論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	A Homotopy Method for Solving Systems of Polynomial Equations
著者(和文)	水野眞治
Author(English)	SHINJI MIZUNO
出典(和文)	学位:理学博士, 学位授与機関:東京工業大学, 報告番号:甲第1584号, 授与年月日:1984年3月26日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor of Science, Conferring organization: Tokyo Institute of Technology, Report number:甲第1584号, Conferred date:1984/3/26, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

A-82

A HOMOTOPY METHOD FOR SOLVING SYSTEMS OF POLYNOMIAL EQUATIONS

by

Shinji MIZUNO

D 1538

Supervisor: Associate Prof. Masakazu KOJIMA

December, 1983

Department of Systems Science

Tokyo Institute of Technology

A HOMOTOPY METHOD FOR SOLVING SYSTEMS OF POLYNOMIAL EQUATIONS

Contents

Notations

R : the set of real numbers.

R^m : the m-dimensional Euclidean space with the norm

$$\|z\| = (\sum_{j=1}^{m} z_j^2)^{1/2} \text{ for every } z=(z_1,z_2,\ldots,z_m) \in R^m.$$

C : the set of complex numbers.

$|Z|$: the absolute value of a number $Z \in C$.

i : the imaginary unit , i.e., $i^2 = -1$.

$\bar{Z}$: the conjugate complex number of a number $Z \in C$.

re Z : the real part of a number $Z \in C$.

img Z : the imaginary part of a number $Z \in C$.

C^m : the m-dimensional complex space with the norm

$$\|Z\| = (\sum_{j=1}^{m} Z_j\bar{Z}_j)^{1/2} \text{ for every } Z=(Z_1,Z_2,\ldots,Z_m) \in C^m.$$

int A : the interior of a subset A of R^m or C^m.

cl A : the closure of a subset A of R^m or C^m.

bd A : the boundary of a subset A of R^m or C^m.

co A : the convex hull of a subset A of R^m or C^m.

A\B : the difference set $\{ x \in A : x \notin B \}$.

$W(q,d)=r\{\cos(2\pi q/d) + i \sin(2\pi q/d)\} \in C$ for a positive constant r,

every positive integer d, and q=1,2,...,d.

$w(q,d)=(r \cos(2\pi q/d),\ r \sin(2\pi q/d) \in R^2$ for a positive constant r,

every positive integer d, and q=1,2,...,d.

$V(q,d)=\{x \in R^2\colon x=(a \cos\theta,\ a \sin\theta),\ 0 \leqq a \leqq 2r,\ |\theta-2\pi q/d| \leqq \pi/2d\}$

for a positive constant r, every positive integer d, and

q=1,2,...,d.

$J_1(k)$: the Union Jack triangulation of R^k with the grid size 1 [29].

J_3 : the Union Jack triangulation of $R^{2m} \times (0,1]$ with the continuous refinement of grid size [29].

P_d : the class of monic polynomials of degree d in one complex variable with real coefficients.

rank M: the rank of a matrix M.

det M : the determinant of a matrix M.

Chapter 1. Introduction

We shall employ the symbols R^m and C^m for the m-dimensional Euclidean and complex spaces, respectively. When m=1, we simply write R (the set of all real numbers) and C (the set of all complex numbers) instead of R^1 and C^1. We say that a map $Q:C^m \rightarrow C$ is a polynomial if Q(Z) is the sum of terms each of which has the form

$$AZ_1^{b_1}Z_2^{b_2}\ldots Z_m^{b_m}$$

where A is a real or complex constant, b_k a non-negative integer (k=1,2,...,m), and $Z=(Z_1,Z_2,\ldots,Z_m)$ is a vector of complex valued variables. The degree of the term is defined to be the sum of the b_k's; the degree of the polynomial Q is the maximum of the degrees of the terms.

Let $F=(F_1,F_2,\ldots,F_m):C^m \rightarrow C^m$ be a map such that each j-th component F_j is a polynomial. We call such a map F a polynomial map. We shall consider the system of equations

$$(1.1) \qquad F(Z)=0, \qquad Z \in C^m.$$

If we take m=1, then we have an ordinary algebraic equation in one complex variable Z, i.e.,

$$(1.1)' \quad A_0Z^d+A_1Z^{d-1}+\ldots+A_{d-1}Z+A_d = 0,$$

where the A_k $(k=0,1,\ldots,d)$ are real or complex numbers. In this paper, we study a homotopy method for finding all solutions to the system of equations (1.1) or the equation (1.1)′. The method developed in the field of fixed point and complementarity theory which was originated by Lemke and Howson [19] and Scarf [27]. For a survey of this field see, for example, Allgower and Georg [2,3]. There are two types of homotopy methods: a continuation (predictor-corrector) method and a simplicial (integer labeling or vector labeling) method. A vector labeling method is also called a PL (piecewise linear) method. We have three purposes in this paper. First, we propose an efficient PL method for finding all solutions to (1.1). Second, we report some computational results for the PL method. Third, we study a continuation method for finding all complex solutions to (1.1) in the case that all the coefficients are real. These results have already been published in Kojima and Mizuno [15], and Mizuno [21,22,23]. The paper [23] is a revised version of [22].

Kuhn reported the first application of a fixed point computing method to an algebraic equation in one complex variable. He proposed an integer labeling method for approximating a solution of an algebraic equation in [17], and later modified and improved the method so that it could efficiently approximate all solutions in [18]. Kojima, Nishino, and Arima [16] later introduced a PL homotopy method and reported some numerical experiments.

For approximating all solutions of a system of polynomial equations in several complex variables, Drexler [6] and Chow, Mallet-Paret, and Yorke [5] proposed a homotopy continuation method. Garcia and Zangwill [10,11] also showed that the basic idea commonly used in fixed point computing methods could be applicable to a certain class of systems of equations which includes systems of polynomial equations. They proposed a PL homotopy method for approximating all solutions. Their method is purely theoretical and not suitable for computer implementation.

Now we shall show an outline of a homotopy method [5,10] for finding all solutions to (1.1). Let $G=(G_1,G_2,...,G_m):C^m \rightarrow C^m$ be an auxiliary polynomial map such that all the solutions to the system of equations

$$(1.2) \quad G(Z)=0, \quad Z \in C^m$$

are known. We give an example of G in Section 2.1 (see (2.3)). The system (1.2) is then continuously deformed into the system (1.1). To achieve this, we define a homotopy $H:C^m \times [0,1] \rightarrow C^m$ between the two maps F and G by

$$(1.3) \quad H(Z,t)=(1-t)F(Z)+tG(Z) \text{ for every } (Z,t) \in C^m \times [0,1],$$

and consider the system of equations

(1.4) $H(Z,t)=0, \quad (Z,t) \in C^m \times [0,1]$.

Thus the system (1.2) is continuously deformed into the system (1.1), through the system (1.4) as the homotopy parameter t decreases from 1 to 0. Let Π denote the set of all solutions to the system (1.4), i.e.,

(1.5) $\Pi = \{(Z,t) \in C^m \times [0,1] : H(Z,t)=0\}$.

Suppose that

(1.6) 0 is a regular value of H,

and

(1.7) the set Π is bounded.

Chow, Mallet-Paret, and Yorke [5] showed that the condition (1.6) holds for almost all the complex coefficients of the polynomial maps F and G. In Section 2.1, we will propose a sufficient condition for (1.7). Under the condition (1.6), the set Π consists of a disjoint union of smooth paths. Chow, Mallet-Paret, and Yorke [5] and Garcia and Zangwill [10] proved that those paths are monotone with respect to t because the map $H(.,t)$ is analytic. Since each path is bounded and monotone, it has one end point in $C^m \times \{0\}$ and the other end point in $C^m \times \{1\}$. We immediately see that $(Y,1) \in \Pi$ for every solution Y of (1.2), and that if $(Z,0) \in \Pi$, then Z is a solution of (1.1). Therefore, if we trace Π from a known point (Y,1) until the homotopy parameter t attains 0, we can calculate a solution Z of (1.1). Repeating the same procedure from all solutions of (1.2), we compute all solutions of (1.1). This is an outline of a homotopy method.

Generally, the set Π can be very complicated. So we need a numerical approximation procedure for tracing Π. Garcia and Zangwill [10] proposed a PL approximation of Π. In the case of the PL method, we do not need the condition (1.6). In Section 2.1, we explain their idea. Although their idea gives us a theoretical foundation for approximating all the solutions of (1.1), their method is difficult to implement on a computer.
Using their idea, Mizuno [21] extended the PL homotopy method of Kojima, Nishino, and Arima [16] so that it could apply to the case of several variables. He also gave a first report of computational results for the PL homotopy method of finding all solutions to (1.1). Then Kojima and Mizuno [15] improved the PL homotopy method by utilizing a carefully chosen auxiliary map. The computational results of [15] and [21] indicate that the new method [15] is superior to the old one [21].

The above mentioned methods are applicable to polynomial maps with complex coefficients. Additionally, Mizuno

[22,23], Saigal [26], and Zangwill [34] have examined the case where all the coefficients are real. Such a case often occurs. For example, the stationary condition for minimization of a real valued polynomial objective function in several real variables turns out to be a system of equations, and global and local minima correspond to certain real solutions of the system (see Problems 3 and 4 of Section 2.4). Zangwill [34] applied a homotopy method to the minimization problem of a polynomial function. But, as pointed out in [3], his idea has a weakness. Saigal [26] proposed a PL homotopy method for finding all real solutions to (1.1)′ in some interval. Mizuno [22,23] studied the solution set

$$S = \{(Z,t) \in C^m \times (0,1): H(Z,t)=0 \}$$

in the case that all the coefficients of polynomial maps F and G are real. Note that S is the solution set of (1.4) in the open set $C^m \times (0,1)$. There are two distinctive features of the solution set S. First, it can happen that the solution set S does not form a disjoint union of smooth paths and has bifurcation points, i.e., the condition (1.6) does not hold, for any small real perturbation to the coefficients of the polynomial maps F and G. Second, the solution set enjoys a certain symmetry, i.e., if a point belongs to S then its complex conjugate does. When we trace a path numerically, the first property may cause some difficulty but the second will be useful. Mizuno [22,23] analyzed S in detail and proposed an efficient homotopy continuation method for finding all complex solutions of (1.1)′. Kawada [12] reported that the new homotopy continuation method consumed about a half of CPU time of the usual one.

The remainder of this paper is organized as follows: In Chapter 2, we review the results of Kojima and Mizuno [15]. In Section 2.1, we show an outline of the method proposed by Garcia and Zangwill [10] and a basic idea of our PL homotopy method. The PL homotopy method utilizes a carefully chosen auxiliary map and a new triangulation. In Section 2.2, we provide the auxiliary map. The map is neither analytic nor continuous. But it has trivial zero points and is linear in a neighborhood of each zero point. The linearity guarantees that zero points are unchanged by a PL approximation of the auxiliary map. In Section 2.3, we construct the new triangulation which consists of two copies of the triangulation J_3 [29]. The new triangulation was devised to increase computational efficiency. A new PL homotopy is also defined in Section 2.3. In Section 2.4, we report some computational results. In Section 2.5, we give some remarks on our algorithm and point out the difference between the PL homotopy methods of Mizuno [21] and Kojima and Mizuno [15].

In Chapter 3, we review the results of Mizuno [22,23]. In Section 3.1, we investigate the solution set S in the case of m=1 and real coefficients. The homotopy map H is represented as $H(x+iy)=h_1(x,y,t)+ih_2(x,y,t)$, where h_1 and h_2 are polynomial maps from $R^2\times[0,1]$ into R. Since all the coefficients of F and G are real, there is a polynomial map h_3 from $R^2\times[0,1]$ into R such that $h_2(x,y,t)=yh_3(x,y,t)$. Then the solution set S is divided into two sets:

$$S^R = \{(x+iy,t)\in C\times(0,1):\ h_1(x,0,t)=0\},$$
$$S^C = \{(x+iy,t)\in C\times(0,1):\ h_1(x,y,t)=0,\ h_3(x,y,t)=0\}.$$

Using this division we make the structure of S clear. In Section 3.2, we propose a new homotopy method for finding all solutions of F(Z)=0 by using the nice structure of S. In Section 3.3, we study the solution set S in the case of several variables and real coefficients.

Chapter 2. A PL method for finding all solutions

2.1. Background and basic idea

In this Section, we first outline the method proposed by Garcia and Zangwill [10], and then present the basic idea of our method. Suppose that each component F_j of a map $F=(F_1,F_2,\dots,F_m):C^m \to C^m$ has the following form:

$$(2.1)\qquad F_j(Z) = Z_j^{d_j} + P_j(Z) \quad \text{for every } Z=(Z_1,Z_2,\dots,Z_m) \in C^m,$$

where d_j is a positive integer and $P=(P_1,P_2,\dots,P_m):C^m \to C^m$ is an analytic map; P needs not be a polynomial map for the time being. Garcia and Zangwill [10] imposed the following condition on the map P:

(2.2) If $|Z_j| \to +\infty$ on a sequence, then $|P_j(Z)/Z_j^{d_j}| \to 0$ on an infinite subsequence.

For example, if each P_j is a polynomial map with a degree less than d_j, then (2.2) is satisfied.

Let r be a positive number. We define an auxiliary polynomial map $G=(G_1,G_2,\dots,G_m)$ by

$$(2.3) \qquad G_j(Z) = Z_j^{d_j} - r^{d_j} \quad \text{for } j=1,2,\ldots,m,$$

and consider the system of equations (1.2). Each j-th component equation $G_j(Z)=0$ involves only the complex variable Z_j, and it has d_j distinct solutions

$$r\{\cos(2\pi q/d_j) + i \sin(2\pi q/d_j)\} \text{ for } q=1,2,\ldots,d_j.$$

Hence the entire system (1.2) has $d_1d_2\ldots d_m$ distinct solutions. For every positive integer d and q=1,2,...,d, let

$$(2.4) \qquad W(q,d)=r\{\cos(2\pi q/d) + i \sin(2\pi q/d)\}.$$

Let

$$(2.5) \qquad \Lambda=\{(q_1,q_2,\ldots,q_m): q_j=1,2,\ldots,d_j \ (j=1,2,\ldots,m)\},$$

$$(2.6) \qquad Y_I=(W(q_1,d_1),W(q_2,d_2),\ldots,W(q_m,d_m))$$
$$\text{for every } I=(q_1,q_2,\ldots,q_m) \in \Lambda.$$

Then the set of solutions to the system (1.2) can be written as $\{Y_I : I \in \Lambda\}$. Each Y_I will serve as an initial point of the method.

Now we consider the homotopy (1.3), the system of equations (1.4), and the solution set (1.5). The condition (2.2) which we have imposed on the map F ensures the boundedness (1.7). As pointed out in the Introduction, we need a numerical approximation procedure for tracing the solution set Π. Garcia and Zangwill [10] proposed a PL approximation of Π. In their method, the m-dimensional complex space C^m is identified with the 2m-dimensional Euclidean space R^{2m}, and the homotopy H, the maps F and G in C^m are converted into the homotopy h, the maps f and g in R^{2m}, respectively. That is, we define

$$h(z,t)=(h_1(z,t),h_2(z,t),\dots,h_m(z,t)),$$
$$h_j(z,t)=(\text{ re } H_j(Z,t),\ \text{img } H_j(Z,t)) \quad (j=1,2,\dots,m),$$

for every $(z,t)=(z_1,z_2,\dots,z_m,t) \in R^{2m}\times[0,1]$ such that

$$z_j=(\text{ re } Z_j,\ \text{img } Z_j) \in R^2 \quad (j=1,2,\dots,m),$$

where re Z and img Z denote the real and imaginary part of $Z \in C$, respectively. The maps $f:R^{2m} \to R^{2m}$ and $g:R^{2m} \to R^{2m}$ are similarly induced from the maps $F:C^m \to C^m$ and $G:C^m \to C^m$. Thus we obtain the systems of equations

$$(2.7) \qquad f(z)=0, \quad z \in R^{2m},$$
$$(2.8) \qquad g(z)=0, \quad z \in R^{2m},$$
$$(2.9) \qquad h(z,t)=0, \quad (z,t) \in R^{2m}\times[0,1],$$

which are equivalent to (1.1), (1.2) and (1.4), respectively. Specifically, we see that

(2.10) the system (2.8) has exactly $d_1 d_2 \ldots d_m$ distinct solutions
$y_I = (w(q_1,d_1), w(q_2,d_2), \ldots, w(q_m,d_m))$
$(I=(q_1,q_2,\ldots,q_m) \in \Lambda)$

where

(2.11) $w(q,d) = (\text{ re } W(q,d), \text{ img } W(q,d))$
$= (r \cos(2\pi q/d), r \sin(2\pi q/d)).$

We also see that the map $h: R^{2m} \times [0,1] \rightarrow R^{2m}$ is a homotopy between the maps $f: R^{2m} \rightarrow R^{2m}$ and $g: R^{2m} \rightarrow R^{2m}$, i.e.,

$$h(z,t) = (1-t)f(z) + tg(Z) \quad \text{for every } (z,t) \in R^{2m} \times [0,1].$$

The conversion above from the systems (1.1), (1.2) and (1.4) in the complex space C^m into the systems (2.7), (2.8) and (2.9) in the Euclidean space R^{2m} makes it possible to apply the results and techniques developed in the fixed point and complementarity theory. The homotopy h has a distinctive feature: for every $t \in [0,1]$ the Jacobian of the map $h(.,t): R^{2m} \rightarrow R^{2m}$ evaluated at each $z \in R^{2m}$ is nonnegative. We can derive this feature from the fact that for every $t \in [0,1]$ the map $H(.,t): C^m \rightarrow C^m$ is analytic. We shall assume that

(2.12) 0 is a regular value of the map $f:R^{2m} \to R^{2m}$, i.e., the Jacobian of f is nonzero at each solution z of (2.7).

Since the map f=h(.,0) has a nonnegative Jacobian everywhere, (2.12) is equivalent to

(2.13) the Jacobian of f is positive at each solution z of (2.7).

The assumption (2.12) also implies that each solution of (2.7) is isolated. On the one hand, from the boundedness (1.7), we have that

(2.14) the solution set of (2.9) is bounded.

Specifically all the solutions of (2.7) lie in a compact set. Hence we obtain

(2.15) the system (2.7) has at most a finite number of solutions.

By the construction of the map g=h(.,1), we can also show that

(2.16) the Jacobian of g is positive at each solution y_I of (2.8).

The properties (2.13), (2.14), (2.15) and (2.16) will play important roles in the discussion below.

Let

$$(2.17)\quad \Delta=\{(z,t)\in R^{2m}\times[0,1] : h(z,t)=0\}.$$

Obviously, Δ can be written as

$$\Delta=\{(z_1,z_2,\ldots,z_m,t)\in R^{2m}\times[0,1] : z_j=(\text{re } Z_j,\ \text{img } Z_j)$$
$$(j=1,2,\ldots,m),\ (Z_1,Z_2,\ldots,Z_m,t)\in\Pi\},$$

where Π is defined by (1.5). If we impose the regularity condition (1.6), we can prove that Δ consists of $d_1d_2\ldots d_m$ distinct smooth paths each of which connects a known solution y_I of (2.8) with an unknown solution of (2.7). See Fig.1. In this case, one can employ continuation methods (see, for example, Allgower and Georg [2] and Chow, Mallet-Paret, and Yorke [4]). Li and Yorke [20] have reported a computational experiment of a continuation method applied to Wilkinson's polynomial. We shall approximate Δ by disjoint PL paths below. The regularity condition (1.6) will be unnecessary.

Using a uniform triangulation K^{δ} of $R^{2m}\times[0,1]$ such that each vertex lies in either $R^{2m}\times\{0\}$ or $R^{2m}\times\{1\}$, we approximate the homotopy $h:R^{2m}\times[0,1]\to R^{2m}$ by a PL homotopy $h^{\delta}:R^{2m}\times[0,1]\to R^{2m}$. Here δ represents

$$\sup_{\sigma}\max\{|u_j-v_j| : \sigma\in K^{\delta},\ (u_1,u_2,\ldots,u_{2m},t)\in\sigma,$$
$$(v_1,v_2,\ldots,v_{2m},s)\in\sigma,\ j=1,2,\ldots,2m\}.$$

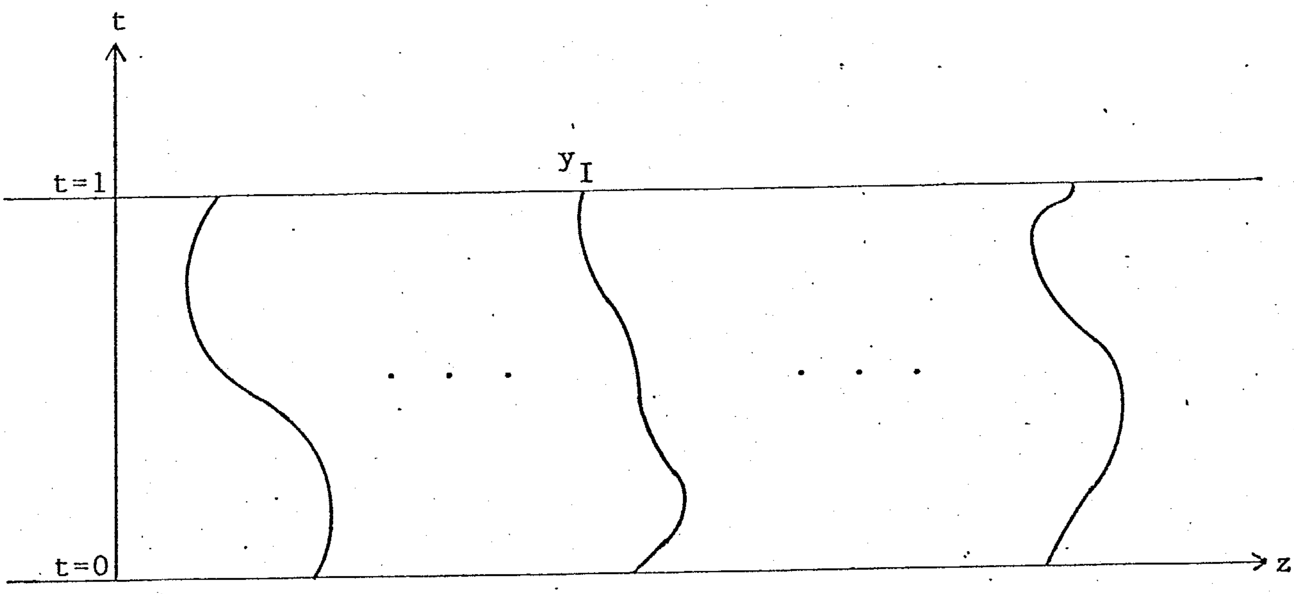


Fig.1. Solution set Δ .

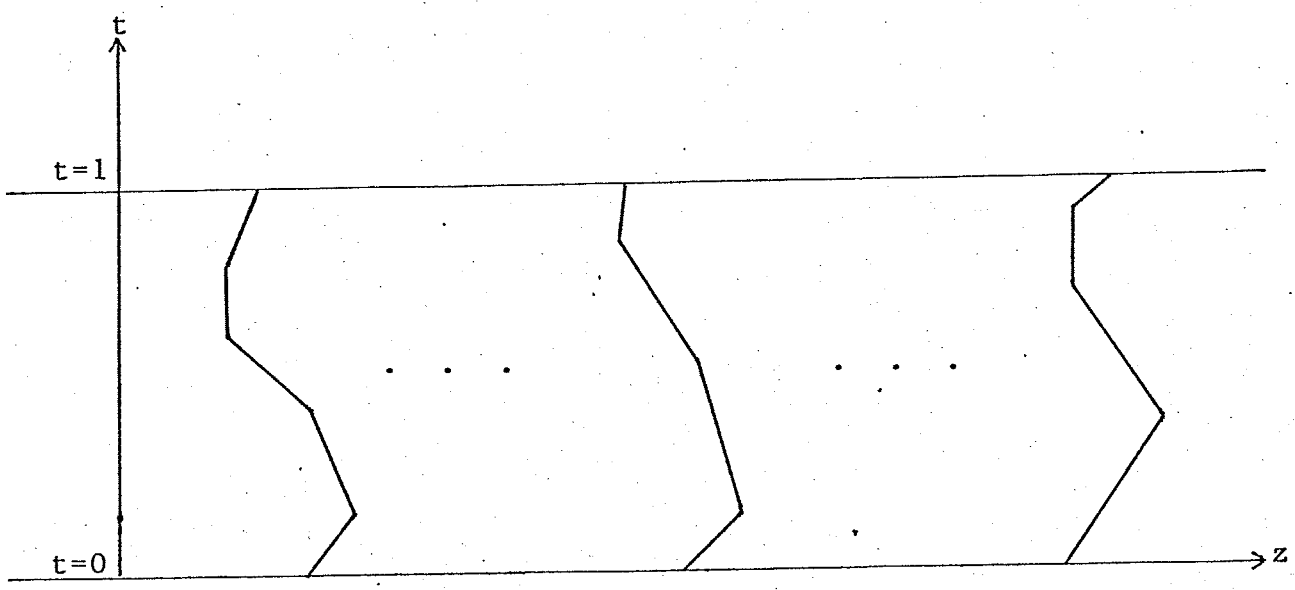


Fig.2. Solution set Δ^* .

We call δ the grid size of K^{δ}. The PL homotopy h^{δ} satisfies

(2.18) $\lim_{\delta \downarrow 0} h^{\delta}(z,t)=h(z,t)$ for every $(z,t) \in R^{2m}\times[0,1]$.

Hence, if $\delta>0$ is sufficiently small then the solution set Δ^{*} of the system

(2.19) $h^{\delta}(z,t)=0, \quad (z,t) \in R^{2m}\times[0,1]$

approximates the solution set Δ of the system (2.9). We now assume that 0 is a regular value of the PL map h^{δ}. Then, as is well known in the fixed point and complementarity theory, Δ^{*} consists of PL paths and loops which are disjoint.

Suppose that δ is sufficiently small. Then we can foresee from (2.18) that the properties (2.14) and (2.15) will be inherited from the system (2.9) by its PL approximation (2.19). More precisely, we have, for each k=0 and 1,

(2.20) the system $h^{\delta}(z,k)=0$ has the same number of solutions as the system $h(z,k)=0$, and a small neighborhood of a solution of the latter system contains a unique solution of the former system,

and

(2.21) the solution set Δ^{*} of the system (2.19) is bounded.

Furthermore, by using the properties (2.13) and (2.16) we can prove that

(2.22) each solution of $h^{\delta}(y,1)=0$ is connected with a solution of $h^{\delta}(z,0)=0$, and vice versa, by a PL path in Δ^{*} (see Fig.2).

Consequently tracing all the PL paths which start from solutions of $h^{\delta}(y,1)=0$ until we attain the hyperplane $R^{2m}\times\{0\}$, we approximate all the solutions of (2.7). This is an outline of the PL method proposed by Garcia and Zangwill [10].

In order to implement the method described above on a computer, we need to determine a grid size δ for which (2.20), (2.21), and (2.22) hold. However, they did not show how to determine such a grid size. Since the map $h^{\delta}(.,1):R^{2m} \rightarrow R^{2m}$ is a PL approximation of the map $g:R^{2m} \rightarrow R^{2m}$ which has been induced from the simple auxiliary polynomial map G given by (2.3), it is possible to calculate a grid size δ such that (2.20) holds for k=1. In fact, we can apply the results on PL approximation of PC^{1} maps given by Kojima [14]. See also Kojima, Nishino, and Arima [16] and Mizuno [21].

It seems difficult, however, to calculate a grid size δ which ensures (2.20) for k=0 and/or (2.22) unless we know detailed information about the Jacobian matrix of the map f in a neighborhood of each solution of the system f(z)=0 (see Kojima [14] and Saigal [25]). Furthermore, if two distinct solutions of f(z)=0 are close to each other, we are forced to choose the grid size δ small enough to separate them. In such a case, we have to consume a great number of pivot operations to trace PL paths. Therefore, we must say that the method involves these difficulties and computational inefficiencies although it certainly gives us a theoretical foundation for approximating all the solutions of F(Z)=0. In their paper [11], Garcia and Zangwill extended the method to a more general class of systems of equations. But the difficulties pointed out above have not been solved.

In our approach, we shall overcome those difficulties by utilizing Eaves-Saigal continuous deformation technique [7]. This technique was effectively used in the method proposed by Kojima, Nishino, and Arima [16] for approximating all solutions of an algebraic equation in one complex variable. Our method is based on the Master Thesis by Mizuno [21] in which he extended the method to systems of polynomial equations in several variables. Another important feature of our method is the use of a carefully chosen auxiliary map $\hat{g}:R^{2m} \rightarrow R^{2m}$ with the property that the system $\hat{g}(z)=0$ has the same solutions as the system $g(z)=0$. The map $\hat{g}$ is neither analytic nor continuous. But it is linear in a neighborhood of each solution of the system $\hat{g}(z)=0$. In Mizuno [21], $g(z)=0$ has been used as an auxiliary system. Compared with the map $g:R^{2m} \rightarrow R^{2m}$ naturally induced from the simple polynomial map (2.3), we can take a larger grid size δ such that (2.20) holds for k=1 (see Section 2.5).

Now suppose that $F=(F_1,F_2,\ldots,F_m):C^m \to C^m$ is a polynomial map. For each $j=1,2,\ldots,m$, let d_j denote the degree of F_j and $\hat{F}_j$ the polynomial map which is the sum of all terms of F_j with the degree d_j. In the remainder of this chapter, we shall assume the condition that

(2.23) for each fixed $t \in [0,1]$, the system of equations
$$tZ_j^{d_j} + (1-t)\hat{F}_j(Z) = 0 \quad (j=1,2,\ldots,m)$$
has only the trivial solution $Z=0$.

It can be readily verified that if the polynomial map F has the form (2.1) for some polynomial map $P=(P_1,P_2,\ldots,P_m)$ then the condition (2.2) implies (2.23) above.

Taking $t=0$ in (2.23), we see that the system $\hat{F}(Z)=0$ has only the trivial solution $Z=0$. This ensures that the system $F(Z)=0$ has exactly $d_1 d_2 \cdots d_m$ distinct solutions if the assumption (2.12) is satisfied (Garcia and Li [9]). The condition (2.23) also implies (2.14).

In our method, we shall construct a PL homotopy map $\phi^{\delta}:R^{2m}\times(0,s^*] \to R^{2m}$ which satisfies the properties (2.24), (2.25), (2.27), and (2.28) below, where s^* is a positive number.

(2.24) The system $\phi^{\delta}(z,s^*)=0$ has exactly $d_1 d_2 \cdots d_m$ distinct known solutions y_I $(I \in \Lambda)$.

(2.25) $\lim_{t \downarrow 0} \phi^{\delta}(z,t)=f(z)$ for every $z \in R^{2m}$.

Let

(2.26) $X=\{(z,t) \in R^{2m}\times(0,s^*]:\ \phi^{\delta}(z,t)=0\}.$

By (2.24), we see $(y_I,s^*) \in X$ for every $I \in \Lambda$. Let X_I denote the connected component of X which contains (y_I,s^*). As is well known, each X_I forms a PL path under a regularity assumption.

(2.27) Each path X_I starting from (y_I,s^*) converges to some $(z^*,0)$.

It follows from (2.25) that z^* is a solution of $f(z)=0$.

(2.28) For every solution z^* of $f(z)=0$, there is a unique path X_I which converges to $(z^*,0)$.

Therefore, tracing all the PL paths by applying complementary pivoting to the PL system $\phi^{\delta}(z,t)=0$, we can calculate approximations of all the solutions to $f(z)=0$.

2.2. An auxiliary map and its PL approximation

For every positive integer d and q=1,2,...,d, let

$$V(q,d)=\{x \in R^2 : x=(a\cos\theta,\ a\sin\theta),\ 0\leqq a\leqq 2r,\ |\theta-2\pi q/d|\leqq \pi/2d\}.$$

Fig.3 shows the sets V(q,d) for d=4 and q=1,2,3,4. Each V(q,d) forms a closed fan shaped neighborhood of the point w(q,d) defined by (2.11). It is easily verified that the minimum distance from the point w(q,d) to the boundary of V(q,d) is $r\sin(\pi/2d)$, i.e.,

$$(2.29)\quad \inf\{\|x-w(q,d)\| : x \notin V(q,d)\} = r\sin(\pi/2d).$$

Let

$$(2.30)\quad \psi(x;1)=x-(r,0) \quad \text{for every } x \in R^2,$$

and for every integer $d\geqq 2$, define the map $\psi(.;d):R^2 \to R^2$ by

$$\psi(x;d)=\begin{cases}(a\cos(\theta-2\pi q/d)-r,\ a\sin(\theta-2\pi q/d)) \\ \quad \text{if } x=(a\cos\theta,\ a\sin\theta) \in V(q,d) \text{ for some } q=1,2,\dots,d, \\ (a^d\cos(d\theta),\ a^d\sin(d\theta)) \\ \quad \text{if } x=(a\cos\theta,\ a\sin\theta) \notin V(q,d) \text{ for any } q=1,2,\dots,d.\end{cases}$$

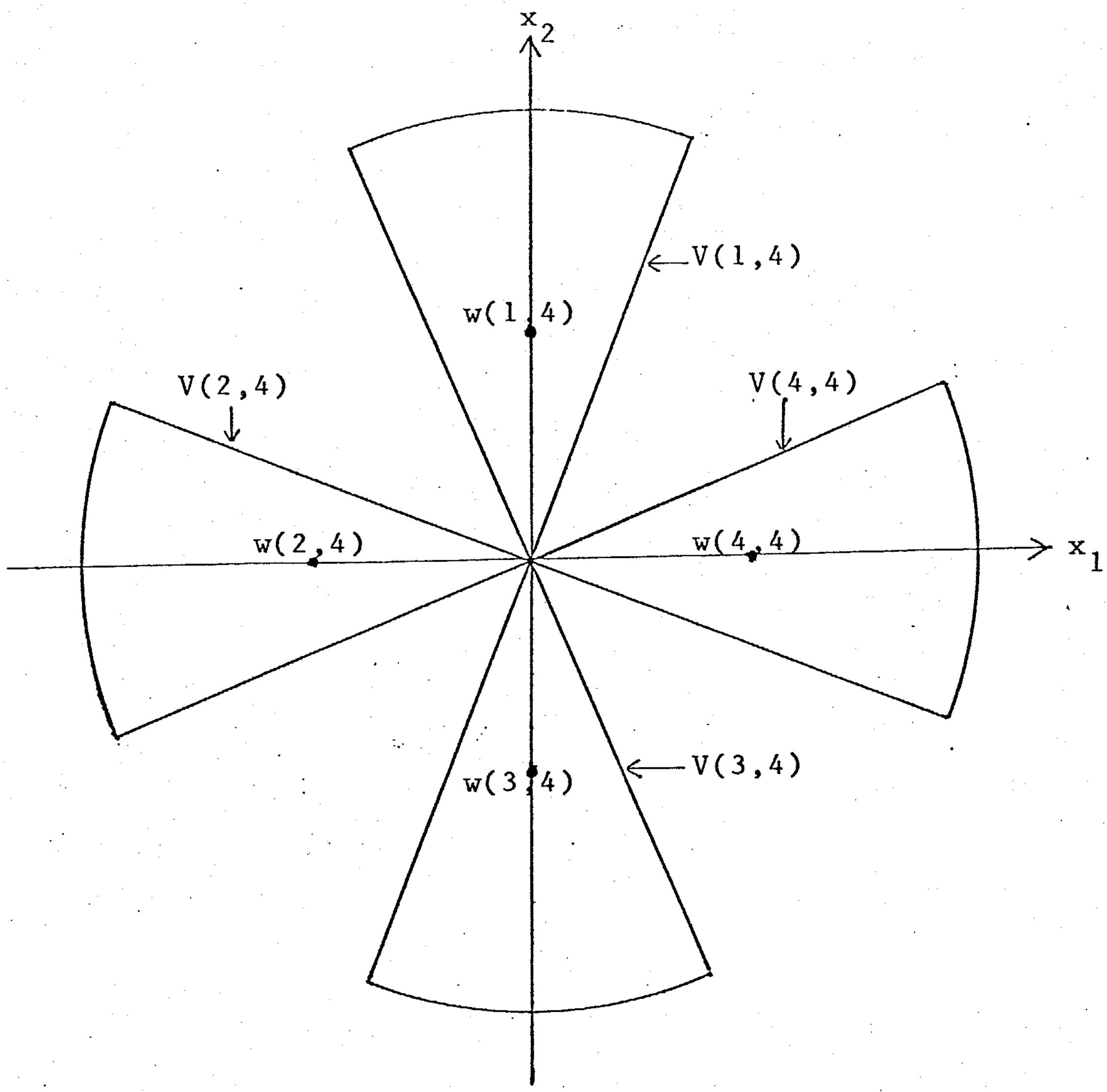


Fig.3. $V(q,4)$, $q=1,2,3,4$.

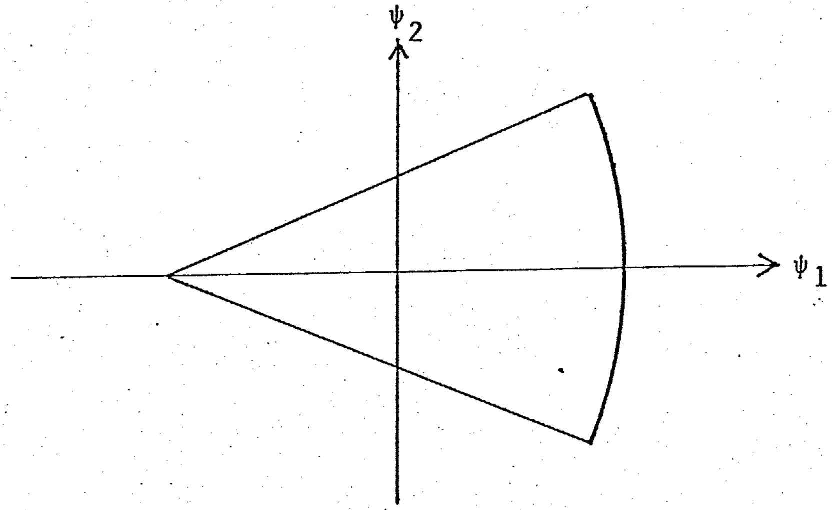


Fig.4. The image of $V(q,4)$ $(q=1,2,3,4)$ under the map $\psi(\cdot;4)$.

Fig.4 illustrates the image of the sets V(q,4) for q=1,2,3,4 under the map $\psi(.;4)$; each V(q,d) is mapped onto the fan shaped region which is congruent to V(q,d) itself. More precisely, we have

$$(2.31)\quad \psi(x;d)=\begin{bmatrix}\cos(2\pi q/d) & \sin(2\pi q/d)\\ -\sin(2\pi q/d) & \cos(2\pi q/d)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\begin{bmatrix}r\\ 0\end{bmatrix}$$

for every $x=(x_1,x_2)\in V(q,d)$

(d=1,2,... ; q=1,2,...,d).

Hence $\psi(.;d)$ is affine on each V(q,d). Note that the 2X2 matrix appearing in the right side of (2.31) is an orthogonal matrix with the determinant 1. As for points x outside of V(q,d) for any q=1,2,...d, we have that

$$(2.32)\quad \psi(x;d)=(\mathrm{re}\ X^d,\ \mathrm{img}\ X^d)\quad \text{if } x=(\mathrm{re}\ X, \mathrm{img}\ X),\ X\in C$$

(d=2,3,...).

By (2.30), (2.31) and (2,32), we further obtain that

$$(2.33)\quad \{x\in R^2:\ \psi(x;d)=0\} = \{w(q,d):\ q=1,2,\dots,d\}\quad (d=1,2,\dots)$$

and that

(2.34) the Jacobian of the map $\psi(.;d)$ at $w(q,d)$ is $+1$

$(d=1,2,\ldots\ ;\ q=1,2,\ldots,d)$.

Now we are ready to construct the map $\hat{g}:R^{2m} \to R^{2m}$. Let d_j be the degree of the j-th component F_j of the polynomial map $F=(F_1,F_2,\ldots,F_m):C^m \to C^m$ $(j=1,2,\ldots,m)$. For every $z=(z_1,z_2,\ldots,z_m) \in R^{2m}$, define

(2.35) $$\hat{g}_j(z)=\psi(z_j;d_j) \quad (j=1,2,\ldots,m),$$
$$\hat{g}(z)=(\hat{g}_1(z),\hat{g}_2(z),\ldots,\hat{g}_m(z)).$$

Let $J_1(k)$ denote the Union Jack triangulation [29] of the k-dimensional Euclidean space with the grid size 1. For each $\rho>0$, let

$$\rho J_1(k)=\{\{\rho x:\ x \in \sigma\}:\ \sigma \in J_1(k)\}.$$

Using the triangulation $\rho J_1(2m)$ of R^{2m}, we shall approximate the map $\hat{g}:R^{2m} \to R^{2m}$ by a PL map $\hat{g}^{\rho}:R^{2m} \to R^{2m}$ in a standard way; let $\hat{g}^{\rho}(v)=\hat{g}(v)$ for every vertex v of $\rho J_1(2m)$ and then extend the map $\hat{g}^{\rho}$ affinely on each 2m-dimensional simplex σ of $\rho J_1(2m)$. The remainder of this section is devoted to proving the following result.

Theorem 1. Suppose

(2.36) $0 < \rho \leqq (r/\sqrt{2}) \sin(\pi/2d_j)$ $(j=1,2,\ldots,m)$.

Then we have

(2.37) $\{z \in R^{2m}: \hat{g}^{\rho}(z)=0\} = \{y_I: I \in \Lambda\}$

and that

(2.38) if a 2m-dimensional simplex $\sigma \in \rho J_1(2m)$ contains a y_I for some $I \in \Lambda$, then the Jacobian of the affine map $\hat{g}^{\rho} | \sigma$ (the restriction of $\hat{g}^{\rho}$ to σ) is one.

(See (2.5), (2.10) and (2.11) for the definition of Λ and y_I)

Proof. Let $\phi^{\rho}(.;d_j):R^2 \to R^2$ denote the PL approximation of the map $\phi(.;d_j)$ on the triangulation $\rho J_1(2)$ of R^2 $(j=1,2,\ldots,m)$. Then we have

(2.39) $\hat{g}^{\rho}_j(z)=\phi^{\rho}(z_j;d_j)$
for every $z=(z_1,z_2,\ldots,z_m) \in R^{2m}$ $(j=1,2,\ldots,m)$.

This equality follows from the structure of the Union Jack triangulation $J_1(2m)$. Hence we see

$$\{z \in R^{2m}: \hat{g}^{\rho}(z)=0\} = \{z \in R^{2m}: z=(z_1,z_2,\ldots,z_m), \phi^{\rho}(z_j;d_j)=0,\ j=1,2,\ldots,m\}.$$

On the other hand, it follows from the definition of y_I $(I \in \Lambda)$ (see (2.10)) that

$$\{y_I: I \in \Lambda\} = \{z \in R^{2m}: z=(z_1,z_2,\dots,z_m),\ z_j=w(q,d_j),$$
$$q=1,2,\dots,d_j,\quad j=1,2,\dots,m\}.$$

Therefore, in order to establish (2.37) and (2.38), we only need to show

(2.37)′ $\{x \in R^2: \psi^{\rho}(x;d)=0\} = \{w(q,d): q=1,2,\dots,d\}$ $(d=1,2,\dots)$

and that

(2.38)′ if a 2-dimensional simplex $\sigma \in \rho J_1(2)$ contains a $w(q,d)$ for some $q=1,2,\dots,d$, then the Jacobian of the affine map $\psi^{\rho}(.;d)|\sigma$ is one $(d=1,2,\dots)$.

If $d=1$, then, by (2.30), we have

$$\psi^{\rho}(x;1)=\psi(x;1)=x-(r,0) \quad \text{for every } x \in R^2.$$

Thus (2.37)′ and (2.38)′ follow directly. Hence we have only to deal with the case $d \geqq 2$. By (2.31), we first observe that

(2.40) if $\sigma \in \rho J_1(2)$ is contained in $V(q,d)$ for some $q=1,2,\dots,d$, then $\psi^{\rho}(x;d)=\psi(x;d)$ for every $x \in \sigma$.

Since the diameter of each $\sigma \in \rho J_1(2)$ is not greater than $r\sin(\pi/2d)$ by the assumption (2.36) and the minimum distance from $w(q,d)$ to the boundary of $V(q,d)$ is $r\sin(\pi/2d)$ (see(2.29)), we see if $w(q;d) \in \sigma \in \rho J_1(2)$, then $\sigma \subset V(q,d)$. Hence, by (2.34) and (2.40), we obtain (2.38)′. Furthermore, it follows from (2.33) and the argument above that

$$\{x \in R^2: \ \psi^{\rho}(x;d)=0\} \supset \{w(q,d): \ q=1,2,\ldots,d\}.$$

What we have left is to show

$$\{x \in R^2: \ \psi^{\rho}(x;d)=0\} \subset \{w(q,d): \ q=1,2,\ldots,d\}.$$

Suppose that $\psi^{\rho}(x';d)=0$. Let σ' be a 2-dimensional simplex of $\rho J_1(2)$ which contains x'. If $\sigma' \subset V(q,d)$ for some $q=1,2,\ldots,d$, then by (2.33) and (2.40), x' must coincide with $w(q,d)$. Hence, it suffices to derive a contradiction from the assumption

(2.41) $\quad \sigma' \not\subset V(q,d) \quad$ for any $q=1,2,\ldots,d$.

Let

$$U(p)=\{x \in R^2: \ \psi(x;d)\cdot p > 0\} \quad \text{for every } p \in R^2,$$

where $p \cdot q$ denotes the inner product of $p \in R^2$ and $q \in R^2$. Then for every nonzero $p \in R^2$, we have

$$(2.42)\quad U(p) \bigcup \mathrm{cl}\ U(-p) = \mathrm{cl}\ U(p) \bigcup U(-p) = R^2,$$
$$\mathrm{bd}\ U(p) = \mathrm{bd}\ U(-p),$$

where cl A and bd A denote the closure and boundary of a set A, respectively. On the one hand, if $\sigma' \subset U(p)$ for some nonzero $p \in R^2$, then we would have $\psi^{\rho}(x';d) \cdot p > 0$, a contradiction to $\psi^{\rho}(x';d)=0$. Hence $\sigma' \not\subset U(p)$ and $\sigma' \not\subset U(-p)$ for any nonzero $p \in R^2$, which together with (2.42) imply

$$(2.43)\quad \sigma' \bigcap \mathrm{bd}\ U(p) \neq \phi \text{ for any nonzero } p \in R^2.$$

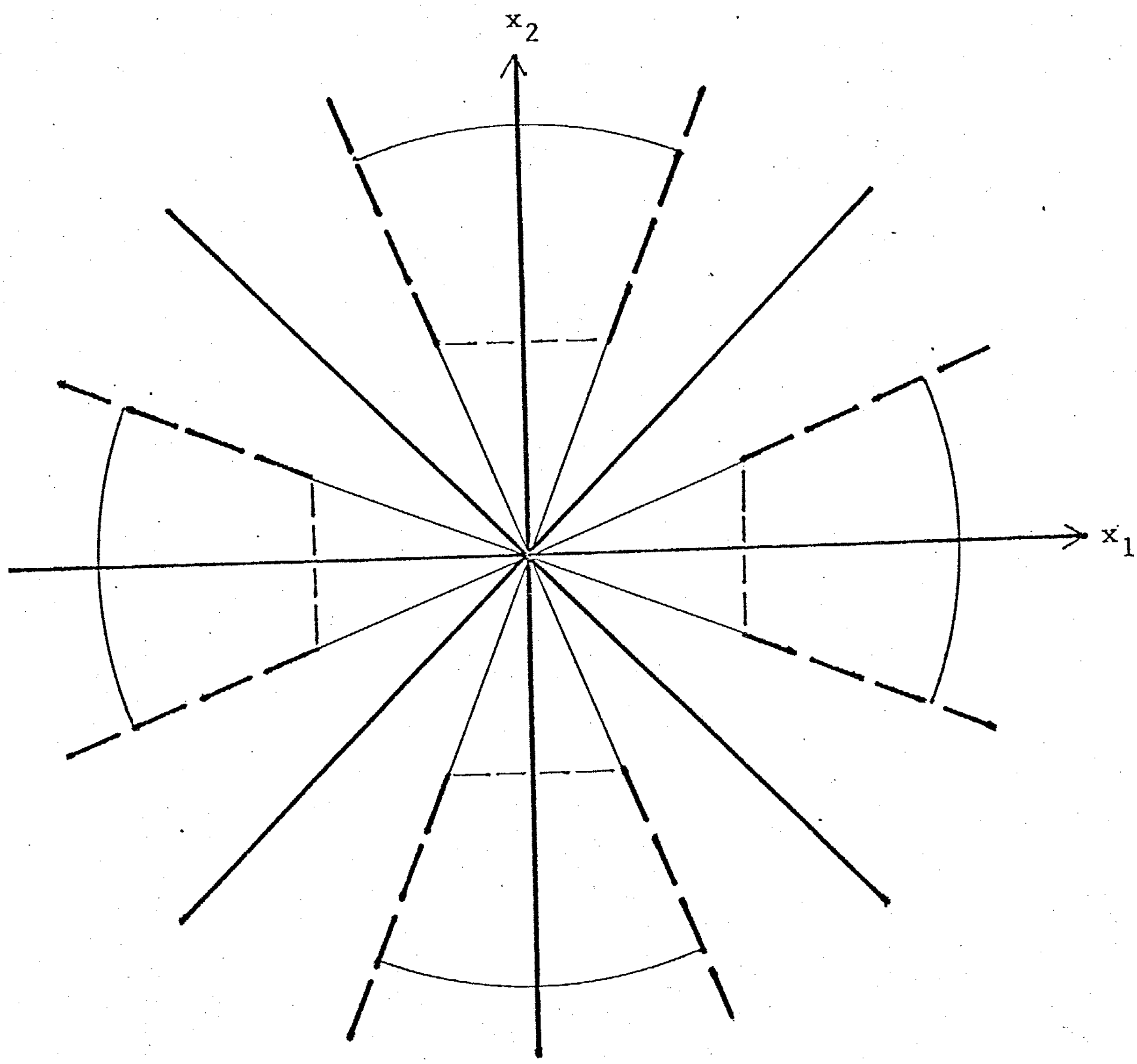
Specifically, we have

$$(2.44)\quad \sigma' \bigcap \mathrm{bd}\ U((1,0)) \neq \phi \text{ and } \sigma' \bigcap \mathrm{bd}\ U((0,1)) \neq \phi.$$

Let

$$B_1 = \mathrm{bd}\ U((1,0)) \bigcap \{\bigcup_{q=1}^{d} \mathrm{int}\ V(q,d)\},$$
$$B_2 = \mathrm{bd}\ U((1,0)) \setminus B_1.$$

See Fig.5. By a simple calculation, we see



: bd U((0,1)).

: B_1.

: B_2.

Fig.5. bd U((0,1)), B_1 and B_2 (d=4).

$$\min\{\|x-y\| : x \in \mathrm{bd}\, U((0,1)),\ y \in B_2\}$$
$$= r \tan(\pi/2d) > r \sin(\pi/2d).$$

Since the diameter of each $\sigma \in \rho J_1(2)$ is not greater than $r \sin(\pi/2d)$, it follows from the inequality above that if $\sigma \cap \mathrm{bd}\, U((0,1)) \neq \phi$, then $\sigma \cap B_2 = \phi$. Hence, by (2.44) and the definition of B_1 and B_2 above, we must have

$$\sigma' \cap \mathrm{bd}\, U((0,1)) \neq \phi \quad \text{and} \quad \sigma' \cap B_1 \neq \phi.$$

From $\sigma' \cap B_1 \neq \phi$, we can assume without loss of generality that

$$\sigma' \cap \mathrm{bd}\, U((1,0)) \cap \mathrm{int}\, V(d,d) \neq \phi.$$

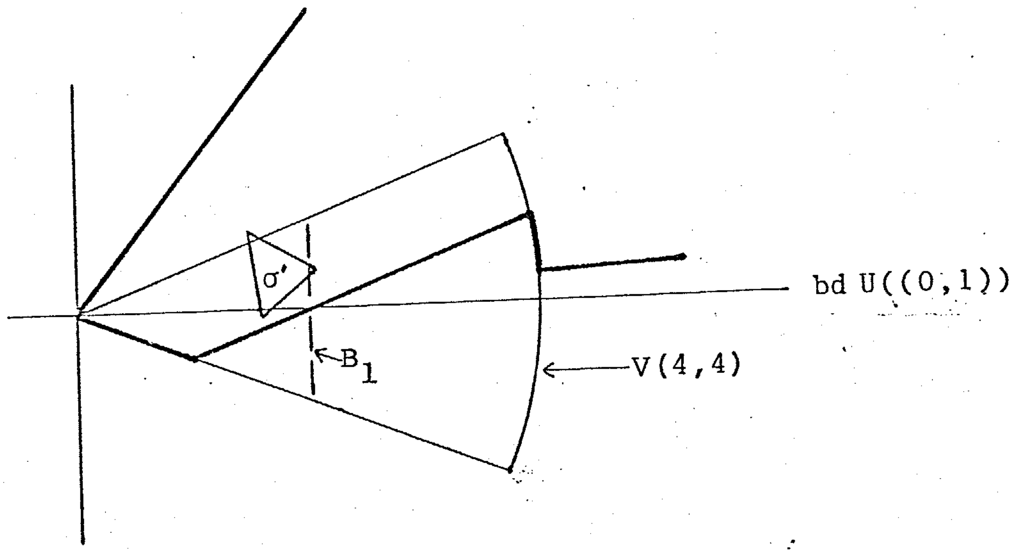
Thus we obtain the situation illustrated by (a) or (b) of Fig.6. Finally, we can easily verify that if (a) occurs, then

$$\sigma' \cap \mathrm{bd}\, U((-\sin(\pi/2d),\ \cos(\pi/2d))) = \phi$$

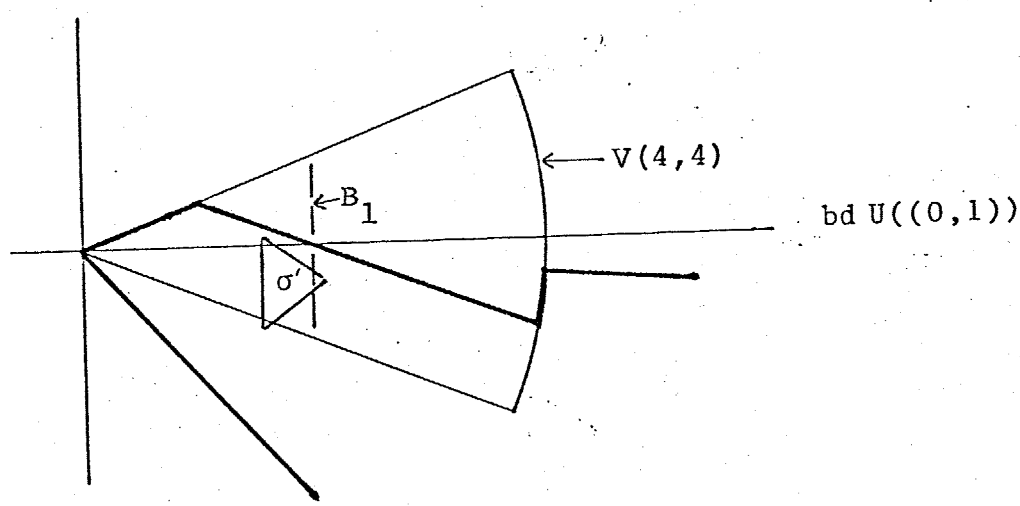
and that if (b) occurs then

$$\sigma' \cap \mathrm{bd}\, U((\sin(\pi/2d),\ \cos(\pi/2d))) = \phi.$$

This contradicts (2.43), and completes the proof.



(a) ——— : bd U((−sin(π/2d) , cos(π/2d))).



(b) ——— : bd U((sin(π/2d), cos(π/2d))).

Fig.6.

2.3. A PL homotopy

In order to construct the PL homotopy $\phi^{\delta}: R^{2m} \times (0, s^*] \rightarrow R^{2m}$ which satisfies (2.24), (2.25), (2.27) and (2.28), we need a triangulation with a continuous refinement of grid size. Here we shall make use of the triangulation J_3 of $R^n \times (0,1]$ given by Todd [29]. First we introduce some notations. Let

$$t_k = 2^{-k} \quad (k=0,1,2,\dots),$$
$$s_k = 3-t_k \quad (k=0,1,2,\dots),$$
$$S^1 = \{s_k : k=0,1,2,\dots\},$$
$$S^2 = \{1,2\},$$
$$S^3 = \{t_k : k=0,1,2,\dots\},$$
$$S = S^1 \bigcup S^2 \bigcup S^3.$$

Let σ be an (n+1)-dimensional simplex of J_3. Then there exists a nonnegative integer k such that every vertex of σ lies in either the hyperplane $R^n \times \{t_k\}$ or $R^n \times \{t_{k+1}\}$. Hence the vertices of σ are written as

$$(u^0, t_k), \dots, (u^p, t_k),$$
$$(u^{p+1}, t_{k+1}), \dots, (u^{n+1}, t_{K+1}).$$

Let σ^* denote the convex hull of the points

$$(u^0, s_k), \ldots, (u^p, s_k),$$
$$(u^{p+1}, s_{k+1}), \ldots, (u^{n+1}, s_{k+1}).$$

Obviously, we see that σ^* forms an (n+1)-dimensional simplex in R^{n+1}. If J is a triangulation of a subset of R^{n+1} and a, b are real numbers with a<b, we shall employ the symbols J[a,b] for the collection $\{\sigma \in J: \sigma \subset R^n \times [a,b]\}$ of (n+1)-dimensional simplices.

Let k_1 be a nonnegative integer and $s^* = s_{k_1}$. Define the collection L^1, L^2, L^3 and L of (n+1)-dimensional simplices as follows:

$$L^1 = \{\sigma^*: \sigma \in J_3[t_{k_1}, 1]\},$$
$$L^2 = J_1(n+1)[1,2],$$
$$L^3 = J_3,$$
$$L = L^1 \cup L^2 \cup L^3.$$

Fig.7 illustrates L for the case n=1 and k_1=2. L^1, L^2, L^3 and L are triangulations of $R^n \times [2, s^*]$, $R^n \times [1,2]$, $R^n \times (0,1]$ and $R^n \times (0, s^*]$, respectively. (When k_1=0, we have $L^1 = \phi$.) Each vertex of L^j lies in the hyperplane $R^n \times \{s\}$ for some $s \in S^j$ (j=1,2,3). We also see that the intersection of the triangulation L with each hyperplane $R^n \times \{s\}$ $(s \in S)$ induces a Union Jack triangulation with the dimension n, which we will

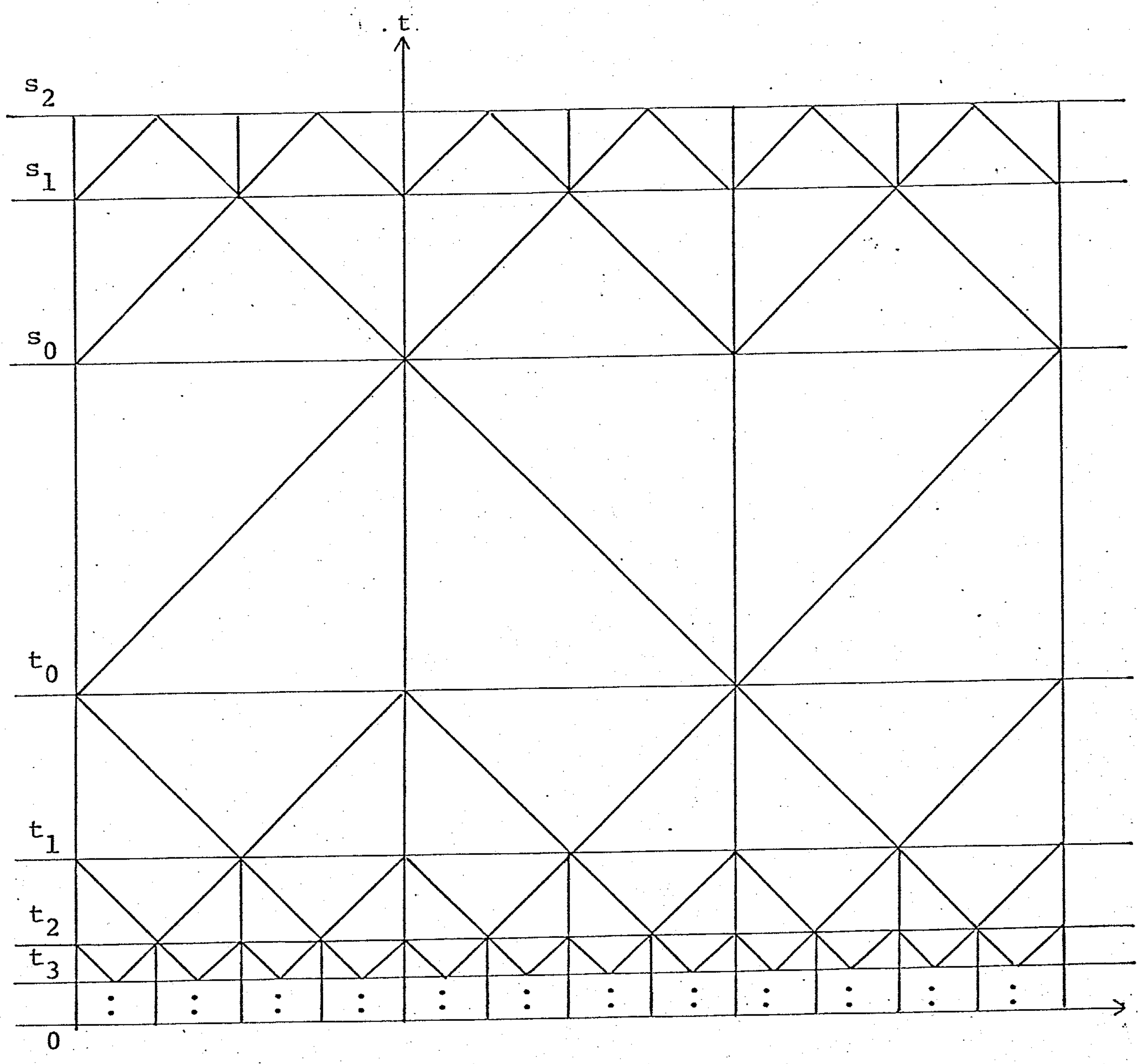


Fig.7. The triangulation L for the case $n=1$ and $k_1=2$.

denote by $L(s)$. Specifically, $L(s^*)$ coincides with the Union Jack triangulation with the grid size 2^{-k_1}, i.e.,

$$\text{(2.45)} \quad \text{grid } L(s^*) = 2^{-k_1}.$$

It should be noted that the grid size of $L(s)$ increases as s decreases from s^* to 2, and it decreases to 0 as s decreases from 1 to 0;

$$\lim_{s \downarrow 0} \text{grid } L(s) = 0.$$

These properties will contribute to the computational efficiency of our method.

For every positive number δ, let

$$\delta L = \{\{(\delta z, t): (z,t) \in \sigma\}: \sigma \in L\},$$
$$\delta L(s) = \{\{(\delta z, t): (z,t) \in \sigma\}: \sigma \in L(s)\} \quad (s \in S),$$
$$\delta L^j = \{\{(\delta z, t): (z,t) \in \sigma\}: \sigma \in L^j\} \quad (j=1,2,3).$$

That is, δL ($\delta L(s)$ or δL^j) is a triangulation which is a reduction (if $\delta<1$) or expansion (if $\delta>1$) of L ($L(s)$ or L^j) along the first n axes $z_1, z_2, \ldots, z_n$. It follows from (2.45) that

$$\text{grid } \delta L(s^*) = 2^{-k_1} \delta.$$

We are now ready to construct the PL homotopy map ϕ^{δ}: $R^{2m} \times (0, s^{*}] \to R^{2m}$. Let n=2m. For every vertex (u,s) of δL, define

$$\phi^{\delta}(u,s) = \begin{cases} \hat{g}(u) & \text{if } s \in S^1, \\ f(u) & \text{if } s \in S^3, \end{cases}$$

and extend the map ϕ^{δ} affinely on each simplex σ of δL;

$$\phi^{\delta}(z,t) = \sum_{j=0}^{2m+1} \lambda_j \phi^{\delta}(u^j, s^j)$$

for every $(z,t) \in \sigma$ satisfying

$$(z,t) = \sum_{j=0}^{2m+1} \lambda_j (u^j, s^j), \quad \sum_{j=0}^{2m+1} \lambda_j = 1,$$

$$\lambda_j \geqq 0 \quad (j=0,1,\ldots,2m+1),$$

where $(u^0,s^0),(u^1,s^1),\ldots,(u^{2m+1},s^{2m+1})$ are vertices of σ.

Theorem 2. Suppose that conditions (2.12), (2.23), and

(2.46) $0 < \rho = 2^{-k_1}\delta \leqq (r/\sqrt{2})\sin(\pi/2d_j) \quad (j=1,2,\ldots,m)$,

hold and also that the set X defined by (2.26) does not intersect with any face of dimension less than 2m of any $\sigma \in \delta L$ (regularity condition). Then (2.24), (2.25), (2.27), and (2.28) hold.

Proof. By construction, we know

$$(2.47)\quad \hat{g}^{\rho}(z) = \phi^{\delta}(z,s^*) \quad \text{for every } z \in R^{2m}.$$

Hence (2.24) follows directly from Theorem 1. Suppose that $(z,s) \in \sigma$ and $s \in (0,1]$. Then, by the definition of the map ϕ^{δ}, we see

$$\phi^{\delta}(z,s) \in \mathrm{co}\{f(u): (u,t) \in \sigma\},$$

where co A denotes the convex hull of a set A. Hence $\phi^{\delta}(z,s)$ converges to f(z) as $s \to +0$ because the diameter of the simplex σ converges to zero as $s \to +0$ and f is continuous.

Since $(y_I,s^*) \in X$ is a boundary point of the set $R^{2m} \times (0,s^*]$, we know, under the regularity condition, that each connected component X_I of X which contains (y_I,s^*) forms a PL path. By using a similar argument as in [16] we can prove (2.27) and (2.28). We shall outline the proof below. The details are omitted here.

(i) We shall show that $X \subset \Gamma \times (0,s^*]$ for some compact subset Γ of R^{2m}. Assume on the contrary that there exists a sequence $\{(z^p,s^p)\} \in X$ such that $\|z^p\| \to +\infty$ as $p \to +\infty$. Let σ^p be a (2m+1)-dimensional simplex of δL which contains (z^p,s^p) (p=1,2,...), and let

$$\tau^p = \{z \in R^{2m}: (z,s) \in \sigma^p\} \quad (p=1,2,\ldots).$$

Then, by the construction of the PL homotopy ϕ^{δ}, we can find a sequence $\{t^p\} \in [0,1]$ such that

$$0 \in t^p\{co\ \hat{g}(\tau^p)\} + (1-t^p)\{co\ f(\tau^p)\}$$

or equivalently

$$(2.48) \quad 0 \in t^p\{co\ \hat{g}_j(\tau^p)\} + (1-t^p)\{co\ f_j(\tau^p)\} \qquad (j = 1,2,\ldots,m).$$

For $p=1,2,\ldots$, let $Z(p)=(Z(p)_1,Z(p)_2,\ldots,Z(p)_m)$ be a point in C^m such that $z_j^p=(\text{re } Z(p)_j, \text{ img } Z(p)_j)$ $(j=1,2,\ldots,m)$, and T^p be a subset of C^m such that $\tau^p=\{(z_1,z_2,\ldots,z_m):$ $z_j=(\text{re } Z_j, \text{img } Z_j)$, $(Z_1,Z_2,\ldots,Z_m) \in T^p\}$. For every $j=1,2,\ldots,m$, let $\hat{G}_j$ be a map from C^m into C such that $\hat{g}_j(z)=(\text{re } \hat{G}_j(Z), \text{ img } \hat{G}_j(Z))$ if $z=(z_1,z_2,\ldots,z_m) \in R^{2m}$, $Z=(Z_1,Z_2,\ldots,Z_m) \in C^m$ and $z_k=(\text{re } Z_k, \text{ img } Z_k)$ for $k=1,2,\ldots,m$. Then we can convert the inclusion relation (2.48) in the Euclidean space into the relation in the complex space

$$(2.49) \quad 0 \in t^p\{co\ \hat{G}_j(T^p)\} + (1-t^p)\{co\ F_j(T^p)\} \qquad (j = 1,2,\ldots,m).$$

Since the diameter of each subset T^p of C^m is bounded by $\sqrt{2m}\,\delta$ and $t^p \in [0,1]$ ($p=1,2,\ldots$), by taking an appropriate subsequence if necessary, we can assume without loss of generality that all points in the set $\{Z/\|Z(p)\| : Z \in T^p\}$ converge to a common nonzero $\tilde{Z}$ as $p\to+\infty$ and that $\{t^p\}$ converges to some $\tilde{t} \in [0,1]$ as $p\to+\infty$. Recall that the degree of each F_j is d_j and $\hat{F}_j$ denotes the polynomial which is the sum of all the terms of F_j with the degree d_j. Also we have $\hat{G}_j(Z)=Z_j^{d_j}$ (if $d_j>1$) or Z_j-r (if $d_j=1$) for every $Z \in C^m$ with $|Z_j|>2r$. Hence, dividing (2.49) by $\|Z(p)\|^{d_j}$ and taking the limit as $p\to+\infty$, we obtain

$$\tilde{t}\tilde{Z}_j^{d_j} + (1-\tilde{t})\hat{F}_j(\tilde{Z}) = 0 \quad (j=1,2,\ldots,m).$$

This contradicts the assumption (2.23). Thus we have shown

(2.50) $\quad X \subset \Gamma \times (0,s^*]$ for some compact subset Γ of R^{2m}.

(ii) We have seen in Section 2.1 that (2.13) and (2.15) hold under the assumptions (2.12) and (2.23). Let $z_1,z_2,\ldots,z_p$ denote all the distinct solutions to the system $f(z)=0$. By applying the results on PL approximations of smooth mappings given in Section 3 of Kojima [14], we can find a $\hat{s} \in S^3$ such that

(2.51) $\quad X \cap R^{2m}\times(0,\hat{s}]$ consists of p paths, say $X_1', X_2', \ldots, X_p'$ which are disjoint with each other, (2.52) each path X_j' converges to $(z_j, 0)$,

and that

(2.53) if $(z, \hat{s}) \in X_j'$ for some $j=1,2,\ldots,p$ (or equivalently $\phi^{\delta}(z,\hat{s})=0$), then the Jacobian of the map $\phi^{\delta}(.,\hat{s})$: $R^{2m} \to R^{2m}$ at z is positive.

Let X_j^* denote the connected component of X which contains X_j' $(j=1,2,\ldots,p)$. Then it follows from (2.53) and the regularity assumption that

$$X_j^* \cap X_k^* = \phi \quad \text{if} \quad j \neq k.$$

This can be proved by applying the index (or orientation) theory (Eaves and Scarf [8]). Furthermore, since $X_j^* \subset X \subset \Gamma \times (0, s^*]$, each path X_j^* must originate from a point in $R^{2m} \times \{s^*\}$. This implies that each X_j^* coincides with some X_I, and we have shown (2.28).

(iii) By Theorem 1 and the equality (2.47), we see that the Jacobian of the map $\phi^{\delta}(.,s^*):R^{2m} \to R^{2m}$ is positive at every y_I $(I \in \Lambda)$. Hence, by using the same index theoretic argument as above and $X_I \subset X \subset \Gamma \times (0,s^*]$, we obtain

$$X_I \cap X_{I'} = \phi \quad \text{if} \quad I \neq I' \quad \text{and (2.27).}$$

The theorem above provides us with an algorithm that approximates all the solutions to the system $f(z)=0$. Namely, trace all the PL paths X_I starting from the known point (y_I, s^*) $(I \in \Lambda)$ until they hit the hyperplane $R^{2m} \times \{t_k\}$ for a sufficiently large k; if $(z, t_k) \in X_I$, then z is an approximate solution of $f(z)=0$. The practical procedure for tracing X_I is done by applying complementary pivoting to the PL system $\phi^{\delta}(z,s)=0$, $(z,s) \in R^{2m} \times (0, s^*]$.

The algorithm involves the parameters r, ρ and k_1; δ is determined by $\delta = 2^{k_1}\rho$. We shall explain the role of these parameters. For the time being, we assume that r and ρ are fixed, and focus our attention to the parameter k_1. First we observe that as the homotopy parameter t decreases from 2 to 1, the map $\phi^{\delta}(.,t): R^{2m} \rightarrow R^{2m}$ is continuously deformed from a PL approximation $\phi^{\delta}(.,2)$ of $\hat{g}$ into a PL approximation $\phi^{\delta}(.,1)$ of f. Hence, we will trace a path X_I from an approximate solution of $\hat{g}(z)=0$ to an approximate solution of $f(z)=0$. If the grid size of the triangulation δL^2 of $R^{2m} \times [1,2]$ is too small, we have to traverse a great number of small simplices along the path X_I until it hits the hyperplane $R^{2m} \times \{1\}$. This causes a computational inefficiency. The role of the parameter k_1 is expanding the small initial grid size ρ up to the grid size $\delta = 2^{k_1}\rho$ of $\delta L[1,2]$ in order to increase the computational efficiency when ρ is too small.

Because of the same reason as above, we want to take ρ as large as possible. But Theorem 2 requires us to take ρ not greater than the number $\min_j\{(r/\sqrt{2})\sin(\pi/2d_j)\}$. Hence we shall fix ρ at this number.

By the construction of the PL homotopy ϕ^{δ}, we also see that each map $\phi^{\delta}(.,t):R^{2m} \to R^{2m}$ $(t \in [2,s^*])$ approximates the common map $\hat{g}:R^{2m} \to R^{2m}$ although the accuracy may become lower as the homotopy parameter t decreases from s^* to 2. Hence, unless k_1 is too large, each path X_I starting from the point (y_I,s^*) is expected to penetrate each layer $R^{2m}\times[s_k,s_{k-1}]$ $(k=1,2,...,k_1)$ of the triangulation $\delta L[2,s^*]$ almost vertically and encounters a relatively small number of simplices.

The parameter r determines the location of all the initial points (y_I,s^*) $(I \in \Lambda)$ from which the path X_I will start. It should be also noted that the initial grid size ρ which we have fixed at $\min_j\{(r/\sqrt{2})\sin(\pi/2d_j)\}$ changes in proportion to the parameter r.

2.4. Computational results

The algorithm was coded in Extended FORTRAN (MELCOM-COSMO 700), and was applied to six problems (Problems 1-6 described below). We used double precision arithmetic (mantissa of length 15 decimal places and exponent in the range 10^{-78} through 10^{74}) for Problems 2-6, and quadruple precision arithmetic (32 decimal places and 10^{-78} through 10^{74}, respectively) for Problem 1. Changing the values of the parameters r and k_1 for some of the problems, we examined their effect upon the computational efficiency. We stopped the iteration for tracing a path X_I when the path attained the hyperplane $R^{2m} \times (t_k)$ such that the grid size of its triangulation $L(t_k)$ is less than 10^{-7} or when all the absolute values of the components of the map f evaluated at the latest approximate solution become less than 10^{-10}.

In Tables 1-9, we employ the following notation:

$\#\hat{g}$: the total function evaluation of the map $\hat{g}$.

$\#f$: the total function evaluation of the map f.

PIV^j $(j=1,2,3)$: the number of $(2m+1)$-dimensional simplices of triangulations δL^j $(j=1,2,3)$ which were met by the paths X_I $(I \in \Lambda)$, i.e., the total number of pivot iterations which were consumed in $R^{2m} \times [2, s^*]$, $R^{2m} \times [1,2]$ and $R^{2m} \times (0,1]$, respectively.

T.PIV: the sum of PIV^j's (j=1,2,3), and the number in the bracket denotes the mean value per a path, i.e., $T.PIV/(d_1 d_2 \ldots d_m)$.

The first three examples are algebraic equations in one complex variable.

Problem 1 (Wilkinson's polynomial of the degree 20 [32]).

$$F(Z) = \prod_{j=1}^{20} (Z+j) + 2^{-23} Z^{19} = 0$$

Li and Yorke [20] also solved this problem. They reported that more than 9000 derivative evaluations were consumed to calculate all solutions. We spent about 4000 evaluations of f and $\hat{g}$ (see Table 1). But this comparison may not be fair because the starting points were different.

Problem 2 (Degree 80 Taylor expansion of cos(Z) centered at Z=0).

$$F(Z) = \sum_{k=0}^{40} (-1)^k Z^{2k}/(2k)! = 0.$$

This problem was suggested by Dr. Tanabe [28]. We got 20 real approximate solutions and 60 complex ones using the algorithm (see Table 2).

Problem 3.

$$F(Z) = \sum_{k=0}^{500} Z^k = 0.$$

Obviously, we see

$$F(Z) = (Z^{501} - 1)/(z - 1) \quad \text{if } Z \neq 1.$$

Hence the solutions are

$$\cos(2\pi k/501)+i \sin(2\pi k/501) \quad (k=1,2,...,500).$$

We were able to approximate all the solutions by the algorithm (see Table 3).

From the results given in Tables 1 and 2, we may conclude:

(2.54) the total number PIV^1 of pivot operations consumed when the paths lie in $R^{2m} \times [2,s^*]$ increases as k_1 does.

(2.55) the total number PIV^2 of pivot iterations consumed when the paths lie in $R^{2m} \times [1,2]$ is much affected by the choice of the parameter r and the grid size δ of δL^2, and if r is fixed and δ is not too large, then it is inversely proportional to δ.

(2.56) the total number PIV^3 of pivot iterations consumed when the paths lie in $R^{2m} \times (0,1]$ increases as δ does.

When δ is too small, the number PIV^2 becomes very large. In that case, we should increase $\delta = 2^{k_1}(r/\sqrt{2})\sin(\pi/2d)$ to some extent by controlling the parameters k_1 and r.

Next we shall show computational results on unconstrained nonconvex minimization problems (Problems 4 and 5). Let $\Theta(z)$ denote a objective function to be minimized in the m-dimensional Euclidean space, and $D\Theta(z)$ the gradient vector at z. As is well known, if the objective function Θ attains the minimum value at z then it must be a solution of the system of equations:

(2.57) $D\Theta(z) = 0, \quad z \in R^m.$

We have computed all the solutions to the system (2.57) in the complex space, i.e., all the solutions to the system:

(2.57)′ $D\Theta(Z) = 0, \quad Z \in C^m.$

If a minimum point exists, it must be a real solution of the system (2.57)′.

Problem 4 (Minimization problem in two variables).

$$\text{Minimize } \Theta(x,y) = a_1x^4+a_2y^4+a_3x^3+a_4x^2y+a_5xy^2 +a_6y^3+a_7x^2+a_8xy+a_9y^2+a_{10}x+a_{11}y,$$

where the constants $a_1,a_2,\dots,a_{11}$ are given in Table 4.

The system $D\Theta(X,Y)=0$ has nine solutions for each data (4-1, 4-2, 4-3) in Table 4. In Table 5, we have shown only the real solutions obtained and the values of the objective function corresponding to them. These points are either local (global) minimum (L.MIN or G.MIN), local (global) maximum (L.MAX or G.MAX) or saddle point (SADDLE) solutions. The type of each stationary point is shown in the column 'TYPE'. We took $r=4$ and $k_1=0$.

Problem 5 (Minimization problem in three variables).

$$\text{Minimize } \Theta(x,y,z) = x^4+y^4+z^4+(x+y+z+1)^3.$$

We took $r=4$ and $k_1=0$. We got 27 complex solutions to $D\Theta(X,Y,Z)=0$ among which three are considered to be real. These three points are shown in table 6.

Problem 6 (System of polynomial equations in 5 complex variables).

$$F_j(Z) = Z_j^2+(Z_1+Z_2+Z_3+Z_4+Z_5)-2Z_j-a = 0$$
$$(j=1,2,3,4,5).$$

The computational results are shown in Tables 7, 8 and 9 when a=10, a=4, and a=4.1, respectively. In all cases, the system F(Z)=0 has 32 real solutions and no complex solutions.

When we take a=4, the Jacobian matrix of F at a solution (1,1,1,1,1) has rank 1. Thus the system is highly degenerate around this point, and we can not guarantee theoretically that the algorithm computes all the solutions. Nevertheless, we did obtain all the solutions. Among the 32 paths, 16 ones converge to the degenerate solution (1,1,1,1,1). We required a large number of pivot iterations because these 16 paths are very complicated.

When we took a=4.1, we got 16 solutions around the point (1,1,1,1,1). In this case, the total number of iterations is much less than the case a=4.

r	k_1	δ	$\#\hat{g}$	#f	PIV^1	PIV^2	PIV^3	T.PIV
4	0	0.222	21666	22415	0	43114	887	44001 (2200)
4	5	7.101	1809	2674	1317	940	2146	4403 (220)
4	8	56.81	2186	3546	2093	125	3434	5652 (283)
16	0	0.888	5032	6178	0	10121	1009	11130 (557)
16	3	7.101	1140	2778	637	961	2240	3838 (192)
16	6	56.81	1630	3614	1540	115	3509	5164 (258)
64	0	3.551	925	2992	0	1923	1914	3837 (192)
64	1	7.101	516	2815	44	867	2340	3251 (163)
64	4	56.81	1023	3699	934	115	3593	4642 (232)

Table 1. Problem 1.

r	k_1	δ	$\#\hat{g}$	#f	PIV^1	PIV^2	PIV^3	T.PIV
32	0	0.444	12548	15783	0	24096	3828	27924 (349)
32	2	1.777	5234	8149	2679	5548	4749	12976 (162)
64	0	0.888	5338	11215	0	12184	3971	16155 (202)
64	1	1.777	2989	8762	167	5797	5389	11353 (142)
128	0	1.777	6699	14471	0	15299	5464	20763 (260)
128	1	3.554	3640	16697	168	6536	13226	19930 (249)

Table 2. Problem 2.

r	k_1	δ	$\#\hat{g}$	#f	PIV^1	PIV^2	PIV^3	T.PIV
1	1	.0044	10107	26630	1137	14346	16491	31974 (64)

Table 3. Problem 3.

Problem	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
4 - 1	1	1	-1	-1	1	-1	-1	1	-1	1	-1
4 - 2	1	1	-4	-2	2	4	-6	-8	-6	0	0
4 - 3	4	8	-1	2	1	-2	-16	-8	-16	2	-1

Table 4. The coefficients of Problem 4.

Problem	T.PIV	Real Solutions		Function Value	TYPE
		X	Y		
4 - 1	1091 (121)	1.0000	1.0000	-1.0000	L.MIN
		0.7913	1.0422	-0.9873	SADDLE
		-1.2502	1.7236	-7.9683	G.MIN
4 - 2	1103 (123)	4.2389	1.7852	-173.50	G.MIN
		-0.5000	0.5000	-0.3750	SADDLE
		2.0000	-2.0000	-16.000	SADDLE
		0.0000	0.0000	0.0000	L.MAX
		-1.7852	-4.2389	-173.50	G.MIN
4 - 3	1041 (116)	1.4961	1.1631	-34.134	L.MIN
		-0.2113	1.0936	-10.823	SADDLE
		-1.1372	0.9465	-15.321	L.MIN
		1.4659	-0.3805	-14.749	SADDLE
		0.0744	-0.0503	-0.0996	L.MAX
		-1.2721	0.3870	-13.622	SADDLE
		1.4544	-0.5983	-14.839	L.MIN
		0.2704	-0.8445	-4.0241	SADDLE
		-1.5781	-1.1541	-39.575	G.MIN

Table 5. Problem 4.

T.PIV	Real Solutions			Function Value	TYPE
	x	y	z		
7037 (261)	-0.4492	-0.4492	-0.4492	0.0801	SADDLE
	-0.2772	-0.2772	-0.2772	0.0225	L.MIN
	-6.0236	-6.0236	-6.0236	-1025.1	G.MIN

Table 6. Problem 5.

r	k_1	δ	$\#\hat{g}$	#f	PIV^1	PIV^2	PIV^3	T.PIV
1	0	0.5	3475	7959	0	6137	4812	10949 (342)
1	2	2.0	1989	6965	1011	1789	5669	8469 (265)
2	0	1.0	1178	6713	0	2148	5249	7397 (231)
2	1	2.0	1199	6819	408	1372	5744	7524 (235)
4	0	2.0	995	7110	0	1918	5698	7616 (238)
4	1	4.0	929	7201	408	899	6334	7641 (239)

Table 7. Problem 6 (a=10.0).

r	k_1	δ	$\#\hat{g}$	#f	PIV^1	PIV^2	PIV^3	T.PIV
2	0	1.0	1718	49063	0	4420	45780	50200 (1569)

Table 8. Problem 6 (a=4.0).

r	k_1	δ	$\#\hat{g}$	#f	PIV^1	PIV^2	PIV^3	T.PIV
2	0	1.0	1677	16905	0	4226	13803	18029 (563)

Table 9. Problem 6 (a=4.1).

2.5. Remarks

As we have observed in Section 2.2, the auxiliary map $\hat{g}:R^{2m} \to R^{2m}$ and its PL approximation have a separable structure (see (2.35) and (2.39)). If we make use of this structure effectively, we can save some of the pivots and function evaluations of $\hat{g}$ which will be used when a PL path X_I traverses $R^{2m} \times [2,s^*]$. See Kojima [13] or Todd [30,31] for more detail. In the computational results given in Section 2.4, we did not utilize this structure.

In the algorithm developed in Mizuno [21], the auxiliary map $g:R^{2m} \to R^{2m}$ which is induced from the simple polynomial $G:C^m \to C^m$ defined by (2.3) was used instead of $\hat{g}:R^{2m} \to R^{2m}$. Except for this point, his algorithm is the same as ours. It was shown that if the initial grid size ρ is not greater than $\min_j(0.5r/d_j)$ then his algorithm approximates all solutions of the system of polynomial equations $F(Z)=0$ under the same hypothesis as assumed in this paper. It should be noted that if $\max_j d_j \geqq 2$, then the initial grid size $\rho=\min_j\{(r/\sqrt{2})\sin(\pi/2d_j)\}$ of our algorithm satisfies the inequalities

$$\min_j(1.0r/d_j) \leqq \rho \leqq \min_j(1.12r/d_j).$$

Hence the initial grid size given in [21] is about a half of ours. Another disadvantage of his algorithm lies in the fact that solutions of the PL system $g^{\rho}(z)=0$ are not known explicitly although it has $d_1d_2\cdots d_m$ distinct solutions in neighborhood of y_I $(I \in \Lambda)$. So we have to compute the initial points from which the PL path X_I $(I \in \Lambda)$ start.

We could modify our algorithm in some ways. First, recall that the parameter r determines the location of solutions to the system $\psi(z_j;d_j)=0$ and its PL approximation $\psi^{\rho}(z_j;d_j)=0$. We have taken a common r for all $j=1,2,\ldots,m$. This is only for simplicity of notation, and we can take distinct r_j corresponding to each j. Also, we may choose distinct initial grid sizes ρ_j for each coordinate j.

For the numerical stability of the algorithm, it is useful to employ the maps $f':R^{2m} \to R^{2m}$ and $\hat{g}':R^{2m} \to R^{2m}$ defined by

$$f'(z) = \begin{cases} f(z) & \text{if } \|f(z)\| \leqq 1, \\ f(z)/\|f(z)\| & \text{otherwise,} \end{cases}$$

$$\hat{g}'(z) = \begin{cases} \hat{g}(z) & \text{if } \|\hat{g}(z)\| \leqq 1, \\ \hat{g}(z)/\|\hat{g}(z)\| & \text{otherwise,} \end{cases}$$

instead of using $f:R^{2m} \to R^{2m}$ and $\hat{g}:R^{2m} \to R^{2m}$ themselves. It can be easily shown that the sequence of simplices which will be generated remains unchanged theoretically. If the system of equations to be solved has a larger degree, the norm of the maps f and $\hat{g}$ may become very large along a path X_I. In such a case, this technique stabilizes the numerical pivot operation. We used this technique in the numerical experiments whose results are shown in Section 2.4.

Chapter 3. Polynomials with real coefficients

3.1. The case of one variable

Let d be a positive integer and P_d be the class of monic polynomials of degree d with real coefficients, that is, for each $F \in P_d$, there are $a_j \in R$ $(j=1,2,\ldots,d)$ such that

$$F(Z) = Z^d + a_1 Z^{d-1} + \ldots + a_{d-1}Z + a_d$$
$$\text{for each } Z \in C.$$

We define a standard homotopy $H: C\times[0,1] \to C$ between two polynomials F and $G \in P_d$ by (1.3). Let (h_1, h_2) be the map of $R^2\times[0,1]$ into R^2 such that $H(x+iy,t) = h_1(x,y,t) + ih_2(x,y,t)$ for each $(x,y,t) \in R^2\times[0,1]$. We use the symbol $\bar{Z}$ for the conjugate complex number of $Z \in C$. Since the coefficients of F and G are real, we have $H(\bar{Z},t) = \bar{H}(Z,t)$ for each $(Z,t) \in C\times[0,1]$, which implies

(3.1) $h_2(x,0,t) = 0$ for each $(x,t) \in R\times[0,1]$.

Hence we can define a map $h: R\times[0,1] \to R$ by $h(x,t) = H(x,t)$ for each $(x,t) \in R\times[0,1]$. From (3.1), there is a polynomial $h_3: R^2\times[0,1] \to R$ such that

(3.2) $h_2(x,y,t) = yh_3(x,y,t)$ for each $(x,y,t) \in R^2 \times [0,1]$.

We define some solution sets as follows:

(3.3) $S = \{(Z,t) \in C \times (0,1) : H(Z,t) = 0\}$,

(3.4) $S^R = \{(x,t) \in R \times (0,1) : h(x,t) = 0\}$,

(3.5) $S^C = \{(x+iy,t) \in C \times (0,1) : h_1(x,y,t)=0,\ h_3(x,y,t)=0\}$,

(3.6) $S^0 = \{(Z,t) \in S : D_Z H(Z,t) = 0\}$,

where D_Z denotes a partial derivative with respect to Z. If we regard R as a subset of C, S^R is a subset of S. From the above definitions, we easily see that

(3.7) $S = S^R \bigcup S^C$,

(3.8) $(Z,t) \in S^C$ if and only if $(\bar{Z},t) \in S^C$.

In Theorem 3, we shall show some other distinctive features of the solution sets (3.3), (3.4), (3.5), and (3.6) under the following two conditions.

(3.9) $F(Z)=0$ and $G(Z)=0$ have no common solutions.

(3.10) $0 \in R^2$ is a regular value of the map $(h_1,h_3):R^2 \times [0,1] \to R^2$ on $R^2 \times (0,1)$.

Remark. We shall show that (3.10) holds for each $F \in P_d$ and almost all $G \in P_d$. We define a polynomial $\tilde{G} : C \times R^2 \to C$ by $\tilde{G}(Z,a,b)=G(Z)+aZ+b$ and a homotopy $\tilde{H} : C \times [0,1] \times R^2 \to C$ by $\tilde{H}(Z,t,a,b)=(1-t)F(Z)+t\tilde{G}(Z,a,b)$. Since $\tilde{H}$ is a polynomial with real coefficients for each $(t,a,b) \in [0,1] \times R^2$, we have a map $(\tilde{h}_1,\tilde{h}_3):R^2 \times [0,1] \times R^2 \to R^2$ such that

$$\tilde{H}(x+iy,t,a,b)=\tilde{h}_1(x,y,t,a,b)+iy\tilde{h}_3(x,y,t,a,b)$$
$$\text{for each } (x,y,t,a,b) \in R^2 \times [0,1] \times R^2.$$

As in [4], the transversality theorem of [1] shows that $0 \in R^2$ is a regular value of $(\tilde{h}_1,\tilde{h}_3)$ on $R^2 \times (0,1)$ for almost all $(a,b) \in R^2$.

Let w be a homeomorphism (resp. a diffeomorphism) of the interval (0,1) into $C^n \times R^k$, then the set $\{w(p):0<p<1\}$ is called <u>a path</u> (resp. <u>a smooth path</u>). If the point $\lim_{p \downarrow 0} w(p)$ (or $\lim_{p \uparrow 1} w(p)$) exists, it is said to be <u>an end point</u> of the path. If $D_p w_i(p)$ has a (nonzero) common sign for all the points on a path, the path is said to be (<u>strictly</u>) <u>monotone</u> with respect to the i-th component. Let w be a homeomorphism (resp. a diffeomorphism) of the unit circle $I=\{(x,y):x^2+y^2=1\}$ into $C^n \times R^k$, then the set $\{w(p):p \in I\}$ is called <u>a loop</u> (resp. <u>a smooth loop</u>). We define the following sets:

$$S^{R+} = \{(x,t) \in S^R : D_x h(x,t) > 0\},$$
$$S^{R-} = \{(x,t) \in S^R : D_x h(x,t) < 0\},$$
$$S^{C+} = \{(x+iy,t) \in S^C : y > 0\},$$
$$S^{C-} = \{(x+iy,t) \in S^C : y < 0\}.$$

Theorem 3.. Under the conditions (3.9) and (3.10), the following results (3.11)-(3.18) hold:

(3.11) The difference set $S \setminus S^0$ consists of a disjoint union of smooth paths which are strictly monotone with respect to t. We call a connected component of $S \setminus S^0$ an arc. Moreover each arc has two end points in the set $\Omega = \{(Z,1): G(Z)=0\} \cup \{(Z,0): F(Z)=0\} \cup S^0$.

(3.12) S^0 consists of at most $2(d-1)$ points.

(3.13) S^R consists of a disjoint union of smooth paths.

(3.14) S^C consists of a disjoint union of smooth paths and loops.

(3.15) $S^0 = S^R \cap S^C$.

(3.16) For each point $(Z^0,t^0) \in S^0$, there are exactly four arcs $S_1 \subset S^{R+}$, $S_2 \subset S^{R-}$, $S_3 \subset S^{C+}$ and $S_4 \subset S^{C-}$ having the end point (Z^0,t^0) such that either

(a) $S_1 \cup S_2 \subset R \times (t^0,1)$ and $S_3 \cup S_4 \subset C \times (0,t^0)$

or

(b) $S_1 \cup S_2 \subset R \times (0,t^0)$ and $S_3 \cup S_4 \subset C \times (t^0,1)$.

Moreover $S_4 = \{(\bar{Z},t): (Z,t) \in S_3\}$.

(3.17) For each solution α of $G(Z)=0$ (resp. $F(Z)=0$) with multiplicity m, there are m arcs in $S\setminus S^0$ with the end point $(\alpha,1)$ (resp. $(\alpha,0)$).

(3.18) The number of arcs in $S\setminus S^0$ is at most $5d-4$.

The solution sets S, S^R, S^C and S^0 have a structure such as illustrated in Fig.8.

Proof of (3.11). From the definition of S^0, 0 is a regular value of H on the difference set $C\times(0,1)\setminus S^0$. Hence $S\setminus S^0$ consists of a disjoint union of smooth paths which are strictly monotone with respect to t, because the map H is analytic ([5],[10]). Since $H(.,t)\in P_d$ for each $t\in[0,1]$, the solution set S is bounded and each arc has two end points which do not belong to $S\setminus S^0$ but satisfy $H(Z,t)=0$. Hence the two end points lie in the set Ω.

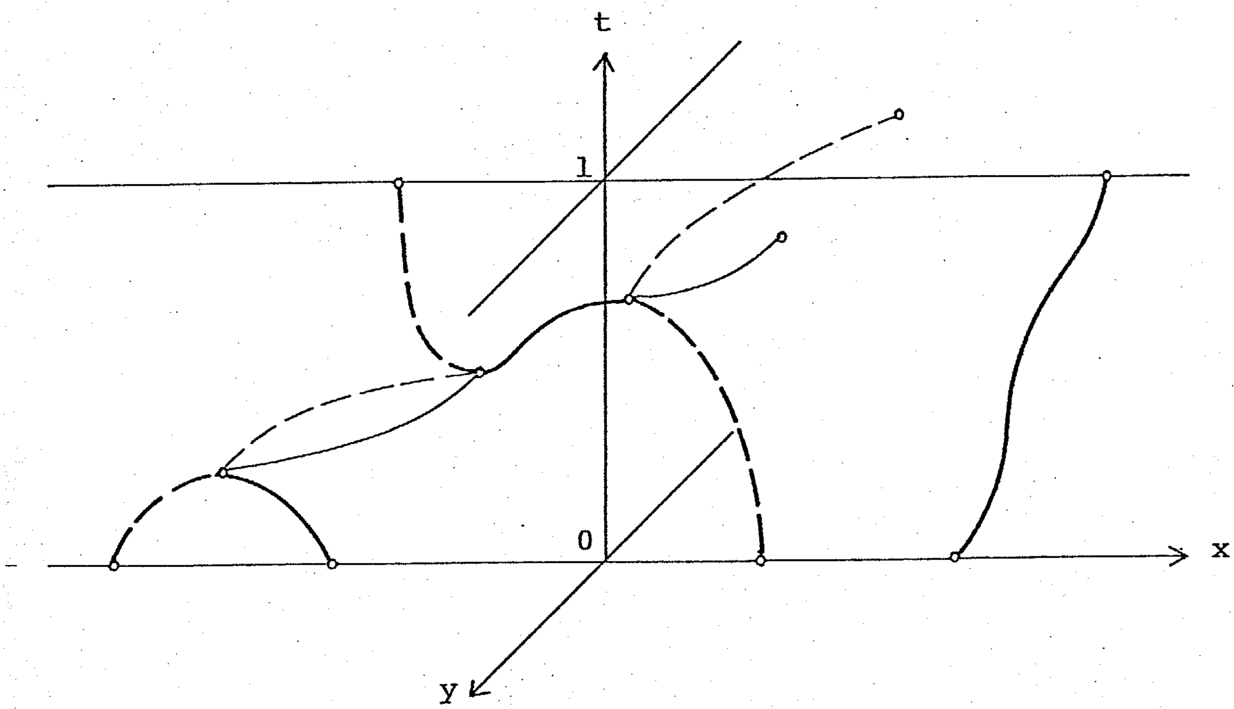
Proof of (3.12). Suppose that $(Z,t)\in S^0$, then we have

(3.19) $H(Z,t) = (1-t)F(Z) + tG(Z) = 0,$

(3.20) $D_ZH(Z,t) = (1-t)D_ZF(Z) + tD_ZG(Z) = 0.$

From these equalities, we have

(3.21) $F(Z)D_ZG(Z) - G(Z)D_ZF(Z) = 0.$



$S^0 \cup \{(Z,1):G(Z)=0\} \cup \{(Z,0):F(Z)=0\}$: ∘

S^{R+} : ———— S^{R-} : — — — —

S^{C+} : ———— S^{C-} : – – – –

Fig.8.

We denote the left side of (3.21) by $Q(Z)$. By the condition (3.9), the polynomial $Q(Z)$ is not identically zero. Since F and G are monic polynomials of degree d, the degree of $Q(Z)$ is at most $2(d-1)$. Thus the number of solutions to (3.21) can not be greater than $2(d-1)$. For each solution to (3.21), at most one $t \in (0,1)$ satisfies (3.19) and (3.20) under the condition (3.9).

Proof of (3.13). We have $t=F(x)/(F(x)-G(x))$ for each $x \in R$ such that $F(x) \neq G(x)$. By the condition (3.9), we have the result.

Proof of (3.14). (3.14) is a direct consequence of the condition (3.10).

Proof of (3.15). From (3.2) and the Cauchy-Riemann conditions, we have

$$(3.22) \quad \begin{aligned} D_Z H(x+iy,t) &= D_y h_2(x,y,t) + iD_x h_2(x,y,t) \\ &= yD_y h_3(x,y,t) + h_3(x,y,t) + iyD_x h_3(x,y,t) \\ &\quad \text{for each } (x,y,t) \in R^2 \times [0,1]. \end{aligned}$$

Suppose that $(x^0+iy^0,t^0) \in S^0$, then we have

$$(3.23) \quad y^0 h_3(x^0,y^0,t^0) = 0,$$
$$(3.24) \quad D_Z H(x^0+iy^0,t^0) = 0.$$

Assume that $y^0 \neq 0$, then from (3.22), (3.23) and (3.24), we see that

$$D_y h_3(x^0,y^0,t^0) = D_x h_3(x^0,y^0,t^0) = 0.$$

Since we also have $D_x h_1(x^0,y^0,t^0)=0$ and $D_y h_1(x^0,y^0,t^0)=0$ from (3.24), it follows that

$$\text{rank } D_{(x,y,t)}(h_1(x^0,y^0,t^0),h_3(x^0,y^0,t^0)) \leqq 1.$$

This contradicts the condition (3.10). Thus we have $y^0=0$, i.e., $(x^0,t^0) \in S^R$. From (3.22), (3.24) and $y^0=0$, we have $h_3(x^0,y^0,t^0)=0$, i.e., $(x^0,t^0) \in S^C$.

Now suppose that $(x^0+iy^0,t^0) \in S^R \cap S^C$, then $y^0=0$ and $h_3(x^0,y^0,t^0)=0$. From (3.22), we have $D_Z H(x^0+iy^0,t^0)=0$, i.e., $(x^0+iy^0,t^0) \in S^0$.

Proof of (3.16). Let $(x^0,t^0) \in S^0$. Then from (3.13) and (3.15), there are exactly two arcs S_1 and S_2 in $S^R \setminus S^0$ with the end point (x^0,t^0). Similarly, from (3.14) and (3.15) there are exactly two arcs S_3 and S_4 in $S^C \setminus S^0$ with the end point (x^0,t^0). Since $(Z,t) \in S^C$ if and only if $(\bar{Z},t) \in S^C$, we have $S_4=\{(\bar{Z},t):(Z,t) \in S_3\}$. From (3.11), the arcs are strictly monotone with respect to t. Hence there are two cases:

(a) $S_3 \bigcup S_4 \subset C \times (0,t^0)$,

(b) $S_3 \bigcup S_4 \subset C \times (t^0,1)$.

The homotopy invariance theorem [24] shows that if (a) occurs, then $S_1 \bigcup S_2 \subset R \times (t^0,1)$, and if (b) occurs, then $S_1 \bigcup S_2 \subset R \times (0,t^0)$. It is easily verified that one of the sets S_1 and S_2 is contained in S^{R+} and the other in S^{R-}.

<u>Proof of (3.17)</u>. Using the homotopy invariance theorem, we can also prove (3.17).

<u>Proof of (3.18)</u>. From (3.11), each arc has two end points in the sets S^0, $\{(Z,1): G(Z)=0\}$ or $\{(Z,0): F(Z)=0\}$. From (3.12) and (3.16), the number of end points in S^0 is at most $2(d-1)\times 4$. From (3.17), the number of end points in $\{(Z,1): G(Z)=0\}$ and $\{(Z,0): F(Z)=0\}$ is 2d. Hence the number of arcs is at most

$$(2(d-1)\times 4 + 2d)/2 = 5d - 4.$$

3.2. A computational method

Suppose that all solutions of $G(Z)=0$ are known and single. Using the nice structure of the solution set S which we have shown in the preceding section, we propose an efficient method for finding all solutions of $F(Z)=0$. The method consists of two phases, Phase I and Phase II. In Phase I, we find all complex (not real) solutions and a half of real solutions by tracing the arcs in S^{R+} and S^{C+}. In Phase II, we find the other half of real solutions by tracing the arcs which belong to S^{R-} and have end points in $R\times\{0\}$. Note that no complex arithmetic is needed when we trace the arcs in S^{R+} and S^{R-}.

We define the sets

$$E_0(R^+) = \{Z: (Z,0) \text{ is an end point of an arc in } S^{R+}\},$$
$$E_1(R^+) = \{Z: (Z,1) \text{ is an end point of an arc in } S^{R+}\}.$$

In the same way, we also define the sets $E_0(R^-)$, $E_1(R^-)$, $E_0(C^+)$, $E_1(C^+)$, $E_0(C^-)$ and $E_1(C^-)$. Let q_j $(j=1,2,\ldots,d_1)$ be real solutions of $G(Z)=0$ and q_j $(j=d_1+1,d_1+2,\ldots,d_1+d_2)$ be complex ones such that img $q_j>0$. Since $\bar{q}_j$ $(j=d_1+1,d_1+2,\ldots,d_1+d_2)$ are also solutions of $G(Z)=0$, we have $d_1+2d_2=d$. Suppose that

$$q_1 > q_2 > \ldots > q_{d_1} .$$

Then we can easily show that

$$D_Z G(q_j) > 0 \quad \text{if } j \text{ is odd and } 1 \leqq j \leqq d_1,$$
$$D_Z G(q_j) < 0 \quad \text{if } j \text{ is even and } 1 \leqq j \leqq d_1.$$

Hence we have

$$E_1(R^+) = \{ q_j : j \text{ is odd and } 1 \leqq j \leqq d_1 \},$$
$$E_1(R^-) = \{ q_j : j \text{ is even and } 1 \leqq j \leqq d_1 \},$$
$$E_1(C^+) = \{ q_j : d_1+1 \leqq j \leqq d_1+d_2 \},$$
$$E_1(C^-) = \{ \bar{q}_j : d_1+1 \leqq j \leqq d_1+d_2 \}.$$

We denote the number of elements in $E_1(R^+)$ by d_0, that is, $d_0=d_1/2$ (if d is even) or $d_0=(d_1+1)/2$ (if d is odd).

First we explain Phase I in which we will find the points in $E_0(R^+)$, $E_0(C^+)$ and $E_0(C^-)$. From (3.11), (3.16), and (3.17) in Theorem 3, we can easily verify the following corollary.

Corollary 4. Let $S^+=S^{R+} \bigcup S^{C+} \bigcup S^0$ then S^+ consists of a disjoint union of d_0+d_2 piecewise smooth paths Q_j ($j=1,2,\ldots,d_0+d_2$) each of which is monotone with respect to t and has one end point in $(E_1(R^+) \bigcup E_1(C^+)) \times \{1\}$ and the other end point in $(E_0(R^+) \bigcup E_0(C^+)) \times \{0\}$.

Let $q \in E_1(R^+) \bigcup E_1(C^+)$, then there is a piecewise smooth path $Q \subset S^+$ and an $r \in E_0(R^+) \bigcup E_0(C^+)$ such that $(q,1)$ and $(r,0)$ are end points of Q. Let

$$(z_0,t_0) = (q,1),$$
$$Q \bigcap S^0 = \{(z_k,t_k) : k=1,2,\ldots,m-1 \text{ and } t_1 > t_2 > \ldots > t_{m-1}\},$$
$$(z_m,t_m) = (r,0)$$

for a positive integer m. For each k ($1 \leqq k \leqq m$), there is an arc S_k in S^{C+} or S^{R+} having the end points (z_{k-1},t_{k-1}) and (z_k,t_k). Assume that (z_{k-1},t_{k-1}) is a known point. Since the arc S_k is a subset of a smooth path or loop S^* in S^R or S^C, we will trace the arc from the starting point (z_{k-1},t_{k-1}) to the direction in which t decreases. By (3.11) and (3.16), the arc S_k attains the point (z_k,t_k) if and only if t attains 0 ($k=m$) or t increases ($k \leqq m-1$) while we move along S^*. Since the point (z_0,t_0) is known, we can find all the points (z_k,t_k) ($k=1,2,\ldots,m$) by repeating the above procedure. Note that if $S_k \subset S^{C+}$ for some $k<m$, then $S_{k+1} \subset S^{R+}$. In such a case, two arcs in $S^R \setminus S^0$ have the end point (z_k,t_k). We shall show which arc is S_{k+1}. If $F(z_k)>0$ then $S_{k+1} \subset (-\infty,z_k) \times (0,t_k)$, and if $F(z_k)<0$ then $S_{k+1} \subset (z_k,+\infty) \times (0,t_k)$ (see Fig.9, where + (resp. −) denotes the region on which $h(z,t)$ has a positive (resp. negative) value). Tracing all the piecewise smooth path

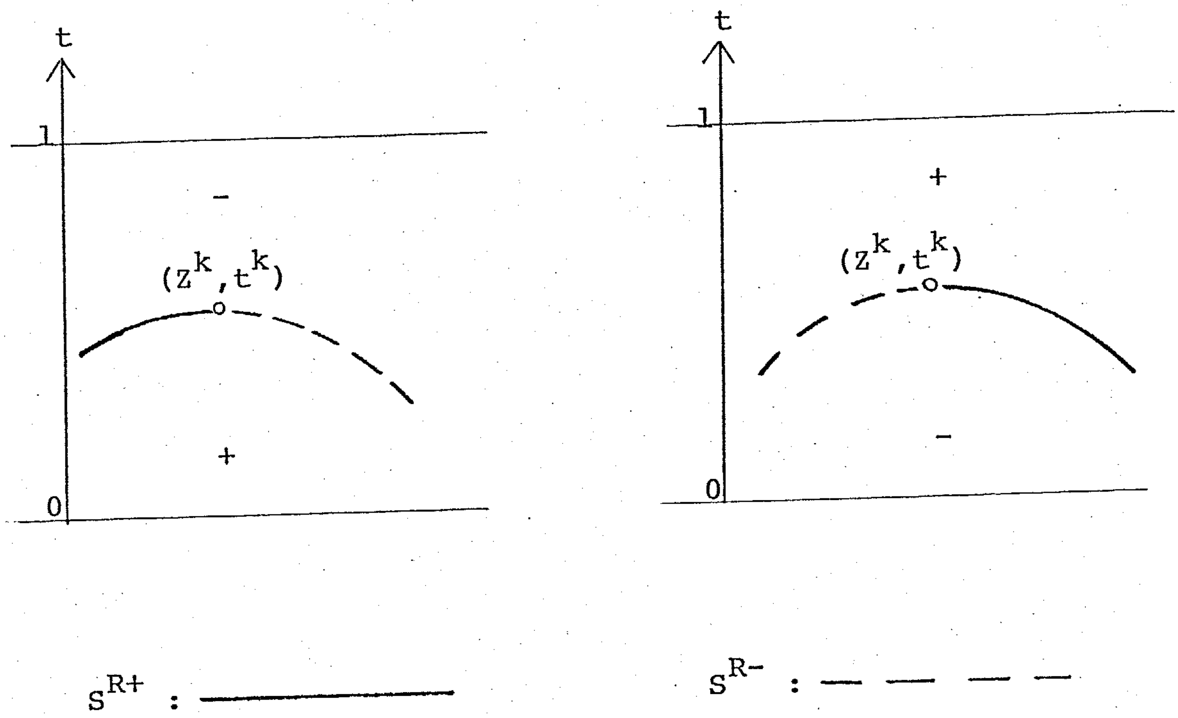


Fig.9.

Q's from the points in $E_1(R^+)$ and $E_1(C^+)$, we can obtain all the points in $E_0(R^+)$, $E_0(C^+)$, and S^0. Since $z \in E_0(C^-)$ if and only if $\bar{z} \in E_0(C^+)$, we can also obtain all the points in $E_0(C^-)$.

Now we shall show Phase II of our method. In this phase, we shall find all points in $E_0(R^-)$. For each point $x \in E_0(R^-)$, there is an arc $S_1 \subset S^{R-}$ having the end point $(x,0)$. The other end point (x^0,t^0) of S_1 lies in either S^0 or $E_1(R^-)\times\{1\}$. Hence if we trace all the arcs in S^{R-} from the starting points in S^0 and $E_1(R^-)\times\{1\}$, we will find all the points in $E_0(R^-)\times\{0\}$. However, some of the arcs have no end end points in $E_0(R^-)\times\{0\}$. Now we show how to determine which arcs in S^{R-} have an end point in $E_0(R^-)\times\{0\}$. Let ω be an orthogonal projection of $R\times[0,1]$ onto R, i.e.,

$$\omega(x,t) = x \text{ for every } (x,t) \in R\times[0,1],$$
$$\omega(U) = \{\, x : (x,t) \in U \,\} \quad \text{for every} \quad U \subset R\times[0,1].$$

First we observe that each connected component of $\omega(S^{R+})$ or $\omega(S^{R-})$ forms an open bounded interval. Since we have traced all the piecewise smooth paths Q_j's in Phase I, we already know the set $\omega(S^{R+})$. Let

$$\omega(S^{R+}) = \bigcup_{j=1}^{m} (a_{2j-1}, a_{2j})$$

where m is a positive integer and $a_1 < a_2 < \ldots < a_{2m}$ (see Fig. 10).

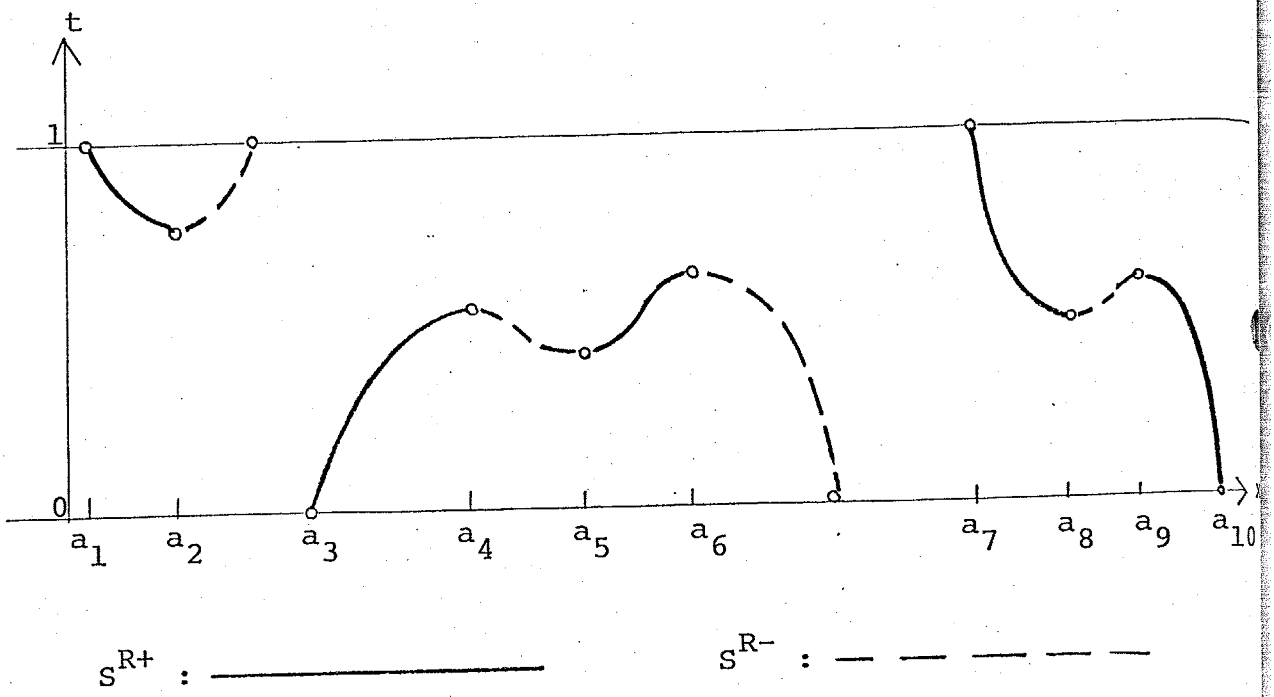


Fig.10.

Then we have

$$(3.25)\quad \{a_1, a_2, \ldots, a_{2m}\} = E_1(R^+) \bigcup E_0(R^+) \bigcup \omega(S^0).$$

We easily see that

$$\omega(S^{R-}) \bigcap (a_{2j-1}, a_{2j}) \text{ is empty for every } j=1,2,\ldots,m,$$
$$\omega(S^{R-}) \bigcap (a_{2j}, a_{2j+1}) \text{ is an open interval for every } j=1,2,\ldots,m-1.$$

Fig. 11 shows cases that can not be occur.

Lemma 5. For each fixed j $(1 \leqq j \leqq m-1)$, let $S_j \subset S^{R-}$ be a unique arc in $(a_{2j}, a_{2j+1}) \times (0,1) \subset R \times [0,1]$. Let (x_1, t_1) and (x_2, t_2) be end points of S_j such that $t_1 > t_2$. Then $x^2 \in E_0(R^-)$ if and only if the set

$$A = \{E_1(R^-) \bigcap (a_{2j}, a_{2j+1})\} \bigcup \{\omega(S^0) \bigcap (a_{2j}, a_{2j+1})\}$$

consists of the single point x_1.

Proof. Note that x_1 and x_2 are members of $E_0(R^-)$, $E_1(R^-)$, or $\omega(s^0)$. From (3.25), we have that $(a_{2j}, a_{2j+1}) \bigcap \omega(S^0) = \phi$. So the two end points lie in $bd\{(a_{2j}, a_{2j+1}) \times (0,1)\}$. Thus x_1 and x_2 are members of the set

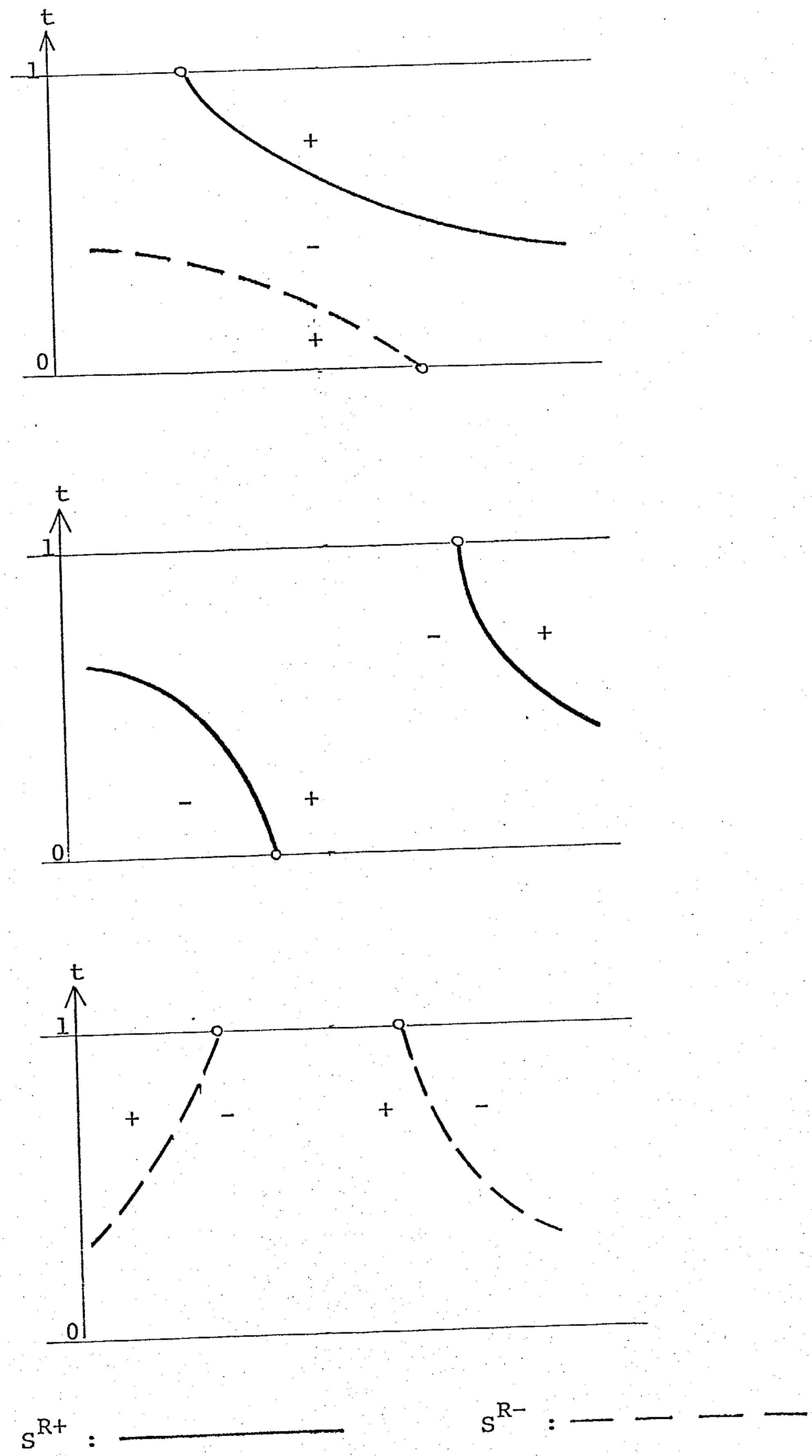


Fig. 11.

$$A^* = A \bigcup \{E_0(R^-) \bigcap (a_{2j}, a_{2j+1})\}.$$

Conversely each member of A^* is an end point of S_j. Hence A^* consists of the two points x_1 and x_2. Therefore we have the result.

In Lemma 5, we have shown a necessary and sufficient condition an arc in $S^{R-} \bigcap (a_1, a_{2m}) \times (0,1)$ to have an end point in $E_0(R^-) \times \{0\}$. We have the following lemma about arcs in $S^{R-} \bigcap \{(-\infty, a_1) \bigcup (a_{2m}, \infty)\} \times (0,1)$.

<u>Lemma 6.</u> (i) If d is odd, then we have

$$\omega(S^{R-}) \bigcap \{(-\infty, a_1) \bigcup (a_{2m}, +\infty)\} = \phi.$$

(ii) If d is even, then we have

$$\omega(S^{R-}) \bigcap (a_{2m}, +\infty) = \phi,$$

$\omega(S^{R-}) \bigcap (-\infty, a_1)$ is an open bounded interval.

Let $S_1 \subset S^{R-}$ be a unique arc in $(-\infty, a_1) \times (0,1) \subset R \times [0,1]$. Let (x_1, t_1) and (x_2, t_2) be two end points of S_1 such that $t_1 > t_2$. Then $x_2 \in E_0(R^-)$ if and only if the set

$$A = \{E_1(R^-) \bigcap (-\infty, a_1)\} \bigcup \{ \{a_1\} \bigcap \omega(S^0)\}$$

consists of the single point x_1.

Proof. (i) Assume that d is odd. Then there are $c_1 \in R$ and $c_2 \in R$ such that

$$h(x,t) < 0 \quad \text{for every} \quad x < c_1 \text{ and } t \in [0,1],$$
$$h(x,t) > 0 \quad \text{for every} \quad x > c_2 \text{ and } t \in [0,1].$$

Hence the result is evident (see Fig.12 (i)).

(ii) Assume that d is even. Then there are $c_1 \in R$ and $c_2 \in R$ such that

$$h(x,t) > 0 \quad \text{for every} \quad x < c_1 \text{ and } t \in [0,1],$$
$$h(x,t) > 0 \quad \text{for every} \quad x > c_2 \text{ and } t \in [0,1].$$

Hence the first result is evident (see Fig. 12 (ii)). In the same way as we have obtained Lemma 5, we can show the second result.

From the above two lemmas, we will determine which arcs in S^{R-} have end points in $E_0(R^-) \times \{0\}$. Tracing these arcs until they hit t=0, we will find all the points in $E_0(R^-)$. Note that we also know the interval $[a_{2j}, a_{2j+1}]$ which contains a point of $E_0(R^-)$. So we will find every points of $E_0(R^-)$ by any method for finding a solution of an equation in a given interval, for example, the bisection method.

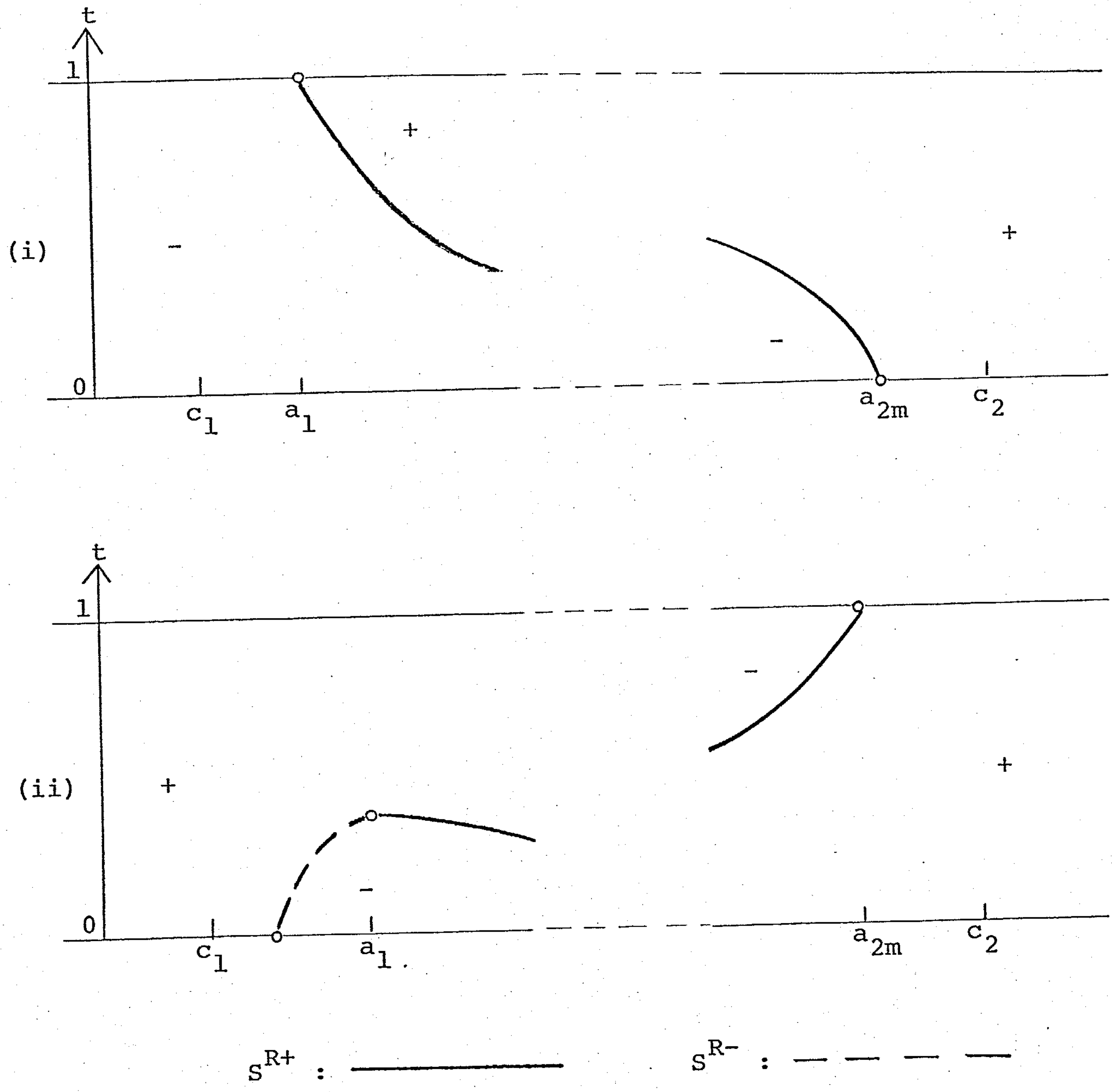


Fig. 12.

3.3. The case of several variables

In this section, we list some properties of the solution set to a homotopy equation between polynomial systems in m variables. Let F and $G: C^m \to C^m$ be polynomial maps with real coefficients. We define a homotopy H between F and G by (1.3). Since the coefficients of F and G are real, we can define a polynomial h of $R^m \times [0,1]$ into R^m by $h(x,t) = H(x,t)$ for each $(x,t) \in R^m \times [0,1]$. We define some subsets of $C^m \times (0,1)$ as follows:

$$S = \{ (Z,t) \in C^m \times (0,1) : H(Z,t) = 0 \},$$
$$S^R = \{ (x,t) \in R^m \times (0,1) : h(x,t) = 0 \},$$
$$S^0 = \{ (Z,t) \in S : \det D_Z H(Z,t) = 0 \},$$
$$S^C = \{ (Z,t) \in S : (Z,t) \notin S^R \setminus S^0 \}.$$

We assume that

(3.26) $0 \in R^m$ is a regular value of h on $R^m \times (0,1)$,

(3.27) $0 \in C^m$ is a regular value of H on $(C^m \setminus R^m) \times (0,1)$,

(3.28) the solution set Π is bounded.

By adding a linear term $AZ+b$ to G, the transversality theorem [1] shows that (3.26) and (3.27) hold for almost all $A \in R^{m \times m}$ and $b \in R^m$. A sufficient condition for (3.28) has been given in Chapter 2. Under the above three conditions, we can prove the properties (3.11), (3.17) in Theorem 3 and the following results:

(3,29) S^0 consists of a finite number of points.

(3.30) S^R consists of a disjoint union of smooth paths and loops.

(3.31) $S^0 \subset S^R$.

(3.32) For each $(Z,t) \in S^0$, let V be an open neighborhood of Z in C^m such that $\{cl\ V\} \times \{t\} \cap S=(Z,t)$, where cl V denotes the closure of V. Assume that the degree of the map $H(.,t)$ on V with respect to 0 is k $(k>1)$, then there are exactly 2k arcs $S_1, S_2, \ldots, S_{2k}$ having the end point (Z,t) such that

$$S_1,\ S_2 \subset S^R,$$

$$S_3,\ S_4, \ldots,\ S_{2k} \subset S^C,$$

$$S_{2j} = \{(\bar{Z}_1, \ldots, \bar{Z}_m, t) : (Z_1, \ldots, Z_m, t) \in S_{2j-1}\}$$

$$j=2,3,\ldots,k.$$

(3.33) Assume that F and G are gradient maps of polynomial functions. Then the signs of the eigenvalues of $D_x h$ stay constant on each arc in $S^R \setminus S^0$ ([2]).

Since the map H is analytic, (3.27) implies (3.31) ([5]).
The proof of (3,32) is outlined as follows.
The homotopy invariance theorem [24] shows that there are k arcs, having the end point (Z,t), in $C^m X(0,t)$ and $C^m X(t,1)$, respectively. From (3.30) and (3.31), S^R contains two of the arcs. Since the coefficients of F and G are real, we have the last k-1 equalities.

Acknowledgement

The author would like to thank Associate Prof. Masakazu Kojima and Dr. Komei Fukuda for their helpful advice and suggestions. He is also grateful to Mr. James Edward Brucker for correcting the English in this paper.

References

[1] R. Abraham and J. Robbin, Transversal mappings and flows, Benjamine, New York, (1967).

[2] E. Allgower and K. Georg, "Simplicial and continuation methods for approximating fixed points and solutions to systems of equations", SIAM Review, 22(1) (1980) 28-85.

[3] E. Allgower and K. Georg, "Predictor-corrector and simplicial methods for approximating fixed-points and zero-points of nonlinear mappings", in: A. Bachem, M. Grotschel and B. Korte, eds., Mathematical Programming, Bonn 1982 (Springer, Berlin, 1983).

[4] S. N. Chow, J. Mallet-Paret, and J. A. Yorke, "Finding zeros of maps: Homotopy methods that are constructive with probability one", Mathematics of Computation, 32 (1978) 887-899.

[5] S. N. Chow, J. Mallet-Paret, and J. A. Yorke, "A homotopy method for locating all zeros of a system of polynomials", in: H. O. Peigen and H. O. Walther, eds., Functional differential equations and approximate fixed points (Springer-Verlag, Berlin, New York, 1979)

[6] F. J. Drexler, "A homotopy method for the calculation of all zero-dimensional polynomial ideals", in: H. Wacker, ed., Continuation methods (Academic Press, New York, 1978).

[7] B. C. Eaves and R. Saigal, "Homotopies for computation of fixed points on unbounded regions", Mathematical Programming 3 (1972) 225-237.

[8] B. C. Eaves and H. Scarf, "The solution of systems of piecewise linear equations", Mathematics of Operations Research 1 (1976) 1-27.

[9] C. B. Garcia and T. Y. Li, "On the number of solutions to polynomial systems of equations", SIAM Journal on Numerical Analysis 17 (1980) 540-546.

[10] C. B. Garcia and W. I. Zangwill, "Determining all solutions to certain systems of nonlinear equations", Mathematics of Operations Research 4 (1979) 1-14.

[11] C. B. Garcia and W. I. Zangwill, "Finding all solutions to polynomial systems and other systems of equations", Mathematical Programming 16 (1979) 159-176.

[12] N. Kawada, "Saikin no takoushiki kaihou ni kansuru ikutsuka no zikken to hyoka (in Japanese)", Master Thesis, Department of Information Sciences, Tokyo Institute of Technology (1983).

[13] M. Kojima, "On the homotopic approach to systems of equations with separable mappings", Mathematical Programming Study 7 (1978) 170-184.

[14] M. Kojima, "Studies on PL approximations of piecewise-C^1 mappings in fixed points and complementarity theory", Mathematics of Operations Research 3 (1978) 17-36.

[15] M. Kojima and S. Mizuno, "Computation of all solutions to a system of polynomial equations", Mathematical Programming 25 (1983) 131-157.

[16] M. Kojima, H. Nishino, and N. Arima, "A PL homotopy for finding all the roots of a polynomial", Mathematical Programming 16 (1979) 37-62.

[17] H. W. Kuhn, "A new proof of the fundamental theorem of algebra", Mathematical Programming Study 1 (1974) 148-158.

[18] H. W. Kuhn, "Finding roots of polynomials by pivoting", in: S. Karamardian, ed., Fixed points: Algorithms and applications (Academic Press, New York, 1977).

[19] C. E. Lemke and J. T. Howson, "Equilibrium points of bimatrix games", Society of Industrial and Applied Mathematics 12 (1964) 413-423.

[20] T. Y. Li and J. A. Yorke, "A simple reliable numerical algorithm for following homotopy paths", in: S. M. Robinson, ed., Analysis and computation of fixed points (Academic Press, New York, 1980).

[21] S. Mizuno, "A simplicial algorithm for finding all solutions to polynomial systems of equations", Master Thesis, Department of Systems Sciences, Tokyo Institute of Technology (1981).

[22] S. Mizuno, "A structure of the solution set to a homotopy equation between polynomials with real coefficients", Research Reports B-124, Department of Information Sciences, Tokyo Institute of Technology (1982).

[23] S. Mizuno, "An analysis of the solution set to a homotopy equation between polynomials with real coefficients", to appear in Mathematical Programming.

[24] J. M. Ortega and W. G. Rheinboldt, Iterative solution of nonlinear equations in several variables, (Academic Press, New York, 1970).

[25] R. Saigal, "On piecewise linear approximations to smooth mappings", Mathematics of Operations Research 4 (1979) 153-161.

[26] R. Saigal, "On computing all real roots of a polynomial with real coefficients", Technical Report, Northwestern University, (1982).

[27] H. Scarf, "The approximation of fixed points of a continuous mapping", SIAM Journal on Applied Mathematics 15 (1967) 1328-1343.

[28] K. Tanabe, private communication.

[29] M. J. Todd, "Union Jack triangulations", in:S. Karamardian, ed., Fixed points: Algorithm and applications (Academic Press, New York, 1977).

[30] M. J. Todd, "Traversing large pieces of linearity in algorithms that solves equations by following PL paths", Mathematics of Operations Research 5 (1980) 242-257.

[31] M. J. Todd, "Exploiting structure in PL homotopy algorithms for solving equations", Mathematical Programming 18 (1980) 233-247.

[32] J. H. Wilkinson, "The evaluation of the zeros of ill-conditioned polynomials. Part I", Numerische Mathematik 1 (1959) 150-166.

[33] A. H. Wright, "Finding all solutions to a system of polynomial equations", Technical Report, Western Michigan University (1982).

[34] W. I. Zangwill, "Determining all minima of certain functions", Preprint, University of Chicago, (1981).