論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	Fixed Point Theory for Amenable Semigroups of Various Transformations
著者(和文)	高橋渉
Author(English)	WATARU TAKAHASHI
出典(和文)	学位:理学博士, 学位授与機関:東京工業大学, 報告番号:甲第401号, 授与年月日:1971年3月26日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor of Science, Conferring organization: Tokyo Institute of Technology, Report number:甲第401号, Conferred date:1971/3/26, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

FIXED POINT THEORY FOR AMENABLE SEMIGROUPS OF VARIOUS TRANSFORMATIONS

BY WATARU TAKAHASHI

JANUARY 1971

CONTENTS

0. INTRODUCTION

Brouwer [2] proved the famous Brouwer's fixed point theorem: Let K be the closed unit sphere of E^n and T be a continuous mapping of K into itself. Then T has a fixed point in K.

Schauder [37] extended this theorem: Let K be a compact convex subset of a normed linear space X and T be a continuous mapping of K into K. Then T has a fixed point in K.

Next, Tychonoff [40] generalized the Schauder's fixed point theorem: Let K be a compact convex subset of a locally convex linear topological space X and T be a continuous mapping of K into itself. Then T has a fixed point in K.

Up to the present, it was not known, even for the one dimensional case, whether every two commuting continuous functions has a common fixed point. Recently, Huneke [21] showed that it is not true.

However, if we consider a commutative family of linear continuous mappings, we know the following theorem proved by Kakutani [24] and Markov [30]: Let K be a compact convex subset of a locally convex linear topological space X and F be a commuting family of linear continuous mappings of K into itself. Then F has a common fixed point in K.

This was generalized by Day [9]: Let K be a compact convex subset of a locally convex linear topological space X and F be a semigroup of linear continuous mappings of K into itself. If F, when regarded as an abstract semigroup, is amenable, or even if it has a left invariant mean, then there exists in K a common fixed point of the family F.

On the other hand, recently, Browder [3] [4] [5] [6] [7], Edelstein [14] [15] [16], Kirk [26] [27] [28], de Marr [10], Petryshyn [33] [34], and others proved the fixed point theorems for nonexpansive mappings (i.e. mappings which do not increase distances) in a Banach space. Among them, de Marr proved a fixed point theorem for commutative families of nonexpansive mappings of a compact convex subset of a Banach space into itself.

In chapter 1 of this paper, we extend de Marr's fixed point theorem: Let K be a nonempty compact convex subset of a Banach space B and S be an amenable semigroup of nonexpansive mappings of K into itself. Then S has a common fixed point in K. The method of the proof of this theorem was afterward employed by Mitchell in his paper [31].

In chapter 2 of this paper, we shall introduce a concept of convexity in a metric space and study the general properties of the space, by name a convex metric space. Paticularly we check whether convex metric spaces possess the properties which a convex subset of a Banach space must do. And also, as an extension of a strictly convex Banach space, we introduce a concept of strict convexity in a convex metric space and study the properties of the space. Furthermore we formulate fixed point theorems for nonexpansive mappings in a convex metric space defined above and prove them. Consequently, these generalize fixed point theorems which have been proved by Browder [3], Kirk [26], de Marr [10] and the author; see chapter 1.

In chapter 3 of this paper, we investigate more detailed properties of a convex metric space. In the previous chapter, we introduced concepts of a condition (C), normal structure and strict convexity having connection with fixed point theorems for nonexpansive mappings. So, one of the aims of this chapter is to obtain a sufficient condition for a condition (C), normal structure and strict convexity. In fact, we show that if a convex metric space is uniformly convex, then the space possesses these conditions. Furthermore we discuss fixed point theorems for nonexpansive mappings in a uniformly convex metric space, too.

If we turn our attention to that a nonexpansive mapping is connected with only the metric of the operating space, the question naturally arises as to whether this is true if one consider fixed point theorems for nonexpansive mappings in a usual metric space.

In chapter 4 of this paper, we show that the answer is affirmative for the case when a family of nonexpansive mappings is finite and commutative.

In chapter 5 of this paper, we discuss the properties of invariant ideals for amenable semigroups of Markov operators on C(X), where C(X) is the Banach algebra of continuous complex valued functions on a compact Hausdorff space X. Recently, by Schaefer [35] [36] and Sine [38], some properties on invariant ideals have been investigated for the case when a family of Markov operators is the semigroup generated by a single operator. The results obtained by Schaefer and Sine can be extended in obvious way to an amenable semigroup of Markov operators on C(X). For example, we can extend the notation of ergodicity of Markov operator T on $C(X)$, defined first for the case of the semigroup generated by T; that is, an amenable semigroup $\Sigma = \{T\}$ is ergodic if and only if for $f \in C(X)$, the convex closure $\overline{co}\{Tf : T \in \Sigma\}$ of $\{Tf : T \in \Sigma\}$ contains an invariant function g for all T in Σ. Thus, in §2 of this chapter, we give a representation theorem for maximal ideals invariant under each T in Σ. In §3, we prove that an amenable semigroup $\Sigma = \{T\}$ is ergodic if and only if invariant functions under each T in Σ separate invariant probabilities under each adjoint operator T^* of T in Σ. Finally, we obtain a characterization of maximal ideals invariant under each T in Σ. This is a generalization of Schaefer's result [35].

In chapter 6 of this paper, we investigate the ergodic theory for amenable semigroups Σ of positive contractions T on $L^1(X,\mathfrak{F},m)$ where $(X,\mathfrak{F},m)$ is a measure space. So far, various necessary and sufficient conditions for the existence of invariant measure equivalent to m have been obtained by several authors for the case when a family of positive contractions on $L^1(X,\mathfrak{F},m)$ is the semigroup generated by a single operator; that is, there are several conditions obtained by Hopf [19], Dowker [11] [12], Calderon [8], Hajian-Kakutani [17], Sucheston [39], and others for the case of an operator which arises from a measurable transformation, and by Ito [22] and Hajian-Ito [18] for the case of an operator which arises from a Markov process. Furthermore, these have been extended elegantly by Neveu [32] for the case of a positive contraction operator on $L^1(X,\mathfrak{F},m)$. On the other hand, the pointwise ergodic theorem and the mean ergodic theorem also have been obtained by several authors for the case when a family of positive contractions is the semigroup generated by a single operator; that is, at first, Birkhoff [1] proved it for point transformations with an invariant σ-finite measure. For Markov processes, Kakutani [25] proved it for a finite invariant measure and for bounded functions. Hopf [20] extended it to a finite invariant measure and functions in $L^1(X, \mathfrak{F}, m)$. Dunford-Schwartz [13] proved it for a σ-finite invariant measure and functions in $L^1(X, \mathfrak{F}, m)$.

Main results in this chapter are the following; at first, we find necessary and sufficient conditions for the existence of a strictly positive element which is invariant under each T in Σ. Secondly we find several equivalent conditions for no existence of non trivial and non negative element in $L^1(X, \mathfrak{F}, m)$ which is invariant under each T in Σ. Finally, we extend the ergodic theorem obtained by Birkhoff, to an amenable semigroup, which has been proved for a single positive contraction T with a strictly positive invariant function in $L^1(X, \mathfrak{F}, m)$.

In chapter 7 of this paper, we turn our attention to the mean ergodic theorem. That is, at first, we generalize the above theorem to extend the result obtained by Hopf [20]. Furthermore we will find a sufficient condition for ergodicity of an amenable semigroup of positive contractions on $L^1(X, \mathfrak{F}, m)$, which is weaker than the condition of the above theorem. And we obtain a projection which is useful for investigating the ergodic theory. This is a generalization of Ito's result [23]. Finally, by using the above theorem, we shall obtain a characterization of extreme points of $\{ f \in L^1 : f \geq 0, \|f\| = 1, Tf = f, T \in \Sigma \}$. This is a generalization of the result obtained by Schaefer [35] for the case of a single positive contraction operator.

In chapter 8 of this paper, we shall prove the adjoint ergodic theorem for amenable semigroups of operators on a Banach space. In chapter 6, we introduced an order on an amenable semigroup by using the property which the semigroup must possess. By using this, we shall prove the above adjoint ergodic theorem. This is a generalization of the result obtained by Lloyd [29].

The author wishes to express his hearty thanks to Professor H. Umegaki, Professor T. Shimogaki, and the menbers of their seminars for many kind suggestions and advices in the course of preparing the present paper.

January 1971

REFERENCES FOR INTRODUCTION

[1] BIRKHOFF, G. D., Proof of the ergodic theorem. Proc. Nat. Acad. Sci. U.S.A. 17 (1931), 656-660.

[2] BROUWER, L. E. J., Über abbildung von mannigfaltigkeiten. Math. Ann. 71 (1911), 97-115.

[3] BROWDER, F. E., Nonexpansive nonlinear operators in a Banach space. Proc. Nat. Acad. Sci. U.S.A. 54 (1965), 1041-1044.

[4] BROWDER, F. E., Fixed point theorems for noncompact mappings in Hilbert space. Proc. Nat. Acad. Sci. U.S.A. 53 (1965), 1272-1276.

[5] BROWDER, F. E., Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch. Rat. Mech. Anal. 24 (1967), 82-90.

[6] BROWDER, F. E., AND W. V. PETRYSHYN, The solution by iteration of linear functional equations in Banach spaces. Bull. Amer. Math. Soc. 72 (1966), 566-570.

[7] BROWDER, F. E., AND W. V. PETRYSHYN, The solution by iteration of nonlinear functional equations in Banach spaces. Bull. Amer. Math. Soc. 72 (1966), 571-575.

[8] CALDERON, A. P., Sur les mesures invariantes. C. R. Acad. Sci. Paris 240 (1955), 1960-1962.

[9] DAY, M. M., Fixed point theorem for compact convex sets. Illinois J. Math. 5 (1961), 585-590.

[10] DE MARR, R., Common fixed points for commuting contraction mappings. 13 (1963), 1139-1141.

[11] DOWKER, Y. N., On measurable transformations on finite measure spaces. Ann. of Math. (2) 62 (1955), 504-516.

[12] DOWKER, Y. N., Sur les applications mesurables. C. R. Acad. Sci. Paris 242 (1956), 329-331.

[13] DUNFORD, N., AND J. T. SCHWARTZ, Convergence almost everywhere of operator averages. J. Rational Mech. Anal. 5 (1956), 129-178.

[14] EDELSTEIN, M., On nonexpansive mappings of Banach space. Proc. Camb. Phil. Soc. 60 (1964), 439-447.

[15] EDELSTEIN, M., A theorem on fixed points under isometrics. Amer. Math. Monthly 70 (1963), 298-300.

[16] EDELSTEIN, M., On nonexpansive mappings. Proc. Amer. Math. Soc. 15 (1964), 689-695.

[17] HAJIAN, A. B., AND S. KAKUTANI, Weakly wandering sets and invariant measures. Trans. Amer. Math. Soc. 110 (1964), 136-151.

[18] HAJIAN, A. B., AND Y. ITO, Iterates of measurable transformations and Markov operators. Trans. Amer. Math. Soc. 117 (1965), 371-386.

[19] HOPF, E., Theory of measures and invariant integrals. Trans. Amer. Math. Soc. 34 (1932), 373-393.

[20] HOPF, E., The general temporally discrete Markoff process. J. Rational Mech. Anal. 3 (1964), 13-45.

[21] HUNEKE, J. P., Two counterexample to a conjecture on commutaing continuous functions of the closed unit interval. Abstract 67T-231, Notices Amer. Math. Soc. 14 (1967), 284.

[22] ITO, Y., Invariant measures for Markov processes. Trans. Amer. Math. Soc. 110 (1964), 152-184.

[23] ITO, Y., Uniform integrability and the pointwise ergodic theorem. Proc. Amer. Math. Soc. 16 (1965), 222-227.

[24] KAKUTANI, S., Two fixed-point theorems concerning bicompact convex set. Proc. Imp. Acad. Jap. 14 (1938),242-245.

[25] KAKUTANI, S., Ergodic theorems and the Markoff process with a stable distribution. Proc. Imp. Acad. Tokyo 16 (1940), 49-54.

[26] KIRK, W. A., A fixed point theorem for mappings which do not increase distances. Amer. Math. Monthly 72 (1965), 1004-1006.

[27] KIRK, W. A., AND L. P. BELLUCE, Fixed-point theorems for family of contraction mappings. Pac. J. Math. 18 (1966) 213-217.

[28] KIRK, W. A., AND L. P. BELLUCE, Nonexpansive mappings and fixed points in Banach space. Illinois J. Math. 11 (1967), 474-479.

[29] LLOYD, S. P., An adjoint theorem. Ergodic theory. Academic press. New York. (1963), 195-201.

[30] MARKOV, A. A., Quelques theoremes sur les ensembles abeliens. C. R. (Doklady) Acad. Sci. URSS(N. S.) 1 (1936), 311-313.

[31] MITCHELL, T., Fixed points of reversible semigroups of nonexpansive mappings. Kodai Math. Sem. Rep. 22 (1970)

[32] NEVEU, J., Existence of bounded invariant measures in ergodic theory. Fifth Berkeley Symposium on Math. Statistics vol II, Part 2 (1967), 461-472.

[33] PETRYSHYN, W. V., Construction of fixed points of demi-compact mappings in Hilbert space. J. of Math. Anal. and Appl. 14 (1966) 274-284.

[34] PETRYSHYN, W. V., Fixed-point theorems involving P-compact, semicontractive, and accretive operators not defined on all of a Banach space. J. of Math. Anal. and Appl. 23 (1968), 336-354.

[35] SCHAEFER, H. H., Invariant ideals of positive operators in C(X). I. Illinois J. Math. 11 (1967), 703-715.

[36] SCHAEFER, H. H., Invariant ideals of positive operators in C(X). II. Illinois J. Math. 12 (1968), 525-538.

[37] SCHAUDER, J., Der fixpunktsalt in funktionalräumer, Studia Math. 2 (1930), 171-180.

[38] SINE, R., Geometric theory of a single Markov operator. Pac. J. Math. 27 (1968), 155-166.

[39] SUCHESTON, L., On existence of finite invariant measures. Math. Z. 86 (1964), 327-336.

[40] TYCHONOFF, A., Ein fixpunktsalt, Math. Ann. 111 (1935), 767-776.

Chapter 1

FIXED POINT THEOREM FOR AMENABLE SEMIGROUPS OF NONEXPANSIVE MAPPINGS

FIXED POINT THEOREM FOR AMENABLE SEMIGROUPS OF NONEXPANSIVE MAPPINGS

1. Introduction.

Let K be a subset of a Banach space B. A mapping s of K into B is said to be nonexpansive if for each pair of elements x and y of K, we have $\| sx - sy \| \leqq \| x - y \|$.

Kakutani [5] and Markov [7] proved the following theorem: Let K be a compact convex subset of a locally convex linear topological space B and S be a commuting family of linear continuous mappings of K into itself. Then S has a common fixed point in K. Day [2] showed that this is true even if S is an amenable semigroup.

On the other hand, de Marr [3] proved a fixed point theorem for a family of nonlinear mappings: Let K be a nonempty compact convex subset of a Banach space B. If S is a nonempty commutative family of nonexpansive mappings of K into itself, then the family S has a common fixed point in K.

The question naturally arises as to whether this is true if one considers an amenable semigroup of nonexpansive mappings.

In this paper, we shall show that the answer is affirmative.

2. Preliminaries.

Let S be an abstract semigroup and m(S) be the space of all bounded real valued functions of S, where m(S) has the supremum norm. An element $\lambda \in m(S)^*$ (the dual space of m(S)) is mean on m(S) if $\lambda(e) = \|\lambda\| = 1$ where e denotes the constant 1 function on S. A mean λ is left [right] invariant if

$$\lambda(l_s f) = \lambda(f) \quad [\ \lambda(r_s f) = \lambda(f)\ ]$$

for all $f \in m(S)$ and $s \in S$, where the left [right] translation l_s [r_s] of $m(S)$ by s is given by $(l_s f)(s') = f(ss')$ [$(r_s f)(s') = f(s's)$]. An invariant mean is a left and a right invariant mean. A semigroup that has a left invariant mean [right invariant mean] is called left amenable [right amenable]. A semigroup that has an invariant mean is called amenable.

Let M be a nonempty compact Hausdorff space and $C(M)$ be the space of bounded continuous real valued functions on M. The norm will be the supremum norm. Let S be a semigroup of continuous mappings of M into M and define a mapping U_s for each s in S from $C(M)$ into $C(M)$ by attaching to each $f \in C(M)$, the function $U_s f$ on M such that $(U_s f)(x) = f(sx)$ for each x in M.

We shall prove the following Lemma by using Day's fixed point theorem [2].

LEMMA 1. Let M be a nonempty compact Hausdorff space and S be an amenable semigroup of continuous mappings of M into M. Then, there exists $L^* \in C(M)^*$ (the dual space of $C(M)$) such that $L^*(e) = \|L^*\| = 1$ where e is the constant 1 function on M and $L^*(U_s f) = L^*(f)$ for all $f \in C(M)$ and $s \in S$.

Proof. Let $K[C(M)] = \{L \in C(M)^* : L(e) = \|L\| = 1\}$. Since U_s for each s in S is a linear mapping of $C(M)$ into itself such that $U_s(e) = e$ and $\|U_s\| = 1$, a mapping $U_s{}^*$ that is given by $(U_s{}^*L)(f) = L(U_s f)$ for all $L \in C(M)^*$ and $f \in C(M)$ is a weak*-continuous affine mapping of $K[C(M)]$ into itself.

If $\{ U_s^* : s \in S \}$ is an amenable semigroup, from Day's fixed point theorem [2], there exists $L^* \in K[C(M)]$ such that $(U_s^* L^*)(f) = L^*(f)$ for all $f \in C(M)$. We shall show that $\{ U_s^* : s \in S \}$ is an amenable semigroup. Since the mapping σ of S onto $\{ U_s^* : s \in S \}$ that is given by $\sigma(s) = U_s^*$ for each s in S is a homomorphism, $\{ U_s^* : s \in S \}$ is an amenable semigroup from [1].

The following lemma was proved by de Marr in [3].

LEMMA 2. (de Marr). Let B be a Banach space and let M be a nonempty compact subset of B and let $\overline{co}\, M$ be the closed convex hull of M. Let ρ be the diameter of M. If $\rho > 0$, then there exists an element $u \in \overline{co}\, M$ such that

$$\sup \{ \|x-u\| : x \in M \} < \rho .$$

3. Main theorem.

THEOREM 1. Let K be a nonempty compact convex subset of a Banach space B and S be an amenable semigroup of nonexpansive mappings of K into itself. Then there exists an element z in K such that $sz = z$ for each s in S.

Proof. By using Zorn's lemma, we can find a minimal nonempty compact convex set $X \subset K$ such that X is invariant under each s in S. If X consists of a single point, then the theorem is proved.

By using Zorn's lemma again, we can find a minimal nonempty compact set $M \subset X$ such that M is invariant under each s in S. We will now show that $M = \{ sx : x \in M \}$ for each s in S.

Since the semigroup of restrictions of all mappings s in S to M is amenable [1], by Lemma 1. there exists an element L^* in $K[C(M)]$

such that $L^*(U_s f) = L^*(f)$ for all $f \in C(M)$. The Riesz theorem asserts that to the element L^*, there corresponds a unique probability measure m on M such that

$$L^*(f) = \int_M f \, dm$$

for each f in C(M).

Since M is a compact metric space and m is a probability measure on M, there exists a unique closed set $F \subset M$ called support of m satisfying (i) $m(F) = 1$, (ii) if D is any closed set such that $m(D) = 1$, then $F \subset D$. Moreover F is the set of all point $x \in M$ having the property that $m(G) > 0$ for each open set G containing x. It is obvious that F is contained in s(M) for each s in S, since each s in S is a measurable transformation of M into M and hence $m(sM) = m(M) = 1$. Let 1_F be the characteristic function of the closed subset F in M. Since for each s in S

$$1 = m(F) = \int_M 1_F(x) \, dm$$

$$= \int_M 1_F(sx) \, dm = m(s^{-1}F),$$

it is clear that F is contained in $s^{-1}(F)$ for each s in S. Therefore F is invariant under each s in S.

If M contains more than one point, by using Lemma 2. there exists an element u in the closed convex hull of M such that

$$\rho = \sup \{ \|u - x\| \ : \ x \in M \} < \delta(M)$$

where $\delta(M)$ is the diameter of M.

Let us define

$$X_0 = \bigcap_{x \in M} \{ y \in X \ : \ \|x - y\| \leq \rho \},$$

then X_0 is the nonempty closed convex proper subset of X such that

$s(X_0) \subset X_0$ for each s in S. This is a contradiction to the minimality of X. Therefore M contains only one point which is a common fixed point for the semigroup of nonexpansive mappings of K into itself.

COROLLARY 1. (de Marr). Let K be a compact convex subset of a Banach space B and S be a family of commutative nonexpansive mappings of K into itself. Then S has a common fixed point in K.

Proof. Since a commutative semigroup is an amenable semigroup, Corollary is obvious from Theorem 1.1.

REMARK 1. Theorem 1. is true even if S is a left amenable semigroup. We can discuss the above by using purely metric methods; see [6] [8].

REFERENCES

[1] DAY, M. M., Amenable semigroup. Illinois J. Math. 1 (1957), 509-544.

[2] DAY, M. M., Fixed point theorem for compact convex sets. Illinois J. Math. 5 (1961), 585-590.

[3] DE MARR, R., Common fixed points for commuting contraction mappings. Pacific J. Math. 13 (1963), 1139-1141.

[4] DUNFORD, N., AND J. T. SCHWARTZ, Linear operators, Part I. Interscience, New York (1958).

[5] KAKUTANI, S., Two fixed point theorems concerning bicompact convex set. Proc. Japan Acad. 14 (1938), 242-245.

[6] KIJIMA, Y., AND W. TAKAHASHI, A fixed point theorem for nonexpansive mappings in metric space. Kōdai Math. Sem.

Rep. (1969), 326-330.

[7] MARKOV, A. A., Quelques theorems sur les ensembles abeliens. C.R. (Doklady). Acad. Sci. URSS (N.S.) 1 (1936), 311-313.

[8] TAKAHASHI, W., A convexity in metric space and nonexpansive mappings, I. Kōdai Math. Sem. Rep. 22 (1970), 142-149.

[9] TAKAHASHI, W., Fixed point theorem for amenable semigroups of nonexpansive mappings. Kōdai Math. Sem. Rep. 21 (1969), 383-386.

Chapter 2

A CONVEXITY IN METRIC SPACE AND NONEXPANSIVE MAPPINGS, I

A CONVEXITY IN METRIC SPACE AND NONEXPANSIVE MAPPINGS, I

1. Introduction.

Recently, Browder [1], Kirk [6], de Marr [4] and the author [7] proved some fixed point theorems for nonexpansive mappings (i.e. mappings which do not increase distances) in a Banach space.

In this paper, we shall discuss them in certain metric space. At first, we shall introduce a concept of convexity in a metric space and study the properties of the space which we call a convex metric space. Furthermore, we formulate some fixed point theorems for nonexpansive mappings in the space and prove them.

Consequently, these generalize fixed point theorems which have been proved by Browder, Kirk, de Marr and the author in the papers listed above.

2. Definitions and propositions.

Let X be a metric space and K be a subset of X. A mapping T of K into X is said to be nonexpansive if for each pair of elements x and y of K, we have $d(Tx,Ty) \leqq d(x,y)$.

Definition 1. A convex metric space X is a metric space such that satisfies the following conditions:

(1) there exists an operation $W(x,y,\lambda) \in X$ for all $x, y \in X$ and λ ($0 \leqq \lambda \leqq 1$),

(2) $d(u,W(x,y,\lambda)) \leqq \lambda d(u,x) + (1-\lambda)d(u,y)$ for all $u \in X$.

A Banach space and each of its convex subsets are of course, convex metric spaces. But a Fréchet space is not necessary a convex metric space. There are many examples of convex metric spaces which are not imbedded in any Banach space. We give two preliminary examples here.

EXAMPLE 1. Let I be the unit interval [0,1] and X be the family of closed intervals $[a_i,b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i,b_i]$, $I_j = [a_j,b_j]$ and λ ($0 \leqq \lambda \leqq 1$), we define a mapping W by

$$W(I_i,I_j,\lambda) = [\lambda a_i+(1-\lambda)a_j, \lambda b_i+(1-\lambda)b_j]$$

and define a metric d in X by the Hausdorff distance, i.e.

$$d(I_i,I_j) = \sup_{a \in I} \{ |\inf_{b \in I_i} \{|a-b|\} - \inf_{c \in I_j} \{|a-c|\}| \}.$$

EXAMPLE 2. We consider a linear space L which is also a metric space with the following properties:

(1) For $x, y \in L$, $d(x,y) = d(x-y,0)$;

(2) For $x, y \in L$ and λ ($0 \leqq \lambda \leqq 1$),

$$d(\lambda x+(1-\lambda)y,0) \leqq \lambda d(x,0)+(1-\lambda)d(y,0) .$$

A subset K of a convex metric space X is said to be convex if $W(x,y,\lambda) \in K$ for all $x, y \in K$ and λ ($0 \leqq \lambda \leqq 1$). The following three Propositions are easy.

PROPOSITION 1. Let X be a convex metric space and

$\{K_\alpha : \alpha \in A\}$ be a family of convex subsets of X, then $\bigcap_{\alpha\in A} K_\alpha$ is also a convex subset of X.

PROPOSITION 2. Let X be a convex metric space, then

$$S(x,r) = \{ y \in X : d(x,y) < r \}$$

and

$$\overline{S}(x,r) = \{ y \in X : d(x,y) \leqq r \}$$

are convex subsets of X.

PROOF. For $y, z \in S(x,r)$ and λ ($0 \leqq \lambda \leqq 1$), there exists $W(y,z,\lambda) \in X$. Since X is a convex metric space,

$$\begin{aligned} d(x,W(y,z,\lambda)) &\leqq \lambda d(x,y) + (1-\lambda)d(x,z) \\ &< \lambda r + (1-\lambda)r = r. \end{aligned}$$

Therefore $W(y,z,\lambda) \in S(x,r)$.

Similarly, $\overline{S}(x,r)$ is a convex set.

PROPOSITION 3. Let X be a convex metric space, then

$$d(x,y) = d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y)$$

for $x, y \in X$ and λ ($0 \leqq \lambda \leqq 1$).

PROOF. Since X is a convex metric space, we obtain

$$\begin{aligned} d(x,y) &\leqq d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y) \\ &\leqq \lambda d(x,x) + (1-\lambda)d(x,y) + \lambda d(x,y) + (1-\lambda)d(y,y) \\ &= \lambda d(x,y) + (1-\lambda)d(x,y) = d(x,y) \end{aligned}$$

for $x, y \in X$ and λ. Therefore

$$d(x,y) = d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y)$$

for $x, y \in X$ and λ.

Let E be a nonempty bounded closed convex subset of a convex metric space X. Let

$$R_x(E) = \sup \{ d(x,y) : y\varepsilon E \},$$
$$R(E) = \inf \{ R_x(E) : x\varepsilon E \},$$
$$E_c = \{ x\varepsilon E : R_x(E)=R(E) \}.$$

DEFINITION 2. A convex metric space X will be said to satisfy a condition (C) if every bounded decreasing net of non-empty closed convex subsets of X has a nonempty intersection.

We obtain the following Proposition from Definition 2, Proposition 1 and Proposition 2.

PROPOSITION 4. If X satisfies a condition (C), then E_c is nonempty, closed and convex.

PROOF. Let $E_n(x) = \{ y\varepsilon E : d(x,y) \leqq R(E)+1/n \}$ for $n = 1, 2, 3, \cdots$ and $x \varepsilon X$. It is easily seen that the sets $C_n = \bigcap_{x\varepsilon E} E_n(x)$ form a decreasing sequence of nonempty closed convex sets, and hence $\bigcap_{n=1}^{\infty} C_n$ is nonempty, closed and convex. On the other hand $E_c = \bigcap_{n=1}^{\infty} C_n$.

For $E \subset X$, we denote the diameter of E by $\delta(E)$. A point $x \varepsilon E$ is a diametral point of E provided

$$\sup \{ d(x,y) : y\varepsilon E \} = \delta(E).$$

DEFINITION 3. A convex metric space is said to be have normal structure if for each closed bounded convex subset E of X which contains more than one point there exists $x \varepsilon E$ which is not a diametral point of E.

It is obvions from the following Proposition that a compact convex metric space has normal structure.

PROPOSITION 5. Let X be a convex metric space and let M be a nonempty compact subset of X and let K be the least closed convex set containing M. If the diameter of M, denoted by $\delta(M)$, is positive, then there exists an element $u \in K$ such that
$$\sup\{d(x,u) : x \in M\} < \delta(M).$$

PROOF. Since M is compact, we may find $x_1, x_2 \in M$ such that $d(x_1,x_2) = \delta(M)$. Let $M_0 \subset M$ be maximal so that $M_0 \supset \{x_1,x_2\}$ and $d(x,y) = 0$ or $\delta(M)$ for all $x, y \in M_0$. It is obvious that M_0 is finite. Let us assume $M_0 = \{x_1,x_2,\cdots,x_n\}$. Since X is a convex metric space, we can define

$$y_1 = W(x_1,x_2, 1/2),$$
$$y_2 = W(x_3,y_1, 1/3),$$
$$\vdots$$
$$y_{n-2} = W(x_{n-1},y_{n-3}, 1/n-1),$$
$$y_{n-1} = W(x_n,y_{n-2}, 1/n) = u .$$

Since M is compact, we can find $y_0 \in M$ such that
$$d(y_0,y_{n-1}) = \sup\{d(x,y_{n-1}) : x \in M\}.$$
Now, by using the condition (2) of convex metric space, we obtain

$$\begin{aligned} d(y_0,y_{n-1}) &\leq \frac{1}{n}d(y_0,x_n) + \frac{n-1}{n}d(y_0,y_{n-2}) \\ &\leq \frac{1}{n}d(y_0,x_n) + \frac{n-1}{n}\left(\frac{1}{n-1}d(y_0,x_{n-1}) + \frac{n-2}{n-1}d(y_0,y_{n-3})\right) \\ &= \frac{1}{n}d(y_0,x_n) + \frac{1}{n}d(y_0,x_{n-1}) + \frac{n-2}{n}d(y_0,y_{n-3}) \end{aligned}$$

$$\vdots$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} d(y_o, x_k) \leq \delta(M).$$

Therefore if $d(y_o, y_{n-1}) = \delta(M)$, then we must have

$$d(y_o, x_k) = \delta(M) > 0$$

for all $k = 1,2,3,\cdots,n$, which means that $y_o \in M_o$ by definition of M_o. But, then we must have $y_o = x_k$ for some $k = 1,2,3,\cdots,n$, which is a contradiction. Therefore

$$\sup \{ d(x, y_{n-1}) : x \in M \} = d(y_o, y_{n-1}) < \delta(M).$$

As an extension of the case in Banach space, we introduce a concept of strict convexity in a convex metric space.

DEFINITION 4. A convex metric space X is said to be strictly convex if there exists a unique element $z \in X$ such that $\lambda d(x,y) = d(z,y)$ and $(1-\lambda)d(x,y) = d(x,z)$ for all $x, y \in X$ and λ ($0 \leqq \lambda \leqq 1$).

We have seen from Proposition 3 that

$$d(x,y) = d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y)$$

for each pair of elements x and y of a convex metric space and all real numbers λ ($0 \leqq \lambda \leqq 1$). Furthermore, from

$$d(x, W(x,y,\lambda)) \leqq \lambda d(x,x) + (1-\lambda) d(x,y)$$
$$= (1-\lambda) d(x,y)$$

and $$d(W(x,y,\lambda), y) \leqq \lambda d(x,y),$$

it is obvious that $W(x,y,\lambda)$ is an element of X such that satisfies

$$(1-\lambda)d(x,y) = d(x,W(x,y,\lambda))$$

and $$\lambda d(x,y) = d(W(x,y,\lambda),y) \ .$$

2. FIXED POINT THEOREMS.

Now, we will prove fixed point theorems for nonexpansive mappings in convex metric spaces.

We can show the validity of the following Theorem by the method of Kirk [5]. However, for the sake of completeness, we give the proof.

THEOREM 1. Let X be a convex metric space, and suppose that X satisfies a condition (C). Let K be a nonempty bounded closed convex subset of X, and supposed that K has normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

PROOF. Let Φ be a family of all nonempty closed and convex subsets of K, each of which is mapped into itself by T. By (C) and Zorn's lemma, Φ has a minimal element E. We show that E consists of a single point.

Let $x \in E_c$. Then $d(Tx,Ty) \leqq d(x,y) \leqq R_x(E)$ for all $y \in E$, and hence T(E) is contained in the spherical ball $\overline{S}(T(x),R(E))$. Since $T(E\cap\overline{S}) \subset E\cap\overline{S}$, the minimality of E implies $E \subset \overline{S}$.. Hence $R_{T(x)}(E) \leqq R(E)$. Since $R(E) \leqq R_x(E)$ for all $x \in E$, $R_{T(x)}(E) = R(E)$. Hence $T(x) \in E_c$ and $T(E_c) \subset E_c$. By Proposition 4 $E_c \in \Phi$.

If $z, w \in E_c$, then $d(z,w) \leqq R_z(E) = R(E)$. Hence,

$\delta(E_c) \leqq R(E) \leqq R_u(E) < \delta(E)$. Since this contradicts the minimality of E, $\delta(E) = 0$ and E consists of a single point.

We prove the following

THEOREM 2. Let X be a strictly convex metric space which satisfies a condition (C), and K be a nonempty bounded closed convex subset of X, and suppose that K has normal structure. If $\mathcal{F}$ is a commuting family of nonexpansive mappings of K into itself, then the family has a common fixed point in K.

PROOF. If T is a nonexpansive mapping in a strictly convex metric space, the fixed point F of T is a nonempty closed convex set. In fact, as $W(x,y,\lambda) \in K$ for $x, y \in F$ and λ $(0 \leqq \lambda \leqq 1)$, by Proposition 3

$$\begin{aligned} d(Tx,Ty) &\leqq d(Tx,T(W(x,y,\lambda))) + d(T(W(x,y,\lambda)),Ty) \\ &\leqq d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y) \\ &= d(x,y) \end{aligned}$$

and hence by strict convexity of the space $T(W(x,y,\lambda)) = W(x,y,\lambda)$. This implies that F is convex.

Let F_α be the fixed point sets of $T_\alpha \in \mathcal{F}$.

If $u \in F_\alpha$, then for any α'

$$T_\alpha T_{\alpha'} u = T_{\alpha'} T_\alpha u = T_{\alpha'} u$$

i.e.,, $T_{\alpha'} u$ lies in F_α and each $T_{\alpha'}$ maps F_α into itself.

If we are given a finite sequence $\alpha_1, \alpha_2, \cdots, \alpha_m$, and consider T_{α_m} as a nonexpansive mapping of $F_{\alpha_1} \cap F_{\alpha_2} \cap \cdots \cap F_{\alpha_m}$ into itself, it follows from Theorem 1 that $\bigcap_{k=1}^{m} F_{\alpha_k} \neq \phi$. Hence by a condition (C), the family $\{ F_\alpha \}$ has a nonempty intersection, but this consists of the common fixed point.

In the general case the fixed point set of a nonexpansive mapping is not convex. However, we will prove the following Theorem by assuming compactness. Before the proof of Theorem, we define the following

DEFINITION 5. Let K be a compact convex metric space. Then a family $\mathcal{F}$ of nonexpansive mappings T of K into itself is said to have fixed point property in K if there exists a compact subset M of E such that $M = \{ T(x) : x \varepsilon M \}$ for each $T \varepsilon \mathcal{F}$ whenever there exists a compact convex subset E of K such that E is invariant under each $T \varepsilon \mathcal{F}$.

THEOREM 3. Let X be a convex metric space and K be a compact convex subset of X. If $\mathcal{F}$ is a family of nonexpansive mappings with fixed point property in K, then the family $\mathcal{F}$ has a common fixed point in K.

PROOF. By using Zorn's lemma, we can find a minimal nonempty compact convex set $E \subset K$ such that E is invariant under each $T \varepsilon \mathcal{F}$. If X consists of a single point, then Theorem is proved.

By hypothesis, there exists a compact subset M of E such that $M = \{ T(x) : x \varepsilon M \}$ for each $T \varepsilon \mathcal{F}$.

If M contains more than one point, by Proposition 5 there exists an element u in the least convex set containing M such that $\rho = \sup \{ d(u,x) : x \varepsilon M \} < \delta(M)$

where $\delta(M)$ is the diameter of M.

Let us define

$$E_0 = \bigcap_{x \varepsilon M} \{ y \varepsilon E : d(x,y) \leqq \rho \},$$

then E_0 is the nonempty closed convex proper subset of E such that $T(E_0) \subset E_0$ for each T in $\mathcal{F}$. . This is a contradiction to the minimality of E.

De Marr [4] showed that a commutative family of nonexpansive mappings of K into itself has fixed point property in K. The following Theorem asserts that this is true even if one considers a left amenable semi group of nonexpansive mappings.

THEOREM 4. Let K be a compact convex metric space. If $\mathcal{F}$ is a left amenable semigroup of nonexpansive mappings T of K into K, then the family $\mathcal{F}$ has fixed point property in K.

PROOF. Let E be a compact convex subset of K suct that E is invariant under each T in $\mathcal{F}$. By using Zorn's lemma, we can find a minimal nonempty compact set $M \subset E$ such that M is invariant under each T in $\mathcal{F}$.

Let C(M) be the space of bounded continuous real valued functions on M and $C(M)^*$ be the dual space of C(M) and $K[C(M)] = \{ L \varepsilon C(M)^* : L(e)=\|L\|=1 \}$ where e denotes the constant 1 function on M.

Since the semigroup of restrictions of mappings T to M is left amenable , by [3] there exist an element $L^* \varepsilon K[C(M)]$ such that $L^*(U_T f) = L^*(f)$ for all $f \varepsilon C(M)$ and $T \varepsilon \mathcal{F}$ where $U_T f$ denotes an element of C(M) such that $(U_T f)(x) = f(Tx)$ for each x in M. The Riesz theorem asserts that to the element L^*, there corresponds a unique probability measure m on M such that

$$L^*(f) = \int_M f \, dm$$

for each f in $C(M)$.

Since M in a compact metric space and m is a probability measure on M, there exists a unique closed set $F \subset M$ called support of m satisfying (i) $m(F) = 1$, (ii) if D is any closed set such that $m(D) = 1$, then $F \subset D$.

It is obvious that F is contained in $T(M)$ for each T in $\mathcal{F}$, since each T in $\mathcal{F}$ is a measurable transformation of M into M and hence $m(T(M)) = m(M) = 1$.

Let χ_F is the characteristic function of the closed subset F in M. Since for each T in

$$1 = m(F) = \int_M \chi_F(x)\, dm = \int_M \chi_F(T(x))\, dm$$

$$= m(T^{-1}F) ,$$

it is clear that F is contained in $T^{-1}(F)$ for each T in $\mathcal{F}$. Therefore F is invariant under each T in $\mathcal{F}$. This implies $T(M) = M$ for each T in $\mathcal{F}$.

3. APPLICATION.

Let K be a compact convex subset of a Banach space X and Σ be the family of all nonexpansive mappings of K into itself. Then, for each pair of elements U and V of Σ the expression D(U,V), defined by

$$D(U,V) = \sup \{ \| Ux - Vx \| : x \in K \}$$

is a metric on Σ and also for any real number λ such that $0 \leqq \lambda \leqq 1$, the expression $W(U,V,\lambda)$, defined by

$$W(U,V,\lambda)(x) = \lambda(Ux) + (1-\lambda)(Vx)$$

is a nonexpansive mapping of K into itself.

LEMMA. The family Σ with metric D and operation W is a cmpact convex metic space.

PROOF. We will show that the family Σ with metric D and operation W satisfies the condition (2) of convex metric space. In fact, for elements U, V and T of Σ and any real number λ such that $0 \leqq \lambda \leqq 1$,

$$\begin{aligned} D(T,W(U,V,\lambda)) &= \sup \{ \| Tx - W(U,V,\lambda)x \| : x \in K \} \\ &\leqq \sup \{ \lambda \| Tx - Ux \| \\ &\qquad + (1-\lambda) \| Tx - Vx \| : x \in K \} \\ &\leqq \lambda D(T,U) + (1-\lambda) D(T,V) . \end{aligned}$$

We will show that the family Σ with metric D is compact. Let $\{ U_n \}$ be a sequence of nonexpansive mappings of Σ Then we show that there exists a subsequence $\{ U_k \}$ of $\{ U_n \}$ such that U_k converges to a point of Σ.

Let A_n be a $1/n$ net of K and $N = \bigcup \{ A_n : n = 1, 2, \cdots \}$.

Then it is obvious that there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that $U_k x$ for every x in N converges to a point of K. We show that $U_k z$ converges to a point Uz for every z of K. Let z any point of K and ε be any given positive number, then there exists $x \in N$ such that $\| z - x \| \leqq \varepsilon/3$. Now, since $U_k x$ converges, there exists a positive integer k_0 such that

$$\begin{aligned}
\| U_{k_1} z - U_{k_2} z \| &\leqq \| U_{k_1} z - U_{k_1} x \| \\
&\quad + \| U_{k_1} x - U_{k_2} x \| + \| U_{k_2} x - U_{k_2} z \| \\
&\leqq \| z - x \| + \| U_{k_1} x - U_{k_2} x \| + \| x - z \| \\
&\leqq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

if $k_1, k_2 > k_0$. Hence $U_k z$ converges to a point of K. Let us define $Uz = \lim U_k z$ for every z of K. Then it is obvious that U is a nonexpansive mapping. We will show that the convergence is uniform. Let ε be any positive number and choose n_0 such that $1/n_0 \leqq \varepsilon/3 < \varepsilon$, then A_{n_0} contains a point x such that $\| x - z \| \leqq \varepsilon/3$. Now k_0 exists such that $\| U_{k_1} x - U_{k_2} x \| \leqq \varepsilon/3$ for all x in A_{n_0} when $k_1, k_2 > k_0$. Thus k_0 is independent of z and $\| U_{k_1} z - U_{k_2} z \| \leqq \varepsilon$ when $k_1, k_2 > k_0$. This shows uniformity of the convergence. Therefore, there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that U_k converges.. This completes the proof.

THEOREM.5. Let K be a compact convex subset of a Banach space and Σ be the compact convex metric space of all non-expansive mappings of K into itself and $\mathcal{F}$ be a family of non-expansive mappings of Σ into itself. If $\mathcal{F}$ has fixed point property in Σ , then $\mathcal{F}$ has a common fixed nonexpansive mapping in Σ .

PROOF. Theorem is obvious from Theorem 3 and Lemma.

The author wishes to express his hearty thanks to Professor H. Umegaki for many kind suggestions.

REFERENCES.

[1] BROWDER, F. E., Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U. S. A. 54 (1965) 1041-1044.

[2] BUSEMANN, H., The geometry of geodesics, Academic Press INC., New York, N. Y., 1955.

[3] CONVERSE, G., NAMIOKA, I. and R. P. PHELPS, Extreme invariant positive operator, Preprint (1967) 1-22.

[4] DE MARR, R., Common fixed-points for commuting contraction mappings, Pac. J. Math. 13 (1963) 1139-1141.

[5] KIRK, W. A., A fixed pont theorem for mappings which do not increase distances, Amer. Math. manthly 72 (1965) 1004-1006.

[6] TAKAHASHI, W., Fixed point theorem for amenable semigroup of nonexpansive mappings, to appear.

TOKYO INSTITUTE OF TECHNOLOGY

Chapter 3

A CONVEXITY IN METRIC SPACE AND NONEXPANSIVE MAPPINGS, II

A CONVEXITY IN METRIC SPACE AND NONEXPANSIVE MAPPINGS, II

1. Introduction.

In the previous chapter, we defined a space with a convexity in a metric space, by name a convex metric space, and investigated the properties of the space. Furthermore, we considered concepts of a condition (C), normal structure and strict convexity having connection with fixed point theorems for nonexpansive mappings.

One of the aims of this chapter is to obtain a sufficient condition for a condition (C), normal structure and strict convexity; that is, we shall show that if a convex metric space is uniformly convex, then the space satisfies these conditions. Consequently, fixed point theorems for nonexpansive mappings obtained in the previous chapter will be proved in a uniformly convex metric space, too.

The author wishes to express his hearty thanks to Professor H. Umegaki, Professor T. Shimogaki and Mr. Y. Kijima for many kind suggestions and advices.

2. Uniformly convex metric space.

Let C be a subset of a convex metris space. Then $\overline{co}(C)$ is the closure of the least convex set containing C. The following definition is the most important one in this chapter.

DEFINITION. A convex metric space X is said to be uniformly convex if for $\varepsilon > 0$, $x \in X$, $y \in X$, $z \in X$, and $r > 0$, there exists $\alpha(\varepsilon) > 0$ such that

$$d(z, W(x,y;1/2)) \leqq r(1-\alpha) < r$$

whenever $d(z,x) \leqq r$, $d(z,y) \leqq r$, and $d(x,y) \geqq r\cdot\varepsilon$.

The following Theorem is the most important theorem in this chapter.

THEOREM 1. Let X be a uniformly convex metric space, and suppose that X is complete. Then X satisfies a condition (C).

Proof. Let $\{C_n\}$ be a decreasing sequence of nonempty bounded closed convex subsets of X. If for every n, $\delta(C_n) \neq 0$, there exist $x_n, y_n \in C$ such that

$$d(x_n, y_n) \geqq \delta(C_n)/2 .$$

Since $d(z_n, x_n) \leqq \delta(C_n)$, $d(z_n, y_n) \leqq \delta(C_n)$ for all $z_n \in C_n$ and the space is uniformly convex,

$$d(z_n, W(x_n, y_n; 1/2)) \leqq \delta(C_n)(1-\alpha) < \delta(C_n) ,$$

i.e., there exist $u_n^1 \in C_n$ such that

$$d(x_n, u_n^1) \leqq \delta(C_n)(1-\alpha)$$

for all $x_n \in C_n$. Let

$$C_n^1 = \{u_n, u_{n+1}, u_{n+2}, \cdots\}.$$

Then it is obvious that $C_n^1 \neq 0$, $C_n^1 \supset C_{n+1}^1$ for every n. And hence

$$\overline{co}(C_n^1) \neq 0 , \ \overline{co}(C_n^1) \supset \overline{co}(C_{n+1}^1) .$$

Similarly, there exist $u_n^2 \in \overline{co}(C_n^1) \subset C_n$ such that

$$d(y_n, u_n^2) \leqq \delta(C_n)(1-\alpha)^2$$

for all $y_n \in \overline{co}(C_n^1)$.

Thus we obtain u_n^3, u_n^4, ... and $\overline{co}(C_n^2)$, $\overline{co}(C_n^3)$, ... It is obvious that $C_n \supset \overline{co}(C_n^1) \supset \overline{co}(C_n^2) \supset \cdots\cdots$ and $\delta(\overline{co}(C_n^m)) \to 0$ as $m \to \infty$. Since X is complete, there exist $c_n \in X$ such that

$$\bigcap_m \overline{co}(C_n^m) = \{ c_n \}$$

for all n. By

$$\bigcap_m \overline{co}(C_n^m) \supset \bigcap_m \overline{co}(C_{n+1}^m) ,$$

we obtain $c_1 = c_2 = c_3 = \cdots\cdots$. Therefore, $c \in C_n$ for all n, and hence $\bigcap_n C_n \neq \phi$. Even if $\{ C_\kappa : \kappa \in A \}$ is a decreasing net, we can prove Theorem 1 by using the same method.

The above Theorem and the following Lemma are used in the proof of Theorem 2.

LEMMA 1. Let X be a convex metric space, and suppose that X satisfies a condition (C). Let C be a nonempty bounded closed convex subset of X. Then for all $x \in X$, there exists $u \in C$ such that

$$d(x,u) = \inf \{ d(x,y) : y \in C \} .$$

Proof. Let $r = \inf \{ d(x,y) : y \in C \}$ and

$$U_n = \{ v \in X : d(x,v) \leqq r+1/n \} .$$

It is easy that the sets $C_n = U_n \cap C$ form a decreasing sequence of nonempty bounded closed convex sets, and hence by a condition (C) $\bigcap_n C_n \neq \phi$. Let $u \in \bigcap_n C_n$. Since $d(x,u) \leqq r + 1/n$ for every n, $d(x,u) = r$. This completes the proof.

We also obtain the following Corollary from the above Lemma.

COROLLARY 1. Let X be a convex metric space, and suppose that X satisfies a condition (C). Let C be a nonempty closed convex subset of X. Then for $x \in X$, there exists $u \in C$ such that

$$d(x,u) = \inf \{ d(x,y) : y \in C \} .$$

Proof. Let $y \in C$ and $r = d(x,y)$. Then $U(x,r) \cap C$ is a nonempty bounded closed convex subset of X. The Corollary is obvious from Lemma 1.

THEOREM 2. Let X be a uniformly convex metric space, and suppose that X is complete. Let C be a nonempty closed convex subset of X. Then for $x \in X$, there exists a unique element $u \in C$ such that

$$d(x,u) = \inf \{ d(x,y) : y \in C \} .$$

Proof. The existence of $u \in C$ is obvious from Theorem 1 and Corollary 1. That $u \in C$ is unique is evident from a uniformly convexity of the space. In fact, if there exist u_1, $u_2 \in C$ such that

$$d(x,u_1) = d(x,u_2) = \inf \{ d(x,y) : y \in C \} ,$$

then

$$d(x,W(u_1,u_2;1/2)) < d(x,u_1) = d(x,u_2).$$

The following Lemma will be helpful in proving Theorem 3.

LEMMA 2. A uniformly convex metric space X has normal structure.

Proof. Let C be a bounded closed convex subset of X which contains more than one point. Let $\delta(C)$ be a diameter of C. Then there exist $x, y \in C$ such that $d(x,y) \geq \delta(C)/2$. Since $d(z,x) \leqq \delta(C)$, $d(z,y) \leqq \delta(C)$ for all $z \in C$ and the space is uniformly convex,

$$d(z, W(x,y;1/2)) \leqq \delta(C)(1-\alpha) < \delta(C)$$

for some $\alpha > 0$. Therefore the space has normal structure.

Now, we formulate a fixed point theorem for nonexpansive mapping in a uniformly convex metric space.

THEOREM 3. Let X be a uniformly convex metric space, and suppose that X is complete. Let K be a nonempty bounded closed convex subset of X. If T is a nonexpansive mapping of K into itself then T has a fixed point in K.

Proof. It is obvious from Theorem 1, Lemma 3, and Theorem 1 in the previous chapter.

We see from following Lemma that a uniformly convex metric space is a special type of a strictly convex one.

LEMMA 3. A uniformly convex metric space X is strictly convex.

Proof. Let us assume that there exist $x_1, x_2 \in X$ and λ $(0 \leqq \lambda \leqq 1)$

such that

$$\lambda d(x_1, x_2) = d(x_1, y_1) = d(x_1, y_2) ,$$

$$(1-\lambda) d(x_1, x_2) = d(x_2, y_1) = d(x_2, y_2)$$

for some distinct points $y_1 , y_2 \varepsilon X$. It is obvious that there exists $\varepsilon > 0$ such that

$$d(y_1, y_2) \geqq \lambda d(x_1, x_2) \cdot \varepsilon .$$

By

$$d(x_1, y_1) \leqq \lambda d(x_1, x_2) , \quad d(x_1, y_2) \leqq \lambda d(x_1, x_2)$$

and a uniform convexity of the space, we obtain

$$(1) \quad d(x_1, W(y_1, y_2; 1/2)) \leqq (1-\lambda) d(x_1, x_2)(1-\alpha) < \lambda d(x_1, x_2)$$

for some $\alpha > 0$. Similarly, there exists $\alpha' > 0$ such that

$$(2) \quad d(x_2, W(y_1, y_2; 1/2)) \leqq (1-\lambda) d(x_1, x_2)(1-\alpha') < (1-\lambda) d(x_1, x_2) .$$

By (1) and (2),

$$\begin{aligned} d(x_1, W(y_1, y_2; 1/2)) &+ d(W(y_1, y_2; 1/2), x_2) \\ &< \lambda d(x_1, x_2) + (1-\lambda) d(x_1, x_2) \\ &= d(x_1, x_2) . \end{aligned}$$

This is a contradiction. Therefore, we complete the proof.

Now, by using Theorem 1, Lemma 2, and Theorem 2 in the previous chapter, we shall obtain the following fixed point theorem.

THEOREM 4. Let X be a uniformly convex metric space, and suppose that X is complete. Let K be a nonempty bounded closed convex subset of X. If $\mathfrak{F}$ is a commuting family of nonexpansive mappings of K into itself, then $\mathfrak{F}$ has a common fixed point in K.

COROLLARY 2 (Browder [1]). Let B be a uniformly convex Banach space and K be a bounded closed convex subset of B. If $\mathfrak{F}$ is a commuting family of nonexpansive mappings of K into itself, then $\mathfrak{F}$ has a common fixed point in K.

Proof. Since K is a bounded closed convex subset of a uniformly convex Banach space, K satisfies all the conditions of Theorem 4. Hence Corollary 2 is true.

REFERENCES

[1] BROWDER, F. E., Nonexpansive nonlinear operators in a Banach space. Proc. Nat. Acad. Sci. U.S.A. 54 (1965), 1041-1044.

[2] DUNFORD, N., AND J. T. SCHWARTZ, Linear operators, Part I. Interscience, New York (1958).

[3] TAKAHASHI, W., A convexity in metric space and nonexpansive mappings, I. Kōdai Math. Sem. Rep. 22 (1970), 142-149.

[4] TAKAHASHI, W., On the fixed point theorems of nonexpansive mappings. 3-rd. Functional Analisis Congress Rep. (1968) 51-71.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.

Chapter 4

FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS IN METRIC SPACE

FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS

IN METRIC SPACE

1. Introduction.

Let K be a subset of a metric space X with a metric d. A mapping T of K into X is said to be nonexpansive if for each pair x, y of elements in K, $d(Tx,Ty) \leqq d(x,y)$.

Recently, several fixed point theorems for nonexpansive mappings in a Banach space have been derived by Belluce and Kirk [1] [2], Browder [3], de Marr [4], Kirk [6] and the author [7]. In the above papers, they have proved the fixed point theorems by assuming the convexity for domains of nonexpansive mappings.

In this paper, we shall show fixed point theorems for nonexpansive mappings without assumption of convexity; that is, we shall prove fixed point theorems for nonexpansive mappings in a metric space under certain conditions. At first, we shall prove a fixed point theorem for a single nonexpansive mappings. Seco ndly, by using this theorem, we shall obtain a fixed point theorem for a family of finite commutative nonexpansive mappings.

The author wishes to express his hearty thanks to Professor H. Umegaki, Professor T, Shimogaki and Mr. Y. Kijima for many kind suggestions and advices in the course of preparing the present paper.

2. Definitions and lemmas.

Let X be a metric space. A subset F of X is said to be admissible if it is an intersection of closed spheres $\{x \in X : d(x,y) \leqq c\}$, $y \in X$, $0 < c \leqq \infty$. For this admissible set, we define the following;

$$R_x(F) = \sup\{d(x,y) : y \in F\},$$

$$R(F) = \inf\{R_x(F) : x \in F\},$$

$$F_c = \{x \in F : R_x(F) = R(F)\}.$$

DEFINITION 1. A metric space X is said to have a condition (C) if every bounded decreasing net of nonempty admissible subsets of X has a nonempty intersection.

We obtain the following Lemma from the above definition.

LEMMA 1. Let F be a nonempty bounded admissible subset of X. If X has a condition (C), then F_c is nonempty and admissible.

Proof. Let $F(x,n) = \{y \in F : d(x,y) \leqq r+1/n\}$. It is obvious that the set $C_n = \bigcap_{x \in F} F(x,n)$ from a decreasing sequence of nonempty admissible sets, and hence $\bigcap_{n=1}^{\infty} C_n = F_c$ is admissible and by (C) nonempty.

DEFINITION 2. An admissible set $S \subset X$ is said to have normal structure if for each bounded admissible subset H of S which contains more than one point, there is some point $x \in H$ which is not a diametral point of H.

The following Lemma will be helpful in proving Theorem 1.

LEMMA 2. Let F be a bounded admissible subset of X which contains more than one point. If F has normal structure, then $\delta(F_c) < \delta(F)$, where $\delta(S)$ denotes the diameter of a set S.

Proof. By normal structure, F contains at least one nondiametral point x. Hence $R_x = \delta(F)$. If z and w are any two points of F_c, then

$$d(z,w) \leqq \sup \{ d(z,y) : y \in F \} = R_x(F) = R(F) .$$

Hence we have $\delta(F_c) \leqq R(F) \leqq R_x(F) < \delta(F)$.

3. Fixed point theorems.

THEOREM 1. Let X be a metric spase, and suppose that X has a condition (C). Let K be a nonempty bounded admissible subset of X, and suppose that K has normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

Proof. Let Φ be a family of all nonempty admissible subsets of K, each of which is mapped into itself by T. Then Φ has a minimal element F.

Let us assume that F contains more than one point.

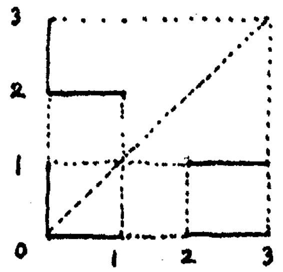
Let $x \in F_c$. Then

$$d(Tx,Ty) \leqq d(x,y) \leqq R_x(F) = R(F)$$

for all $y \in F$, and hence T(F) is contained in the spherical ball U(Tx,R(F)). Since $T(F \cap U) \subset F \cap U$, the minimality of F implies $F \subset U$. Hence $Tx \in F_c$ and F_c is mapped into itself; by Lemma 1, $F_c \in \Phi$. If $\delta(F) > 0$, then F_c is properly contained in F. This is a contradiction. Hence F consists of a single point.

Theorem 1 asserts that a sufficient condition for which a nonexpansive mapping has a fixed point is that the domain of the mapping has normal structure. However, it is not known whether normal structure is a necessary condition for the existence of a fixed point or not. We shall conjecture from the following examples that normal structure is a necessary condition for the existence of a fixed point.

EXAMPLE 1. Let $K = [0,1] \cup [2,3]$. It is obvious that K has not normal structure. In fact, $\{1,2\}$ is admissible, but $\{1,2\}$ has not a nondiametral point in it. Let T be a mapping of K into K such that $Tx = 2$ for $x \in [0,1]$ and $Tx = 1$ for $x \in [2,3]$. Then T is a nonexpansive mapping of K into itself. Since K is compact, it is obvious that K has a condition (C). Clearly T has not a fixed point.

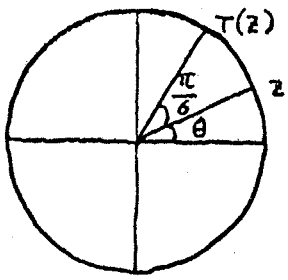


EXAMPLE 2. Let $K = \{z = (\cos\theta, \sin\theta) : 0 \leqq \theta < 2\pi\}$ and

$$d(z_1, z_2) = \sqrt{(\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2}.$$

K has not normal structure. But, since K is compact, K has a condition (C). Let

$Tz = (\cos(\theta + \pi/6), \sin(\theta + \pi/6))$. Then T is a nonexpansive mapping of K into itself. Clearly, T has not a fixed point in K.



The following Theorem is a generalization of Theorem 1. Consequently, it is a generalization of Belluce and Kirk's result [1] obtained in a Banach space.

THEOREM 2. Let X be a metric space, and suppose that X has a condition (C). Let K be a nonempty bounded admissible subset of X, and suppose that K has normal structure. If $\mathfrak{F}$ is a finite commuting family of nonexpansive mappings of K into itself, then $\mathfrak{F}$ has a common fixed point in K.

Proof. Let $\mathfrak{X}$ be a family of nonempty bounded admissible subsets of K, each of which is invariant under each $T \in \mathfrak{F}$. By Zorn's lemma and a condition (C), we can find a minimal element X^* of $\mathfrak{X}$.

Let $\mathfrak{F} = \{T_1, T_2, \cdots, T_n\}$ and $W = \{x \in X^* : T_1 T_2 \cdots T_n x = x\}$. By Theorem 1, $W \neq \phi$. Furthermore $T_i W = W$ for $i=1,2,\cdots,n$. In fact, if $x \in W$, $T_i x = T_i T_1 T_2 \cdots T_n x = T_1 T_2 \cdots T_n T_i x$, i.e., $T_i x \in W$. Inversely, if $x \in W$, $T_1 T_2 \cdots T_{i-1} T_{i+1} \cdots T_n x \in W$, and hence

$$x = T_i T_1 T_2 \cdots T_{i-1} T_{i+1} \cdots T_n x \in W .$$

Let H be the least admissible set containing W. Since X^* is admissible, $H \subset X^*$. By normal structure, H contains a point x such that

$$\sup \{ d(x,z) : z \in H \} = r < \delta(H)$$

provided $\delta(H) > 0$. Let

$$C = \bigcap_{z \in H} \{ x \in X^* : d(x,z) \leqq r \} .$$

Then C is a nonempty admissible subset of X^* and moreover

$$C = \bigcap_{z \in W} \{ x \in X^* : D(x,z) \leqq r \} = C'.$$

In fact, if $x \in C'$, $d(x,z) \leqq r$ for all $z \in W$, and hence $U(x, r) \supset W$. Since H is the least admissible set containing W,

$U(x,r) \supset H$, and hence $d(x,z) \leqq r$ for all $z \in H$. Inversely, it is obvious that $C \subset C'$.

By $C = C'$ and

$$d(T_i c, z) = d(T_i c, T_i w) \leqq d(c,w) \leqq r$$

for $c \in C$, $z = T_i w$, $z, w \in W$, and $i = 1, 2, \cdots, n$, we have $T_i C \subset C$ and hence by the minimality of X^*, $C = X^*$. Since

$$\delta(H) = \delta(C \cap H) \leqq r < \delta(H) ,$$

$C = X^*$ is impossible. Hence $\delta(H) = 0$, so H consists of the desired fixed point.

COROLLARY 1.(Belluce and Kirk). Let B be a Banach space, and K be a nonempty weakly compact convex subset of B and suppose that K has normal structure. If $\mathfrak{F}$ is a finite commutative family of nonexpansive mappings of K into itself, then $\mathfrak{F}$ has a common fixed point in K.

Proof. Since K is a nonempty weakly compact convex subset of B and has normal structure, K satisfies all the conditions of Theorem 2. Hence, Corollary 1 is true.

REFERENCES

[1] BELLUCE, L. P., AND W. A. KIRK, Fixed point theorems for family of contraction mappings. Pac. J. Math. 18 (1966) 213-217.

[2] BELLUSE, L. P., AND W. A. KIRK, Nonexpansive mappings and fixed points in Banach space. Illinois J. Math. 11 (1967), 474-479.

[3] BROWDER, F. E., Nonexpansive nonlinear operators in a Banach space. Proc. Nat. Acad. Sci. U.S.A. 54 (1965), 1041-1044.

[4] DE MARR, R., Common fixed-points for commuting contraction mappings. Pac. J. Math. 13 (1963), 1139-1141.

[5] DUNFORD, N., AND J. T. SCHWARTZ, Linear operators, Part I. Interscience, New York (1958).

[6] KIRK, W. A., A fixed point theorem for mappings which do not increase distances. Amer. Math. Monthly 72 (1965), 1004-1006.

[7] TAKAHASHI, W., Fixed point theorem for amenable semigroup of nonexpansive mappings. Kodai Math sem. Rep. 21 (1969), 383-386.

[8] TAKAHASHI, W., A convexity in metric space and nonexpansive mappings, I. Kodai Math. Sem. Rep. 22 (1970), 142-149.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.

Chapter 5

INVARIANT IDEALS FOR AMENABLE SEMIGROUPS OF MARKOV OPERATORS

INVARIANT IDEALS FOR AMENABLE SEMIGROUPS OF MARKOV OPERATORS

BY WATARU TAKAHASHI

1. Introduction.

Let X be a compact Hausdorff space and let $C(X)$ be the Banach algebra of continuous real or complex valued functions on X, with supremum norm. We denote by $C(X)^*$ the strong dual of Banach space $C(X)$. A Markov operator on $C(X)$ is a continuous linear mapping of $C(X)$ into itself such that $Te = e$ and $Tf \geq 0$ whenever $f \geq 0$, where e denotes the constant 1 function on X. Let Σ be an amenable semigroup of Markov operators T of $C(X)$ into itself. Some properties on invariant ideals have been investigated by Schaefer [5], [6] and Sine [7] for the case when Σ is the semigroup generated by a single Markov operator T. These results can be extended in obvious way to an amenable semigroup of Markov operators on $C(X)$. For example, we can extend the notion of ergodicity of Markov operator T on $C(X)$, defined first for the case of the semigroup generated by T in [5]; that is, an amenable semigroup $\Sigma = \{T\}$ is ergodic if and only if for each $f \in C(X)$, the convex closure $\overline{co}\{Tf : T \in \Sigma\}$ of $\{Tf : T \in \Sigma\}$ contains an invariant function g for all $T \in \Sigma$. In fact, this invariant function is unique in $\overline{co}\{Tf : T \in \Sigma\}$. Thus, in this paper, we generalize essentially results in [5] and [6] by a modification of previous papers; that is, in §2 we give a representation theorem for maximal ideals invariant under each element T of an amenable semigroup Σ. In §3 we prove that an amenable semigroup $\Sigma = \{T\}$ is ergodic if and only if invariant functions under each T in Σ separate invariant probabilities under each the adojoint operator T^* of T in Σ and then prove the bijective correspondence, for ergodic amenable semigroup $\Sigma = \{T\}$, between the family of maximal ideals invariant under each T in Σ and the extreme points of the set of probabilities on X invariant under each T^*.

The author wishes to express his hearty thanks to Professor H. Umegaki and Professor T. Shimogaki for many kind suggestions and advices.

2. Representation theorem.

Let Σ be an abstract semigroup and $m(\Sigma)$ be the space of all bounded real valued functions of Σ, with supremum norm. An element $\mu \in m(\Sigma)^*$ (the dual space of $m(\Sigma)$) is mean on $m(\Sigma)$ if $\mu(e) = \|\mu\| = 1$. A mean μ is left [right] invariant if $\mu(l_s f) = \mu(f)$ [$\mu(r_s f) = \mu(f)$] for all $f \in m(\Sigma)$ and $s \in \Sigma$, where the left [right] translation l_s [r_s] of $m(\Sigma)$ by s is given by $(l_s f)(s') = f(ss')$ [$(r_s f)(s') = f(s's)$]. An invariant mean is a left and a right invariant mean. A semigroup that has a left invariant mean [right invariant mean] is called left amenable [right amenable]. A semigroup that has a invariant mean is called amenable.

At first, we shall prove the following Lemma by using Day's fixed point theorem [2].

LEMMA 1. Let $\{T\} = \Sigma$ be an amenable semigroup of Markov operators on $C(X)$, then there exists $\phi \in C(X)^*$ such that $\|\phi\| = 1$, $\phi \geq 0$ and $T^*\phi = \phi$ for each $T \in \Sigma$, where T^* is the adjoint of T.

Proof. $K = \{ \psi \in C(X)^* : \|\psi\| = 1, \ \psi \geq 0 \}$ is a compact and convex set. Since $\{T^*\} = \Sigma^*$ is an amenable semigroup of affine w^*-continuous mappings of K into itself, from Day's fixed point theorem, there exists $\phi \in C(X)^*$ such that $\|\phi\| = 1$, $\phi \geq 0$ and $T^*\phi = \phi$ for all $T \in \Sigma$.

In imitation of [5], we obtain the following two Definition and two Lemmas.

DEFINITION 1. Let $\{T\} = \Sigma$ be an amenable semigroup of Markov operators on $C(X)$. A $\{T\}$-ideal is a closed proper ideal in $C(X)$ which is invariant under each $T \in \Sigma$. A $\{T\}$-ideal is said to be maximal if it is not properly contained in any other $\{T\}$-ideal.

DEFINITION 2. $\{T\} = \Sigma$ is said to be irreducible if there exist no $\{T\}$-ideals distinct from (o).

LEMMA 2. $\{T\} = \Sigma$ possesses at least one maximal $\{T\}$-ideal, and each $\{T\}$-ideal is contained in some maximal $\{T\}$-ideal.

LEMMA 3. Let J be a $\{T\}$-ideal and denote by q the canonical mapping of $C(X)$ onto $C(X)/J$. Then $I \rightarrow q(I)$ is a bijective map of the set of all $\{T\}$-ideals containing J onto the set of all $\{T_J\}$-ideals, where T_J is a operator induced by T on $C(X)/J$ and a $\{T\}$-ideal I is maximal if and only if $\{T_J\}$ is irreducible.

Since the above two Lemmas are clear, we do not give the proofs.

If ϕ is an element of $C(X)^*$, I_ϕ denotes a ideal $\{ f \in C(X) : \phi(|f|) = o \}$, where $|f|$ is a function such that $|f|(x) = |f(x)|$ for $x \in X$.

THEOREM 1. Let $\{T\} = \Sigma$ be an amenable semigroup of Markov operators on $C(X)$ and let I be a maximal $\{T\}$-ideal, then there exists a normalized positive measure $\phi \in C(X)^*$ such that $I = I_\phi$ and $T^*\phi = \phi$ for all $T \in \Sigma$.

Proof. Since $\{T_I\}$ is the amenable semigroup of Markov operators on $C(S_I)$ where S_I denotes the support of I, it follows from Lemma 1 that there exists a normalized positive measure $\hat{\phi} \in C(S_I)^*$ such that $T_I{}^*\hat{\phi} = \hat{\phi}$ for $T \in \Sigma$.

Now, since I is maximal and hence $\{T_I\}$ is irreducible, $I_{\hat{\phi}} = (o)$. Therefore, if q denotes the canonical mapping of $C(X)$ onto $C(X)/I$, $\phi = \hat{\phi} \circ q$ is a positive measure on X such that $I = I_\phi$. $T^*\phi = \phi$ for each $T \in \Sigma$ follows from

$$\begin{aligned} T^*\phi(f) &= \phi(Tf) = \hat{\phi}\circ q(f) = \hat{\phi}(Tf+I) \\ &= \hat{\phi}(T_J(f+I)) = T_J^*\hat{\phi}(f+I) = \hat{\phi}(f+I) \\ &= \hat{\phi}\circ q(f) = \phi(f) \end{aligned}$$

for all $f \in C(X)$.

3. Ergodic amenable semigroup of Markov operators.

Schaefer in [5] defined that a bounded operator T on a Banach space E is called ergodic if for each $x \in E$, the convex closure $K(x)$ of the orbit $(x, Tx, T^2x, \cdots)$ contains a fixed vector x_0 of T. We extend this and give the following Definition.

DEFINITION 3. Let $\{T\} = \Sigma$ be a semigroup of bounded operators on $C(X)$. $\{T\} = \Sigma$ is said to be ergodic if for each $f \in C(X)$, the convex closure $\overline{co}\{Tf : T \in \Sigma\}$ of $\{Tf : T \in \Sigma\}$ contains an invariant function g for all $T \in \Sigma$.

If a semigroup $\{T\} = \Sigma$ is amenable and $\{Tf : T \in \Sigma\}$ weakly compact, then $\{T\} = \Sigma$ is of course ergodic.

THEOREM 2. Let $\{T\} = \Sigma$ be an amenable semigroup of Markov operators on $C(X)$ and ergodic. Then there exists a positive projection P from $C(X)$ onto

$$F = \{ f \in C(X) : Tf = f \text{ for each } T \in \Sigma \}$$

such that $Pe = e$ and $PT = TP = P$ for all $T \in \Sigma$.

Proof. Let $f \in C(X)$. Since $\{T\} = \Sigma$ is ergodic, there exists $g \in \overline{co}\{Tf : T \in \Sigma\}$ such that $Tg = g$ for all $T \in \Sigma$. For $\varepsilon > 0$, there exists $T_1, T_2, \cdots, T_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $\alpha_i > 0$, $\sum \alpha_i = 1$ such that

$$\| g - \sum \alpha_i T_i f \| \leq \varepsilon ,$$

and we have

$$| g(x) - \sum \alpha_i TT_i f(x) | \leq \varepsilon$$

for $x \in X$ and $T \in \Sigma$.

If μ is an invariant mean on $m(\Sigma)$, we denote $\mu_T(h(T)) = \mu(h)$ where $h \in m(\Sigma)$ and we obtain

$$\begin{aligned}
\varepsilon &\geq \sup_T | g(x) - \sum \alpha_i TT_i f(x) | \\
&= | \mu_T(g(x) - \sum \alpha_i TT_i f(x)) | \\
&= | g(x) - \sum \alpha_i \mu_T(TT_i f(x)) | \\
&= | g(x) - \sum \alpha_i \mu_T(Tf(x)) | \\
&= | g(x) - \mu_T(Tf(x)) | .
\end{aligned}$$

Therefore, $g(x) = \mu_T(Tf(x))$ for each $x \in X$.

Defining $(Pf)(x) = \mu_T(Tf(x))$, we obtain Theorem. In fact, for $T_0 \in \Sigma$, $PT_0f(x) = \mu_T(TT_0f(x)) = \mu_T(Tf(x)) = Pf(x)$ and hence $PT_0 = P$.

COROLLARY 1. Let P be the above projection, then P^* is the mapping of $C(X)^*$ onto

$$\{ \phi : T^*\phi = \phi \text{ for each } T \in \Sigma \}.$$

Moreover, $\phi \in C(X)^*$ is invariant under each $T^* \in \Sigma^*$ if and only if it is invariant under P^*.

Proof. Since $PT = TP = P$ for each $T \in \Sigma$ and $Pf \in \overline{co}\{Tf : T \in \Sigma\}$ for each $f \in C(X)$, we obtain Corollary.

We recall that K is the set of positive normalized measures of $C(X)^*$ and F is the set of functions invariant under each $T \in \Sigma$. An element of K is called probability.

COROLLARY 2. Let P the above projection and let

$$\Phi = \{ \phi \in K : T^*\phi = \phi \text{ for each } T \in \Sigma \}$$

Then, for distinct elements $\phi, \psi \in \Phi$, there exists an invariant function $g \in F$ such that $\phi(g) \neq \psi(g)$.

Proof. If $\phi \neq \psi$, there exists $f \in C(X)$ such that $\phi(f) \neq \psi(f)$. From

$$\phi(Pf) = P^*\phi(f) = \phi(f) \neq \psi(f) = P^*\psi(f) = \psi(Pf),$$

if $Pf = g$, we obtain Corollary.

We can prove the converse of Corollary 2.

THEOREM 3. Let $\{T\} = \Sigma$ be an amenable semigroup of

Markov operators on $C(X)$. Suppose that for distinct elements $\phi, \psi \in \Phi$, there exists an invariant function $f \in F$ such that $\phi(f) \neq \psi(f)$. Then $\Sigma = \{T\}$ is ergodic.

Proof. For $x \in X$, δ_x denotes the point measure at x. $\{T^*\delta_x : T \in \Sigma\}$ is invariant under each $T_0 \in \Sigma$ and so is the weak*-closed convex hull $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$ of $\{T^*\delta_x : T \in \Sigma\}$. Since $\Sigma^* = \{T^*\}$ is amenable, by Day's fixed point theorem [2], $\Sigma^* = \{T^*\}$ has an invariant probability measure ϕ_x in $W^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$. From $T^*\delta_x(f) = Tf(x) = f(x)$ for all $f \in F$, we obtain that the invariant measure ϕ_x is unique in $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$. The weak*-continuity of the mapping $x \longrightarrow \phi_x$ follows from the facts that $f(x) = T^*\delta_x(f) = \phi_x(f)$ for $f \in F$ and that invariant functions separate invariant probability measures. Defining $Pf(x) = \phi_x(f)$ for each $f \in C(X)$, we obtain $Pf \in C(X)$. Now, we show that for each $f \in C(X)$, Pf is a $\{T\}$-invariant function and Pf is contained in $\overline{co}\{Tf : T \in \Sigma\}$. In fact, for $\phi \in C(X)^*$, let $Q\phi$ be a unique invariant measure in $w^*\overline{co}\{T^*\phi : T \in \Sigma\}$. Then we obtain $P^*\phi = Q\phi$ from that invariant functions separate invariant probability measures. On the other hand, we obtain $T^*Q = QT^* = Q$ for all T in Σ. Hence, we have $T^*P^* = P^*T^* = P^*$. By using this, it follows that for each $f \in C(X)$, Pf is a $\{T\}$-invariant function. If Pf is not contained in $\overline{co}\{Tf : T \in \Sigma\}$, there exists $\phi \in C(X)^*$ such that $\phi(Pf) > \sup[\phi(Tg) : g \in \overline{co}\{Tf : T \in \Sigma\}]$. From

$$\sup[\phi(g) : g \in \overline{co}\{Tf : T \in \Sigma\}] \geq \phi(Pf),$$

we obtain a contradiction. Hence for each $f \in C(X)$, Pf is contained in $\overline{co}\{Tf : T \in \Sigma\}$.

Since this is true for each $T \in \Sigma$, it follows that $\{T\} = \Sigma$ is ergodic.

The following Theorem is an extension of theorem 2 in [5].

THEOREM 4. Let $\{T\} = \Sigma$ be an amenable semigroup of Markov operators on $C(X)$ and ergodic and let

$$\Phi = \{ \phi \in K : T^{*}\phi = \phi \text{ for each } T \in \Sigma \}$$

Then, $\phi \rightarrow I_{\phi}$ is a bijective mapping of the set ex Φ of extreme points of Φ onto the set of maximal $\{T\}$-ideals. Φ is simplex inthe sense of [4] and ex Φ is weak*-closed. Moreover, every $\{T\}$-ideal of the form I_{ϕ} ($\phi \in \Phi$) is the intersection of all maximal $\{T\}$-ideals containing it.

Proof. To show that Φ is simplex in the sense of [4], it is sufficient that ϕ^{+} is P^{*}-invariant whennever ϕ is. Since $\phi^{+} \geq 0$ and $\phi^{+} \geq \phi$, $P^{*}\phi^{+} \geq 0$ and $P^{*}\phi^{+} \geq P^{*}\phi = \phi$, and hence $P^{*}\phi^{+} \geq \phi^{+}$. Therefore, $(P^{*}\phi^{+} - \phi^{+})(e) = 0$ shows $P^{*}\phi^{+} = \phi^{+}$.

Since an element of Φ is invariant under P^{*}, it follows that ex Φ is weak*-compact.

The remainder is obvious from Theorem 1, Corollary 1 and [5].

REFERENCES

[1] DAY, M. M., Amenable semigroup. Illinois J. Math.

1 (1957), 509-544.

[2] DAY, M. M., Fixed point theorems for compact convex sets. Illinois J. Math. 5 (1961), 585-590.

[3] DUNFORD, N., AND J. T. SCHWARTZ, Linear operator, Part 1. Interscience, New York (1958).

[4] PHELPS, R. R., Lectures on Choquet's theorem. Van Nostrand (1966).

[5] SCHAEFER, H. H., Invariant ideals of positive operators in C(X). 1. Illinois J. Math. 11 (1967), 703-715.

[6] SCHAEFER, H. H., Invariant ideals of positive operators in C(X). II. Illinois J. Math. 12 (1968), 525-538.

[7] SINE, R., Geometric theory of a single Markov operator. Pacific J. Math. 27 (1968), 155-166.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.

Chapter 6

INVARIANT FUNCTIONS FOR AMENABLE SEMIGROUPS OF POSITIVE CONTRACTIONS ON L^1

INVARIANT FUNCTIONS FOR AMENABLE SEMIGROUPS OF POSITIVE CONTRACTIONS ON L^1

1. Introduction.

Let $(X, \mathfrak{F}, m)$ be a σ-finite measure space and Σ be an amenable semigroup of positive contractions on $L^1(X, \mathfrak{F}, m) = L^1$. In this paper, we are interested in finding necessary and sufficient conditions for the existence of a strictly positive element which is invariant under every element T in an amenable semigroup of positive contractions on L^1 and obtaining a generalization of well known ergodic theorem proved for the case when Σ is the semigroup generated by a single positive contraction T on L^1. So far various necessary and sufficient conditions for the existence of invariant measure equivalent to m have been obtained by several authors for the case when Σ is the semigroup generated by a single positive contraction T on L^1; that is, there are several conditions obtained by Hopf [12], Dowker [5] [6], Calderón [2], Hajian-Kakutani [10] and Sucheston [17] for the case of an operator which arises from a measurable transformation, and by Ito [14] and Hajiān-Ito [11] for the case of an operator which arises from a Markov process. Furthermore, these have been extended elegantly by Neveu [16] for the case of a positive contraction on L^1. In this paper, we extend some results obtained by Neveu [16] to an amenable semigroup of positive contractions on L^1. On the other hand, the ergodic theorem also has been obtained by several authors for the case when Σ is the semigroup generated by a single operator. It was first proved by Birkoff [1] for point transformations with an invariant σ-finite measure. For Markov processes, Kakutani [15] proved it for a finite invariant measure and for bounded functions. Hopf [13] extended it to a finite invariant measure and functions in L^1. Dunford-Schwarz [7] proved it for a σ-finite invariant measuse and functions in L^1.

Main results in this paper are the following; at first, we find necessary and sufficient conditions for the existence of a strictly positive element which is invariant under every T in Σ; see Theorem 1. Secondly, we find necessary and sufficient conditions for the existence of no positive element in L^1 which is invariant under every T in Σ; see Theorem 2. Finally, we extend the well known ergodic theorem to an arbitrary amenable semigroup which has been proved for a single positive contraction T with a strictly positive invariant function in L^1; see Theorem 3. It is interesting to note that essentially same results (Theorem 1) were performed by Y. Ito (Brown University) in Japan.

The author wishes to express his hearty thanks to Professor H. Umegaki and Professor T. Shimogaki for many kind suggestions and advices in the course of preparing the present paper.

2. Preliminaries.

Let Σ be an abstract semigroup and $m(\Sigma)$ be the space of all bounded real valued functions on Σ with the supremum norm. For each $s \in \Sigma$ and $f \in m(\Sigma)$, we define elements f_s and f^s in $m(\Sigma)$ given by $f_s(t) = f(st)$ and $f^s(t) = f(ts)$ for all $t \in \Sigma$. An element $\mu \in m(\Sigma)^*$ (the dual space of $m(\Sigma)$) is called mean on $m(\Sigma)$ if $\|\mu\| = \mu(1) = 1$. A mean μ is called left [right] invariant if $\mu(f_s) = \mu(f)$ [$\mu(f^s) = \mu(f)$] for all $f \in m(\Sigma)$ and $s \in \Sigma$. An invariant mean is a left and a right invariant mean. A semigroup that has a left invariant mean [right invariant mean] is called left amenable [right amenable]. A semigroup that has an invariant mean is called amenable. Let Σ be an amenable semigroup, then $\Sigma s \cap \Sigma t \neq \phi$ and $s\Sigma \cap t\Sigma \neq \phi$ for all $s, t \in \Sigma$. So, if we define an order $t \geq s$ by $t \in \Sigma s \cup \{s\}$, Σ is a directed set.

LEMMA 1. Let Σ be a semigroup and M be the closed linear span of the subset $\{ f_s - f, f^s - f : f \in m(\Sigma) \text{ and } s \in \Sigma \}$ of $m(\Sigma)$. Then, Σ is amenable if and only if 1 is not contained in M. If Σ is an amenable semigroup with the order defined by the above and f is an element of $m(\Sigma)$, then we have

$$\sup_{s} \inf_{s \leq t} f(t) \leq \mu(f) \leq \inf_{s} \sup_{s \leq t} f(t)$$

for any invariant mean μ on $m(\Sigma)$.

Proof. If Σ is amenable, by definition there exists an invariant mean μ on $m(\Sigma)$. Since $\mu(f) = 0$ for all $f \in M$, it is obvious that 1 is not contained in M. On the other hand, if 1 is not contained in M, then there exists an element $\mu \in m(\Sigma)^*$ such that $\|\mu\| = \mu(1) = 1$ and $\mu(f) = 0$ for all $f \in M$. Therefore, Σ

has an invariant mean. Let c be a real number satisfying

$$c < \sup_{s} \inf_{s \leq t} f(t).$$

Then there exists an element u such that $c < f(t)$ for all $t \geq u$. Since $f^{u}(t) = f(tu) > c$ for all $t \in \Sigma$ and μ is an invariant mean on $m(\Sigma)$, we have $\mu(f) = \mu(f^{u}) \geq \mu(c) = c$. Therefore,

$$\sup_{s} \inf_{s \leq t} f(t) \leq \mu(f).$$

Similarly, we obtain

$$\mu(f) \leq \inf_{s} \sup_{s \leq t} f(t).$$

Throughout this paper, let $(X, \mathfrak{F}, m)$ be a finite or σ-finite measure space and let $L^1 = L^1(X, \mathfrak{F}, m)$ and $L^\infty = L^\infty(X, \mathfrak{F}, m)$ be Banach spaces with their respective norms defined as usual. Since L^∞ is the dual space of L^1, we use this duality to write $\langle f, h \rangle$ for $\int f \cdot h \, dm$, where $f \in L^1$ and $h \in L^\infty$. A notation 1_F is the characteristic function of a measurable set F. Let T be a linear operater on L^1, then we denote the adjoint operater by T^*. If T is a positive contraction on L^1, then T^* is also the positive contraction on L^∞. The following Lemma was proved by Neveu [16].

LEMMA 2. Let λ be a positive linear form defined on L^1; that is, let $\lambda \in (L^\infty)^*_+$. Then there exists the largest element $g \in L^1$ such that the form induced by it on L verifies $g \leq \lambda$. Moreover, the complement $G = \{ x : g(x) = 0 \}$ of the support of g is the largest set in $\mathfrak{F}$ for which there exists a positive function $h \in L^\infty_+$ with $h > 0$ on G and $\lambda(h) = 0$. In particular, if $g > 0$, then $\lambda(h) > 0$ for every $h \in L^\infty_+$, $h \neq 0$. If $g = 0$, then $\lambda(h) = 0$ for at least one $h \in L^\infty$ such that $h > 0$.

3. Finite invariant meamures.

The main part of the following Theorem was proved by Neveu [16] for the case when Σ is the semigroup generated by a single positive contraction T on L^1. The proof is similar to that of Neveu.

THEOREM 1. Let Σ be an amenable semigroup of positive contractions on L^1 and f be an arbitrary but fixed element of L^1 such that $f > 0$ (a.e.). Then, the following conditions are equivalent:

(1) there exists $g \in L^1$ such that $g > 0$ (a.e.) and $Tg = g$ for all T in Σ ;

(2) if $h \in L^\infty_+$ and $\inf_T \langle Tf, h \rangle = 0$, then $h = 0$ (a.e.) ;

(3) if $h \in L^\infty_+$ and $\sup_S \inf_{S \le T} \langle Tf, h \rangle = 0$, then $h = 0$ (a.e.) ;

(4) if $h \in L^\infty_+$ and $\mu_T \langle Tf, h \rangle = 0$ for an invariant mean μ on $m(\Sigma)$, then $h = 0$ (a.e.) ;

(5) if $h \in L^\infty_+$ and $0 \in \overline{co} \{ T^*h : T \in \Sigma \}$, then $h = 0$ (a.e.) (here $\overline{co}$ B is the closed convex hull of $B \subset L^\infty$ in the sense of L^∞-norm);

(6) if $h \in L^\infty_+$ and $\sum_{i=0}^\infty T_i^* h < 2$ for some sequence $\{T_i\}$ in Σ, then $h = 0$;

(7) if $h \in L^\infty_+$ and $\sum_{i=0}^\infty T_i^* h < \infty$ for some sequence $\{T_i\}$ in Σ, then $h = 0$;

(8) $\sum_{i=0}^\infty T_i f = \infty$ for any sequence $\{T_i\}$ in Σ ;

(9) if $F \in \mathfrak{F}$ and $\sum_{i=0}^\infty T^* 1_F \le 1 + \varepsilon$ for some sequence $\{T_i\}$ in Σ, then $F = \phi$ (here ε denotes an arbirary but fixed strict positive real number).

Proof. (1)$\Rightarrow$(2). Let f and f_0 be strictly positive functions in L^1. Since $f_0 \le af + (f_0 - af)^+$ for any real number,

$$\langle Tf_0, h \rangle \leq a \langle Tf, h \rangle + \|(f_0 - af)^+\|_1 \|h\|_\infty$$

and hence if $\inf \langle Tf, h \rangle = 0$, it is seen that

$$\inf_T \langle Tf_0, h \rangle = 0 ..$$

If we take $f_0 = g$, we obtain $\langle g, h \rangle = 0$ and hence $h = 0$ (a.e.).

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(4) is obvious from Lemma 1.

(4)$\Rightarrow$(5). Let $h \in L_+^\infty$ and $0 \in \overline{co}\{T^*h : T \in \Sigma\}$. Then, for $\varepsilon > 0$, there exists an element $\sum_{i=1}^n \alpha_i T_i^* h$ ($\sum_{i=1}^n \alpha_i = 1$ and $0 \leq \alpha_i$ for each i) such that

$$\varepsilon \geq \|\sum_{i=1}^n \alpha_i T_i^* h\|_\infty \|f\|_1 .$$

Now, we have

$$\begin{aligned}
\varepsilon &\geq \|f\|_1 \|\sum_{i=1}^n \alpha_i T_i^* h\|_\infty \\
&\geq \sup_T \|Tf\|_1 \|\sum_{i=1}^n \alpha_i T_i^* h\|_\infty \\
&\geq \sup_T \langle Tf, \sum_{i=1}^n \alpha_i T_i^* h \rangle \\
&\geq \mu_T \langle Tf, \sum_{i=1}^n \alpha_i T_i^* h \rangle \\
&= \sum_{i=1}^n \alpha_i \mu_T \langle T_i Tf, h \rangle \\
&= \sum_{i=1}^n \alpha_i \mu_T \langle Tf, h \rangle = \mu_T \langle Tf, h \rangle .
\end{aligned}$$

Therefore, $\mu_T \langle Tf, h \rangle = 0$. We obtain $h = 0$ (a.e.) by (4).

(5)$\Rightarrow$(6). Let $h \in L_+^\infty$ and $\sum_{i=0}^\infty T_i^* h < 2$ for some sequence $\{T_i\}$ in Σ. From the following inequalities

$$\frac{1}{n} \sum_{i=0}^{n-1} T_i^* h \leq \frac{1}{n} \sum_{i=0}^\infty T_i^* h \leq \frac{2}{n} ,$$

it is seen that $0 \in \overline{co}\{T^*h : T \in \Sigma\}$. Therefore, we obtain $h = 0$ by (5).

(4)$\Rightarrow$(1). We define λ by $\lambda(h) = \mu_T \langle Tf, h \rangle$ for all $h \in L^\infty$. It is obvious that λ is a positive linear form satisfying

$$|\lambda(h)| \leq \|f\|_1 \|h\|_\infty .$$

Also, for any $h \in L^\infty$ and $T_0 \in \Sigma$,

$$\lambda(T_0^* h) = \mu_T \langle Tf, T_0^* h \rangle = \mu_T \langle T_0 Tf, h \rangle$$

$$= \mu_T \langle Tf, h \rangle = \lambda(h) .$$

Since $\lambda \in (L^\infty)^*_+$, there exists the largest element g in L^1 by lemma 2. This element g is invariant under each T in Σ. In fact,

$$\langle Tg, h \rangle = \langle g, T^* h \rangle \leq \lambda(T^* h) = \lambda(h)$$

holds for every positive $h \in L^\infty_+$ and hence $Tg \leq \lambda$. This implies $Tg \leq g$. On the other hand, since $T^* 1 \leq 1$ and $\lambda(T^* 1) = 1$, we have $(\lambda - g)(T^* 1) \leq (\lambda - g)(1)$ and hence $\langle Tg, 1 \rangle \geq \langle g, 1 \rangle$. Therefore, $Tg = g$ for all T in Σ. Now, we show $g > 0$ (a.e.). By (4), if $h \in L^\infty_+$ and $h \neq 0$,

$$0 < \mu_T \langle Tf, h \rangle = \lambda(h) .$$

On the other hand, suppose that $G = \{ x : g(x) = 0 \}$ is nonempty. Then, by Lemma 2, there exists $h \in L^\infty_+$ such that $h > 0$ on G and $\lambda(h) = 0$. This is a contradiction. Therefore, $G = \phi$ and hence we have $g > 0$ (a.e.).

(6)$\Rightarrow$(2). The proof has need of the following Lemma which is a generalization of lemma 3 in [16].

LEMMA 3. Let h be an element in L^∞ such that $0 \leq h \leq 1$ and

$$\sup_S \inf_{S \leq T} \langle Tf, h \rangle = 0 ,$$

then, there exists for each $\delta > 0$ an element h_δ in L^∞_+ such that $h_\delta \leq h$, $\langle f, h - h_\delta \rangle < \delta$ and $\sum_{i=0}^{\infty} T_i^* h_\delta \leq 1$ for some sequence $\{ T_i : I = T_0 \leq T_1 \leq \cdots \}$ in Σ.

Proof. Let f and f_0 be strictly positive elements in L^1. Then, since the condition $\sup \inf \langle Tf, h \rangle = 0$ implies

$$\sup_S \inf_{S \leq T} \langle Tf_0, h \rangle = 0 ,$$

we can choose $U_j \in \Sigma$ inductively such that $\langle f, U_1^* h \rangle < \delta/2$ and

$$\langle (U_{j-1}U_{j-2}\cdots U_1+U_{j-1}\cdots U_2+\cdots+U_{j-1}U_{j-2}+U_{j-1}+I)f,\ U_j^* h\rangle < 2^{-j}\delta .$$

Define

$$h_0 = \sum_{j=0}^{\infty}(U_{j+1}U_j\cdots U_1+U_{j+1}U_j\cdots U_2+\cdots+U_{j+1}U_j+U_{j+1})^* h$$

$$= \sum_{j=0}^{\infty}(U_jU_{j-1}\cdots U_1+U_jU_{j-1}\cdots U_2+\cdots+U_j+I)^* U_{j+1}^* h$$

and then $h_\delta = (h-h_0)^+$. Obviously, $0 \le h_\delta \le h$ and $h_\delta \ge h-h_0$. We will show that $\langle f, h-h_\delta\rangle < \delta$. In fact,

$$\langle f, h-h_\delta\rangle$$

$$\le \langle f, \sum_{j=0}^{\infty}(U_jU_{j-1}\cdots U_1+U_j\cdots U_2+\cdots+U_j+I)^* U_{j+1}^* h\rangle$$

$$= \sum_{j=0}^{\infty}\langle (U_jU_{j-1}\cdots U_1+U_j\cdots U_2+\cdots+U_j+I)f,\ U_{j+1}^* h\rangle$$

$$< \sum_{j=0}^{\infty} 2^{-j}\delta = \delta .$$

To finish the proof of Lemma 3, it suffices to show that

$$F_{i,k} = h_\delta+U_{i+1}^* h_\delta+(U_{i+2}U_{i+1})^* h_\delta+\cdots+(U_{i+k}\cdots U_{i+1})^* h_\delta \le 1$$

for all nonnegative integers i, k. The sufficiency of the above inequality is clear by taking $I=T_0$, $T_1=U_1$, $T_2=U_2U_1$, ..., $T_j=U_jU_{j-1}\cdots U_1$, ... and $i=0$ and letting $k\to\infty$.

It is obvious that $F_{i,0}\le 1$ for all i. Assume that the inequality is true for all i and for the value k-1. From

$$F_{i,k} = h_\delta+U_{i+1}^*(h_\delta+U_{i+2}^* h_\delta+\cdots+(U_{i+k}\cdots U_{i+2})^* h_\delta)$$

$$= h_\delta+U_{i+1}^*(h_\delta+U_{i+1+1}^* h_\delta+\cdots+(U_{i+1+k-1}\cdots U_{i+1+1})^* h_\delta)$$

$$= h_\delta+U_{i+1}^* F_{i+1,k-1},$$

we obtain that $F_{i,k}\le 1$ on $\{x : h_\delta(x)=0\}$. On the other hand, we have that on $\{x : h_\delta(x)>0\}$, $h_\delta = h-h_0$ and hence

$$F_{i,k} = h_\delta+U_{i+1}^* h_\delta+\cdots+(U_{i+k}\cdots U_{i+1})^* h_\delta$$

$$\le h_\delta+U_{i+1}^* h+\cdots+(U_{i+k}\cdots U_{i+1})^* h$$

$$\le h_\delta+h_0 = h \le 1 .$$

This completes the proof of Lemma 3.

We prove that (6)$\Rightarrow$(3). Let h be an element in L_+^∞ such that

$$\sup_S \inf_{S\le T} \langle Tf, h\rangle = 0 .$$

We can assume without loss of generality that $0\le h\le 1$ and

$$\sup_S \inf_{S\le T} \langle Tf, h\rangle = 0 .$$

By Lemma 3, there exists $h_\delta \in L_+^\infty$ such that $h_\delta \le h$, $\langle f, h-h_\delta\rangle < \delta$ and $\Sigma_{i=0}^\infty T_i^* h_\delta \le 1$ for some sequence $\{T_i\}$ in Σ. Since (6) implies $h_\delta = 0$, we obtain $\langle f, h\rangle < \delta$ for all $\delta > 0$. Therefore, we have $h=0$.

(2)$\Rightarrow$(8). Let $\{T_i\}$ be a sequence in Σ and f_0 be a positive element in $L^1 \wedge L^\infty$. If we define $h\in L_+^\infty$ by $h=f_0(1+\Sigma_{i=0}^\infty T_i f)^{-1}$ with the convention $(+\infty)^{-1}=0$, then obviously $h(\Sigma_{i=0}^\infty T_i f)\le f_0$ with the convention $0\cdot\infty=0$ and hence

$$\int h \sum_{i=0}^\infty T f \, dm = \sum_{i=0}^\infty \int h\cdot T_i f \, dm = \sum_{i=0}^\infty \langle T_i f, H\rangle < \infty .$$

Therefore, $\inf_i \langle T_i f, h\rangle = 0$ and hence we obtain $h=0$ by (2).

(8)$\Rightarrow$(7). Let h be an element in L_+^∞ such that $\Sigma_{i=0}^\infty T_i^* h < \infty$ for some sequence $\{T_i\}$ in Σ and f_0 be a strictly positive element in $L^1 \wedge L^\infty$. If we define $f' \in L_+^1$ by

$$f' = f_0(1+\Sigma_{i=0}^\infty T_i^* h)^{-1},$$

then f' is strictly positive and $f'(\Sigma_{i=0}^\infty T_i^* h)\le f_0$. Therefore, we obtain

$$\int f'(\Sigma_{i=0}^\infty T_i^* h)\, dm = \int (\Sigma_{i=0}^\infty T_i f')\cdot h \, dm < \infty$$

and hence $h=0$ by (8).

(7)$\Rightarrow$(6) and (7)$\Rightarrow$(9) are obvious.

(9)$\Rightarrow$(3). Let be an element in L_+^∞ such that

$$\sup_S \inf_{S\le T} \langle Tf, h\rangle = 0 .$$

If $F=\{x : h(x) > a\}$ where a is a strictly positive number, then we obtain

$$\sup_S \inf_{S \le T} \langle Tf, 1_F \rangle = 0 .$$

Let $\varepsilon, \varepsilon' > 0$ and $\delta = \varepsilon\varepsilon'/1+\varepsilon$. From Lemma 3, there exists $h_\delta \in L_+^\infty$ such that $h_\delta \le h$, $\langle f, h-h_\delta \rangle < \delta$ and $\sum_{i=0}^\infty T_i^* h_\delta \le 1$ for some sequence $\{T_i\}$ in Σ. If $F_{\varepsilon,\varepsilon'} = \{ x : h_\delta(x) > 1/1+\varepsilon \}$, then $F_{\varepsilon,\varepsilon'}$ is a subset of F satisfying $\langle f, 1_F - 1_{F_{\varepsilon,\varepsilon'}} \rangle < \varepsilon'$ and

$$\sum_{i=0}^\infty T_i^* 1_{F_{\varepsilon,\varepsilon'}} \le 1 + \varepsilon .$$

Therefore, we obtain $F_{\varepsilon,\varepsilon'} = \phi$ from (9) and hence $F = \phi$. Finally, since a was arbitrary, we obtain $h = 0$.

The following Theorem is a counterpart to Theorem 1; that is, we obtain necessary and sufficient conditions for the existence of no invariant measure weaker than m.

THEOREM 2. Let Σ be an amenable semigroup of positive contractions on L^1 and f be an arbitrary but fixed element in L^1 such that $f > 0$ (a.e.) . Then, the following conditions are equivalent:

(1) if $g \in L_+^1$ and $Tg = g$ for all T in Σ, then $g = 0$;

(2) there exists $h \in L^\infty$ such that $h > 0$ and

$$\inf_T \langle Tf, h \rangle = 0 ;$$

(3) there exists $h \in L^\infty$ such that $h > 0$ and

$$\sup_S \inf_{S \le T} \langle Tf, h \rangle = 0 ;$$

(4) there exists $h \in L^\infty$ such that $h > 0$ and

$$\mu_T \langle Tf, h \rangle = 0$$

where μ is an invariant mean on $m(\Sigma)$;

(5) there exists $h \in L^\infty$ such that $h > 0$ and

$$0 \in \overline{co} \{ T^* h : T \in \Sigma \} ;$$

(6) there exists $h \in L^\infty$ such that $h > 0$ and $\sum_{i=0}^\infty T_i^* h < 2$ for some sequence $\{T_i\}$ in Σ ;

(7) there exists $h \in L^\infty$ such that $h > 0$ and $\sum_{i=0}^\infty T_i^* h < \infty$

for some sequence $\{T_i\}$ in Σ ;

(8) $\sum_{i=0}^{\infty} T_i f < \infty$ for some seqence $\{T_i\}$ in Σ ;

(9) there exist positive real numbers $M_n \uparrow \infty$ and elements $F_n \uparrow X$ in $\mathfrak{F}$ such that $\sum_{i=0}^{\infty} T_i^* 1_{F_n} \leq 1 + M_n$ for some sequence $\{T_i\}$ in Σ .

Proof. (1)$\Rightarrow$(2), (3) or (4). Let $\lambda(h) = \mu_T \langle Tf, h \rangle$ for all $h \in L^\infty$. Then $\lambda \in (L^\infty)^*_+$. From Lemma 2, we can find $g \in L^1_+$ such that $g \leq \lambda$ and $Tg = g$ for all $T \in \Sigma$. Since (1) is valid, $g = 0$. Therefore, by Lemma 2, there exists $h \in L^\infty$ such that $h > 0$ on $X = \{x : g(x) = 0\}$ and

$$0 = \lambda(h) = \mu_T \langle Tf, h \rangle$$
$$\geq \sup_S \inf_{S \leq T} \langle Tf, h \rangle \geq \inf_T \langle Tf, h \rangle \geq 0 .$$

(4)$\Rightarrow$(3)$\Rightarrow$(2) is obvious.

(2)$\Rightarrow$(1). Let f, $f_0 > 0$ be elements in L^1. Since the condition $\inf \langle Tf, h \rangle = 0$ implies

$$\inf_T \langle Tf, h \rangle = 0 ,$$

by taking $f_0 = g$, we obtain

$$\inf_T \langle Tg, h \rangle = \langle g, h \rangle = 0 .$$

Therefore we have $g = 0$.

(5)$\Rightarrow$(4) and (6)$\Rightarrow$(5) are obvious from (4)$\Rightarrow$(5) and (5)$\Rightarrow$(6) in Theorem 1.

(3)$\Rightarrow$(6). The proof has need of the follwing Lemma.

LEMM 4. Let h be a element in L^∞ such that $0 < h \leq 1$ and

$$\sup_S \inf_{S \leq T} \langle Tf, h \rangle = 0 ,$$

then there exists an element $h' \in L^\infty$ such that $0 < h' \leq h$ and $\sum_{i=0}^{\infty} T_i^* h' \leq 1$ for some sequence $\{T_i ;\ I = T_0 \leq T_1 \leq \cdots\}$ in Σ .

Proof. Let f, f_0 be strictly positive elements in L^1. Then, since the condition $\sup \inf \langle Tf, h \rangle = 0$ implies

$$\sup_{S} \inf_{S \leq T} \langle Tf_0, h \rangle = 0,$$

we can find $U_j \in \Sigma$ inductively such that $\langle f, U_1^* h \rangle < 1/2^2$ and

$$\langle (U_{j-1} U_j \cdots U_1 + U_{j-1} \cdots U_2 + \cdots + U_{j-1} U_{j-2} + U_{j-1} + I) f, U_j^* h \rangle < 2^{j+1}.$$

For $i = 1, 2, \cdots$, define

$$h_i = \sum_{j=i}^{\infty} (U_{j+1} U_j \cdots U_1 + U_{j+1} \cdots U_2 + \cdots + U_{j+1} U_j + U_{j+1})^* h$$
$$= \sum_{j=i}^{\infty} (U_j U_{j-1} \cdots U_1 + U_j \cdots U_2 + \cdots + U_j + I)^* U_{j+1}^* h$$

and $h_{2^{-i}} = (h - h_i)^+$. Obviously, $0 \leq h_{2^{-i}} \leq h$ and $h_{2^{-i}} \geq h - h_i$ for all i. We can find show by the method of Lemma 3 that $\langle f, h - h_{2^{-i}} \rangle \leq 2^{-i}$ for all i. Define $h' \in L_+^\infty$ by

$$h' = \sum_{i=0}^{\infty} \frac{1}{2^i} h_{2^{-i}} .$$

Then, we have $\{ x : h'(x) > 0 \} = \bigcup_i \{ x : h_{2^{-i}}(x) > 0 \}$ and

$$\int_{\{h_{2^{-i}}=0\}} f \cdot h \, dm \leq \int f(h - h_{2^{-i}}) \, dm \leq \frac{1}{2^i}.$$

Therefore, we obtain $h' > 0$ (a.e.). Also, we can show by the method of Lemma 3 that

$$F_{i,k,p} = h_{2^{-p}} + U_{i+1}^* h_{2^{-p}} (U_{i+2} U_{i+1})^* h_{2^{-p}} + \cdots$$
$$+ (U_{i+k} \cdots U_{i+1})^* h_{2^{-p}} \leq 1$$

for all nonnegative integers i, k and a fixed nonnegative integer p. Therefore, taking $i = 0$ and $T_0 = I$, $T_1 = U_1$, $\cdots$, $T_j = U_j U_{j-1} \cdots U_1, \cdots$ and letting $k \to \infty$, we obtain $\sum_{i=0}^{\infty} T_i^* h_{2^{-p}} \leq 1$ for all p. Therefore, $\sum_{i=0}^{\infty} T_i^* h' \leq 1$. This completes the proof of Lemma 4.

The proof of (3)$\Rightarrow$(6) is obvious from Lemma 4. (6)$\Rightarrow$(7) is also clear. (7)$\Rightarrow$(8) is obvious from (8)$\Rightarrow$(7) in Theorem 1.

(8)$\Rightarrow$(2). Let $\{T_i\}$ be a sequence in Σ such that $\sum_{i=0}^{\infty} T_i f$

If we define $h = f_0(1 + \sum_{i=0}^{\infty} T_i f)^{-1}$ where f_0 is a strictly positive element in $L^1 \cap L^{\infty}$, then h is strictly positive and

$$\inf_T \langle Tf, h \rangle = 0 .$$

In fact, from $h(\sum_{i=0}^{\infty} T_i f) \leq f_0$, we obtain

$$\int h(\sum_{i=0}^{\infty} T_i f)\, dm = \sum_{i=0}^{\infty} \langle T_i f, h \rangle < \infty .$$

(3)$\Rightarrow$(9). Let h be an element in L^{∞} such that $0 < h \leq 1$ and

$$\sup_S \inf_{S \leq T} \langle Tf, h \rangle = 0 .$$

Then by Lemma 4, there exists $h' \in L^{\infty}$ such that $0 < h' \leq h$ and $\sum_{i=0}^{\infty} T_i^* h' \leq 1$ for some $I = T_0 \leq T_1 \leq \cdots$. Since h' is strictly positive, there exist positive real numbers $M_n \uparrow \infty$ such that

$$F_n = \{ x : h'(x) > 1/1 + M_n \} \uparrow X .$$

We can also show that $\sum_{i=0}^{\infty} T_i^* 1_{F_n} \leq 1 + M_n$. In fact, since $(1 + M_n)h' \geq 1_{F_n}$,

$$\sum_{i=0}^{\infty} T_i^* 1_F \leq (1 + M_n) \sum_{i=0}^{\infty} T_i^* h' \leq 1 + M_n .$$

(9)$\Rightarrow$(6). Let $M_n \uparrow \infty$ and $F_n \uparrow X$ be positive real numbers and elements in $\mathfrak{F}$ such that $\sum_{i=0}^{\infty} T_i^* 1_{F_n} \leq 1 + M_n$ for sequence $\{T_i\}$ in Σ . If we choose k_n such that $1 + M_n < 2^{k_n}$ and define

$$h = \sum_{n=0}^{\infty} \frac{1}{2^{k_n + n}} 1_{F_n - F_{n-1}} ,$$

then h is strictly positive and $\sum_{i=0}^{\infty} T_i^* h < 2$. In fact,

$$\sum_{i=0}^{\infty} T_i^* h = \sum_{i=0}^{\infty} T_i^* (\sum_n \frac{1}{2^{k_n + n}} 1_{F_n - F_{n-1}})$$

$$= \sum_n \frac{1}{2^{k_n+n}} (\sum_i T_i^* 1_{F_n - F_{n-1}}) \leq \sum_n \frac{1}{2^{k_n+n}} (\sum_i T_i^* 1_F)$$

$$\leq \sum_n \frac{1}{2^{k_n+n}} (1 + M_n) < \sum_n \frac{1}{2^n} = 2 .$$

This completes Theorem 2.

4. Ergodic theorem.

In this section, let $(X, \mathfrak{F}, m)$ be a finite measure space. The following Theorem is a generalization of the well known ergodic theorem for the case when Σ is the semigroup generated by a single positive contraction T in L^1.

THEOREM 3. Let $(X, \mathfrak{F}, m)$ be a finite measure space and Σ be an amenable semigroup of positive contractions T on L^1 and soppose that $T1 = 1$ for all T in Σ. If $f \in L^1$ and $\mathfrak{B} = \{ A : T^* 1_A = 1_A, T \in \Sigma \}$, then the conditional expectation $E(f|\mathfrak{B})$ of f with respect to $\mathfrak{B}$ is contained in $\overline{co}\{Tf : T \in \Sigma\}$ where $\overline{co}\, B$ is the closed convex hull of $B \subset L^1$ in the sense of L^1-norm.

Proof. Since $T^*1 = 1$ for all T in Σ, it is obvious that $X \in \mathfrak{B}$. That $\mathfrak{B}$ is a σ-field is obvious from that $\mathfrak{B}_T = \{ A : T1_A = 1_A \}$ for each T in Σ is a σ-field. We will show that $\{ Tf : T \in \Sigma \}$ is weakly sequentially compact. To show this, it suffices to show that the countable additivity of the integrals Tf dm is uniform with respect to T in Σ. (See p.292 in [8].)

Let $f_n = \min (f, n1)$ for all $n = 1, 2, \cdots$ and $\varepsilon > 0$. Since $f_n \uparrow f$, by the Lebesque's convergence theorem, there exists a positive integer n_0 such that $\|f - f_{n_0}\|_1 < \varepsilon/2n_0$. Fix this positive integer n_0 and determine a positive number $\delta = \varepsilon/2n_0$. If $m(E) < \delta$, we have

$$\begin{aligned} < Tf, 1_E > &\leqq < Tf_{n_0}, 1_E > + < Tf - Tf_{n_0}, 1_E > \\ &\leqq < T(n_0 1), 1_E > + \|T(f - f_{n_0})\| \\ &\leqq n_0 m(E) + \|f - f_{n_0}\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all T in Σ. Therefore, it follows that the countable additivity of the integrals $\int_E Tf\, dm$ is uniform with respect to T in Σ. Since $\{Tf : T \in \Sigma\}$ is weakly sequentially compact, it follows that $\overline{co}\,\{Tf : T \in \Sigma\}$ is weakly compact. (See p.430 and p.434 in [8].) On the other hand, since $\{Tf : T \in \Sigma\}$ is invariant under each T in Σ and each T in Σ is weakly continuous and linear, $\overline{co}\,\{Tf : T \in \Sigma\}$ is also invariant under each T in Σ. Now, by using Day's fixed point theorem [4], we can find an element $u \in \overline{co}\,\{Tf : T \in \Sigma\}$ such that $Tu = u$ for all T in Σ. We will show that this element u is $\mathcal{B}$-measurable. Let a be a real number, then it is obvious that $T(u - a1) = u - a1$ for all T in Σ. Therefore,

$$(u - a1)^+ - (u - a1)^- = T(u - a1)^+ - T(u - a1)^- .$$

By positivity of T, we obtain that

$$(u - a1)^+ \leqq T(u - a1)^+ \quad \text{and} \quad (u - a1)^- \leqq T(u - a1)^- .$$

Hence, it follows by $\|T\| \leqq 1$ that

$$(u - a1)^+ = T(u - a1)^+ \quad \text{and} \quad (u - a1)^- = T(u - a1)^- .$$

Therefore,

$$T \min(1, n(u - a1)^+) \leqq \min(1, n(u - a1)^+)$$

for all $n = 1, 2, \cdots$ and hence

$$T \min(1, n(u - a1)^+) = \min(1, n(u - a1)^+) .$$

(See p.16 in [9].) Since $\min(1, n(u-a1)^+) \uparrow 1_{\{u>a\}}$ as $n \to \infty$, we obtain that $T1_{\{u>a\}} = 1_{\{u>a\}}$ for all T in Σ. Therefore, u is $\mathcal{A}$-measurable. By using this, we will obtain that u is $\mathcal{G}$-measurable. In fact, since $T^*1_{\{u>a\}} \leqq T^*1 = 1$, we obtain

$$1_{\{u>a\}} T^* 1_{\{u>a\}} \leqq 1_{\{u>a\}}$$

and hence

$$\int (1_{\{u>a\}} - 1_{\{u>a\}} T^* 1_{\{u>a\}})\, dm$$

$$= m(\{u>a\}) - \int 1_{\{u>a\}} T^* 1_{\{u>a\}}\, dm$$

$$= m(\{u>a\}) - \int T1_{\{u>a\}} 1_{\{u>a\}}\, dm$$

$$= m(\{u>a\}) - m(\{u>a\}) = 0 .$$

Therefore, we have

$$1_{\{u>a\}} = 1_{\{u>a\}} T^* 1_{\{u>a\}} .$$

On the other hand, since

$$\int T^* 1_{\{u>a\}}\, dm = \int T1 \cdot 1_{\{u>a\}}\, dm$$

$$= \int 1_{\{u>a\}}\, dm = m(\{u>a\}) ,$$

we obtain $T^*1_{\{u>a\}} = 1_{\{u>a\}}$. Therefore u is $\mathcal{A}$-measurable. Now, we will show that $u = E(f|\mathcal{G})$. Let $A \in \mathcal{G}$ and $\sum_{i=1}^{n} \alpha_i T_i f$ be an elemen in $\overline{co}\{Tf : T \in \Sigma\}$ where $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geqq 0$ for $i = 1, 2, \cdots$. Then

$$< \sum_{i=1}^{n} \alpha_i T_i f, 1_A > = \sum_{i=1}^{n} \alpha_i < f, T_i^* 1_A >$$

$$= \sum_{i=1}^{n} \alpha_i < f, 1_A > = < f, 1_A > .$$

If $\Sigma_{i=1}^{n} \alpha_i T_i f \to u$ in the sense of L_1-norm , it follows that $\langle u, 1_A \rangle = \langle f, 1_A \rangle$. On the other hand, we know that

$$\langle f, 1_A \rangle = \langle E(f|\mathcal{B}), 1_A \rangle .$$

Therefore,

$$\langle u, 1_A \rangle = \langle E(f|\mathcal{B}), 1_A \rangle$$

for all $A \in \mathcal{B}$. Since u is $\mathcal{B}$-measurable, we obtain that

$$u = E(f|\mathcal{B}) .$$

REMARK. It is obvious that u is a unique invariant function in $\overline{co}\{Tf : T \in \Sigma\}$. For the case when Σ is the semigroup generated by a single positive contraction T on L^1, $1/n \Sigma_{i=0}^{n-1} T^i f$ tends to $E(f|\mathcal{B})$ in the sense of L^1-norm.

REFERENCES

[1] BIRKHOFF, G. D., Proof of the ergodic theorem. Proc. Nat. Acad. Sci. U.S.A. 17 (1931), 656-660.

[2] CALDERÓN, A. P., Sur les mesures invariantes. C. R. Acad. Sci. Paris 240 (1955), 1960-1962.

[3] DAY, M. M., Amenable semigroup. Illinois J. Math. 1 (1957), 509-544..

[4] DAY, M. M., Fixed point theorem for compact convex sets. Illinois J. Math. 5 (1961), 585-590.

[5] DOWKER, Y. N., On measurable transformations on finite measure spaces. Ann. of Math. (2) 62 (1955), 504-516.

[6] DOWKER, Y. N., Sur les applications mesurables. C. R. Acad. Sci. Paris 242 (1956), 329-331.

[7] DUNFORD, N., AND J. T. SCHWARTZ, Convergence almost everywhere of operator averages. J. Rational Mech. Anal. 5 (1956), 129-178.

[8] DUNFORD, N., AND J. T. SCHWARTZ, Linear operators,

Part 1. Interscience, New York (1958).

[9] FOGUEL, S. R., Thr ergodic theory of Markov processes. Van Nostrand (1969).

[10] HAJIAN, A. B., AND S. KAKUTANI, Weakly wandering sets and invariant measures. Trans. Amer. Math. Soc. 110 (1964), 136- 151.

[11] HAJIAN, A. B., AND Y. ITO, Iterates of measurable transformations and Markov operators. Trans. Amer. Math. Soc. 117 (1965), 371-386.

[12] HOPF, E., Theory of measures and invariant integrals. Trans. Amer. Math. Soc. 34 (1932), 373-393.

[13] HOPF, E., The general temporally discrete Markoff process. J. Rational Mech. Anal. 3 (1954), 13-45.

[14] ITO, Y., Invariant measures for Markov processes. Trans. Amer. Math. Soc. 110 (1964), 152-184.

[15] KAKUTANI, S., Ergodic theorems and the Markoff process with a stable distribution. Proc. Imp. Acad. Tokyo 16 (1940), 49-54.

[16] NEVEU, J., Existence of bounded invariant measures in ergodic theory. Fifth Berkeley Symposium on Math. Statistics vol II, Part 2 (1967), 461-472.

[17] SUCHESTON, L., On existence of finite invariant measures. Math. Z. 86 (1964), 327-336.

[18] TAKAHASHI, W., Invariant ideals for amenable semigroups of Markov operators. To appear.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.

Chapter 7

ERGODIC THEOREMS FOR AMENABLE SEMIGROUPS OF POSITIVE CONTRACTIONS ON L^1

ERGODIC THEOREMS FOR AMENABLE SEMIGROUPS OF POSITIVE CONTRACTIONS ON L^1

1. Introduction.

In [8], Schaefer defined that a bounded operator T on a Banach space B is called ergodic if for each $f \in B$, the convex closure $\overline{co}\{f, Tf, \cdots\}$ of the orbit $\{f, Tf, \cdots\}$ contains a fixed vector g of T. It is well known that this fixed vector is unique in $\overline{co}\{f, Tf, \cdots\}$ whenever $\{T^n : n \in N\}$ is equicontinuous [7, p.8-11]. Now, we can extend the notation of ergodicity defined for the case of the semigroup generated by a single operator T to an amenable semigroup $\Sigma = \{T\}$ of operators on B. However, it is not known whether for $f \in B$, a fixed vector g in $\overline{co}\{Tf : T \in \Sigma\}$ is unique or not.

In this paper, at first when $(X, \mathcal{F}, m)$ is a finite measure space, we prove that an amenable semigroup $\Sigma = \{T\}$ of positive contractions on $L^1(X, \mathcal{F}, m)$ with a strictly positive function g invariant under T in Σ is ergodic and for each $f \in L^1(X, \mathcal{F}, m)$, a fixed vector u in $\overline{co}\{Tf : T \in \Sigma\}$ is unique and it can be represented as the conditional expectation of f relative to a subfield of $\mathcal{F}$. This theorem has been already proved by the author [11] for the case when g is the constant function 1. Seco ndly, we shall find a sufficient condition for ergodicity of an amenable semigroup of positive contractions on L^1, which is weaker than the conditin of the above theorem. Finally, we obtain a characterization of extreme points of

$\{ f \in L^1 : f \geq 0, \|f\|=1, Tf=f, T \in \Sigma \}$ of $L^1(x, \mathfrak{F}, m)$. This is a generalization of the result obtained by Schaefer [8] for the case of the semigroup generated by a single operator T.

The author wishes to express his hearty thanks to Professor H. Umegaki and Professor T. Simogoki for many kind suggestions and advices in the course of preparing the present paper.

2. Ergodic theorems.

We extend the notation of ergodicity defined for the case of the semigroup generated by a single operator T on a Banach space to an amenable semigroup of operators on B; that is, an amenable semigroup $\Sigma = \{ T \}$ of operators on B is ergodic if and only if for each $f \in B$, the convex closure $\overline{co}\{Tf : T \in \Sigma\}$ of $\{Tf : T \in \Sigma\}$ contains an invariant vector u for all T in Σ.

Throughout this paper, let $(x, \mathfrak{F}, m)$ be a finite measure space and let $L^1 = L^1(X, \mathfrak{F}, m)$ and $L^\infty = L^\infty(X, \mathfrak{F}, m)$ be Banach spaces of all real valued $\mathfrak{F}$-measurable functions on X, with their respective norms defined as usual. Since L^∞ is the dual space of L^1, we use this duality to write $< f, h >$ for $\int f \cdot h \, dm$, where $f \in L^1$ and $h \in L^\infty$. A notation 1_F is the characteristic function of a measurable set F. A positive contraction operator T on L^1 is a linear mapping of L^1 into itself such that $\|T\| \leqq 1$ and $Tf \geq 0$ for $f \geq 0$ and we will denote by $\Sigma = \{ T \}$ an amenable semigroup of positive contractions on L^1.

The following Theorem was proved by the author in [11]. We will use this for proving Theorem 2.

THEOREM 1. Let $(X, \mathcal{F}, m)$ be a finite measure space and $\Sigma = \{T\}$ be an amenable semigroup of positive contractions on L^1 and suppose that $T1 = 1$ for all T in Σ. Then $\Sigma = \{T\}$ is ergodic. If $f \in L^1$ and $\mathcal{B} = \{A \in \mathcal{F} : T^* 1_A = 1_A,\ T \in \Sigma\}$, then a fixed function u is unique in $\overline{co}\{Tf : T \in \Sigma\}$ and the function u is the conditional expectation $E(f|\mathcal{B})$ of f with respect to $\mathcal{B}$.

We sketch the proof. It is obvious that $\mathcal{B}$ is σ-field. By showing that for each $f \in L^1$, $\{Tf : T \in \Sigma\}$ is weakly sequentially compact, we can obtain that $\overline{co}\{Tf : T \in \Sigma\}$ is weakly compact. Now by using Day's fixed point theorem, we can find a function $u \in \overline{co}\{Tf : T \in \Sigma\}$ such that $Tg = g$ for all T in Σ. Finally, we will show that this function u is $\mathcal{B}$-measurable and

$$< u, 1_A > = < E(f|\mathcal{B}), 1_A > .$$

Therefore, we obtain that $u = E(f|\mathcal{B})$; see [11].

THEOREM 2. Let $(X, \mathcal{F}, m)$ be a finite measure space and let $\Sigma = \{T\}$ be an amenable semigroup of positive contractions on L^1 and suppose that there exists $g \in L^1$ such that $g > 0$ and $Tg = g$ for all T in Σ. Then $\Sigma = \{T\}$ is ergodic. If $f \in L^1$, then a fixed function u is unique in $\overline{co}\{Tf : T \in \Sigma\}$ and the function u can be represented as the function $g \cdot E(f|\mathcal{B})/E(g|\mathcal{B})$ where $\mathcal{B} = \{A \in \mathcal{F} : T^* 1_A = 1_A,\ T \in \Sigma\}$.

Proof. If $\nu(B) = \int_B g\, dm$ for $B \in \mathcal{F}$, ν is a measure equivalent to m. Now, we define operators U by

$$U(h) = T(hg) \cdot g^{-1}$$

for $h \in L^1(X, \mathcal{F}, \nu)$, we obtain that $\{U\}$ is an amenable semigroup of positive contractions on $L^1(X,\mathcal{F},\nu)$. In fact, since $T^*1 = 1$,

$$\int Uh\, d\nu = \int T(hg)\cdot g^{-1}\cdot g\, dm$$

$$= \int h\cdot g\, dm$$

$$= \int h\, d\nu$$

for all positive functions $h \in L^1(X,\mathcal{F},\nu)$ and hence U is a contraction on $L^1(X,\mathcal{F},\nu)$. It is obvious that U is positive and $\{U\}$ is an amenable semigroup; see [1].

Now, by using Theorem 1, we obtain that $\{U\}$ is ergodic and if $h \in L^1(X,\mathcal{F},\nu)$, a fixed point v is unique in $\overline{co}\,\{Uh : U \in \{U\}\}$ and the function v can be represented as the conditional expectation $E(h|\mathcal{B}_U)$ of h relative to $\mathcal{B}_U$ where $\mathcal{B}_U = \{A \in \mathcal{F} : U^*1_A = 1_A,\ U \in \{U\}\}$. By using this, we have $\Sigma = \{T\}$ is ergodic and if $f \in L^1$, a fixed point u is unique in $\overline{co}\,\{Tf : T \in \Sigma\}$. Finally, we show that u can be represented as the function $g\cdot E(f|\mathcal{B})/E(g|\mathcal{B})$. In fact, if $A \in \mathcal{B}$, from

$$\int h\cdot U^*1_A\, d\nu = \int T(hg)\cdot g^{-1}\cdot 1_A\cdot g\, dm$$

$$= \int hg\cdot T^*1_A\, dm$$

$$= \int h\cdot 1_A\cdot g\, dm$$

$$= \int h\cdot 1_A\, d\nu$$

for all $h \in L^1(X,\mathcal{F},\nu)$, we obtain that $\mathcal{B} \subset \mathcal{B}_U$. Similarly, we have that if $A \in \mathcal{B}_U$, $A \in \mathcal{B}$. Therefore, we obtain $\mathcal{B} = \mathcal{B}_U$. Since $E(h|\mathcal{B}_U) = E(h|\mathcal{B})$ is $\mathcal{B}$-measurable, for our proof it is sufficient to show that $E(h|\mathcal{B})\,E(g|\mathcal{B}) = E(hg|\mathcal{B})$. This equality is obvious from

$$\int_A E(h|\mathcal{B})\cdot E(g|\mathcal{B})\ dm = \int_A E(h|\mathcal{B})\cdot g\ dm$$

$$= \int_A E(h|\mathcal{B})\ d\nu$$

$$= \int_A h\ d\nu$$

$$= \int_A h\cdot g\ dm$$

$$= \int_A E(hg|\mathcal{B})\ dm$$

for all $A \in \mathcal{B}$. This completes the proof.

The above Theorem is a generalization of Hopf's result and the following Theorem is a generalization of Ito's result obtained for the case when $\Sigma = \{T\}$ is the semigroup generated by a single operator T on L^1.

THEOREM 3. Let $(X, \mathcal{F}, m)$ be a finite measure space and let $\Sigma = \{T\}$ be an amenable semigroup of positive contractions on L^1 and suppose that there exists $g \in L^1$ such that $g > 0$ and $\{Tg : T \in \Sigma\}$ is uniformly integrable, then $\Sigma = \{T\}$ is ergodic. If $f \in L^1$, then a fixed point u is unique in $\overline{co}\{Tf : T \in \Sigma\}$. If $u = Pf$, P is a positive projection of L^1 onto

$$\{ f \in L^1 : Tf = f,\ T \in \Sigma \}$$

such that $P = TP = PT$ for all T in Σ .

Proof. For each $f \in L^1$, we will show that $\{Tf : T \in \Sigma\}$ is weakly sequentially compact. To show this, it suffices to show that the countable additivity of the integrals $\int_E Tf\ dm$ is uniform with respect to T in Σ . (See p.292 in [3].) Let g be an

element in L^1 such that $g > 0$ and $\{ Tg : T \in \Sigma \}$ is uniformly integrable and $f_n = \min(f, ng)$ for all $n=1, 2, \cdots$. Let $\varepsilon > 0$. Since $f_n \uparrow f$, by the Lebesque's convergence theorem, there exists a positive integer n_0 such that $\| f - f_{n_0} \| < \varepsilon/2$. Fix this positive integer n_0 and determine a positive number $\delta = \delta_\varepsilon > 0$ such that $< Tg, 1_E > < \varepsilon/2n_0$ whenever $m(E) < \delta$.
Then, we have

$$\begin{aligned} < Tf, 1_E > &\leqq < Tf_{n_0}, 1_E > + < Tf-Tf_{n_0}, 1_E > \\ &\leqq < T(n_0 g), 1_E > + \| T(f-f_{n_0}) \|_1 \\ &\leqq n_0 < Tg, 1_E > + \| f-f_{n_0} \|_1 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all T in Σ . Therefore, it follows that the countable additivity of the integrals $\int_E Tf\, dm$ is uniform with respect to T in Σ . Since $\{ Tf : T \in \Sigma \}$ is weakly sequentially compact, it follows that $\overline{co}\{ Tf : T \in \Sigma \}$ is weakly compact. (See p.430 and p.434 in [3].) On the other hand, since $\{Tf : T \in \Sigma \}$ is invariant under each T in Σ and each T in Σ is weakly continuous and linear, $\overline{co}\{ Tf : T \in \Sigma \}$ is also invariant under each T in Σ . Now, by using Day's fixed point theorem [2], we can find an element $u \in \overline{co}\{ Tf : T \in \Sigma \}$ such that $Tu = u$ for all T in Σ . We will show that this fixed element u is unique in $\overline{co}\{ Tf : T \in \Sigma \}$. Let $\varepsilon > 0$ and $h \in L^\infty$. Then there exists an element

$$\sum_{i=1}^{n} \alpha_i T_i f \quad (\sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \geqq 0 \text{ for each } i)$$

such that

$$\| \sum_{i=1}^{n} \alpha_i T_i f - u \|_1 \cdot \| h \|_\infty < \varepsilon .$$

Now, for an invariant mean μ on Σ, we have

$$\varepsilon \geq \sup_T \| \sum_{i=1}^{m} \alpha_i TT_i(f-u) \|_1 \cdot \| h \|_\infty$$

$$\geq \sup_T | < \sum_{i=1}^{m} \alpha_i TT_i(f-u), h > |$$

$$> | \mu_T < \sum_{i=1}^{m} \alpha_i TT_i(f-u), h > |$$

$$= | \sum_{i=1}^{m} \alpha_i \mu_T < TT_i(f-u), h > |$$

$$= | \sum_{i=1}^{m} \alpha_i \mu_T < T(f-u), h > |$$

$$= | \mu_T < Tf-u, h > |$$

$$= | \mu_T < Tf, h > - < u, h > | \, .$$

Therefore, $\mu_T < Tf, h > = < u, h >$ for $h \in L^\infty$ and we obtain that u is unique in $\overline{co}\{ Tf : T \in \Sigma \}$. That $P^2 = P$ is obvious from

$$< PPf, h > = \mu_T < TPf, h > = \mu_T < Pf, T^*h >$$

$$= \mu_T \mu_{T_0} < T_0 f, T^*h > = \mu_T \mu_{T_0} < T_0 f, h >$$

$$= \mu_{T_0} < T_0 f, h > = < Pf, h > .$$

Finally, we obtain $P = TP = PT$ from

$$< TPf, h > = < Pf, T^*h > = \mu_{T_0} < T_0 f, T^*h >$$

$$= \mu_{T_0} < TT_0 f, h > = \mu_{T_0} < T_0 f, h >$$

$$= < Pf, h >$$

and

$$< PTf, h > = \mu_{T_0} < T_0 Tf, h > = \mu_{T_0} < T_0 f, h >$$

$$= < Pf, h > .$$

3. A characterization of extreme points.

A subset A of L^1 is called solid if $f \in A$ and $|g| \leqq |f|$, $g \in L^1$, imply that $g \in A$. A band A in L^1 is a closed solid vector subspace of L^1. The smallest band B_A containing a subset A of L^1 is called the band generated by A [7, p.209]. The following Theorem ia a generalization of Schaefer's result [8] obtained for the case when $\Sigma = \{T\}$ is the semigroup generated by a single positive contraction operator T on L^1.

THEOREM 3. Let $\Sigma = \{T\}$ be an amenable semigroup of positive contractions on L^1 and ergodic and define

$$\Phi = \{ f \in L^1 : f \geqq 0, \|f\|=1, Tf=f, T \in \Sigma \} .$$

There exist minimal bands $\neq (0)$ invariant under T in Σ if and only if the set Φ has extreme points, and the latter are in one-to-one correspondence with the former by virtue of $g \to B_g$, where B_g denotes the band in L^1 generated by $\{g\}$.

Proof. We need two Lemmas.

LEMMA 1. Let $\Sigma = \{T\}$ be an amenable semigroup of positive contractions on L^1 and suppose that there exist no non-trivial bands invariant under T in Σ. Then the space of fixed points for all T in Σ is at most one-dimensional.

Proof. Let $Tf = f$ for all T in Σ. Since T is positive and

$$f^+ - f^- = f = Tf = Tf^+ - Tf^- ,$$

we obtain that $f^+ \leqq Tf^+$ and $f^- \leqq Tf^-$ for all T in Σ. Hence it follows by $\|T\| \leqq 1$ that $Tf^+ = f^+$ and $Tf^- = f^-$ for all T in Σ. In fact, we have that

$$0 \leqq \int (Tf^{+}-f^{+})\, dm = \int Tf^{+}\, dm - \int f^{+}\, dm$$

$$\leqq \int f^{+}\, dm - \int f^{+}\, dm = 0 .$$

Since the bands B_{f^+} and B_{f^-} generated by $\{f^+\}$ and $\{f^-\}$ are invariant under T in Σ and lattice disjoint, we obtain that either $f^+ = 0$ or else $f^- = 0$. Thus the fixed space is a totally orderd vector lattice and hence, since it is Archimedean, at most one-dimensional.

LEMMA 2. Let $\Sigma = \{T\}$ be an amenable semigroup of positive contractions on L^1 and ergodic. If g is an extreme point of Φ and f is a fixed function under T in Σ and contained in the weakly closed band $\overline{B}_g$ generated by $\{g\}$, then $f = cg$ for some real number c .

Proof. Let $0 \leqq f \leqq g$ and $f \neq 0$. Since g is an extreme point and

$$g = \|f\| \cdot \frac{f}{\|f\|} + \|g-f\| \cdot \frac{g-f}{\|g-f\|} ,$$

we obtain that $f = cg$ for some c. More generally, suppose that $0 \leqq |f| \leqq kg$ for some real number k. From

$$f^{+} + f^{-} = |f| \leqq kg ,$$

we have $f^{+}/k \leqq g$ and $f^{-}/k \leqq g$. Hence we obtain $f = cg$ for some real number c.

Now, since $\Sigma = \{T\}$ is ergodic, there exists the projection P as in the proof of Theorem 3. This projection P maps B_g onto $\{cg : c \in R\}$ and hence $P(\overline{B}_g) = \{cg : c \in R\}$.

Now, we prove Theorem 4. Let g be an extreme point of Φ and P be the projection as in the proof of Theorem 3. Suppose that D is a band invariant under T in Σ and $(0) \neq D \subset B_g$. From Lemma 2,

$$P(B_g) = \{cg : c \in R\}.$$

Since $D \neq 0$ and $\Sigma = \{T\}$ is ergodic, it follows that $D \cap \Phi = \{g\}$. Hence $g \in D$ and $D = B_g$.

Conversely, let B be a minimal band $\neq (0)$ invariant under T in Σ. The restriction of T to B satisfies the hypothesis of the Lemma 1 and hence $B \cap \Phi$ is a singleton g. If $g = \alpha f_1 + \beta f_2$ for some $f_1, f_2 \in \Phi$ and $\alpha + \beta = 1$, $\alpha, \beta \geqq 0$, we have $g \geqq f_1$ and $g \geqq f_2$. Since B is solid, we obtain $f_1, f_2 \in B$. This is a contradiction. Therefore, g is an extreme point.

REFERENCES

[1] DAY, M. M., Amenable semigroup. Illinois J. Math. 1 (1957), 509-544.

[2] DAY, M. M., Fixed point theorem for compact convex sets. Illinois J. Math. 5 (1961), 585-590.

[3] DUNFORD, N., AND J. T. SCHWARTZ, Linear operators, Part I. Interscience, New York (1958).

[4] FOGUEL, S. R., The ergodic theory of Markov processes. Van Nostrand (1969).

[5] HOPF, E., The general temporally discrete Markoff process. J. Rational Mech. Anal. 3 (1954), 13-45.

[6] ITO, Y., Uniform integrability and the pointwise ergodic theorem. Proc. Amer. Math. Soc. 16 (1965), 222-227.

[7] JACOBS, K., Neuere methoden und ergebnisse der ergoden theorie. Berlin-Göttingen-Heidelberg (1960).

[8] SCHAEFER, H. H., Invariant ideals of positive operators in C(X). I. Illinois J. Math. 11 (1967), 703-715.

[9] SCHAEFER, H. H., Topological vector space. Macmillan, New York (1966).

[10] TAKAHASHI, W., Invariant ideals for amenable semigroups of Markov operators. Kodai Math. Sem. Rep. 23 (1971), in press.

[11] TAKAHASHI, W., Invariant functions for amenable semigroups of positive cotractions on L^1. Kodai Math. Sem. Rèp. (1971), in press.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.

Chapter 8

ADJOINT ERGODIC THEOREM FOR AMENABLE SEMIGROUPS OF OPERATORS

ADJOINT ERGODIC THEOREM FOR AMENABLE SEMIGROUPS OF OPERATORS

1. Introduction.

In [6], Lloyd proved the following theorem:

Let $\{ T(t) : 0 < t < \infty \}$ be a semigroup of operators on a Banach space B such that

$$M = \overline{\lim_{t \to \infty}} \| T(t) \| < \infty .$$

Then there exist operators $P^* \in L(B^*)$ (not necessarily adjoints of operators in $L(B)$) in the closed convex hull of $\{ T(t)^* : 0 < t < \infty \}$ in $L(B^*, B)$ with the property

$$T(t)^* P^* = P^* T(t)^* = P^* , \quad 0 < t < \infty .$$

This theorem is very useful in the ergodic theory of Markov processes.

In this paper, we shall extend Lloyd's theorem to the case of an amenable semigroup by considering an order that the semigroup must possess. Consequently, this is very useful in the ergodic theory for amenable semigroups of positive contractions on L^1.

The author wistes to express his hearty thanks to Professor H. Umegaki and Professor T. Shimogaki for many kind suggestions and advices.

2. Preliminaries.

i. Let Σ be an amenable semigroup, then it is well known that $\Sigma s \cap \Sigma t \neq \phi$ and $s\Sigma \cap t\Sigma \neq \phi$ for all $s, t \in \Sigma$. So, if we define an order $s \leqq t$ by $t \in s\Sigma \uplus \{s\}$ on Σ, Σ is a directed set [8].

Throughout this paper, we shall consider an amenable semigroup with the order defined by the above.

ii. Let B be a Banach space and B* be the dual space of B, then we shall denote by $< f, h >$ the value of $f \in B$ at $h \in B^*$ and denote by $L(B)$ [$L(B^*)$] the set of all bounded linear operators on B [B*], with operator norm topology. Paticularly, we denote by $L(B,B^*)$ [$L(B^*,B)$] the set $L(B)$ [$L(B^*)$] with the weak operator topology determined by B* [B].

LEMMA 1 (Arens [1]). The closed unit ball of $L(B^*)$ is a compact set in $L(B^*,B)$.

3. Adjoint ergodic theorem.

At first, we shall prove the adjoint ergodic theorem for an amenable semigroup of operators on a Banach space by considering the order induced above.

THEOREM 1. Let $\Sigma = \{T\}$ be an amenable semigroup in $L(B)$ such that

$$M = \inf_{S} \sup_{S \leqq T} \| T \| < \infty .$$

Then there exists an operator $P^* \in L(B^*)$ in the closed convex hull of $\{T^* : T \in \Sigma\}$ in $L(B^*,B)$ such that $T^*P^* = P^*$ for all $T \in \Sigma$. Such an operator is a projection onto the subspace of vectors invariant under all T^* in Σ^*.

Proof. Let $\varepsilon > 0$, then there exists $S_0 \in \Sigma$ such that

$$\| T \| < M + \varepsilon \ , \quad S_0 \leqq T \ .$$

Denote by $K(S_0)$ the convex hull of the set $\{T^* : S \leqq T, T \in \Sigma\}$, and by $\overline{K}(S_0)$ the closure of $K(S_0)$ in $L(B^*,B)$. By Lemma 1, $\overline{K}(S_0)$ is a compact convex subset of the multiple $M+\varepsilon$ of the closed unit ball in $L(B^*,B)$.

Since $U^*T_0^* = (T_0U)^*$ for $T_0^* \in \{T^* : S_0 \leqq T, T \in \Sigma\}$ and $U \in \Sigma$ it follows that the set $\{T^* : S_0 \leqq T, T \in \Sigma\}$ is invariant under all $U^* \in \Sigma^*$. And since each U^* in Σ^* is linear and continuous with $L(B^*,B)$-topology, $\overline{K}(S_0)$ is also invariant under $U^* \in \Sigma^*$. Now, by using Day's fixed point theorem, we obtain $P^* \in \overline{K}(S_0)$ such that $T^*P^* = P^*$ for all T in Σ. We shall show that such an operator is a projection. Let $\{V_\alpha\}$ be a generalized sequence in $K(S_0)$ converging in $L(B^*,B)$ to $P^* \in \overline{K}(S_0)$. That is,

$$\lim_\alpha < f, V_\alpha h > = < f, P^*h >$$

for $f \in B$ and $h \in B^*$. From $V_\alpha P^* = P^*$, we obtain that

$$\begin{aligned} < f, P^{*2}h > &= \lim_\alpha < f, V_\alpha P^*h > \\ &= \lim_\alpha < f, P^*h > \\ &= < f, P^*h > . \end{aligned}$$

Therefore, P^* is a projection. Suppose that h_0 is such that $T^*h_0 = h_0$ for $T^* \in \Sigma^*$ and $< f, V_\alpha h > \longrightarrow < f, P^*h >$ for $f \in B$

and $h \in B^*$. Then from

$$< f, P^*h_0 > = \lim_\alpha < f, V_\alpha h_0 >$$

$$= \lim_\alpha < f, h_0 >$$

$$= < f, h_0 > ,$$

it follows that P^* is a projection onto the subspace of vectors invariant under all T^* in Σ^*.

COROLLARY 1. Let $\Sigma = \{T\}$ be a commutative semigroup in $L(B)$ such that

$$M = \inf_S \sup_{S \leqq T} \| T \| < \infty .$$

Then there exists an operator $P^* \in L(B^*)$ in the closed convex hull of $\{T^* : T \in \Sigma\}$ in $L(B^*,B)$ such that $P^*T^* = T^*P^* = P^*$ for all T in Σ. Such an operator is a projection onto the subspace of vectors invariant under all T^* in Σ^* and has the norm $\|P^*\| \leqq M$.

Proof. It is sufficient to show that $\|P^*\| \leqq M$. In fact, let $\{V_\alpha\}$ be a generalized sequence in $K(S_0)$ converging in $L(B^*,B)$ to $P^* \in \overline{K}(S_0)$. Then from

$$\lim_\alpha < f, V_\alpha T^*h > = < f, P^*T^*h > = < f, P^*h >$$

it follows that $\{V_\alpha T^*\}$ is a generalized sequence in $L(B^*,B)$ to P^*. Therefore it follows that $\|P^*\| \leqq M$.

The above Theorem and Corollary have many applications in the ergodic theory of various fields.

REFERENCES

[1] ARENS, R., Operations in function classes. Monatshefte fur Math. U. Physik. 55 (1951), 1-19.

[2] DAY, M. M., Amenable semigroup. Illinois J. Math. 1 (1957), 509-544.

[3] DAY, M. M., Fixed point theorem for compact convex sets. Illinois J. Math. 5 (1961), 585-590.

[4] GRANIRER, E., On amenable semigroups with a finite-dimensional set of invariant means I. Illinois J. Math. 7 (1963), 32-48.

[5] GRANIRER, E., A theorem on amenable semigroups. Trans. Amer. Math. Soc. 111 (1964), 367-379.

[6] LLOYD, S. P., An adjoint theorem. Ergodic theory. Academic press. New York. (1963), 195-201.

[7] TAKAHASHI, W., Invariant ideals for amenable semigroups of Markov operators. Kodai Math. Sem. Rep. 23 (1971), in press.

[8] TAKAHASHI, W., Invariant functions for amenable semigroups of positive contractions on L^1. Kodai Math. Sem. Rep. 23 (1971), in press.

[9] TAKAHASHI, W., Ergodic theorems for amenable semigroups of positive contractions on L^1, Kodai Math. Sem. Rep. 23 (1971), in press.

[10] TAKAHASHI, W., On amenable semigroups of positive contractions on L^1-space. 8-th. Functional Analysis Simposium (1970), 68-81.