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The Behavior of Solutions of Some Semi-Linear Diffusion Equations for Large Time

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応用物理学専攻

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Preface

The equation of reaction and diffusion in one space variable is

$$(*) \qquad u_t = \frac{1}{2} u_{xx} + F(u) \qquad -\infty < x < +\infty$$

where F is a real valued function with the continuous first derivative. Such an equation arises in population dynamics, in flame propagation problems in chemical reactor theory and in other areas. The diffusion term $\frac{1}{2} u_{xx}$ may represent the random migration of the individuals of a population or chemical species in a mixture and the non-linear term $F(u)$ the reproduction rate of individuals or the speed of chemical reaction. The dependent variable u may be a population density or the temparature of a mixture. In these applications it is useally assumed, as we do, that there are two trivial steady states $u = 0$ (there is no disturbance) and $u = 1$ (the saturated state) and that all states lie between them: $F(0) = F(1) = 0$, $0 \leqq u(t,x) \leqq 1$.

The equation (*) was first introduced by R.A.Fisher in 1937 to model the spread of advantageous genetic traits in a population. In the same year the mathematical treatment of it was given in some details by A.N.Kolmogolov, I.G.Petrovsky and N.S.Piskunov who are also motivated by the application to the population genetics. They all assumed, at least, that $F(u) > 0$ for $0 < u < 1$ and $F'(0) > 0$. Ya.I.Kanel' considered, among others, the combustion case where $F(u) = 0$ for $0 < u < \mu$ and $F(u) > 0$ for $\mu < u < 1$ with $0 < \mu < 1$ (1961 to 1964). D.G.Aronson and H.F.Weinberger interpreted the case of sign change: $F(u) < 0$ for $0 < u < \mu$ and

$F(u) > 0$ for $\mu < u < 1$, as the model of population dynamics where the heterozygote is inferior (1975).

The central subject concerning the equation (*) is to find features of solutions that do not depend on details of initial states and to understand how the disturbance spreads. The simplest one of such features is the convergence to a stable steady state. In the case of Fisher or Kolmogorov et al. the saturated state is the only stable steady state and it is true that for any initial disturbance the solution approaches to unity as time t tends to infinity. But this does not fully explain the behavior of the solutions, because if the disturbance disappears at some time, so does at every later time. In this paper a more clear picture of solutions is obtained in the case

$$F(u) > 0 \quad \text{for} \quad 0 < u < 1.$$

It will be proved in this case that if the system is initialy disturbed in a finite interval and will be ultimately saturated, then we can chose functions $m(t)$ and $n(t)$ so that after sufficient time has past the solution is getting close to unity as going into the x-interval $(-n(t), m(t))$, decreasing to zero apart from it and forming fronts, called propagating fronts, around both ends of it. The function $m(t)$ (or $n(t)$) which becomes large unboundedly, will be calculated at least within $O(\log t)$ under the additional assumption: $F(u) = F'(0)u + o(u^{1+p})$ $(p > 0)$.

The phenomenon of this kind was first observed by Kolmogorov *et al*. when the initial function is the indicater function of the negative real axis. Kanel' improved this by allowing more general initial condition, though he still required them to be either monotone and zero or unity outside a finite interval, or a perturbation of a propagating front. Recently the problem has attached an increasing amount of attention and many authors obtained partial answers to it, but they were far from satisfactory in the sense that the case of initially disturbed in a bounded domain was excluded.

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The Behavior of Solutions of Some Non-Linear Diffusion Equations for Large Time

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0. Introduction.

Consider the semi-linear diffusion equation

$$(1) \qquad u^{\cdot} = \frac{1}{2} u'' + F(u) \qquad t > 0, \quad -\infty < x < \infty$$

$$\left(u = u(t,x), \quad u^{\cdot} = \frac{\partial u}{\partial t}, \quad u'' = \frac{\partial^2 u}{\partial x^2} \right)$$

with the initial condition

$$(2) \qquad u(0,\cdot) = f.$$

The function F is always assumed in this paper to satisfy

$$(3) \qquad F \in C^1[0,1], \quad F(0) = F(1) = 0 \quad \text{and} \quad F(u) > 0 \quad 0 < u < 1$$

and the initial function f to be measurable and compatible to F, i.e. $0 \leqq f \leqq 1$. Our interest is in the behavior of the solution for large time t.

We mean by the solution of (1) and (2) such a function $u(t,x)$ defined on the upper half plane $[0,\infty) \times (-\infty,\infty)$ that (i) $0 \leqq u \leqq 1$, (ii) u has continuous derivatives $u^{\cdot}$ and u'' and satisfies (1) in $(0,\infty) \times (-\infty,\infty)$, and (iii) $u(t,\cdot)$ converges to f as $t \downarrow 0$ in locally L^1 sense. It is well known that such a solution exists and is unique. We denote it by $u(t,x;f)$. It is clear that $u(t,x;u(s,\cdot;f)) = u(t+s,x;f)$ (Huygens property) and $u(t,x;f(\cdot+y)) = u(t,x+y;f)$. We sometimes consider the equation (1) with different F's and in such cases use the notation $u(t,x;f;F)$ in order to elucidate the dependence on F. There are just two trivial solutions of (1): $u \equiv 0$ and $u \equiv 1$. We always consider our problem for non-trivial initial functions f; $f \not\equiv 0$ and $\not\equiv 1$. Such initial functions are called data. We will mainly deal with such data that $f(x) \to 0$ as $x \to \infty$.

The behavior of a solution $u(t,x;f)$ is closely related to solutions of ordinary differential equations

$$(4) \qquad \frac{1}{2} w'' + c w' + F(w) = 0 \qquad (0 \leqq w \leqq 1)$$

where c is a real constant. This equation is formally obtained if we substitute the wave form $u(t,x) = w(x-ct)$ in (1). A non-trivial solution of (1) with such form is called, if exists, a travelling wave with speed c. An associated function (or, equivalently, global solution of (4)) w is called a front of a travelling wave with speed c or simply a c-front, which will be denoted by w_c. Since (4) is trasformed to $1/2\, w'' - c\, w' + F(w) = 0$ by inverting the sign of x, we always assume $c \geqq 0$.

In many articles ([1], [6], etc.) it is shown, under the restriction $F'(0) > 0$, that there exists the minimal speed, denoted by c_0, such that a c-front exists if and only if $|c| \geqq c_0$. We will give, for completeness, a proof of this assertion under present situation, though the proof is essentially the same as those given in papers cited above. Since the equation (4) is invariant under the translation along the x-axis and w_c has the corresponding ambiguity, we set the normalization: $w_c(0) = 1/2$ except in § 1 and § 2. Under this convention just one w_c corresponds to each $c \geqq c_0$.

General solutions of (1) and (2) are related with c-fronts in the following manner: if a datum f satisfies certain conditions, then

$$(5) \qquad u(t, x + m(t)) \longrightarrow w_c(x) \qquad \text{as} \quad x \longrightarrow \infty$$

where $u = u(t,x;f)$ and $m(t) = \sup\{x;\ u(t,x) = 1/2\}$ (any number, e.g. zero, may be assigned to $m(t)$ when the set expressed with braces is void). This phenomenon was observed by Kolmogorov, Petrovsky and Piscounov [13]; they proved that (5) is valid with $c = c_0$ if we set $f = I_{(-\infty,0)}$ (I_S is the indicator function of a set S). Kametaka [10] found a certain criterion on a datum for (5) to hold which is satisfied with many data but not easily checked for a given one. The main purpose of this article is to prove (5) for sufficiently general datum, e.g. any data with compact support (Theorems 8.1, 8.2 and

8.3; Theorem 8.3 contains Kametaka's result).

The method of the proofs is similar to that used by the authors mentioned above and summarized in the following. Let $u(t,x)$ be a solution of (1) and (2) and suppose that for each positive t, $u'(t,x) < 0$ on a right half x-axis, i.e. an infinite interval $\{x;\ x > N\}$. Define

$$M(t) = \sup\{u(t,x);\ u'(t,y) < 0 \text{ for all } y > x\}$$

and define for $0 \leqq w \leqq M(t)$

$$x(t,w) = \sup\{x;\ u(t,x) = w\}$$

$$\phi(t,w) = u'(t,x(t,w)).$$

Considering ϕ as functional of datum f, we denote it by $\phi(t,w;f)$. Then, since $u(t,x;w_c) = w_c(x-ct)$, $\phi(t,w;w_c)$ is independent of t: this function is denoted by $\tau_c(w)$. We will prove (5) by showing that $\phi(t,w)$ converges to $\tau_c(w)$. This will be carried out at first, in § 6, for data which is subject to several restrictions (Lemmas 6.1 to 6.4) and then, in § 8, for general data by using this result and by applying some comparison theorem on a parabolic equation. The section 7 is devoted to estimate the order of $u(t,x;f)$ decreasing to zero as t tends to infinity which justifies the application of the comparison theorem.

The section 1 is devoted to prove the existence of c-fronts. The case in which $c_0 > \sqrt{2F'(0)}$ is illustlated by examples in which an explicit form of w_{c_0} is given. Also some comparison lemmas about the equation (4) are proved. In the section 2 asymptotic behaviors of w_c for large x are investigated. Results are refinements of those obtained from the standard theory of ordinary differential equations but will play minor roles in the main story of this paper. In the section 3 we introduce comparison theorems concerninig parabolic equations which will play important roles in later arguments together with results of § 1 and § 5. In the sections 4 and 5 some properties of $u(t,x;f;F)$ which are well known or readily proved are explained. The main theorems are proved through § 6 to § 8 and formulated in § 8.

In the section 9 we will investigate the speed of $m(t)$ tending to infinity. It will be proved that if (5) occurs then $m^{\cdot}(t)$ converges to c as t tends to infinity. If $F'(0) \geqq F'(u)$ we will get, under additionl assumptions, a fine estimate;

$$c_0 t - m(t) \sim \frac{3}{2c_0} \log t \qquad \text{as} \quad t \to \infty. \text{(*)}$$

The question of when we may replace $m(t)$ by ct + const. in (5) will be answered.

In the last section an alternative method, which is a modification of that used in P.C.Fife and J.B.McLeod [3b](**), is applied to the problem described by (5) in case $c_0 > \sqrt{2F'(0)}$.

Notations. We will use throughout the article the following notations in addition to those introduced above:

$$\alpha = F'(0), \quad c^* = \sqrt{2\alpha},$$

$$\beta = \sup \frac{F(u)}{u}, \quad \gamma = \sup |F'(u)|, \quad \gamma^* = \sup F'(u)$$

(the supremum of a function is taken with respect to all arguments for which it is defined unless otherwise specified); for a real number A, $A^+ = \max\{0,A\}$, $A^- = \min\{0,A\}$; if A is a real function of z, A^+ is a function defined by $A^+(z) = A(z)^+$; $R = (-\infty,\infty)$ whole real line, $E = (0,\infty) \times R$ open half plane: $E_t = (0,t) \times R$, $\ell_t = \{t\} \times R$ $(t > 0)$;

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \qquad t > 0, \quad x \in R$$

(*) " $a(t)$ $b(t)$ as $t \to s$ " means that $\lim_{t\to s} a(t)/b(t) = 1$.

(**) Their situation is different from ours, where F changes its sign at least one time.

($p(t,x-y)$ are fundamental solutions for the heat equation $\dot{u} = 2^{-1}u''$); for $t > 0$ and a measurable function g on R we write

$$P_t g = P_t g(x) = \int_R p(t,x-y)g(y)dy$$

if this integral converges absolutely.

Some terminologies, which are used throughout this paper, are introduced in the beginning of the section 1.

Most of the results of the present paper were announced in [17].

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1. Fronts of Travelling Waves.

In this section we find all non-trivial solutions for the equation (1), called travelling wave, which has the wave form

$$u(t,x) = w(x-ct)$$

where c is a constant called a speed and w is a function on R called a front or a c-front. We denote any c-front by w_c to elucidate the speed. A c-front is characterized as a non-trivial solution of the equation (4) on R. It will be shown that if a global solution exists it is unique up to the translation along x-axis. Thus w_c corresponds at most one to each c except this ambiguity. As mentioned in §0 we treat only the case $c \geqq 0$. This amounts, as far as global solutions are concerned, to set the boundary condition

$$(1.1) \qquad w(\infty) = 0 \quad \text{and} \quad w(-\infty) = 1$$

to the equation (4).

We often consider the equation (4) in the phase plane:

$$(1.2) \qquad \begin{cases} w' = p \\ p' = -2cp - 2F(w). \end{cases}$$

The range of (w,p) is restricted to the strip $0 \leqq w \leqq 1$. Any solution of (1,2) which stays in this strip and terminates at its boundary $w = 0$ or 1 is called, for convenience, a c-<u>manifold</u>. We call a corresponding solution of (4) a c-<u>solution</u>. Thus a c-solution is a function defined and satisfying (4) on a (finite or infinite) interval, at the end points of which it attains 0 or 1. A manifold to which a c-front corresponds is also called a c-<u>front</u>. Let $c \geqq \sqrt{2\alpha}$ and put

$$(1.3) \qquad \underline{b} = c - \sqrt{c^2-2\alpha} \quad \text{and} \quad \overline{b} = c + \sqrt{c^2-2\alpha}.$$

It will be proved that there exist c-manifolds which enter the origine along a line $p = -\underline{b}w$ or a line $p = -\bar{b}w$. A c-manifold entering the origine along $p = -\underline{b}w$ (resp. $p = -\bar{b}w$) is called $(c,\underline{b})$-manifold (resp. $(c,\bar{b})$-manifold).

We will mean also by a c-manifold a corresponding curve drawn in (w,p)-plane. Parametrizing the part of this curve under w-axis with its w-coordinates, we denote its p-coordinate by $\tau(w)$. Then τ satisfies

$$(1,4) \qquad \tau' = -2c - \frac{2F(w)}{\tau}$$

in its domain of definition.

The prospects of the vecter field defined by the right-hand side of (1,2) is important in the arguments of later sections as well as of this section. It will be explained in the proof of Theorem 1.1 and illustrated in Appendix. We often use explicitely or implicitely an argument described below. Let $Q_1(x) = (w_1(x), p_1(x))$ and $Q_2(x) = (w_2(x), p_2(x))$ be smooth curves in R^2. Suppose $(w_o,p_o) = (w_1(0),p_1(0)) = (w_2(0),p_2(0))$. Then according as

$$\det \begin{pmatrix} w_1'(0) & w_2'(0) \\ p_1'(0) & p_2'(0) \end{pmatrix} < 0 \quad \text{or} \quad > 0,$$

the angle measured from $Q_2(x)$ toward $Q_1(x)$ around the point (w_o,p_o) lies in the interval $(0,\pi)$ or in the interval $(-\pi,0)$ for all sufficiently small x. If the former case (resp. the latter case) occurs we will say that the curve Q_2 crosses the curve Q_1 (at (w_o,p_o)) from the left-(resp. right-) hand side of Q_1. For example let $g(x)$ be defined and twice continuously differentiable on an interval (x_1,x_2) with $0 \leqq g \leqq 1$. Then the curve $\{(g(x), g'(x)); x_1 < x < x_2\}$ crosses c-manifolds from the right or left according as

$$-g'\left(\frac{1}{2}g'' + cg' + F(g)\right) < 0 \quad \text{or} \quad > 0$$

at intersecting points, since for a solution (w,p) of (1.2)

$$\det \begin{pmatrix} g' & w' \\ g'' & p' \end{pmatrix} = -g'\{\tfrac{1}{2}g'' + c\,g' + F(g)\} \quad \text{at} \quad (w,p) = (g,g').$$

The next theorem follows from standard arguments concerning with the 2-dimentional autonomous system. The proof is given for completeness.

Theorem 1.1. (i) There exists a positive constant c_0 such that a c-front exists if and only if $c \geq c_0$. The c-front is unique up to the translation along x-axis. (ii) c_0 satisfies that $\sqrt{2\alpha} \leqq c_0 \leqq \sqrt{2\beta}$. (iii) Let $c \geq c_0$. Then for a c-front w_c there exists $\lim_{x\to\infty} \frac{w_c'(x)}{w_c(x)} = -b$, $b = \underline{b}$ if $c > c_0$, and $b = \overline{b}$ if $c = c_0$, where $\underline{b}$ and $\overline{b}$ are defined by (1,3). (Especially $w_c(\log x)$ is regularly varying at infinity with exponent $-b$ $\div$ $w_c(x+x_0)/w_c(x) \longrightarrow \exp\{-bx_0\}$ as $x \to \infty$.)

Proof. Step 1. Consider the fields ~~for (1,2)~~ for different c's, say c and c', c' > c. Since

$$\text{Det} \begin{pmatrix} p & p \\ -2F(w)-2cp & -2F(w)-2c'p \end{pmatrix} = 2(c - c')p^2$$

is negative, c'-manifolds never cross each c-manifold from the right hand of the c-manifold (c-manifolds are considered to be directed). (cf. Fig.I) Note that the field points downward on the w-axis and that its w-component directs right in the upper half and left in the lower half of the strip $0 \leqq w \leqq 1$ (Fig.I). Clearly the c-front lies always under the w-axis if it exists.

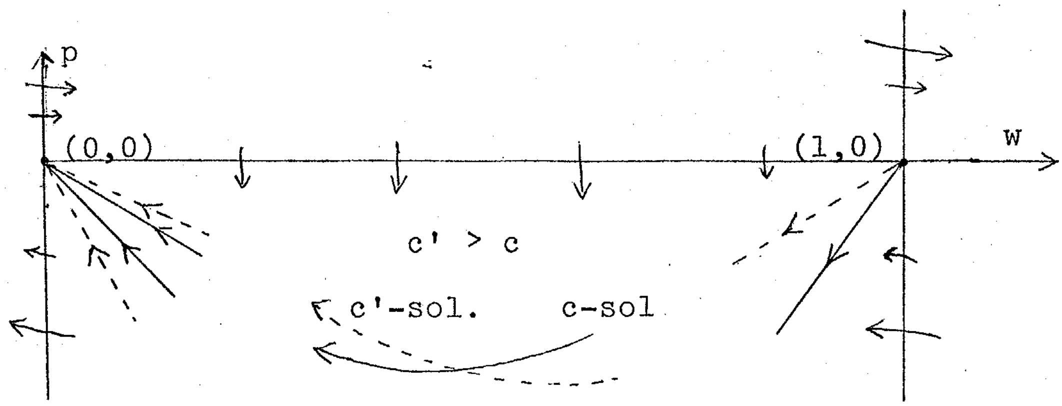

Fig.I

There exists the unique c-manifold issuing from (1,0). It is routine to prove the existance. To prove the uniqueness assertion let (w^*,p^*) and (w^{**},p^{**}) be two such manifolds. Regarding the difference $p = p^* - p^{**}$ as a function of w, we see that the derivative $dp/dw = p2F(w)/p^*p^{**}$ has the same sign with p, which implies $p \equiv 0$ since p converges to 0 as $w \to 1$.

The c'-manifold issuing from (1,0) lies over the c-manifold issuing from (1,0) if $c' > c$. This is proved by assuming the contray and by tracing back a c'-manifold passing through a point between two manifold untill getting to w-axis.

Step 2. Let a c-manifold pass a point $(w_0,-bw_0)$ where $b > 0$. The sine of the angle, made by a tangent vector of it at this point and a half line $p = -bw$, $w > 0$, which is directed to the origin, and taken from the former toward the latter, is equal to the ratio of

$$(1,5) \qquad \det \begin{pmatrix} -1 & p \\ b & -2F(w)-2cp \end{pmatrix}_{\substack{w=w_0 \\ p=-bw_0}} = [b^2 - 2bc + \frac{2F(w_0)}{w_0}]\, w_0$$

to $\sqrt{1+b^2}\sqrt{b^2+4(F(w_0)/w_0+cb)^2}\, w_0$. This ratio is less than a negative constant, say $-\varepsilon$, if $\overline{b} > b > \underline{b}$ $(c \geqq c^*)$ and larger than a positive constant, say ε, if $b < \underline{b}$, $\overline{b} < b$ or $c^* > c\ (\geqq 0)$ for $0 < w < \delta$, where ε or δ become small unrestrictedly only if $c \geqq c^*$ and b approaches to $\underline{b}$ or $\overline{b}$ (c being fixed). It is easy to see that if $0 \leq c < c^*$ every c-manifold reachs the negative p-axis with finite x and that if $c > c^*$ there exists a c-manifold which enters the origin along $p = -\bar{b}w$ (called $(c,\bar{b})$-manifold) and those along $p = -\underline{b}w$(called $(c,\underline{b})$-manifold). These exhaust all c-manifolds entering the origin (in case $c > c^*$).(*)

Step 3. From Step 1 it follows that if the c-manifold issuing from (1,0) enters (0,0) then the situation is same for any $c' > c$. Let c_0 be the infimum of such c's. By Step 2 $c^* \leqq c_0 \leqq \sqrt{2\beta}$ (the right side of (1,5) is negative for $c > \sqrt{2\beta} = b$). Since $\bar{b}$ is increasing with c and every c'-front lies over the c-front for $c' > c > c_0$, the c-front enters (0,0) along $p = -\underline{b}w$ if $c > c_0$. This proves the first half of (iii). The c_0-manifold issuing from (1,0) enters (0,0), because c-fronts with $w(0) = 1/2$ converges to a c_0-solution with $w(-\infty) = 1$ increasingly for $x < 0$ and decreasingly for $x > 0$ as $c \downarrow c_0$. Thus the c_0-front exists and (i) is proved. The second half of (iii) is trivial if $c_0 = c^*$. Let $c_0 > c^*$ and $c_0 > c > c^*$. The c_0-front is obtained as the limit of c-manifolds issuing from (1,0) as $c \uparrow c_0$. Since each of these manifolds is bounded from the above by the $(c,\bar{b})$-manifold which moves monotonously, as $c \uparrow c_0$, to the $(c_0,\bar{b})$-manifold, the c_0-front is the $(c_0,\bar{b})$-manifold. Thus (iii) is proved.

The proof of the theorem is completed.

<u>Remark.</u> <u>The $(c,\bar{b})$-manifold</u>, whose existance has been proved in Step 2 of the above proof, <u>is unique if</u> $c > c^*$ as is shown below. Parametrizing any two $(c,\bar{b})$-manifolds with w-coordinates, denote by $p = p(w)$ the difference of their p-coordinates. Assume

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(*) According to the behavior of F near zero, there occur both cases that a c*-manifold entering the origine exists and that such one does not exists (see Remark of Lemma 2.2).

$p > 0$. By $2\alpha/\bar{b}^2 = 2c/\bar{b} - 1 < 1$, we derive from (1,4) that $(w/p)\cdot dp/dw < r < 1$ for small w. This implies $p > w^r$ which contradicts to $p = o(w)$, and we have $p \leqq 0$. Similarly $p \geqq 0$. Thus $p = 0$.

In order to illustlate that when $\alpha < \beta$ both the case $c_0 > c^*$ and the case $c_0 = c^*$ occurs according as the shape of F, we give examples which are generalizations of Fisher's population genetic model for the migration of advantageous genes. The results are simi-lare to what K.P.Hadeler and F.Rothe obtained for $F(u) = u(1-u)(1+\nu u)$, $\nu > -1$ (cf. [6]).

Let G(u) be a function defined in $0 \leqq u \leqq 1$, having the continuous derivative which is continuously defferentiable in $0 < u \leqq 1$, and satisfying conditions;

$$G(0) = G(1) = 0, \quad G'(0) > 0, \quad G''(u) = o(u^{-1}),$$

$$G'(0) \geqq G'(u) \quad \text{and} \quad G(u) > 0 \quad \text{for} \quad 0 < u < 1.$$

Put $F(u) = G(u)H(u;\kappa)$, $H(u;\kappa) = 1 + 2\kappa^{-2}(G'(0) - G'(u))$ $\kappa > 0$. Then the function $w = w(x;\kappa)$ given in the inverse form

$$(1,6) \qquad x = \int_{1/2}^{w} \frac{-\kappa du}{2G(u)}$$

is a front with an associated speed

$$c = \frac{\kappa}{2} + \frac{\alpha}{\kappa} \quad (\alpha = G'(0) = F'(0)).$$

If $\kappa = \sqrt{2\alpha}$, then $c = \sqrt{2\alpha}$ and hence $c_0 = \sqrt{2\alpha}$. As κ increases, F(u) decreases and c_0 does not increase. Since $c_0 \geqq \sqrt{2\alpha}$, we get

$$c_0 = \sqrt{2\alpha} \qquad \text{for } \kappa \geqq \sqrt{2\alpha}.$$

If $\kappa < \sqrt{2\alpha}$, then

$$\lim_{x\to\infty} \frac{w'(x;\kappa)}{w(x;\kappa)} = -\frac{2}{\kappa}\lim_{w\to 0}\frac{G(w)}{w} = -\frac{2}{\kappa}\alpha = -c - \sqrt{c^2 - 2\alpha}.$$

From (iii) of Theorem 1.1 it therefore follows that $w(x;\kappa)$ is the c_0-front. This means that

$$c_0 = \frac{\kappa}{2} + \frac{\alpha}{\kappa} \qquad \text{for } \kappa < \sqrt{2\alpha}\,.$$

If $G'(0) - G'(u) \sim u^p L(u)$ where L is slowly varying at zero and $p > 0$, then $\alpha < \beta$ but $c_0 = \sqrt{2\alpha}$ for $\sqrt{2\alpha} \leqq \kappa < \sqrt{2\alpha(p+1)}$.

Similar arguments are available in the case $F'(0) = 0$. Let $G(u)$ be as above and put

$$F(u) = G(u)M(u;\kappa), \quad M(u;\kappa) = \frac{2}{\kappa^2}(G'(0)-G'(u)).$$

Then $w(x;\kappa)$ given by (1.6) is the c_0-front and $c_0 = G'(0)/\kappa$.

The first example may seem to suggest that whether $c_0 > c^*$ or $c_0 = c^*$ does not depend only on the behavior of $F(u)$ near $u = 0$. But there is an exceptional case of Remark to Lemma 2.2 presented later, in which $c_0 > c^*$ is implied only by a behavior of F near $u = 0$.

Let us state a lemma for use in the next section.

Lemma 1.1. Let F^* be a function satisfying the same conditions as imposed to F, and denote by w^*, $\overline{b}^*$, $\underline{b}^*$, etc. the corresponding quantities. Assume that $F < F^*$ for $0 < u < 1$ and that $c > \sqrt{2F^{*\prime}(0)}$. Then, (i) the $(c,\overline{b}^*)$-manifold lies over the $(c,\overline{b})$-manifold as far as they are under the w-axis, and (ii) for every $(c,\underline{b}^*)$-manifold [resp. $(c,\underline{b})$-manifold] there exists a $(c,\underline{b})$-manifold [resp. $(c,\underline{b}^*)$-manifold] such that the $(c,\underline{b}^*)$-manifold lies under the $(c,\underline{b})$-manifold near the origine.

Proof. First note that $\underline{b} \leqq \underline{b}^*$, $\overline{b}^* \leqq \overline{b}$. Since

$$\text{Det}\begin{pmatrix} w' & w^{*\prime} \\ p' & p^{*\prime} \end{pmatrix} = -2(\, pF^*(w^*) - p^*F(w)\,)$$

is positive at $(w,p) = (w^*,p^*)$, (ii) is clear. (i) follows from the fact that a c-manifold for F passing a point below the $(c,\overline{b})$-manifold must reach the negative p-axsis (see Fig II). q.e.d.

Corollary. Under the assumptions of Lemma 1.1 (i) if w and w^* are the $(c,\bar{b})$- and $(c,\bar{b}^*)$-solutions, respectively, with $w(0) \leqq w^*(0)$, then $w(x) < w^*(x)$ for $x > 0$, and (ii) if w [resp. w^*] is a $(c,\underline{b})$- [resp. $(c,\underline{b}^*)$-] solution, then there exists a $(c,\underline{b}^*)$- [resp. $(c,\underline{b})$-] solution w^* [resp. w] such that $w(x) > w^*(x)$ for $x > 0$.

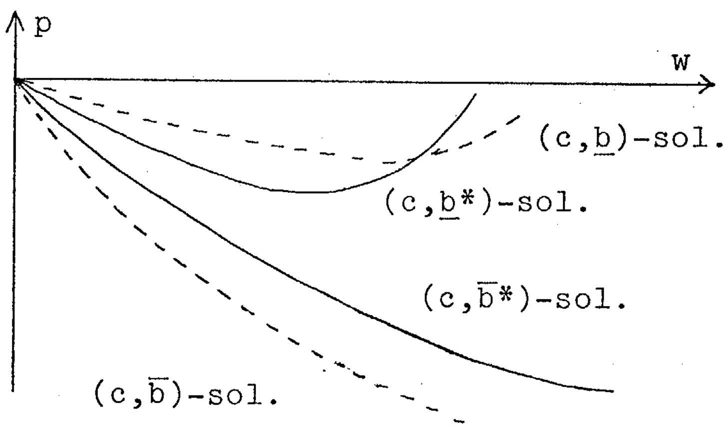

Fig II. $(F \leqq F^*)$

2. Asymptotic Behaviors of c-Fronts as $x \to +\infty$.

The tail of a c-front for large x is nicely approximated by that of a solution for the linear equation $2^{-1}w'' + cw' + \alpha w = 0$, if F behaves regularly (in some sense) near zero. Indeed a theorem in the stability theory says that the error of the aproximation is a small order of e^{-px} with some $p > 0$ if $\alpha u - F(u) = o(u^{1+q})$ with some $q > 0$ (cf. [2]). Here we find (weaker) conditions sufficient and almost necessary for certain estimations about the approximation to hold. Symbols and terminologies introduced in the previous section are used also here (and later sections).

Let us introduce a function $\xi(u)$ defined by

$$F(u) = \alpha u + \tfrac{1}{2}\xi(u) \qquad (\ \alpha = F'(0)\ ).$$

Theorem 2.1. Let $c > \sqrt{2\alpha}$ and $c \geqq c_o$. Assume

$$\int_{0+} \frac{|\xi(u)|}{u^2} du < \infty \quad . \tag{2,1}$$

Then the c-front satisfies

$$w_c(x) = ae^{-bx}(1 + o(1)) \qquad \text{as } x \to \infty, \tag{2,2}$$

where $b = -\lim_{x\to\infty}(w_c'(x)/w_c(x))$ and a is a positive constant and given by

$$a = \begin{cases} \frac{1}{2\sqrt{c^2-2\alpha}} \{\bar{b}w_c(0) + w_c'(0) - \int_0^\infty e^{\underline{b}s} \xi(w_c(s))ds\} & \text{if } c > c_o \\ \frac{-1}{2\sqrt{c^2-2\alpha}} \{\underline{b}w_c(0) + w_c'(o) - \int_0^\infty e^{\bar{b}s} \xi(w_c(s))ds\} & \text{if } c = c_o. \end{cases}$$

Theorem 2.2. Assume $c_o = \sqrt{2\alpha}$ and

$$\int_{0+} \frac{|\xi(u)|}{u^2} |\log u| du < \infty \ . \tag{2,3}$$

Then the c_o-front satisfies either (2,2) with $c = c_o$, $b = \sqrt{2\alpha}$ or

$$w_{c_o}(x) = a_1 x e^{-\sqrt{2\alpha}\, x}(1 + o(1)) \qquad \text{as } x \to \infty, \tag{2,4}$$

where a or a_1 are positive constants and given by

$$a = w_{c_o}(0) + \int_0^\infty se^{\sqrt{2\alpha}\, s} \xi(w_{c_o}(s))ds,$$

$$a_1 = \sqrt{2\alpha}\, w_{c_o}(0) + w_{c_o}'(o) - \int_0^\infty e^{\sqrt{2\alpha}\, s} \xi(w_{c_o}(s))ds.$$

Remark. If $\xi(u)$ has a definite sign near $u = 0$, the condition (2,1) is also necessary for (2,2) to hold. The similar statement is asserted to the case of Theorem 2.2 as well. (See Lemmas 2.1 and 2.2)

Theorem 2.3. Under the assumptions of Theorem 2.2 a sufficient condition for (2,2) to hold is that there exist no F^*, satisfying (3) and not identically equal to F, such that $F^* \geqq F$ and $c_o^* = \sqrt{2\alpha}$ (where c_o^*

is the minimal speed corresponding to F*). If ξ satisfies that

(2,5) $\int_{0+} \frac{|\xi'(u)|}{u} |\log u| du < \infty$ or $\xi(u) = o(u^{1+p}) \quad p > 0$

then this condition is also necessary.

If $\xi \leqq 0$, then (2,4) holds under (2,3).

Applying the last theorem to examples of the section 1, we have that if $\int_{0+} [|G''(u)| + |G'(0)-G'(u)|/u] |\log u| du < \infty$ or $G'(0) - G'(u) = o(u^p)$, $p > 0$, then the c_0-fronts for $F = G \cdot H$ with $\kappa > \sqrt{2G'(0)}$ satisfies (2,4).

The proofs of these theorems follow from Theorem 1.1 and lemmas presented below. The proofs of lemmas are somewhat complicated and may be skipped if the reader is contented with the result for $\xi(u) = o(u^{1+p})$, $p > 0$ or is little interested in the problem all its own.

Let us write $z(x) = e^{bx} w(x)$ with $b = -\underline{\lim}_{x\to\infty}(w'/w)$ for a c-solution which satisfies $w(+\infty) = 0$ and write $z_\infty = \lim_{x\to\infty} z(x)$ if the limit exists.

Lemma 2.1. Suppose $c > \sqrt{2\alpha}$ and at least one of the following conditions holds;

(2.6) $$\int_{0+} \frac{\xi(u)^+}{u^2} du < \infty \quad ,^{(*)}$$

(2,7) $$\int_{0+} \frac{\xi(u)^-}{u^2} du > -\infty .^{(*)}$$

Then for any c-solution w with $w(+\infty) = 0$ there exists $z_\infty = \lim_{x\to\infty} z(x)$ with $0 \leqq z_\infty \leqq \infty$. If $b = \bar{b}$ the condition (2,6) [resp. (2,7)] is necessary and sufficient that $z_\infty < \infty$ [resp. $z_\infty > 0$].

(*) $x^+ = \sup\{x,0\}$, $x^- = \inf\{x,0\}$.

<u>If</u> $b = \underline{b} > 0$, <u>the condition</u> (2,6) [<u>resp</u>. (2,7)] <u>is necessary and sufficient that</u> $z_\infty > 0$ [<u>resp</u>. $z_\infty < \infty$].

<u>Proof</u>. Write the equation (1,2) in the form

$$\left(\frac{d}{dx} + \bar{b}\right)\left(\frac{d}{dx} + \underline{b}\right) w = -\xi(w),$$

then apply the formula $\int_0^x e^{bs}\left(\frac{d}{dx} + b\right) f(s)ds = e^{bx}f(x) - f(0)$ twice, and you have the integral equation

$$(2,8) \qquad w(x) = Ae^{-\underline{b}x} - Be^{-\bar{b}x} + \frac{1}{\bar{b} - \underline{b}} \int_0^x [e^{-\bar{b}(x-s)} - e^{-\underline{b}(x-s)}]\xi(w(s))ds,$$

where

$$A = \frac{\bar{b}w(0) + w'(0)}{\bar{b} - \underline{b}}, \qquad B = \frac{\underline{b}w(0) + w'(0)}{\bar{b} - \underline{b}}.$$

Since $\log w(x) = \int_0^x \frac{w'}{w} ds + \text{const.} = -bx + o(x)$ and since $\bar{b} > \underline{b}$, for the $(c,\bar{b})$- solution we have

$$(2,9) \qquad w(x) = -Be^{-\bar{b}x} + \frac{e^{-\bar{b}x}}{\bar{b} - \underline{b}} \int_0^x e^{\bar{b}s}\xi(w)ds + \frac{1}{\bar{b}-\underline{b}} \int_x^\infty e^{-\underline{b}(x-s)}\xi(w)ds,$$

and

$$(2,10) \qquad \bar{b}w(0) + w'(0) = \int_0^\infty e^{\underline{b}s}\xi(w(s))ds.$$

Let $b = \underline{b}$, i.e. $w'/w \to -\underline{b}$. It follows from (2,8) that

$$(2,11) \qquad z(x) = A - Be^{-(\bar{b}-\underline{b})x} + \frac{1}{\bar{b}-\underline{b}} \int_0^x e^{-(\bar{b}-\underline{b})(x-s)} z \frac{\xi(w)}{w} ds$$

$$- \frac{1}{\bar{b}-\underline{b}} \int_0^x z \frac{\xi(w)}{w} ds,$$

and

$$(2,12) \qquad z'(x) = (\bar{b}-\underline{b})Be^{-(\bar{b}-\underline{b})x} - \int_0^x e^{-(\bar{b}-\underline{b})(x-s)} z \frac{\xi(w)}{w} ds.$$

Since $z'(x) = o(z(x))$ (which follows from $w' + bw = e^{-bx} z' = o(w)$),

$$(2,13) \qquad z(x) = (1+o(1))[\, A - \frac{1}{\bar{b}-\underline{b}} \int_0^x z(s)\frac{\xi(w(s))}{w(s)}\, ds \,].$$

First assume $\int_{0+} \frac{\xi(w)^-}{w^2}\, dw > -\infty$. From the inequality

$$z(x) \leqq A' + D \int_0^x z(s)P(s)ds$$

where A', D are some constants and $P(s) = -\xi(w(s))^- / w(s)$, we can easily deduce the boundedness of $z(x)$ because of the integrability:

$$\int_0^\infty P(s)ds = \int_0^{w(0)} \frac{|\xi(w)^-|}{w^2} \frac{w}{|w'|}\, dw < \infty.$$

The integral in the right side of (2,13) converges, for the left side must be non-negative and z is bounded. Now we get that there exists $z_\infty = A - (\bar{b}-\underline{b})^{-1} \int_0^\infty z\frac{\xi(w)}{w}\, ds$, which is positive only if (2,6) holds.

Next assume $\int_{0+} \frac{\xi(w)^+}{w^2}\, dw < \infty$. Let us prove that $\underline{\lim}_{x\to\infty} z(x) > 0$

Without loss of generality, by virtue of Corollary of Lemma 1.1, we may assume $\xi \geqq 0$ near $u = 0$, which guarantees the existance of $\lim z(x) = z_\infty$. Then $z_\infty = 0$, by (2,13), leads to the contradiction;

$$z(x) = (1+o(1))\frac{1}{\bar{b}-\underline{b}} \int_x^\infty z\, \frac{\xi(w)}{w}\, ds = o(z(x)).$$

Thus $\underline{\lim}\, z(x) > 0$. Now it suffices only to prove that if $\int_{0+} \frac{\xi(w)^-}{w^2} dw = -\infty$ then $\lim z(x) = \infty$. Let $\int_{0+} \frac{\xi(w)^-}{w^2}\, dw = -\infty$.

Let x_1 and x_2 are two points such that $z(x_1) \gtreqq z(x)$ for $x_1 < x < x_2$. Then, by (2,11), (2,12) and that $z'(x) = o(z(x))$,

$$z(x) \geq z(x_1) - \frac{z(x_1)}{\bar{b} - \underline{b}} \int_{x_1}^x \frac{\xi(w)^+}{w} ds - o(z(x_1))$$

for $x_1 < x < x_2$. If x_1 is large, $\int_{x_1}^{x} \frac{\xi(w)^+}{w} ds$ is small and $z(x)$ is little less than $z(x_1)$ for $x > x_1$. Since $\overline{\lim}\, z(x) = \infty$, this proves $\lim z(x) = \infty$. The proof of the lemma in the case $b = \underline{b}$ is completed.

In the case $b = \overline{b}$ we can proceed similary as above starting from (2,9) instead of (2,8). q.e.d.

In the case $c = \sqrt{2\alpha}$, $\alpha > 0$, we get, instead of (2,8),

$$w(x) = [\, w(0) + (w'(0) + bw(0))x \,]\, e^{-bx} - \int_0^x (x-s)e^{-b(x-s)}\, \xi(w(s))ds, \qquad b = \sqrt{2\alpha}. \tag{2,14}$$

Lemma 2.2. Suppose $\alpha > 0$ and at least one of the following conditions holds;

$$\int_{0+} \frac{\xi(u)^+}{u^2} |\log u|\, du < \infty, \tag{2,15}$$

$$\int_{0+} \frac{\xi(u)^-}{u^2} |\log u|\, du > -\infty. \tag{2,16}$$

Then for every c*-solution ($c^* = \sqrt{2\alpha}$) w with $w(+\infty) = 0$ there exists $y_\infty = \lim_{x\to\infty} x^{-1}e^{bx}w(x)$ ($b = \sqrt{2\alpha}$) with $0 \leq y_\infty \leq \infty$. If both (2,15) and (2,16) hold, there then occur two and only two cases: $0 < y_\infty < \infty$ ($z_\infty = \infty$); $0 < z_\infty < \infty$ ($y_\infty = 0$),(*) and a c*-solution to the latter case (or corresponding c*-manifold) is obtained as the limit of $(c,\overline{b})$-solutions (or $(c,\overline{b})$-manifolds) as $c \downarrow c^*$. Conversely if one of these cases occurs to some c*-solution, then both (2.15) and (2,16) hold. The cases $y_\infty = \infty$ occur if and only if (2,16) does not hold. If (2,16) [resp. (2,15)] fails to hold, then $y_\infty = 0$ implies $z_\infty = 0$ [resp. $\overline{\lim}\, z(x) = \infty$].

(*) The use of the symbol z_∞ implies the existence of $\lim z(x)$.

Remark. In the above lemma if $\int_{0+} \frac{\xi(u)^+}{u^2}\,du = \infty$ (which implies (2,15) fails and hence (2,16) holds by the assumption of the lemma) then there is no c^*-solution with $w(+\infty) = 0$. In such a case we have $c_0 > c^*$.

Proof. Let w be a c^*-solution with $w(+\infty) = 0$. First we note that $x = -\frac{1}{b}\log w + o(\log w)$ as $x \to \infty$ and that

$$\int_0^x s\,\frac{\xi(w)^\pm}{w}\,ds = \int_{w(x)}^{w(0)} b^{-2}\,\frac{\xi(u)^\pm}{u^2}\,|\log u|\,du\,(1+o(1)).$$

From (2,14) it follows that

$$(2,17) \qquad z(x) = w(0) + (w'(0)+b\,w(0))x - \int_0^x (x-s)\,z\,\frac{\xi(w)}{w}\,ds,$$

or, introducing the notation $y(x) = z(x)/x$,

$$(2,18) \qquad y(x) = \frac{w(0)}{x} + w'(0) + bw(0) - \int_0^x (1-\tfrac{s}{x})\,y\,s\,\frac{\xi(w)}{w}\,ds.$$

Assume $y(x)$ to be bounded as $x\to\infty$. (Notice this holds always under (2,16) as is easily seen (see the proof of the next lemma).) Then we have, by the hyposesis of the lemma,

$$(2,19) \qquad \int_0^\infty z\,\frac{|\xi(w)|}{w}\,ds < \infty,$$

which further implies that $\int_0^x s\,z\,\frac{\xi(w)}{w}\,ds = o(x)$ and hence that

$$(2,22) \qquad y_\infty = w'(0) + bw(0) - \int_0^\infty z\,\frac{\xi(w)}{w}\,ds.$$

It is clear, by (2,18), that $y_\infty > 0$ implies both (2,15) and (2,16).

Let $y_\infty = 0$. Then we have

$$(2,23) \qquad z(x) = w(0) + x\int_x^\infty z\,\frac{\xi(w)}{w}\,ds + \int_0^x s\,z\,\frac{\xi(w)}{w}\,ds,$$

or, by dividing by z

$$(2,24) \qquad 1 = \frac{w(0)}{z(x)} + \int_x^\infty \frac{z(s)/s}{z(x)/x}\,s\,\frac{\xi(w(s))}{w(s)}\,ds + \int_0^x \frac{z(s)}{z(x)}\,s\,\frac{\xi(w(s))}{w(s)}\,ds.$$

Assume (2,15) to be true. Then $z(x)$ is bounded. Because, assuming the contrary, we can chose a sequence $x_1, x_2, \ldots$ such that $z(x) \leqq z(x_n)$ for $x \leqq x_n$, $z(x_n)/x_n > z(x)/x$ for $x > x_n$ and $z(x_n) \to \infty$, which leads to the contradiction, for the right side of (2,24) tends to zero along this sequence. The boundedness of $z(x)$ implies, by (2,23), that $\int_0^\infty s\, z \frac{|\xi(w)|}{w}\, ds < \infty$, and hence that

$$(2,25) \qquad z_\infty = w(0) + \int_0^\infty s\, z \frac{\xi(w)}{w}\, ds.$$

Clearly $z_\infty > 0$ only if (2,16) holds. If $z_\infty = 0$, we have $z(x) = -\int_x^\infty (s-x)\, z \frac{\xi(w)}{w}\, ds$, from which we deduce that (2,16) is spoiled. Thus we have proved that under (2,15) $y_\infty = 0$ implies $0 \leqq z_\infty < \infty$ where $z_\infty > 0$ is equivalent to (2,16).

Under (2,15) and (2,16) a c^*- solution with $y_\infty = 0$ is obtained as the limit of $(c,\bar{b})$-solutions, with $w(0)$'s taking the same value, as $c \downarrow c^*$. For the proof it suffices to show that $z(x;c) = e^{\bar{b}x} w(x;c)$ are bounded uniformly as $c \downarrow c^*$. Prove this for positive ξ, and then apply Corollary of Lemma 1.1.

Now we prove the existance of a c^*-solution with $y_\infty > 0$ under (2,15). Corollary of Lemma 1.1 (which must be modified appropriately to the present case ; $c = c^*$) allows us to assume $\xi \geqq 0$. Since, for such ξ, $z(x)$ is less than $w(0)(1+bx)$, we have

$$\int_0^\infty z \frac{\xi(w)}{w}\, ds \leqq w(0)(b+1) \left[\int_0^1 \frac{\xi(w)}{w}\, ds + \int_1^\infty s \frac{\xi(w)}{w}\, ds \right]$$

$$= o(w(0)) \qquad \text{as} \quad w(0) \downarrow 0.$$

Since there exists a c^*-solution with $y_\infty = 0$ as already proved we can take initial values $w(0)$ and $w'(0)$ so small that $w(+\infty) = 0$ and $bw(0) + w'(0) > \int_0^\infty z \frac{\xi(w)}{w}\, ds$. By (2,18) we see that this is desired one.

Let (2,15) be spoiled. Since (2,16) implies the boundedness of $y(x)$, by (2,18) we see $y_\infty = 0$ and have (2,23). By Corollary of Lemma 1.1 we see that w is bounded from the below on $x > 0$ by $\hat{w}$ with $\hat{y}_\infty = 0$ where $\hat{w}$ is a c*-solution for $\hat{\xi} \leq \xi$ with $\int_{0+} \frac{|\hat{\xi}(u)|}{u^2} |\log u| du < \infty$. Therefore $\underline{\lim}\, z(x) > 0$, which, by (2,23), turns into $\overline{\lim}\, z(x) = \infty$. (Remark follows from these and (2,19).)

If $\overline{\lim}\, y(x) = \infty$ (which occurs only if (2,16) fails), then we can prove that $\lim y(x) = \infty$ as in the last part of the proof of Lemma 2.1. Now the proof of Lemma 2.2 is completed.

Lemma 2.3. Assume (2,3) and (2,5). Let $\alpha > 0$. Then there exists uniquely a c*-manifold with $y_\infty = 0$.

Proof. When the latter one of (2,5) is assumed, we can apply a theorem in the stability theory (cf. [2]) to get the result by fixing $z_\infty = w(0) + \int_0^\infty s\, z \frac{\xi(w)}{w} ds$. Therefore we assume the other one of (2,5). A c*-solution with $y_\infty = 0$ satisfies (2,17) and

$$(2,26) \qquad w'(0) + bw(0) = \int_0^\infty \xi(w) e^{bs} ds.$$

Assume there are two such solutions with a common $w'(0)$, say δ, and different $w(0)$'s, say ε_1, ε_2. Then we have $y_\infty = 0$ for any c*-solutions with $w'(0) = \delta$ and $\varepsilon_1 < w(0) < \varepsilon_2$. Put $z(0) = \varepsilon$ and regard z as a function of x and ε : $z = z(x;\varepsilon)$. By (2.17) we have that $\eta = \partial z/\partial \varepsilon$ satisfies

$$(2,27) \qquad \eta = 1 + bx - \int_0^x (x-s)\xi'(w)\, \eta\, ds.$$

It follows from this and from $\eta \geq 0$ (ε_2 is assumed to be small) that

$$(2,28) \qquad \eta \leq 1 + bx + x\int_0^x |\xi'(w)|\, \eta\, ds.$$

By (2,28) η is bounded from the above on $x \geqq 0$ by the solution $\hat{\eta}$ of the linear equation

$$(2,29) \qquad \hat{\eta} = 1 + bx + x\int_0^x |\xi'(w)|\, \hat{\eta}\, ds$$

which has the unique solution with the bound: $\hat{\eta}(x) \leqq Ax + B$ where A, B are constants chosen independently of ε. Now differentiate the both sides of (2,26) with respect to ε and we have, by the Fubini's theorem, that $b = \int_0^\infty \xi'(w)\, \eta\, ds$, the right side of which tends to zero if we let ε small. But this is absurd since ε_2 may be arbitrarily small. q.e.d.

Lemmas 2.1 to 2.3 and results of the section 1 prove Theorems 2.1, 2.2 and 2.3 except the last statement in Theorem 2.3. But since, for a front, $w'(0) + bw(0)$ can be assumed to be positive, $\xi \leqq 0$ implies, by (2,22), $y_\infty > 0$ as desired.

Theorem 2.4 *Let* $\alpha = 0$ *and* $c > c_0$. *Then for any small* $\varepsilon > 0$ *we can find constants* C_1, C_2 *and* N *such that*

$$q\left(\frac{x}{c-\varepsilon} + C_1\right) \leqq w_c(x) \leqq q\left(\frac{x}{c+\varepsilon} + C_2\right) \qquad \text{for } x > N$$

where $q(x)$ *is the inverse function of*

$$x(w) = \int_w^{1/2} \frac{du}{F(u)}.$$

Proof. Put $\tau(w_c) = w_c'$. Then $F(w)/\tau(w) \to -c$ as $w \downarrow 0$. For any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$(-c-\varepsilon)\frac{1}{F(w)} \leqq \frac{1}{\tau(w)} \leqq (-c+\varepsilon)\frac{1}{F(w)} \qquad 0 < w < \delta.$$

By integrating each part of this inequality, we get for $x > w_c^{-1}(\delta)$

$$(c+\varepsilon)\int_{w_c(x)}^{\delta} \frac{du}{F(u)} \geqq x - w_c^{-1}(\delta) \geqq (c-\varepsilon)\int_{w_c(x)}^{\delta} \frac{du}{F(u)},$$

or equivalently

$$q\left(\frac{x}{c+\varepsilon}-\frac{w_c^{-1}(\delta)}{c+\varepsilon}+q^{-1}(\delta)\right) \geqq w_c(x) \geqq q\left(\frac{x}{c-\varepsilon}-\frac{w_c^{-1}(\delta)}{c-\varepsilon}+q^{-1}(\delta)\right).$$

q.e.d.

To illustrate what Theorem 2.3 says we put $F(u) = u^{1+p}L(u)$ with $p > 0$ and L slowly varying at zero. Then for $c > c_0$

$$w_c(x) \sim c^{\frac{1}{p}} q(x) \quad \text{as} \quad x \to \infty$$

and $q(x)$ is regularly varying at infinity with exponent $\frac{-1}{p}$. If we take $F(u) = u(-\log u)^{-1-r}L(-\log u)$ with $r > 0$ and L slowly varying at infinity, then for $c > c_0$

$$\log w_c(x) \sim c^{\frac{-1}{2+r}} \log q(x) \quad \text{as} \quad x \to \infty,$$

and $|\log q(x)|$ is regularly varying at infinity with exponent $\frac{1}{2+r}$. (In these cases (with additional conditions on L) w_{c_0} satisfies (2,2).)

The next lemma will be used in the proof of Theorem 9.3.

Lemma 2.4 Let $c \geqq c^*$ and $\alpha > 0$. Assume the condition of Theorem 2.1 if $c > c^*$ and that of Theorem 2.2 if $c = c^*$. Let S be the part of the half strip $0 < w \leqq 1$, $p \leqq 0$ sweeped out by all c-manifolds that enter the origin. Let (w,p) be a c-manifold starting from $(w_0,p_0) \in S$ at $x = 0$. Consider the quantities

$$a = \lim_{x\to\infty} e^{bx} w(x) \qquad \text{if} \quad c > c^*$$

$$a_1 = \lim_{x\to\infty} e^{c^*x} x^{-1} w(x) \qquad \text{if} \quad c = c^*$$

to be functions of $(w_0,p_0) \in S$. Then they are continuous. Especially if (w_0,p_0) approaches to a boundary point of S which is not on $\{(w,p);\ w = 1 \text{ or } p = 0\}$, then a or a_1 tends to zero in each cases.

Proof. When $c > c^*$, the statement is clear by (2,10) and by the first expression of a in Theorem 2.1 which is valid for any (c,$\underline{b}$)-solution, becauce $\{w/w'; w < \delta/2\}$ and $\exp\{\underline{b}x\}w(x)$ are uniformly bounded as long as $(w(0),w'(0))$ moves in the intersection of S and $w > \delta$. In case $c = c^*$ use (2,22) and the expression of a_1 in Theorem 2.2.

Corollary. Let w be a c-solution with $(w(x_0),w'(x_0))$ being a inner point of S. Then, under the assumption of Lemma 2.4, if $(w(x_0), w'(x_0))$ approaches to a boundary point of S not on $\{(w,p); w = 1 \text{ or } p = 0\}$ as a (or a_1) and $w(x_0)$ being fixed, x_0 tends to infinity.

3. Parabolic Equations.

We exibit here comparison theorems on the parabolic equation

$$(3,1) \qquad u^{\cdot} = a u'' + b u' + c u + Q \qquad\qquad u = u(t,x)$$

where a, b, c and Q are functions of $(t,x) \in \bar{E} = [0,\infty) \times R$. It is assumed throughout this section that $a \geqq 0$ and $Q \geqq 0$. Most of the results presented below are standard and proofs of some of them are omitted (see e.g. [4], [8]). When we say u satisfies (3,1) in an open set, it means that $u^{\cdot}$, u' and u'' exist, are continuous and satisfy (3,1) together with u in it. Let D is an open set of E_T, $T > 0$. We denote by $\bar{D}$ the closure of D in R^2 and by ∂D its boundary. We will further impose on solutions in D the continuity on $\bar{D}$.

Proposition 3.1. Let u satisfy (3,1) in an open set D of E_T, $T > 0$ and be continuous on $\bar{D}$. Assume there exists a constant M such that

(3,2) $a(t,x) \leqq M, \quad |b(t,x)| \leqq M(|x|+1), \quad c(t,x) \leqq M(x^2+1)$

and

(3,3) $$u(t,x) \geqq -M e^{Mx^2}$$

for $(t,x) \in D$. Then $u \geqq 0$ in D if $u \geqq 0$ on $D - \ell_T$.(*)

Proposition 3.2. Let D be an open set contained in the rectangle $(0,T) \times (0,1)$. Let u satisfy (3,1) in D and be continuous on $\bar{D}$. Suppose there exists a constant M such that

(3,4) $$a(t,x) \leqq Mx^2(1+|\log x|), \quad |b(t,x)| \leqq Mx(1+|\log x|)$$
$$\text{and} \quad c(t,x) \leqq M(1+|\log x|) \quad \text{for} \quad (t,x) \in D.$$

Then $u \geqq 0$ in D if $u \geqq 0$ on $D - \ell_T$.

Proof. Putting $u_*(t,x) = u(t,\exp\{(1-x^2)/2\})$, apply Proposition 3.1 to u_*.

Proposition 3.3. Let D be a rectangle $(0,T) \times (0,L)$ with $0 < L \leqq \infty$. Let u satisfy (3.1) in D and be continuous and nonnegative on $\bar{D}$. Suppose that (3,2) and (3,3) are satisfied, that $b_* = \sup_D b$ and $c_* = \inf_D c$ are finite and $a_* = \inf_D a > 0$, that $\delta_1 = \inf_{0<t<T} u(t,0) > 0$, and that $\delta_2 = \inf_{0<t<T} u(t,L) > 0$ if $L < \infty$. There then exists a function v defined and continuous on $\bar{D}$, which is positive on $\bar{D} - \ell_0$ and depends only on and continuously on δ_1, δ_2, a_*, b_*, c_*, T and L such that $u \geqq v$ on $\bar{D}$.

Proof. We prove the proposition only when $L < \infty$. Set

$$v(t,x) = \varepsilon e^{c_* t} \int_{-\infty}^{0} p(a_* t, x + b_* t - y) y^2 dy$$

where ε is a positive constant chosen so small that

$$v(t,o) \leqq \delta_1 \quad \text{and} \quad v(t,L) \leqq \delta_2 \qquad \text{for} \quad o < t < T.$$

(*) $\ell_T = \{(T,x);\ x \in R\}$.

Noticing v is a solution of

$$v^{\cdot} = a_* v'' + b_* v' + c_* v \quad \text{with} \quad v(0,x) = x^2 I_{(-\infty,0)}(x),$$

we see that $w = u - v$ satisfies (3,1) with Q replaced by Q_* $= (a-a_*)v'' + (b-b_*)v' + (c-c_*)v + Q$ and the boundary condition: $w \geqq 0$ on $\partial D - \ell_T$. It is easily seen that $v'' \geqq 0$ and $v' \leqq 0$, and hence $Q_* \geqq 0$. Therefore by Proposition 3.2 we have $w \geqq 0$ in D as desired.

Proposition 3.4. Let u satisfy (3,1) with $Q \equiv 0$ in E and be continuous on $\bar{E}$. Suppose that (3,2) and (3,3) are satisfied in E_t for each $t > 0$ and that c is bounded on each compact set of $\bar{E}$. Suppose $g(x) = u(0,x)$ satisfies

(3.5) $$g(x) \leqq 0 \ \text{if} \ x_1 < x < x_2; \ \geqq 0 \ \text{if} \ x < x_1 \ \text{or} \ x > x_2$$

with some extended real constants x_1 and x_2: $-\infty \leqq x_1 < x_2 \leqq \infty$. Then there exist extended real functions $X_1(t)$ and $X_2(t)$ of $t > 0$ with $-\infty \leqq X_1(t) \leqq X_2(t) \leqq \infty$ such that

$$u(t,x) \begin{cases} \leqq 0 & \text{if} \ X_1(t) < x < X_2(t) \\ > 0 & \text{if} \ x < X_1(t) \ \text{or} \ x > X_2(t). \end{cases}$$

If $x_1 = -\infty$ [resp. $x_2 = \infty$], we may set $X_1(t) \equiv -\infty$ [resp. $X_2(t) \equiv \infty$].

Proof. By virtue of Proposition 3.1 it suffices to prove that if $u(T,\bar{x}_1) < 0$ and $u(T,\bar{x}_2) < 0$ with $\bar{x}_1 < \bar{x}_2$, $T > 0$ then $u(T,x) \leqq 0$ for $\bar{x}_1 < x < \bar{x}_2$. Let D_1 and D_2 are connected components of $\{(t,x);\ u(t,x) < 0,\ 0 < t < T\}$ whose boundary contains $(T,\bar{x}_1)$ and $(T,\bar{x}_2)$, respectively. Define an open set D contained in E_T by the relation that

$$(t,x) \in D \quad \text{iff} \quad \begin{cases} y_1 < x < y_2 \ \text{for some} \ y_1 \ \text{and} \ y_2 \ \text{with} \\ (t,y_1) \in D_1 \ \text{and} \ (t,y_2) \in D_2. \end{cases}$$

Since both $\bar{D}_1 \cap \ell_0$ and $\bar{D}_2 \cap \ell_0$ contain points of the segment

$\{0\} \times [x_1, x_2]$, $g \leqq 0$ on $\overline{D} \cap \ell_0$ by (3,5) and hence $-u \geqq 0$ on $\partial D - \ell_T$. Then Proposition 3.1 is applied to $-u$ to get $u \leqq 0$ in D (we may assume $\overline{D}$ is compact by curtailing it if necessary). Thus the proposition is proved.

4. Fundamental Properties of $u(t,x;f;F)$.

It is well known that our Cauchy problem (1) and (2) is reduced to finding the solution of the integral equation

$$(4,1) \qquad u(t,x) = P_t f(x) + \int_0^t ds \int_R p(t-s,x-y)F(u(s,y))dy$$

such that $0 \leqq u \leqq 1$ (this will be proved in the following).

The solution of (4,1) is obtained by the usual method of the successive approximation. The uniqueness of the (bounded) solution is proved by usual method (cf. [13]). Let u be a solution of (4,1) (with $0 \leqq u \leqq 1$). Then we have equations for $t > 0$

$$(4,2) \qquad u'(t,x) = (P_t f)'(x) + \int_0^t ds \int_R p'(t-s,x-y)F(u(s,y))dy$$

$$(4,2)' \qquad u'(t,x) = (P_t f)'(x) + \int_0^t ds \int_R p(t-s, x-y)F'(u(s,y))u'(s,y)dy$$

and

$$(4,3) \qquad u''(t,x) = (P_t f)''(x) + \int_0^t ds \int_R p'(t-s,x-y)F'(u(s,y))u'(s,y)dy.$$

We will use formulas

$$(4,4) \qquad \int_R \frac{|y|}{t} p(t,y)dy = \frac{2}{\sqrt{2\pi t}},$$

$$(4,5) \qquad \int_R \frac{y^2}{t^2} p(t,y)dy = \frac{1}{t}.$$

From these equations or formulas it follows that

$$(4,6) \qquad |u'(t,x)| \leqq \frac{1}{\sqrt{\pi/2}} \{ \frac{1}{\sqrt{t}} + 2 \,\|F\|\, \sqrt{t} \}$$

and

(4.7) $$|u''(t.x)| \leqq \frac{1}{t} + 2\gamma(||F||t + 1)$$

where $||F|| = \sup_u |F(u)|$. We remark that these inequalities imply in particular, by Fuygens property of u, that u' and u'' are bounded on $t > 1$, $x \in R$. (Similar boundedness assertion of u''' is deduced from (4,8) which follows.) The existance and continuity of $u^{\cdot}$ follows from (4,1) and the inequality (derived from (4,6))

$$|\int_R p^{\cdot}(t-s,x-y)F(u(s,y))dy| = |\int_R p'(t-s,x-y)F'(u)u'dy|$$

$$\leqq \text{const.}\{\frac{1}{\sqrt{s}} + \sqrt{s}\}\frac{1}{\sqrt{t-s}},$$

Now the derivation of the equation (1) and (2) from (4,1) is immediate. The uniqueness of the continuous solution of (1) and (2) follows from Proposition 3.1 if f is continuous. For a measurable f, putting $f_n = u(1/n,\cdot)$ and $u_n(t,x) = u(t+1/n,x)$ with u a solution of (1) and (2), each u_n is the unique solution of (1) and (2) with f_n in place of f and hence u_n is the solution of (4,1) where f is replaced by f_n. Letting n tend to infinity we see that u satisfies (4,1). Therefore uniqueness assertion for the equations (1) and (2) follows from that for the equation (4,1).

Let u be a solution of (1) and (2): $u(t,x) = u(t,x;f)$. Then u satisfies (4,1) as just proved. By differentiating the both sides of (4,1) with respect to t we obtain

(4,8) $$u^{\cdot}(t,x) = (P_t f)^{\cdot} + P_t F(f)(x) + \int_0^t ds \int_R p(t-s,x-y)F'(u(s,y))u^{\cdot}(s,y)dy$$

from which the existances of u''' and $u^{\cdot\prime}$ follow. Putting $v = u'$

(4,9) $$v^{\cdot} = \frac{1}{2}v'' + F'(u)v.$$

Suppose f is Lipshitz continuous on R. Then by (4,2) u' is bounded on E_T, $T < \infty$, and converges to f' at any points where f' exists, since $(P_t f)'$ has these properties.

Suppose F'' exists and is continuous on $[0,1]$. Then from

(4,8), by considering $u^*(t,x) = u(t+1/n,x) = u(t,x;u(1/n,\cdot))$ if necessary, we see as before that $v = u^{\cdot}$ satisfies (4,9). If we further assume that f' exists and is Lipshitz continuous on R, $u^{\cdot}$ is bounded on each E_T and converges to $\frac{1}{2}f'' + F(f)$ at any point at which f'' exists.

The next lemma will be used repeatedly.

Lemma 4.1 Let $k(t,x)$ and $Q(t,x)$ are bounded measurable functions on E. Then for each bounded measurable function g on R the integral equation

(4,10) $$u(t,x) = P_t g(x) + \int_0^t ds\, P_{t-s}\{k(s,\cdot)u(s,\cdot) + Q(s,\cdot)\}(x)$$

has the unique solution which is bounded and continuous on E_T, $T < \infty$. Such a solution satisfies

(4,11) $$e^{k^*t}P_t g^-(x) + \int_0^t e^{k^*(t-s)}P_{t-s}Q^-(s,\cdot)(x)\,ds \leqq u(t,x) \leqq e^{k^*t}P_t g^+(x) + \int_0^t e^{k^*(t-s)}P_{t-s}Q^+(s,\cdot)(x)\,ds$$

where $k^* = \sup k(t,x)$.

Proof. For a measurable function v on E we write

(4,12) $$\mathbb{K}_\lambda v = \mathbb{K}_\lambda v(t,x) = \int_0^t e^{\lambda(t-s)}P_{t-s}v(s,\cdot)(x)\,ds,$$

where λ is a real constant, if the double integral for $|v|$ is finite. Then formulas

(4,13) $$\sum_{n=1}^{\infty}(\mu\mathbb{K}_\lambda)^n v = \mu\,\mathbb{K}_{\lambda+\mu}v,$$

(4.14) $$\lambda\,\mathbb{K}_\lambda\{P_\cdot g\}(t,\cdot) = e^{\lambda t}P_t g - P_t g, \qquad \mathbb{K}_0\{P_\cdot g\}(t,\cdot) = tP_t g,$$

(4,15) $$(\lambda-\mu)\,\mathbb{K}_\lambda \circ \mathbb{K}_\mu = \mathbb{K}_\lambda - \mathbb{K}_\mu$$

are valid as far as the both sides of each equation have the meaning. Rewrite the equation (4,10) in the form $u = P_\cdot g + \mathbb{K}_0\{ku+Q\}$, and apply $\mathbb{K}_{-\lambda}$ to the both sides of it, then we obtain, by (4,14) (4,15), the equation

$$u = e^{-\lambda t}P_t g + \mathbb{K}_{-\lambda}\{(k+\lambda)u+Q\}.$$

Iterating this equation, we see that the solution of (4,10) is necessarily given by

$$u = \sum_{n=1}^{\infty} [\mathbb{K}_{-\lambda}\circ(k+\lambda)]^n \{[e^{-\lambda t}P_t g]_{t=\cdot} + \mathbb{K}_{-\lambda}Q\}$$

where $\mathbb{K}_{-\lambda}\circ(k+\lambda)$ is the mapping: $v \mapsto \mathbb{K}_{-\lambda}\{(k+\lambda)v\}$. Chosing λ so large that $k+\lambda \geqq 0$, we have

$$\begin{aligned} u &\leqq \sum_{n=0}^{\infty} [(k^*+\lambda)\,\mathbb{K}_{-\lambda}]^n\{[e^{-\lambda t}P_t g^+]_{t=\cdot} + \mathbb{K}_{-\lambda}Q^+\} \\ &= (k^*+\lambda)\,\mathbb{K}_{k^*}[e^{-\lambda t}P_t g^+]_{t=\cdot} + \mathbb{K}_{k^*}Q^+ \\ &= e^{k^* t}P_t g^+ + \mathbb{K}_{k^*}Q^+. \end{aligned}$$

This is the same as the second inequality of (4,11). The first inequality is similarly proved.

<u>Remark 1</u>. In Lemma 4.1 if k and Q are uniformly Lipshitz continuous in x, the solution of (4,10) gives the unique solution, which is bounded on E_T, for the equation

$$u^{\cdot} = \frac{1}{2}u'' + ku + Q.$$

<u>Remark 2</u>. Let F^* and f^* be a function on [0,1] satisfying (3) and a datum, respectively. Put $u^* = u(t,x;f^*,F^*)$, $u = u(t,x;f;F)$ and $w = u^* - u$. Then w satisfies (4.10) in which $g = f^* - f$, $k = (F(u^*) - F(u))/(u^*-u)$ (*) and $Q = F^*(u^*) - F(u^*)$. Terefore, by the first inequality of (4,11), <u>if</u> $F^* \geqq F$ <u>and</u> $f^* \geqq f$ <u>then</u> $u^* \geqq u$.

Let g be a bounded nonnegative measurable function on R. Then it follows from the inequality, valid for $|x| < M$,

$$(4,16) \qquad P_t g \geqq \|g\| \int_{|y|>N} p(1,y)dy + \frac{1}{\sqrt{2\pi t}} \int_{|y|<M+\sqrt{t}N} g(y)dy$$

(*) If $u^* = u$, we put $k = F'(u)$.

that

(4,17) <u>if</u> $g_n \to g$ <u>in locally</u> L^1 <u>sense and boundedly, then</u> $\sqrt{t}P_t g_n \to \sqrt{t}P_t g$ <u>uniformly on</u> $(0,T)\times(-M,M)$ <u>for each</u> $T<\infty$ <u>and</u> $M<\infty$.

Similarly we see that, for $|x| < M$

$$|(P_t g)'(x)| \leqq \frac{1}{\sqrt{t}}\|g\| \int_{|y|>N} p(1,y)|y|dy \tag{4,18}$$
$$+ \frac{1}{\sqrt{2\pi}\, t}\int_{|y|<M+\sqrt{t}\,N} g(y)dy.$$

<u>Lemma 4.2</u>. <u>Let</u> f_n, $n = 1, 2, \dots$, <u>and</u> f <u>be data and</u> u_n <u>and</u> u <u>corresponding solutions of</u> (1) <u>and</u> (2). <u>Suppose</u> $f_n \to f$ <u>in locally</u> L^1 <u>sense</u>. <u>Then</u> $\sqrt{t}\,u_n \to \sqrt{t}\,u$, $tu_n' \to tu'$, <u>and</u> $t\sqrt{t}\,u_n'' \to t\sqrt{t}\,u''$ <u>as</u> $n\to\infty$ <u>uniformly on</u> $(0,T)\times(-M,M)$ <u>for each pair of finite constants</u> M <u>and</u> T.

<u>Proof</u>. Putting $w_n = u_n - u$, we see that w_n satisfies (4.10) with $g = g_n = f_n - f$, $k = (F(u_n)-F(u))/(u_n-u)$ and $Q \equiv 0$. Therefore that $\sqrt{t}\,w_n \to 0$ in the desired sense follows from (4,11) and (4,17). By (4,2) we have

$$|w_n'| \leqq |(P_n g_n)'| + \int_0^t ds\int_R |p'(t-s,x-y)[F(u_n(s,y))-F(u(s,y))]|dy.$$

The first term multiplied by t tends to zero by virtue of (4,18). The second term is bounded, for $|x| < M$, by

$$2\,\|F\| \int_{0<s<\varepsilon \text{ or } t-\varepsilon<s<t} ds \int_{-\infty}^{\infty} p(t-s,y)\,\frac{|y|}{t-s}\,dy$$
$$+ 4\,\|F\| \int_{\varepsilon}^{t-\varepsilon} ds\int_N^{\infty} p(t-s,y)\frac{y}{t-s}\,dy + \frac{4\sqrt{t}\,\gamma}{\sqrt{2\pi}} \sup_{\substack{|y|<N+M \\ \varepsilon<s<t}} |w_n(s,y)|.$$

Chose ε so small and N so large that the first two terms are less than an arbitrarily given positive constant and then let n tend to infinity so that the last term tends to zero. This proves that $tw_n' \to 0$. The last convergence assertion is proved similarly by using (4,3).

Lemma 4.3. Let F_n, $n = 1, 2, \ldots,$ and F be functions on $[0,1]$ satisfying (3) and u_n and u corresponding solutions with common initial datum f; $u_n = u(t,x;f;F_n)$, $u = u(t,x;f;F)$. Suppose $F_n \to F$ uniformly. Then $u_n \to u$ and $u_n' \to u'$ uniformly on E_T for each $T < \infty$. Further suppose $F_n' \to F'$ uniformly. Then $u_n^{\cdot} \to u^{\cdot}$ and $u_n'' \to u''$ in the same sense.

Proof. Set $w_n = u_n - u$. Then w_n satisfies (4,10) with $g = 0$, $k = (F(u_n)-F(u))/(u_n-u)$ and $Q = F_n(u_n) - F(u_n)$. Putting $\delta_n = \|F_n - F\|$, we have $|w_n| \leqq \delta_n \gamma^{-1}(e^{\gamma t}-1)$. This proves $u_n \to u$ in the required sense. Remaining assertions are similarly proved by (4,2) or (4,3).

Lemma 4.4. Let a datum f have the continuous first derivative on R which is Lipshitz continuous there. Suppose there exists extended real constants x_1 and x_2; $-\infty \leqq x_1 \leqq x_2 \leqq \infty$, and $c > 0$ such that

$$(4,19) \qquad \frac{1}{2}f'' + cf' + F(f) \begin{cases} \geqq 0 & \text{if } x < x_1 \text{ or } x > x_2 \\ \leqq 0 & \text{if } x_1 < x < x_2 \end{cases}$$

where x's are those points at which f'' exist. Then there exist extended real functions X_1 and X_2 of $t > 0$ with $-\infty \leqq X_1(t) \leqq X_2(t) \leqq \infty$ such that

$$(4,20) \qquad z^{\cdot}(t,x) \begin{cases} > 0 & \text{if } x < X_1(t) \text{ or } x > X_2(t) \\ \leqq 0 & \text{if } X_1(t) < x < X_2(t) \end{cases}$$

where $z(t,x) = u(t,x+ct;f)$. If $x_1 = x_2$ we may put $X_1 \equiv X_2$, if $x_1 = -\infty$ then $X_1 \equiv -\infty$ and if $x_2 = \infty$ then $X_2 \equiv \infty$.

Proof. At first assume F'' exists and is continuous. Then by the equation (4,9) and remarks mentioned just after it the function $v(t,x) = z^{\cdot}(t,x-ct) = u^{\cdot}(t,x) + c\,u'(t,x)$, where $u = u(t,x;f)$, satisfies (4,9) and that, as $t \downarrow 0$,

$$v(t,x) \to g(x) = \frac{1}{2} f''(x) + c\,f'(x) + F(f(x)) \quad \text{a.s.}$$

and is bounded on E_T, $T < \infty$. It is proved as before that $v(t,x)$ can be approximated uniformly on each finite rectangle $[T^{-1},T]\times[-M,M]$ by solutions of (4,9) v_n such that v_n are continuous on $t \geqq 0$ and $g_n = v_n(0,\cdot)$ satisfy that $g_n(x) \geqq 0$ if $x < x_1$ or $x > x_2$; $\leqq 0$ if $x_1 < x < x_2$. Therefore we may assume that g is continuous to apply Proposition 3.4 which proves (4,20) (see also Proposition 3.3). In the case that F'' does not exist, use Lemma 4.3 and notice Proposition 3.3 to see the strict inequality in (4,20). q.e.d.

Lemma 4.5. Suppose two data f and f_* satisfy

$$f_*(x) \leqq f(x) + O(e^{-bx})$$

where b is a positive constant. Set $v(t,x) = u(t,x;f_*) - u(t,x;f)$. Then for each constant c

$$v(t,x+ct) \leqq O(e^{-\kappa t-bx})^{(*)} \quad \text{with} \quad \kappa = b\left(c - \frac{b}{2} - \frac{\gamma^*}{b}\right),$$

and if $c < b$, for each finite N,

$$v(t,x+ct) \leqq O\left(\frac{1}{\sqrt{t}}\, e^{-\left(\frac{c^2}{2}-\gamma^*\right)t}\right) \quad \text{uniformly in } x > N.$$

Proof. By Lemma 4.1 (see Remark 2 for it) we have

$$v(t,x) \leqq e^{\gamma^* t} P_t[f_*-f]^+(x+ct).$$

(*) If $f_*(x) \leqq f(x) + o(e^{-bx})$, then this can be replacd by " $v(t,x+ct) \leqq o(e^{-\kappa t-bx})$ as $t \to \infty$ uniformly in $x > N$.

Set

$$g(x) = 1 \quad \text{if} \quad x < 0, \quad = e^{-bx} \quad \text{if} \quad x > 0.$$

Then

$$P_t g(x) = \int_{x/\sqrt{t}}^{\infty} p(1,y)dy + e^{-bx+\frac{b^2}{2}t} \int_{-\infty}^{(x-bt)/\sqrt{t}} p(1,y)dy.$$

The lemma follows from these and the formula $\int_x^{\infty} p(1,y)dy \sim \frac{1}{x} p(1,x)$ as $x \to \infty$.

The follwing theorem asserts that if $c \geqq \sqrt{2\gamma^*}$ c-fronts are stable in a certain sense.

Theorem 4.1. Let $c \geqq \sqrt{2\gamma^*}$ and $f(x) = w_c(x+x_o) + O(e^{-bx})$. with some constants b and x_o. Then

$$u(t,x+ct;f) = w_c(x+x_o) + O(e^{-\kappa t-bx})$$

where κ is defined in Lemma 4.5; and if $c < b$

$$u(t,x+ct;f) = w_c(x+x_o) + O(\frac{1}{\sqrt{t}} e^{-(\frac{c^2}{2}-\gamma^*)t})$$

uniformly in $x > N > -\infty$.

Proof. Immediate from Lemma 4.5 and the stationarity of c- fronts: $u(t,x+ct;w_c(\cdot+x_o)) = w_c(x+x_o)$.

Let $\alpha > 0$. It is proved in McKean [14] (in case $F(u) = u(1-u)$) that if $f(x) \sim aw_c(x)$ as $x \to \infty$ with $a > 0$ and $c \geqq \sqrt{2\gamma^*}$ then $u(t,x+ct;f) \to w_c(x+x_o)$ uniformly in $x > N$, where $x_o = b^{-1}\log a$, $b = -\lim_{x\to\infty} [w_c'/w_c]$.

Here is a proof of this assertion under our setting. It will be not wasteful to remark that Theorem 4.1 is not directly available for the present problem since $w_c(x)$ decays as $x \to \infty$ little more rapidly than e^{-bx}, $b = c-\sqrt{c^2-2\alpha} \leqq c$ and $\kappa = \alpha-\gamma^* \leqq 0$.

Now we return to the proof. By the relation $f(x) \sim w_c(x+x_o)$, for any fixed $\delta > 0$, we have $w_c(x+x_o+\delta) \leqq f(x) \leqq w_c(x+x_o-\delta)$ for all sufficiently large x, and by Lemma 4.5 we see

$$w_c(x+x_o+\delta) - Q(t) \leqq u(t,x+ct;f) \leqq w_c(x+x_o-\delta) + Q(t)$$

for $x > N$ with $Q(t) = O(t^{-1/2}\exp\{-(c^2/2 - \gamma^*)t\}$. In paticular $u(t,x+ct;) \to w_c(x+x_o)$ as desired.

5. Limits of $u(t,x+ct;f)$.

We will investigate in this section the problem: what is the limit of

$$z(t,x) = u(t,x+ct;f)$$

as $t \to \infty$? The limit $w(x) = \lim_{t\to\infty} z(t,x)$, if exists, must be a solution of (4) on R. In fact, in the equation

$$z(t+s,x) = u(t,x+ct;z(s,\cdot))$$

letting s tend to infinity, we have, by Lemma 4.2, the equation

$$w(x) = u(t,x+ct;w),$$

from which we see that w satisfies the equation (4) on R. In particular if $0 \leqq c < c_o$, $w \equiv 1$ or $w \equiv 0$.

The following lemma is due to Aronson and Weinberger [1] except some additional statements, The proofs given here are based on their idears.

begin paragraph here

Lemma 5.1 *Let* $q(x)$ *be a c-solution* ($c \geq 0$) *defined on a interval* (L_1, L_2), $-\infty \leqq L_1 < L_2 \leqq \infty$, *such that* $q(L_1) = 0$ *or* $= 1$ *and that* $q(L_2) = 0$.

(i) *Let* $q(L_1) = 1$ *and* $L_2 = \infty$ (*these implies* $c \geqq c_0$). Set

(5.1) $$f(x) = 1 \text{ if } x < L_1, = q(x) \text{ if } L_1 < x$$

and set $z(t,x) = u(t,x+ct;f)$. *Then* $z(t,x)$ *decreases with* t. *The limit* $w(x) = \lim z(t,x)$ *is zero if* $\lim_{x\to\infty} q(x)/w_c(x) = 0$, *and it is a c-front if otherwise and* $\alpha > 0$.

(ii) *Let* $q(L_1) = 0$. *Put*

(5,2) $$f(x) = 0 \text{ if } x < L_1 \text{ or } x > L_2, = q(x) \text{ if } L_1 < x < L_2$$

and set $z(t,x) = u(t,x+ct;f)$. *Then* $z(t,x)$ *increases with* t. *Its limit is unity if* $c < c_0$ *or* $\overline{\lim}_{t\to\infty} q(x)/w_c(x) = \infty$, *and it is a c-front if otherwise and* $\alpha > 0$.

Proof. We prove only (i), since the proof of (ii) is very similar. Let $q(L_1) = 1$, $L_1 > -\infty$ and $L_2 = \infty$. Noticing $q'(L_1+0) < 0$, define for each constant $a > \|F\|$

$$f^*(x) = \begin{cases} 1 & x < L_1 - \delta \\ 1 - a(x-L_1+\delta)^2 & L_1-\delta < x < L_1+\delta \\ q(x) & x > L_1+\delta \end{cases}$$

where δ is a positive constant possibly chosen so that $q'(L_1+\delta) = -4a\delta$ and that $1 - a(2\delta)^2 = q(L_1+\delta)$. Then f^* is continuous, has the continuous first derivative and satisfies $\frac{1}{2} f^{*''} + c f^{*'} + F(f^*) \leqq 0$ at any $x \neq L_1 \pm \delta$. Thus Lemma 4.4 says that $z^*(t,x) = u(t,x+ct;f^*)$ decreases with t. It is clear that z has the same property by virtue of Lemma 4.2, since f^* converges to f as $a \to \infty$. If $\underline{\lim}\, q(x)/w_c(x) > 0$, we have $f(x) \geqq w_c(x+x_0)$ with some constant x_0 and hence $z(t,x) \geqq$

$u(t,x+ct;w_c(\cdot+x_0)) = w_c(x+x_0)$. Thus $w(x) = \lim z(t,x) \geqq w_c(x+x_0)$. This proves that w is a c-front. If $\underline{\lim}\ q/w_c = 0$, we have $\underline{\lim}\ w/w_c = 0$, but this implies $w = 0$ because $w_c(x+x_0)/w_c(x)$ converges to e^{-bx_0} as $x \to \infty$. q.e.d.

The information on the behavior of $u(t,x+ct;f)$ may be roughly gathered by Theorem 5.1 stated below. Results will be somwhat sharpend in Theorems 9.3 and 9.4. We will need the following condition on f

Condition [G]: $u(t,x;f) \to 1$ as $t \to \infty$ locally uniformly.

If $\alpha > 0$, this is the case for any data. We will discuss about Condition [G] at the end of this section.

Theorem 5.1. Let f be a datum and set $z(t,x) = u(t,x+ct;f)$.
(i) Let $f(x) = O(e^{-bx})$ as $x \to \infty$. Suppose either that $b > c - \sqrt{c^2 - 2\alpha}$ and $c > c_0$ or that $b > c^*$, $c = c_0 = c^*$, and (2,3) and (2,4) are valid. Then for each $N > -\infty$

$$z(t,x) \to 0 \quad \text{as} \quad t \to \infty \qquad \text{uniformly in} \quad x > N.$$

(ii) Suppose either that $0 \leqq c < c_0$ and Condition [G] is satisfied or that $c \geq c_0$ and $\underline{\lim}\ f(x)e^{bx} > 0$ with $b < c - \sqrt{c^2 - 2\alpha}$. Then for each $N > -\infty$

$$z(t,x) \to 1 \quad \text{as} \quad t \to \infty \qquad \text{uniformly in} \quad x > N.$$

Remark. Assertions of Theorem 5.1 are obtained by several authors ([13], [10]) in a special case under some restriction on F and by Aronson and Weinberger in case that f has compact support(mainly) and $c \neq c_0$.

Proof of Theorem 5.1. Let $f(x) = o(e^{-bx})$, $b > c - \sqrt{c^2 - 2\alpha}$ and $c > c_0$. We can chose a constant c_* such that $c > c_* > c_0$

and $b > c_* - \sqrt{c_*^2 - 2\alpha}$. Then there exists a c_*-solution q which satisfies the conditions of Lemma 5.1 (i) and for which $f \leqq f_*$ where f_* is defined by the right side of (5,1). In the inequality $z(t,x) \leqq u(t,x+ct;f_*)$ the right hand side tends to zero, since, by Lemma 5.1, $u(t,x+c_*t;f)$ tends to zero or to a c_*-front. Thus $z(t,x) \to 0$. The required uniformity of convergence is obvious. In the case $c = c_0$ we can similarly proceed, but taking as $q(x)$ a c_0-solution which corresponds to the extremal one in all c_0-manifolds that enter the origine (see the last diagram of Appendix). These prove (i).

Let $\underline{\lim}\, f(x)e^{bx} > 0$ with $b < c - \sqrt{c^2 - 2\alpha}$, $c \geqq c_0$. We can find a function F_* satisfying (3) such that $F_* \leqq F$ and $F_*' \leqq \alpha$; e.g. set, for small u,

$$(5,3) \qquad F_*(u) = \int_0^u (F'(v) \wedge \alpha)\, dv\,.$$

Then, putting $u_* = u(t,x+ct;f;F_*)$, we have $z \geqq z_*$ and hence $z(t,\cdot) \to 1$, since, by Lemma 4.5 and Lemma 5.1 (ii), $z_*(t,\cdot) \to 1$. In the case $c < c_0$, we can proceed as in the proof of (i) using Lemma 5.1 (ii). Thus (ii) is proved. q. e. d.

In the case $c = c_0 > c^*$, which is excluded from the above theorem, the situation is much simpler (see also § 10):

<u>Lemma 5.2.</u> <u>Let</u> $c_0 > c^*$. <u>For any couple of constants</u> b <u>and</u> η <u>with</u> $c_0 - \sqrt{c_0^2 - 2\alpha} < b < c_0 + \sqrt{c_0^2 - 2\alpha}$ (<u>i.e.</u> $b^2/2 - c_0 b + \alpha < 0$) <u>and</u> $b^2/2 - c_0 b + \alpha < -\eta < 0$, <u>there exist positive constants</u> A <u>and</u> A' <u>such that if we set</u>

$$U_*(t,x) = w_{c_0}(x - Ae^{-\eta t}) - e^{-\eta t - bx}$$

$$U^*(t,x) = w_{c_0}(x - A'(1-e^{-\eta t})) + e^{-\eta t - bx}$$

<u>and if a datum</u> f <u>satisfies</u>

(5,4) $$U_*(t_1,x+x_1) \leqq f(x) \leqq U^*(t_2,x+x_2)$$

<u>for some constants</u> t_1, t_2, x_1, x_2, <u>then</u>

(5,5) $$U_*(t+t_1,x+x_1) \leqq u(t,x+c_o t;f) \leqq U^*(t+t_2,x+x_2)$$

<u>for all</u> $t > 0$, $x \in R$.

<u>Proof</u>. Extending F to a continuously differentiable function $\overline{F}$ on R so that $\overline{F}'(u) \leqq \alpha$ for all $u \notin [0,1]$, and setting $v(t,x) = u(t,x+c_o t;f) - U_*(t+t_1,x+x_1)$, we have, by the mean value theorem,

$$v^{\cdot} = \frac{1}{2} v'' + c_o v' + F'(\theta)v + Q_*(t+t_1,x+x_1)$$

where

$$Q_* = \frac{1}{2} U_*'' + c_o U_*' + \overline{F}(U_*) - U_*^{\cdot} .$$

Since (5,4) implies $v(0,x) \geqq 0$, for the proof of the right hand side inequality in (5,5) it suffices to show that $Q_* \geqq 0$ or E, by virtue of Proposition 3.1. Set $w(t,x) = w_{c_o}(x-Ae^{-\eta t})$ and $h(t,x) = e^{-\eta t-bx}$. Then, putting $\overline{\alpha} = -b^2/2 + c_o b - \eta$,

$$Q_*(t,x) = -\overline{F}(w) + \overline{F}(w-h) + \overline{\alpha}\, h - A\eta e^{-\eta t} w'.$$

Since $\overline{\alpha} > \alpha$, we can find a positive constant $\delta > 0$ so small that

$$0 < w < \delta \quad \text{or} \quad 1-\delta < w < 1 \quad \text{implies} \quad \overline{\alpha} - \frac{1}{h}\,[\overline{F}(w) - \overline{F}(w-h)] > 0$$

for all $h > 0$. Then choose constants N and $a > 0$ such that

$$\delta < w_{c_o}(x) \leqq 1-\delta \quad \text{implies} \quad w_{c_o}'(x) < -a \quad \text{and}$$

$$w_{c_o}(x) \leqq 1-\delta \quad \text{implies} \quad x > -N.$$

Note that $w(t,x) \leqq 1-\delta$ implies $w_{c_o}(x) \leqq 1-\delta$. Now we may define A by the equation

$$A\eta a - \gamma^* e^{bN} = 0$$

so that $Q_* \geqq 0$, for δ, N and a are chosen independently of A.

Noticing that $w^* \equiv w_{c_o}(x-A'(1-e^{-\eta t})) \geqq w_{c_o}(x) \geqq h(t,x)$ for large x, and following the procedure similar to that taken in the above, we can find a constant A' such that

$$Q^*(t,x) \equiv \frac{1}{2} U^{*''} + c_o U^{*'} + \overline{F}(U^*) - U^*$$

$$= -\overline{F}(w^*) + \overline{F}(w^*+h) - \overline{\alpha}\, h + A' e^{-\eta t} w^{*'} \qquad (U^* = w^* + h)$$

$$\leqq 0$$

which proves the right hand side inequality of (5,5).

These complete the proof of Lemma 5.2.

As the direct consequence of Lemma 5.2 we have

Lemma 5.3. Let $c_o > c^*$ and b and η make up a couple of constants in Lemma 5.2.

(i) Suppose that a datum f satisfies Condition [G] and $f(x) = O(e^{-bx})$. Then for some constants x_1, x_2, and K

$$w_{c_o}(x+x_1) - Ke^{-\eta t-bx} \leqq u(t,x+c_o t;f) \leqq w_{c_o}(x+x_2) + Ke^{-\eta t-bx}$$

for all $t > 0$, $x \in R$.

(ii) For any $\varepsilon > 0$ there exists a positive constant δ such that if $|f(x) - w_{c_o}(x)| < \delta e^{-bx}$ for all $x \in R$ then $|u(t,x+c_o t;f) - w_{c_o}(x)| < \varepsilon e^{-bx}$ for all $t > 0$, $x \in R$.

Remarks to Condition [G]. Functions F satisfying (3) are classified into two classes according as Condition [G] is satisfied for all data or otherwise. In the former [resp. latter] case we say F belongs to the class I [resp. class II]. The class II is not empty. Some criteria for F to belong to the class I are obtained by several authors: Fujita [5], Hayakawa [7], Kobayashi-Sirao-Tanaka [12] (they all deal with the problem in the multidimentional case). Hayakawa [7] says that if $\underline{\lim}_{u \downarrow 0} F(u)/u^3 > 0$ (*) then F belongs to the class I and that if $F(u) = o(u^p)$ (*) with some $p > 3$ then F belongs to the class II. Kobayashi *et al.*'s results are sharpenings

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(*) In n-dimentional case these must be replaced by $\underline{\lim}\, F(u)/u^{1+2/n} > 0$ and by $F(u) = o(u^p)$ with $p > 1+2/n$, respectively.

of these consequences. Here is a rapid proof of the assertion: <u>if</u> $F(u)/u^3 \to \infty$ <u>as</u> $u \downarrow 0$ <u>then</u> F <u>belongs to the class</u> I (the proof is good for the multidimentional case). Let f be any datum and set $u = u(t,x;f;F)$. Then there exists $\varepsilon > 0$ and $t_o > 0$ such that

$$u(1,x) \geqq \varepsilon p(t_o,x) \qquad x \in R.$$

It suffices to prove that $u_* = u(t,x;f_*) \to 1$ where $f_* = \varepsilon p(t_o,\cdot)$. Noticing $u_*(t,x) \geqq P_t f_*(x) = \varepsilon p(t+t_o,x)$, it suffices, in turn, to prove that $g(x) = \varepsilon p(t+t_o,x)$ satisfies, with some $t > 0$,

$$(5,6) \qquad \frac{1}{2} g''(x) + F(g(x)) > 0 \qquad x \in R,$$

because this inequality implies that $u(t,x;g)$ increases with t and hence tends to unity by what is remarked at the beginning of this section. Since $g''(x) = g(x)(x^2 - t_1)/t_1^{\,2}$, $t_1 = t+t_o$ and since $\varepsilon/\sqrt{2\pi t_1} \geqq g(x) \geqq \varepsilon/\sqrt{2\pi t_1 e}$ for $|x| < \sqrt{t_1}$, we have

$$\frac{1}{2} g'' + F(g) \begin{cases} > 0 & \text{if } |x| \geqq \sqrt{t_1} \\ > F(g) - \dfrac{g}{2t_1} \geqq \left[\dfrac{F(g)}{g^3} - \dfrac{\pi e}{\varepsilon^2}\right] g^3 & \text{if } |x| < \sqrt{t_1}. \end{cases}$$

The right hand side of the last inequality is positive for some large t. Thus (5,6) is obtained.

We note that for any $\varepsilon > 0$ there exists a datum $f < \varepsilon$ with compact support for which [G] is valid. This follows from Lemma 5.1. It is also obtained that if $\underline{\lim}_{t\to\infty}\, u(t,x_o) > 0$ for some x_o then [G] is valid, as is proved below. The idea of the proof comes from Kanel' [11]. We may assume $x_o = 0$. It is easily checked that

$$u(t,x) = \int_0^\infty p^*(t,x,y)f(y)dy + \int_0^t ds\int_0^\infty p^*(t-s,x,y)F(u(s,y))dy$$

$$+ \int_0^t u(s,0)p'(t-s,x)ds \qquad x > 0,$$

where $p^*(t,x,y) = p(t,x-y) - p(t,x+y)$. The first two integrals are positive and the last integral, which is equal to $2\int_{x/\sqrt{t}}^\infty u(t-x^2/v^2,0)p(1,v)dv$, is bounded below by $(1/2)\underline{\lim}\ u(t,0)$ for $0 < x < N$ and for all sufficiently large t. Thus the fact remarked at the beggining of this paragraph proves $u(t,x) \to 1$ in the desired sense.

6. Behavior of Front of $u(t,x;f)$ (Special Case).

In this section we treat the problem expressed by (5) when datum f has a certain special form. Suppose $u(t,x) = u(t,x;f)$ decreases as x increases on some right half of the x-axis and tends to zero as $x \to \infty$ for each $t > 0$. Put

$$L(t) = \sup\{x;\ u'(t,x) = 0\} \quad (= -\infty \text{ if } \{\cdot\} \text{ is empty}),$$

$$M(t) = u(t,L(t))$$

$$x(t,w) = \sup\{x;\ u(t,x) = w\} \qquad 0 \leqq w \leqq M(t)$$

$$\phi(t.w) = u'(t,x(t,w)).$$

Note that ϕ is determined only by the shape of the tail of u (i.e. invariant under the transform: $f \to f(\cdot+\text{const.})$) and conversely restored to it through the inverse form

$$x(t,w) = \int^{w} \frac{du}{\phi(t,u)}. \tag{6,1}$$

We write $\phi = \phi(t,w;f)$ to express that ϕ is determined by f and $M(t) = M(t;\phi)$ for convenience. We also write $M = M\{\tau\}$ if τ is a nonnegative function defined on an interval $[0,M]$. Thus $M(t;\phi) = M\{\phi(t,\cdot)\}$. For each t the graph of $\phi(t,\cdot)$ is identical to the orbit of $(w,p) = (u(t,x),u'(t,x))$, $L(t) \leqq x < \infty$. We will use abbreviations $\phi^{\cdot} = \partial\phi/\partial t$, $\phi' = \partial\phi/\partial w$, etc. By formulas $\partial x/\partial w = 1/u'$, $\partial x/\partial t = -u^{\cdot}/u'$, $\phi' = u''/u'$ and

$$\phi'' = \frac{u'u''' - (u'')^2}{(u')^3}, \tag{6,2}$$

we derive from (1) the equation

$$\phi^{\cdot} = \frac{1}{2}\phi^2\phi'' - F\phi' + F'\phi \qquad 0 < w < M(t). \tag{6,3}$$

Let g be another initial datum and set $\psi = \phi(t,w;g)$ and

$\omega = \phi - \psi$. Then it follows from (6,3) that ω satisfies

(6,4) $$\omega^{\cdot} = \frac{1}{2}\phi^2\omega'' - F\omega' + [F' + \frac{1}{2}(\phi+\psi)\psi'']\omega$$

in the domain $\{(t,w);\ 0 < w < \min\{M(t,\phi),M(t,\psi)\},\ t > 0\}$. This equation is fundamental in the later arguments. We will sonsider it as a parabolic equation discussed in § 3 by regarding ϕ, ψ, ϕ'' or ψ'' as given functions. Let τ be a solution of (1,4). Then $\omega = \phi - \tau$ satisfies (6,4) where ψ is replaced by τ, since (1,4) implies

$$0 = \frac{1}{2}\tau^2\tau'' - F\tau' + F'\tau \qquad 0 < w < M\{\tau\}.$$

We denote by τ_c $(c \geqq c_o)$ the solution of (1,4) corresponding to the c-front, i.e. the unique solution solving (1,4) on the interval $[0,1]$ with $\tau(0) = \tau(1) = 0$. The equation (6,4) will be sometimes cited in the alternative form

(6,4)' $$\omega^{\cdot} = \frac{1}{2}\psi^2\omega'' - F\omega' + [F' + \frac{1}{2}(\phi+\psi)\phi'']\omega .$$

Lemma 6.1. Let $c_1 \geqq c_o$. Let a datum f satisfy the assumption of Lemma 4.4 with $c = c_1$. Suppose that there exists a constant x_3 such that

(6,5) $$f' \geqq 0 \quad \text{if } x \leqq x_3; \quad < 0 \quad \text{if } x > x_3,$$

that there exists a function $\varepsilon(t) > 0$, $t > 0$ such that

(6,6) $$\phi(t,w;f) < \tau_{c_1}(w) \qquad 0 < w < \varepsilon(t), \quad t > 0^{(*)}$$

and that $u(t,x) = u(t,x;f)$ satisfies

(6,7) $$u(t,x) \to 1 \quad \text{as } t \to \infty$$

and

(6,8) $$u(t,x+c_1t) \to 0 \quad \text{as } t \to \infty.$$

(*) This condition can be removed if $x_2 = \infty$ where x_2 appears in (4,19).

<u>Then</u> $\phi = \phi(t,w;f)$ <u>satisfies</u>

(6,9) $$\phi(t,w) \leqq \tau_{c_1}(w) + o(1) \qquad 0 \leqq w \leqq M(t;\phi)$$

<u>uniformly as</u> $t \to \infty$.

<u>Proof</u>. Applying Proposition 3.4 to the equation (4,9) satisfied by $v = u'$, we see, by (6,5), that $u'(t,\cdot) < 0$ on a right half line and $\geqq 0$ on the other half. Especialy $\phi = \phi(t,w;f)$ is well defined and $M(t;\phi) = \max_{x \in R} u(t,x)$. Put $z(t,x) = u(t,x+c_1 t)$. Then by Lemma 4.4 there exists extended real functions $X_1(t)$, $X_2(t)$, $-\infty \leqq X_1 \leqq X_2 \leqq \infty$, with which (4,20) holds. We will examine the evolution of the orbits of the vector functions $(w(x),p(x)) = (z(t,x),z'(t,x))$ $x \in R$ in the half strip $D = \{(w,p);\ 0 < w < 1,\ p \leqq 0\}$. Parts of these orbits contained in D are denoted by S_t. For the proof of the lemma we assume that $-\infty < X_1(t) < X_2(t) < \infty$ for any $t > 0$, since the other case is easy to deal with. Denote by A_t a point in D that has coordinates $(z(t,X_1(t)),z'(t,X_1(t))) = (z(t,X_1(t)),\phi(t,z(t,X_1(t))))$ or coordinates $(M(t;\phi), 0)$ according as $z'(t,X_1(t)) < 0$ or $\geqq 0$. Denote also by B_t a point with coordinates $(z(t,X_2(t)),z'(t,X_2(t)))$. By the equation $z^{\cdot} = \frac{1}{2} z'' + c_1 z' + F(z)$ and by what is remarked just before Theorem 1.1, c_1-manifolds cross S_t from the right or the left hand of S_t according as $z^{\cdot} > 0$ or < 0 at intersecting points (see Fig III). Notice that $z^{\cdot} \leqq 0$ on the closed arc of S_t between A_t and B_t and $z^{\cdot} > 0$ on the other parts of S_t. From these and the hypothesis (6,6) it follows that S_t lies under the c_1-manifold passing through A_t for each $t > 0$. We denote this manifold by T_t. A_t and T_t lies over the c_1-front.

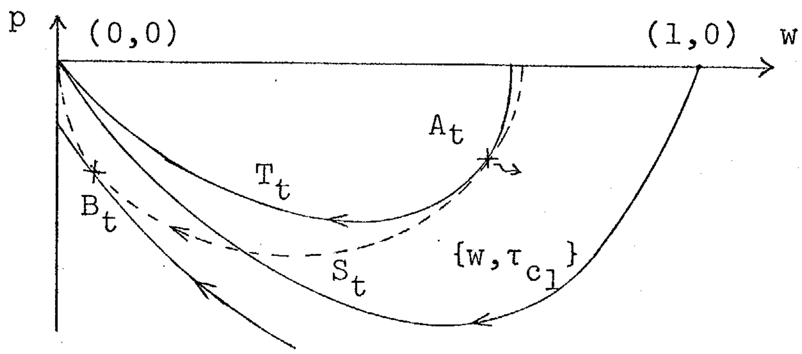


Fig III.

It is shown below that if $z'(t,X_1(t)) < 0$ then A_s, $s > t$ does not enter into the open domain bounded by T_t and the w-axis untill $z'(s,X_1(s))$ vanishes first after t. Let $\tau(u)$ be a solution of (1,4) corresponding to T_t and set $\omega = \phi - \tau$. Then ω satisfies (6,4) with $\psi = \tau$ and by Proposition 3.1 $\omega(t+s,w) \geqq 0$, $0 < w < M\{\tau\}$ for sufficiently small s. This proves the desired assertion. Especially we proved that if $z'(t,X_1(t)) < 0$ for $t_1 < t < t_2$ then T_{t_1} lies over S_t for $t_1 < t < t_2$.

To prove the assertion of the lemma first assume that there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $z'(t_n,X_1(t_n)) \geqq 0$, $n = 1, 2, \ldots$. By the hypothesis (6,7), for each $\varepsilon > 0$ we can find n_o such that $M(t;\phi) > 1 - \varepsilon$ for $t > t_{n_o} = t_*$. Let T be a c_1- manifold passing through $(1-\varepsilon,0)$. If $t > t_*$ and $z'(t,X_1(t)) \geqq 0$, then T lies over T_t and hence over S_t. If $t > t_*$ and $z'(t,X_1(t)) < 0$, then, by what has proved in the previous paragraph and by the continuity of S_t with respect to t, we can find a time t' with $t_* < t' \leqq t$ such that T lies over $T_{t'}$ and $T_{t'}$ lies over S_t. Consequently T lies over T_t for any $t > t_*$. Since T converges to the c_1-front as $\varepsilon \downarrow 0$, we have (6,9).

Next assume the remaining case: $z'(t,X_1(t)) < 0$ for $t > t_*$

with some constant t_*. Then T_t lies over $S_{t'}$ if $t' \geqq t > t_*$. There exists an unbounded sequence t_n such that w-coordinate of A_{t_n} tends to unity as $n \to \infty$. Because in the opposite case we can find $\delta \in (0,1)$ such that $z(t_o,x_o) > \delta$ implies $z^{\cdot}(t_o,x_o) > 0$, which in turn implies $z(t,x_o) > \delta$ for $t > t_o$ and contradicts to (6,7) and (6,8). Since A_{t_n} tends to the point (1,0), T_{t_n} converges to the c_1-front. Consequenyly we have (6,9). The proof of the lemma is completed.

Lemma 6.2. Let $f_o = I_{(-\infty,0]}$. Then for any data f for which ϕ is well defined, we have

$$\phi(t,w;f_o) \leqq \phi(t,w;f) \qquad t > 0, \quad 0 < w < M(t)$$

where $M(t) = M(\phi(t,\cdot;f))$.

Proof. Set $u = u(t,x;f)$ and $u_o = u(t.x;f_o)$. We prove a stronger assertion: for any $s > 0$

$$u_o'(s,x_o) \leqq u'(s,x_1) \quad \text{if} \quad u(s,x_1) = u_o(s,x_o).$$

Fix $s > 0$. Let $u(s,x_1) = u_o(s,x_o)$. Putting

$$v(t,x) = u(t,x-x_o+x_1) - u_o(t,x),$$

we have $v(s,x_o) = 0$ and $v'(s,x_o) = u'(s,x_1) - u_o'(s,x_o)$. Therefore it suffices to prove $v(s,x_o+x) \geqq 0$ for $x \geqq 0$. Since v solves the equation

$$v^{\cdot} = \frac{1}{2} v'' + F'(u_o + \theta v)v \qquad 0 \leqq \theta \leqq 1$$

with the initial condition

$$\lim_{t \downarrow 0} v(t,x) = f(x-x_o+x_1) - f_o(x) \begin{cases} \leqq 0 & \text{if } x < 0 \\ \geqq 0 & \text{if } x > 0, \end{cases}$$

we see, by the Proposition 3.1 and by the method of approximation as used in the proof of Lemma 4.4, that $v(t,\cdot) \geq 0$ on a right half x-axis and < 0 on the other half for each t. This proves the required assertion.

By the same method as used above Kolmogorov *et al*. showed that $\phi(t,w;f_o)$ increase with t. But this fact now clear by Huygens property: $\phi(t+s,w;f) = \phi(t,w;u(s,\cdot;f))$ and the lemma just proved. Thus there exists $\tau(w) = \lim_{t\to\infty} \phi(t,w;f_o)$. Since $\tau_{c_o}(w) = \phi(t,w;w_{c_o})$ we have $\tau(w) \leq \tau_{c_o}(w)$. These prove that $u(t,x+m(t);f_o)$ converges to some function, say $\hat{w}(x)$, which is decreasing with $\hat{w}(\infty) = 0$ and $\hat{w}(-\infty) = 1$. But by Lemma 4.2 $\phi(t,w;\hat{w}) = \lim_{s\to\infty} \phi(t,w;u(s,x+m(s);f_o)) = \lim_{s\to\infty} \phi(t+s,w;f_o) = \tau(w)$, from which we see, using (6,3), that τ satisfies (1,4) for some c. Since $\tau \leq \tau_{c_o}$, we have $\tau = \tau_{c_o}$. Consequently we have

Lemma 6.3 (Kolmogorov *et al*.). *Let* $f_o = I_{(-\infty,0]}$. *Then*

$$\phi(t,w;f_o) \uparrow \tau_{c_o}(w) \quad \text{as} \quad t \uparrow \infty.$$

The next lemma is complementary to Lemmas 6.1, 6.2 and 6.3.

Lemma 6.4. *Let* $c_2 > c_o$. *Suppose a datum* f *has the Lipshitz continuous first derivative with* $f' \leq 0$ *and satisfies that*

$$\frac{1}{2} f'' + c_2 f' + F(f) \begin{cases} \leq 0 & \text{if } x < 0 \\ \geq 0 & \text{if } x > 0 \end{cases}$$

where x's *are those points at which* f'' *exist. Further suppose that* $u(t,x+c_2t;f) \to 1$ *as* $t \to \infty$. *Then* $\phi = \phi(t,w;f)$ *satisfies*

$$\phi(t,w) \geq \tau_{c_2}(w) + o(1) \qquad 0 < w < M(t;\phi) \tag{6,10}$$

uniformly as $t \to \infty$.

Proof. The proof is very similar to that of Lemma 6.1 and here only the outline is given. Let S_t be the orbit of the vector function $(z(t,x), z'(t,x))$ of $x \in R$, where $z(t,x) = u(t,x+c_2t;f)$. As in the proof of Lemma 6.1 we can take a point A_t on S_t such that $z' < 0$ iff z is larger than the w-coordinate of A_t. Then c_2-manifold T_t passing through A_t bounds $S_{t'}$ below for $t' > t$. There exists a sequence $\{t_n\}$ along which p-coordinates of A_t tends to unity. Since A_t is bounded below by the graph of $\phi(t,\cdot;f_o)$ by virtue of Lemma 6.2, A_{t_n} approaches to the point (1,0). Thus T_{t_n} converges to the c_2-front. This implies (6.10).

7. Asymptotic Behavior of $u(t,x;f)$ for large x.

In order to apply Proposition 3.3 to the equations (6,4) or (6,4)' we must know the behavior of $\phi = \phi(t,w;f)$ and ϕ'' as $w \downarrow 0$, which are involved in that of $u = u(t,x;f)$, u', u'' and u''' as $x \to \infty$. Roughly speaking, the behavior of $u(t,x;f)$ and of its derivatives are asymptoticaly same as that of $e^{\alpha t}P_t f$ for data f belonging to certain classes.

Definition. (i) Let μ be a non-negative constant. A datum f is said to belong to the class $[E_\mu]$ if

$$f(x) = 0 \quad \text{for} \quad x > x_o \quad \text{with some cnstant } x_o \quad \text{in case } \mu = 0$$

and

$$f(x) \sim A(x)p(\mu,x) \quad \text{as} \quad x \to \infty \qquad \text{in case } \mu > 0,$$

where p is defined in § 0 and A is such a function that $A(\log x)$ is slowly varying at infinity, i.e. $A > 0$ and

$$A(x+x_o) \sim A(x) \quad \text{as} \quad x \to \infty \quad \text{for each constant } x_o.$$

(ii) Let λ be a positive constant. A datum f is said to belong to the class $[F_\lambda]$ if

$$f(x) \sim A(x)e^{-\lambda x} \qquad \text{as} \quad x \to \infty,$$

where A is same as in (i).

What we want to prove in this section is stated in the next two lemmas.

Lemma 7.1. Let f be a datum belonging to the class $[E_\mu]$ $(\mu \geqq 0)$.

(i) Set $u = u(t,x;f)$. Then following relations hold

$$(7,1) \qquad \log u(t,x) \sim -\frac{x^2}{2(\mu+t)}$$

$$(7,2) \qquad \frac{\partial^j u(t,x)}{\partial x^j} \sim \left(-\frac{x}{t+\mu}\right)^j u(t,x) \qquad j = 1,\ 2,\ 3$$

as $x \to \infty$ uniformly in $t \in (1/T,T)$ for each (finite) $T > 1$.

(ii) Set $\phi = \phi(t,w;f)$. Then

$$\phi(t,w) \sim \frac{\sqrt{2}}{\sqrt{t+\mu}} \sqrt{|\log w|}\, w \quad \text{and} \quad \phi''(t,w) = o\left(\frac{1}{\sqrt{t+\mu}} \frac{\sqrt{|\log w|}}{w}\right)$$

as $w \downarrow 0$ uniformly in $t \in (1/T,T)$ for each $T > 1$.

If $\mu = 0$ all these relations hold uniformly in $t \in (0,T)$.

Lemma 7.2. Let f be a datum belonging to the class $[F_\lambda]$ $(\lambda > 0)$.

(i) Set $u = u(t,x;f)$. Then the following relations hold

$$\frac{\partial^j u(t,x)}{\partial x^j} \sim (-\lambda)^j e^{\left(\frac{\lambda^2}{2}+\alpha\right)t} A(x) e^{-\lambda x} \qquad j = 0, 1, 2, 3 \tag{7,3}$$

as $x \to \infty$ uniformly in $t \in (1/T,T)$ for each $T > 1$.

(ii) We have

$$\phi''(t,w;f) = o(w^{-1})$$

as $w \downarrow 0$ uniformly in $t \in (1/T,T)$ for each $T > 1$.

Remark 1. The second parts of Lemmas 7.1 or 7.2 are readily derived from the first parts of them and from (6,2). It is also clear by Lemma 7.1 (ii) combined with Lemma 6.2 that for any datum f, for which $\phi(t,w;f)$ is well defined,

$$\phi(t,w;f) = O\left(\frac{1}{\sqrt{t}} \sqrt{|\log w|}\, w\right) \tag{7,4}$$

as $w \downarrow 0$ uniformly in $t \in (0,T)$.

Remark 2. By the fact that $v(t,x) = 1 - u(t,x;f)$ is a solution of the Cauchy problem

$$v^{\cdot} = \frac{1}{2} v'' - F(1-v), \quad v(0+,\cdot) = 1 - f,$$

we can derive similar results on the behaviors of $1 - u(t,x;f)$ and its derivatives as $x \to -\infty$ to those obtained above. We will not, however, use them later except the following simplest case: if $1 - f(-x)$ belongs to the class $[E_\mu]$ $(\mu \geqq 0)$, then

$$\log(1 - u(t,x)) \sim -\frac{x^2}{2(\mu+t)}$$

$$u'(t,x) \sim \frac{-x}{\mu+t}(1 - u(t,x))$$

as $x \to -\infty$ for each $t > 0$, where $u(t,x) = u(t,x;f)$.

For the proofs of Lemmas 7.1 and 7.2 we prepare several lemmas.

Lemma 7.3. Let g be a locally bounded measurable function with $\int_R p(t,x)|g(x)|dx < \infty$ for any $t > 0$ and ess. $\sup\{x ; |g(x)| > 0\} = x_1 < \infty$. Then

$$P_t g(x) = o\left(\exp\left\{-\frac{(x-x_1)^2}{2t}\right\}\right) \quad \text{as } x \to \infty$$

uniformly in $t \in (0,T)$ for each $T < \infty$.

Proof. Immediate from

$$\exp\left\{\frac{(x-x_1)^2}{2t}\right\} P_t g(x) = \int_{-\infty}^{x_1} e^{-\frac{(x-x_1)(x_1-y)}{t}} p(t,x_1-y)g(y)dy$$

$$= \int_0^\infty e^{-\frac{(x-x_1)w}{\sqrt{t}}} e^{-\frac{w^2}{2}} g(x_1-\sqrt{t}\,w)\,dw.$$

Lemma 7.4. In addition to assumptions imposed on g in Lemma 7.3, suppose $g \geqq 0$ on the interval $[x_2, x_1]$ with some $x_2 < x_1$. Then for any constant x_3 with $x_3 < x_1$

(7,5) $$\exp\left\{\frac{(x-x_3)^2}{2t}\right\} P_t g(x) \to \infty,$$

especially

$$P_t g(x) \sim P_t\{g \cdot I_{(x_3,x_1)}\}(x)$$

and especially

(7,6) $$\frac{\partial^n}{\partial x^n} P_t g(x) \sim \left(\frac{-x}{t}\right)^n P_t g(x) \qquad n = 1, 2, \ldots$$

as $x \to \infty$ *uniformly in* $t \in (0,T)$.

Proof. The divergence in (7,5) follows from

$$\exp\{\frac{(x-x_3)^2}{2t}\} P_t g(x) \geqq \exp\{\frac{(x-a)^2}{2t}\} P_t g(x)$$

$$\geqq \int_a^{x_1} e^{\frac{(x-a)(y-a)}{t}} p(t,a-y)g(y)dy\cdot(1+o(1))$$

as $x \to \infty$, where a is a constant which satisfies $\max\{x_2,x_3\} < a < x_1$, $g(a) > 0$ and $\int_a^x g(y)dy = (x-a)g(a) + o(x-a)$ as $x \downarrow a$ so that $\int_{a+t}^{x_1} p(t,a-y)g(y)dy \to g(a)$ as $t \downarrow 0$ (cf. Widder [17]).

Lemma 7.5. *Let* f *be a datum with* $\lim_{x\to\infty} f(x) = 0$. *Then*

$$u(t,x;f) = e^{\alpha t} P_t f(x)(1+t\cdot o(1))$$

where $o(1) \to 0$ *as* $x \to \infty$ *uniformly in* $t \in (0,T)$ *for each* T.

Proof. Define $F(u) = 0$ for $u > 1$. Putting $v(t,x) = e^{\alpha t}P_t f(x)$, $u = u(t,x;f)$ and $w = u - v$, we have

$$w = \mathbb{K}_0\{kw + F(v) - \alpha v\}$$

where $k = (F(u)-F(v))/(u-v)$ and $\mathbb{K}_0$ is defined by (4,12). By Lemma 4.1

$$|w(t,x)| \leqq \int_0^t e^{\gamma(t-s)} P_{t-s}|F(v(s,\cdot)) - \alpha v(s,\cdot)|(x)ds.$$

Since $v(t,x) \to 0$ as $x \to \infty$ uniformly in $t \in (0,T)$, for any $\varepsilon > 0$ we can chose constants M and L so that

$$|F(v(s,x)) - \alpha v(s,x)| \leqq \frac{\varepsilon}{2} v(t,x) + MP_{s-\delta} I_{(-\infty,L)}(x)$$

if $\delta < s < T$, where $\delta = \varepsilon/2\beta$. Then, using the inequality $|F(v) - \alpha v| < \beta v$ $(v > 0)$, we see that if $0 < t < T$

$$|w(t,x)| \leqq t e^{\gamma t}\{\varepsilon P_t f(x) + M P_{(t-\delta)\vee 0} I_{(-\infty,L)}(x)\}.$$

This proves $w = t \cdot o(P_t f(x))$ uniformly in $t \in (0,T)$, since the second term in the braces is small order of $P_t f(x)$ as $x \to \infty$ uniformly in $t \in (0,T)$ by virtue of Lemmas 7.3 and 7. .

Lemma 7.6. Let g be same as in Lemma 7.4. Let T and n be a positive constant and a non-negative integer, respectively. Then we can find such constant $K_n(T,g) = K_n(T,x_1)$ depending only on T, n and $x_1 = \text{ess. sup}\{x; g \neq 0\}$ that

$$\int_R |p'(s,x-y)| P_t|g|(y)|y|^n dy$$

$$\leqq \frac{K_n(T,g)}{t+s} \left\{ \frac{1}{\sqrt{s}} |x|^n + |x|^{n+1} \right\} P_{t+s}|g|(x)$$

for $x > 1$, $t \geqq 0$, $s > 0$, $t+s < T$.

Proof. Setting

$$J(z,t,x) = \int_R |p'(s,x-y)| p(t,y-z)|y|^n dy$$

we have, for $x > 1$,

$$J(z,t,x) = p(t+s,x-z) \int_R \left| \sqrt{\frac{t}{(t+s)s}}\, w + \frac{x-z}{t+s} \right| e^{-\frac{w^2}{2}}$$

$$\times \left| x - \sqrt{\frac{ts}{t+s}}\, w - \frac{s}{t+s}(x-z) \right|^n dw$$

$$\leqq \frac{K_n(T)}{t+s} p(t+s,x-z) \left\{ \left(\sqrt{\frac{t(t+s)}{s}} + |z|^n \right)(|x|^n + |z|) + |x|^{n+1} \right\},$$

where $K_n(T)$ is a constant depending only on n and T, and

$$\int_R J(z,s,x)|g(z)|dz \leqq \frac{K_n(T,g)}{t+s} \left\{ \frac{1}{\sqrt{s}} |x|^n + |x|^{n+1} \right\} P_{t+s}|g|(x),$$

which is the desired inequality.

Proof of Lemma 7.1 in case $\mu = 0$. Let f belong to the class $[E_0]$. The relation (7,1) is clear by Lemmas 7.3, 7.4 and 7.5. For the estimation of u' we rewrite (4,2)' as follows

$$u'(t,x) = e^{\alpha t}(P_t f)'(x) + \int_0^t ds \int_R p'(t-s,x-y)J(s,y)dy$$

where $J(s,y) = F(u(s,y)) - \alpha e^{\alpha s}P_s f(y)$. Then, using Lemmas 7.5 and 7.6, we see, as in the proof of Lemma 7.5, that the last term in the above equation is small order of $\frac{x}{t}P_t f(x)$ as $x \to \infty$ uniformly in $t \in (0,T)$. Since $(P_t f)'(x) \sim -\frac{x}{t}P_t f(x) \sim -\frac{x}{t}e^{-\alpha t}u(t,x)$, the case $j = 1$ in (7,2) is obtained.

Estimation of u'' is carried out as follows: Set $f_* = u(t,\cdot)$, $u_*(t,x) = u(t,x;f_*) = u(2t,x)$. To prove is that $u_*''(t,x) \sim e^{2\alpha t}(x/2t)^2 P_{2t}f(x)$ uniformly in $t \in (0,T/2)$. By (4,3)

$$(7,7) \quad u_*''(t,x) = e^{\alpha t}(P_t f_*)''(x) + \int_0^t ds \int_R p'(t-s,x-y)J(s,y)dy$$

where

$$J(s,y) = F'(u_*(s,y))u_*'(s,y) - \alpha e^{\alpha s}(P_s f_*)'(y).$$

Since $f_*'(x) \sim -e^{\alpha t}\frac{t}{x}P_t f(x) \sim e^{\alpha t}(P_t f)'(x)$ uniformly in $t < T/2$, we see, as in the proof of Lemma 7.5, using (7,6), that

$$(P_s f_*)'(x) = P_s f_*'(x) \sim -e^{\alpha t}\frac{x}{t+s}P_{t+s}f(x)$$

$$\sim e^{-\alpha s}u_*'(s,x)$$

uniformly in $0 < s \leqq t < T/2$. By this relation and by the inequality (4,6), for any $\varepsilon > 0$ we can find constants M and L depending only on ε, T and f such that

$$|J(s,y)| \leqq \varepsilon\frac{|y|}{t+s}P_{t+s}f(y) + \frac{M}{\sqrt{t+s}}P_s I_{(-\infty,L)}(y)$$

for $0 < s \leqq t < T/2$. Therefore Lemma 7.6 says that the last term in (7,7) is bounded by

$$K_1(T,f)\left(\frac{x}{t\sqrt{t}} + \frac{x^2}{t} \right) P_{2t}f(x) \quad + \quad K\,x\, P_t I_{(-\infty,L)}(x)$$

for $x > 1$, $t \in (0,T/2]$, where K is a constant depending only on L, M, and T, while we see also by Lemma 7.6 that

$$(P_t f_*)''(x) \quad = \quad (P_t f_*')'(x) \quad \sim \quad e^{\alpha t}\left(\frac{x}{2t} \right)^2 P_{2t} f(x)$$

uniformly in $t \in (0,T/2]$. Thus we have $u(2t,x) = u_*(t,x) \sim e^{2\alpha t}(x/2t)^2 P_{2t} f(x)$ with required uniformity.

Noticing that $u^{\cdot} \sim \frac{1}{2} u''$ and $u''' \sim 2u^{\cdot\prime}$, and using the equation $u^{\cdot\prime} = \frac{1}{2}(P_t f)''' + (P_t F(f))' + (\mathbb{K}_0\{F'(u)u^{\cdot}\})'$ ($\mathbb{K}_0$ is defined by (4,12)), we can estimate the tail of u''' at infinity as in the case of u''. Now Lemma 7.1 has been proved in the case $\mu = 0$.

For the proof of Lemma 7.2 and of the rest of Lemma 7.1 we prepare the next

Lemma 7.7. Let $A(x)$ be a function as appears in Definition of the classes $[E_\mu]$ and $[F_\lambda]$, and T a positive constant.

(i) Let $\{g_t(x)\}_{0 \leq t < T}$ be a family of bounded functions such that $g_t(x) \sim A(x\mu/(\mu+t))p(\mu+t,x)$ $(\mu > 0)$ as $x \to \infty$ uniformly in $t \in (0,T)$, then

$$(7,8) \qquad \frac{\partial^n}{\partial x^n} P_t g_s(x) \quad = \quad g_{t+s}(x)\left(- \frac{x}{\mu+t+s}\right)^n \left(1 + o(1)\, \frac{1}{\sqrt{t}^{\,n}} \right) \qquad n = 0,\ 1$$

where $o(1) \to 0$ as $x \to \infty$ uniformly in $0 \leq s < t < T$.

(ii) If $g(x) \sim A(x)e^{-\lambda x}$ $(\lambda > 0)$ and $g(x)$ is bounded, then

$$\frac{\partial^n}{\partial x^n} P_t g(x) \quad = \quad A(x)\, e^{-\lambda x}\{(-\lambda)^n + \frac{1}{t^{n-1}} \cdot o(1)\} \qquad n = 0,\ 1,\ 2$$

<u>where</u> $o(1) \to 0$ <u>as</u> $x \to \infty$ <u>uniformly in</u> $t \in (0,T)$.

<u>Proof</u>. First we prove (i). Write $g_t(x) = A_t(x)p(\mu+t,x)$. Then $A_t(x) = A(\frac{\mu}{\mu+t}x)(1+o(1))$ as $x \to \infty$ uniformly in t and

$$\int_R (\frac{x-y}{t})^n p(t,x-y)g_s(y)dy$$

$$= \int_{-\infty}^{x} A_s(x-y)p(\mu+s,x-y)p(t,y)(\frac{y}{t})^n dy + O(\frac{x^{n-1}}{\sqrt{t}^{2n-1}} e^{-\frac{x^2}{2t}}) \qquad x > 1.$$

Let J denote the first term in the right hand side of this equation . Then

$$J = p(\mu+t+s,x) \int_{-\infty}^{w_1} A_s(\frac{\mu+s}{\mu+t+s}x - \sqrt{\frac{(\mu+s)t}{\mu+t+s}}w) \times$$

$$\times (\sqrt{\frac{\mu+s}{\mu+s+t}}\frac{w}{\sqrt{t}} + \frac{1}{\mu+t+s}x)^n e^{-\frac{w^2}{2}} dw/\sqrt{2\pi},$$

where w_1 is defined by $\frac{\mu+s}{\mu+t+s}x - \sqrt{\frac{(\mu+s)t}{\mu+t+s}}w_1 = 0$. Since $A(\log x)$ is slowly varying at infinity, $A(x)$ is expressed in the form

$$A(x) = a(x)\exp\{\int_1^{e^x} \frac{\varepsilon(y)}{y} dy\} \qquad x > 0$$

where $a(x) \to a_o$, $0 < a_o < \infty$ and $\varepsilon(x) \to 0$ as $x \to \infty$. Especially

$$\frac{A(x-x_o)}{A(x)} = \frac{a(x-x_o)}{a(x)} \exp\{\int_{e^x}^{e^{x-x_o}} \frac{\varepsilon(y)}{y} dy\} = o(e^{-x_o}) \quad \text{as} \quad x_o \to -\infty$$

uniformly in $x > 0$. Then it is not difficult to see

$$J = p(\mu+t+s) \int_{-\infty}^{w_1} A(\frac{\mu}{\mu+t+s}(x-\sqrt{\frac{t(\mu+t+s)}{\mu+s}}w))$$

$$\times (\frac{x}{\mu+s+t})^n e^{-\frac{w^2}{2}} dw/\sqrt{2\pi}\cdot(1+o(1)/\sqrt{t}^n)$$

$$= g_{t+s}(x)\left(\frac{x}{\mu+t+s}\right)^n\left(1+o(1)\frac{1}{\sqrt{t}^n}\right).$$

These combined with the fact that $t^{-n}p(t,x) = o(g_t(x))$ as $x \to \infty$ uniformly in t proves (7,8).

The second part of the lemma follows from

$$\int_R p(t,x-y)g(y)\left(\frac{x-y}{t}\right)^n dy$$

$$= \frac{e^{-\lambda x}}{\sqrt{t}^n}\int_{-\infty}^{\frac{x}{\sqrt{t}}} A(x-\sqrt{t}y)\,\frac{e^{-\frac{y^2}{2}}}{2}\,e^{\lambda\sqrt{t}y}y^n dy + O\left(\frac{x^{n-1}}{t^n/\sqrt{t}}e^{-\frac{x^2}{2t}}\right)$$

$$= e^{-\lambda x}\frac{1}{t^n}\frac{\partial^n}{\partial\lambda^n}e^{\lambda^2 t/2}A(x)\left(1+o(1)\frac{1}{\sqrt{t}^n}\right).$$

The proof of Lemma 7.7 is completed.

Now we prove Lemma 7.1 in case $\mu > 0$. Fix $t_1 > 0$. By (7,8) and (4,2)' and by Lemma 7.5 it is easy to see that (7,2) with $j = 1$ holds uniformly in $t \in (t_1,T)$. Using this, (7,8) and the equation

$$u''(t_1+t,x) = (P_t u'(t_1,\cdot))'(x) + \int_0^t (P_{t-s}g_s)'(x)ds$$

where $g_s(x) = F'(u(t_1+s,x))u'(t_1+s,x)$, we see (7,2) with $j = 2$. The case of $j = 3$ is similarly proved. The proof of Lemma 7.1 (i) is completed.

Lemma 7.2 (i) is similarly proved by Lemma 7.7 (ii).

8. Approach of Front of $u(t,x;f)$ to Front of Travelling Wave.

Here we will clarify the behavior of the front of $u = u(t,x;f)$, i.e. the function $u(t,\cdot + m(t))$ where

$$m(t) = \sup\{x \; ; \; u(t,x) = \frac{1}{2}\},$$

for large t. We will use symbols $\phi(t,u;f)$, τ_c, $M(t;\phi)$, which were introduced in §6, in this section too. We will mainly deal with data which satisfy the following

Condition [W]: there exists $t_o > 0$ and finite number N such that

(8.1) $\lim_{x\to\infty} u(t_o,x) = 0, \quad u'(t_o,x) < 0 \quad \text{for} \quad x > N$

and

(8,2) $\varliminf_{x\to-\infty} u(t_o,x) > 0 \quad \text{or} \quad u'(t_o,x) > 0 \quad \text{for} \quad x < -N$

where $u(t,x) = u(t,x;f)$: if F belongs to the class II (see Remark to [G] in § 5 for the definition) we assume $\varliminf_{x\to-\infty} u(t_o,x) > 0$.

This condition scarcely narrows the class of data to be dealt with. For example if f does not increase for large values of x and tends to zero as $x \to \infty$ then (8,1) is satisfied. Data which belong to the class $[E_\mu]$ or $[F_\lambda]$ also satisfy (8,1). The condition (8,1) guarantees the existance of $\phi(t,w;f)$. The condition (8,2) is imposed in order to apply Lemmas 8.1 or 8.2, given later, which prove $M\{\phi(t,\cdot;f)\} \to 1$ as $t \to \infty$ for data f satisfying [W]. Especially [W] implies Condition [G] (see § 5) and that $m(t)$ takes a definite value for every sufficiently large t. If F belongs to class I any datum with compact support satisfies [W]. We can remove Condition [W] under a certain restriction on F (Theorem 8.5).

Now we state the main theorems, from which it will be seen that the behavior of the front of $u(t,x;f)$ depends mainly on the behavior of f for large x which is inherited to $u(t,\cdot;f)$ as was seen in the previous section.

Theorem 8.1. *Let a datum f belong to the class $[E_\mu]$ ($\mu \geqq 0$) or to the class $[F_\lambda]$ with $\lambda > c_0 - \sqrt{c_0^2 - 2\alpha}$.*(*) *Suppose Condition [W] is satisfied. Then $u = u(t,x;f)$ satisfies*

$$(8,3) \qquad \lim_{t\to\infty} u(t,x+m(t)) = w_{c_0}(x)$$

uniformly in $x > -m(t)$.

Corollary. *Let f be a datum with compact support. Suppose F belongs to the class I. Then $u = u(t,x;f)$ satisfies*

$$u(t,x) - w_{c_0}(x-m(t))I_{(0,\infty)}(x) - w_{c_0}(-x+m_*(t))I_{(-\infty,0)}(x)$$

$$\longrightarrow 0 \quad \text{as } t \to \infty \quad \text{uniformly in } x \in R$$

where $m_*(t) = \inf\{x;\ u(t,x) = 1/2\}$.

Theorem 8.2. *Let a datum f belong to the class $[F_\lambda]$ with $0 < \lambda \leqq c_0 - \sqrt{c_0^2 - 2\alpha}$ ($\alpha > 0$). Suppose Condition [W] is satisfied. Then $u = u(t,x;f)$ satisfies*

$$(8,4) \qquad \lim_{t\to\infty} u(t,x+m(t)) = w_c(x) \qquad c = \frac{\lambda}{2} + \frac{\alpha}{\lambda}$$

uniformly in $x > -m(t)$.

Remark. Let f be a datum such that $f(-x)$ belongs to the class $[E_\mu]$ or $[F_\lambda]$ as well ws $f(x)$. Then assertions analogous to Corollary of Theorem 8.1 hold if F belongs to the class I. The condition that F belongs to class I can be replaced by Condition [G] (cf. § 10).

(*) The classes $[E_\mu]$ and $[F_\lambda]$ are defined in § 7.

In Theorems 8.1 or 8.2 the condition that f belongs to the class $[E_\mu]$ or $[F_\lambda]$ should be weakened. This is suggested by the next theorem.

Theorem 8.3. Let f be a differentiable datum for which Condition [W] holds. Suppose

(8,5) $\quad -f'(x) \geqq f(x)(b+o(1)) \quad$ as $x \to \infty$

and

(8,6) $\quad f(x)e^{-b_* x} \to \infty \quad$ as $x \to -\infty$

for a positive constants $b \geqq c_o - \sqrt{c_o^2 - 2\alpha}$ and $b_* < b$. Then for $u = u(t,x;f)$, (8,3) holds uniformly in $x > -m(t)$.

The next theorem is complementary to these theorems.

Theorem 8.4. Let $\alpha > 0$. Let f be a datum which is differentiable and positive for large x and satisfies that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} f'(x)/f(x) = 0$. Then under Condition [W]

$$\lim_{t\to\infty} u(t,x+m(t)) = \frac{1}{2}$$

uniformly on each compact set of R.

For proofs of these theorems we need two more lemmas.

Lemma 8.1. Let f be a continuously differentiable datum. Suppose a constant $\delta > 0$ is related with f in such a manner that the equation $f(x) = \delta$ has just two roots, say x_1, x_2, $x_1 < x_2$, and $f'(x) < 0$ for $x \geqq x_2$, > 0 for $x \leqq x_1$. Let $\delta(t)$ be a solution of the equation $\delta^\cdot(t) = F(\delta(t))$ with $\delta(0) = \delta$. Then for each t, $\delta(t)$ is related with $u(t,\cdot;f)$ in the same manner of how δ is related with f, as long as $\delta(t) < \sup_{x\in R} u(t,x;f)$.

Proof. Let $g(x)$ be a continuously differentiable function such that $0 \leqq g \leqq 1$, $g' \leqq 0$ and g is not a constant. Suppose that the equation $f(x) = g(x)$ has just two roots, say y_1, y_2, $y_1 < y_2$, and that $f'(x) < 0$ and $f(x) \leqq g(x)$ for $x \geqq y_2$. Put $u = u(t,x;f)$, $v = u(t,x;g)$ and $T = \sup\{t ; u(t,x) > v(t,x)$ for some $x \in R\}$. We show here that $u'(t,x) < 0$ for $x \geqq \sup\{x ; u(t,x) = v(t,x)\}$ if $0 < t < T$. Since $w = u - v$ satisfies

$$w^{\cdot} = \frac{1}{2} w'' + F'(v+\theta w)w' \qquad (0 \leqq \theta = \theta(t,x) \leqq 1),$$

the set $\{(t,x) ; T > t \geqq 0,\ u(t,x) < v(t,x)\}$ has just two connected components by virtue of Proposition 3.4. Let D be one of them which contains a right half x-axis. By Lemmas 7.3, 7.4 and 7.5 $\bar{D}$,the closure of D in $\bar{E}$, contains a right half of $\ell_T = \{T\}\times R$ if $T < \infty$. Assume for simplicity that T is finite and set $\partial D = \bar{D} - D$ and $\Gamma = \partial D - (\ell_T \cup \ell_0)$. Then $u' \leqq v' < 0$ on Γ because the intersection of D and ℓ_t is connected for each positive $t < T$. Thus $u' < 0$ on Γ, which implies, by Proposition 3.3, $u' < 0$ in D. This is the same as what was announced to be shown.

Now the lemma is easily proved. Let t be such a time that $\delta(t) < \sup_x u(t,x)$. Clearly we can find a function g for which $g \geqq \delta$, conditions stated at the beginning of this proof are satisfied and $t < T$ where T is defined in the previous paragraph. Since $\delta(t) < u(t,x;g)$ $x \in R$, we have $u'(t.x) < 0$ for $x \geqq X_2(t) = \sup\{x; \delta(t) = u(t,x)\}$. Similarly we get $u'(t,x) > 0$ for $x \leqq X_1(t) = \inf\{x; \delta(t) = u(t,x)\}$. It is clear that $u(t,x) > \delta(t)$ if $X_1(t) < x < X_2(t)$. Thus $\delta(t)$ and $u(t,\cdot)$ are related in the required manner.

The proof of the lemma is completed.

The similar method proves

Lemma 8.2. *Let f be a continuously differentiable datum Suppose the set $\{x;\ f(x) = \delta\}$ consists of just one point and $f'(x) < 0$ if $f(x) \leqq \delta$. Let $\delta(t)$ be defined as in Lemma 8.1. Then $\delta(t)$ has the same relation to $u(t,\cdot;f)$ as δ does to f for each $t > 0$.*

Proof of Theorem 8.1. Step 1. Set $u = u(t,x;f)$ and $\phi(t,w) = \phi(t+t_o,w;f) = \phi(t,w;u(t_o,\cdot))$ where t_o appears in Condition [w]. We will prove

$$(8,7) \qquad \phi(t,w) = \tau_{c_o}(w) + o(1) \qquad 0 \leqq w < M(t;\phi)$$

as $t \to \infty$ uniformly. Since we know that $\phi \geqq \tau_{c_o} + o(1)$ as a direct consequence of Lemmas 6.2 and 6.3, for the proof of (8,7) it suffices to show that

$$(8,8) \qquad \phi(t,w) \leqq \tau_{c_o}(w) + o(1) \qquad 0 \leqq w < M(t;\phi).$$

Let $c_1 > c_o$. Condition [W] enables us to find a constant $\delta > 0$ which is related with $u(t_o,\cdot)$ in the manner stated in Lemma 8.1 or 8.2. We will show in Step 2 that there exists a datum f^* and a constant $t_1 > t_o$ such that f^* satisfies conditions imposed in Lemma 6.1 and inequalities

$$f^*(x) < \min\{g(x),\ \delta\} \qquad x \in R$$

and

$$\phi(0,w;f^*) > \phi(0,w;g) \qquad 0 < w < M(\phi(0,\cdot;f^*))$$

where $g(x) = u(t_1,x)$. Let such f^* and t_1 be found for each $c_1 > c_o$ sufficiently near c_o. Set $\phi^* = \phi(t,w;f^*)$ and

$\psi = \phi(t,w;g) = \phi(t+(t_1-t_o),w)$. Then we have, by Lemma 8.1 or 8.2,

$$M(t;\phi^*) < M(t;\psi) \qquad t \geqq 0$$

and hence $\omega = \psi - \phi^*$ is defined and satisfies (6,4), where ϕ is replaced by ϕ^*, in the domain $D = \{(t,w);\ t > 0,\ 0 < w < M(t;\phi^*)\}$. Since $M(t;\phi^*) = \sup_x u(t,x;f^*)$ and $\omega(t,M(t;\phi^*)) < 0$, there exists a continuous function $M(t)$ such that $0 < M(t) < M(t;\phi^*)$ $t \geqq 0$ and $\omega < 0$ in $D - D_*$ where $D_* = \{(t,w);\ t > 0,\ 0 < w < M(t)\}$. Then $\omega \leqq 0$ on ∂D_*. Check that Proposition 3.2 is applicable to the equation (6,4) in D^* for the present ω, using Lemma 7.1 (ii) or Lemma 7.2 (ii). Then we have $\omega \leqq 0$ in D_*. Consequently $\psi \leqq \phi^*$ in D. Since $\phi^* \leqq \tau_{c_1} + o(1)$ by Lemma 6.1 and since $\tau_{c_1} \downarrow \tau_{c_o}$ as $c_1 \downarrow c_o$ we get (8,8).

Step 2. Now we construct f^*. We carry out this only in the case $\underline{\lim}_{x \to -\infty} f(x) = 0$ (in the other case the construction is much simpler). Thus we assume that F belongs to the class I. Set $h(x) = \delta \exp\{-x^2\}$ and take $t_1 > t_o$ such that $h(x) \leqq \min\{g(x), \delta\}$ $\quad x \in R$ where $g(x) = u(t_1,x;f)$ (see Lemma 7.4).

First we assume that f belongs to the class $[E_\mu]$ or $[F_\lambda]$ with $\lambda > c_o + \sqrt{c_o^2 - 2\alpha}$. Let $c_1 > c_o$ and let $\frac{\lambda}{2} + \frac{\alpha}{\lambda} > c_1 > c_o$ if f belongs to the class $[F_\lambda]$. Then, by Lemma 7.1 or Lemma 7.2 there exists a constant x_2 such that $g' < 0$ for $x > x_2$ and

$$\frac{1}{2}g'' + c_1 g' + F(g) > 0 \quad \text{and} \quad g < \frac{\delta}{2e} \qquad \text{for} \quad x > x_2 - 1.$$

Set $k(x) = -a(x-x_2)^2 + g'(x_2)(x-x_2) + g(x_2)$ where the constant $a > 0$ is chosen so large that

$$\frac{1}{2} k'' + c_1 k' + F(k) \quad < \quad 0 \qquad \text{if} \quad k > 0,$$

$$\max_{x \in R} k(x) \ < \ \frac{\delta}{e} \quad \text{and} \quad \sqrt{g'(x_2)^2 + 4ag(x_2)}\ \frac{1}{a} \ < \ 1.$$

Since $(1/2)h'' + c_1 h' + F(h) > 0$ for $x < -1$ and $h(-1) = \delta/e$, two trajectories $\{(h(x),h'(x));\ x < -1\}$ and $\{(k(x),k'(x)); k(x) > 0,\ k'(x) < 0\}$ drawn in the vertical half strip $(0,\delta) \times (0,\infty)$ cross each other at just one point, say (w,p). Let $x_1{}^*$ and x_1 be values of parmeter at which they pass though it: $h(x_1^*) = k(x_1\ ) = w$, $h'(x_1^*) = k'(x_1\ ) = p$. Now we may put

$$f^*(x) \ = \ \begin{cases} g(x) & \text{if} \quad x > x_2 \\ k(x) & \text{if} \quad x_1 < x < x_2 \\ h(x-x_1+x_1{}^*) & \text{if} \quad x < x_1. \end{cases}$$

By Theorem 5.1 $u(t,x+c_1t;f^*) \to 0$ as $t \to \infty$ and by Lemmas 7.1 and 7.2 $\overline{\lim}_{w \downarrow 0}\ \phi(t,w;f^*)/w \ < \ -c_1 - \sqrt{c_1{}^2 - 2\alpha}$ which implies (6,6) for $\phi = \phi(t,w;f^*)$. Other requirements for f^* are clear by the construction and hypotheses.

When f belongs to the class $[F_\lambda]$ with $c_o - \sqrt{c_o{}^2 - 2\alpha} \leqq \lambda \leqq c_o + \sqrt{c_o{}^2 - 2\alpha}$, we can find x_2' such that

$$\frac{1}{2} g'' + c_1 g' + F(g) < 0 \quad \text{and} \quad g < \frac{\delta}{2e} \qquad \text{for} \quad x > x_2'$$

and then construct f^* as above, but in this case f^* satisfies the condition (4,19) with $x_2 = \infty$. Thus f^* is constructed.

Step 3. The inverse function of $u(t,\cdot+m(t))$ is given by

$$\int_{1/2}^{u} \frac{dw}{\phi(t,w;f)} \ = \ x(t,u) - m(t)$$

and converges to $w_{c_o}{}^{-1}(u)$; $w_{c_o}{}^{-1}(w_{c_o}(x)) = x$. The desired assertion follows from the inequality

$$|u(t,x+m(t)) - w_{c_o}(x)| = |u(t,x+m(t)) - u(t,x(t,w_{c_o}(x)))|$$

$$\leqq K_t |w_{c_o}^{-1}(w) - x(t,w) + m(t)| \quad \text{if} \quad w = w_{c_o}(x) \leqq M(t,\phi)$$

where $K_t = \sup\{|u'(t,x)|; x \in R\}$ is bounded for large t by the remark following (4,6). This completes the proof of Theorem 8.1.

Proof of Theorem 8.2. Set $\phi = \phi(t,w;f)$. It is proved as in the proof of Theorem 8.1 that $\phi \leqq \tau_c + o(1)$. Thus it suffices to prove that $\phi \geqq \tau_c + o(1)$. This follows from Lemma 7.2 and the next lemma.

Lemma 8.3. Let $\alpha > 0$. Let a datum f be positive and differetiable on a right half x-axis. Suppose $\lim_{x\to\infty} f(x) = 0$ and

(8,9) $$0 \leqq -f'(x) \leqq (b + o(1))f(x) \quad \text{as} \quad x \to \infty \quad (*)$$

with $0 < b < c_o - \sqrt{c_o^2 - 2\alpha}$. Then

(8,10) $$\phi(t,w;f) \geqq \tau_c(w) + o(1) \quad \text{as} \quad t \to \infty$$

uniformly where $c = \frac{b}{2} + \frac{\alpha}{b}$.

Proof. Set $u = u(t,x;f)$. First it is proved that (8,9) implies

(8,11) $$0 \leqq -u'(t,x) \leqq (b + o(1))u(t,x) \quad \text{as} \quad x \to \infty$$

for each $t > 0$. Put $v(t,x) = \exp\{\alpha t\}P_t f(x)$. It is easy to see $o \leqq -v'(t,x) \leqq (b + o(1))v(t,x)$. By Lemma 7.5 $v(t,x) \sim u(t,x)$ as $x \to \infty$. Set $w = v - u$. Then $w' = \mathbb{K}_0\{F'(u)w' + \alpha v' - F'(u)v'\}$ ($\mathbb{K}_0$ is defined by (4,12)) and, by Lemma 4.1,

$$|w'(t,x)| \leqq \mathbb{K}_\gamma\{|\alpha - F'(u)||v'|\}(t,x) = o(v(t,x))$$

(*) " $a(t) \leqq b(t)$ as $t \to \infty$ " means that $a(t) \leqq b(t)$ $t > N > 0$.

as $x \to \infty$. Thus we have (8,11).

Set $\phi(t,w) = \phi(t+1,w;f)$. From (8,11) it follows that $\phi(t,w) \geqq -bw+o(w)$ as $w \downarrow 0$. Take a constant c_2 with $c_o < c_2 < c$. It is not difficult to construct a continuous function $\psi_o(w)$ $0 \leqq w \leqq 1$ which has a piece-wise continuous derivative bounded on each compact set of the half open interval $[0,1)$ and satisfies

$$\psi_o(w) < 0 \quad 0 < w < 1, \qquad \psi_o(0) = \psi_o(1) = 0$$

$$\psi_o(w) < \phi(0,w) \quad 0 < w < M(0;\phi)$$

$$-2c_2 - \frac{2F(w)}{\psi_o(w)} \begin{cases} \geqq \psi_o'(w) & 0 < w < \frac{1}{2} \\ \leqq \psi_o'(w) & \frac{1}{2} < w < 1 \end{cases}$$

and

$$-\psi_o'(0) < c_2 - \sqrt{c_2^2 - 2\alpha}$$

(ψ_o may be taken to be equal to a $(c',\underline{b})$-manifold with $c_2 < c' < c$ near $w = 0$ and equal to $\phi(1/2,w;f_o)$, $f_o = I_{(-\infty,0)}$ near $u = 1$). Let $g(x)$ be a non-trivial solution of the differetial equation $g' = \psi_o(g)$ on R and set $\psi = \phi(t,w;g)$. By Theorem 5.1 $u(t,x+c_2t;g) \to 1$ as $t \to \infty$, because the last condition imposed on ψ_o implies $g(x)\exp\{b^*x\} \to \infty$ as $x \to \infty$ if $-\psi_o'(0) < b^* < c_2 - \sqrt{c_2^2 - 2\alpha}$. It is easily checked that g satisfies conditions imposed on f in Lemma 6.4. Thus $\psi(t,w) \geqq \tau_{c_2}(w) + o(1)$, while (6,4) combined with boundary conditions:

$$\phi(0,w) > \psi(0,w) = \psi_o(w) \qquad 0 < w < M(0;\phi),$$

$$\phi(t,w) > \psi(t,w) \quad \text{for } w \text{ near } 0 \text{ or } M(t;\phi) \; (*)$$

implies $\phi(t,w) > \psi(t,w)$ $0 < w < M(t;\phi)$. Therefor we have

(*) If $M(t;\phi) = 1$, to get this strict inequality we may use Remark 2 of Lemmas 7.1 and 7.2.

$\phi(t,w) \geqq \tau_{c_2}(w) + o(1)$ by virtue of Lemma 6.4. This proves (8,10) because $\tau_{c_2}(w) \uparrow \tau_c(w)$ as $c_2 \uparrow c$. q.e.d.

Proof of Theorem 8.3. Set $\phi(t,w) = \phi(t+t_o,w;f)$ where t_o is a constant which appears in [W]. As in the proof of Lemma 8.3 we see that $\phi(t,w) \leqq -bw + o(w)$ and that for each b_1, $b_* < b_1 < b$ there exists a smooth datum g such that

$$g(x) \sim e^{-b_1 x}$$

$$g'(x) < 0 \qquad x < 0$$

and

$$\psi(0,w) > \phi(0,w) \qquad 0 < w < M(0;\psi)$$

where $\psi = \phi(t,w;g)$. Let δ be a positive constant which is related with $u(t_o,\cdot;f)$ in the manner stated in Lemmas 8.1 or 8.2. Conditions (8,2) and (8,6) allow us to assume that g satisfies in addition $g(-x) < \min\{u(t_o,x;f),\delta\}$ so that $M(t;\psi) < M(t;\phi)$ $t \geqq 0$ and $M(t;\psi) \to 1$ as $t \to \infty$. Then as before we have $\phi < \psi$, while, as was shown in the proof of Theorem 8.1,

$$\psi(t,w) \leqq \tau_{c'}(w) + o(1) \qquad c' = \max\{c_o, \frac{b_1}{2} + \frac{\alpha}{b_1}\}.$$

This implies $\phi(t,w) = \tau_{c_o}(w) + o(1)$ and proves (8,1) as before.

Proof of Theorem 8.4. Set $\phi = \phi(t,w;f)$. Condition [W] and Lemma 8.1 implies $M(t;\phi) \to 1$ as $t \to \infty$, while Lemma 8.3 says that $\phi(t,w) = o(1)$ as $t \to \infty$. From these the assertion of the theorem is obvious.

Under an additional restriction on F, Condition [W] can be removed:

Theorem 8.5. *Suppose* $F(u)/u$ *is non-increasing. Then in each of Theorems 8.1 to 8.4 Condition* [W] *may be removed from assumptions of each one. In Theorem 8.3 the condition* (8,6) *may be also removed simultaneously.*

For the proof we use the following lemmas

Lemma 8.4. *Suppose* $F(u)/u$ *is non-increasing. Let* f_1, f_2 *and* f *be so related that* $f = f_1 + f_2$. *Let* u, u_1 *and* u_2 *be corresponding solutions of* (1) *and* (2). *Then* $u \leqq u_1 + u_2$.

Proof. Set $k(t,x) = F(u(t,x))/u(t,x)$ and let u_1^* and u_2^* be solutions of the equation

$$u^{\cdot} = \frac{1}{2} u'' + ku$$

with $u_1^*(0+,\cdot) = f_1$ and $u_2^*(0+,\cdot) = f_2$, respectively. Then $u = u_1^* + u_2^*$. Since $F(u_i)/u_i \geqq F(u)/u$ for any $(t,x) \in E$, we have $u_i \geqq u_i^*$ $(i = 1, 2)$. Thus $u \leqq u_1 + u_2$.

Lemma 8.5. *Suppose* $F(u)/u$ *is non-increasing. Let* f_1 *and* f_2 *belong to the class* $[E_0]$ *i.e.* $\sup\{x; f_i(x) > 0\} < \infty$ *and set* $m_i(t) = \sup\{x; u(t,x;f_i) = 1/2\}$ $(i = 1, 2)$. *Then* $m_1(t) - m_2(t)$ *is bounded for large* t.

Proof. Clearly we can assume $f_1 = I_{(-1,0]}$ and $f_2 = I_{(-\infty,0]}$. Since $f_1 = f_2 - f_2(\cdot+1)$, by Lemma 8.4 and Theorem 8.1

$$\underline{\lim}_{t\to\infty} u(t,m_2(t);f_1) \geqq w_{c_0}(x) - w_{c_0}(x+1) \sim (1-e^{-c^*})w_{c_0}(x) \quad (x \to \infty).$$

Hence $m_2(t) - m_1(t)$ $(\geqq 0)$ is bounded.

Proof of Theorem 8.5. We deal with only the case that f belongs to the class $[E_\mu]$ for some $\mu > 0$. The other cases are analogously treated and omitted here.

Let f belong to the class $[E_\mu]$. For each positive integer n define

$$f_n = 1 \quad \text{for} \quad x < -n; \quad = f(x) \quad \text{for} \quad x < -n$$

and set $m_n(t) = \sup\{x;\ u(t,x;f_n) = 1/2\}$. Since $f \leqq f_n + f_o(\cdot+n)$, where $f_o = I_{(-\infty,0]}$, by Lemma 8.4

$$|u(t,x+m(t)) - w_{c_o}(x+m(t)-m_n(t))|$$

$$\leqq u(t,x+m(t)+n;f_o) + |u(t,x+m(t);f_n) - w_{c_o}(x+m(t)-m_n(t))| \ .$$

By Theorem 8.1 the last term in this inequality tends to zero as $t \to \infty$ uniformly in $x \in R$. Thus, writing $m_o(t) = \sup\{x;u(t,x;f_o) = 1/2\}$,

$$\overline{\lim_{t\to\infty}}\ |u(t,x+m(t))-w_{c_o}(x+m(t)-m_n(t))| \leqq w_{c_o}(x+\underline{\lim}_{t\to\infty}(m(t)-m_o(t))+n).$$

Since $\underline{\lim}(m(t)-m_o(t)) > -\infty$ by Lemma 8.5, the left side quantity in the above inequality become arbitrarily small uniformly in $x > -N$ for each real N when we let n large. Therefor (8,3) holds uniformly in $x > -N$. The required uniformity in $x > -m(t)$ is obtained if we bound $u(t,x+m(t))$ below by $u(t,x+m(t);f_*)$ where

$$f_*(x) = 0 \quad \text{for} \quad x < 0; \quad = f(x) \quad \text{for} \quad x > 0$$

and apply Theorem 8.1 and Lemma 8.5. q.e.d.

In the proof carried out in the above we needed Theorems 8.1 to 8.4 applied to data with $\underline{\lim}_{x\downarrow-\infty} f(x) > 0$. But for such data the proofs of these theorems are much simplified. Indeed we need only the comparison argument based on Proposition 3.1 in the phase space and Theorem 4.1 in addition to Lemmas 6.2, 6.3 and 8.2 (see the proof of Theorem 8.3). Correspondingly Theorem 8.5 can be obtained more easily than Theorems 8.1 or 8.2.

9. Speed of Propagation.

We have seen in the previous section that the front of $u(t,x;f)$ propagates with speed $m^{\cdot}(t)$ as forming the shape of the c-front with some constant c, provided that the tail of f at (positive) infinity behaves regularly in a certain sense. The purpose of this section is to get nice estimations of $m(t)$.

Theorem 9.1. Let f be a datum. Set $u = u(t,x;f)$. Suppose, for some continuous function $k(t)$, there exists

$$\lim_{t\to\infty} u(t,x+k(t)) = g(x) \quad \text{in locally } L_1 \text{ sense,}$$

where g is not a constant. Then g is a c-front with some speed c, $|c| \geq c_o$. If $m(t)$ is defined by (for large t)

$$u(t,m(t)) = \frac{1}{2} \quad \text{and} \quad m(t) - k(t) \quad \text{being bounded,}$$

then m is continuously defferentiable and $m^{\cdot}(t) \to c$ as $t \to \infty$. Furthermore $v(t,x) \equiv u(t,x+m(t))$, v' and v'' converge to w_c, w_c' and w_c'', respectively, as $t \to \infty$ locally uniformly.

Proof. Set, for $\mu > 0$, $f_\mu = u(\mu,\cdot)$, $g_\mu = u(\mu,\cdot;g)$ and

$$v_\mu(t,x) = u(t,x+k(t);f_\mu).$$

In the identity

$$v_\mu(t,x) = u(\mu,x;u(t,\cdot+k(t))),$$

letting t tend to infinity, we have, by Lemma 4.2, that

$$(9,1) \qquad v_\mu \to g_\mu, \quad v_\mu' \to g_\mu' \quad \text{and} \quad v_\mu'' \to g_\mu'' \quad \text{as } t \to \infty$$

locally uniformly for each $\mu > 0$. Fix any $\mu > 0$ and set $f_* = f_\mu$, $g_* = g_\mu$. Let J be a connected component of the open set $\{x;\ g_*'(x) \neq 0\}$, which is not empty, for g is not constant. Without loss of generality we assume $g_*' < 0$ on J. Let $x_1 \in J$ be fixed. Then we can define a continuous function $k_*(t)$ (for large t) by

$$u(t,k_*(t);f_*) = g_*(x_1) \quad \text{and} \quad \lim_{t\to\infty} (k_*(t) - k(t)) = x_1.$$

Set $v_*(t,x) = u(t,x+k_*(t);f_*)$. Then by (9.1)

(9,2) $\quad v_* \to g_*, \quad v_*' \to g_*' \quad \text{and} \quad v_*'' \to g_*'' \quad \text{as} \quad t \to \infty$

locally uniformly. Note that $v_*(t, 0) = g_*(x_1)$ is constant. Letting $x = 0$ be fixed and t tend to infinity in the equation

$$v_*^{\cdot} = \frac{1}{2} v_*'' + cv_*' + F(v_*),$$

we have

$$k_*^{\cdot}(t) \to c \equiv - \frac{(1/2)g_*''(x_1) + F(g_*(x_1))}{g_*'(x_1)} \quad \text{as} \quad t \to \infty.$$

Integrating the both sides of the same equation by t from n to $n+1$ and letting n tend to infinity, we have $0 = (1/2)g_*'' + cg_*' + F(g_*)$. Hence g_* is a c-front. By $g_{\mu+s} = u(s,\cdot;g_*)$, we see that g_μ is a c-front for each $\mu > 0$ with common speed c. Consequently g is a c-front.

The last statement follows from (9,2) and the identity $u(t+\mu,x+k_*(t+\mu)) = v_*(t,x+k_*(t+\mu)-k_*(t))$. That $m^{\cdot}(t) \to c$ is obtained as in the case of $k_*(t)$.

Lemma 9.1. Suppose $F(u)/u$ is non-increasing. Let f_1, f_2 and f be data so related that $f = f_1 + f_2$. Let u, u_1 and u_2 be corresponding solutions of (1) and (2). Then $u \leqq u_1 + u_2$.

Proof. Set $k(t,x) = F(u(t,x))/u(t,x)$ and let $u_1{}^*$ and $u_2{}^*$ be solutions of the equation

$$u^{\cdot} = \frac{1}{2} u'' + k u$$

with $u_1{}^*(0,\cdot) = f_1$ and $u_2{}^*(0,\cdot) = f_2$, respectively. Then $u = u_1{}^* + u_2{}^*$. Since $F(u_i)/u_i \geqq F(u)/u$ for any $(t,x) \in E$, we have $u_i \geqq u_i{}^*$ $i = 1, 2$. Thus $u \leqq u_1 + u_2$.

From Lemma 9.1 it follows that if $F(u)/u$ is non-increasing the behavior of f as $x \to -\infty$ produces no other effect on $m(t)$ than the difference of a bounded function. In fact, letting $m_o(t)$ and $m_1(t)$ be corresponding to $f_o = I_{(-\infty,0)}$ and to $f_1 = I_{(-1,0)}$, respectively, by Lemma 9.1, as $x \to \infty$,

$$\varliminf_{t\to\infty} u(t,m_o(t);f_1) \geqq w_{c_o}(x) - w_{c_o}(x+1) \sim (1-e^{-c^*})w_{c_o}(x)$$

and hence $m_o(t) - m_1(t)$ ($\geqq 0$) is bounded.

McKean [14] found that if $F(u)$ has the special form $\alpha u(1-u)$ and intial datum f is the indicater function of negative real axis then $m(t) \leqq c^*t - \sigma(t) + \text{const.}$, where we write

$$\sigma(t) = \frac{\log t}{2c^*}. \tag{9,3}$$

This is easily extended to the case that $F(u)/u \leqq \alpha$, i.e. $\beta = \alpha$, and is readily derived from

<u>Proposition 9.1.</u> <u>Let</u> v <u>be a solution of the linear equation</u>

$$v^\cdot = \frac{1}{2} v'' + \alpha v \tag{9,4}$$

<u>with</u> $v(0+,\cdot) = g$ a.e. <u>and with</u> $v(t,x) = O(\exp\{x^2\})$ <u>uniformly in</u> $t \in (0,T)$ <u>for each</u> $T < \infty$, <u>where</u> g <u>is measurable and satisfies</u> $\int_R e^{c^*x}|g(x)|dx < \infty$. <u>Then</u>

$$v(t,x+c^*t-\sigma(t)) \to \frac{1}{\sqrt{2\pi}}\int_R e^{c^*y}g(y)dy\, e^{-c^*x} \quad \underline{\text{as}} \quad t \to \infty.$$

<u>Proof</u>. By the equation $(x+c^*t-\lambda\log t-y)^2 = (x-y-\lambda\log t)^2 + 2c^*t(x-y-\lambda\log t) + 2\alpha t^2$, we see

$$v(t,x+c^*t-\lambda\log t) = e^{\alpha t}P_t g(x+c^*t-\lambda\log t)$$

$$= \frac{e^{c^*\lambda\log t}}{\sqrt{2\pi t}}\int_R \exp\{-\frac{(x-y-\lambda\log t)^2}{2t} + c^*y\}g(y)dy\, e^{-c^*x}.$$

Then substitute $\lambda = 1/2c^*$ to get the result.

Suppose $F(u)/u \leqq \alpha$. Let f be a datum and set $u = u(t,x;f)$, $m(t) = \sup\{x;\ u(t,x) = 1/2\}$. Then

$$m(t) \leqq c^*t - \sigma(t) + \text{const.} \quad \text{if} \quad \int_R e^{c^*x} f(x)dx < \infty.$$

This is immediate from Proposition 9.1 and the inequality $u(t,x) \leqq e^{\alpha t}P_t f(x)$. Let A be a positive function such that

$$\int_R A(x)dx < \infty \quad \text{and} \quad A(\log x) \text{ varies slowly at infinity.}$$

Then under the same restriction on F

$$c^*t - \sigma(t) - m(t) \to \infty \qquad \text{if} \quad f(x) = O(A(x)e^{-c^*x}) \quad \text{as } x \to \infty.$$

For the proof it suffices to show that if $f(x) \sim A(x)\exp\{-c^*x\}$ as $x \to \infty$ and $f(x) = 1$ for $x < 0$ then $u(t,c^*t-\sigma(t)) \to 0$ as $t \to \infty$. But for such f we have seen, in Theorem 8.1, $u(t,x+m(t)) \to w_{c_o}(x)$, while $\overline{\lim}\, u(t,x+c^*t-\sigma(t)) \leqq \text{const.}\exp\{-c^*x\} = o(w_{c_o}(x))$, since, by Lemma 2,2 (see also (2,22)), $\lim_{x\to\infty} w_{c_o}(x)\exp c^*x/x > 0$. Thus $\overline{\lim}\, u(t,x+c^*t-\sigma(t)) = 0$.

We get here a more exact estimation under some additional restrictions on F and f.

<u>Theorem 9.2</u>. <u>Suppose</u> $F(u) \leqq \alpha u$ <u>for</u> $0 < u < 1$ <u>and</u> $\alpha u - F(u) = o(u^{1+\delta})$ <u>with some</u> $\delta > 0$. <u>Let</u> f <u>be a datum with</u> $\sup\{x;\ f(x) > 0\} < \infty$. <u>Then</u>

$$c^*t - 3\sigma(t) + \text{const.} \leqq m(t) \leqq c^*t - 3\sigma(t) + O(\log\log t).$$

<u>Proof</u>. Step 1. We can find a function F^* and F_* satisfying (3) such that $F^* \geqq F \geqq F_*$ and $F^*(u)/u$ and $F_*(u)/u$ are decreasing. Therefore there is no loss of generality in assuming that $F(u)/u$ is non-increasing and that $f = f_o = I_{(-\infty,0)}$ by virtue of Lemma 8.5.

Set $u = u(t,x;f_o)$ and $v(t,x) = u(t,x+m(t))$. Then, by Lemma 6.3, we see $v(t,x) \downarrow w_{c_o}(x)$ for each $x < 0$ and $\uparrow w_{c_o}(x)$ for each $x > 0$ as $t \to \infty$.

Let $\underline{m}(t)$ be the maximal convex function on $t > 0$ that bounds $m(t)$ below. In the remaining part of this step we prove that $m(t) - \underline{m}(t)$ is bounded. Define a function $a(t,s,N)$ $t, s, N > 0$ by

$$m(t+N) - m(t) = m(t+s+N) - m(t+s) + a(t,s,N).$$

Since $c^*t - m(t) \to \infty$ and $m^\cdot(t) \to c^*$, there exists an unbounded sequence $\{t_n\}$ such that $m(t_n) = \underline{m}(t_n)$. It is easily seen that

$$0 \leqq m(t) - \underline{m}(t) \leqq \sup_{\substack{s,N>0 \\ r>t_n}} a(r,s,N) \quad \text{if} \quad t > t_n.$$

Therefore for our present purpose it is sufficient to prove that

(9,5) $\qquad a(t,s,N) \to 0$ as $t \to \infty$ uniformly in s, N.

Set $h(t,x) = [v(t,x) - v(t+N,x)]^+$. Then $h(t,x) \leqq v(t,x+M(t))$ with some function M such that $M(t) \to \infty$ as $t \to \infty$. By Lemma 8.4 and the monotonicity of $F(u)/u$

$$\begin{aligned} \tfrac{1}{2} &= u(s,m(t+s)-m(t);v(t,\cdot)) \\ &\leqq u(s,m(t+s)-m(t);h(t,\cdot)) + u(s,m(t+s)-m(t);v(t+N,\cdot)). \end{aligned}$$

Since $u(s,m(t+s)-m(t);h(t,\cdot)) \leqq v(t+s, M(t)) \to 0$ as $t \to \infty$ uniformly in s, N, we have

$$u(s,m(t+s)-m(t);v(t+N,\cdot)) + o(1)$$

$$\geqq \tfrac{1}{2} = u(s,m(t+s+N)-m(t+N);v(t+N,\cdot))$$

and hence $m(t+s+N) - m(t+N) \geqq m(t+s) - m(t) + o(1)$ where $o(1) \to 0$ as $t \to \infty$ uniformly in s, N. This is the same as the statement (9,5).

Step 2. Set $k(t,x) = F(v(t,x))/v(t,x)$. Then we have

$$v^\cdot = \tfrac{1}{2} v'' + m^\cdot v' + kv.$$

In terms of the standard 1-dimentional Brownian motion $\{B_t, t \geqq 0; P_x, x \in R\}$ (cf. [9]), we have by Kac's formula

$$v(t,x) = E_x[\, e^{\int_0^t k(s,B_{t-s}+m(t)-m(s))ds} f_o(B_t+m(t))],$$

where $E_x[\cdot]$ stands for the expectation of Brownian motion B_t starting from a position x. Let q_* be a function on R defined by

$$q_*(x) = Me^{-bx} \qquad M > 0, \quad b > 0,$$

where constants M and b are chosen so that $k \geqq \alpha - q_*$. This is possible, because $v(t,x) \uparrow w_{c_o}(x)$ as $t \uparrow \infty$ for $x > 0$, $\log w_{c_o}(x) \sim -c^*x$ as $x \to \infty$ and $F(u)/u = \alpha + o(u^\delta)$. Then

$$v(t,x) \geqq e^{\alpha t} E_x[\, e^{-\int_0^t q_*(B_s+m(t)-m(t-s))ds} f_o(B_t+m(t))].$$

Write

$$m(t) = c^*t - n(t).$$

Setting $L = \inf_{t>s>0} \{m(t) - m(t-s) - \frac{s}{t} m(t)\}$, we see, by the convexity of $\underline{m}(t)$ and the boundedness of $m(t) - \underline{m}(t)$, that L is finite and we have

$$v(t,x) \geqq e^{\alpha t} E_x[\, e^{-\int_0^t q_*(B_s+ m(t)\frac{s}{t}+L)ds} f_o(B_t+m(t))]$$

and

$$v(t,-1) \geqq e^{\alpha t} E_{-1}[\, e^{-\int_0^t q_*(B_s+ m(t)\frac{s}{t}+L)ds} \mid B_t+m(t) = -1]$$

$$\times\ P_{-1}[-1 < B_t + m(t) < 0].$$

where $E_\cdot[\cdot|\cdot]$ stands for the conditional expectation. Since $\{B_s + m(t)\frac{s}{t}\ ;\ 0 < s < t\}$ conditioned on $B_t + m(t) = -1$ has the same conditional law as $\{B_s;\ 0 < s < t\}$ conditioned on $B_t = -1$, the right hand side of the last inequality is equal to

$$e^{\alpha t}\ E_{-1}[\ e^{-\int_0^t q_*(B_s+L)ds}\ |\ B_t = -1]\int_0^1 p(t,y-m(t))dy$$

$$= E_{-1+L}[\ e^{-\int_0^t q_*(B_s)ds}\ |\ B_t = -1+L]\ \frac{e^{c^*n(t)}}{\sqrt{2\pi t}}\ \left(\int_0^1 e^{c^*y}dy + o(1)\right)$$

$$= \frac{e^{c^*}-1}{c^*}\ p_*(t,-1+L,-1+L)e^{c^*n(t)}(1+o(1))$$

where $p_*(t,x,y)$ is the fundamental solution of the parabolic equation

$$u^{\cdot} = \frac{1}{2}u'' - q_*u \qquad t > 0,\ \ x \in R.$$

In order to estimate $p_*(t,x,x)$ let $\hat{p}(t,x,y)$ be the fundamental solution of $u^{\cdot} = u'' - \hat{q}\,u$ where $\hat{q}(x) = \exp\{-2x\}$. Then

$$\hat{p}(t,x,y) = \int_0^\infty e^{-t\lambda}\ \frac{1}{\pi^2}K_{is}(e^x)K_{is}(e^y)\sinh(s\pi)d\lambda \qquad s^2 = \lambda,$$

where $i = \sqrt{-1}$ and K_μ are modified Bessel functions of the second kind:

$$K_{is}(z) = \int_0^\infty e^{-z\cosh t}\ \cos(st)\,dt$$

(cf. [15]). It is easy to obtain the corresponding integral representation of $p_*(t,x,y)$ from which we have the asmptotic formula:

$$p_*(t,x,x) = \frac{4\sqrt{2}}{\sqrt{\pi}\,b^2}\ \{K_0(\frac{2\sqrt{2M}}{b}\ e^{-bx/2})\}^2\ \frac{1}{\sqrt{t^3}}\ (1 + O(\frac{1}{\sqrt{t}})).\ ^{(*)}$$

(*) We need here only " $0 < C_1 \leqq \sqrt{t^3}p_*(t,x,x) \leqq C_2 < \infty\ \ (t\uparrow\infty)$". This is obtained under the assumption that $\int^{+\infty} q_*(x)xdx < \infty$, $q_* \geqq 0$, $q_* \not\equiv 0$ and q_* is locally bounded.

Therefore

$$v(t,-1) \geqq \text{const.} \frac{e^{c^*n(t)}}{\sqrt{t^3}} (1+o(1))$$

and

$$n(t) \leqq \frac{3}{2c^*} \log t + \text{const.}$$

This proves the first inequality in the theorem.

Step 3. We may assume without loss of generality that $F(u)/u < \alpha - \eta$ for $1/2 < u < 1$ with some positive constant η ($< \alpha$) (if this is not the case, consider $\sup\{x;\ u(t,x) = 1-\varepsilon\}$ instead of $m(t)$). Then, setting

$$q^*(x) = \eta \text{ if } x < 0 \text{ and } = 0 \text{ if } x > 0,$$

we have $k \leqq \alpha - q^*$ for all $t > 0$, $x \in R$, and

$$v(t,x) \leqq e^{\alpha t} E_x[e^{-\int_0^t q^*(B_s+m(t)-m(t-s))ds} f_o(B_t+m(t))].$$

Since for large t, by Step 2, $m(t) - m(t-s) - m(t)\frac{s}{t} = n(t-s) - n(t)(1-\frac{s}{t}) \leqq 4\sigma(t)$, we see , as before,

$$\frac{1}{2} = v(t,0) \leqq e^{\alpha t} E_0[\, e^{-\int_0^t q^*(B_s+m(t)\frac{s}{t}+4\sigma(t))ds} \mid B_t+m(t) = 0]$$

$$\times P_0[\, B_t+m(t) < 0]$$

$$= p^*(t,4\sigma(t),4\sigma(t))e^{c^*n(t)}(\frac{1}{c^*}+o(1)),$$

where $p^*(t,x,y)$ is the fundamental solution of

$$u^{\cdot} = \frac{1}{2}u'' - q^*u.$$

The Laplace transform of $p^*(t,x,x)$ is explicitely caluculated:

$$G(x,x) = \frac{1}{\sqrt{2}}\cdot\{\frac{1}{\sqrt{\lambda}}(1-e^{-2\sqrt{2\lambda}x}) - \frac{2}{\eta}(\sqrt{\lambda}-\sqrt{\lambda+\eta})e^{-2\sqrt{2\lambda}x}\} \qquad x > 0,$$

and inverting the transform we have, setting $\hat{\sigma} = 4\sigma$,

$$p^*(t,\hat{\sigma}(t),\hat{\sigma}(t))$$

$$= \frac{1}{\sqrt{2\pi t}}\{1-e^{-\frac{8\hat{\sigma}^2}{4t}}\} + \frac{1}{\eta\sqrt{2}}\{\frac{1}{2\sqrt{\pi t}\,t}(1-e^{-\eta t})\}*\{\frac{\sqrt{2}\,\hat{\sigma}}{\sqrt{\pi t}\,t}e^{-\frac{2\hat{\sigma}^2}{t}}\} \quad (*)$$

$$= \frac{1}{\sqrt{2\pi t}}\frac{2\hat{\sigma}^2}{t}(1+o(1)) \qquad \text{as} \quad t \to \infty.$$

Thus we have

$$\frac{1}{2} \leqq \frac{2\hat{\sigma}(t)^2}{\sqrt{2\pi}\sqrt{t^3}}e^{c^*n(t)}\left(\frac{1}{c^*}+o(1)\right)$$

or

$$n(t) \geqq \frac{3}{2c^*}\log t - \frac{2}{c^*}\log\log t + \text{const.}$$

and the second inequality of the theorem has been proved. The proof of Theorem 9.2 is completed.

Remark. If we put $w(t,x) = v(t,x) - w_{c_o}(x)$, $v(t,x) = u(t,x\ f)$, then $w^\cdot = \frac{1}{2}w'' + cw' + F'(\theta)w - (c^*-m^\cdot)v'$ and especially

$$c^* - m^\cdot(t) = \frac{(1/2)w''(t,0) + c^*w'(t,0)}{v'(t,0)}.$$

From this it is reasonable to expect that under the assumptions of Theorem 9.2 the order of the decay expressed in (8,3) is not more rapid than that of $\frac{1}{t}$.

It is interesting to compare the results as in Theorem 9.2

(*) $a(t)*b(t) = \int_0^t a(t-s)b(s)ds$.

with the result obtained in case $F(u) = u(u-a)(1-u)$ where $0 < a < 1$ (as typical example): Fife and McLeod [3a] says that with such F there exists the unique speed c for which the differential equation (4) has a global solution w such that $0 \leqq w \leqq 1$, $w(-\infty) = 1$ and $w(\infty) = 0$ and that for any continuous f with $\underline{\lim}_{x\to-\infty} f(x) > a$ and $\overline{\lim}_{x\to\infty} f(x) < a$ it holds that

$$(*) \qquad |u(t,x+ct) - w(x+x_o)| \to 0 \quad \text{as} \quad t \to \infty$$

where u is the solution of (1) and (2) with present F and x_o is some constant.

Next three theorems give an answer to the question of when $m(t) - ct$ is bounded and such formula as described by (*) holds.

Theorem 9.3. *Let $\alpha > 0$ and $c > c_o$. Let f be such a datum that there exists $\lim_{x\to\infty} e^{bx} f(x) = a \leqq \infty$ where $b = c - \sqrt{c^2 - 2\alpha}$. Suppose $\int_{0+} |\alpha - F'(u)|u^{-1} du < \infty$. Set $u = u(t,x;f)$. Then $m(t) - ct$ is bounded if and only if $0 < a < \infty$. If this is the case and if Condition [W] is satisfied, then*

$$(9,6) \qquad u(t,x+ct) \to w_c(x+x_o) \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad x > N,$$

where $x_o = b^{-1}\log(a_o/a)$, $a_o = \lim_{x\to\infty} w_c(x)e^{bx}$, *for each* $N > -\infty$.

Proof. It suffices to prove (9,6) assuming that $0 < a < \infty$. Set $w_* = \underline{\lim}_{t\to\infty} u(t,\cdot+ct)$ and $w^* = \overline{\lim}_{t\to\infty} u(t,\cdot+ct)$. We will prove that if $0 < a < \infty$

$$(9,7) \qquad w^*(x) \sim w_*(x) \sim f(x) \qquad \text{as} \quad x \to \infty.$$

Note that these are immediate consequences of Lemma 4.5 if $c^2/2 \geqq \gamma^*$. First we prove $w_*(x) \geqq f(x)(1+o(1))$. For this purpose, as in the proof of Theorem 5.1 (ii), take a function $\hat{F}$ satisfying (3) such that $\int_{0+} |\hat{F}(u) - \alpha u| u^{-2} du < \infty$, $\hat{F}'(0) = \alpha$, $\hat{F}' \leqq \alpha$ and $\hat{F} \leqq F$. Then $\hat{u}(t,x) \equiv u(t,x;f;\hat{F}) \leqq u(t,x)$ and, since $c^2/2 > \alpha = \sup \hat{F}'$, $\lim \hat{u}(t,x) = \hat{w}_c(x+\hat{x}_o)$ where $\hat{w}_c$ is a c-front corresponding to $\hat{F}$ and $\hat{x}_o$ is determined by $\hat{w}_c(x+\hat{x}_o) \sim f(x)$ as $x \to \infty$. Thus $w_*(x) \geqq f(x)(1+o(1))$. Next we prove $w^*(x) \leqq w_c(x+x_o)$. By Corollary of Lemma 2.4, for any $\delta > 0$ we can find a continuous datum g such that with some constant L

$$\frac{1}{2}g'' + cg' + F(g) = 0 \quad \text{for} \quad x > L \quad \text{and} \quad g = 1 \quad \text{for} \quad x < L$$

and such that $g(x) \sim f(x)$ as $x \to \infty$ and $f(x) \leqq g(x-\delta)$ $x \in R$. Let $\hat{u} = u(t,x;g)$. Then by Lemma 5.1 $\lim \hat{u}(t,x+ct) = w_c(x+x_o)$. Since $u(t,x) \leqq \hat{u}(t,x-\delta)$, we have $w^*(x) \leqq w_c(x+x_o-\delta)$. Hence $w^* \leqq w_c(x+x_o)$. Since $\lim w_c(x+x_o)e^{bx} = a_o e^{-bx_o}$, $w^* \leqq f\cdot(1+o(1))$. Consequently (9,7) has been proved.

If Condition [W] is satisfied, we have $u(t,x+m(t)) \to w_c(x)$, and hence (9,7) implies (9.6). The proof of Theorem 9.3 is completed.

Similarly we obtain

Theorem 9.4. *Let* $c_o = c^*$. *Assume* (2,4) *and that*

$$\int_{0+} |F'(u) - \alpha| \, |\log u| \, u^{-1} du < \infty \quad \text{or} \quad F(u) - \alpha u = o(u^p) \quad p > 1.$$

Suppose $\lim f(x)x^{-1}e^{c^*x} = a$ *exists and is positive and finite. Then under Condition* [W]

$$(9,8) \qquad u(t,x+c_ot;f) \to w_{c_o}(x+x_o)$$

where $x_o = \log(a_o/a)/c^*$, $a_o = \lim w_{c_o}(x)x^{-1}e^{c^*x}$.

In case $c = c_o > c^*$ we have

Theorem 9.5. Assume the hypotheses of Lemma 5.3 (i). Then $m(t) - c_o t$ is bounded. If we assume in addition the hypotheses of Theorem 8.1, then (9,8) holds where x_o is some constant.

Proof. The boundedness of $m(t) - c_o t$ is clear by Lemma 5.3 (i). Let f belong to the class $[F_\lambda]$ with $\lambda > c_o - \sqrt{c_o^2 - 2\alpha}$. Then if $c_o - \sqrt{c_o^2 - 2\alpha} < b < \min\{\lambda, c_o + \sqrt{c_o^2 - 2\alpha}\}$ we have

$$\sup_{t>0} |u(t,x+m(t);f) + w_{c_o}(x)| e^{bx} \to 0 \quad \text{as} \quad x \to \infty$$

and hence, by Theorem 8.1,

$$\sup_{x \in R} |u(t,x+m(t);f) - w_{c_o}(x)| e^{bx} \to 0 \quad \text{as} \quad t \to \infty.$$

which combined with Lemma 5.3 (ii) deduces (9,8). When f belongs to the class $[E_\mu]$, we can similarly proceed with any $c_o - \sqrt{c_o^2 - 2\alpha} < b < c_o + \sqrt{c_o^2 - 2\alpha}$. q.e.d.

10. Supplement to The Case $c_o > c^*$.

Here is given an alternative proof of Theorem 8.1 in the case $c_o > c^*$, which provides a better consequence. The proof is a modification of a proof given in P.C. Fife and J.B.McLeod [3.b] to the assertion cited in § 9 and simpler than that was given through § 6 to § 8.

Theorem 10.1. Assume $c_o > c^*$. Let f be a datum such that $f(x) = O(e^{-bx})$ for a constant $b > c_o - \sqrt{c_o^2 - 2\alpha}$. It is assumed in addition that Condition [G] in § 5 is satisfied (this is automatic when $\alpha > 0$). Then (9.8) holds.

Proof. Set $u = u(t,x:f)$ and $z(t,x) = u(t,x+c_o t)$. Obserbing that $v = u$ or $v = u'$ satisfies that, with $k = F(u)/u$ or $k = F'(u)$, respectively,

$$v(t+1,x+y) = \int_0^1 p'(1-s,y)v(t+s,x)ds + \int_0^\infty p^*(1,y,r)v(t,x+r)dr$$
$$+ \int_0^1 ds\int_0^\infty p^*(1-s,y,r)k(t+s,x+r)v(t+s,x+r)dr$$

for $x \in R$, $y > 0$ and $t > 0$, where p^* is defined just after the equation appeared in the last paragraph of § 5, differentiating the both sides of this equation with respect to y, and then putting $y = 1$, we deduce the estimates: for $t > 0$, $x \in R$

$$|u'(t+1,x+1)| \leqq K|u|_{t,x}^{t+1}, \quad |u''(t+1,x+1)| \leqq K|u'|_{t,x}^{t+1}$$

where $|v|_{t,x}^{t+1} = \sup_{t<s<t+1,y>x} |v(s,y)|$ and K is a constant independent of t and x. Then, by the equality $u^{\cdot} = 2^{-1}u'' + F(u)$ and by Lemma 5.3 (i), we see that for $t > 1$, $x \in R$

$$(10,1) \qquad z(t,x),\ |z'(t,x)|,\ |z^{\cdot}(t,x)| < K_1\min\{e^{-b_1x}+e^{-\eta t}e^{-bx},1\}$$

where $b_1 = c_o + 2^{-1}\sqrt{c_o^2-2\alpha}$ and K_1 is a constant independent of t and x. Let ε be a positive constant so small that $(c_o-b)\varepsilon < \eta$, and set

$$E(t) = \int_{-\varepsilon t}^{\varepsilon t} e^{2c_ox}[\ \tfrac{1}{4}\ z'(t,x)^2 - \int_0^{z(t,x)}F(r)dr]dx.$$

Then, by (10,1), $E(t)$ is bounded as t tends to infinity and

$$(1o,2) \qquad E^{\cdot}(t) = o(1) - \int_{-\varepsilon t}^{\varepsilon t} e^{2c_ox}[\ \tfrac{1}{2}z'' + c_oz' + F(z)]^2dx.$$

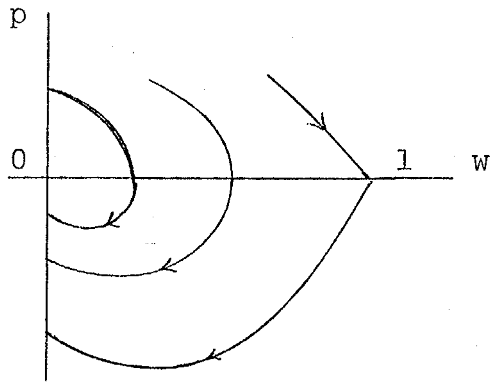
From these it follows that there exists an unbounded sequence $\{t_n\}$ along which $E(t_n) \to 0$. Since z, z', z'' and z''' are bounded for $x \in R$, $t > 1$ (see the remark following (4,6) and (4,7)), we can find a subsequence $\{t_{n'}\} \subset \{t_n\}$ such that $z(t_{n'},x)$ converges in the norm of $C^2(-N,N)$ for each $N > 0$. Let $w(x) = \lim z(t,x)$. Then,by (1o,2) and $\lim E(t_{n'}) = 0$, $2^{-1}w''+c_ow'+F(w) =0$ and, by Lemma 5.3 (i), w dose not degenerate. Therefore

w is a c_o-front. Since any c_o-front is stable in the sense of Lemma 5.3 (ii), we have actually $\lim z(t,x) = w(x)$ (see the proof of Theorem 9.5). Thus the theorem is proved.

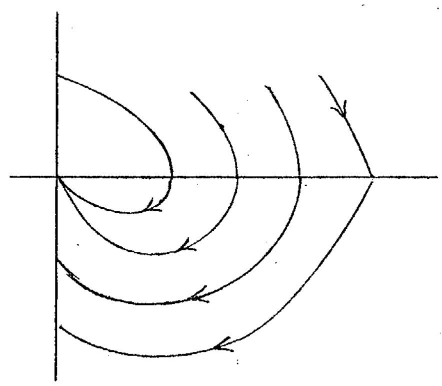
Appendix.

Following diagrams illustlate solutions of the equation (1,2).

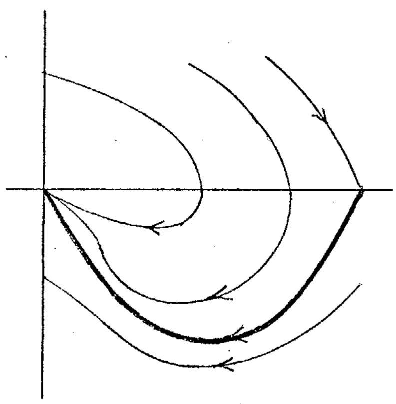
(a) $0 \leqq c < \sqrt{2\alpha}$



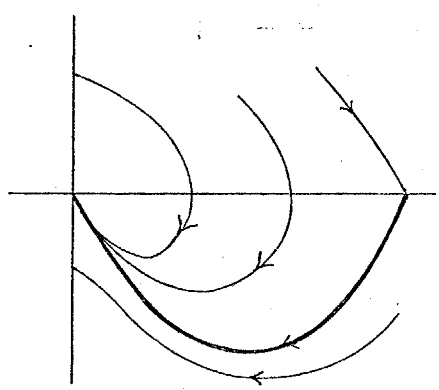
(b) $\sqrt{2\alpha} \leqq c < c_o$, $c > 0$



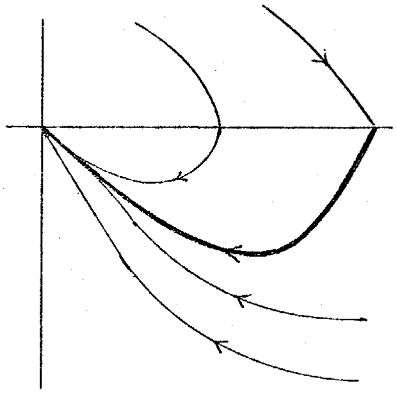
(c) $c = c_o > \sqrt{2\alpha}$



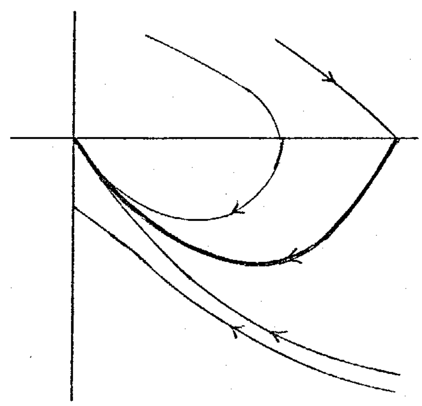
(d) $c = c_o = \sqrt{2\alpha}$ (case 1)



(e) $c > c_o$



(f) $c = c_o = \sqrt{2\alpha}$ (case 2)



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A REMARK ON THE NON-LINEAR DIRICHLET PROBLEM OF BRANCHING MARKOV PROCESSES

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1. Let A be the Dynkin's characteristic operator for a continuous strong Feller process (X_t, P_x) on a locally compact Hausdorff space with a countable open base. We will call the process a base process. Let D be an open subset of the state space and ψ be a bounded measurable function on the boundary ∂D of D. We assume $\|\psi\| \leqq 1$. Then the non-linear Dirichlet problem of a branching Markov process is

$$(1) \quad \begin{aligned} &Au(x)+c(x)\left(\sum_{n=0}^{\infty} q_n(x)u(x)^n - u(x)\right) = 0 \quad \text{in } D, \\ &u(b) = \psi(b) \quad \text{on } \partial D,^{(*)} \end{aligned}$$

where c is a nonnegative bounded continuous function on D ($c=0$ on D^c) and q_n is a bounded continuous function on D ($q_n=0$ on D^c) satisfying

$$\sum_{n=0}^{\infty} |q_n(x)| = 1, \quad \text{for } x \in D.$$

For simplicity, we assume $q_n(x) \geqq 0$ in the following. Let $\bar{X}_t$ be the stopped process of the base process at the boundary of D. Let $(\mathbf{X}_t, \mathbf{P}_x)$ be the branching Markov process on

$$\mathbf{S} = \bigcup_{n=0}^{\infty} \bar{D}^n \cup \{\Delta\},^{(**)}$$

determined by $\bar{X}_t$, c and q_n (cf. [1]).

For a bounded measurable function f on $\bar{D}$, we define $\hat{f}$ on $\mathbf{S}$ by

$$(2) \quad \begin{aligned} &\hat{f}(\mathbf{x}) = f(x_1)\cdots f(x_n), \quad \text{when } \mathbf{x} = (x_1,\cdots,x_n), \\ &\hat{f}(\delta) = 1, \\ &\hat{f}(\Delta) = 0. \end{aligned}$$

Then if $\|f\| \leqq 1$, $\hat{f}$ is bounded on $\mathbf{S}$.

Let τ be the killing time of the base process by means of the multiplicative functional $\exp(-\int_0^t c(X_s)ds)$ and T be the first hitting time of X_t to the boundary ∂D, and we assume

$$(3) \quad P_x[T < \tau] \geqq \varepsilon > 0, \quad \text{for all } x \in \bar{D}.$$

(*) The equality should be understood in a suitable sense.

(**) $\bar{D}^n$ is the n-fold cartesian product of $\bar{D}$ and $\bar{D}^0 = \{\delta\}$, where δ is an extra point. Δ is another extra point.

THEOREM. For a boundary function ψ, set

$$(4) \qquad f = \begin{cases} \psi & \text{on } \partial D, \\ 0 & \text{in } D. \end{cases}$$

Then under the assumption (3), there exists

$$(5) \qquad u(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(X_t)], \quad x \in \bar{D},$$

which is a solution of the non-linear Dirichlet problem (1) satisfying

$$\lim_{\substack{x\in D \\ x\to b\in\partial D}} u(x) = \psi(b),$$

if b is a regular point(*) of the boundary ∂D and if ψ is continuous at b (cf.[2]).

Instead of (4), let us take

$$(6) \qquad f = \begin{cases} \psi & \text{on } \partial D, \\ g & \text{in } D, \end{cases}$$

as an initial data in (5), where g is a measurable function on D with $\|g\| \leqq 1$. If $\|g\| < 1$, the limit in (5) exists and does not depend on the choice of the initial value g on D. In precise, let n_t^D be the number of particles in D at t, then

$$(7) \qquad u(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(X_t);\ X_s^i \in \partial D \text{ for all } i, \text{ or } X_s = \delta \text{ at some } s<\infty]$$
$$+ \lim_{t\to\infty} \mathbf{E}_x[\hat{f}(X_t);\ n_s^D \uparrow \infty \text{ when } s\uparrow\infty].$$

Therefore, the second term vanishes when $\|g\| < 1$, and the first term does not depend on g. However, if

$$\mathbf{P}_x[n_t^D \uparrow \infty \text{ when } t\uparrow\infty] > 0,$$

at some point $x_0 \in D$, then the second term does not vanish, if we take e.g., $g \equiv 1$.

Take $\psi \equiv 1$, for simplicity. If we take $f_1 \equiv 1$ on $\bar{D}$, then

$$u_1(x) = \lim_{t\to\infty} \mathbf{E}_x[\hat{f}_1(X_t)] = 1, \quad \text{for all } x \in \bar{D},^{(**)}$$

while if we take $f_0 = 1$ on ∂D (= 0 in D),

$$u_0(x_0) = \lim_{t\to\infty} \mathbf{E}_{x_0}[\hat{f}_0(X_t)] < 1.$$

u_0 is known to be the extinction probability of the branching Markov process with absorbing boundary (cf.[3],[4]).

(*) The regularity is for the base process.

(**) We assume the branching Markov process does not explode.

2. The solution of the non-linear Dirichlet problem is not unique, as is seen in §1, but how many solution do we have? To discuss the problem, let us take the simplest case of one dimensional branching Brownian motion, $\bar{D} = [-\ell, \ell]$, and $\psi \equiv 1$. The problem in this case is

$$(8) \qquad \begin{aligned} &u''(x) + c(h(u) - u)(x) = 0, \quad x \in (-\ell,\ell), \quad c;\text{constant} > 0, \\ &u(-\ell) = u(\ell) = 1, \end{aligned}$$

where

$$h(u) = \sum_{n=0}^{\infty} q_n u^n, \quad q_n \geqq 0, \quad \sum_{n=0}^{\infty} q_n = 1.$$

We assume

$$1 < h'(1)$$

and $h(u)$ is analytic on R^1.

We will prove, for example, when $h(u) = u^2$, there is a critical length $\ell_0 = \pi/2\sqrt{c}$, and <u>if</u> $\ell > \ell_0$, <u>the number of solution of</u> (8) <u>is</u> "<u>approximately</u>" (cf. (14) and (15) in precise)

$$n \sim 2[\ell/\ell_0],^{(*)} \qquad \ell \gg \ell_0,$$

where $[a]$ denotes the greatest integer <u>strictly</u> less than a.

To solve (8) put

$$(9) \qquad C(u) = 2c(u - h(u)).$$

Then we can write the equation (8) as

$$(10) \qquad 2u'' = C(u).$$

Introducing

$$f(u) = \int_0^u C(r)dr$$

and taking the value b of $u(x)$ at $u'(x) = 0$ as a parameter, we have

$$(11) \qquad (u')^2 = f(u) - f(b).$$

Therefore the formal solution for (8) is

$$(12) \qquad x = \int^u \frac{du}{\sqrt{f(u) - f(b)}}.$$

Put

$$(13) \qquad F(b) = \int_0^1 \frac{du}{\sqrt{B}}, \qquad B = \frac{f(1+(b-1)u) - f(b)}{(b-1)^2}.$$

<u>LEMMA</u> 1. (1) $F'(b) < 0$, (ii) $F(1) = \frac{1}{\sqrt{c(h'(1)-1)}} \frac{\pi}{2}$, <u>and</u> (iii) $\lim_{\substack{b \to q \\ b > q}} F(b) = \infty$, <u>where</u> $0 \leqq q < 1$ <u>is the root of</u>

$$h(u) - u = 0.^{(+)}$$

(*) We count the trivial solution $u \equiv 1$.

(+) ~~b is necessarily ...~~

PROOF. Because

$$B(u,b) = -\int_u^1 f'(1+(b-1)r)(b-1)^{-1}dr = \int_u^1 C(1+(b-1)r)(b-1)^{-1}dr$$

$$= -\int_u^1 dr \int_0^r C'(1+(b-1)s)ds,$$

we have

$$\frac{\partial B}{\partial b} = -\int_u^1 dr \int_0^r C''(1+(b-1)s)\cdot s\,ds.$$

Since

$$F'(b) = -\frac{1}{2}\int_0^1 \frac{B'}{B^{3/2}}\,du, \text{ and } \quad C''(u) = -2ch''(u) < 0,$$

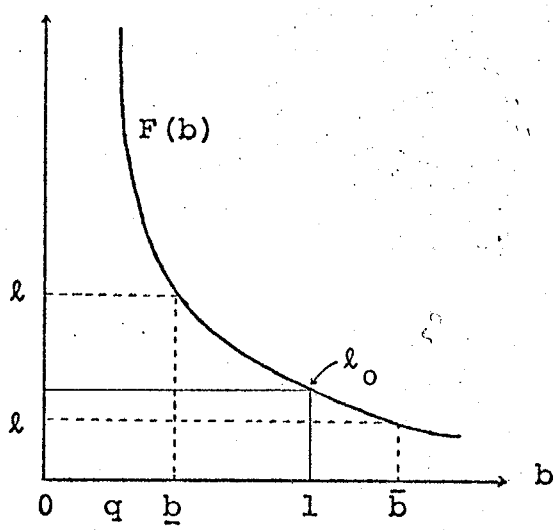
we have $B' > 0$ and hence $F'(b) < 0$, proving (i).

Becuase $B(u,1) = c(h'(1)-1)(1-u^2)$, we have

$$F(1) = \frac{1}{\sqrt{c(h'(1)-1)}}\int_0^1 \frac{du}{\sqrt{1-u^2}},$$

proving (ii). Because $C(u) \sim u-q$, $f(u)-f(q) \sim (u-q)^2$. Therefore, $1/\sqrt{B(u,q)} \sim 1/(1-u)$, implying (iii) by monotone convergence theorem.

Fig.1.



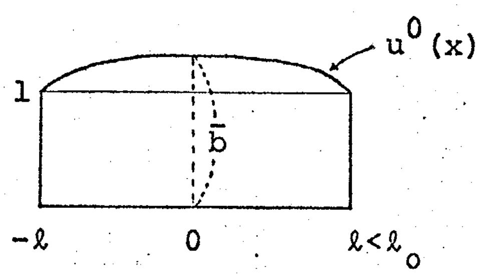
Put

$$\ell_o = \pi/2\sqrt{c(h'(1)-1)},$$

then this is the critical length of the domain as will be seen in the following.

Case 1. When $\ell < \ell_o$, we have the solution $u^0 \geqq 1$ as in Fig.2.

Fig.2.



Because

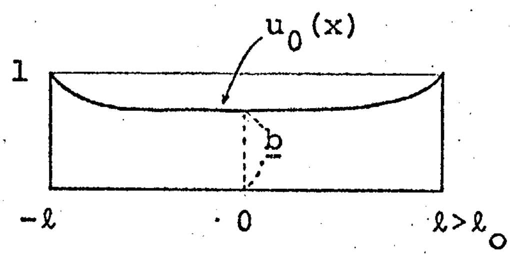
$$\int_1^{\bar{b}} \frac{du}{\sqrt{f(u)-f(\bar{b})}} = F(\bar{b}),$$

to have the solution u^0, $\ell = F(\bar{b})$. It is clear that such $\bar{b} > 1$ exists when and only when $\ell < \ell_o$ (cf.Fig.1).

Case 2. When $\ell = \ell_o$, the solution of (8) is unique, i.e., we have just the trivial solution $u \equiv 1$.

Case 3. When $\ell > \ell_o$, we have the solution u_0 as in Fig.3.

Fig.3.



Because

$$\int_{\underline{b}}^{1} \frac{du}{\sqrt{f(u)-f(\underline{b})}} = F(\underline{b}),$$

$\underline{b}$ is determined by $\ell = F(\underline{b})$. It is clear that such $q < \underline{b} < 1$ exists when and only when $\ell > \ell_o$ (cf.Fig.1).

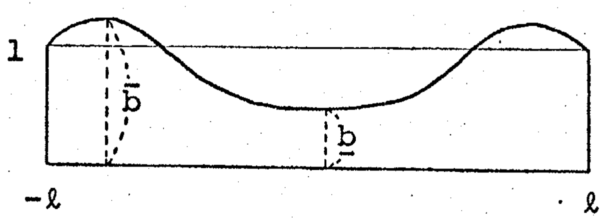
Let $u_0(x) = \underline{b}+\varepsilon$, then

$$|x| = \int_{\underline{b}}^{\underline{b}+\varepsilon} \frac{du}{\sqrt{f(u)-f(\underline{b})}} \longrightarrow \infty, \quad \text{if } \underline{b} \longrightarrow q,$$

where ε is arbitrary and hence $u_0(x) \longrightarrow q$, $x \in R^1$. This is understandable because the efect of absorbing boundary decreases when $\ell \to \infty$ and the extinction probability u_0 converges to that of the branching Brownian motion without boundary.

Now, if the length ℓ of the domain is large enough, we have, for example, the solution of two nodes as in Fig.4.

Fig.4.



The sufficient condition for existence of such solution is

$$3F(1) < \ell.$$

In general, if

$$(n+1)F(1) < \ell,$$

there exist solutions with nodes up to n. However, it may happen that the solution of the same nodes is not unique for a given length of the domain. For example, take the solution of Fig.4, then

$$F(\underline{b}) + 2F(\bar{b}) = \ell.$$

Put

$$K_3(\underline{b}) = F(\underline{b}) + 2F(\bar{b}).$$

Since $f(\bar{b}) = f(\underline{b})$, $\bar{b}$ is a function of $\underline{b}$, $q < \underline{b} \leq 1$, satisfying

<u>LEMMA</u> 2.

(i) $\quad 0 \geq \dfrac{d\bar{b}}{d\underline{b}} \geq -1$, and $\dfrac{d\bar{b}}{d\underline{b}} = -1$ at $\underline{b} = 1$,

(ii) $\quad \bar{b} \to 1$, when $\underline{b} \to 1$.

<u>PROOF</u>. It is clear that $\bar{b} \to 1$, when $\underline{b} \to 1$. By differentiating $f(\underline{b}) = f(\bar{b})$ with respect to $\underline{b}$, $\bar{b}' = d\bar{b}/d\underline{b} = f'(\underline{b})/f'(\bar{b})$. We have $-f'(\underline{b})/(\bar{b}-\underline{b}) = (f(\bar{b})-f(\underline{b})-f'(\underline{b})(\bar{b}-\underline{b}))/(\bar{b}-\underline{b})^2 \sim (1/2)f''(1)$, and similarly $f'(\bar{b})/(\bar{b}-\underline{b}) \sim (1/2)f''(1)$. Since $f''(1) \neq 0$, we have $\bar{b}' \to -1$, when $\underline{b} \to 1$. To show $\bar{b}' \geq -1$, assume the contrary; $\inf \bar{b}' < -1$. Since $\bar{b}'|_{\underline{b}=q} = 0$, there is a point b_o, $q < b_o < 1$, and at which $\bar{b}'$ attains the minimum at b_o. Since $\bar{b}'' = d^2\bar{b}/d\underline{b}^2 = (f''(\underline{b})-f''(\bar{b})(\bar{b}')^2)/f'(\bar{b})$ and f'' is monotone, it follows that $\bar{b}'' < 0$ at b_o, but this contradicts our setting of b_o.

By Lemma 2 and by the equation

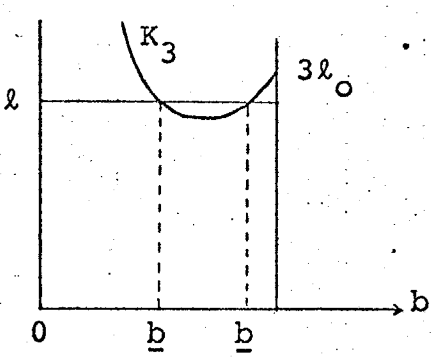
$$K_3'(\underline{b}) = F'(\underline{b}) + 2F'(\underline{b})\frac{d\bar{b}}{d\underline{b}},$$

we have

$$K_3'(1) = -F'(1) > 0.$$

Clearly $\lim_{b\to q} K_3(b) = \infty$, on the other hand. This means if we take ℓ like in Fig.5, we can choose two different values of $\underline{b}$.

Fig.5.



Put $\quad F_1 = F$, $\; F_{2n}(\underline{b}) = nF(\underline{b})+nF(\bar{b})$, $\; F_{2n+1}(\underline{b}) = (n+1)F(\underline{b})+nF(\bar{b})$,

$\quad K_{2n}(\underline{b}) = F_{2n}(\underline{b})$, $\; K_{2n+1}(\underline{b}) = nF(\underline{b})+(n+1)F(\bar{b})$.

Then, for a given length ℓ of the domain, the number of solutions is the number of crossing of F_n and K_n with ℓ as in Fig.6.(*)

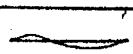
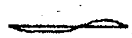
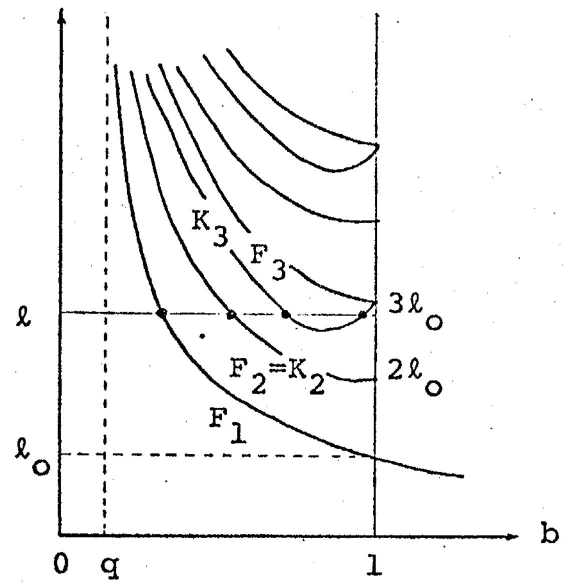
(*) We distinguish  and .

Fig.6.



<u>PROPOSITION</u>. <u>The number of solutions is bounded from below as</u>

(14) $$2[\ell/\ell_o] \leqq n, \qquad \ell > \ell_o = \pi/2\sqrt{c(h'(1)-1)}.$$

<u>In the special case of</u> $h(u) = u^2$,

(15) $$2[\ell/\ell_o] \leqq n \leqq 2[\ell/\ell_o]+2, \qquad \ell > \ell_o = \pi/2\sqrt{c},$$

where [a] denotes the greatest integer <u>strictly</u> less than a.

We have the upper bound of (15) because F_n and K_n are convex in this case.

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ON LIMIT THEOREMS FOR NON-CRITICAL GALTON-WATSON PROCESSES WITH $EZ_1 \log Z_1 = \infty$

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1. Let $Z_0 = 1, Z_1, Z_2, \cdots$ denote a non-critical Galton-Watson processes with a non-degenerate offspring distribution $p_j = P[Z_1 = j]$, $j = 0,1,2,\cdots$, having a finite mean $m = EZ_1 < \infty$. Write $f_0(s) = s$, $f_1(s) = f(s) = \sum_{j=0}^{\infty} p_j s^j$, $f_2(s) = f(f_1(s))$, $f_3 = f(f_2(s)),\cdots$, $0 \leqq s \leqq 1$. When $\sum_{j=0}^{\infty} p_j(j\log j) < \infty$ it is known (cf. Athreya and Ney [1]) that in case $m > 1$ Z_n grows up to infinity as m^n a.s. on $\{Z_n \to \infty$ as $n \to \infty\}$, and in case $m < 1$ $1 - f_n(0) = P[Z_n \neq 0]$ decays to zero as m^n. Stating more minutely;

(i) when $m < 1$, $\lim_{n\to\infty}(1 - f_n(0))^{-1}P[Z_n = j]$ exist for $j = 1,2,3,\cdots$ and define a probability distribution on $\{1,2,3,\cdots\}$; and furthermore $m^{-n}(1-f_n(0))$ converges to a positive constant if and only if $\sum p_j(j\log j) < \infty$; and

(ii) when $m > 1$, for some sequence of positive constants c_n, $c_n^{-1}Z_n$ converges almost surely to a proper non-degenerate random variable, and furthermore $m^{-n}c_n$ converges to a positive number if and only if $\sum p_j(j\log j) < \infty$.

If $\sum p_j(j\log j) = \infty$, we have $\lim(1 - f_n(0))m^{-n} = 0$ in case $m < 1$ and $\lim c_n m^{-n} = 0$ in case $m > 1$. In this note we will investigate the rate of decay for $1 - f_n(0)$ or of growing up for Z_n in certain cases with $\sum p_j(j\log j) = \infty$.

2. Let us introduce a condition:

(1) $$\sum_{j=n}^{\infty} jp_j \sim (\log n)^{-\alpha} L(\log n) \qquad \text{as } n \to \infty,^{(*)}$$

where $L(x)$ is a function slowly varying at infinity and α a real constant.

<u>THEOREM</u>. <u>Let</u> $L(x)$ <u>be as above and</u> $0 \leqq \alpha < 1$.

(a) <u>If</u> $m < 1$, <u>then the condition</u> (1) <u>is equivalent to</u>

(2) $$\log(m^{-n}(1 - f_n(s))) \sim -An^{1-\alpha}L(n) \qquad \text{as } n \to \infty$$

<u>for</u> $0 \leqq s < 1$, <u>where</u> $A = |\log m|^{-\alpha}/m(1-\alpha)$.

(b) <u>If</u> $m > 1$, <u>then the condition</u> (1) <u>is equivalent to</u>

(3) $$\log \frac{Z_n}{m^n} \sim -An^{1-\alpha}L(n) \qquad \text{as } n \to \infty$$

<u>a.s. on</u> $\{Z_n \to \infty$ as $n \to \infty\}$,

(*) "$a(x) \sim b(x)$ as $x \to c$" means that $\lim_{x\to c} a(x)/b(x) = 1$.

where A is defined as in (a).

REMARK 1. Let (1) hold with $\alpha \geq 0$. Then $\sum p_j(j\log j) = \infty$ if and only if either $0 \leq \alpha < 1$, or $\alpha = 1$ and $\int_1^\infty L(y)y^{-1}dy = \infty$. This follows from LEMMA 3 and 4 which are stated in the following section. When $\alpha = 1$, the assertions of THEOREM are also true if we replace $-An^{1-\alpha}L(n)$ by $-\int_1^n L(y)y^{-1}dy/(m\log m)$ in (2) and (3).

REMARK 2. Suppose $\alpha > 0$. Then (1) follows from

$$\sum_{j=1}^{n} j^2 p_j \sim n(\log n)^{-\alpha-1}L(\log n) \quad \text{as } n \to \infty,$$

which is in turn implied by

$$\sum_{j=n}^{\infty} p_j \sim n^{-1}(\log n)^{-\alpha-1}L(\log n) \quad \text{as } n \to \infty.$$

These conditions are satisfied with $p_n \sim n^{-2}(\log n)^{-\alpha-1}L(\log n)$.

3. Before going into the proof of THEOREM we state several preliminary lemmas.

LEMMA 1. *If* $m < 1$, *then*

$$G(s) = \lim_{n\to\infty} \frac{1 - f_n(s)}{1 - f_n(0)} \quad \textit{exists for } 0 \leq s < 1,$$

and

$$-G'(s) = \lim_{n\to\infty} \frac{f_n'(s)}{1 - f_n(0)} > 0 \quad \textit{for } 0 < s < 1.$$

PROOF. It is known that $G(s)$ is well defined in the above formula because of the increasingness and the boundedness of the defining sequence of functions (c.f.[1]). Since $1 - f_n(s)$ is analytic we get the latter formula except $G' < 0$. By the definition of G we find the functional equation $G(f(s)) = mG(s)$, which shows $G(s)$ is not constant because $m < 1$. Therefore $-G'(s) > 0$ in the interval $(0,1)$, because $-G'(s)$ is non-negative, analytic and non-decreasing.

LEMMA 2. $\sum p_j(j\log j) = \infty$ *if and only if* $\int_0^\infty (m - f'(1-r^t))dt = \infty$ *for some* $0 < r < 1$.

PROOF. Since $\int_0^\infty (m - f'(1-r^t))dt = \sum_{k=1}^\infty kp_k \int_0^\infty (1 - (1-r^t)^{k-1})dt$, it suffices to see that $\int_0^\infty (1 - (1-r^t)^{k-1})dt \sim \text{const.}\log k$ as $k \to \infty$. But $(-\log r) \int_0^\infty (1-(1-r^t)^k)dt = \int_0^1 (1-(1-v)^k)v^{-1}dv = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \sim$
$\sim \log k$ as $k \to \infty$.

The next lemma is part of the standard textbook literature (see W.Feller[2] $_p422, _p237$).

LEMMA 3. *Suppose that* $L(x)$ *is slowly varying at infinity and* $u(x)$ *a non-negative function on* $[0\ \infty)$. *Put*

$$U(t) = \int_0^t u(x)dx \quad \textit{and} \quad U^*(t) = \int_t^\infty u(x)dx.$$

Then the condition that $u(x) \sim t^d L(t)$ as $t \to \infty$ implies

$$U(t) \sim (d+1)^{-1} t^{d+1} L(t) \quad \text{as} \quad t \to \infty \quad \text{when} \quad d+1 > 0,$$

and

$$U^*(t) \sim |d+1|^{-1} t^{d+1} L(t) \quad \text{as} \quad t \to \infty \quad \text{when} \quad d+1 < 0.$$

Moreover if $u(x)$ is monotonic, the converse assertions also hold.

LEMMA 4. (1) is equivalent to

$$(4) \qquad m - f'(1-v) \sim (-\log v)^{-\alpha} L(-\log v) \qquad \text{as} \quad v \downarrow 0$$

where α may be an arbitrary real number.

PROOF. Observe the equation

$$m - f(s) = \sum_{j=2}^{\infty} j p_j (1 - s^j) = (1-s) \sum_{k=0}^{\infty} \Big(\sum_{j=k+2}^{\infty} j p_j \Big) s^k$$

and then apply a Tauberian theorem to coefficients $a_k = \sum_{j=k+2}^{\infty} j p_j$.

4. We now prove THEOREM. Let $m < 1$. Suppose $m^{-n}(1 - f_n(0)) \to 0$ as $n \to \infty$. Then we can see, by LEMMA 1,2 and the equation

$$\frac{f_n'(s)}{m^n} = \prod_{k=0}^{n-1} \frac{f'(f_k(s))}{m},$$

that as $n \to \infty$

$$(5) \qquad -\log \frac{1 - f_n(s)}{m^n} \sim -\sum_{k=0}^{n-1} \log\Big(1 - \Big(1 - \frac{f'(f_k(s))}{m}\Big)\Big) \sim \sum_{k=0}^{n-1} \Big(1 - \frac{f'(f_k(s))}{m}\Big) \qquad 0 \leq s < 1.$$

Taking any tow numbers r and s, $0 < r < m$, $0 \leq s < 1$, fixed,

$$1 - m^k \leq f_k(s) \leq 1 - r^k \qquad k > k_o$$

with some integer k_o, and hence

$$(6) \qquad \int_0^n \Big(1 - \frac{f'(1 - m^t)}{m}\Big) dt \geq \sum_{k=k_o}^{n} \Big(1 - \frac{f'(f_k(s))}{m}\Big) \geq \int_{k_o}^{n} \Big(1 - \frac{f'(1-r^t)}{m}\Big) dt.$$

By LEMMA 2 we get $\sum p_j (j \log j) = \infty$. Conversely, from the above argument, we see that $\sum p_j (j \log j) = \infty$ implies $\lim_{n\to\infty} (1 - f_n(s)) m^{-n} = 0$. Therefore , for the proof of THEOREM these two conditions may be always assumed.

From (6) it follows that

$$(7) \qquad \sum_{k=0}^{n} \Big(1 - \frac{f'(f_k(s))}{m}\Big) \sim \int_0^n \Big(1 - \frac{f'(1 - m^t)}{m}\Big) dt \qquad \text{as} \quad n \to \infty.$$

Indeed, setting $a_n(x) = \int_0^n \Big(1 - \frac{f'(1-x^t)}{m}\Big) dt = \int_{x^n}^{1} \Big(1 - \frac{f'(1-v)}{m}\Big) \frac{dv}{v \log v}$,

$0 < x < 1$, and $b_n(s) = \sum_{k=0}^{n} \Big(1 - \frac{f'(f_k(s))}{m}\Big)$, we get $a_n(r)/a_n(m) \geq$

$\geq \log m/\log r$ and hence, by (6), $1 \geq \overline{\lim}_{n\to\infty} b_n / a_n(m) \geq \underline{\lim}\, b_n / a_n(m) \geq \log m/\log r$, which leads to $\lim b_n / a_n(m) = 1$ since r can be taken arbitrarily near m.

Because $1 - f'(1-r^x)$ is a monotone function of x, the converse part of LEMMA 3 is applicable to it. Using LEMMA 4 it is seen that (1) is equivalent to

$$(8) \qquad \int_0^n \left(1 - \frac{f'(1-r^t)}{m}\right) dt \sim n^{1-\alpha} L(n) \frac{(-\log r)^{-\alpha}}{m(1-\alpha)} \qquad \text{as } n \to \infty.$$

By (5) and (7), we have that (2) is also equivalent to (8). Thus (a) has been proved.

The proof of (b) is performed along the same lines as in the above. Let $m > 1$. Denote by $g(x)$ the inverse function of $f(x)$ defined for $q < x < 1$ where q is the smallest non-negative root of $f(s) = s$, and by $g_n(x)$ its n-times iteration. Let s_o be any fixed number between q and 1. Then c_n in section 1 can be taken as $(-\log g_n(s_o))^{-1}$. Thus $c_n^{-1} \sim 1 - g_n(s_o)$, and (3) is equivalent to

$$m^n \log(1 - g_n(s_o)) \sim A n^{1-\alpha} L(n).$$

Corresponding to LEMMA 1 we have that for $q < s < 1$

$$\lim \frac{1 - g_n(s)}{1 - g_n(s_o)} = H(s) \text{ exists and } \lim \frac{g_n'(s)}{1 - g_n(s_o)} = -H'(s) > 0.$$

Observe that

$$g_n'(s) = 1 / \prod_{k=1}^{n} f'(g_k(s))$$

and that

$$r^k \geq 1 - g_k(s_o) \geq m^{-k} \quad \text{for } k > k_o$$

where r is an arbitrarily fixed number in $(m^{-1}, 1)$ and k_o some constant. Then imitate the proof of (a) to see that

$$\log[(1 - g_n(s_o))m^n] \sim \sum_{k=0}^{n-1} \left(1 - \frac{f'(g_n(s_o))}{m}\right)$$

$$\sim \int_0^n \left(1 - \frac{f'(1-m^t)}{m}\right) dt \qquad \text{as } n \to \infty.$$

It is clear that (b) follows from these formulas as in the case $m < 1$.

References.

[1] K.B.Athreya-P.E.Ney, Branching Processes. Springer, Berlin.(1973)

[2] W.Feller, An introduction to probability theory and its application. Vol.2, Wiley, New York.(1966)

非線形拡散方程式 $u^{\cdot} = \frac{1}{2}u'' + F(u)$ の解の travelling wave への収束

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1、単独の非線形方程式

$$(1) \qquad u_t = \frac{1}{2}u_{xx} + F(u) \qquad \left(u = u(t,x),\ t > 0,\ x \in R^1\right)$$

を考える。ここで F は区間 [0, 1] 上で定義された滑かな函数で $F(0) = F(1) = 0$ を満たすとする。初期条件

$$(2) \qquad u(0+, x) = f(x) \qquad 0 \leqq f \leqq 1$$

(常に $f \not\equiv 0$ とする) を満たす (1) の古典的解で $0 \leqq u \leqq 1$ となるものが唯一存在する (cf. [4])。F として次の二通りの場合を考える。

I. $F(u) > 0 \quad 0 < u < 1, \quad \alpha \equiv F'(0) > 0$

II. $F'(0) < 0,\ F'(1) < 0$ かつ次のような定数 μ がある:

$F(u) < 0 \quad 0 < u < \mu; \quad F(u) > 0 \quad \mu < u < 1, \quad 0 < \mu < 1$。

これらの条件の意味については [1] または [3] を参照されたい。なおこれらの文献では，u は拡散しながら増殖する生物群の個体群密度を表わすとされている。

2．本稿の目的は (1)−(2) の解の $t \to \infty$ とした時の行動を調べることにあるが，それは次の方程式

(3) $$0 = \frac{1}{2} w'' + c w' + F(w) \qquad \left(w = w(x),\ x \in R^1 \right)$$
c は実定数

の解と関連づけられる，但し w は常に F に適合するとする (i.e. $0 \leq w \leq 1$)。(3) は $w(x - ct)$ が (1) を満たすことと同じである。この形の (1) の解を *travelling wave (solution)*，w を *travelling wave front* と呼ぶ（但し自明解 $w \equiv 0$ 及び $w \equiv 1$ を除く）。$x \mapsto -x$ なる変換を考えることにより，$c \geq 0$ の場合だけ調べれば十分である。以下で問題になるのは (3) の

(4) $$\lim_{x \to -\infty} w(x) = 1, \quad \lim_{x \to +\infty} w(x) = 0$$

を満たす解である。$w(0) = \frac{1}{2}$ と規格化すればそのような解は高々一つである。

3．次の結果は Aronson-Weinberger [1] による。

(i) $F \in I$ とする。ある正の定数 c_0 があって，(3) の自明

でない解が存在するための必要十分条件が $|c| \geq c_0$ と書かれる。$c_0 \leq \sup_u F(u)/u$ である。対応する解 w を $w(0)=\frac{1}{2}$ と規格化して, 以後これを w_c と書く。$c \geqq c_0$ の時 w_c は (4) をみたし, $w_c' < 0$ かつ

$$(5) \qquad \lim_{x\to\infty} \frac{w_c'(x)}{w_c(x)} = -b, \quad b = \begin{cases} c + \sqrt{c^2-2\alpha} & c = c_0 \\ c - \sqrt{c^2-2\alpha} & c > c_0 \end{cases}$$

が成り立つ。

(ii) $F \in \mathrm{II}$ とする。ある定数 c_* があって (4) を満たす (3) の解は $c = c_*$ の時存在しかつその時に限る。また

$$(6) \qquad c_* > 0 \quad と \quad \int_0^1 F(u)\,du \quad は同値である。$$

対応する解を w_* と書く。$w_*' < 0$ である。

上の結果の証明は, (6) を除けば, (3) を

$$\begin{cases} w' = p \\ p' = -2c - 2F(w) \end{cases}$$

と書いて, これに二次元自励系の理論を適用すれば比較的容易になされる。(6) の証明も容易である。実際 (3) の右辺に w' をかけて x に関して積分すれば, $\int^x w''w'\,dx = \frac{1}{2}(w')^2$ より

$$c_0 \int_{-\infty}^{\infty} (w')^2\,dx = \int_0^1 F(w)\,dw$$

を得る。したがって (6) が成り立つ。

4、 $F \in I$ とする。$f \equiv 0$ でなければ (1)−(2) の解 u は

$$u(t,x) \longrightarrow 1 \qquad (t \to \infty, \text{広義一様})$$

を満たす (cf. [1])。また $f(x) \to 0$ $(x \to \infty)$ であれば各 $t \geq 0$ に対して $u(t,x) \to 0$ $(x \to \infty)$ である。これらが満たされる時，十分大きなすべての t に対して

$$(7) \qquad m(t) \equiv \sup\{x \,;\, u(t,x) = \tfrac{1}{2}\}$$

は有限値をとる。定理を述べる前に次の条件を用意する：

$$(8) \qquad f(x) = 0 \qquad x > N \qquad (N \text{ はある実数}),$$

(9) $A > 0$, $\displaystyle\lim_{x \to \infty} \frac{A(x+x_0)}{A(x)} = 1$ がすべての $x_0 \in R^1$ に対して成立する函数 A 及び実定数 $\lambda > 0$ をもって

$$f(x) = e^{-\lambda x} A(x)$$

と書ける。

定理 1、 f は上の (7) または (8) の他に次を満たすとする

$$(10) \qquad \lim_{x \to -\infty} f(x) = 0 \quad \text{または} \quad f(x) \text{ はある左半直線上で非増加。}$$

この時 (1)−(2) の解 u に対し，$x > 0$ に関して一様に

$$(11) \qquad |u(t,x) - w_c(x - m(t))| \longrightarrow 0 \qquad (t \to \infty)$$

が成り立つ，但し $m(t)$ は (7) で定義され c は

$$c = \begin{cases} c_0 & (8) \text{ または } (9) \text{ で } \lambda \geqq c_0 - \sqrt{c_0^2 - 2\alpha}, \\ \dfrac{\lambda}{2} + \dfrac{\alpha}{\lambda} & (9) \text{ で } \lambda < c_0 - \sqrt{c_0^2 - 2\alpha} \text{ の時.} \end{cases}$$

で与えられる。$m(t)$ は十分大きな t に対して微分可能で $dm(t)/dt \longrightarrow c \quad (t \to \infty)$ が成立する。もし $F(u)/u$ が非増加函数であれば上で条件(10)は除かれる。

証明は [5] を参照。

5、$F \in \mathrm{II}$ とする。次は Fife－McLeod [2] による。

<u>定理2</u>、(i) $\underline{\lim}_{x \to -\infty} f(x) > \mu$ かつ $\overline{\lim}_{x \to +\infty} f(x) < \mu$ であれば，ある $x_0 \in R^1$ があって $x \in R$ に関し一様に

$$\left| u(t,x) - w_*(x - c_* t + x_0) \right| \to 0 \qquad t \to \infty.$$

(ii) 任意の $\bar{\mu} > \mu$ に対しある $L > 0$ があって，

$$f(x) > \bar{\mu} \qquad |x| < L$$

かつ f の台が有界であれば，ある $x_1, x_2 \in R^1$ があって一様に

$$\left| u(t,x) - w_*(x - c_* t + x_1) - w_*(-x + c_* t + x_2) \right| \to 0 \quad (t \to \infty).$$

6、$F \in \mathrm{I}$ で特に $c_0 > \sqrt{2\alpha}$ の場合には定理1の比較的簡単な証明があり，結果も精密化される。

<u>定理3</u>、$F \in \mathrm{I}$ で $c_0 > \sqrt{2\alpha}$ とする。ある $b_1 > c_0 - \sqrt{c_0^2 - 2\alpha}$ に

対し　$f(x) = O(e^{-bx})$ であれば，ある $x_0 \in R^1$ があって $x > 0$ に関し一様に

$$|u(t,x) - w_{c_0}(x - c_0 t + x_0)| \longrightarrow 0 \qquad t \to \infty 。$$

証明の概略。2つの正定数 $\bar{b}$, γ を

$$c_0 - \sqrt{c_0^2 - 2\alpha} < \bar{b} < c_0 + \sqrt{c_0^2 - 2\alpha}, \quad \bar{b} \leq b,$$
$$\frac{\bar{b}^2}{2} - c_0 \bar{b} + \alpha < -\gamma$$

となるようにとっておく。十分大きな正定数Aに対し

$$\overline{U}^*(t,x) = w_{c_0}(x - A(1 - e^{-\gamma t})) + e^{-\gamma t - \bar{b}x}$$
$$\overline{U}_*(t,x) = w_{c_0}(x - Ae^{-\gamma t}) - e^{-\gamma t - \bar{b}x}$$

とおけば，次が成立：f がある定数 t_1, t_2, x_1, x_2 をもって

$$\overline{U}_*(t_1, x + x_1) \leq f(x) \leq \overline{U}^*(t_2, x + x_2) \qquad x \in R$$

を満たせば，(1)－(2) の解 u はすべての $x \in R$, $t > 0$ に対し

$$\text{(12)} \qquad \overline{U}_*(t + t_1, x + x_1) \leq u(t, x + c_0 t) \leq \overline{U}^*(t + t_2, x + x_2)$$

を満たす。これは次のような方針で証明される。F を R^1 上の滑らかな函数 $\bar{F}$ に拡張しておく，但し $\bar{F}' \leq \alpha$ とする。

$V(t,x) = u(t, x + c_0 t) - \overline{U}_*(t + t_1, x + x_1)$ とおくと

$$V_t = \frac{1}{2} V_{xx} + c_0 V_x + \bar{F}'(0) V + Q(t+t_1; x+x_1)$$
$$Q(t,x) \equiv \frac{1}{2} \bar{U}_{*xx} + c_0 \bar{U}_{*x} + \bar{F}(\bar{U}_*) - \bar{U}_{*t}$$

を得る。$V(0,x) \geq 0$ であるから $V \geq 0$ を得るには $Q \geq 0$ を云えばよい。そこでAをこれが成り立つように選んでおく（詳細は省略 c.f. [5]）。こうして (12) の最初の不等式が得られる。残りの不等式も同様に証明される。

$w_{c_0}(x)$ はほぼ $e^{-b_0 x}$ $(b_0 = c_0 - \sqrt{c_0^2 - 2\alpha})$ の速さで，$x \to \infty$ の時0に近づく。したがって各 $t > 0$ に対し $\bar{U}_*(t,\cdot)$ の台は有界である。上に述べたことから次はほとんど明らかである。

<u>補題1</u>. (i) 定理の仮定の下に，ある定数 x_1, x_2 及び K があって，次が成立

(13) $$w_{c_0}(x+x_1) - Ke^{-\gamma t - \bar{b}x} \leq u(t, x+c_0 t) \leq w_{c_0}(x+x_2) + Ke^{-\gamma t - \bar{b}x}.$$

(ii) 任意の正数 $\varepsilon > 0$ に対し次のような $\delta > 0$ が存在する：

$$|f(x) - w_{c_0}(x)| < \delta e^{-\bar{b}x} \qquad x \in R$$

であれば

$$|u(t, x+c_0 t) - w_{c_0}(x)| < \varepsilon e^{-\bar{b}x} \qquad t > 0,\ x \in R.$$

次の補題は放物型方程式に関する Shauder の評価として知

られているものの特別な場合である。

<u>補題2</u>. (1)−(2) の解を u とする。u_x, u_{xx} 及び u_{xxx} は $[1,\infty)\times R^1$ 上で有界である。また (t,x) の函数 V に対し

$$|V|_{t,x}^{t+1} = \sup_{t<s<t+1,\, y>x} |V(s,y)|$$

と書くことにすれば,

$$|u_x(t+1,x+1)| \leqq K|u|_{t,x}^{t+1} \quad , \quad |u_{xx}(t+1,x+1)| \leqq K|u_x|_{t,x}^{t+1}$$

が成り立つ, 但し K は F にのみ依存する定数である。

これらの準備があれば定理3の証明は容易である。補題2と(13)より, $z(t,x) \equiv u(t,x+c_0t)$ は次を満たす。

(14) $$z,\ |z_x|,\ |z_t| \leqq K_1 \min\{e^{-b_1x}+e^{-\gamma t-\bar{b}x},\ 1\}, \quad (t>1)$$

ここに $b_1 = c_0+\sqrt{c_0^2-2\alpha}/2$, また K_1 は t, x によらない定数。今正の数 ε を $(c_0-\bar{b})\varepsilon<\gamma$ととり

$$E(t) = \int_{-\varepsilon t}^{\varepsilon t} e^{2c_0x}\left[\frac{1}{4}z_x(t,x)^2 - \int_0^{z(t,x)} F(r)\,dr\right]dx$$

とおく。(11) より, $E(t)$ は有界 ($t\to\infty$ の時), また

$$\frac{dE}{dt} = \varepsilon e^{2c_0\varepsilon t}\left(\frac{1}{4}(z_x)^2 - \int_0^z F(r)dr\right)(t,\varepsilon t) + O(e^{-2c_0\varepsilon t})$$

$$+\int_{-\varepsilon t}^{\varepsilon t} e^{2c_0x}\left[\frac{1}{2}z_x z_{tx} - F(z)z_t\right]dx$$

$$= o(1) + \left[\frac{1}{2} e^{2c_0 x} z_x z_t \right]_{-\varepsilon t}^{\varepsilon t}$$

$$- \int_{-\varepsilon t}^{\varepsilon t} \left[\frac{1}{2} (e^{2c_0 x} z_x)_x - F(z) \right] z_t \, dx$$

$$= o(1) - \int_{-\varepsilon t}^{\varepsilon t} e^{2c_0 x} \left[\frac{1}{2} z_{xx} + c_0 z_x + F(z) \right] dx$$

である。したがって次のような列 $\{t_n\}$ $(t_n \to \infty)$ がとれる：

$$E(t_n) \longrightarrow 0 \qquad n \to \infty \text{。}$$

補題2の前半により，$\{t_n\}$ の部分列 $\{t_{n'}\}$ を $\{z(t_{n'}, x)\}$ が $C^2[-N, N]$ $(N > 0)$ の位相で収束するようにとれる。$w(x) = \lim z(t_{n'}, x)$ とおけば $\lim E(t_{n'}) = 0$ より $\frac{1}{2} w'' + c_0 w' + F(w) = 0$ を得る。一方補題1より w は自明でない。したがってある x_0 があって $w(x) = w(x + x_0)$ と書ける。w_{c_0} は補題1 (ii) の意味で安定であるから $\lim_{t \to \infty} z(t, x) = w(x)$ である。定理3の証明終。

注意1．上の証明と同様な筋道で定理2が証明できる。

注意2．定理3と同様のことは多次元の場合にも証明できる。証明の方針は同じである。結果は次のようになる：$\xi^2 = 1$ なる $\xi \in R^n$ を一つとる。ある $b > c_0 - \sqrt{c_0^2 - 2\alpha}$ に対し

$$f(x) = O(e^{-b x \xi}) \qquad x \in R \quad (x\xi = \Sigma x_i \xi_i)$$

であれば $u_t = \frac{1}{2}\Delta u + F(u)$, $u(0,x) = \bar{f}(x)$ の解は $\lambda > 0$ に関し一様に

$$|u(t,\lambda\xi) - w_{c_0}(\lambda - c_0 t + L)| \longrightarrow 0 \qquad (t \to \infty)$$

を満たす。

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Kolmogorov-Petrovsky-Pishunov の方程式について

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次の初期値問題を考える。

(1) $$\dot{u} = \frac{1}{2}u'' + F(u)$$

$$\left(\begin{array}{ll} u = u(t,x) & t>0,\ x\in R^1 \\ \dot{u} = \partial u/\partial t, & u'' = \partial^2 u/\partial x^2 \end{array} \right)$$

(2) $$u(0,x) = f(x),$$

ここで $F(u)$ は区間 $0 \leq u \leq 1$ 上の函数で次を満たす:

(3) $$F \in C^1[0,1],\ F(0) = F(1) = 0,\ F(u) > 0 \quad 0<u<1,$$
$$F(u) \leq \alpha u \quad 0 \leq u \leq 1 \quad (\alpha \equiv F'(0)),$$

また f は $0 \leq f \leq 1$ なる R^1 上の可測函数である。

方程式 (1) は R. A. Fisher [1] により、ある種の一地域的に移動する一生物集団の個体群密度 u が満たすべき方程式として導入された。初期値問題 (1)-(2) については、K-P-P [2] によって数学的な考察が与えられ(解の存在と一意性の証明)、次の結果が得られた。

<u>定理(K-P-P)</u> (i) 微分方程式

(4) $$\frac{1}{2}w'' + \sqrt{2\alpha}\,w' + F(w) = 0 \qquad x \in R$$

の解で $0 \leq w \leq 1$, $w(0) = \frac{1}{2}$ を満たすものが唯一存在する。
(ii) $u(t,x)$ を

$$f_0(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

を初期値とする (1) の解とすると、$u'(t,x) < 0$ $(t>0)$ であり、

$m(t)$ を $u(t, m(t)) = 1/2$ で定義すれば $t\uparrow\infty$ とした時，

(5) $\quad u(t, x+m(t)) \downarrow w(x) \quad (x\leq 0)\ ;\ \uparrow w(x) \quad (x>0)$

である。ただし $w(x)$ は (i) で述べられた (4) の解である。

$w(x)$ が (4) の解であることは w よりつくった進行波型の函数 $u(t,x) = w(x-\sqrt{2\alpha}\,t)$ が (1) を満足することに他ならない。以下 (i) における (4) の解を w で表わす。w に関しては次のことが証明できる：

$$w'(x) < 0 \quad x\in R, \quad w(-\infty) = 1, \ w(+\infty) = 0,$$

(6) $$\frac{w'(x)}{w(x)} \longrightarrow -\sqrt{2\alpha} \qquad (x\to\infty),$$

(7) $$\log w(x) = -\sqrt{2\alpha}\,x + o(x) \qquad (x\to\infty)。$$

(ii) と類似のことは $f(x)$ の台が有界 $(f\not\equiv 0)$ の時にも成立する。すなわちそのような f を初期値とする (1) の解 u は $u(t, x+m(t)) \longrightarrow w(x)$ （広義一様）をみたす，但し $m(t)$ は

(8) $$m(t) = \sup\{x;\ u(t,x) = \frac{1}{2}\}$$

で定義される。（ $u(t,x)\longrightarrow 1 \quad (t\to\infty)$ が証明できる。したがって十分大きな t に対しては $m(t)$ は有限な値をとる。）このことは特に，時刻 $t=0$ に有界な領域にだけ存在していた生物集団の，時刻 t での個体群密度が，およそ $-m(t) < x < m(t)$ の範囲ではほぼ上限に達し，この領域を離れると指数函数的に小さくなることを示している。そして領域の広がる速さは $m'(t) \sim \sqrt{2\alpha}$ である。本稿の目的は $m(t)$ のより詳しい評価を得ることにある：

<u>定理</u>　F は (3) の他に

(9) $$\int_{0+} (\alpha u - F(u))\frac{|\log u|}{u}\,du < \infty$$

を満たし，さらに $F(u)/u$ が非増加であるとする。このとき

$$\sup\{x\,;\,f(x)>0\} < \infty$$

を満たす f $(0\leq f\leq 1,\ f\neq 0)$ を初期値とする (1) の解 u に対し $m(t)$ を (8) で定義すれば次が成立する：

$$m(t) = \sqrt{2\alpha}\,t - \frac{3}{2\sqrt{2\alpha}}\log t + O(\log\log t)\,。$$

注意　参考のために線型の場合の対応する結果を述べておく： $V^{\cdot} = \frac{1}{2}V'' + \alpha V$, $V(0,x) = g(x)$ の解 $V(t,x)$ は $\int |g(x)|\,e^{\sqrt{2\alpha}\,x}dx < \infty$ であれば次を満たす。

$$V\left(t,\ x+\sqrt{2\alpha}\,t - \frac{1}{2\sqrt{2\alpha}}\log t\right) \longrightarrow A\,e^{-\sqrt{2\alpha}\,x} \qquad (t\to\infty)$$

$$A = \frac{1}{\sqrt{2\pi}}\int g(y)\,e^{\sqrt{2\alpha}\,y}dy\ 。$$

次の補題は初期値問題 (1)－(2) を扱う上で基本的である。

補題　F_1 , F_2 は R^1 上の有界連続な微係数をもつ函数とする。u_i を，(1) で F を F_i で置き代えた方程式の $u_i(0,x) = f_i(x)$ (f_i は有界可測）なる解とする ($i=1,2$)。この時，$F_1\leq F_2$ 及び $f_1\leq f_2$ から $u_1\leq u_2$ が出る。

証明　差 $u = u_2 - u_1$ の満たす方程式

$$u^{\cdot} = \frac{1}{2}u'' + F_1'(\theta)u + Q$$

($Q(t,x) = F_2(u_2(t,x)) - F_1(u_2(t,x))$) に放物型偏微分方程式に関する最大値原理を適用すればよい。

以下において $f = f_0$ (K-P-P の定理の (ii) を見よ) の場合の定理の証明を述べる。一般の場合はこの場合に帰着されるがここでは省略する。簡単のため次を仮定する。

(10) $F'(u) \geqq 0 \quad 0<u<\frac{1}{2}\ ;\ F'(u) \leqq 0 \quad \frac{1}{2}<u<1$。

$u(t,x)$ を f_0 を初期値とする (1) の解とする。$V(t,x) = u(t, x+m(t))$ とおく。

$$T_t g(x) = \int_{-\infty}^{\infty} p(t, x-y) g(y) dy, \quad p(t,x) = \frac{\exp\{-x^2/2t\}}{\sqrt{2\pi t}}$$

と書けば，補題から $u(t,x) \leqq e^{\alpha t} T_t f_0(x)$ 及び $1 - u(t,x) \leqslant T_t(1-f_0)(x)$ が得られる。（Appendix I を (1) 及び (1) の両辺を x で微分した式に適用すれば）$|u'(t,x)||x|$, $|u^{\cdot}(t,x)|$ は x の函数として R^1 上で可積分である。これらに注意すれば

$$m(t) = -\int_{-\infty}^{\infty} u'(t, x+m(t))(m(t)+x)dx + \int_{-\infty} u'(t, x+m(t))x\,dx$$

$$= -\int_{-\infty}^{\infty} u'(t,x)x\,dx + \int_{-\infty}^{\infty} w'(x)x\,dx + o(1),$$

$$\dot{M}(t) \equiv -\int_{-\infty}^{\infty} u'(t,x)x\,dx = \int_0^{\infty} u\,dx - \int_{-\infty}^{0} (1-u)\,dx$$

とおいて

$$M^{\cdot}(t) = \int \left(\frac{1}{2}u'' + F(u)\right)dx = \int F(V)dx$$

が得られる。(5) 及び (10) より $M^{\cdot}(t)$ は増加函数である。
$m(t) = M(t) + \text{const.} + o(1)$ と書けたから

(11) $$L \equiv \inf_{t>s>0} \left[m(t) - m(t-s) - \frac{s}{t} m(t) \right]$$

は有限である。

さて，$k(t,x) \equiv F(V(t,x))/V(t,x)$ とおけば (1) は

$$V^{\cdot} = \frac{1}{2}V'' + m^{\cdot}V' + kV$$

となる。$\{B_t, t \geqq 0\ ;\ P_x, x \in R\}$ を 1 次元ブラウン運動とすると，Kac の公式により

(12) $$V(t,x) = E_x\left[e^{\int_0^t k(s, B_{t-s} + m(t) - m(s))ds} f_0(B_t + m(t)) \right]。$$

$$g_*(x) \equiv \alpha - \frac{F(w(x))}{w(x)} \quad x>0 \; ; \quad \equiv \alpha \quad x \leq 0$$

とおけば $F(u)/u$ の単調性により g_* は非増加，$V(t,x) \leq w(x)$ $(x>0)$ であるから $k \geq \alpha - g_*$，また (6), (7) 及び (9) により

$$(13) \qquad \int^{+\infty} g_*(x)\, x\, dx < \infty 。$$

(11) の L を使えば，(12) より

$$V(t,x) \geq e^{\alpha t} E_x[\, e^{-\int_0^t g_*(B_s + m(t)\frac{s}{t} + L)ds} ; B_t + m(t) < 0] 。$$

したがって g_* の単調性から

$$V(t,-1) \geq e^{\alpha t} E_{-1}[\, e^{-\int_0^t g_*(B_s + m(t)\frac{s}{t} + L)ds} \mid B_t + m(t) = -1]$$

$$\times\ P_{-1}[\, -1 < B_t + m(t) < 0 \,] 。$$

条件：$B_t + m(t) = -1$ の下での $\{ B_s + m(t)\frac{s}{t} ; 0<s<t \}$ の法則は条件：$B_t = -1$ の下での $\{ B_s ; 0<s<t \}$ の法則に等しいから上の不等式の右辺は

$$n(t) = \sqrt{2\alpha}\, t - m(t)$$

を用いて次のように書ける：

$$e^{\alpha t} E_{-1}[\, e^{-\int_0^t g_*(B_s + L)ds} \mid B_t = -1] \int_0^1 p(t, y - m(t))\, dy$$

$$= E_{-1+L}[\, e^{-\int_0^t g_*(B_s)ds} \mid B_t = -1+L] \frac{e^{\sqrt{2\alpha}\, n(t)}}{\sqrt{2\pi t}} \left(\int_0^1 e^{\sqrt{2\alpha}\, y} dy + o(1) \right)$$

$$= \frac{e^{\sqrt{2\alpha}} - 1}{\sqrt{2\alpha}}\, p_*(t, -1+L, -1+L)\, e^{\sqrt{2\alpha}\, n(t)} (1 + o(1)),$$

ここに $p_*(t,x,y)$ は $u^{\cdot} = \frac{1}{2} u'' - g_* u$ $(t>0, x \in R)$ の基本解である。(13) により；固定した x に対して

$$p_*(t,x,x) \sim A(x) \frac{1}{\sqrt{t}^{\,3}} \qquad (t \to \infty)$$

である (Appendix II) から，

$$V(t,-1) \geq \text{const.} \frac{e^{\sqrt{2\alpha}\, n(t)}}{\sqrt{t^3}} (1+o(1))$$

したがって

(14) $$n(t) \leq \frac{3}{2\sqrt{2\alpha}} \log t + \text{const.}$$

を得る。

次に

$$g^*(x) = 2F(\tfrac{1}{2}) \quad x<0; \quad = 0 \quad x>0$$

とおき，$p^*(t,x,y)$ で $\dot{u} = \frac{1}{2}u'' - g^*u$ の基本解を表わせば，上と同じようにして（(11) の代わりに (14) を使う）

(15) $$V(t,0) \leq p^*(t,\sigma(t),\sigma(t))\, e^{\sqrt{2\alpha}\, n(t)} (\text{const.} + o(1))$$

を得る，但し $\sigma(t) = 4\frac{1}{2\sqrt{2\alpha}} \log t$ とおいた。Appendix II により

$$p^*(t,\sigma(t),\sigma(t)) \sim \sqrt{\frac{2}{\pi}} \frac{\sigma(t)^2}{\sqrt{t^3}} \qquad (t\to\infty)$$

である。したがって (15) より

$$n(t) \geq \frac{3}{2\sqrt{2\alpha}} \log t - \frac{2}{\sqrt{2\alpha}} \log\log t + \text{const.}$$

を得る。(14) と合わせて定理が証明された。

Appendix I

函数 $G(t,x,y)$ は3変数に関し連続で，$|G(t,x,u_1) - G(t,x,u_2)| \leq K|u_1-u_2|$，$G(t,x,0)=0$ を満たすとする。領域 $T \leq t \leq T+1$，$x \geq L$ で連続な函数 $u(t,x)$ がそこで $(\dot{u} - \frac{1}{2}u'')(t,x) = G(t,x,u(t,x))$ を満たすならば

$$|u'(T+1,L+1)| \leq M \sup\{|u(t,x)|;\ T<t<T+1,\ x>L\}$$

が成り立つ。M は K にのみ依存する定数である。

証明 $p^*(t,x,y) = p(t,x-y) - p(t,x+y)$ $\left(p = e^{-\frac{x^2}{2t}}/\sqrt{2\pi t}\right)$ とおく。

$$u(T+1,L+y) = \int_0^1 p'(1-s,y)\,u(T+s,L)\,ds + \int_0^\infty p^*(1,y,r)\,u(T,L+r)\,dr$$

$$+\int_0^1 ds \int_0^\infty p^*(1-s, y, r)\, G(T+s, L+r, u(T+s, L+r))\, dr$$

の両辺を y で微分し，$y=1$ とおいて，右辺の積分を評価すれば求める不等式が得られる。

Appendix II

定理　$q(x)$ は R^1 上の非負連続函数で次を満たすとする。

$$0 < \int_0^\infty q(x)\, x\, dx < \infty$$

この時，$\dot{u} = u'' - qu$ (*) の基本解系 $p(t,x,y)$ に対し次が成立する。

$$p(t,x,y) = A(x,y)(1+o(1))/\sqrt{t}^3,$$

但し，$o(1)$ は，各 $N\in R$ に対し，$t\to\infty$ とした時 $x, y > -N$，$(|x|+|y|)/\sqrt{t} \to 0$ なる限り一様に小さくなる；また $A(x,y)$ は正，連続な函数で，片方の変数をとめると $A'' - qA = 0$ を満たし，

$$\lim_{x,y\to\infty} \frac{A(x,y)}{xy} = \frac{1}{2\sqrt{\pi}}$$

が成り立つ。

次は Titchmarsh [3] § 5.7 による。

補題 1　$\hat{q}(x)$ は $(0,\infty)$ 上の非負連続函数で，$\int_0^\infty \hat{q}(x)\,dx < \infty$ とする。境界条件 $u(t,0)=0$ に対応する $\dot{u} = u'' - \hat{q}u$ の基本解系 $\hat{p}(t,x,y)$ $(x, y>0)$ は

(1) $$\hat{p}(t,x,y) = \frac{1}{\pi}\int_0^\infty e^{-t\lambda}\psi(x,\lambda)\psi(y,\lambda)\,k(\lambda)\,d\lambda$$

と書ける。但し ψ は，$\lambda = s^2$ として，

(*) $\dot{u} = \frac{1}{2}u'' - qu$ に対応する結果を得たければ，以下で，t を $t/2$ で，q を $2q$ でおき換えればよい。

$$\psi(x,\lambda) = \frac{\sin sx}{s} + \frac{1}{s}\int_0^x \sin\{s(x-y)\}\hat{g}(y)\psi(y,\lambda)\,dy$$

の解として定まり，

$$a(\lambda) = -\frac{1}{s}\int_0^\infty \sin(sy)\hat{g}(y)\psi(y,\lambda)\,dy ,$$

$$b(\lambda) = \frac{1}{s} + \frac{1}{s}\int_0^\infty \cos(sy)\hat{g}(y)\psi(y,\lambda)\,dy,$$

$$k(\lambda) = [\sqrt{\lambda}\,(a^2(\lambda) + b^2(\lambda))]^{-1};$$

(2) $$k(\lambda) \sim \sqrt{\lambda} \quad (\lambda \to \infty) , \quad a^2(\lambda) + b^2(\lambda) > 0$$

である。

上でさらに $\int^{+\infty} \hat{g}(x)\,x\,dx < \infty$ を仮定する。

$$\left|\frac{\psi(x,\lambda)}{x}\right| \leqq 1 + \int_0^x \left|\frac{\psi(y,\lambda)}{y}\right| \hat{g}(y)\,y\,dy \qquad x > 0$$

より，

$$\frac{|\psi(x,\lambda)|}{x} \leqq e^M \qquad M \equiv \int_0^\infty \hat{g}(y)\,y\,dy$$

である。$\hat{\varphi}(x)$ を

$$\hat{\varphi}(x) = x + \int_0^x (x-y)\hat{g}(y)\hat{\varphi}(y)\,dy$$

の解とすると，$\psi(x,\lambda) = \hat{\varphi}(x)(1 + \varepsilon(\sqrt{\lambda}\,x))$ ，$\varepsilon(u) = O(u^2)$
と書ける。実際 $w(x,\lambda) = |\hat{\varphi}(x) - \psi(x,\lambda)|/x$ とおけば，

$$w(x,\lambda) \leqq \frac{1}{6}(x\sqrt{\lambda})^2 + \int_0^x \frac{1}{6}(x-y)^2\lambda\;\psi(y,\lambda)\hat{g}(y)dy + \int_0^x w(y,\lambda)\hat{g}(y)\,y\,dy$$

だから

$$w(x,\lambda) \leqq \frac{1}{6}(x\sqrt{\lambda})^2(1 + Me^M)e^M .$$

(3) $$k(\lambda) = \frac{\sqrt{\lambda}}{K^2}(1 + o(1)) \quad (\lambda \downarrow 0), \quad K \equiv 1 + \int_0^\infty \psi(y)\hat{g}(y)dy < \infty$$

である。任意の $N > 0$ に対し (1) を

$$\hat{p}(t,x,y) = \frac{1}{\pi}\left\{\int_0^{N/t} + \int_{N/t}^\infty\right\} e^{-\lambda t}\psi(x,\lambda)\psi(y,\lambda)k(\lambda)d\lambda$$

と分けて積分すれば，(2) と (3) から $k(\lambda) \leqq M_1\sqrt{\lambda}$ とおさえられることに注意して，

第二項 $\leqq e^{2M} M_1 \dfrac{xy}{\sqrt{t^3}} \int_N^\infty e^{-u}\sqrt{u}\,du$

第一項 $= \int_0^N e^{-u}\left\{1+\varepsilon\left(\frac{x}{\sqrt{t}}\sqrt{u}\right)\right\}\left\{1+\varepsilon\left(\frac{y}{\sqrt{t}}\sqrt{u}\right)\right\}\sqrt{u}(1+o(1))du\, \dfrac{\hat{\varphi}(x)\hat{\varphi}(y)}{K^2}\dfrac{1}{\sqrt{t^3}}$。

$\int_0^\infty e^{-u}\sqrt{u}\,du = \sqrt{\pi}/2$ であるから以上より次が成立する。

(4) $$\hat{p}(t,x,y) = \frac{1}{2\sqrt{\pi}}\frac{\hat{\varphi}(x)\hat{\varphi}(y)}{K^2}\frac{1}{\sqrt{t^3}}(1+o(1)),\quad x,y>0$$

但し，$o(1)$ は $t\to\infty$ の時，$(x+y)/\sqrt{t}\to 0$ なる限り一様に小さくなる。

補題 2　$g(x)\geqq 0\quad x\in R$，$\int_{-\infty}^0 g(x)|x|dx = \infty$ とすると，

$$Q_L(-L)=1;\ Q_L(x)>0,\ Q_L'(x)\leqq 0,\ Q_L''-gQ_L=0 \qquad x>-L$$

によって定まる函数 $Q_L(x)$ $(x>-L)$ は次を満たす。

(5) $$L\,Q_L(x)\longrightarrow 0\quad (L\to\infty)\qquad x\in R。$$

証明　$x=0$ に対して (5) を示せば十分である。θ_L，ψ_L を

$$\theta_L(x) = 1+\int_{-L}^x (x-y)\theta_L(y)g(y)dy \qquad (x>-L)$$
$$\psi_L(x) = (L+x)+\int_{-L}^x (x-y)\psi_L(y)g(y)dy$$

の解とする。$\psi_L(x) = \theta_L(x)\int_{-L}^x \theta_L(y)^{-2}dy$ に注意して

$$D_L(x) \equiv C_L\theta_L(x)-\psi_L(x) = \theta_L(x)\int_x^\infty\frac{dy}{\theta_L^2(y)},\quad C_L\equiv\int_{-L}^\infty\frac{dy}{\theta_L^2(y)}$$

とおけば $D_L\geqq 0$，$D_L'\leqq 0$ $(x>-L)$ であるから

$$Q_L(x) = C_L^{-1}D_L(x)。$$

$$D_L(0) = \theta_L(0)\int_0^\infty\frac{dy}{\theta_L^2(y)} \leqq \theta_L(0)\int\frac{dx}{[\theta_L(0)+\theta_L'(0)x]^2} = \frac{1}{\theta_L'(0)}$$
$$= 1\Big/\int_{-L}^0\theta_L(y)g(y)dy$$

及び $C_L\theta_L \geqq \psi_L$ により

$$Q_L(0) \leq 1 \Big/ \int_{-L}^{0} \psi_L(y) g(y)\, dy ,$$

すなわち

$$\text{(6)} \qquad \frac{L}{Q_L(0)} \geq \int_{-L}^{0} \frac{\psi_L(y)}{L} g(y)\, dy$$

を得る。 $\frac{\psi_L(x)}{L} \geq 1 + \frac{x}{L}$ に注意して

$$\frac{\psi_L(0)}{L} \geq \int_{-L}^{0} \left(1 + \frac{x}{L}\right) g(y)|y|\, dy$$

$$\geq \int_{-L/2}^{0} \frac{1}{2} g(y)|y|\, dy \to \infty \qquad (L \to \infty)$$

したがって

$$\text{(7)} \qquad \frac{\psi_L'(0)}{L} = \frac{1}{L} + \int_{-L}^{0} \frac{\psi_L(y)}{L} g(y)\, dy \to \infty \qquad (L \to \infty)$$

実際, $\psi_L(y) \geq \psi_L(0) + \psi_L'(0) y \quad (-L < y < 0)$ だから $\frac{\psi_{L_n}'(0)}{L_n} < C$ ならば

$$\frac{\psi_{L_n}'(0)}{L_n} \geq \int_{y_0}^{0} g(y)\, dy \, \frac{\psi_{L_n}(0)}{L_n} - C \int_{y_0}^{0} g(y)|y|\, dy \to \infty$$

となって矛盾。(6) と (7) より (5) が得られる。

<u>定理の証明</u> $\int_{-\infty}^{0} g(x)|x| < \infty$ の時は (4) を得たと同様の方法で定理が証明できる（詳細は省略する）。以下,定理の仮定に加えて

$$\int_{-\infty}^{0} g(x)|x|\, dx = \infty$$

を仮定する。 $u(t, -L) = 0$ に対応する $\dot{u} = u'' - g u \quad (x > -L)$ の基本解系を $p^{(L)}(t, x, y)$ で表わす。(4) より

$$\text{(8)} \qquad p^{(L)}(t, x, y) = \frac{1}{2\sqrt{\pi}} \frac{\psi_L(x)\psi_L(y)}{K_L^2} \frac{1}{\sqrt{t}^3} (1 + o(1)) \qquad x, y > -L$$

$$K_L = 1 + \int_{-L}^{\infty} \psi_L(y) g(y)\, dy .$$

$p^{(L)}$ は L に関し増加だから ψ_L / K_L もそうである。

$$\psi(x) = \lim_{L \to \infty} \phi_L(x), \qquad \phi_L(x) = \psi_L(x) / K_L$$

とおくと, (7) より $L / K_L \to 0$ だから,

$$\psi(x) = \int_{-\infty}^{x} (x-y)\psi(y) g(y) dy, \quad \int_{-\infty}^{0} \psi(y) g(y) |y| dy < \infty。$$

$1 = \lim_{x\to\infty} \phi_L(x)/x = 1/K_L + \int_{-L}^{\infty} \phi_L(y) g(y) dy$ だから

$$\lim_{x\to\infty} \frac{1}{x}\psi(x) = \int_{-\infty}^{\infty} \psi(y) g(y) dy = 1。$$

このことに注意して次が得られる。

(9) $\quad \phi_L(x) = \psi(x)(1+o(1)),\quad o(1)\to 0$ は $x>-N$ に関し一様。

さて

(10) $\quad p(t,x,y) = p^{(L)}(t,x,y) + \int_0^t p(t-s,-L,y) d_s Q_L(s,x)$

である。但し Q_L は $Q_L(t,-L)=1$, $Q_L(0,x)=0$ なる $\dot{Q}_L = Q_L'' - g\,Q_L$ $(x>-L)$ の解。(10) の右辺第2項を J_L とおけば

$$\overline{\lim_{t\to\infty}} \sqrt{t}^3 J_L \leqq Q_L(x)\,\overline{A}(-L,y),$$

但し $\overline{A}(-L,y) = \overline{\lim}_{t\to\infty} p(t,-L,y)/\sqrt{t}^3$, また $Q_L(x) \equiv \lim_{t\to\infty} Q_L(t,x)$ は補題2のものと同じである。$\int_{-\infty}^{\infty} g(x)|x|dx < \infty$ の場合の結果と比較すれば $\overline{A}(-L,y) = O(L)$ $(L\to\infty,\ y>-N$ に関し一様) である。したがって，補題2から $Q_L(x)\overline{A}(-L,y) \to 0$ $(x, y>-N$ に関し一様)。ゆえに

$$\lim_{L\to\infty} \overline{\lim_{t\to\infty}} \sqrt{t}^3 [p(t,x,y) - p^{(L)}(t,x,y)] = 0 \quad \text{収束は } x,y>-N \text{ に関し一様。}$$

(8), (9) と合わせれば，次が得られる：

$$p(t,x,y) = \frac{1}{2\sqrt{\pi}} \psi(x)\psi(y) \frac{1}{\sqrt{t}^3} (1+o(1))。$$

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107. *Limit Theorems for Poisson Branching Processes*

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1. The process treated here is a model of the population growth in a biological system in which each object gives births at various times of its life length and new born objects behave as their parents independently of others. The process is specified by two nonnegative continuous functions on $[0, \infty)$ $\lambda(x)$, $\mu(x)$ and a probability generating function $h(s)=\sum_{n=1}^{\infty} h_n s^n$, $\sum_{n=1}^{\infty} h_n=1$, $h_n \geqq 0$ $(n=1, 2, \cdots)$; *a living object of age x gives births to j objects before it reaches age $x+dx$ without dying itself with a probability $h_j \lambda(x)dx$ and dies before age $x+dx$ with a probability $\mu(x)dx$* where these probabilities are independent of each other and of past history. This process appeared in [2] as a special case of general age dependent branching processes and was called a *Poisson branching process.* In this paper limit theorems will be given for probability generating functions of the population size at time t of Poisson branching processes. Limit theorems of such type are studied by Ryan [5] for subcritical general age dependent branching processes. His results contain a part of ours as a special case. The forms and proofs of theorems given here are simpler than Ryan's and almost parallel with ones of age dependent branching processes given in [1].

2. Let $Z(t)$ be the population size at time t of a Poisson branching process specified by $\lambda(x)$, $\mu(x)$ and $h(s)$ as in the first section and let $F(s, t)$ be its generating function; $F(s, t)=E[s^{Z(t)}]$, $0 \leqq s \leqq 1$. We always assume that the process starts with a single object of age 0. Let L be the time when the initial object dies and $G(t)$ be the distribution function of L; $G(t)=\int_0^t \mu(u) \exp\left(-\int_0^u \mu(r)dr\right)du$. By conditioning on L we get

$$F(s, t)=s(1-G(t))E\left[\exp\left\{\int_0^t \log F(s, t-u)dN(u)\right\} \middle| L>t\right] + \int_0^t E\left[\exp\left\{\int_0^u \log F(s, t-v)dN(v)\right\} \middle| L=u\right]dG(u),$$

in which we denote by $N(t)$ the number of direct children of the initial particle that have been ever born untill time t. Then we have

$$F(s, t)=s(1-G(t)) \exp\left\{\int_0^t (h(F(s, t-u))-1)\lambda(u)du\right\} + \int_0^t \exp\left\{\int_0^u (h(F(s, t-v))-1)\lambda(v)dv\right\}dG(u). \tag{1}$$

When $m=h'(1-)<\infty$, $F(s,t)$ is continuous in $t\in[0,\infty)$ for each $s\in[0,1]$ and is the unique solution of (1) with $0\leqq F\leqq 1$ (see [3] for the proof).

From now on we assume $m=h'(1-)<\infty$.

Put $g(s)=\int_0^\infty \exp\left(\int_0^t \lambda(u)du(h(s)-1)\right)dG(t)$ and let q be the smallest root in $[0,1]$ of $g(s)=s$. Then q is the extinction probability for $Z(t)$; $q=\lim_{t\to\infty} F(0,t)$ ([3]), and $q=1$ is equivalent to $g'(1)<1$ and $g(1)=1$. We call our process *subcritical* if $g'(1)<1$ and $g(1)=1$, *critical* if $g'(1)=g(1)=1$ and *supercritical* if $g'(1)<1$ or $g(1)<1$.

Let $M(t)$ denote $E[Z(t)]=F'(1,t)$. From (1) it follows that

$$M(t)=m\int_0^t M(t-u)\lambda(u)(1-G(u))du+(1-G(t)).$$

We can see that $M(t)$ is bounded on each finite interval and the standard renewal theorem deduces ([5], [3]) the following.

Lemma 1. *Suppose there exists* α *such that* $m\int_0^\infty \exp(-\alpha t)(1-G(t))\lambda(t)dt=1$ *and* $\int_0^\infty \exp(-\alpha t)(1-G(t))dt<\infty$. *Then* $M(t)\sim a\cdot\exp(\alpha t)$ *as* $t\to\infty$ *where* $a=\int_0^\infty \exp(-\alpha t)(1-G(t))dt\left(m\int_0^\infty \exp(-\alpha t)(1-G(t))\lambda(t)tdt\right)^{-1}$.

3. In this section we study the asymptotic behavior of $F(s,t)$ as $t\to\infty$ for subcritical processes under the condition of Lemma 1. Let α be a number in Lemma 1. Note that α is necessarily negative when $g(1)=1$ and $g'(1)<1$.

Lemma 2. *If* $g'(1)<1$, $g(1)=1$ *and the condition of Lemma* 1 *is satisfied, then* $\sup_{t>0,1\geqq s\geqq 0}\exp(-\alpha t)(1-F(s,t))\equiv K<\infty$.

Before going into the proof of the lemma we state the main theorem.

Theorem 1. *If* $g'(1)<1$, $g(1)=1$, *the condition of Lemma* 1 *is satisfied and*

$$\int_0^\infty te^{-\alpha t}\lambda(t)(1-G(t))dt<\infty, \tag{2}$$

then limit $\inf_{t\to\infty}(1-F(s,t))\exp(-\alpha t)>0$ *iff* $\sum h_j(j\log j)<\infty$ *and* $E[X\log X]<\infty$, *where* $X\equiv\int_0^L \exp(-\alpha t)\lambda(t)dt$. *In this case* $\text{limit}_{t\to\infty}(1-F(s,t))\exp(-\alpha t)\equiv Q(s)$ *exists and defines a positive analytic function of* $s\in[0,1)$ *with* $Q'(1-)<\infty$.

Remark 1. We can see that $\sum h_j(j\log j)<\infty$ and $E[X\log X]<\infty$ iff $E[Y\log Y]<\infty$ where $Y\equiv\int_0^L \exp(-\alpha t)dN(t)$. Therefore, by the inequality

$$E[Y\log Y]\geqq E[E[Y\,|\,N(L)]\log E[Y\,|\,N(L)]]$$
$$\geqq \textstyle\sum_{k=1}^{\infty} p_k k\log k\displaystyle\int_0^\infty \exp(-\alpha t)dL_k(t)$$

where $p_k=P[N(L)=k]$ and $L_k(t)=k^{-1}E[N(t)\,|\,N(L)=k]$, the sufficiency part of the theorem is reduced to the result of [5].

The next theorem is an immediate corollary of Theorem 1.

Theorem 2. *Suppose that conditions of Theorem 1 is satisfied and that $E[X\log X]$ and $\sum h_j(j\log j)$ are both finite where X is defined in Theorem 1. Then $\lim_{t\to\infty} P[Z(t)=k\,|\,Z(t)>0]=b_k$ exist and $\{b_k\}_{k=1}^\infty$ is a probability distribution with mean $\sum kb_k=Q'(1-)(Q(0))^{-1}$.*

Now we prove Lemma 2. After simple calculations (1) is rewritten as follows;

$$(3)\qquad \begin{aligned}1-F(s,t)&=\xi^1(t)-\xi^2(t)-\xi^3(t)-\xi^4(t)\\&\quad+m\int_0^t(1-F(s,t-u))\lambda(u)(1-G(u))du\end{aligned}$$

where

$$\xi^1(t)=(1-G(t))\exp\left\{-\int_0^t(1-h(F(s,t-u)))\lambda(u)du\right\},$$
$$\xi^2(t)=(1-G(t))A\left(\int_0^t(1-h(F(s,t-u)))\lambda(u)du\right),$$
$$\xi^3(t)=\int_0^t A\left(\int_0^u(1-h(F(s,t-v)))\lambda(v)dv\right)dG(u),$$
$$\xi^4(t)=\int_0^t\{m(1-F(s,t-u))-(1-h(F(s,t-u)))\}\lambda(u)(1-G(u))du,$$

with $A(x)=x-1+\exp(-x)$. Let us write $\xi_\alpha^1(t)=\exp(-\alpha t)\xi^1(t)$ etc. Put $H(t)=1-F(s,t)$ and $R_\alpha(t)=\int_0^t m\exp(-\alpha u)\lambda(u)(1-G(u))du$. Then $H_\alpha(t)\leqq\exp(-\alpha t)(1-G(t))+\int_0^t H_\alpha(t-u)dR_\alpha(u)$ since ξ^2, ξ^3 and ξ^4 are all non-negative. Lemma 2 now follows from the Renewal theorem ([4]).

For the proof of Theorem 1 we need the following lemmas.

Lemma 3. *$\xi^2(t)$, $\xi^3(t)$ and $\xi^4(t)$ are all Riemann integrable.*

Proof. Taking Laplace transforms, it follows from (3) that

$$\hat{H}_\alpha(x)(1-\hat{R}_\alpha(x))=\hat{\xi}_\alpha^1(x)-\hat{\xi}_\alpha^2(x)-\hat{\xi}_\alpha^3(x)-\hat{\xi}_\alpha^4(x)$$

where we set $\hat{\xi}_\alpha^1(x)=\int_0^\infty\exp(-xt)\xi_\alpha^1(t)dt$ etc. Since $\hat{\xi}_\alpha^1(0+)<\infty$, by compairing the signs of terms in both sides, we see that $\hat{\xi}_\alpha^i(0+)<\infty$, $i=2,3,4$.

The next lemma furnishes a key for the proof of Theorem 1.

Lemma 4. *Let Y be a non-negative random variable with $E[Y]=1$. Then for any $\delta>0$ $\int_0^\delta E[A(uY)]u^{-2}du<\infty$ iff $E[Y\log Y]<\infty$.*

Corollary. *Let $f(s)=\sum_{i=0}^\infty q_i s^i$ be a probability generating function with $c=f'(1)<\infty$. Then for $0<\delta<1$, $\int_0^\delta[c-u^{-1}(1-f(1-u))]u^{-1}du$*

$<\infty$ *iff* $\sum_{i=0}^{\infty} q_i(i \log i)<\infty$.

We omit the proof of these results (see [1] for a proof).

Proof of Theorem 1. Let $\liminf (1-F(0,t)) \exp(-\alpha t)>0$. We first note that there exists a positive constant C such that for all $t\geqq 0$

$$(1-F(0,t))>Ce^{\alpha t} \quad \text{and} \quad 1-h(F(0,t))>Ce^{\alpha t}.$$

By Lemma 3, using the inequality $A(x)>x-1$, $x>0$,

$$\infty>\int_0^\infty \xi_\alpha^2(t)dt$$
$$>C\int_0^\infty (1-G(t))dt\int_0^t e^{-\alpha v}\lambda(v)dv-\int_0^\infty e^{-\alpha t}(1-G(t))dt,$$

and then the hypothesis of the theorem implies

$$\int_0^\infty (1-G(t))dt\int_0^t e^{-\alpha v}\lambda(v)dv<\infty. \tag{4}$$

From (2) and (4) it follows

$$\int_0^\infty dt\int_t^\infty dG(u)\int_0^u e^{-\alpha v}\lambda(v)dv<\infty. \tag{5}$$

We see similarly that

$$\infty>\int_0^\infty \xi_\alpha^3(t)dt>\int_0^\infty e^{-\alpha t}dt\int_0^t A\left(C\int_0^u e^{\alpha(t-v)}\lambda(v)dv\right)dG(u).$$

Since $A(x)<x$, this inequality, combined with (5), leads to

$$\infty>\int_0^\infty e^{-\alpha t}dt\int_0^\infty A\left(C\int_0^u e^{\alpha(t-v)}\lambda(v)dv\right)dG(u)$$
$$=(-\alpha)^{-1}C\int_0^C E[A(uX)]u^{-2}du.$$

Consequently, by Lemma 4, we obtain $E[X\log X]<\infty$ since $E[X]=1$.

We deduce in a similar way, using Corollary instead of Lemma 4 and using integrability of ξ_α^4, that $\sum h_j(j\log j)<\infty$.

To prove the converse part, we assume that $E[X\log X]<\infty$ and $\sum h_j(j\log j)<\infty$. Since $E[X\log X]<\infty$ implies (4), we see that $\xi_\alpha^2(t)$ is directly Riemann integrable. ξ_α^1, ξ_α^3 and ξ_α^4 are also directly Riemann integrable: for example

$$\sup_{n\leq t<n+1} \xi_\alpha^3(t)<e^{-\alpha}\inf_{n-1<u<n} \xi_\alpha^3(u)+Km\int_{n-1}^{n+1} dG(u)\int_0^u e^{-\alpha v}\lambda(v)dv$$

and the sum of the right hand sides over n converges. The renewal theorem therefore can be applied and then

$$\lim_{t\to\infty} e^{-\alpha t}(1-F(s,t))=\frac{\int_0^\infty(\xi_\alpha^1(t)-\xi_\alpha^2(t)-\xi_\alpha^3(t)-\xi_\alpha^4(t))dt}{\int_0^\infty tdR_\alpha(t)}$$

exists. We denote this limit by $Q(s)$. Now we claim that

$$\lim_{s\uparrow 1}\frac{Q(s)}{1-s}=\frac{\int_0^\infty(1-G(t))e^{-\alpha t}dt}{\int_0^\infty tdR_\alpha(t)}. \tag{6}$$

If we prove this equation, since $Q(s)$ is non-increasing, we obtain that

$Q(s)>0$, $0\leqq s<1$ and the proof is completed.

For the proof of (6) it suffices to show that

$$\lim_{s\uparrow 1}\frac{1}{1-s}\int_0^\infty (\xi_\alpha^2(t)+\xi_\alpha^3(t)+\xi_\alpha^4(t))dt=0.$$

If we take a convention such that $F(s,t)=1$ for $t<0$, then $(1-s)^{-1}(\xi_\alpha^2(t)+\xi_\alpha^3(t))$ is written as

$$\int_0^\infty\int_0^u \frac{1-h(F(s,t-v))}{1-s}\lambda(v)dv \times\left[1-\frac{1-\exp\left(\int_0^u (1-h(F(s,t-v)))\lambda(v)dv\right)}{\int_0^u (1-h(F(s,t-v)))\lambda(v)dv}\right]dG(u).$$

From Lemma 1 and Lemma 2 it follows that

$$\frac{(\xi_\alpha^2(t)+\xi_\alpha^3(t))}{1-s}<\text{const.}e^{-\alpha t}E[A(mKe^{\alpha t}X)].$$

Since the right hand side of this inequality is integrable on $[0,\infty)$ by Lemma 4, the dominated convergence theorem is now applied to obtain

$$\lim_{s\uparrow 1}\frac{1}{1-s}\int_0^\infty (\xi_\alpha^2(t))+\xi_\alpha^3(t))dt=0.$$

We can argue similarly to get that $\sum h_j(j\log j)<\infty$ implies

$$\lim_{s\uparrow 1}\frac{1}{1-s}\int_0^\infty \xi_\alpha^4(t)dt=0.$$

Thus the theorem is proved.

Remark 2. Evaluation (4) is not implied by conditions of Theorem 1, i.e. there exists a triple $\lambda(t)$, $\mu(t)$ and $h(s)$ for which conditions of Theorem 1 are satisfied but (4) fails to hold.

4. For the supercritical processes we get only an unsatisfactory result.

Theorem 3. *Let $g'(1-)>1$. If there exists a number β such that $1=\gamma\int_0^\infty \exp(-\beta t)\lambda(t)(q-J(t))dt$ and $\int_0^\infty \exp(-\beta t)(q-J(t))dt<\infty$, where $J(t)=\int_0^t \exp\left\{(h(q)-1)\int_0^u \lambda(v)dv\right\}dG(u)$ and $\gamma=h'(q)$, then $(q-F(0,t))\cdot\exp(-\beta t)$ is bounded on $[0,\infty)$.*

For the proof the same method as in the proof of Lemma 2 is applied.

In order to demonstrate that a number β defined in the above will be proper one, we give a simple example. If we take $\lambda(t)\equiv\lambda$, $\mu(t)\equiv\mu$ where λ and μ are positive constants, then our process is a Markov branching process determined by the backward equation $\frac{\partial}{\partial t}F(s,t)=u(F(s,t))$, $u(s)=(\lambda+\mu)\left(\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}s\cdot h(s)-s\right)$ with $\beta=\gamma\lambda q-\mu q^{-1}$

which coincides with an usual parameter β_0 determined by the equation $1=u'(q)\int_0^\infty \exp(-(\lambda+\mu+\beta_0)t)dt$.

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