論文 / 著書情報
Article / Book Information

題目(和文)	3次元多様体のライデマイスタートージョンと基本群の表現空間
Title(English)	Reidemeister torsion of 3-manifolds and the spaces of representations of fundamental groups
著者(和文)	北野晃朗
Author(English)	TERUAKI KITANO
出典(和文)	学位:博士（理学）, 学位授与機関:東京工業大学, 報告番号:甲第2696号, 授与年月日:1994年3月26日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor of Science, Conferring organization: Tokyo Institute of Technology, Report number:甲第2696号, Conferred date:1994/3/26, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Reidemeister torsion of 3-manifolds and the spaces of representations of the fundamental groups

(3次元多様体のライデマイスタートージョンと基本群の表現空間)

Teruaki KITANO

北野 晃朗

Department of Mathematics

Tokyo Institute of Technology

February 1994

To My Parents

CONTENTS

Introduction

In this paper, we study the Reidemeister torsion in the context of low-dimensional topology. Reidemeister torsion is originally defined by Reidemeister, Franz, and de Rham. It is a piecewise-linear invariant of certain finite cell complexes with respect to a representation of its fundamental group in certain Lie groups. In this paper, we deal the Reidemeister torsion of 3-manifolds for $SL(2;\mathbf{C})$ and $SU(2)$-representations. We may consider this invariant as a function over the space of the representations of fundamental groups. In this case the space of representations of fundamental groups is related to the geometric structures and topological invariants of 3-manifolds, for example, Casson's invariant and Witten's invariants.

The actual definition of the Reidemeister torsion will be given in chapter 1, where we give necessary definitions and propositions. As an application of the Reidemeister torsion, Reidemeister and Franz obtained the complete piecewise-linear classification of the lens spaces in all dimensions. These examples are explained in chapter 1. The concept of the torsion is well-explained in [3] and [13].

In chapters 2 and 3, we deal the Reidemeister torsion of 3-manifolds for the $SL(2;\mathbf{C})$-representations. Recently Johnson [5] derived an explicit formula for the Reidemeister torsion of Brieskorn homology 3-spheres for $SL(2;\mathbf{C})$-irreducible representations as follows. Let M_n be a 3-manifold obtained by performing the $\frac{1}{n}$-Dehn surgery on the torus (p,q)-knot. The manifold M_n is diffeomorphic to the Brieskorn homology 3-sphere $\Sigma(p,q,pqn\pm 1)$. In this case, the fundamental group $\pi_1 M_n$ admits a presentation as follows.

$$\pi_1 M_n = \langle x, y \mid x^p = y^q, ml^n = 1\rangle$$

where m is a meridian of the torus knot which is a word of x and y and l is similarly a londitude. Then Johnson proved the following theorem.

THEOREM(JOHNSON). *The distinct conjugacy classes of the $SL(2;\mathbf{C})$-irreducible representations of $\pi_1 M_n$ are given by $\rho_{(a,b,k)}$ such that*

(1) $0 < a < p, 0 < b < q, a \equiv b \mod 2.$

(2) $0 < k < N = |pqn+1|, k \equiv na \mod 2.$

(3) $tr\rho_{(a,b,k)}(x) = 2\cos\dfrac{\pi a}{p},$

(4) $tr\rho_{(a,b,k)}(y) = 2\cos\dfrac{\pi b}{q},$

(5) $tr\rho_{(a,b,k)}(m) = 2\cos\dfrac{\pi k}{N}.$

In this case the Reidemeister torsion $\tau_{(a,b,k)}$ *for* $\rho_{(a,b,k)}$ *is given by*

$$\tau_{(a,b,k)} = \begin{cases} 2(1-\cos\frac{\pi a}{p})(1-\cos\frac{\pi b}{q})(1+\cos\frac{\pi kpq}{N}) & a \equiv b \equiv 1, k \equiv n \quad mod\ 2 \\ 0 & a \equiv b \equiv 0 \text{ or } k \not\equiv n \quad mod\ 2. \end{cases}$$

His methods can be applied to the investigation of more general Seifert fibered spaces and give a way to compute the Reidemeister torsion of them. The main result in chapter 2 is the following theorem. Let M^3 denote the orientable Seifert fibered space given by the following Seifert index ;

$$M = \{b, (\varepsilon, g); (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}.$$

THEOREM. *Let* $\rho : \pi_1 M \to SL(2; \mathbf{C})$ *be an irreducible representation. Then the Reidemeister torsion* $\tau(M; V_\rho)$ *is given by*

$$\tau(M; V_\rho) = \begin{cases} 0 & \text{if } H = I \\ 2^{4-m-4g} \prod_{i=1}^{m} (1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}) & \text{if } H \neq I, \varepsilon = o \\ (2 - 2\cos\frac{s\pi}{N+1})^{4-m-2g} \prod_{i=1}^{m} (1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}) & \text{if } H \neq I, \varepsilon = n \end{cases}$$

where

(1) $H = \rho(h)$,

(2) h *is a representative element of generic fiber in* $\pi_1 M$,

(3) $\rho_i, \nu_i \in \mathbf{Z}$ *such that* $\begin{vmatrix} \alpha_i & \rho_i \\ \beta_i & \nu_i \end{vmatrix} = -1$ *and* $0 < \rho_i < \alpha_i$,

(4) $k_i(\rho) \in \mathbf{Z}$ *such that* $0 \leq k_i \leq \alpha_i$, *and* $k_i(\rho) \equiv \beta_i$ *mod 2.*

(5) $N = \dfrac{\beta_1}{\alpha_1} + \cdots \dfrac{\beta_m}{\alpha_m}$,

(6) $s \in \mathbf{Z}$ *such that* $0 \leq s \leq 2N + 2$.

This is an extension of Johnson's result mentioned above. In particular the sets of values of the Reidemeister torsion are again finite subsets in **R**. It follows that it has no continuous variations, although the dimension of the space of the representations of the fundamental group of these manifolds is generally positive.

In chapter 3, we consider the following problem.

PROBLEM. *Does there exist a closed 3-manifold with continuous variations of the Reidemeister torsion for* $SL(2; \mathbf{C})$*-representations ?*

In order to attack this problem, we first need to investigate the spaces of the $SL(2; \mathbf{C})$-representations for given manifolds. Applying the method, due to Riley [16], to the Wirtinger presentation of the figure-eight knot, we determine the space of representations of the fundamental group of the figure-eight knot exterior. Then by the method of Johnson, we obtain an explicit formula of the Reidemeister torsion of this manifold. The formula shows that it has continuous variations. The main result in chapter 3 is the following.

THEOREM. *Let $K \subset S^3$ denote the figure-eight knot, and E its exterior, that is, the complement of an open tubular neighborhood of K. Let M denote the double $E \cup_{id} E$ of E. Then the set of values of the Reidemeister torsion $\tau(M; V_\rho)$ of M for $SL(2;\mathbf{C})$-representations is the set of all nonzero complex numbers. Therefore $\tau(M; V_\rho)$ has continuous variations.*

In chapter 4, we deal the Reidemeister torsion for $SU(2)$-representations. In particular we study the connection between the Casson's invariant and the Reidemeister torsion.

In 1985 Casson defined a topological invariant for homology 3-spheres. Let M^3 be a closed, oriented homology 3-sphere. By the classical theorem of Heegaard, M can be decomposed into two handlebodies of a certain genus g. We respectively denote these handlebodies by N_1 and N_2 and the common boundary surface by Σ_g. Let R_g be the space of SU(2)-irreducible representations of $\pi_1\Sigma_g$. Similarly R^1, R^2 and R_M are defined as the space of SU(2)-irreducible representations of each fundamental group of N_1, N_2, or M. Then SU(2) acts on these spaces of representations by conjugation. We denote the orbit spaces respectively by $\hat{R}_g, \hat{R}^1, \hat{R}^2$ and $\hat{R}_M$. It is easy to see that $\hat{R}_M$ coincides with the intersection of $\hat{R}^1$ and $\hat{R}^2$ by van Kampen's theorem, when we consider $\hat{R}^1$ and $\hat{R}^2$ as a submanifold in $\hat{R}_g$.

Casson's invariant $\lambda(M)$ is defined to be the half of the algebraic intersection number of $\hat{R}^1$ and $\hat{R}^2$ in $\hat{R}_g$. Casson's invariant is explained in [1].

Later Johnson [5] defined a natural volume form on $\hat{R}_g$ in the above situations. Let V be an n-dimensional vector space over $\mathbf{R}$. We denote the n-dimensional exterior product $\bigwedge^n V$ by $\det V$. A volume on V is defined to be a nonzero element of $\det V$. Johnson's theorem is the following.

THEOREM(JOHNSON).

(1) *There are natural volumes $\hat{v}_1, \hat{v}_2$ and $\hat{v}_g$ on the tangent spaces $T_{[\rho]}\hat{R}^1, T_{[\rho]}\hat{R}^2$ and $T_{[\rho]}\hat{R}_g$ for $^\forall[\rho] \in \hat{R}^1 \cap \hat{R}^2 = \hat{R}_M$.*

(2) *Suppose $\hat{R}^1$ and $\hat{R}^2$ have a transverse intersection at $[\rho]$. A nonzero real number $t_{[\rho]}$ is defined by*

$$\hat{v}_1 \wedge \hat{v}_2 = t_{[\rho]} \cdot \hat{v}_g.$$

Then a sign of $t_{[\rho]}$ coincides with the sign of the Casson's one. In particular if $\hat{R}^1$ and $\hat{R}^2$ are transversal for $^\forall[\rho] \in \hat{R}^1 \cap \hat{R}^2$, then the following holds,

$$2\lambda(M) = \sum_{[\rho]} sign(t_{[\rho]}).$$

(3) *We consider the Lie algebra $su(2)$ as a $\pi_1 M$-module using a representation ρ and the adjoint representation of SU(2). We denote the $\pi_1 M$-module $su(2)$ by*

$su(2)_\rho$. Then the Reidemeister torsion $\tau_\rho(M)$ of M with $su(2)_\rho$-coefficients is defined. In this case, the following holds up to sign ;

$$t_{[\rho]} = \tau_\rho(M).$$

This result is a very interesting but it only points out a vague connection between Casson's invariant and the Reidemeister torsion. We would like to understand geometrically the meaning of this connection. Our main result in chapter 4 is the following theorem.

THEOREM. *Johnson's volumes $\hat{v}_1, \hat{v}_2$ and $\hat{v}_g$ are respectively the Reidemeister torsion of N_1, N_2, and Σ_g up to sign, that is,*

$$\hat{v}_1 = \epsilon_1 \tau_\rho(N_1) \in det H^1(N_1; su(2)_\rho),$$

$$\hat{v}_2 = \epsilon_2 \tau_\rho(N_2) \in det H^1(N_2; su(2)_\rho),$$

$$\hat{v}_g = \epsilon_g \tau_\rho(\Sigma_g) \in det H^1(\Sigma_g; su(2)_\rho)$$

where $\epsilon_1, \epsilon_2, \epsilon_g \in \{\pm 1\}$.

By this theorem, we can consider the relation

$$\hat{v}_1 \wedge \hat{v}_2 = t_{[\rho]} \cdot \hat{v}_g$$

as a relation with the Reidemeister torsion. Hence we have the following relation of the Reidemeister torsion of these manifolds, up to sign,

$$\tau_\rho(N_1)\tau_\rho(N_2) = \tau_\rho(M)\tau_\rho(\Sigma_g).$$

Now we have the following natural exact sequence which is derived from the Heegaard decomposition

$$M = N_1 \cup_{\Sigma_g} N_2,$$

$$0 \to C_*(\Sigma_g; su(2)_\rho) \to C_*(N_1; su(2)_\rho) \oplus C_*(N_2; su(2)_\rho) \to C_*(M; su(2)_\rho) \to 0.$$

Then we get the following equality by well-known fact in the theory of the Reidemeister torsion,

$$\tau_\rho(N_1)\tau_\rho(N_2) = \tau_\rho(M)\tau_\rho(\Sigma_g)\tau(\mathbf{H}),$$

which holds up to sign where the chain complex $\mathbf{H}$ is the following homology exact sequence ;

$$\mathbf{H} : 0 \to H_1(\Sigma_g; su(2)_\rho) \to H_1(N_1; su(2)_\rho) \oplus H_1(N_2; su(2)_\rho) \to 0.$$

Suppose that a volume on $H_1(\Sigma_g; su(2)_\rho)$ is given by the product of the volumes on $H_1(N_1; su(2)_\rho)$ and $H_1(N_2; su(2)_\rho)$. Then it holds that

$$\tau(\mathbf{H}) = 1 \text{ up to sign.}$$

Hence we have the following, up to sign,

$$\tau_\rho(N_1)\tau_\rho(N_2) = \tau_\rho(M)\tau_\rho(\Sigma_g).$$

Therefore we have naturally the relation

$$\hat{v}_1 \wedge \hat{v}_2 = t_{[\rho]} \cdot \hat{v}_g$$

from a well-known relation of torsion invariants.

In chapter 5, we deal the Reidemeister torsion of knots in S^3. In 1928, Alexander introduced a new knot invariant which is now called the Alexander polynomial. In 1962, Milnor [12] proved that the Alexander polynomial of a link in S^3 is equal to a certain Reidemeister torsion of the exterior of the link. As an application of his interpretation, he derived the well known symmetry of the coefficients of the Alexander polynomial. The connection between the Alexander polynomial and the Reidemeister torsion is explained in [17].

In 1992, Wada [18] defined the twisted Alexander polynomial for finitely presentable groups. We consider the case of the group of a knot. Let $K \subset S^3$ be a knot and E its exterior. We denote the fundamental group $\pi_1 E$ by Γ and the canonical abelianization of Γ by

$$\alpha : \Gamma \to T = < t > .$$

Then we will assign a Laurent polynomial $\Delta_{K,\rho}(t)$ with R-coefficients where R is a unique factorization domain to each linear representation $\rho : \Gamma \to GL(n; R)$. We call it the twisted Alexander polynomial of K associated to ρ. For simplicity, we suppose that R is the real number field $\mathbf{R}$ and the image of ρ is included in $SL(n; \mathbf{R})$. It is a generalization of the Alexander polynomial $\Delta_K(t)$ of K in the following sense. The Alexander polynomial $\Delta_K(t)$ of K is written as

$$\Delta_K(t) = (1-t)\Delta_{K,\mathbf{1}}(t)$$

where $\mathbf{1} : \Gamma \to SL(1; \mathbf{R}) = \{1\}$ is the 1-dimensional trivial representation of Γ.

We consider the following problem, the analogy of the Milnor's theorem.

PROBLEM. *Can we consider the twisted Alexander polynomial of K as a certain Reidemeister torsion of its exterior E ?*

For the representation $\rho : \Gamma \to SL(n; \mathbf{R})$, we define the representation

$$\rho \otimes \alpha : \Gamma \to GL(n; \mathbf{R}(t))$$

by

$$(\rho \otimes \alpha)(x) = \rho(x)\alpha(x) \text{ for } {}^\forall x \in \Gamma.$$

Then our main theorem in chapter 5 is the following.

THEOREM. *The twisted Alexander polynomial* $\Delta_{K,\rho}(t)$ *associated to* ρ *is the Reidemeister torsion* $\tau_{\rho\otimes\alpha}E$ *for* $\rho \otimes \alpha$ *; that is,*

$$\Delta_{K,\rho}(t) = \tau_{\rho\otimes\alpha}E.$$

As an application of this interpretation, we obtain the symmetry of the twisted Alexander polynomial in the following sense.

THEOREM. *If* ρ *is equivalent to an* $SO(n)$*-representation, then we have*

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1})$$

up to a factor ϵt^{nk} *where* $\epsilon \in \{\pm 1\}$ *and* $k \in \mathbf{Z}$.

Acknowledgements

The author would like to express his sincere gratitude to Professor Shigeyuki Morita and Professor Yoshihiko Mitsumatsu for their continuous encouragement. He should be grateful to Professor Sadayoshi Kojima and Professor Masaaki Wada for their useful communications and correspondence. He also thanks all peoples of "Morita Seminar" at Tokyo Institute of Technology. Finally he wish to thank his wife, Noriko Kitano for her supporting.

Chapter 1

Definition of Reidemeister torsion

§1. Reidemeister torsion

Let us describe the definition of the Reidemeister torsion. See [3], [5], and [13].

Let V be an n-dimensional vector space over a field $\mathbf{F}$. Let $\mathbf{b}=(b_1,\ldots,b_n)$ and $\mathbf{c}=(c_1,\ldots,c_n)$ be two different bases for V. Setting

$$c_i = \sum_{j=1}^{n} a_{ij} b_j \text{ for } {}^{\forall} i \in \{1,\ldots,n\},$$

we obtain a nonsingular matrix $A = (a_{ij}) \in M(n;\mathbf{F})$ and denote the determinant of A by $[\mathbf{b}/\mathbf{c}]$.

Suppose

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is a chain complex of finite dimensional vector spaces over $\mathbf{F}$.

Let $Z_q(C_*) \subset C_q$ denote the kernel of ∂_q and $B_q(C_*) \subset C_q$ the image of ∂_{q+1}. The q-th homology group $H_q(C_*)$ of C_* is defined by the quotient vector space $Z_q(C_*)/B_q(C_*)$. For ${}^{\forall} q \in \{0.\ldots,m\}$, assume the preferred basis $\mathbf{c}_q$ for $C_q(C_*)$ and $\mathbf{h}_q$ for $H_q(C_*)$. Choose any basis $\mathbf{b}_q$ for $B_q(C_*)$ and its lift $\tilde{\mathbf{b}}_q$ in C_{q+1}. Furthermore we choose the representative element $\tilde{\mathbf{h}}_q$ of $\mathbf{h}_q$ in $Z_q(C_*)$.

Since

$$0 \longrightarrow B_q(C_*) \longrightarrow Z_q(C_*) \longrightarrow H_q(C_*) \longrightarrow 0$$

is an exact sequence, the vectors $(\mathbf{b}_q, \tilde{\mathbf{h}}_q)$ is a basis for $Z_q(C_*)$. Since the next sequence

$$0 \to Z_q(C_*) \to C_q(C_*) \to B_q(C_*) \to 0$$

is similarly exact, the vectors $(\mathbf{b}_q, \tilde{\mathbf{h}}_q, \tilde{\mathbf{b}}_q)$ is a basis for $C_q(C_*)$. It is easily shown that $[\mathbf{b}_q, \tilde{\mathbf{h}}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]$ depends only on $\mathbf{b}_q, \mathbf{h}_q, \mathbf{b}_{q-1}$ and does not depend on the choices of liftings. Hence we simply denote it by $[\mathbf{b}_q, \mathbf{h}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]$.

DEFINITION 1.1. *The torsion $\tau(C_*)$ of the chain complex C_* is defined by the alternating product*

$$\prod_{q=0}^{m}[\mathbf{b}_q,\mathbf{h}_q,\mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^q}.$$

LEMMA 1.2. *If $\{\mathbf{b}'_0,\ldots,\mathbf{b}_{m-1}\}$ are the other bases, then we have*

$$\prod_{q=0}^{m}[\mathbf{b}_q,\mathbf{h}_q,\mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^q}=\prod_{q=0}^{m}[\mathbf{b}'_q,\mathbf{h}_q,\mathbf{b}'_{q-1}/\mathbf{c}_q]^{(-1)^q}.$$

Proof.

It is obvious that

$$[\mathbf{b}'_q,\mathbf{h}_q,\mathbf{b}'_{q-1}/\mathbf{c}_q]=[\mathbf{b}'_q/\mathbf{b}_q][\mathbf{b}'_{q-1}/\mathbf{b}_q][\mathbf{b}_q,\mathbf{h}_q,\mathbf{b}_{q-1}/\mathbf{c}_q].$$

Then we have

$$\begin{aligned}\prod_{q=0}^{m}[\mathbf{b}'_q,\mathbf{h}_q,\mathbf{b}'_{q-1}/\mathbf{c}_q]^{(-1)^q}&=\prod_{q=0}^{m}[\mathbf{b}_q,\mathbf{h}_q,\mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^q}[\mathbf{b}'_q/\mathbf{b}_q]^{(-1)^q}[\mathbf{b}'_{q-1}/\mathbf{b}_{q-1}]^{(-1)^q}\\&=\prod_{q=0}^{m}[\mathbf{b}_q,\mathbf{h}_q,\mathbf{b}_{q-1}]^{(-1)^q}.\end{aligned}$$

REMARK.

(1) *If the chain complex C_* is acyclic, then $\tau(C_*)$ depend on only the bases for chain modules.*

(2) *If C_* is not acyclic, the torsion $\tau(C_*)$ is defined to be zero. However we may consider $\tau(C_*)$ as a nonzero linear functional on $\bigotimes_{q=0}^{m} detH_q(C_*)^{(-1)^q}$ where $detH_q(C_*)^{-1}$ is the dual vector space of $detH_q(C_*)$. In chapter 4, we consider this type torsion as a volume form on the spaces of the representations.*

Now we apply the torsion to the geometric situations. Let X be a finite cell complex and $\tilde{X}$ the universal covering of X with $\pi_1 X$ acting as deck transformations. The chain complex $C_*(\tilde{X};\mathbf{Z})$ becomes a chain complex of free $\mathbf{Z}[\pi_1 X]$-modules. Let $\rho:\pi_1 X\to SL(n;\mathbf{F})$ be a representation. We may consider $V=\mathbf{F}^n$ as a $\pi_1 X$-module by using this representation ρ and denote it by V_ρ. We define the chain complex $C_*(X;V_\rho)$ by

$$C_*(\tilde{X};\mathbf{Z})\otimes_{\mathbf{Z}[\pi_1 X]}V_\rho.$$

Then we choose a preferred basis $\{\tilde{\sigma}_j\otimes e_i\}$ of $C_q(\tilde{X};V_\rho)$ for $^\forall q$ where $\{e_1,\ldots,e_n\}$ is some fixed basis of V and $\{\tilde{\sigma}_j\}$ is the preferred basis over $\mathbf{Z}[\pi_1 X]$ of $C_q(\tilde{X};\mathbf{Z})$ consisting of the lifts of the q-cells of X.

We consider the following case that $C_*(X;V_\rho)$ is acyclic, namely, all homology groups vanish : $H_*(X;V_\rho)=0$. Then we call the representation ρ the acyclic representation.

REMARK. *If the representation ρ is acyclic, then Euler characteristic of the local homology*

$$\chi(X; V_\rho) = 0.$$

Since

$$\chi(X; V_\rho) = \chi(K) \cdot n,$$

we have Euler number $\chi(X) = 0$.

DEFINITION 1.3. *Let $\rho : \pi_1 X \to SL(n; \mathbf{F})$ be an acyclic representation. Then the Reidemeister torsion $\tau(X; V_\rho)$ is defined by the torsion of $C_*(X; V_\rho)$.*

REMARK.

(1) *The Reidemeister torsion $\tau(X; V_\rho)$ depends on several choices.*

  (a) *The choices of lifts of the cells to $\tilde{X}$.*

  (b) *The orientation and the ordering of cells of X.*

  (c) *The choices of a basis of V.*

We restrict the case of oriented closed manifolds and then can prove the well-definedness as follows.

(a) If we change the lift $\tilde{\sigma}_i$ to $x \cdot \tilde{\sigma}_i$ where $x \in \pi_1 X$, then the original basis changes at $\{\tilde{\sigma}_i \otimes \mathbf{e}_1, \ldots, \tilde{\sigma}_i \otimes \mathbf{e}_n\}$. In this case the change of the Reidemeister torsion is the same as the change caused by the action of x on V. But we consider only $SL(n; \mathbf{F})$-representation, that is, $\det\rho(x) = 1$. Hence the indetermination due to lifting cells differently disappears.

(b) Johnson has proved the existence of the natural orientation and the ordering of cells of X. See [5] for details.

(c) The following lemma is straightforward.

LEMMA 1.4. *Let $\{\mathbf{e}'_i\}$ be another basis of V and $\{\mathbf{c}'_q\}$ the corresponding the new basis of C_q for ${}^\forall q$. Then*

$$[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}'_q] = C^{d_q}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]$$

where $C = det(\mathbf{e}'_1, \ldots, \mathbf{e}'_n)$ and $d_q = dimC_q$.

Apply this lemma to the following,

$$\begin{aligned}\prod_{q=0}^{m}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}'_q]^{(-1)^q} &= C^{d_0 - d_1 + \cdots + (-1)^m d_m}\prod_{q=0}^{m}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^q} \\ &= C^{\chi(X)}\prod_{q=0}^{m}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^q} \\ &= \prod_{q=0}^{m}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^q}.\end{aligned}$$

Hence $\tau(X;V_\rho)$ does not depend on the choices of a basis for V.

§2. Invariance under subdivisions

In this section we prove its invariance under subdivisions of the cell structure, up to $\mathbf{Z}_2$. Here $\mathbf{Z}_2 = \{\pm 1\}$ acts naturally on $\mathbf{F}$ as a natural reflection. This fact is based on two lemmas. See [11], and [13].

LEMMA 2.1. *Let* $0 \to C'_* \to C_* \to C''_* \to 0$ *be an exact sequence of n-dimensional chain complexes with preferred basis* $\{\mathbf{c}'_i\}, \{\mathbf{c}_i\}$ *and* $\{\mathbf{c}''_i\}$ *such that* $[\mathbf{c}'_i, \mathbf{c}''_i/\mathbf{c}_i] = 1$ *for* $^\forall i$. *Suppose two of complexes are acyclic. Then the third one is also acyclic and the Reidemeister torsions are all well-defined. In this case the next formula holds that*

$$\tau(C_*) = (-1)^{\sum_{i=0}^{n} \beta'_{i-1}\beta''_i} \tau(C'_*)\tau(C''_*)$$

where $\beta'_i = dim\partial C'_{i+1}$ *and* $\beta''_i = dim\partial C''_{i+1}$.

Proof. It is easy to show the acyclicity of the third complex from the long exact sequence of $0 \to C'_* \to C_* \to C''_* \to 0$.

We consider the next diagram for each i.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \partial C'_{i+1} & \longrightarrow & \partial C_{i+1} & \longrightarrow & \partial C''_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C'_i & \longrightarrow & C_i & \longrightarrow & C''_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \partial C'_i & \longrightarrow & \partial C_i & \longrightarrow & \partial C''_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We choose the bases $\mathbf{b}'_i$ in $\partial C'_{i+1}$ and $\mathbf{b}''_i$ in $\partial C''_{i+1}$ and make $\mathbf{b}_i = (\mathbf{b}'_i, \mathbf{b}''_i)$ for ∂C_{i+1}. We will show that

$$\tau(C'_*)\tau(C''_*)\tau(C_*)^{-1} = (-1)^{\sum_{i=0}^{n} \beta'_{i-1}\beta''_i}$$

Here

$$\tau(C'_*)\tau(C''_*)\tau(C_*)^{-1} = \prod_{i=0}^{n}[\mathbf{b}'_i, \mathbf{b}'_{i-1}/\mathbf{c}'_i]^{(-1)^i}[\mathbf{b}''_i, \mathbf{b}''_{i-1}/\mathbf{c}''_i]^{(-1)^i}[\mathbf{b}_i, \mathbf{b}_{i-1}/\mathbf{c}_i]^{(-1)^{i+1}}.$$

Because this value does not depend on the choices of $\mathbf{b}'_i$ and $\mathbf{b}''_i$, we may assume that

$$[\mathbf{b}'_i, \mathbf{b}'_{i-1}/\mathbf{c}'_i] = [\mathbf{b}''_i, \mathbf{b}''_{i-1}/\mathbf{c}''_i] = 1.$$

From the assumptions, we may choose identifications

$$\begin{aligned}
\partial C_{i+1} &\cong \partial C'_{i+1} \oplus \partial C''_{i+1}, \\
C_i &\cong C'_i \oplus C''_i, \\
\partial C_i &\cong \partial C'_i \oplus \partial C''_i, \\
C'_i &\cong \partial C'_{i+1} \oplus \partial C'_i, \\
C''_i &\cong \partial C''_{i+1} \oplus \partial C''_i.
\end{aligned}$$

Then we can identify

$$C_i \cong \partial C'_{i+1} \oplus \partial C'_i \oplus \partial C''_{i+1} \oplus \partial C''_i.$$

Now we get a basis for C_i,

$$\begin{aligned}
(\mathbf{b}'_i, \mathbf{b}'_{i-1}, \mathbf{b}''_i, \mathbf{b}''_{i-1}) &= (\mathbf{c}'_i, \mathbf{c}''_i) \\
&= \mathbf{c}_i.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
(\mathbf{b}'_i, \mathbf{b}'_{i-1}, \mathbf{b}''_i, \mathbf{b}''_{i-1}) &= (-1)^{\beta'_i\beta''_{i-1}}(\mathbf{b}'_i, \mathbf{b}''_i, \mathbf{b}'_{i-1}, \mathbf{b}''_{i-1}) \\
&= (-1)^{\beta'_i\beta''_{i-1}}(\mathbf{b}_i, \mathbf{b}_{i-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
[\mathbf{b}'_i, \mathbf{b}'_{i-1}/\mathbf{c}'_i][\mathbf{b}''_i\mathbf{b}''_{i-1}, /\mathbf{c}''_i][\mathbf{b}_i\mathbf{b}_{i-1}, /\mathbf{c}_i]^{-1} &= 1 \cdot 1 \cdot (-1)^{\beta'_{i-1}\beta''_i} \\
&= (-1)^{\beta'_{i-1}\beta''_i}.
\end{aligned}$$

Therefore we have

$$\tau(C'_*)\tau(C''_*)\tau(C_*)^{-1} = (-1)^{\sum_{i=0}^{n}\beta'_{i-1}\beta''_i}$$

Q.E.D.

LEMMA 2.2. *Let* (X, Y) *be a pair of cell complexes and* $(\tilde{X}, \tilde{Y})$ *a pair of universal covering complexes whose cell structures come from base complexes. If the fundamental group* $\pi_1 X$ *permutes the components of* $\tilde{X} - \tilde{Y}$ *as deck transformations and all relative homology groups vanish :* $H_*(K, L; V_\rho) = 0$, *then we have*

$$\tau(K, L; V_\rho) = \pm 1.$$

Proof. Let X_0 denote the union of Y with the one component of $\tilde{X} - \tilde{Y}$. Then the injection

$$C_*(\tilde{X}_0, \tilde{Y}; \mathbf{Z}) \otimes V \longrightarrow C_*(\tilde{X}, \tilde{Y}; \mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1 X]} V$$

is an isomorphism where $\tilde{X}_0$ is the lift of X_0. Thus the torsion $\tau(X, Y; V_\rho)$ is the image of the torsion invariant of $\tau(X_0, Y; V_1)$ where the subscript **1** denotes the trivial representation in V. But this is in turn the image of a corresponding invariant with the vector space V replaced by the ring $\mathbf{Z}$ of integers. Since the only units in $\mathbf{Z}$ are ± 1, it follows that

$$\tau(X, Y; V_\rho) = \pm 1.$$

Q.E.D.

PROPOSITION 2.3. *The Reidemeister torsion* $\tau(X; V_\rho)$ *modulo* $\mathbf{Z}_2$ *is invariant under subdivisions.*

Proof. We fix an universal covering $\tilde{X}$ of X whose cell structure is coming from X. Choose s sequence

$$\phi = X_0 \subset X_1 \subset \cdots \subset X_r = X$$

of subcomplexes of X so that each $X_{i+1} - X_i$ consists a single cell. Let $\tilde{X}_i$ denote the preimage of X_i in $\tilde{X}$. Let I denote the unit interval considered as a CW-complex with $\pi_1 X$ acting trivially. For a given subdivision X', let $\tilde{X}'$ denote the subdivided complex of $\tilde{X}$ corresponding to X' and $\tilde{Y}_i$ the CW complex formed from $\tilde{X} \times I$ by subdividing $\tilde{X}_i \times \{1\}$. Since $\tilde{Y}_0 = \tilde{X} \times \{0\}$,

$$\tau(X; V_\rho) = \tau(Y_0; V_\rho) \text{ where } Y_0 = \tilde{Y}_0 / \pi_1 X.$$

Because each pair $(\tilde{Y}_{i+1}, \tilde{Y}_i)$ clearly satisfies the conditions of Lemma 2.2,

$$\tau(Y_0; V_\rho) \equiv \tau(Y_1; V_\rho) \equiv \cdots \equiv \tau(Y_r; V_\rho) \bmod \mathbf{Z}_2$$

where $Y_i = \tilde{Y}_0 / \pi_1 X$ and $Y_r = Y$.

Thus

$$\tau(X; V_\rho) \equiv \tau(Y; V_\rho) \bmod \mathbf{Z}_2.$$

Now let $\tilde{M}_i$ denote the subcomplex of $\tilde{Y}$ formed from $(\tilde{X}' \times \{1\}) \cup (\tilde{X}_i \times I)$. Then by a similar argument, $\tilde{X}' \times \{1\} = \tilde{M}_0$, we have

$$\tau(X'; V_\rho) = \tau(M_0; V_\rho).$$

In this case,

$$\tau(M_0; V_\rho) \equiv \tau(M_1; V_\rho) \equiv \cdots \equiv \tau(M_r; V_\rho) \qquad \bmod \mathbf{Z}_2.$$

Since $M_r = Y$.

$$\tau(X'; V_\rho) = \tau(Y; V_\rho).$$

Therefore we have

$$\tau(X; V_\rho) \equiv \tau(X'; V_\rho) \bmod \mathbf{Z}_2.$$

Q.E.D.

REMARK. *By the proof of the above lemmas and proposition, if dimension of V is even, then the Reidemeister torsion is really invariant under subdivisions and it is a combinatorial invariant of p.l. manifolds.*

§3. Examples of Reidemeister torsion

In this section, we compute the example of the Reidemeister torsion for the $SL(2;\mathbf{C})$-representations. For simplicity, we have some conventions.

(1) When a representation ρ are given, for $^\forall x$, we denote its image $\rho(x)$ by the corresponding capital letter X.

(2) We denote the 2-dimensional complex vector space by V and the canonical basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ of V by $\{\mathbf{e}_1, \mathbf{e}_2\}$.

First we consider the 1-dimensional sphere S^1. This can be considered as a cell complex with a 0-cell p and 1-cell x. Then we may consider that $\pi_1 S^1 \cong \mathbf{Z}$ generated by x.

PROPOSITION 3.1. *Let $\rho : \pi_1 S^1 \to SL(2;\mathbf{C})$ be a representation. All homology groups vanish : $H_*(S; V_\rho) = 0$ if and only if $det(X - I) \neq 0$. In this case the Reidemeister torsion of S^1 is given by*

$$\tau(S^1; V_\rho) = det(X - I).$$

Proof. The chain complex $C_*(S^1; V_\rho)$ is given by

$$0 \longrightarrow V \xrightarrow{\partial = X - I} V \longrightarrow 0.$$

Hence $C_*(S^1; V_\rho)$ is acyclic if and only if $\det(X - I) \neq 0$, that is, ρ is non parabolic. In this case by easy computation,

$$\begin{aligned}\tau(S^1; V_\rho) &= [\mathbf{b}_0/\mathbf{c}_1]^{-1}[\mathbf{b}_0/\mathbf{c}_0] \\ &= \det(X - I).\end{aligned}$$

Q.E.D.

Next we consider the 3-dimensional lens space. Let $p > q$ be relatively prime integers and π denote the cyclic group of order p with generator t. We consider the 3-dimensional sphere

$$S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbf{C}^2.$$

Then the group π acts on S^3 without fixed points by the rule

$$t \cdot (z_1, z_2) = (\omega z_1, \omega^q z_2)$$

where ω the complex number $\exp(\frac{2\pi\sqrt{-1}}{p})$. The quotient manifold S^3/π is the lens space denoted by $L(p, q)$. Note that $\pi_1 L(p, q)$ is isomorphic to π.

$L(p, q)$ can be considered as a cell complex with only 4 cells, $\{\bar{u}_0, \ldots, \bar{u}_3\}$. They are the image of the natural projection of the cells in S^3 as follows,

(0) 0-cell $u_0 = (1, 0)$,

(1) 1-cell $u_1 = \{(\exp(\theta\sqrt{-1}), 0) \mid 0 < \theta < \frac{2\pi}{p}\}$,

(2) 2-cell $u_2 = \{(z_1, \sqrt{1 - |z_1|^2}) \mid |z_1| < 1\}$,

(3) 3-cell $u_3 = \{(z_1, \exp(\theta\sqrt{-1})\sqrt{1 - |z_1|^2} \mid |z_1| < 1, 0 < \theta < \frac{2\pi}{p}\}$,

Thus $C_*(\tilde{L}(p, q); \mathbf{Z})$ is a free $\mathbf{Z}[\pi]$-module with 4 generators $\{\tilde{u}_0, \ldots, u_3\}$. The boundary operators are easily seen to be as follows.

$$\begin{aligned} \partial_1 \tilde{u}_1 &= (t - 1)\tilde{u}_0. \\ \partial_2 \tilde{u}_2 &= (1 + t + \cdots + t^{p-1})\tilde{u}_1. \\ \partial_3 \tilde{u}_3 &= (t^r - 1)\tilde{u}_2. \end{aligned}$$

where $r \in \mathbf{Z}$ such that $q \cdot r \equiv 1 \bmod p$.

Let $\rho : \pi \to SL(2; \mathbf{C})$ be a representation given by

$$\rho(t) = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}.$$

Since it holds that

$$1 + \omega + \cdots + \omega^{p-1} = 0,$$

the boundary operators in $C_*(L(p, q); V_\rho)$ become

$$\begin{aligned} \partial_1(\tilde{u}_1 \otimes v) &= \begin{pmatrix} \omega & 0 \\ 0 & \omega - 1 \end{pmatrix} \tilde{u}_2 \otimes v, \\ \partial_2(\tilde{u}_2 \otimes v) &= 0, \\ \partial_3(\tilde{u}_3 \otimes v) &= \begin{pmatrix} \omega^{r-1} & 0 \\ 0 & \omega^r - 1 \end{pmatrix} \tilde{u}_2 \otimes v. \end{aligned}$$

Then $C_*(L(p,q);V_\rho)$ is obviously acyclic and $\tau(L(p,q);V_\rho)$ is well-defined. We take the bases $\{\mathbf{c}_0,\ldots,\mathbf{c}_3\}$ coming from the cell decomposition as follows.

$$\begin{aligned}
\mathbf{c}_3 &= \{\tilde{u}_3 \otimes \mathbf{e}_1, \tilde{u}_3 \otimes \mathbf{e}_2\}.\\
\mathbf{c}_2 &= \{\tilde{u}_2 \otimes \mathbf{e}_1, \tilde{u}_2 \otimes \mathbf{e}_2\}.\\
\mathbf{c}_1 &= \{\tilde{u}_1 \otimes \mathbf{e}_1, \tilde{u}_1 \otimes \mathbf{e}_2\}.\\
\mathbf{c}_0 &= \{\tilde{u}_0 \otimes \mathbf{e}_1, \tilde{u}_0 \otimes \mathbf{e}_2\}.
\end{aligned}$$

Then we can take the bases by the following ;

$$\begin{aligned}
\mathbf{b}_2 &= \{(\omega^r - 1)e_1 \otimes \tilde{u}_2, (\bar{\omega}^r - 1)e_2 \otimes \tilde{u}_2\}.\\
\mathbf{b}_1 &= \{0\}.\\
\mathbf{b}_0 &= \{(\omega - 1)e_1 \otimes \tilde{u}_0, (\bar{\omega} - 1)e_2 \otimes \tilde{u}_0\}.
\end{aligned}$$

Therefore $\tau(L(p,q);V_\rho)$ is given by

$$\begin{aligned}
\tau(L(p,q);V_\rho) &= [\mathbf{b}_2/\mathbf{c}_3]^{-1} \cdot [\mathbf{b}_2/\mathbf{c}_2] \cdot [\mathbf{b}_1/\mathbf{c}_1]^{-1} \cdot [\mathbf{b}_0/\mathbf{c}_0]\\
&= \det\begin{pmatrix} \omega^r - 1 & 0 \\ 0 & \bar{\omega}^r - 1 \end{pmatrix} \cdot \det\begin{pmatrix} \omega - 1 & 0 \\ 0 & \bar{\omega} - 1 \end{pmatrix}\\
&= (\omega^r - 1)(\bar{\omega}^r - 1)(\omega - 1)(\bar{\omega} - 1)\\
&= 4(1 - \cos\frac{2\pi}{p})(1 - \cos\frac{2\pi r}{p}).
\end{aligned}$$

Applying this computation to $L(7,1)$ and $L(7,2)$, we obtain

$$\begin{aligned}
\tau(L(7,1)) &= 4(1 - \cos\frac{2\pi}{7})^2\\
\tau(L(7,2)) &= 4(1 - \cos\frac{2\pi}{7})(1 + \cos\frac{\pi}{7})
\end{aligned}$$

Thus the Reidemeister torsion distinguishes $L(7,1)$ from $L(7,2)$. Then $L(7,1)$ is not homeomorphic to $L(7,2)$.

REMARK. *It is well known that $L(7,1)$ has the same homotopy type with $L(7,2)$.*

Chapter 2

Reidemeister torsion of the Seifert fibered spaces for SL(2;C)-representations

§0. Introduction

Recently Johnson [5] derived an explicit formula for the Reidemeister torsion of Brieskorn homology 3-spheres for $SL(2;\mathbf{C})$-irreducible representations. Let M_n be a 3-manifold obtained by the $\frac{1}{n}$-surgery on a torus (p,q)-knot. It is a Brieskorn homology 3-sphere $\Sigma(p,q,pqn\pm 1)$. The fundamental group $\pi_1 M_n$ admits a presentation as follows.

$$\pi_1 M_n = \langle x, y \mid x^p = y^q, ml^n = 1\rangle$$

where m is a meridian of the torus knot which is a word of x and y and l is similarly a londitude. Johnson proved the following theorem.

THEOREM(JOHNSON). *The distinct conjugacy classes of the $SL(2;\mathbf{C})$-irreducible representations of $\pi_1 M_n$ are given by $\rho_{(a,b,k)}$ such that*

$$\begin{array}{ll}(1) & 0<a<p, 0<b<q, a\equiv b \bmod 2.\\ (2) & 0<k<N=|pqn+1|, k\equiv na \bmod 2.\\ (3) & tr\rho_{(a,b,k)}(x) = 2\cos\dfrac{\pi a}{p},\\ (4) & tr\rho_{(a,b,k)}(y) = 2\cos\dfrac{\pi b}{q},\\ (5) & tr\rho_{(a,b,k)}(m) = 2\cos\dfrac{\pi k}{N}.\end{array}$$

In this case the Reidemeister torsion $\tau_{(a,b,k)}$ for $\rho_{(a,b,k)}$ is given by

$$\tau_{(a,b,k)} = \begin{cases} 2(1-\cos\frac{\pi a}{p})(1-\cos\frac{\pi b}{q})(1+\cos\frac{\pi kpq}{N}) & a\equiv b\equiv 1, k\equiv n \quad \mathrm{mod}\ 2\\ 0 & a\equiv b\equiv 0 \text{ or } k\not\equiv n \quad \mathrm{mod}\ 2.\end{cases}$$

Key proposition of this theorem is the following. We have this proposition as a corollary of Lemma 2.1 in chapter 1.

PROPOSITION. *Let M be a closed, oriented 3-manifold with torus decomposition $A\cup_{T^2} B$ and $\rho:\pi_1 M\to SL(2;\mathbf{C})$ a representation whose restriction to $\pi_1 T^2$ is acyclic. Then $H_*(M;V_\rho)=0$ if and only if $H_*(A;V_\rho)=H_*(B;V_\rho)=0$. Moreover in this case,*

$$\tau(M;V_\rho) = \tau(A;V_\rho)\tau(B;V_\rho).$$

This methods can be applied to the investigation of more general Seifert fibered spaces and give a way to compute the Reidemeister torsion of them.

The main result of this paper is the following theorem. Let M^3 denote the orientable Seifert fibered space given by the following Seifert index ;

$$M = \{b, (\varepsilon, g); (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}.$$

MAIN THEOREM. *Let $\rho : \pi_1 M \to SL(2; \mathbf{C})$ be an irreducible representation. Then the Reidemeister torsion $\tau(M; V_\rho)$ is given by*

$$\tau(M; V_\rho) = \begin{cases} 0 & \text{if } H = I \\ 2^{4-m-4g} \prod_{i=1}^{m} (1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}) & \text{if } H \neq I, \varepsilon = o \\ (2 - 2\cos \frac{s\pi}{N+1})^{4-m-2g} \prod_{i=1}^{m} (1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}) & \text{if } H \neq I, \varepsilon = n \end{cases}$$

where

(1) $H = \rho(h)$,
(2) *h is a representative element of generic fiber in $\pi_1 M$,*
(3) *$\rho_i, \nu_i \in \mathbf{Z}$ such that* $\begin{vmatrix} \alpha_i & \rho_i \\ \beta_i & \nu_i \end{vmatrix} = -1$ *and* $0 < \rho_i < \alpha_i$,
(4) *$k_i(\rho) \in \mathbf{Z}$ such that $0 \leq k_i(\rho) \leq \alpha_i$, and $k_i(\rho) \equiv \beta_i$ mod 2.*
(5) $N = \dfrac{\beta_1}{\alpha_1} + \cdots \dfrac{\beta_m}{\alpha_m}$,
(6) *$s \in \mathbf{Z}$ such that $0 \leq s \leq 2N + 2$.*

REMARK. *In general the dimension of the space of $SL(2; \mathbf{C})$-representations of a Seifert fibered space is not zero ; in particular the distinct classes of irreducible representations are not finite. However the set of the Reidemeister torsion turns out to be a finite subset in $\mathbf{R}$ by this theorem ; that is, the Reidemeister torsion is a constant function on each connected component of the space of irreducible representations.*

Now we describe the contents of this chapter. In §1 we examine the Reidemeister torsion for the 2-dimensional torus and the solid torus. These results will be used later for the torus decomposition formula. In §2 we investigate $SL(2; \mathbf{C})$-irreducible representation of Seifert fibered spaces. In §3, we give a proof of Main theorem for the case of $H = -I$. In §4, we prove the non-acyclicity of the chain complex $C_*(M; V_\rho)$ in the case of $H = I$.

§1. Reidemeister torsion of the torus and the solid torus

In this section, we compute the Reidemeister torsion of the torus T^2 and the solid torus S. First we consider the condition of the acyclicity of T^2. When a representation ρ is fixed, we denote the matrix $\rho(x)$ for $^\forall x$ by the corresponding capital letter X. We denote the 2-dimensional complex vector space $\mathbf{C}^2$ by V and the canonical basis of V by $\{\mathbf{e}_1, \mathbf{e}_2\}$.

DEFINITION 1.1. *A parabolic element of $SL(2;\mathbf{C})$ is a nontrivial element which fixes some nonzero vector in V. Equivalently an element is parabolic if it is conjugate to $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $^\exists t \in \mathbf{C} - \{0\}$.*

DEFINITION 1.2. *Let $\rho : \pi_1 T^2 \longrightarrow SL(2;\mathbf{C})$ be a representation. Then it is called a parabolic representation if X is either trivial or a parabolic element in $SL(2;\mathbf{C})$ for $\forall x \in \pi_1 T^2$.*

We can easily prove the following lemma.

LEMMA 1.3. *Let $\rho : \pi_1 T^2 \longrightarrow SL(2;\mathbf{C})$ be a representation. The following statement are equivalent.*

(1) ρ is a parabolic representation.

(2) $\det(X - I) = 0$ for $\forall x \in \pi_1 T^2$ where I is the unit matrix in $SL(2;\mathbf{C})$.

Now we describe the condition of acyclicity.

PROPOSITION 1.4. *Let $\rho : \pi_1 T^2 \longrightarrow SL(2;\mathbf{C})$ be a representation. Then all homology groups vanish : $H_*(T^2, V_\rho) = 0$ if and only if ρ is a non-parabolic representation. In this case, the Reidemeister torsion is given by*

$$\tau(T^2; V_\rho) = 1.$$

Proof. Suppose ρ is a non-parabolic representation. We fix an orientation on T^2. By assumption, there is an element $x \in \pi_1 T^2$ such that $\det(X - I) \neq 0$. We take $y \in \pi_1 T^2$ such that the geometric intersection number $x \cdot y = 1$. We assume that a cell structure of T^2 is given by the following ;

(0) one 0-cell p,

(1) two 1-cells x and y,

(2) one 2-cell w,

with the attaching map given by $\partial w = xyx^{-1}y^{-1}$. By easy computation, this chain complex is given as follows ;

$$0 \longrightarrow w \otimes V \xrightarrow{\partial_2} x \otimes V \oplus y \otimes V \xrightarrow{\partial_1} p \otimes V \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} -(Y-I) \\ X-I \end{pmatrix},$$
$$\partial_1 = (X-I \quad Y-I).$$

Since $\det(X-I) \neq 0$, ∂_1 is surjective and then $\dim(\mathrm{Ker}\partial_1) = 2$. Similarly ∂_2 is injective and $\dim(\mathrm{Im}\partial_2) = 2$. On the other hand, we have

$$\mathrm{Im}\partial_2 \subset \mathrm{Ker}\partial_1$$

by the definition of the boundary operators. Hence

$$\mathrm{Im}\partial_2 = \mathrm{Ker}\partial_1.$$

Therefore this chain complex $C_*(T^2; V_\rho)$ is acyclic. Then $\tau(T^2; V_\rho)$ is well-defined. Since a canonical basis of $V \oplus V$ is given by

$$\{(\mathbf{e}_1, \mathbf{0}), (\mathbf{e}_2, \mathbf{0}), (\mathbf{0}, \mathbf{e}_1)(\mathbf{0}, \mathbf{e}_2)\},$$

we may identify the bases

$$\mathbf{c}_2 = \{\mathbf{e}_1, \mathbf{e}_2\},$$
$$\mathbf{c}_1 = \{(\mathbf{e}_1, 0), (\mathbf{e}_2, 0), (0, \mathbf{e}_1), (0, \mathbf{e}_2)\},$$
$$\mathbf{c}_0 = \{\mathbf{e}_1, \mathbf{e}_2\}.$$

We take a basis $\mathbf{b}_i$ of B_i for ${}^\forall i \in \{0, 1\}$ which satisfies

$$\mathbf{b}_1 = \partial \mathbf{c}_2,$$
$$\mathbf{b}_0 = \partial \mathbf{c}_1.$$

By the definition of Reidemeister torsion,

$$\tau(T^2; V_\rho) = [\mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]^{-1}[\mathbf{b}_0/\mathbf{c}_0].$$

From the straightforward computation,

$$[\mathbf{b}_1/\mathbf{c}_2] = 1,$$
$$[\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_0] = \det \begin{pmatrix} -(Y-I) & 0 \\ X-I & I \end{pmatrix}$$
$$= \det(Y-I),$$
$$[\mathbf{b}_0/\mathbf{c}_0] = \det(Y-I).$$

Therefore the Reidemeister torsion is given by

$$\tau(T^2; V_\rho) = 1.$$

Conversely we assume that ρ is a parabolic representation. If ρ is a trivial representation, it is clear that $C_*(T^2; V_\rho)$ is a usual V-coefficient chain complex and not acyclic. Hence we may assume ρ is nontrivial. Then there is an element

$$x \in \pi_1 T^2 \text{ such that } X = \rho(x) \neq I.$$

Let $v \in V$ denote the fixed vector of X and L the complex line spanned by v. Let $y \in \pi_1 T^2$ be any other element such that $Y = \rho(y) \neq I$. Since Y commutes with X, they have a common eigenvector which must be v or its multiple. Since Y is a parabolic element of $SL(2; \mathbf{C})$, Y also fixes the vector v. Then we have

$$\mathrm{Im}\partial_1 \subset L$$

and ∂_1 is not surjective. Hence we have

$$H_0(T^2; V_\rho) \neq 0.$$

This completes the proof.

REMARK. *If $\tau(M; V_\rho)$ is well-defined for an even dimensional closed orientable manifold M, then the absolute value of the Reidemeister torsion*

$$|\tau(M; V_\rho)| = 1.$$

See Ray-Singer [15] for details.

Next we consider the solid torus $S = S^1 \times D^2$ with $\pi_1 S \cong \mathbf{Z}$ generated by x.

PROPOSITION 1.5. *Let $\rho : \pi_1 S \to SL(2; \mathbf{C})$ be a representation. The representation ρ is non-parabolic if and only if the chain complex $C_*(S; V_\rho)$ is acyclic. In this case the Reidemeister torsion of S is given by*

$$\tau(S; V_\rho) = det(X - I).$$

Proof. It is easy to see that S has the same simple homotopy type as S^1. We may assume that a cell structure of S^1 is given by one 0-cell p and one 1-cell x. Then the corresponding chain complex is given by

$$0 \longrightarrow x \otimes V \overset{\partial = X - I}{\longrightarrow} p \otimes V \longrightarrow 0.$$

Hence $C_*(S; V_\rho)$ is acyclic if and only if $\det(X - I) \neq 0$. Therefore ρ is a non-parabolic representation. If we take a basis $\mathbf{b}_0 = \{\partial \mathbf{e}_1, \partial \mathbf{e}_2\}$ for $B_0(C_*)$, then the Reidemeister torsion is given by

$$\begin{aligned}\tau(S; V_\rho) &= [\mathbf{b}_0/\mathbf{c}_1]^{-1}[\mathbf{b}_0/\mathbf{c}_0] \\ &= 1 \cdot \det(X - I) \\ &= \det(X - I).\end{aligned}$$

This completes the proof of Proposition 1.5.

§2. Irreducible representations of Seifert fibered spaces

In this section, we investigate the $SL(2;\mathbf{C})$-irreducible representation of the Seifert fibered space M given by the Seifert index ;

$$\{b,(\varepsilon,g),(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)\}.$$

It is well known that the fundamental group of M has a presentation as follows. If $\varepsilon=o$, that is, if the orbit surface is orientable, then

$$\pi_1 M=\langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_m,h\ \|[a_i,h]=[b_i,h]=[q_i,h]=1,\\ q_i^{\alpha_i}h^{\beta_i}=1,q_1\ldots q_m[a_1,b_1]\ldots[a_g,b_g]=h^b\rangle.$$

If $\varepsilon=n$, that is, if the orbit surface is nonorientable, then

$$\pi_1 M=\langle v_1,\ldots,v_g,q_1,\ldots,q_m,h\ |v_ihv_i^{-1}=h^{-1},q_ihq_i^{-1}=h,\\ q_i^{\alpha_i}h^{\beta_i}=1,q_1\ldots q_mv_1^2\ldots v_g^2=h\rangle.$$

REMARK. *In the case of* $\epsilon=o$ *generators* a_i,b_i *and* q_i *come from the fundamental group of the orbit surface. Then we can choose the representative closed curves on the orbit surface*

$$q_1,\ldots,q_m \text{ such that } q_1\ldots q_m[a_1,b_1]\ldots[a_g,b_g]=1.$$

Similarly we choose the curves in the case of $\epsilon=n$.

We fix this presentation for π_1M and consider only $SL(2;\mathbf{C})$-irreducible representations. The next lemma gives us a clue to compute the Reidemeister torsion.

LEMMA 2.1. *Let* $\rho:\pi_1M\to SL(2;\mathbf{C})$ *be an irreducible representation. Then the image of the generic fiber* h *is given by*

$$H=\rho(h)=\begin{cases}\pm I & (\varepsilon=o)\\ \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} & (\varepsilon=n)\end{cases}$$

where

$$I \text{ is the unit matrix in } SL(2;\mathbf{C}),$$
$$\lambda\in\mathbf{C} \text{ such that } \lambda^{2N+2}=1,$$
$$N=\frac{\beta_1}{\alpha_1}+\cdots+\frac{\beta_m}{\alpha_m}.$$

Proof. By the irreducibility of ρ, it is easy to see that H is a non-parabolic element.

Case 1: $\varepsilon = o$

Suppose $H \neq \pm I$. Let u be an eigenvector for an eigenvalue λ of H. Since H commutes with $A_i = \rho(a_i), B_i = \rho(b_i)$ and $Q_j = \rho(q_j)$, all vectors $A_i u, B_i u$ and $Q_j u$ is contained in the vector space spanned by u. It contradicts the irreducibility of ρ. Thus $H = \pm I$.

Case 2: $\varepsilon = n$

Since we consider the conjugacy classes of representations, we may suppose H is the diagonal matrix $H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Subcase 1: $m = 0$

In this case M has no exceptional fibers ; it is an S^1-bundle over a non-orientable surface of genus g. By the relation $V_i H = H^{-1} V_i$,

$$\begin{aligned} V_i H \mathbf{e}_1 &= \lambda V_i \mathbf{e}_1 \\ &= H^{-1} V_i \mathbf{e}_1. \end{aligned}$$

Accordingly we get

$$H V_i \mathbf{e}_1 = \lambda^{-1} V_i \mathbf{e}_1$$

and $V_i \mathbf{e}_1$ is contained in the eigenspace for an eigenvalue λ^{-1} as in Case 1. Similarly $V_i \mathbf{e}_2$ is contained in the eigenspace for λ. Thus we may set each

$$V_i = \begin{pmatrix} 0 & a_i \\ b_i & 0 \end{pmatrix} \text{ such that } a_i b_i = -1.$$

By simple computation, we have

$$V_i^2 = -I.$$

The relation of $\pi_1 M$ implies

$$\begin{aligned} H &= V_1^2 V_2^2 \ldots V_g^2 \\ &= (-I)^g \end{aligned}$$

Hence

$$H = \pm I.$$

Subcase 2: m$\geq$1

Then M has the exceptional fibers $q_1, \ldots, q_m$. For $^\forall q_j$, we set the corresponding matrix

$$Q_j = \begin{pmatrix} s_j & t_j \\ u_j & v_j \end{pmatrix}.$$

The condition $HQ_j = Q_j H$ implies

$$\begin{pmatrix} \lambda s_j & \lambda t_j \\ \lambda^{-1} u_j & \lambda^{-1} v_j \end{pmatrix} = \begin{pmatrix} \lambda s_j & \lambda^{-1} t_j \\ \lambda u_j & \lambda^{-1} v_j \end{pmatrix}.$$

If we compare each entry of the left-side with the one of the right-side,

$$\lambda = \lambda^{-1} \quad \text{or} \quad t_j = u_j = 0.$$

If $\lambda = \lambda^{-1}$, then we get $\lambda = \pm 1$ and consequently $H = \pm I$. If $\lambda \neq \lambda^{-1}$, then every Q_j is a diagonal matrix. In this case, the relation $q_j^{\alpha_j} h^{\beta_j} = 1$ implies

$$\begin{pmatrix} s_j^{\alpha_j} & 0 \\ 0 & v_j^{\alpha_j} \end{pmatrix} = \begin{pmatrix} \lambda^{-\beta_j} & 0 \\ 0 & \lambda^{\beta_j} \end{pmatrix}.$$

Hence we get

$$s_j = \lambda^{-\frac{\beta_j}{\alpha_j}}, \text{ and } v_j = \lambda^{\frac{\beta_j}{\alpha_j}}.$$

On the other hand, we get

$$V_i = \begin{pmatrix} 0 & a_i \\ b_i & 0 \end{pmatrix} \text{ such that } V_i^2 = -I$$

as in the subcase 1. The relation $h = q_1 \dots q_m v_1^2 \dots v_g^2$ implies

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = (-I)^g \begin{pmatrix} s_1 \dots s_m & 0 \\ 0 & v_1 \dots v_m \end{pmatrix}$$
$$= (-1)^g \begin{pmatrix} \lambda^{-(\frac{\beta_1}{\alpha_1} + \dots \frac{\beta_m}{\alpha_m})} & 0 \\ 0 & \lambda^{\frac{\beta_1}{\alpha_1} + \dots \frac{\beta_m}{\alpha_m}} \end{pmatrix}.$$

Hence the following holds

$$\lambda^{-(\frac{\beta_1}{\alpha_1} + \dots + \frac{\beta_m}{\alpha_m})} = (-1)^g \lambda.$$

Therefore setting $N = \frac{\beta_1}{\alpha_1} + \dots + \frac{\beta_m}{\alpha_m}$, we get

$$\lambda^{2N+2} = 1.$$

This completes the proof of Lemma 2.1.

From the above lemma, we get easily the following corollary.

COROLLARY 2.2. *$Q_i = \rho(q_i)$ has only eigenvalues which are roots of unity.*

§3. Proof of Main theorem (1)

In this section, we give the proof of Main theorem. Here we decompose M into tubular neighborhoods of exceptional fibers and their complement. Then we compute the Reidemeister torsion for each part and apply Lemma 1.3 in chapter 1 to our situations. Since we can compute the $SL(2;\mathbf{C})$-torsion for $\varepsilon = n$ as in the case of $\varepsilon = o$, we will prove only the case of $\varepsilon = o$.

We put

$$\Sigma^* = \Sigma - (D_0^2 \cup \cdots \cup D_m^2)$$

where Σ is an orientable closed surface of genus g and $D_0^2, \ldots, D_m^2$ are disjoint embedded open 2-disks. Also let M_m denote the trivial S^1-bundle $\Sigma^* \times S^1$. We give a canonical torus decomposition of Seifert fibered space M as follows.

$$M \cong M_m \cup S_0 \cup S_1 \cdots \cup S_m$$

where any S_i is the solid torus. The solid torus S_0 is the one corresponding to the triviality obstruction b and for $^\forall i \in \{1, \ldots, m\}$ S_i is the one corresponding to the exceptional fiber.

LEMMA 3.1. *Let $\rho : \pi_1(M) \to SL(2;\mathbf{C})$ be an irreducible representation. Suppose all homology groups of the boundary vanish : $H_*(\partial M_m; V_\rho) = 0$. Then $H_*(M; V_\rho) = 0$ if and only if $H_*(M_m; V_\rho) = H_*(S_o; V_\rho) = \cdots = H_*(S_m; V_\rho) = 0$. In this case, we have*

$$\tau(M; V_\rho) = \tau(M_m; V_\rho)\tau(S_0; V_\rho) \ldots \tau(S_m; V_\rho).$$

Proof. Apply Lemma 1.3 in chapter 1 to the short exact sequence of the chain complex given by the torus decomposition of M ;

$$0 \to \bigoplus_{i=0}^{m} C_*(\partial S_i; V_\rho) \to C_*(M_m; V_\rho) \oplus \bigoplus_{i=0}^{m} C_*(S_i; V_\rho) \to C_*(M; V_\rho) \to 0.$$

By the proof of Proposition 1.4, $\dim \partial C_*(\partial S_i; V_\rho)$ is even. Therefore we have Lemma 3.1.

PROPOSITION 3.2. *Let $\rho : \pi_1(M) \to SL(2;\mathbf{C})$ be an irreducible representation. We denote the restriction to $\pi_1(M_m)$ by the same symbol ρ. Then all homology groups vanish : $H_*(M_m; V_\rho) = 0$ if and only if $H = \rho(h) = -I$. In this case the $SL(2;\mathbf{C})$-torsion is given by*

$$\tau(M_m; V_\rho) = 2^{2-2m-4g}.$$

Proof. It is easy to see that M_m has the same simple homotopy type as the direct product of the one point union of $2g + m$ circles $S^1 \vee \cdots \vee S^1$ and S^1. We denote this space by $(\bigvee_i S_i) \times S^1$. Then $\bigvee_i S_i$ has a natural cell decomposition given by one 0-cell u and $2g+m$ 1-cells a_i, b_i, q_j. It gives a cell decomposition of $(\bigvee_i S_i) \times S^1$ by

(1) 0-cell u,

(2) 1-cells $a_1, \ldots, a_g, b_1, \ldots, b_g, q_1, \ldots, q_m, h$ corresponding to the generators of $\pi_1 M$.

(3) 2-cells $v_{a_1}, v_{a_2}, \ldots, v_{a_g}, v_{b_1}, \ldots, v_{b_g}, v_{q_1}, \ldots, v_{q_m}$ respectively with boundary a_i, b_i and q_i.

By using this cell structure, we can determine the structure of $C_*(M_m; V_\rho)$. Recall that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a canonical basis of V. The 2-chain module $C_2(M_m; V_\rho)$ is a free $\mathbf{Z}[\pi_1 M_m]$-module on

$$\{v_{a_j} \otimes \mathbf{e}_i, v_{b_j} \otimes \mathbf{e}_i, v_{q_j} \otimes \mathbf{e}_i\}.$$

Similarly $C_1(M_m; V_\rho)$ is a free $\mathbf{Z}[\pi_1 M_m]$-module on

$$\{a_j \otimes \mathbf{e}_i, b_j \otimes \mathbf{e}_i, q_j \otimes \mathbf{e}_i, h \otimes \mathbf{e}_i\}$$

and $C_0(M_m)$ is a free $\mathbf{Z}[\pi_1 M_m]$-module on

$$\{u \otimes \mathbf{e}_i\}.$$

Then the boundary operators are given by

$$\partial_2 = \begin{pmatrix} I-H & 0 & \cdots & & & & & 0 \\ 0 & I-H & 0 & \cdots & & & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & & & & & 0 & I-H \\ A_1 - I & A_2 - I & \ldots & B_1 - I & \ldots & Q_1 - I & \ldots & Q_m - I \end{pmatrix},$$

$$\partial_1 = (\, A_1 - I \quad \ldots \quad B_1 - I \quad \ldots \quad Q_1 - I \quad \ldots \quad Q_m - I \quad H - I \,).$$

It is easy to see that $C_*(M_m; V_\rho)$ is acyclic if and only if $H = -I$. Let $\mathbf{b}_i$ be a basis of the boundary $B_i(M_m; V_\rho)$ for $^\forall i \in \{0, 1\}$. Then the Reidemeister torsion is given by

$$\tau(M_m; V_\rho) = [\mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]^{-1}[\mathbf{b}_0/\mathbf{c}_0].$$

We may choose a lift of $\mathbf{b}_1$ which coincides with $\mathbf{c}_2$ and the one of $\mathbf{b}_0$ which coincides with $\{\mathbf{e}_1 \otimes h, \mathbf{e}_2 \otimes h\}$. By the simple computation,

$$\begin{aligned} \tau(M_m; V_\rho) &= 1 \cdot (\det(I-H))^{-(2g+m+1)} \cdot \det(H-I) \\ &= (\det(I-H))^{-(2g+m)}. \end{aligned}$$

Then we substitute $-I$ for H, we have

$$\begin{aligned}\tau(M_m; V_\rho) &= (\det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix})^{-(2g+m)} \\ &= 2^{-2(2g+m)}.\end{aligned}$$

This completes the proof of Proposition 3.2.

Because ∂M_m is the disjoint union of tori, the fundamental group $\pi_1 M$ is generated by h and $\{q_1, \ldots, q_m\}$. Then $C_*(\partial M_m; V_\rho)$ is acyclic if and only if $H = -I$ by Proposition 1.4.

PROPOSITION 3.3. *If $H = -I$, then the Reidemeister torsion $\tau(S; V_\rho)$ is given by*

$$\tau(S_0; V_\rho) = 2^2.$$

Proof. Let ρ_0 and ν_0 be integers such that $\begin{vmatrix} 1 & \rho_0 \\ b & \nu_0 \end{vmatrix} = -1$. We define an element $l_0 \in \pi_1 M_m$ by $q_0^{\rho_0} h^{\nu_0}$. The sewing of the solid torus S_0 makes the curve $m_0 = q_0 h^b$ on the component of ∂M_m null-homotopic in S_0. On the other hand the closed curve l_0 is the generator in $\pi_1 S_0 \cong \mathbf{Z}$. Then the relation implies

$$\begin{aligned}L_0 &= \rho(l_0) \\ &= Q_0^{\rho_0} H^{\nu_0}.\end{aligned}$$

Since $q_0 = (h^b)^{-1} = (q_1 \ldots q_m [a_1, b_1] \ldots [a_g, b_g])^{-1}$ and $\nu_0 - b\rho_0 = -1$,

$$\begin{aligned}L_0 &= (Q_1 \ldots Q_m [A_1, B_1] \ldots [A_g, B_g])^{-\rho_0} H^{\nu_0} \\ &= H^{-b\rho_0 + \nu_0} \\ &= H^{-1} \\ &= -I.\end{aligned}$$

Therefore the Reidemeister torsion of S_0 is given as follows,

$$\begin{aligned}\tau(S_0; V_\rho) &= \det(L_0 - I) \\ &= \det\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \\ &= 2^2.\end{aligned}$$

This completes the proof.

PROPOSITION 3.4. *If $H = -I$, then the Reidemeister torsion $\tau(S_i; V_\rho)$ is given by*

$$\tau(S_i; V_\rho) = 2(1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}).$$

Proof. Let ρ_i and ν_i be integers such that $\begin{vmatrix} \alpha_i & \rho_i \\ \beta_i & \nu_i \end{vmatrix} = -1$ and $0 < \rho_i < \alpha_i$. We define the generator $l_i \in \pi_1 S_i$ by $q_i^{\rho_i} h^{\nu_i}$. Here the image of l_i is given by

$$\begin{aligned} L_i &= \rho(l_i) \\ &= Q_i^{\rho_i} H^{\nu_i} \\ &= (-1)^{\nu_i} Q_i^{\rho_i}. \end{aligned}$$

By Proposition 1.5, we have

$$\begin{aligned} \tau(S_i; V_\rho) &= \det(L_i - I) \\ &= \det((-1)^{\nu_i} Q_i^{\rho_i} - I) \\ &= 2 - (-1)^{\nu_i} \mathrm{tr} Q_i^{\rho_i}. \end{aligned}$$

In view of relations

$$q_i^{\alpha_i} h^{\beta_i} = 1$$

and

$$H = -I,$$

it holds that

$$Q_i^{\alpha_i} = (-I)^{\beta_i}.$$

Then we denote the eigenvalues of Q_i by

$$\exp(\frac{\sqrt{-1}k_i(\rho)\pi}{\alpha_i})$$

and

$$\exp(-\frac{\sqrt{-1}k_i(\rho)\pi}{\alpha_i})$$

where $0 \leq k_i(\rho) \leq \alpha_i$ and $k_i(\rho) \equiv \beta_i \bmod 2$. Hence we get

$$\tau(S_i; V_\rho) = 2(1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}).$$

This completes the proof of Proposition 3.4.

By Lemma 3.1, $\tau(M; V_\rho)$ is given by

$$\begin{aligned} \tau(M; V_\rho) &= \tau(M_m; V_\rho)\tau(S_0; V_\rho) \dots \tau(S_m; V_\rho) \\ &= 2^{2-2m-4g} \cdot 2^2 \cdot 2^m \cdot \prod_{i=1}^{m} (1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}) \\ &= 2^{4-m-4g} \prod_{i=1}^{m} (1 - (-1)^{\nu_i} \cos \frac{\rho_i k_i(\rho)\pi}{\alpha_i}). \end{aligned}$$

We have a proof of Main theorem for the case of $H = -I$.

§4. Proof of Main theorem (2)

If $H = I$, we cannot apply Lemma 3.1 to our situations because a given representation is not acyclic when we restricts it to the complement of exceptional fibers. However then the representation ρ is not acyclic. Now we prove the following proposition.

PROPOSITION 4.1. *Let $\rho : \pi_1(M) \to SL(2; \mathbf{C})$ be an irreducible representation such that $H = \rho(h) = I$. Then ρ is not acyclic ; that is, $H_*(M; V_\rho) \neq 0$.*

Proof.

The proof is by contradiction. We assume all homology groups of M vanish : $\mathrm{H}_*(M; V_\rho) = 0$. Then the following sequences given by the Mayer-vietoris sequence are exact.

$$
\begin{aligned}
&0 \longrightarrow \mathrm{H}_2(\partial M_m; V_\rho) \longrightarrow \mathrm{H}_2(M_m; V_\rho) \longrightarrow 0, \\
&0 \longrightarrow \mathrm{H}_1(\partial M_m; V_\rho) \longrightarrow \mathrm{H}_1(M_m; V_\rho) \oplus \bigoplus_{i=0}^{m} \mathrm{H}_1(S_i; V_\rho) \longrightarrow 0, \\
&0 \longrightarrow \mathrm{H}_0(\partial M_m; V_\rho) \longrightarrow \mathrm{H}_0(M_m; V_\rho) \oplus \bigoplus_{i=0}^{m} \mathrm{H}_0(S_i; V_\rho) \longrightarrow 0.
\end{aligned}
$$

Case 1: There exists a non-parabolic element in $\{A_i, B_i, Q_j\}$.

From the proof of Proposition 3.2, in the chain complex $C_*(M_m; V_\rho)$,

$$\mathrm{rank}(\partial_2) = \mathrm{rank}(\partial_1) = 2.$$

In this case, by easy computation, the homology groups of M_m are given as follows ;

$$
\begin{aligned}
\mathrm{H}_2(M_m; V_\rho) &\cong V^{2g+m-1} \\
\mathrm{H}_1(M_m; V_\rho) &\cong V^{2g+m-1} \\
\mathrm{H}_0(M_m; V_\rho) &= 0.
\end{aligned}
$$

By the above exact sequences and the Poincare duality, we have the following identifications

$$
\begin{aligned}
\mathrm{H}_0(\partial M_m; V_\rho) &\cong \mathrm{H}_2(\partial M_m; V_\rho) \\
&\cong \mathrm{H}_2(M_m; V_\rho) \\
&\cong V^{2g+m-1}
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathrm{H}_0(\partial M_m; V_\rho) &\cong \mathrm{H}_0(M_m; V_\rho) \oplus \bigoplus_{i=0}^{m} \mathrm{H}_0(S_i; V_\rho) \\
&\cong \{0\} \oplus V^{m+1-k} \\
&\cong V^{m+1-k}
\end{aligned}
$$

where k is the number of the solid tori with non-trivial 0-dimensional homology group. Hence we have

$$k = 2 - 2g.$$

Because k is a non-negative integer, the genus g=0 or 1.

First we assume g=0 ; that is, $k = 2$. In this case,

$$\pi_1 M =< q_1, \ldots, q_m, h | [q_i, h] = 1, q_i^{\alpha_i} h^{\beta_i} = 1, q_1 \ldots q_m = h^b > .$$

Then we have

$$\overset{m}{\underset{i=0}{\oplus}} \mathrm{H}_0(S_i; V_\rho) \cong V^{m-1}$$

by Proposition 3.3 and 3.4. For simplicity, we may assume

$$\mathrm{rank}(L_i - I) = 0 \text{ for } {}^\forall i \in \{0, \ldots, m-2\}$$

and

$$\mathrm{rank}(L_i - I) = 2 \text{ for } {}^\forall i \in \{m-1, m\}.$$

For ${}^\forall i \in \{0, \ldots, m-2\}$, that is, $L_i \in SL(2; \mathbf{C})$ is a parabolic element. On the other hand, from the relations of $\pi_1 M$, we have

$$\begin{aligned} L_i &= Q_i^{\rho_i} H^{\nu_i} \\ &= Q_i^{\rho_i} \\ &= I. \end{aligned}$$

Hence

$$Q_i = I \text{ for } {}^\forall i \in \{0, \ldots, m-2\}$$

and

$$Q_{m-1} Q_m = I.$$

Hence the representation ρ is reducible because Q_{m-1} and Q_m have a common eigenvector.

Next we assume $g = 1$; that is $k = 0$. In this case, we have

$$\begin{aligned} \pi_1 M =< a_1, b_1, q_1, \ldots, q_m, h | [a_1, h] = [b_1, h] = [q_i, h] = 1, \\ q_i^{\alpha_i} h^{\beta_i} = 1, [a_1, b_1] q_1 \ldots q_m = h^b > . \end{aligned}$$

Then we have

$$\overset{m}{\underset{i=0}{\oplus}} \mathrm{H}_0(S_i; V_\rho) \cong V^{m+1}.$$

Then for ${}^\forall i$

$$\mathrm{rank}(L_i - I) = 0$$

and $L_i \in SL(2;\mathbf{C})$ is parabolic or trivial. On the other hand, we have

$$\begin{aligned} L_i &= Q_i^{\rho_i} H^{\nu_i} \\ &= Q_i^{\rho_i} \\ &= I. \end{aligned}$$

Hence we have

$$Q_i = I \text{ for } {}^{\forall} i \in \{0, \ldots, m\}.$$

Then ρ factors through a representation of the group $< a_1, b_1 | [a_1, b_1] = 1 >$. Since this group is abelian, this representation is reducible. This is a contradiction.

Case 2: All A_i, B_i, Q_i are parabolic elements.

In this case we have,

$$Q_i = I \text{ for } {}^{\forall} i \in \{0, \ldots, m\}.$$

Then we have

$$\operatorname{rank}(\partial_2) = 2 \text{ or } 0$$

for $C_*(M_m; V_\rho)$. Hence

$$\mathrm{H}_2(M_m; V_\rho) \cong \begin{cases} V^{2g+m-1} & \text{if } \operatorname{rank}(\partial_2) = 2 \\ V^{2g+m} & \text{if } \operatorname{rank}(\partial_2) = 0. \end{cases}$$

By Poincare duality and the exact sequence, we obtain

$$\begin{aligned} \mathrm{H}_2(M_m; V_\rho) &\cong \mathrm{H}_2(\partial M_m; V_\rho) \\ &\cong H_0(M_m; V_\rho) \oplus V^{m+1}. \end{aligned}$$

Then we get the genus $g = 1$. Hence this representation ρ is reducible since ρ factors through the representation of the group $< a_1, b_1 | [a_1, b_1] = 1 >$ as in Case 1. This completes the proof of Proposition 4.1.

By the lemmas and the propositions, we get a proof of Main theorem in this chapter.

Chapter 3

Reidemeister torsion of the figure-eight knot exterior for $SL(2;\mathbf{C})$-representations

§0. Introduction

In chapter 2, we obtained an explicit formula of the Reidemeister torsion of Seifert fibered spaces for $SL(2;\mathbf{C})$-representations. In particular the sets of values of the Reidemeister torsion are finite subsets in $\mathbf{R}$. It follows that it has no continuous variations, although the dimension of the space of the representations of the fundamental group of these manifolds is generally positive.

In this chapter we consider the problem of determining whether there exists a closed 3-manifold with continuous variations of the Reidemeister torsion for $SL(2;\mathbf{C})$-representations. In order to attack this problem, we first need to investigate the spaces of $SL(2;\mathbf{C})$-representations for given manifolds. Applying the method due to Riley to the Wirtinger presentation of the figure-eight knot, we determine the space of representations of the fundamental group of the figure-eight knot exterior. Then by the method of Johnson, we obtain an explicit formula of the Reidemeister torsion of the exterior of the figure-eight knot. The formula shows that it has continuous variations. The main result of this chapter is the following.

MAIN THEOREM. *Let $K \subset S^3$ denote the figure-eight knot, and E its exterior ; i.e., the complement of an open tubular neighborhood of K. Let M denote the double $E \cup_{id} E$ of E. Then the set of values of the Reidemeister torsion $\tau(M;V_\rho)$ of M for $SL(2;\mathbf{C})$-representations is the set of all nonzero complex numbers. Therefore $\tau(M;V_\rho)$ has continuous variations.*

Now we describe the contents of this chapter briefly. In section 1 we review Johnson's theory. It gives an explicit formula of the Reidemeister torsion of knot exteriors. In section 2 we give a proof of the main theorem. and prove that the exterior of the figure-eight knot has continuous variations. To compute the matrix A_1, which is a generalization of the Alexander matrix, and its determinant, we used a computer(Reduce 2.3).

§1. Reidemeister torsion of the knot exterior

In this section, we give a review of the Reidemeister torsion of a knot exterior. See [5], and [12].

Let $K \subset S^3$ be a knot and E its exterior. We fix a Wirtinger presentation of the knot group $\pi_1 E$ as follows ;

$$\pi_1 E = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_{n-1} \rangle$$

where r_i is the crossing relation for each i. Let $\rho : \pi_1 E \longrightarrow SL(2; \mathbf{C})$ be a representation. When a representation ρ is fixed, we denote the matrix $\rho(x)$ for $x \in \pi_1 E$ by the corresponding capital letter X. For example, for $x_1 \in \pi_1 E$, we denote the matrix $\rho(x_1)$ by X_1. Consider a matrix

$$A = \begin{pmatrix} \rho(\frac{\partial r_1}{\partial x_1}) & \cdots & \rho(\frac{\partial r_1}{\partial x_n}) \\ \vdots & \ddots & \vdots \\ \rho(\frac{\partial r_{n-1}}{\partial x_1}) & \cdots & \rho(\frac{\partial r_{n-1}}{\partial x_n}) \end{pmatrix}$$

where each $\rho(\frac{\partial r_i}{\partial x_j})$ denotes the image of the free derivative $\frac{\partial r_i}{\partial x_j}$ in 2×2-matrixes. More precisely if $\frac{\partial r_i}{\partial x_j} = \sum\limits_k a_k g_k$ where $a_k \in \mathbf{Z}$ and $g_k \in \pi_1 E$, we denote $\sum\limits_k a_k \rho(g_k)$ in 2×2-matrixes by $\rho(\frac{\partial r_i}{\partial x_j})$. We denote by A_1 the matrix obtained by removing the first column from A. Then Johnson has shown the next formula.

THEOREM 1.1. *(Johnson) Let $\rho : \pi_1 E \longrightarrow$ SL(2;C) be a representation such that $det(X_1 - I) \neq 0$. Then all homology groups vanish : $H_*(E; V_\rho) = 0$ if and only if $det A_1 \neq 0$. In this case the Reidemeister torsion is given by*

$$\tau(E; V_\rho) = \frac{det(X_1 - I)}{det A_1}.$$

REMARK. *The definition of $det A_1$ above is analogous to the standard method of computing the Alexander polynomials of knots. Milnor has shown a parallel result for the Alexander polynomial.*

Let W be a 2-dimensional complex constructed from n 1-cells $x_1, \ldots, x_n$ and (n-1) 2-cells $D_1, \ldots, D_{n-1}$ with attaching maps $r_1, \ldots, r_{n-1}$. It is well-known that the knot exterior E collapses to the 2-dimensional complex W. Then it holds that

$$\tau(E; V_\rho) = \tau(W; V_\rho)$$

by the simple homotopy invariance of the Reidemeister torsion. To prove Theorem 1.1, we show that

$$\tau(W; V_\rho) = \frac{\det(X_1 - I)}{\det A_1}.$$

By an easy computation, this chain complex $C_*(W;V_\rho)$ can be described as follows ;

$$0 \longrightarrow V^{n-1} \xrightarrow{\partial_2} V^n \xrightarrow{\partial_1} V \longrightarrow 0$$

where

$$\partial_2 = A$$
$$= \begin{pmatrix} \rho(\frac{\partial r_1}{\partial x_1}) & \cdots & \rho(\frac{\partial r_1}{\partial x_n}) \\ \vdots & \ddots & \vdots \\ \rho(\frac{\partial r_{n-1}}{\partial x_1}) & \cdots & \rho(\frac{\partial r_{n-1}}{\partial x_n}) \end{pmatrix},$$

$$\partial_1 = \begin{pmatrix} X_1 - I \\ X_2 - I \\ \cdots \\ X_n - I \end{pmatrix}.$$

Here we briefly denote by V^k the k-times direct sum of V.

PROPOSITION 1.2. *Let $\rho : \pi_1 W \longrightarrow SL(2;\mathbf{C})$ be a representation such that the determinant $det(X_1 - I) \neq 0$. Then all homology groups vanish : $H_*(W;V_\rho) = 0$ if and only if $detA_1 \neq 0$. In this case, we have*

$$\tau(W;V_\rho) = \frac{det(X_1 - I)}{detA_1}.$$

Proof. It is obvious that $H_0(W;V_\rho)$ is trivial because $\det(X_1 - I) \neq 0$ and hence the boundary map ∂_1 is surjective. For a canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V, we choose lifts $\tilde{\mathbf{e}}_1 = ((X_1 - I)^{-1}\mathbf{e}_1, \mathbf{0}, \ldots, \mathbf{0}), \tilde{\mathbf{e}}_2 = ((X_1 - I)^{-1}\mathbf{e}_2, \mathbf{0}, \ldots, \mathbf{0})$ in V^n. Define the 2n$\times$ 2n matrix $\tilde{A}$ whose first 2n-2 rows are ∂_2 and last two rows are $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$. The matrix $\tilde{A}$ takes the form

$$\tilde{A} = \begin{pmatrix} * & & A_1 & \\ (X_1 - I)^{-1}\mathbf{e}_1 & 0 & \ldots & 0 \\ (X_1 - I)^{-1}\mathbf{e}_2 & 0 & \ldots & 0 \end{pmatrix}.$$

It is obvious that $\det\tilde{A} \neq 0$ if and only if $\det A_1 \neq 0$. If all homology groups vanish : $H_*(W;V_\rho) = 0$, then

$$\begin{aligned} \mathrm{rank}A &= \mathrm{rank}A_1 \\ &= 2n - 2, \end{aligned}$$

hence $\det A_1 \neq 0$. In this case the Reidemeister torsion is given by

$$\begin{aligned} \tau(W;V_\rho) &= \frac{1}{\det\tilde{A}} \\ &= \frac{\det(X_1 - I)}{\det A_1}. \end{aligned}$$

It is obvious that the contrary is also true. Namely if $\det A_1 \neq 0$, then $H_*(W; V_\rho) = 0$. This completes the proof of Proposition 1.2.

By the above propositions, we have Theorem 1.1.

§2. Proof of the main theorem

Let $K \subset S^3$ denote the figure-eight knot and E its exterior. At first we determine the space of the representations of the fundamental group of E by the methods due to Riley [7]. Here we choose a Wirtinger presentation of the fundamental group of E as follows ;

$$\pi_1 E = \langle x, y \mid wx = yw \rangle$$

where $w = x^{-1}yxy^{-1}x^{-1}$.

The following lemma is straightforward. See [7] for the details.

LEMMA 2.1. *Let X and Y be elements of $SL(2;\mathbf{C})$ which are conjugate in $SL(2;\mathbf{C})$ and not commutative. Then there exists an element U of $SL(2;\mathbf{C})$ such that*

$$UXU^{-1} = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \qquad UYU^{-1} = \begin{pmatrix} s & 0 \\ -t & s^{-1} \end{pmatrix}$$

where $s, t \in \mathbf{C} - \{0\}$.

We apply this lemma to irreducible representations of knot groups. Let $\rho : \pi_1 E \to SL(2;\mathbf{C})$ be an irreducible representation. By the above lemma, we may assume that

$$X = \rho(x) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix},$$

$$Y = \rho(y) = \begin{pmatrix} s & 0 \\ -t & s^{-1} \end{pmatrix}.$$

Here we have

$$\begin{aligned} W &= X^{-1}YXY^{-1}X^{-1} \\ &= \begin{pmatrix} s^{-1} & -1 \\ 0 & s \end{pmatrix}\begin{pmatrix} s & 0 \\ -t & s^{-1} \end{pmatrix}\begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}\begin{pmatrix} s^{-1} & 0 \\ t & s \end{pmatrix}\begin{pmatrix} s^{-1} & -1 \\ 0 & s \end{pmatrix} \\ &= \frac{1}{s^3}\begin{pmatrix} s^2t^2 + 2s^2t + s^2 - 1 & s(s^4t + s^4 - s^2t^2 - 2s^2t - 2s^2 + t) \\ -st(s^2t + s^2 - 1) & -s^2(s^4t - s^2t^2 - s^2t - s^2 + t) \end{pmatrix}. \end{aligned}$$

By elementary computation, we can see that

$$WX - YW = \frac{1}{s^3}\begin{pmatrix} 0 & f(s,t)s^2 \\ f(s,t)t & -f(s,t)st \end{pmatrix}$$

where $f(s,t) = s^2t^2 - (s^4 - 3s^2 + 1)t - (s^4 - 3s^2 + 1)$. Therefore $WX = YW$ if and only if $f(s,t) = 0$. Let $\hat{R}$ denote the space of conjugacy classes of $SL(2;\mathbf{C})$-irreducible representations of $\pi_1 E$. Then by the above observation, we have

PROPOSITION 2.2. $\hat{R} = \{(s,t) \in \mathbf{C}^2 \mid f(s,t) = 0, s \neq 0, t \neq 0\}$.

Solving this equation $f(s,t) = 0$ for t, we have

$$t = \frac{s^4 - 3s^2 + 1 \pm \sqrt{s^8 - 2s^6 - s^4 - 2s^2 + 1}}{2s^2}.$$

We denote the right-hand sides of the above expression by t_+ or t_-. If we substitute $t = 0$ for $f(s,t) = 0$, then we get

$$s = \pm\alpha_\pm$$

where

$$\alpha_\pm = \sqrt{(3 \pm \sqrt{5})/2}.$$

Hence $t \neq 0$ implies $s \neq \pm\alpha_\pm$. We apply Theorem 1.1 to compute the Reidemeister torsion of E. Apply the Fox's free derivative $\frac{\partial}{\partial y}$ to a relation $wxw^{-1}y^{-1} = 1$, then

$$\begin{aligned}
\frac{\partial(wxw^{-1}y^{-1})}{\partial y} &= \frac{\partial w}{\partial y} + wx\frac{\partial w^{-1}}{\partial y} + wxw^{-1}\frac{\partial y^{-1}}{\partial y} \\
&= \frac{\partial w}{\partial y} - wxw^{-1}\frac{\partial w}{\partial y} - wxw^{-1}y^{-1} \\
&= (1-y)\frac{\partial w}{\partial y} - 1.
\end{aligned}$$

Here

$$\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\partial(x^{-1}yxy^{-1}x^{-1})}{\partial y} \\
&= x^{-1} - x^{-1}yxy^{-1} \\
&= x^{-1} - wx.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial(wxw^{-1}y^{-1})}{y} &= (1-y)\frac{\partial w}{\partial y} - 1 \\
&= (1-y)(x^{-1} - wx) - 1.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
A_1 &= (I - Y)(X^{-1} - WX) - I \\
&= \begin{pmatrix} \frac{s^3t^2+2s^3t+s^3-s^2t^2-2s^2t-3s^2-st+s+t}{s^2} & \frac{(s^3t+s^3-s^2t-2s+1)}{s} \\ \frac{t(s^3t+s^3-s^2t^2-3s^2t-2s^2+t+1)}{s^2} & \frac{s(s^3t-s^2t^2-2s^2t+s^2-st-3s+t+1)}{s^2} \end{pmatrix}.
\end{aligned}$$

Here the numerator is

$$\begin{aligned}\det(X-I) &= \det\begin{pmatrix} s-1 & 1 \\ 0 & s^{-1}-1 \end{pmatrix} \\ &= (s-1)(s^{-1}-1) \\ &= -\frac{(s-1)^2}{s}.\end{aligned}$$

Then we have $\det(X-I)=0$ if and only if $s=1$. By Theorem 1.1, if $s \neq 1$ and $\det A_1 \neq 0$, then $\tau(E; V_{\rho_{(s,t)}})$ for $\rho_{(s,t)}$ is given by

$$\begin{aligned}\tau(E; V_{\rho_{(s,t)}}) &= \frac{\det(X-I)}{\det A_1} \\ &= \frac{s(s-1)^2}{(t-1)s^4+6s^3-(t^2+3t+11)s^2+6s+t-1}.\end{aligned}$$

If we respectively substitute t_+ and t_- for t, we get

$$\begin{aligned}\tau(E; V_{\rho_{(s,t_+)}}) &= \tau(E; V_{\rho_{(s,t_-)}}) \\ &= -\frac{s}{2(s^2-s+1)}.\end{aligned}$$

We denote this value of the Reidemeister torsion by $\tau_s(E)$. It is obvious that $\tau_s(E)$ is a continuous function for the parameter $s \in \mathbf{C} - \{0, 1, \pm\alpha_\pm\}$.

Let M denote the double $E \cup_{id} E$ of E. Let $\rho : \pi_1 M \to SL(2;\mathbf{C})$ be an irreducible representation such that the restriction on each $\pi_1 E$ is $\rho_{(s,t_+)}$. We have the following proposition as a corollary of Lemma 2.1 in chapter 1.

PROPOSITION 2.3. *Let M be a closed, oriented 3-manifold with torus decomposition $A \cup_{T^2} B$ and $\rho : \pi_1 M \to SL(2;\mathbf{C})$ a representation whose restriction to $\pi_1 T^2$ is acyclic. Then $H_*(M;V_\rho)=0$ if and only if $H_*(A;V_\rho)=H_*(B;V_\rho)=0$. Moreover in this case*

$$\tau(M;V_\rho) = \tau(A;V_\rho)\tau(B;V_\rho).$$

By the above proposition, the Reidemeister torsion of M is given by

$$\begin{aligned}\tau(M;V_\rho) &= (\tau_s E)^2 \\ &= \frac{s^2}{4(s^2-s+1)^2}.\end{aligned}$$

Hence by an elementary computation, the set of nonzero torsion is just the set of nonzero complex numbers. This completes the proof of Main theorem.

Chapter 4

Reidemeister torsion and a volume form of representation spaces

§0. Introduction

Let M^3 be a closed, oriented homology 3-sphere. We can decompose M into two handlebodies of a certain genus g by Heegaard's theorem. We respectively denote these handlebodies by N_1 and N_2 and the common boundary surface by Σ_g. Let R_g be the space of SU(2)-irreducible representations of $\pi_1 \Sigma_g$. Similarly R^1, R^2 and R_M are defined as the space of the SU(2)-irreducible representations of each fundamental group of N_1, N_2, or M. Then SU(2) acts on these spaces of representations by conjugation. We denote the orbit spaces respectively by $\hat{R}_g, \hat{R}^1, \hat{R}^2$ and $\hat{R}_M$. It is easy to see that $\hat{R}_M$ coincides with the intersection of $\hat{R}^1$ and $\hat{R}^2$ by van Kampen's theorem, when we consider $\hat{R}^1$ and $\hat{R}^2$ as a submanifold in $\hat{R}_g$.

In 1985 Casson defined a topological invariant for M using a natural orientation on $\hat{R}_g$. It is defined to be the half of the algebraic intersection number of $\hat{R}^1$ and $\hat{R}^2$ in $\hat{R}_g$. We denote this topological invariant by $\lambda(M)$. Later Johnson defined a natural volume form on $\hat{R}_g$ as follows. Let V be an n-dimensional vector space over $\mathbf{R}$. We denote the n-dimensional exterior product $\bigwedge^n V$ by $\det V$ and the dual space od V by V^{-1}. A volume on V is defined to be a nonzero element in $\det V$. Note that we can write a given volume v as

$$v = \mathbf{e}_1 \wedge \mathbf{e}_2 \cdots \wedge \mathbf{e}_n$$

for some basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V. Johnson's theorem is the following.

THEOREM(JOHNSON).

(1) *There are natural volumes $\hat{v}_1, \hat{v}_2$ and $\hat{v}_g$ on the tangent spaces $T_{[\rho]}\hat{R}^1, T_{[\rho]}\hat{R}^2$ and $T_{[\rho]}\hat{R}_g$ for $^{\forall}[\rho] \in \hat{R}^1 \cap \hat{R}^2 = \hat{R}_M$.*

(2) *Suppose $\hat{R}^1$ and $\hat{R}^2$ have a transverse intersection at $[\rho]$. A nonzero real number $t_{[\rho]}$ is defined by*

$$\hat{v}_1 \wedge \hat{v}_2 = t_{[\rho]} \cdot \hat{v}_g.$$

Then a sign of $t_{[\rho]}$ coincides with the sign of $[\rho]$ in Casson's invariant. In particular if $\hat{R}^1$ and $\hat{R}^2$ are transversal for $^{\forall}[\rho] \in \hat{R}^1 \cap \hat{R}^2$, then the following holds,

$$2\lambda(M) = \sum_{[\rho]} sign(t_{[\rho]}).$$

(3) *We consider the Lie algebra $su(2)$ as a $\pi_1(M)$-module using a representation ρ and the adjoint representation of SU(2). We denote the $\pi_1(M)$-module $su(2)$ by $su(2)_\rho$. Then the Reidemeister torsion $\tau_\rho(M)$ of M with $su(2)_\rho$-coefficients is defined. In this case, the following holds up to sign ;*

$$t_{[\rho]} = \tau_\rho(M).$$

This result is very intersting but vague connection between Casson's invariant and the Reidemeister torsion. We would like to understand geometrically the meaning of this connection. Our main result of this chapter is following theorem.

MAIN THEOREM. *Johnson's volumes $\hat{v}_1, \hat{v}_2$ and $\hat{v}_g$ are respectively the Reidemeister torsion of N_1, N_2, and Σ_g up to sign, that is,*

$$\hat{v}_1 = \epsilon_1 \tau_\rho(N_1) \in det H^1(N_1; su(2)_\rho),$$

$$\hat{v}_2 = \epsilon_2 \tau_\rho(N_2) \in det H^1(N_2; su(2)_\rho),$$

$$\hat{v}_g = \epsilon_g \tau_\rho(\Sigma_g) \in det H^1(\Sigma_g; su(2)_\rho)$$

where $\epsilon_1, \epsilon_2, \epsilon_g \in \{\pm 1\}$.

By this theorem, we can consider the relation

$$\hat{v}_1 \wedge \hat{v}_2 = t_{[\rho]} \cdot \hat{v}_g$$

as a relation with the Reidemeister torsion. Hence we have the following relation, up to sign ;

$$\tau_\rho(N_1)\tau_\rho(N_2) = \tau_\rho(M)\tau_\rho(\Sigma_g).$$

We recall a well-known lemma of the Reidemeister torsion. See Milnor [13] for details. Let

$$0 \to C'_* \to C_* \to C''_* \to 0$$

be an exact sequence of n-dimensional chain complexes with preferred volumes $\{\mathbf{c}'_i\}$, $\{\mathbf{c}_i\}$, and $\{\mathbf{c}''_i\}$ such that

$$\mathbf{c}'_i \mathbf{c}''_i = \mathbf{c}_i$$

for $^\forall i$. Then the homology exact sequence $\mathbf{H}$:

$$0 \to H_n(C'_*) \to H_n(C_*) \to H_n(C''_*) \to \cdots \to H_0(C'_*) \to H_0(C_*) \to H_0(C''_*) \to 0$$

can be thought of as a free acyclic chain complex. When we give the volumes on the homology groups, then $\tau(\mathbf{H})$ is well-defined for their volumes.

LEMMA. *The next formula holds, up to sign,*

$$\tau(C_*) = \tau(C'_*)\tau(C''_*)\tau(\mathbf{H}).$$

We can use the above lemma to our situation. We have a following natural exact sequence which is derived from the Heegaard decomposition of M,

$$0 \to C_*(\Sigma_g; su(2)_\rho) \to C_*(N_1; su(2)_\rho) \oplus C_*(N_2; su(2)_\rho) \to C_*(M; su(2)_\rho) \to 0.$$

Hence we get the following equality by the above lemma,

$$\tau_\rho(N_1)\tau_\rho(N_2) = \tau_\rho(M)\tau_\rho(\Sigma_g)\tau(\mathbf{H}),$$

up to sign. Then the chain complex $\mathbf{H}$ is the following;

$$\mathbf{H} : 0 \to H_1(\Sigma_g; su(2)_\rho) \to H_1(N_1; su(2)_\rho) \oplus H_1(N_2; su(2)_\rho) \to 0.$$

Suppose that a volume on $H_1(\Sigma_g; su(2)_\rho)$ is given by the product of the volumes on $H_1(N_1; su(2)_\rho)$ and $H_1(N_2; su(2)_\rho)$. Then it holds, up to sign,

$$\tau(\mathbf{H}) = 1$$

In this case, we have the following, up to sign ;

$$\tau_\rho(N_1)\tau_\rho(N_2) = \tau_\rho(M)\tau_\rho(\Sigma_g).$$

Therefore we have naturally the relation $\hat{v}_1 \wedge \hat{v}_2 = t_{[\rho]} \cdot \hat{v}_g$ from a well-known relation of torsion invariants.

Now we describe the contents of this chapter. In section 1 we review the cohomology of groups and its connection to the spaces of representations. In section 2 we review the Johnson's theory. In section 3 we define the Reidemeister torsion with $su(2)_\rho$-coefficients. In this chapter, we deal the Reidemeister torsion for non-acyclic cases. Then we define the Reidemeister torsion using volumes instead of bases. Finally in section 4 we establish the equality of Reidemeister torsion and Johnson' volume.

§1. Cohomology of groups.

In this section, we describe the cohomology of groups with $su(2)$-coefficients. See Brown [2], and Weil [19].

Let π be a discrete group. We denote the space of the SU(2)-irreducible representations of π by $R(\pi)$ and the space of the conjugacy classes of them by $\hat{R}(\pi)$. We fix a representation $\rho \in R(\pi)$. The cochain complex $C^*(\pi; su(2)_\rho)$ is defined by the following.

$$\text{If } n \text{ is positive, } C^n(\pi; su(2)_\rho) = \{f : \underbrace{\pi \times \cdots \times \pi}_{n \text{ times}} \to su(2)_\rho\}.$$
$$\text{If } n \text{ is zero, } C^0(\pi; su(2)_\rho) = su(2)_\rho.$$

The coboundary map is

$$\begin{aligned}\delta_n f(x_1, \ldots, x_{n+1}) =& x_1 f(x_2, \ldots, x_{n+1}) - f(x_1 x_2, x_3, \ldots, x_{n+1}) \\ &+ f(x_1, x_2 x_3, x_4, \ldots, x_{n+1}) \ldots (-1)^{n+1} f(x_1, x_2, \ldots, x_n). \\ \delta_0(v)(x) =& (x-1)v.\end{aligned}$$

for $^\forall x_1, \ldots, {}^\forall x_{n+1}, {}^\forall x \in \pi$ and $^\forall v \in su(2)_\rho$. Then

$$\delta_1 f = 0$$

implies that f is a crossed homomorphism from π to $su(2)_\rho$. Similarly

$$f \in \mathrm{Im}\delta_0$$

implies that f is a principal crossed homomorphism. Hence we have

$$\begin{aligned}Z^1(\pi; su(2)_\rho) &= \{f : \pi \to su(2)_\rho;\ \text{crossed homomorphism}\}, \\ B^1(\pi; su(2)_\rho) &= \{f : \pi \to su(2)_\rho;\ \text{principal crossed homomorphism}\}.\end{aligned}$$

The above cochain complex is just the same as the local cohomology of a particular $K(\pi, 1)$-complex using $su(2)_\rho$-coefficients.

We can consider the closed surface Σ_g given by the following as a CW-complex, which has

$$\begin{aligned}&\text{one } 0-\text{cell } p, \\ &2g 1-\ \text{cells } a_1, b_1, \ldots, a_g, b_g, \\ &\text{one } 2-\text{cell } e.\end{aligned}$$

The attaching map is given by

$$[a_1, b_1] \ldots [a_g, b_g] = 1.$$

See Appendixes for details. We can use this complex to calculate the cohomology of the group $H^*(\pi_1\Sigma_g; su(2)_\rho)$, because the closed surface Σ_g is a $K(\pi,1)$-space. Because of this, we identify

$$H^*(\Sigma_g; su(2)_\rho) \cong H^*(\pi_1\Sigma_g; su(2)_\rho).$$

It is well-known that the tangent space $T_\rho R(\pi_1\Sigma_g)$ of $R(\pi_1\Sigma_g)$ at ρ is isomorphic to $Z^1(\pi_1\Sigma_g; su(2)_\rho)$ and $T_{[\rho]}\hat{R}(\pi_1\Sigma_g)$ to $H_1(\pi_1\Sigma_g; su(2)_\rho)$.

For a 3-dimensional handlebody H_g of genus g, we identify

$$H^*(H_g; su(2)_\rho) \cong H^*(\pi_1 H_g; su(2)_\rho).$$

Then the tangent space of the space of the representations, or the space of the conjugacy classes, is isomorphic to the 1-cocycles, or the 1-dimensional cohomology groups.

Here we introduce a π-module to describe the tangent space of $R(\pi)$. Let $d\pi$ be a π-module with a generator dx for $^\forall x \in \pi$ and relations

$$d(xy) = dx + xdy \text{ for } {}^\forall x, {}^\forall y \in \pi.$$

The map

$$d : \pi \longrightarrow d\pi$$

given by

$$x \mapsto dx$$

is a crossed homomorphism, just by the definition. For any given crossed homomorphism

$$\phi : \pi \longrightarrow su(2)_\rho,$$

there is an unique $\mathbf{Z}[\pi]$-homomorphism

$$f : d\pi \longrightarrow su(2)_\rho$$

such that

$$f(dx) = \phi(x).$$

Hence we frequently identify the maps of two types via this composition with d. Here there is a bilinear pairing over $\mathbf{R}$ of $su(2)_\rho^{-1} \otimes d\pi$ with $\mathrm{Hom}_\pi(d\pi, su(2)_\rho)$ as follows. For $^\forall v^{-1} \otimes dx \in su(2)^{-1} \otimes d\pi$, $^\forall f \in \mathrm{Hom}_\pi(d\pi, su(2)_\rho)$, the pairing

$$< v^{-1} \otimes dx, f >$$

is defined by

$$< v^{-1}, f(x) > .$$

It is easy to see that the above pairing is non-singular. Since $\mathrm{Hom}_\pi(d\pi, su(2)_\rho)$ can be identified with $Z^1(\pi; su(2)_\rho)$, we can identify

$$T_\rho R(\pi) \cong su(2)^{-1} \otimes d\pi$$

via the above dual pairing.

§2. Johnson's theory

In this section we give briefly an exposition of Johnson's theory. See Johnson [5] for details. Recall that we can write a volume v on a vector space V as

$$v = \mathbf{e}_1 \wedge \mathbf{e}_2 \cdots \wedge \mathbf{e}_n$$

for some basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V. Now let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of finite dimensional vector spaces over $\mathbf{R}$. For simplicity, we identify A with the image of A in B. Let

$$v_A = a_1 \wedge a_2 \cdots \wedge a_k$$

and

$$v_c = c_1 \wedge c_2 \cdots \wedge c_l$$

be volumes on A and C. We can lift c_j to the element b_j in B and define a volume v_B on B by

$$a_1 \wedge \cdots \wedge a_k \wedge b_1 \wedge \cdots \wedge b_l.$$

It is easy to see that this is independent of choices of the lifting. Then we can briefly denote the volume v_B by

$$v_B = v_A v_C.$$

Similarly if v_A and v_B are given, there is an unique volume v_C on C such that

$$v_C = v_B / v_A.$$

As a special case, we consider $C = \{0\}$. Then

$$v_C = v_B / v_A$$

is just the ratio of v_A and v_B.

We recall Σ_g is a closed, oriented surface of genus g and let π be a fundamental group of Σ_g. At first we construct a volume on the tangent space of the space of the representations of a free group. Let Γ be a free group of rank n and $d\Gamma$ the Γ-module as section 1. We denote an abelianization $\Gamma/[\Gamma, \Gamma]$ of Γ by A and an n-th exterior product of A by det$A$. Let t be an orientation on A, that is, t is a generator over $\mathbf{Z}$ of det$A$. We suppose that $\rho \in R(\Gamma)$ is given. We fix a volume

$$\theta = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

on $su(2)_\rho$. We will construct a natural volume $v(\theta, t)$ on $d\Gamma \otimes su(2)_\rho^{-1}$ which depends on θ and t. Let

$$\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$$

be a free basis of Γ. This gives a basis

$$\{dx_1, \dots, dx_n\}$$

of $d\Gamma$. Furthermore we get a corresponding basis

$$\mathbf{a} = (a_1, \dots, a_n)$$

of A and hence an orientation

$$t_{\mathbf{x}} = a_1 \wedge \cdots \wedge a_n$$

of A. We define a volume $v(\theta, t_{\mathbf{x}})$ on $d\Gamma \otimes su(2)_\rho^{-1}$ by

$$dx_1 \otimes \mathbf{e}_1^{-1} \wedge dx_1 \otimes \mathbf{e}_2^{-1} \wedge dx_1 \otimes \mathbf{e}_3^{-1} \cdots \wedge dx_n \otimes \mathbf{e}_3^{-1}$$

where $\{\mathbf{e}_1^{-1}, \mathbf{e}_2^{-1}, \mathbf{e}_3^{-1}\}$ is the dual basis in $su(2)_\rho^{-1}$. Here we define $\epsilon_{\mathbf{x}} \in \{\pm 1\}$ by

$$t = \epsilon_{\mathbf{x}} t_{\mathbf{x}}.$$

Then we define a volume $v(\theta, t)$ by $\epsilon_{\mathbf{x}} v(\theta, t_{\mathbf{x}})$. It gives us natural volumes on

$$Z^1(\Gamma, su(2)_\rho) \cong T_\rho R(\Gamma)$$

using the dual pairing. We need to see how $v(\theta, t)$ behaves under a change of the free basis. It is well-known that three Nielsen moves generate all basis changes in Γ.

(1) $y_i = x_j, y_j = x_i, y_k = x_k.(k \neq i, j.)$

(2) $y_1 = x_1^{-1}, y_k = x_k.(k \neq 1.)$

(3) $y_1 = x_1 x_2, y_k = x_k.(k \neq 1.)$

By elementary computation, we have the following proposition.

PROPOSITION 2.1. *A volume $v(\theta, t)$ is unchanged under the Nielsen moves.*

The coboundary map

$$\delta_0 : C^0(\Gamma; su(2)_\rho) = su(2)_\rho \to Z^1(\Gamma; su(2)_\rho)$$

is injective because the 0-dimensional cohomology $H^0(\Gamma; su(2)_\rho)$ is vanishing. Then we have an exact sequence

$$0 \to su(2)_\rho \xrightarrow{\delta_0} Z^1(\Gamma; su(2)_\rho) \to H^1(\Gamma; su(2)_\rho) \to 0.$$

Hence we get a natural volume $v(\theta, t)/\theta$ on $H^1(\Gamma, su(2)_\rho) = T\hat{R}_{[\rho]}(\Gamma)$. It is easy to see that $v(\theta, t)/\theta$ is independent of θ and well-defined up to sign. We denote it by $\hat{v}(t)$. This gives volumes $\hat{v}_1$, and $\hat{v}_2$ on the space of the representations of the handlebodies N_1 and N_2 because Γ can be considered as a fundamental group of a handlebody of genus n.

Next we construct a volume on the space of representations of Σ_g. The fundamental group π of Σ_g has the usual presentation corresponding to a structure of the standard surface complex;

$$\mathbf{Z} \xrightarrow{1 \to \xi} \Pi \longrightarrow \pi \longrightarrow 0$$

where Π is the free group of rank $2g$ on geometric generators

$$a_1, \ldots, a_g, b_1, \ldots, b_g$$

and

$$\xi = \prod_{i=1}^{g} [a_i, b_i].$$

We suppose that

$$a_1, \ldots, a_g, b_1, \ldots, b_g$$

becomes a symplectic basis in $H_1(\Sigma_g; \mathbf{Z})$ such that

$$\begin{aligned} a_i \cdot a_j &= b_i \cdot b_j = 0, \\ a_i \cdot b_j &= -b_j \cdot a_i = \delta_{ij} \end{aligned}$$

with respect to the intersection pairing. In this case we have a volume $v_\Pi(\theta)$ on $R(\Pi)$ which depends only on θ because the symplectic basis induces a natural orientation on $\Pi/[\Pi, \Pi] = H_1(\Sigma_g; \mathbf{Z})$. We have a next exact sequence from the standard surface complex after taking the dual into $su(2)_\rho$;

$$0 \to su(2)_\rho \xrightarrow{\delta_0} Z^1(\Pi; su(2)_\rho) \xrightarrow{\delta_1} su(2)_\rho \to 0$$

where the coboundary map δ_1 is the evaluation of $f \in Z^1(\Pi; su(2)_\rho)$ on ξ. The kernel of this map is $Z^1(\pi; su(2)_\rho) = T_\rho R(\Pi)$ and δ_1 is surjective because $H^2(\pi; su(2)_\rho)$ is vanishing. Furthermore δ_0 is injective because $H^0(\pi; su(2)_\rho)$ is vanishing, too. We can then write the following exact sequences.

$$\begin{aligned} &0 \longrightarrow Z^1(\pi; su(2)_\rho) \longrightarrow Z^1(\Pi; su(2)_\rho) \xrightarrow{f \to f(\xi)} su(2)_\rho \longrightarrow 0, \\ &0 \longrightarrow su(2)_\rho \longrightarrow Z^1(\pi; su(2)_\rho) \longrightarrow H^1(\pi; su(2)_\rho) \longrightarrow 0. \end{aligned}$$

Hence the volumes $v_\Pi(\theta)$ and θ give us a natural volume $v_g(\theta)$ on $Z^1(\pi; su(2)_\rho) = T_\rho R_g$. Then using it and θ again gives us $\hat{v}_g(\theta)$ on $H^1(\pi; su(2)_\rho)$.

REMARK. *We fix a standard Killing form on $su(2)_\rho$. It gives an inner product on $det su(2)_\rho$. When we choose a volume θ on $su(2)_\rho$ satisfying its norm is equal to one, $\hat{v}_g(\theta)$ is well-defined up to sign.*

§3. Reidemeister torsion

In this section let us describe the definition of the Reidemeister torsion with $su(2)$-coefficients. We deal the Reidemeister torsion for the non-acyclic case.

At first we define the torsion of chain complexes. Suppose

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is a chain complex of finite-dimensional vector spaces over $\mathbf{R}$. We suppose that a preferred volume $\mathbf{c}_q$ for $C_q(C_*)$ is given for $^\forall q$. Choose any volume $\mathbf{b}_q$ for $B_q(C_*)$ and $\mathbf{h}_q$ for $H_q(C_*)$.

Because

$$0 \longrightarrow B_q(C_*) \longrightarrow Z_q(C_*) \longrightarrow H_q(C_*) \longrightarrow 0$$

is an exact sequence, the volumes $\mathbf{b}_q$ and $\mathbf{h}_q$ give a volume $\mathbf{b}_q\mathbf{h}_q$ for $Z_q(C_*)$. Similarly the sequence

$$0 \to Z_q(C_*) \to C_q(C_*) \to B_{q-1}(C_*) \to 0$$

is exact and the volumes $\mathbf{b}_q\mathbf{h}_q$ and $\mathbf{b}_{q-1}$ give a volume $\mathbf{b}_q\mathbf{h}_q\mathbf{b}_{q-1}$ for $C_q(C_*)$.

DEFINITION 3.1. *The torsion of the chain complex C_* with volumes $\mathbf{c}_*$ is defined by the alternating product*

$$\prod_{q=0}^{m}(\mathbf{b}_q\mathbf{h}_q\mathbf{b}_{q-1}/\mathbf{c}_q)^{(-1)^{q+1}}$$

and we denote it by $\tau(C_*)$.

REMARK. *The torsion $\tau(C_*)$ is independent of the choices of a volume $\mathbf{b}_q$ for $B_q(C_*)$. Then we consider $\tau(C_*)$ as a nonzero linear function on $\bigotimes_{q=0}^{m} \det H_q(C_*)^{(-1)^{q+1}}$, that is $\tau(C_*)$ is an element of $\bigotimes_{q=0}^{m} \det H_q(C_*)^{(-1)^{q}}$.*

Now we apply the torsion invariant of chain complexes to the following geometric situations. Let K be a finite cell complex and $\tilde{K}$ a universal covering of K with the fundamental group $\pi_1 K$ acting on it as deck transformations. Then the integral chain complex $C_*(\tilde{K};\mathbf{Z})$ has a structure of a chain complex of free $\mathbf{Z}[\pi_1 K]$-modules. Let

$$\rho : \pi_1 K \to SU(2)$$

be a representation. Define the chain complex $C_*(K; su(2)_\rho)$ by

$$C_*(\tilde{K};\mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1 K]} su(2)_\rho$$

and choose a preferred volume

$$\sigma_1 \otimes e_1 \wedge \sigma_1 \otimes e_2 \wedge \sigma_1 \otimes e_3 \wedge \cdots \wedge \sigma_{i_q} \otimes e_3$$

of $C_q(K; su(2)_\rho)$ for $^\forall q$ where $\{e_1, e_2, e_3\}$ is some fixed basis of $su(2)$ and the lifts of the q-cells $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{i_q}\}$ is a basis of $C_q(\tilde{K};\mathbf{Z})$ as a free $\mathbf{Z}[\pi_1 K]$-module.

Definition 3.2. *The Reidemeister torsion* $\tau_\rho(K)$ *of* K *with* $su(2)_\rho$*-coefficients is defined by the torsion of the chain complex* $C_*(K; su(2)_\rho)$.

Remark.

(1) *The Reidemeister torsion* $\tau_\rho(K)$ *depends on several choices. However we can prove the well-definedness of Reidemeister torsion, up to sign as in the acyclic case.*

(2) *This Reidemeister torsion is just the inverse of the one defined in chapter 1. In chapters 3 and 4, we deal this type of the Reidemeister torsion.*

We assume K is the closed oriented surface Σ_g of genus g. Then the Reidemeister torsion $\tau_\rho(K)$ is in $\bigotimes_{q=0}^{3} \det H_q(K; su(2)_\rho)^{(-1)^q}$. If ρ is an irreducible representation, then

$$H_0(K; su(2)_\rho) = H_2(K; su(2)_\rho) = 0.$$

Here

$$H_1(K; su(2)_\rho) \cong H^1(K; su(2)_\rho)$$

from the Poincare duality. Furthermore $H^1(K; su(2)_\rho)$ is isomorphic to the tangent space $T_{[\rho]}\hat{R}_g$ of $\hat{R}_g$ at the conjugacy class $[\rho]$. Therefore the Reidemeister torsion $\tau_\rho(K)$ is in $\det T_{[\rho]}\hat{R}_g$ and it is considered as a volume form on $\hat{R}_g$.

§4. Proof of Main theorem

In this section we give a proof of the main theorem. At first, we prove that

$$\tau_\rho(N_1) = \epsilon_1 \hat{v}_1$$

where $\epsilon_1 \in \{\pm 1\}$. We can similarly prove that

$$\tau_\rho(N_2) = \epsilon_2 \hat{v}_2 \text{ where } \epsilon_2 \in \{\pm 1\}.)$$

It is easy to see that the handlebody N_1 collapses to a cell complex X with one 0-cell p and 2g 1-cells $x_1, \ldots, x_g$ which is a one point union of 2g circles. By the invariance of the Reidemeister torsion for simple homotopies, both torsion $\tau_\rho(N_1)$ and $\tau_\rho(X)$ are equal when we identify both of 1-dimensional homologies with $su(2)_\rho$-coefficients.

Lemma 4.1. $\hat{v}_1 = \epsilon_1 \tau_\rho(X)$.

Proof.

By the definition of the Reidemeister torsion, we have the following;

$$\tau_\rho(X) = (\mathbf{h}_1 \mathbf{b}_0 / \mathbf{c}_1)(\mathbf{b}_0 \mathbf{h}_0 / \mathbf{c}_0)^{-1}.$$

Here from the irreducibility of ρ, we have that $H_0(X; su(2)_\rho)$ is vanishing. Then we have the exact sequence

$$0 \to B_0(X; su(2)_\rho) \to C_0(X; su(2)_\rho) \to 0.$$

Hence the volume $\mathbf{c}_0$ is considered as a volume on $B_0(X; su(2)_\rho)$. We may suppose that

$$\mathbf{b}_0 = \mathbf{c}_0$$

because the torsion is independent of the choices of $\mathbf{b}_0$. Then by the exact sequence,

$$0 \to Z_1(X; su(2)_\rho) \to C_1(X; su(2)_\rho) \to C_0(X; su(2)_\rho) \to 0,$$

volumes $\mathbf{c}_0$ and $\mathbf{c}_1$ give a volume $\mathbf{c}_1/\mathbf{c}_0$ on $Z_1(X; su(2)_\rho)$. Finally from the exact sequence

$$0 \to Z_1(X; su(2)_\rho) \to H_1(X; su(2)_\rho) \to 0,$$

a volume $\mathbf{c}_1/\mathbf{c}_0$ is considered as a volume on $H_1(X; su(2)_\rho)$. When a volume $\mathbf{h}_1$ on $H_1(X; su(2)_\rho)$ is given, $\tau_\rho(X)$ is the ratio of $\mathbf{h}_1$ and $\mathbf{c}_1/\mathbf{c}_0$ because it holds that

$$(\mathbf{h}_1/\mathbf{c}_1/\mathbf{c}_0) = (\mathbf{h}_1\mathbf{c}_o/\mathbf{c}_1).$$

By the definition of $\mathbf{c}_1$,

$$\mathbf{c}_1 = v_\Pi(\theta)$$

and similarly

$$\mathbf{c}_0 = \theta.$$

Hence we have

$$\tau_\rho(X) = (\mathbf{c}_1/\mathbf{c}_0)^{-1} \in (\det H_1(X; su(2)_\rho))^{-1}.$$

Therefore we have

$$\tau_\rho(X) = \hat{v}_1$$

up to sign. This completes the proof.

Here we have the following by the argument in §0,

$$\tau_\rho(N_1)\tau_\rho(N_2) = \tau_\rho(M)\tau_\rho(\Sigma_g).$$

Then by the results of Johnson, we have

$$\hat{v}_1 \wedge \hat{v}_2 = t_\rho \cdot \hat{v}_g$$

and

$$t_\rho = \tau_\rho(M).$$

Therefore we have the following by the above lemma, up to sign,

$$\hat{v}_g = \tau_\rho(\Sigma_g).$$

This completes the proof of Main theorem.

Chapter 5

Twisted Alexander polynomial and Reidemeister torsion

§0. Introduction

In 1992, Wada [18] defined the twisted Alexander polynomial for finitely presentable groups. Let Γ be a finitely presentable group. We suppose that the abelianization $\Gamma/[\Gamma,\Gamma]$ is a free abelian group $T_r =< t_1, \ldots, t_r |\ t_i t_j = t_j t_i >$ of rank r. Then we will assign a Laurent polynomial $\Delta_{\Gamma,\rho}(t_1, \ldots, t_r)$ with a unique factorization domain R-coefficients to each linear representation $\rho : \Gamma \to GL(n; R)$. We call it the twisted Alexander polynomial of Γ associated to ρ. For simplicity, we suppose that R is the real number field $\mathbf{R}$ and the image of ρ is included in $SL(n; \mathbf{R})$.

Because we are mainly interested in the case of the group of a knot, hereafter we suppose that Γ is a knot group. Let $K \subset S^3$ be a knot and E its exterior of K. We denote the canonical abelianization of Γ by $\alpha : \Gamma \to T =< t >$ and the twisted Alexander polynomial $\Delta_{\Gamma,\rho}(t)$ for $\Gamma = \pi_1 E$ by $\Delta_{K,\rho}(t)$. It is a generalization of the Alexander polynomial $\Delta_K(t)$ of K in the following sense. The Alexander polynomial $\Delta_K(t)$ of K is written as

$$\Delta_K(t) = (1-t)\Delta_{K,\mathbf{1}}(t)$$

where $\mathbf{1} : \Gamma \to SL(2; \mathbf{R}) = \{1\}$ is the 1-dimensional trivial representation of Γ.

In 1962, Milnor [12] proved the following theorem about the connection between the Alexander polynomial and the Reidemeister torsion. We consider the abelianization $\alpha : \Gamma \to T$ as a representation of Γ over $\mathbf{R}(t)$ where $\mathbf{R}(t)$ is the rational function field over $\mathbf{R}$. Then Milnor's theorem is the following.

THEOREM(MILNOR). *The Alexander polynomial $\Delta_K(t)$ of K is the Reidemeister torsion $\tau_\alpha(E)$ of E for α; that is,*

$$\Delta_K(t) = (t-1)\tau_\alpha E.$$

This Reidemeister torsion is just the inverse of the one defined in chapter 1. In this chapter we deal this type of the Reidemeister torsion as in chapter 4. We consider the following problem.

PROBLEM. *Can we consider the twisted Alexander polynomial of K as a Reidemeister torsion of its exterior E ?*

To state the main theorem, we define the tensor representation $\rho \otimes \alpha$ by the following. For the representation $\rho : \Gamma \to SL(n; \mathbf{R})$, we define the representation $\rho \otimes \alpha : \Gamma \to GL(n; \mathbf{R}(t))$ by

$$(\rho \otimes \alpha)(x) = \rho(x)\alpha(x) \text{ for } {}^\forall x \in \Gamma.$$

Then our main theorem is the following.

Main Theorem. *The twisted Alexander polynomial* $\Delta_{K,\rho}(t)$ *associated to* ρ *is the Reidemeister torsion* $\tau_{\rho\otimes\alpha}E$ *for* $\rho\otimes\alpha$ *; that is,*

$$\Delta_{K,\rho}(t) = \tau_{\rho\otimes\alpha}E.$$

As an application of this interpretation, we obtain the symmetry of the twisted Alexander polynomial in the following sense.

Corollary. *If* ρ *is equivalent to an* $SO(n)$*-representation, then*

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1})$$

up to a factor ϵt^{nk} *where* $\epsilon \in \{\pm 1\}$ *and* $k \in \mathbf{Z}$.

Now we describe the contents of this chapter briefly. In section 1 we review the theory of the twisted Alexander polynomial. We restrict the definition to the case of the group of a knot. In section 2 we give a proof of Main theorem. In section 3 as an application of Main Theorem, we proof the symmetry of the twisted Alexander polynomial in our contexts.

§1. Twisted Alexander polynomial

Let $K \subset S^3$ be a knot and Γ the knot group $\pi_1 E$. Let $F_k =< x_1, \ldots, x_k >$ denote a free group of rank k and $T =< t >$ an infinite cyclic group. The group ring T over $\mathbf{R}$ is the Laurent polynomial ring $\mathbf{R}[t^{\pm 1}]$ over $\mathbf{R}$. A homomorphism

$$\alpha : \Gamma \to T$$

is the canonical abelianization. This α induces a ring homomorphism of the integral group ring

$$\tilde{\alpha} : \mathbf{Z}[\Gamma] \to \mathbf{Z}[t^{\pm 1}].$$

We choose and fix a Wirtinger presentation

$$P(\Gamma) =< x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} >$$

of Γ and

$$\phi : F_k \to \Gamma$$

the associated homomorphism of the free group F_k to Γ. Similarly ϕ induce a ring homomorphism

$$\tilde{\phi} : \mathbf{Z}[F_k] \to \mathbf{Z}[\Gamma].$$

Let $\rho : \Gamma \to SL(n; \mathbf{R})$ be a representation. The corresponding ring homomorphism of the integral ring $\mathbf{Z}[\Gamma]$ to the matrix algebra $M_n(\mathbf{R})$ is denoted by $\tilde{\rho} : \mathbf{Z}[\Gamma] \to M_n(\mathbf{R})$. The composition of the ring homomorphism $\tilde{\phi}$ and the tensor product homomorphism

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbf{Z}[\Gamma] \to M_n(\mathbf{R}[t^{\pm 1}])$$

will be used so often that we introduce a new symbol

$$\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} : \mathbf{Z}[F_k] \to M_n(\mathbf{R}[t^{\pm 1}]).$$

Let us consider the $(k-1) \times k$ matrix $A_{\rho\otimes\alpha}$ whose (i,j)-component is the $n \times n$ matrix $\Phi(\frac{\partial r_i}{\partial x_j}) \in M_n(\mathbf{R}[t^{\pm 1}])$. This matrix $A_{\rho\otimes\alpha}$ is called the generalized Alexander matrix of the presentation $P(\Gamma)$ associated to the representation ρ. By the definition, the classical Alexander matrix A is $A_{\mathbf{1}\otimes\alpha}$ where $\mathbf{1}$ is a 1-dimensional trivial representation of Γ. For $1 \le {}^\forall j \le k$, let us denote by $A^j_{\rho\otimes\alpha}$ the $(k-1)\times(k-1)$ matrix obtained from $A_{\rho\otimes\alpha}$ by removing the j-th column. Now regard $A^j_{\rho\otimes\alpha}$ as a $(k-1)n \times (k-1)n$ matrix with coefficients in $\mathbf{R}[t^{\pm 1}]$. The following two lemmas are the foundation of our definition of the twisted Alexander polynomial.

LEMMA 1.1. *$det\Phi(x_j - 1) \neq 0$ for $1 \le {}^\forall j \le k$.*

Proof. Since we fix a Wirtinger presentation $P(\Gamma)$ as a presentation of Γ, we have

$$\alpha(x_j) = t \neq 1$$

for $1 \le {}^\forall j \le k$. Then $\det\Phi(x_j - 1) = \det(t\rho(x_j) - I)$ is the characteristic polynomial of $\rho(x_j)$ where I is the unit matrix. This completes the proof of Lemma 1.1.

LEMMA 1.2. *$detA^j_{\rho\otimes\alpha}det\Phi(x_{j'} - 1) = \pm detA^{j'}_{\rho\otimes\alpha}det\Phi(x_j - 1)$ for $1 \le {}^\forall j < {}^\forall j' \le k$.*

Proof. We may assume that $j = 1$ and $j' = 2$ without the loss of generality. Since any relator $r_i = 1$ in $\mathbf{Z}[\Gamma]$, it is easy to see that

$$\sum_{l=1}^{k} \frac{\partial r_i}{\partial x_l}(1 - x_l) = 0$$

in $\mathbf{Z}[\Gamma]$. Then apply the homomorphism Φ to this, we have

$$\sum_{l=1}^{k} \Phi(\frac{\partial r_i}{\partial x_l})\Phi(x_l - 1) = 0.$$

Let $\tilde{A}^2_{\rho\otimes\alpha}$ be the matrix obtained from $A^2_{\rho\otimes\alpha}$ by replacing

$$(\Phi(\frac{\partial r_i}{\partial x_1}), \Phi(\frac{\partial r_i}{\partial x_3}), \ldots, \Phi(\frac{\partial r_i}{\partial x_k}))$$

with

$$(\Phi(\frac{\partial r_i}{\partial x_1})\Phi(x_1 - 1), \Phi(\frac{\partial r_i}{\partial x_3})\Phi(x_3 - 1), \ldots, \Phi(\frac{\partial r_i}{\partial x_k})\Phi(x_k - 1)).$$

Then we have

$$\det\tilde{A}^2_{\rho\otimes\alpha} = \pm\det A^2_{\rho\otimes\alpha}\det\Phi(x_1 - 1).$$

Since

$$\Phi(\frac{\partial r_i}{\partial x_1})\Phi(x_1 - 1) = -\sum_{l=2}^{k} \Phi(\frac{\partial r_i}{\partial x_l})\Phi(x_l - 1),$$

we can reduce the matrix $\tilde{A}^2_{\rho\otimes\alpha}$ to $\tilde{A}^1_{\rho\otimes\alpha}$ where the matrix $\tilde{A}^1_{\rho\otimes\alpha}$ can be obtained by multiplying the first column of the matrix $A^1_{\rho\otimes\alpha}$ by $\Phi(x_2 - 1)$. Therefore we have

$$\begin{aligned} \det\tilde{A}^2_{\rho\otimes\alpha} &= \det\tilde{A}^1_{\rho\otimes\alpha} \\ &= \pm\det A^1_{\rho\otimes\alpha}\det\Phi(x_2 - 1). \end{aligned}$$

This completes the proof of this lemma.

By Lemma 1.1 and Lemma 1.2, we can define the twisted Alexander polynomial of K associated to the representation ρ to be the rational expression

$$\Delta_{K,\rho}(t) = \frac{\det A^1_{\rho\otimes\alpha}}{\det\Phi(x_1 - 1)}.$$

THEOREM(WADA). *The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is well-defined up to a factor ϵt^{nk} as an invariant of the oriented knot type of K where $\epsilon \in \{\pm 1\}$, $k \in \mathbf{Z}$ and n is a degree of ρ.*

REMARK. *Two representations ρ and ρ' are said to be equivalent if there is an element $g \in GL(n; \mathbf{R})$ such that $\rho'(x) = g \cdot \rho(x) \cdot g^{-1}$ in $SL(n; \mathbf{R})$ for $^\forall x \in \Gamma$. Then the twisted Alexander polynomials for ρ and ρ' are the same ;*

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$$

up to a factor $\epsilon t^{\alpha n}$ where $\epsilon \in \{\pm 1\}$ and $\alpha \in \mathbf{Z}$.

§2. Proof of Main theorem

In this section, let $\mathbf{F}$ be the rational function field $\mathbf{R}(t)$ and $V = \mathbf{R}(t)^n$ the n-dimensional vector space over $\mathbf{R}(t)$. We recall a Wirtinger presentation $P(\Gamma)$ of the knot group Γ of K is given by as follows ;

$$P(\Gamma) = \langle x_1, x_2, \ldots, x_k \mid r_1, r_2, \ldots, r_{k-1} \rangle$$

where r_i is the crossing relation for each i.

Let W be a 2-dimensional complex constructed from one 0-cell p, k 1-cells $x_1, \ldots, x_k$ and $(k-1)$ 2-cells $D_1, \ldots, D_{k-1}$ with attaching maps given by $r_1, \ldots, r_{k-1}$. It is well-known that the exterior E of K collapses to the 2-dimensional complex W. If a representation $\rho : \Gamma \to SL(n; \mathbf{R})$ is fixed, we have the following by the simple homotopy invariance of the Reidemeister torsion ;

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(W; V_{\rho\otimes\alpha})$$

up to a factor $\epsilon t^{\alpha n}$ where $\epsilon \in \{\pm 1\}$ and $\alpha \in \mathbf{Z}$. In this case, we show that

$$\tau(W; V_{\rho\otimes\alpha}) = \frac{\det A^1_{\rho\otimes\alpha}}{\det\Phi(x_1 - 1)}.$$

By easy computation, this chain complex $C_*(W; V_{\rho\otimes\alpha})$ is as follows;

$$0 \longrightarrow V^{k-1} \xrightarrow{\partial_2} V^k \xrightarrow{\partial_1} V \longrightarrow 0$$

where

$$\partial_2 = A_{\rho\otimes\alpha} = \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_1}) & \cdots & \Phi(\frac{\partial r_1}{\partial x_n}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_{n-1}}{\partial x_1}) & \cdots & \Phi(\frac{\partial r_{n-1}}{\partial x_n}) \end{pmatrix},$$

$$\partial_1 = \begin{pmatrix} \Phi(x_1 - 1) \\ \Phi(x_2 - 1) \\ \vdots \\ \Phi(x_n - 1) \end{pmatrix}.$$

Here we briefly denote by V^l the l-times direct sum of V.

PROPOSITION 2.1. *All homology groups vanish : $H_*(W; V_{\rho\otimes\alpha}) = 0$ if and only if $detA^1_{\rho\otimes\alpha} \neq 0$. In this case, we have*

$$\tau(W; V_{\rho\otimes\alpha}) = \frac{detA^1_{\rho\otimes\alpha}}{det\Phi(x_1 - 1)}.$$

Proof. It is obvious that $H_0(W; V_{\rho\otimes\alpha})$ is trivial because $\det\Phi(x_1 - 1) \neq 0$ and hence the boundary map ∂_1 is surjective. For a canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V, we choose lifts

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= (\Phi(x_1 - 1)^{-1}\mathbf{e}_1, \mathbf{0}, \ldots, \mathbf{0}),\\ &\vdots\\ \tilde{\mathbf{e}}_n &= (\Phi(x_1 - 1)^{-1}\mathbf{e}_n, \mathbf{0}, \ldots, \mathbf{0})\end{aligned}$$

in V^n. Define the $kn \times kn$ matrix M whose first $(kn - n)$ rows are $A_{\rho\otimes\alpha}$ and last n rows are $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n$. The matrix M takes the form

$$M = \begin{pmatrix} * & & A^1_{\rho\otimes\alpha} & \\ \Phi(x_1 - 1)^{-1}\mathbf{e}_1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ \Phi(x_1 - 1)^{-1}\mathbf{e}_n & 0 & \ldots & 0 \end{pmatrix}.$$

It is obvious that $\det M \neq 0$ if and only if $\det A^1_{\rho\otimes\alpha} \neq 0$. If all homology groups $H_*(W; V_{\rho\otimes\alpha}) = 0$, then

$$\begin{aligned}\operatorname{rank} A_{\rho\otimes\alpha} &= \operatorname{rank} A^1_{\rho\otimes\alpha}\\ &= kn - n.\end{aligned}$$

Hence we have

$$\det A^1_{\rho\otimes\alpha} \neq 0.$$

In this case the Reidemeister torsion is given by

$$\begin{aligned}\tau(W; V_{\rho\otimes\alpha}) =& \det M\\ =& \frac{\det A^1_{\rho\otimes\alpha}}{\det\Phi(x_1 - 1)}.\end{aligned}$$

It is clear that the contrary is also true. Namely if $\det A^1_{\rho\otimes\alpha} \neq 0$, then $H_*(W; V_{\rho\otimes\alpha})$ is trivial. This completes the proof.

By the above proposition, we have the proof of Main theorem.

§3. Symmetry of the twisted Alexander polynomial

Hereafter we suppose that ρ is conjugate to an $SO(n)$-representation of Γ. For simplicity, we may suppose that ρ is an $SO(n)$-representation.

We fix a structure of the simplicial complex in the exterior E and assume that each simplex of E has a dual cell. For a q-simplex of E, we can define not only the dual $(3-q)$-cell in E, but also the dual $(2-q)$-cell in the boundary ∂E. Taking the cells of both types, we obtain a dual complex E' with subcomplex $\partial E'$. We denote the universal covering complex of E by $\tilde{E}$ and the one of E' by $\tilde{E}'$. Let $< c', c >$ denote the algebraic intersection number of $c' \in C_{3-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z})$ and $c \in C_q(\tilde{E}; \mathbf{Z})$. Next lemma is well-known fact. See Milnor [12] for details.

LEMMA 3.1. *The left* $\mathbf{Z}[\Gamma]$*-module* $C_{3-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z})$ *is canonically isomorphic to the dual of* $C_q(\tilde{E}, \mathbf{Z})$ *and the dual pairing*

$$[\ \ ,\ \ ] : C_{3-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z}) \times C_q(\tilde{E}; \mathbf{Z}) \to \mathbf{Z}[\Gamma]$$

is given by

$$[c', c] = \sum_{x \in \Gamma} < c', cx^{-1} > x$$

for $\forall c' \in C_{n-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z})$ *and* $\forall c \in C_q(\tilde{E}; \mathbf{Z})$.

Now let us apply this duality to the torsion invariant. Let $V^*_{\rho\otimes\alpha}$ denote the dual vector space of $V_{\rho\otimes\alpha}$. A structure of left $\mathbf{Z}[\Gamma]$-module in $V^*_{\rho\otimes\alpha}$ is given by the following ; for $\forall x \in \Gamma, \forall u^* \in V^*_{\rho\otimes\alpha}$, and $\forall v \in V_{\rho\otimes\alpha}$,

$$(x \cdot u^*)(v) = u^*({}^t(\rho \otimes \alpha)(x)^{-1} \cdot v)$$

Then we denote this dual representation space by $V^*_{\rho\otimes\alpha}$ and define the dual pairing

$$C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha}) \times C_q(E; V_{\rho\otimes\alpha}) \to \mathbf{R}$$

by

$$(c' \otimes u^*, c \otimes v) = u^*([c', c]v)$$

for $\forall c' \otimes u^* \in C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha})$ and $\forall c \otimes v \in C_q(E; V_{\rho\otimes\alpha})$. Hence it is straightforward that $C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha})$ is isomorphic to the dual of $C_q(E; V_{\rho\otimes\alpha})$.

LEMMA 3.2. *Let* C_* *be an acyclic chain complex with preferred basis* $\{c_i\}$ *and* C^* *the dual complex with preferred basis* $\{c_i^*\}$. *Then we have*

$$\tau(C_*) = \tau(C^*)$$

up to a factor $\epsilon \in \{\pm 1\}$.

By this lemma and the invariance of the Reidemeister torsion for the subdivision of the cell complex, we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(E, \partial E; V^*_{\rho\otimes\alpha}).$$

We define a representation $\bar{\alpha} : \Gamma \to T$ by $\bar{\alpha}(x) = \alpha(x)^{-1}$. For the tensor representation $\rho \otimes \alpha$, because ρ is an $SO(n)$-representation, the dual representation $(\rho \otimes \alpha)^* : \Gamma \to GL(n; \mathbf{R}(t))$ is given by the following ;

$$\begin{aligned}(\rho \otimes \alpha)^*(x) &= {}^t\rho(x)^{-1}\bar{\alpha}(x) \\ &= \rho(x)\bar{\alpha}(x) \\ &= (\rho \otimes \bar{\alpha})(x)\end{aligned}$$

for ${}^\forall x \in \Gamma$. Therefore the representation space $V^*_{\rho\otimes\alpha}$ is equivalent to $V_{\rho\otimes\bar{\alpha}}$. Hence from the above observation, we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(E, \partial E; V_{\rho\otimes\bar{\alpha}}).$$

Similarly it is easy to show that

$$\tau(E; V_{\rho\otimes\bar{\alpha}}) = \tau(E, \partial E; V_{\rho\otimes\alpha}).$$

Apply Lemma 2.1 in chapter 2 to the short exact sequence :

$$0 \to C_*(\partial E; V_{\rho\otimes\alpha}) \to C_*(E; V_{\rho\otimes\alpha}) \to C_*(E, \partial E; V_{\rho\otimes\alpha}) \to 0,$$

we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(\partial E; V_{\rho\otimes\alpha})\tau(E, \partial E; V_{\rho\otimes\alpha}).$$

In this chapter 2, we computed the Reidemeister torsion of $\partial E \cong T^2$. From the results of chapter 2, we have

$$\tau(\partial E; V_{\rho\otimes\alpha}) = 1 \text{ up to a factor } \epsilon t^n.$$

Hence combine the above lemmas,

$$\begin{aligned}\tau(E; V_{\rho\otimes\alpha}) &= \tau(E, \partial E; V_{\rho\otimes\alpha}) \\ &= \tau(E; V_{\rho\otimes\bar{\alpha}}).\end{aligned}$$

By the definition of the twisted Alexander polynomial and Main theorem, it is obvious that

$$\tau(E; V_{\rho\otimes\bar{\alpha}}) = \Delta_{K,\rho}(t^{-1}).$$

Therefore we have

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1}).$$

This completes the proof of Corollary.

Chapter 6

Open Problems

§1. Surface bundles over S^1

Let Σ_g denote the closed oriented surface of genus g and $D \subset \Sigma_g$ an fixed embedded 2-disk. Let

$$f : \Sigma_g \to \Sigma_g$$

be an orientation preserving diffeomorphism. We assume that the restriction of f to D is the identity map on D. We define the 3-manifold M_f by

$$M_f = \Sigma_g \times [0,1]/\sim$$

where $\sim$ means the following ; we identify $(x, 1)$ with $(f(x), 0)$ for $^{\forall}x \in \Sigma_g$. We denote

$$\pi : M_f \to S^1 = [0,1]/\sim$$

the natural projection. This is a Σ_g-bundle over S^1 with the monodromy map f. It is easy to see that its fundamental group $\pi_1 M_f$ has a following presentation.

$$\begin{aligned} \pi_1 M_f = < a_1, b_1, \ldots, a_g, b_g, t \mid [a_i, b_i] &= 1, \\ t a_i t^{-1} &= f_*(a_i), \\ t b_i t^{-1} &= f_*(b_i) > \end{aligned}$$

where $f_* : \pi_1 \Sigma_g \to \Sigma_g$ is the induced homomorphism of f. In this presentation we identify the subgroup

$$T =< t >\subset \pi_1 M_f$$

with $\pi_1 S^1$. From the map

$$\pi : M_f \to S^1,$$

we have the representation

$$\alpha : \pi_1 M_f \to T.$$

We consider that $T \subset \mathbf{Q}(t) - \{0\}$ where $\mathbf{Q}(t)$ is the 1-variable rational function field over $\mathbf{Q}$. Let

$$p : \tilde{M}_f \to M_f$$

denote the infinite cyclic covering of M_f. Then we define the chain complex with $\mathbf{Q}(t)$-coefficients $C_*(M_f; \mathbf{Q}(t)_\alpha)$ by

$$C_*(\tilde{M}_f; \mathbf{Z}) \otimes_{\mathbf{Z}[T]} \mathbf{Q}(t).$$

It is easy to see that all homology groups $H_*(E; \mathbf{Q}(t)_\alpha)$ vanish. In this case, we define the Reidemeister torsion

$$\tau_\alpha(M_f) \in \mathbf{Q}(t) - \{0\}.$$

By the definition of the Reidemeister torsion, this is a topological invariant of surface bundles over S^1. Because the restriction of f to D is the identity map, we may consider

$$N = D \times S^1 \subset M_f.$$

Then we have the torus decomposition of

$$M_f = M_{f,0} \cup_{T^2} N$$

where $M_{f,0}$ is given by the closure $\overline{M_f - N}$. By Lemma 4.1 (chapter 1) and Proposition 4.4(chapter 6), we have

$$\tau_\alpha M_f = (\tau_\alpha M_{f,0})(\tau_\alpha N)$$

up to a factor ϵt^k where $\epsilon \in \{\pm 1\}$ and $k \in \mathbf{Z}$. We conjecture that this Reidemeister torsion is just the Alexander polynomial of

$$M_f \text{ with } \alpha : \pi_1 M_f \to T,$$

and

$$M_{f,0} \text{ with } \alpha : \pi_1 M_{f,0} \to T.$$

This type of Alexander polynomial of 3-manifolds with representations in T has been studied in Kawauchi [6]. Furthermore Casson's invariant of knots is the second derivative of the Alexander polynomial at 1. Therefore we conjecture this invariant is also related to the Casson type invariant of surface bundles over S^1.

§2. Reidemeister torsion of 2-knots

Let $S \subset S^4$ be a smooth 2-knot, that is, S is a smooth embedded 2-dimensional sphere in S^4. Let E denote its exterior. Then E is a compact 4-dimensional manifolds with boundary $S^2 \times S^1$. It is easy to see that

$$H_1(E; \mathbf{Z}) = \mathbf{Z}$$

and

$$\chi(E) = 0.$$

Then we identify the first integral homology group $H_1(E; \mathbf{Z})$ with $T = < t >$ and denote the natural abelianization map by

$$\alpha : \pi_1 E \to T.$$

Let

$$\pi : \tilde{E} \to E$$

denote the infinite cyclic covering of E. The group T acts on $\tilde{E}$ as deck transformations. When we consider

$$\alpha : \pi_1 E \to T$$

as a representation over $\mathbf{Q}(t)$, we can define the chain complex $C_*(E; \mathbf{Q}(t)_\alpha)$ by

$$C_*(\tilde{E}; \mathbf{Z}) \otimes_{\mathbf{Z}[T]} \mathbf{Q}(t)$$

where $\mathbf{Q}(t)$ is the 1-variable rational function field over $\mathbf{Q}$. Then we define the Reidemeister torsion

$$\tau_\alpha(E) \in \mathbf{Q}(t) - \{0\}$$

as in chapter 6. In general E is not aspherical, especially,

$$\pi_2 E \neq 0.$$

In this case, the Alexander polynomial of S is defined as in the case of knots in S^3. But this polynomial invariant is determined by only $\pi_1 E$ and carries no information about $\pi_2 E$. On the other hand the Reidemeister torsion $\tau_\alpha(E)$ is not determined by $\pi_1 E$ and has more higher and deeper information of E. Therefore we believe the importance of the Reidemeister torsion in the theory of 2-knots. This problem has been inspired by communications with Professor Wada.

Appendix

§1. Crossed homomorphism

In this section, we describe the crossed homomorphism.

DEFINITION 1.1. *Let π be a group and M a π-module. A map $\varphi : \pi \to M$ is a crossed homomorphism if and only if*

$$\varphi(xy) = \varphi(x) + x\varphi(y) \text{ for } {}^\forall x, {}^\forall y \in \pi.$$

The set of crossed homomorphisms is an abelian group under the pointwise addition. We denote it by $\times\mathrm{Hom}(\pi, M)$. By the above law, we have the following properties.

(1) $\varphi(1) = 0$ where $1 \in \pi$ is a unit element.

(2) $\varphi(x^{-1}) = -x^{-1}\varphi(x)$ for ${}^\forall x \in \pi$.

EXAMPLES.

(1) If $\varphi : \pi \to M$ is a crossed homomorphism and $f : M \to N$ a π-module homomorphism, then $f \circ \varphi : \pi \to N$ is also a crossed homomorphism.

(2) Let $\varphi : \pi \to \mathbf{Z}[\pi]$ be a map defined by $\varphi(x) = x - 1$ where $\mathbf{Z}[\pi]$ is an integral group ring of π, then φ is a crossed homomorphism.

(3) We generalize example (2). Let M be a π-module and $m \in M$ a fixed element. Let $\varphi_m : \pi \to M$ be defined by $\varphi(x) = (x-1)m$. Then φ_m is a crossed homomorphism. Such a map is called the principal crossed homomorphism. Example (2) is just the case of $m = 1 \in \mathbf{Z}[\pi]$.

Let F_n be a free group of rank n generated by $x_1, x_2, \ldots, x_n$. We investigated the crossed homomorphism $\varphi : F_n \to \mathbf{Z}[F_n]$. Let the value of φ at x_i be arbitarily specified to be φ_i. Then there is a unique extension to ${}^\forall x \in F_n$. Clearly it is unique since φ_i determines φ on all words. To see that it is well defined, we note the two words w_1, w_2 are equal in F_n if and only if they differ by moves of the next form ;

$$w_1 w_2 \longleftrightarrow w_1 x_i x_i^{-1} w_2 \longleftrightarrow w_1 x_i^{-1} x_i w_2$$

Applying the proposed definition of φ to the 3-words

$$w_1 w_2,\ w_1 x_i x_i^{-1} w_2, \text{ and } w_1 x_i^{-1} x_i w_2$$

and then the results are equal in $\mathbf{Z}[F_n]$. Thus the crossed homomorphism φ exists and it is unique.

In partiqular, define

$$\frac{\partial}{\partial x_i} : F_n \to \mathbf{Z}[F_n]$$

by

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}$$

for $^\forall i \in \{1, \ldots, n\}$. These crossed homomorphisms are called the Fox's free derivatives. It is easy to see that they form a basis for $\times\mathrm{Hom}(F_n, \mathbf{Z}[F_n])$ as a $\mathbf{Z}[F_n]$-module. The action of the group π on $\times\mathrm{Hom}(F_n, \mathbf{Z}[F_n])$ given by the right multiplication. Namely for $^\forall x \in \pi$ and $^\forall \phi \in \times\mathrm{Hom}(F_n, \mathbf{Z}[F_n])$,

$$x \cdot \varphi : F_n \to \mathbf{Z}[F_n]$$

is given by

$$x \cdot \varphi(w) = \varphi(w)x^{-1}.$$

REMARK. *If $x_1, \ldots, x_n \in \pi$ and w is a word in $x_1, \ldots, x_n$, then $\frac{\partial w}{\partial x_i}$ still makes sense as an element of $\mathbf{Z}[\pi]$. Then for a crossed homomorphism $\varphi : \pi \to M$, we have*

$$\varphi(w) = \sum_i \frac{\partial w}{\partial x_i}\varphi(x_i).$$

We can easily prove the next proposition.

PROPOSITION 1.2. *We take a group π with generator $x_1, x_2, \ldots, x_n$ and relations $r_i(x_1, \ldots, x_n) = 1$ for $^\forall i \in \{1, \ldots, k\}$. Let M be a π-module and $m_1, \ldots, m_n \in M$, then $\varphi(x_i) = m_i$ defines a unique crossed homomorphism from π to M if and only if $\sum_j \frac{\partial r_i}{\partial x_j} m_j = 0$ in M for $^\forall i$.*

We define a natural π-module associated to the group π.

DEFINITION 1.3. *Let $d\pi$ be the π-module with generator dx for $^\forall x \in \pi$ and relations $d(xy) = dx + xdy$ for $^\forall x, {}^\forall y \in \pi$.*

We have directly the following facts from the above definition.

(1) $d(1) = 0$.

(2) $d(x^{-1}) = -x^{-1}dx$.

(3) The map $d : \pi \to d\pi$ given by $x \mapsto dx$ is a crossed homomorphism.

(4) If $x_1, \ldots, x_n \in \pi$ and w is a word in the x_i's, then we have

$$dw = \sum_{i=0}^{n} \frac{\partial w}{\partial x_i} dx_i.$$

This follows from the law

$$\varphi(w) = \sum_{i=0}^{n} \frac{\partial w}{\partial x_i} \varphi(x_i)$$

by applying it to the crossed homomorphism d.

(5) For any crossed homomorphism $\varphi : \pi \to M$, there exists a unique $\mathbf{Z}[\pi]$-homomorphism $f : d\pi \to M$ such that the next diagram is commutative.

$$\begin{array}{ccc} \pi & & \\ {\scriptstyle d}\downarrow & \overset{\varphi}{\searrow} & \\ d\pi & \xrightarrow{f} & M \end{array}$$

Then we must define f by

$$f(dx) = \varphi(x).$$

Hence we have an induced homomorphism

$$d_* : \mathrm{Hom}_\pi(d\pi, M) \longrightarrow \times\mathrm{Hom}(\pi, M).$$

As this homomorphism d_* is the isomorphism as a $\mathbf{Z}[\pi]$-module, we frequently identify these groups by this construction with d.

(6) The principal crossed homomorphism $\pi \to \mathbf{Z}[\pi]$ sending x to $x - 1$ induces the π-module homomorphism

$$\epsilon : d\pi \to \mathbf{Z}[\pi]$$

given by

$$\epsilon(dx) = x - 1 \text{ for } {}^\forall x \in \pi.$$

§2. The homology of local coefficients

We describe a homology theory of local coefficients for cell complexes. Let X be a finite cell complex and fix a universal covering $\tilde{X} \to X$. We give $\tilde{X}$ a cell decomposition coming from X. By the cellular approximation theorem, the fundamental group $\pi_1 K$ acts freely and cellulary on $\tilde{X}$ as the deck transformations. The integral chain complex $C_*(\tilde{X}; \mathbf{Z})$ becomes the chain complex of $Z[\pi_1 X]$-modules. To be more precise, we choose a lifting $\tilde{\sigma} \subset \tilde{X}$ for each n-cell $\sigma \subset X$. We consider the boundary

$$\partial\sigma = \sum_i \epsilon_i \sigma_i$$

where $\epsilon_i \in \{\pm 1\}$ and σ_i is $(n-1)$-cell. Then

$$\partial\tilde{\sigma} = \sum_i \epsilon_i g_i \tilde{\sigma}_i$$

where $g_i \in \pi_1 X$ and $\tilde{\sigma}_i$ is the lift of σ_i. Replace $\tilde{\sigma}_i$ by σ_i, we get a chain complex of $\mathbf{Z}[\pi_1 X]$-modules with one generator for each cell of X.

DEFINITION 2.1. *Let V be a $\mathbf{Z}[\pi_1 X]$-module. Then the local homology of a finite cell complex X with coefficients in V is defined by the homology of the chain complex $C_*(\tilde{X}; \mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1 X]} V$. Similarly the local cohomology of X is defined by the cochain complex $Hom_{\mathbf{Z}[\pi_1 X]}(C_*(\tilde{X}; \mathbf{Z}), V)$ of the $\mathbf{Z}[\pi_1 X]$-module homomorphisms from $C_*(\tilde{X}; \mathbf{Z})$ to V.*

Example 1. Closed surface Σ_g

Let Σ_g denote the closed oriented surface of genus g. The fundamental group $\pi_1 \Sigma_g$ admits a presentation with $2g$ generators

$$a_1, b_1, \ldots, a_g, b_g$$

and a single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

The cells of Σ_g are given by

(0) a 0-cell P.

(1) $2g$ 1-cells $A_1, B_1, \ldots, A_g, B_g$
corresponding to the generators $a_1, b_1 \ldots, a_g, b_g$ of $\pi_1 \Sigma_g$

(2) a 2-cell U.

Figure 1.1

We lift these cells to universal cover $\tilde{\Sigma}_g$.

Figure 1.2

Hence the boundary maps $\partial_2, \partial_1, \partial_0$ of $C_*(\tilde{\Sigma}_g; \mathbf{Z})$ as a $\mathbf{Z}[\pi_1\Sigma_g]$ homomorphism are given by the following.

$$\begin{aligned}
\partial_0 P &= 0. \\
\partial_1 A_i &= (a_i - 1)P \qquad \text{for } {}^\forall i \in \{1, \ldots, g\}. \\
\partial_1 B_i &= (b_i - 1)P \qquad \text{for } {}^\forall i \in \{1, \ldots, g\}. \\
\partial_2 D &= A_1 + a_1 B_1 - a_1 b_1 a_1 A_1 - [a_1, b_1]B_1 + \ldots \\
&\quad \cdots + [a_1, b_1]\ldots[a_g, b_g](A_g + a_g B_g - a_g b_g a_g A_g - [a_g, b_g]B_g).
\end{aligned}$$

Here we can give a compact description of this complex if we replace the symbols

$$A_1, B_1, \ldots, A_g, B_g \text{ by } da_1, db_1, \ldots, da_g, db_g.$$

Then $C_1(\tilde{\Sigma}_g; \mathbf{Z})$ becomes $\mathbf{Z}[\pi_1\Sigma_g] \otimes_{\pi_1} d\mathring{\pi}$ where $\mathring{\pi}$ is the free group on $a_1, b_1, \ldots, a_g, b_g$. In this case,

$$\begin{aligned}
\partial D &= da_1 + a_1 db_1 - a_1 b_1 a_1^{-1} da_1 - [a_1, b_1]db_1 + \ldots \\
&\quad \cdots + [a_1, b_1]\ldots[a_g . b_g](da_g + a_g db_g - a_g b_g a_g da_g - [a_g, b_g]db_g).
\end{aligned}$$

It is just $d\theta$ where $\theta = [a_1, b_1]\ldots[a_g, b_g]$. If we also replace $C_2(\tilde{\Sigma}_g; \mathbf{Z})$ by $\mathbf{Z}[\pi_1\Sigma_g]$ with 1 representing D, the above chain complex becomes our standard one for Σ_g

$$0 \longrightarrow \mathbf{Z}[\pi_1] \xrightarrow{1 \rightarrow d\theta} \mathbf{Z}[\pi_1] \otimes_{\pi_1} d\mathring{\pi} \xrightarrow{\epsilon} \mathbf{Z}[\pi_1] \longrightarrow 0$$

where ϵ sends dx to $x - 1$.

Example 2

We easily generalize the above arguments to find the chain complex for any 2-dimensional cell complex X having only one vertex P.

Let $\{A_1, \ldots, A_k\}$ denote k 1-cells representing the generator of the free group $F = \pi_1(1 - \text{skelton of } X)$ and $\{D_1, \ldots, D_l\}$ l 2-cells attached by the a_i's words $\{r_1, \ldots, r_m\}$. We get a presentation for $\pi = \pi_1 X$

$$R \xrightarrow{j} F \xrightarrow{p} \pi \longrightarrow 0$$

where R is the free group on $D_1, \ldots, D_l$ and $j(D_i) = r_i$ for ${}^\forall i \in \{1, \ldots, l\}$. For $C_*(\tilde{X}; \mathbf{Z})$, we have

$$\begin{aligned}
C_0 &= \mathbf{Z}[\pi], \\
C_1 &= \mathbf{Z}[\pi] \otimes_{\mathbf{Z}[\pi]} dF, \\
\partial_0 &= \epsilon
\end{aligned}$$

as before. Now C_2 is a $\mathbf{Z}[\pi]$-free module with the generator for each D_i and the corresponding relation will be dr_i, just as with a surface. We can use $\mathbf{Z}[\pi] \otimes dR$ for C_2 with generator $\{dD_i\}$ and the boundary map $dD_i \rightarrow dr_i$ is just what we have called dj. Then we get the following discription ;

$$C_*(\tilde{X}) : 0 \longrightarrow \mathbf{Z}[\pi] \otimes_\pi dR \xrightarrow{dj} \mathbf{Z}[\pi] \otimes_\pi dF \xrightarrow{\epsilon} \mathbf{Z}[\pi] \longrightarrow 0.$$

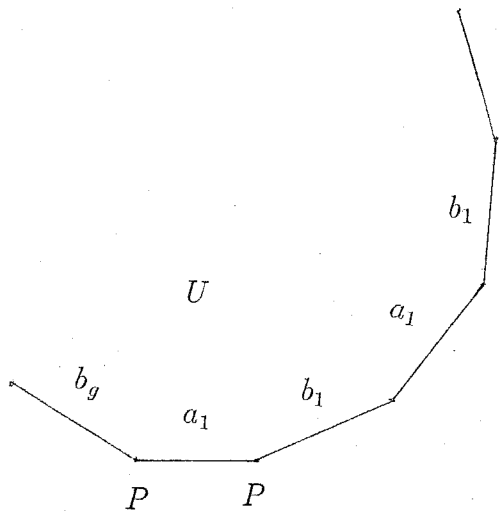

Figure 1.1

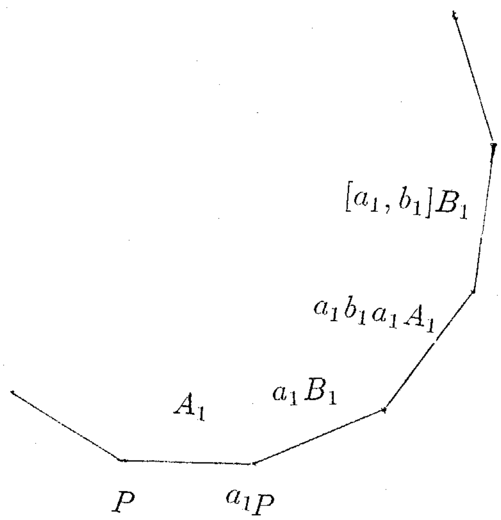

Figure 1.2

References

[1] S. Akbulut and J. D. McCarthy, Casson's invariant for oriented homology 3-spheres, Princeton Univ. Press **36** (1990).

[2] K. Brown, Cohomology of groups, GTM **87**, Springer-Verlag (1982).

[3] M. M. Cohen, A course in simple homotopy theory, GTM **10**, Springer-Verlag (1973).

[4] R. H. Crowell and R. H. Fox, Introduction to knot theory, GTM **57**, Springer-Verlag (1977).

[5] D. Johnson, A geometric form of Casson invariant and its connection to Reidemeister torsion, unpublished lecture notes.

[6] A. Kawauchi, On Quadratic Forms of 3-manifolds, Inv. Math. **43** (1977), 177–198.

[7] T. Kitano, Reidemeister torsion of Seifert fibered spaces for $SL(2;\mathbf{C})$-representations. Preprint (1992).

[8] T.Kitano, Reidemeister torsion of the figure-eight knot exterior for $SL(2;\mathbf{C})$-representations. Preprint (1993).

[9] T.Kitano Reidemeister torsion and a volume form of representation spaces. Preprint (1993).

[10] T. Kitano, Twisted Alexander polynomial and Reidemeister torsion. Preprint (1993).

[11] J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Ann. Math. **74** (1961), 575–590.

[12] J. Milnor, A duality theorem for Reidemeister torsion. Ann. Math. **76** (1962), 137–147.

[13] J. Milnor, Whitehead torsion. Bull. Amer. Math. Soc. **72** (1966), 358–426.

[14] P. Orlik, Seifert Manifolds, Lect. Notes in Math. **761**, Springer-Verlag (1972).

[15] D. B. Ray and I. M. Singer, R-torsion and the Laplacian on Riemannian manifolds, Adv. in Math. **7** (1971), 145–210.

[16] R. Riley, Nonabelian representations of 2-bride knot groups. Quar. J. Oxford (2). **35** (1984), 191–208.

[17] V. G. Turaev, Reidemeister torsion in knot theory. Russian Math. Surveys **41** (1986), 119–182.

[18] M. Wada, Twisted Alexander polynomial for finitely presentable groups. Topology(to appear).

[19] A. Weil, Remarks on the cohomology of groups. Ann. Math. **80** (1964), 149–157.