rho(|x|)]^2 \leq V(x) \leq [\rho(|x|)]^2, \quad x \in \mathbf{R}^n, \quad (7.1)$$

$$c \rho \left(r + \frac{c}{\rho(r)} \right) \leq \rho(r), \quad r \geq 0. \quad (7.2)$$

Assume that

$$\int_1^\infty \frac{dr}{\rho(r)} < \infty. \quad (7.3)$$

Then 1 is a small perturbation of L on $\mathbf{R}^n$. Thus Theorem 1.3 holds true.

Remark. Compare this theorem with a non-uniqueness theorem of [26].

Proof We first note that (7.2) yields

$$c\rho(r) \leq c\rho \left(r - \frac{c}{\rho(r)} + \frac{c}{\rho \left(r - \frac{c}{\rho(r)} \right)} \right) \leq \rho \left(r - \frac{c}{\rho(r)} \right), \quad r \geq \frac{c}{\rho(0)},$$

since ρ is increasing. We show the theorem by using the same approach as in the proof of Theorem 5.1 of [31]. Put $b = c^{-2}$ and

$$\ell = \inf \{j \in \mathbf{Z}; \rho(0) < b^j\}.$$

For $k \geq \ell$, put $r_k = \sup\{r \geq 0; \rho(r) \leq b^k\}$. By the continuity of ρ and (7.3), $\rho(r_k) = b^k$ and $\lim_{k \rightarrow \infty} r_k = \infty$. By (7.2),

$$\rho(r_k + cb^{-k}) \leq c^{-1} \rho(r_k) = b^{1/2} b^k < b^{k+1} = \rho(r_{k+1}).$$

Thus $r_k + cb^{-k} < r_{k+1}$ for $k \geq \ell$. Define a positive continuously differentiable increasing function $\tilde{\rho}$ on $[0, \infty)$ as follows: Put $\tilde{\rho}(r) = b^\ell$ for $r \leq r_\ell$,

$$\tilde{\rho}(r) = b^{k+1} \quad \text{for} \quad r_k + cb^{-k} \leq r \leq r_{k+1} \quad (k \geq \ell);$$

and $\tilde{\rho}(r) = \rho_k(r)$ for $r_k \leq r \leq r_k + cb^{-k}$ ($k \geq \ell$) by choosing a continuously differentiable function ρ_k on $[r_k, r_k + cb^{-k}]$ such that

$$\rho_k(r_k) = b^k, \quad \rho_k'(r_k) = 0, \quad \rho_k(r_k + cb^{-k}) = b^{k+1}, \quad \rho_k'(r_k + cb^{-k}) = 0,$$

and

$$0 \leq \rho_k'(r) \leq B b^{2k}, \quad r_k \leq r \leq r_k + c b^{-k},$$

for some constant $B > 0$ independent of k . Then we have

$$C^{-1} \leq \frac{\tilde{\rho}(r)}{\rho(r)} \leq C, \quad 0 \leq \tilde{\rho}'(r) \leq C \rho(r)^2, \quad r \geq 0, \quad (7.4)$$

for some positive constant C . Introduce a Riemannian metric $g = (g_{ij})_{i,j=1}^n$ by $g_{ij} = \tilde{\rho}(|x|)^2 \delta_{ij}$. Then $M = \mathbf{R}^n$ with this metric g becomes a complete Riemannian manifold. Furthermore, by (7.2) and (7.4), M has the bounded geometry property (1.1) of [4]. The associated gradient ∇ and divergence div are written as

$$\nabla = \tilde{\rho}(|x|)^{-2} \nabla^0, \quad \text{div} = \tilde{\rho}(|x|)^{-n} \circ \text{div}^0 \circ \tilde{\rho}(|x|)^n,$$

where ∇^0 and div^0 are the standard gradient and divergence on $\mathbf{R}^n$. Put

$$\begin{aligned} \mathcal{L} &= \tilde{\rho}(|x|)^{-2} L, \\ m(x) &= \tilde{\rho}(|x|)^{2-n}, \quad A(x) = [a_{ij}(x)]_{i,j=1}^n, \quad \gamma(x) = \tilde{\rho}(|x|)^{-2} V(x). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L}u &= -\frac{1}{m} \text{div}(mA\nabla u) + \gamma \\ &= -\text{div}(A\nabla u) - \left\langle \frac{1}{m} A \nabla^0 m, \nabla u \right\rangle^0 + \gamma, \end{aligned}$$

where $\langle \cdot, \cdot \rangle^0$ is the standard inner product on $\mathbf{R}^n$. Since the inner product $\langle \cdot, \cdot \rangle$ associated with the metric g is written as

$$\langle X, Y \rangle = \langle \tilde{\rho}^2 X, Y \rangle^0,$$

we have

$$\mathcal{L}u = -\text{div}(A\nabla u) - \left\langle \tilde{\rho}^{-2} \frac{A \nabla^0 m}{m}, \nabla u \right\rangle + \gamma. \quad (7.5)$$

By (7.4),

$$|\nabla^0 m(x)| \leq C^3 |n-2| \tilde{\rho}(|x|) m(x).$$

From this we have

$$\begin{aligned} \left\langle \tilde{\rho}^{-2} \frac{A \nabla^0 m}{m}, \tilde{\rho}^{-2} \frac{A \nabla^0 m}{m} \right\rangle &\leq \tilde{\rho}^{-2} \Lambda^2 (C^3 |n-2| \tilde{\rho})^2 \\ &\leq \{\Lambda(C^3 |n-2|)\}^2. \end{aligned}$$

By (7.1) and (7.4),

$$cC^{-2} \leq \gamma(x) \leq C^2.$$

Thus the operator $\mathcal{L} - cC^{-2}/2$ has the Green function; and $\mathcal{L}$ belongs to the class $\mathcal{D}_M(\theta, \infty, \epsilon)$ introduced by Ancona [4], where

$$\theta = \max \left(\Lambda, \Lambda(C^3 |n - 2|), C^2 \right), \quad \epsilon = cC^{-2}/2.$$

Put

$$\mathcal{L}_2 = \tilde{\rho}(|x|)^{-2} (L + 1) = \mathcal{L} + \tilde{\rho}(|x|)^{-2}.$$

In order to apply the results of [4], we proceed to estimate $\tilde{\rho}(|x|)^{-2}$. Let $d(x)$ be the Riemannian distance $\text{dist}(0, x)$ from the origin 0 to x , and put

$$\psi(r) = \int_0^r \tilde{\rho}(s) ds.$$

Then we see that $d(x) = \psi(|x|)$. Denote by ψ^{-1} the inverse function of ψ , and put

$$\Phi(s) = \left[\tilde{\rho}(\psi^{-1}(s)) \right]^{-2}, \quad s \geq 0.$$

Then

$$0 < \tilde{\rho}(|x|)^{-2} = \Phi(d(x)), \quad x \in M.$$

Furthermore,

$$\begin{aligned} \int_0^\infty \Phi(s) ds &= \int_0^\infty \Phi(\psi(r)) \tilde{\rho}(r) dr \\ &= \int_0^\infty \frac{dr}{\tilde{\rho}(r)} \leq C \int_0^\infty \frac{dr}{\rho(r)} dr < \infty. \end{aligned}$$

Hence, by virtue of Corollary 6.1, Theorems 1 and 2 of [4], $\tilde{\rho}(|x|)^{-2}$ is a small perturbation of $\mathcal{L}$ on the manifold M . That is, for any $\varepsilon > 0$ there exists a compact subset K of $D = M$ such that

$$\int_{D \setminus K} H(x, z) \tilde{\rho}(|z|)^{-2} H(z, y) \tilde{\rho}(|z|)^n dz \leq \varepsilon H(x, y), \quad x, y \in D \setminus K,$$

where dz is the Lebesgue measure on $\mathbf{R}^n$, and $H(x, z)$ is the Green function of $\mathcal{L}$ on D with respect to the measure $\tilde{\rho}(|z|)^n dz$. Denote by $G(x, z)$ the Green function of L on D with respect to the measure dz . Since $\mathcal{L} = \tilde{\rho}(|x|)^{-2} L$, we have

$$H(x, z) = G(x, z) \tilde{\rho}(|z|)^{2-n}$$

Thus

$$\int_{D \setminus K} G(x, z) \tilde{\rho}(|z|)^{(2-n)-2} G(z, y) \tilde{\rho}(|y|)^{2-n} \tilde{\rho}(|z|)^n dz \leq \varepsilon G(x, y) \tilde{\rho}(|y|)^{2-n}$$

for any $x, y \in D \setminus K$. Hence 1 is a small perturbation of L on $\mathbf{R}^n$. $\square$

Remark. A sufficient condition for (7.2) is the following: ρ is a positive differentiable function on $[0, \infty)$ satisfying

$$0 \leq \rho'(r) \rho(r)^{-2} \leq C, \quad r \geq 0, \quad (7.6)$$

for some positive constant C . Indeed, from (7.6) we have

$$X(\delta) \equiv \rho \left(r + \frac{\delta}{\rho(r)} \right) \rho(r)^{-1} \leq \exp[C\delta X(\delta)], \quad r \geq 0, \quad \delta > 0.$$

Put $\delta = (2Ce)^{-1}$, and let $\gamma \in (1, e)$ be the solution of the equation

$$\exp[X/2e] = X.$$

Then we get $1 \leq X(\delta) \leq \gamma$. Thus (7.2) holds with $c = \min(\delta, 1/\gamma)$.

The condition (7.3) is sharp, since Theorem 6.2 of [17] yields the following uniqueness theorem.

Theorem 7.2 Suppose that there exists a positive continuous increasing function ρ on $[0, \infty)$ such that

$$|V(x)| \leq \rho(|x|)^2, \quad x \in \mathbf{R}^n. \quad (7.7)$$

Assume that

$$\int_1^\infty \frac{dr}{\rho(r)} = \infty. \quad (7.8)$$

Then [UP] holds. Thus Fact AT holds true.

7.2. Throughout this subsection we assume that D is a bounded domain of $\mathbf{R}^n$. Let L be an elliptic operator on D of the form

$$L = \frac{1}{w(x)} L_0,$$

where w is a positive measurable function on D such that $w, w^{-1} \in L_{\text{loc}}^\infty(D)$.

Theorem 7.3 Let D be a Lipschitz domain. Suppose that there exists a positive function ψ on $(0, \infty)$ such that $s^2\psi(s)$ is increasing and

$$w(x) \leq \psi(\delta_D(x)), \quad x \in D, \quad (7.9)$$

where $\delta_D(x) = \text{dist}(x, \partial D)$. Assume that

$$\int_0^1 s \psi(s) ds < \infty. \quad (7.10)$$

Then 1 is a small perturbation of L on D . Thus Theorem 1.3 holds true.

Remark. (i) The first assertion of this theorem is implicitly shown in [17] (see Theorem 7.11 and Remark 7.12 (ii) there).

(ii) The Lipschitz regularity of the domain D is assumed only for the Hardy inequality to hold for any function in $C_0^\infty(D)$. Thus, for this theorem to hold, it suffices to assume (for example) that D is uniformly Δ -regular John domain or a simply connected domain of $\mathbf{R}^2$ (see [3], [4]).

Proof of Theorem 7.3 For $x \in D$, put

$$D_x = \left\{ y \in D; |x - y| < \frac{\delta_D(x)}{2} \right\}.$$

Then

$$\frac{1}{2} \delta_D(x) \leq \delta_D(y) \leq \frac{3}{2} \delta_D(x), \quad y \in D_x.$$

Thus

$$\begin{aligned} \delta_D(x)^2 w(y) &\leq 4 \delta_D(y)^2 \psi(\delta_D(y)) \\ &\leq 4 \left(\frac{3}{2} \delta_D(x) \right)^2 \psi\left(\frac{3}{2} \delta_D(x) \right). \end{aligned}$$

Put $\Psi(s) = 9s^2\psi((3/2)s)$. Then $\Psi(s)$ is increasing, and satisfies

$$\delta_D(x)^2 \left(\sup_{y \in D_x} w(y) \right) \leq \Psi(\delta_D(x)), \quad \int_0^1 \frac{\Psi(s)}{s} ds < \infty.$$

Hence, by virtue of Proposition 9.2, Theorem 9.1' and Corollary 6.1 of [4], w is a small perturbation of L_0 on D . This implies that 1 is a small perturbation of L on D . $\square$

The condition (7.10) is sharp, since Theorem 7.8 and Lemma 7.6 of [17] yield the following uniqueness theorem.

Theorem 7.4 Suppose that there exists a positive continuous increasing function ψ on $(0, \infty)$ such that

$$c\psi(\delta_D(x)) \leq w(x) \leq \psi(\delta_D(x)), \quad x \in D \quad (7.11)$$

for some positive constant c , and

$$\nu \leq \frac{\psi(\eta s)}{\psi(s)} \leq \nu^{-1}, \quad s > 0, \quad \frac{1}{2} \leq \eta \leq 2, \quad (7.12)$$

for some positive constant ν . Assume

$$\int_0^1 \left[\psi(s) \left(\inf_{s \leq r \leq 1} r^2 \psi(r) \right) \right]^{\frac{1}{2}} ds = \infty. \quad (7.13)$$

Then [UP] holds. Thus Fact AT holds true.

References

- [1] H. Aikawa, *Norm estimate of Green operator, perturbation of Green function and integrability of superharmonic functions*, Math. Ann. **312** (1998), 289–318.
- [2] H. Aikawa and M. Murata, *Generalized Cranston-McConnell inequalities and Martin boundaries of unbounded domains*, J. Analyse Math. **69** (1996), 137–152.
- [3] A. Ancona, *On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^N$* , J. London Math. Soc. **34** (1986), 274–290.
- [4] A. Ancona, *First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains*, J. Analyse Math. **72** (1997), 45–92.
- [5] A. Ancona and J. C. Taylor, *Some remarks on Widder's theorem and uniqueness of isolated singularities for parabolic equations*, in Partial Differential Equations with Minimal Smoothness and Applications, (Dahlberg et al., eds.) Springer, New York, 1992, 15–23.
- [6] D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Sci. Norm. Sup. Pisa **22** (1968), 607–694.

- [7] R. Bañuelos, *Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators*, J. Funct. Anal. **100** (1991), 181–206.
- [8] R. Bañuelos, *Lifetime of conditioned Brownian motion and related estimates for heat kernels, eigenfunctions and eigenvalues in Euclidean domains*, MSRI Lectures, 1998 (<http://www.msri.org/lectures>).
- [9] R. Bañuelos and B. Davis, *Heat kernel, eigenfunctions, and conditioned Brownian motion in planar domains*, J. Funct. Anal. **84** (1989), 188–200.
- [10] R. Bañuelos and B. Davis, *A geometric characterization of intrinsic ultracontractivity for planar domains with boundaries given by the graphs of functions*, Indiana Univ. Math. J. **41** (1992), 885–913.
- [11] R. F. Bass and K. Burdzy, *Lifetimes of conditioned diffusions*, Probab. Theory Relat. Fields, **91** (1992), 405–443.
- [12] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, 1989.
- [13] E. B. Davies and B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and the Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), 335–395.
- [14] A. Grigor'yan and W. Hansen, *A Liouville property for Schrödinger operators*, Math. Ann. **312** (1998), 659–716.
- [15] L. L. Helms, *Introduction to Potential Theory*, Wiley and Sons, New York, 1969.
- [16] K. Ishige, *On the behavior of the solutions of degenerate parabolic equations*, Nagoya Math. J. **155** (1999), 1–26.
- [17] K. Ishige and M. Murata, *Uniqueness of nonnegative solutions of the Cauchy problem for parabolic equations on manifolds or domains*, Ann. Scuola Norm. Sup. Pisa, **30** (2001), 171–223.
- [18] K. Janssen, *Martin boundary and H^p -theory of harmonic spaces*, in Seminar on Potential Theory II, Lecture Notes in Math. Vol. **226**, Springer, Berlin, 1971, 102–151.

- [19] A. Koranyi and J. C. Taylor, *Minimal solutions of the heat equation and uniqueness of the positive Cauchy problem on homogeneous spaces*, Proc. Amer. Math. Soc. **94** (1985), 273–278.
- [20] V. Lin and Y. Pinchover, *Manifolds with group actions and elliptic operators* Memoirs Amer. Math. Soc. Vol. 112, Issue 540, 1994.
- [21] F.-Y. Maeda, *Martin boundary of a harmonic space with adjoint structure and its applications*, Hiroshima Math. J. **21** (1991), 163–186.
- [22] B. Mair and J. C. Taylor, *Integral representation of positive solutions of the heat equation*, in Lecture Notes in Math. Vol. **1096**, Springer, Berlin, 1984, 419–433.
- [23] P. J. Méndez-Hernández, *Toward a geometrical characterization of intrinsic ultracontractivity of the Dirichlet Laplacian*, Michigan Math. J. **47** (2000), 79–99.
- [24] P. J. Méndez-Hernández and K. Wong, *Intrinsic ultracontractivity for Schrödinger operators with mixed boundary conditions*, Potential Anal. **24** (2006), no. 4, 333–355.
- [25] M. Murata, *Uniform restricted parabolic Harnack inequality, separation principle, and ultracontractivity for parabolic equations*, in Functional Analysis and Related Topics, 1991, Lecture Notes in Math. Vol. **1540**, Springer, Berlin, 1993, 277–288.
- [26] M. Murata, *Sufficient condition for non-uniqueness of the positive Cauchy problem for parabolic equations*, in Spectral and Scattering Theory and Applications, Advanced Studies in Pure Math., Vol. **23**, Kinokuniya, Tokyo, 1994, 275–282.
- [27] M. Murata, *Uniqueness and non-uniqueness of the positive Cauchy problem for the heat equation on Riemannian manifolds*, Proc. Amer. Math. Soc. **123** (1995), 1923–1932.
- [28] M. Murata, *Non-uniqueness of the positive Cauchy problem for parabolic equations*, J. Diff. Eq. **123** (1995), 343–387.
- [29] M. Murata, *Non-uniqueness of the positive Dirichlet problem for parabolic equations in cylinders*, J. Func. Anal. **135** (1996), 456–487.

- [30] M. Murata, *Semismall perturbations in the Martin theory for elliptic equations*, Israel J. Math. **102** (1997), 29–60.
- [31] M. Murata, *Martin boundaries of elliptic skew products, semismall perturbations, and fundamental solutions of parabolic equations*, J. Funct. Anal. **194** (2002), 53–141.
- [32] M. Murata, *Heat escape*, Math. Ann. **327** (2003), 203–226.
- [33] M. Murata, *Uniqueness theorems for parabolic equations and Martin boundaries for elliptic equations in skew product form*, J. Math. Soc. Japan, **57** (2005), 387–413.
- [34] M. Murata, *Integral representations of nonnegative solutions for parabolic equations and elliptic Martin boundaries*, J. Funct. Anal. **245** (2007), 177–212.
- [35] R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand, New York, 1966.
- [36] Y. Pinchover, *Representation theorems for positive solutions of parabolic equations*, Proc. Amer. Math. Soc. **104** (1988), 507–515.
- [37] Y. Pinchover, *Criticality and ground states for second-order elliptic equations*, J. Diff. Eq. **80** (1989), 237–250.
- [38] Y. Pinchover, *Maximum and anti-maximum principles and eigenfunction estimates via perturbation theory of positive solutions of elliptic equations*, Math. Ann. **314** (1999), 555–590.
- [39] Y. Pinchover, *Large time behavior of the heat kernel*, J. Funct. Anal. **206** (2004), 191–209.
- [40] R. G. Pinsky, *Positive Harmonic Functions and Diffusion*, Cambridge Univ. Press, Cambridge, 1995.
- [41] R. Smits, *Necessary and sufficient conditions for finiteness of lifetimes*, Potential Theory **5** (1996), 513–521.
- [42] M. Tomisaki, *Intrinsic ultracontractivity and small perturbation for one dimensional generalized diffusion operators*, J. Funct. Anal. **251** (2007), 289–324.