論文 / 著書情報
Article / Book Information

題目(和文)	
Title(English)	Two approaches to fuzzy measure theory: integrals based on pseudo-addition and Choquet's integral
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出典(和文)	学位:理学博士, 学位授与機関:東京工業大学, 報告番号:甲第1860号, 授与年月日:1987年3月26日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor of Science, Conferring organization: , Report number:甲第1860号, Conferred date:1987/3/26, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

TWO APPROACHES TO FUZZY MEASURE THEORY:
INTEGRALS BASED ON PSEUDO-ADDITION AND CHOQUET'S INTEGRAL

by

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A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Science

at

Department of Systems Science

Tokyo Institute of Technology

1987

ACKNOWLEDGEMENTS

The author wishes to express his hearty gratitude for the continuing guidance and encouregement recieved from his supervisor Professor Michio Sugeno. He also wishes to express his thanks to Dr. Takehisa Onisawa for his helpful advice. He is indebted to former and present members of Sugeno Laboratory, especially to Dr. M. Sasaki and Mr. S. Ryumae for their valuable observations, and to Miss K. Miyakawa and Mrs. H. Hayashi for their hospitality. Acknowledgement also is due to his wife, Masako, for her understanding and assistance. He is grateful to his parents and his wife's parents for their understanding and support.

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WITH RESPECT TO FUZZY MEASURES

INTRODUCTION

"Mathematics and practice do not always go hand in hand," some people say. "For instance, in practice one and one does not always make two. Let us consider the situation that two persons work. If they do the work of three persons with a united effort, then one and one makes three. If each is a hindrance to the other and they do the work of one and a half persons, then one and one makes one and a half. In mathematics, however, one and one always makes two." (So mathematics goes for little.(?))

There are certainly the cases where one and one does not make two and it is certain that in mathematics the equation $1 + 1 = 2$ always holds. But the author states that it is possible in mathematics that one and one does not make two, and he states that the concept of fuzzy measures is one of the mathematical tools for that.

A fuzzy measure is a monotone set function which is not always additive. With a fuzzy measure μ, the above case is expressed as follows:

$\mu(\{a\}) = 1$,

$\mu(\{b\}) = 1$,

$\mu(\{a, b\}) = 3$ (in the former case) or 1.5 (in the latter case),

where a and b stand for the two persons. The non-additivity of fuzzy measures, i.e., $\mu(A \cup B) \neq \mu(A) + \mu(B)$, expresses the cases where one and one does not make two.

This dissertation discusses fuzzy measures and fuzzy integrals with respect to fuzzy measures. It consists of two parts. Part I discusses pseudo-additive measures (Sugeno and Murofushi [22]), which are special fuzzy measures with the property:

$$\mu(A \cup B) = \mu(A) \hat{+} \mu(B) \quad \text{whenever} \quad A \cap B = \emptyset,$$

where $\hat{+}$ is a pseudo-additon. A pseudo-addition is an operation with the property that one and one does not necessarily make two, that is, the equation $1 \hat{+} 1 = 2$ does not necessarily hold. Part II discusses general fuzzy measures.

The concept of fuzzy measures was proposed by Sugeno [21] for a mathematical expression of fuzziness in contrast to fuzzy sets. Fuzzy sets are sets without precise boundaries (Zadeh [25]), for instance, "the class of real numbers which are much greater than 1" or "the class of beautiful women." A fuzzy set $\tilde{A}$ on a universal set X, where X is an ordinary set with a precise boundary, is characterized by assigning the grade $f_{\tilde{A}}(x)$ of "$x \in \tilde{A}$" to each point x of X. By contrast, a fuzzy measure on X is characterized by assigning the grade $\mu_x(A)$ of certainty of "$x \in A$" to each subset A of X, where x is an ill-located point of X. Dubois and Prade [6] illustrated these two fuzziness as follows.

"X is supposed to be a set of pieces of furniture. Both points of view correspond to the following situations:

* <u>Fuzzy set</u> : the age of each item is precisely known. $f_{\tilde{A}}(x)$ is an assessment of the answer to the question : 'you know the age of x ; do you consider it is old?' $\tilde{A}$ is the fuzzy set of old pieces of furniture.

* <u>Fuzzy measure</u> : the age of each item is unknown. $\mu_x(A)$ is an assessment of the answer to the question : 'By looking at x, do you consider it is more than 200 years old?' A is the non fuzzy set of pieces of furniture being exactly more than 200 years old."

We now state the mathematical definition of a fuzzy measure. Let $(X, \mathcal{X})$ be a measurable space. In [21] a fuzzy measure on $\mathcal{X}$ is a set function $\mu : \mathcal{X} \to [0, 1]$ with the following properties:

(F1) $\mu(\emptyset) = 0, \quad \mu(X) = 1,$

(F2) $A, B \in \mathcal{X}$ and $A \subset B \implies \mu(A) \leq \mu(B),$

(F3) $\{A_n\} \subset \mathcal{X}$ and $A_n \uparrow A \implies \mu(A_n) \uparrow \mu(A),$

(F4) $\{A_n\} \subset \mathcal{X}$ and $A_n \downarrow A \implies \mu(A_n) \downarrow \mu(A).$

The main features of this measure are non-additivity and monotonicity. (In the sence of the above definition the set function μ on page 1 is not a fuzzy measure since it does not satisfy the condition that $\mu(X) = 1$. But this condition is not essential and we remove it in the text.)

By the definition the fuzzy measure is an extension of the probability measure, and therefore the former is more flexible than the latter. This implies that the fuzzy measure is applicable to ambiguous circumstances.

In this regard, the fuzzy measure interests many scientists, engineers, and mathematicians. They have made many studies concerning theory and applications: classification of fuzzy measures, fuzzy integrals with respect to fuzzy measures, fuzzy modelling of subjective evaluation based on fuzzy measures.

Let us survey classification of fuzzy measures. Since a fuzzy measure is a general set function which has only monotonicity, the concept of fuzzy measures includes various non-additive set functions.

A belief function (Shafer [20]) is a set function b defined as follows.

(1) $b(\emptyset) = 0,\ b(X) = 1; \quad 0 \leq b(A) \leq 1 \quad \forall A.$

(2) $b(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_{i=1}^{n} b(A_i) - \sum_{i<j} b(A_i \cap A_j) + \cdots$

$$+ (-1)^{n+1} b(A_1 \cap A_2 \cap \cdots \cap A_n).$$

b(A) is interpreted as a grade of belief that a given element of X belongs to A.

Shafer [20] defined a plausibility measure Pl by

$$Pl(A) = 1 - b(A^c).$$

A plausibility measure Pl has the property:

$$Pl(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_{i=1}^{n} Pl(A_i) - \sum_{i<j} Pl(A_i \cup A_j) + \cdots$$

$$+ (-1)^{n+1} Pl(A_1 \cup A_2 \cup \cdots \cup A_n).$$

Plausibility measures and belief functions have been introduced by Dempster [5] under the names upper and lower probabilities, induced from a probability measure P on a set Y by a multivalued mapping Γ from Y to X ; for every $A \subset X$ Pl(A) and b(A) are defined by

$$Pl(A) = P(A^*)/P(X^*),$$

$$b(A) = P(A_*)/P(X^*),$$

where $A^* = \{y \in Y | \Gamma y \cap A \neq \emptyset\}$ and $A_* = \{y \in Y | \Gamma y \neq \emptyset, \Gamma y \subset A\}$.

A possibility measure proposed by Zadeh [26] is a set function Π defined by

$$\Pi(A) = \sup_{x \in A} f(x),$$

where f is a function from X into the unit interval [0, 1], i.e., a membership function of a fuzzy set. Π has the following property:

$$\Pi(A \cup B) = \Pi(A) \vee \Pi(B),$$

where $\vee$ stands for maximum. Sugeno [21] called this property F-additivity. A possibility measure is a particular case of plausibility measures.

A necessity measure is a set function N defined by

$$N(A) = 1 - \Pi(A^c);$$

the necessity N(A) of an event is the grade of impossibility of the opposite event. N has the following property:

$$N(A \cap B) = N(A) \wedge N(B),$$

where $\wedge$ stands for minimum. Shafer [20] introduced this measure under the name a consonant belief function; a necessity measure is a particular case of belief functions. Dubois and Prade [7] named this measure a necessity measure.

Sugeno [21] investigated the following fuzzy measure:

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B) \quad \text{whenever} \quad A \cap B = \emptyset,$$

where $-1 < \lambda < \infty$. The operation $(s, t) \mapsto s + t + \lambda st$ is called a λ-addition.

Banon [1] investigated the relation between the above fuzzy measures and summarized it in a figure (Fig. 1).

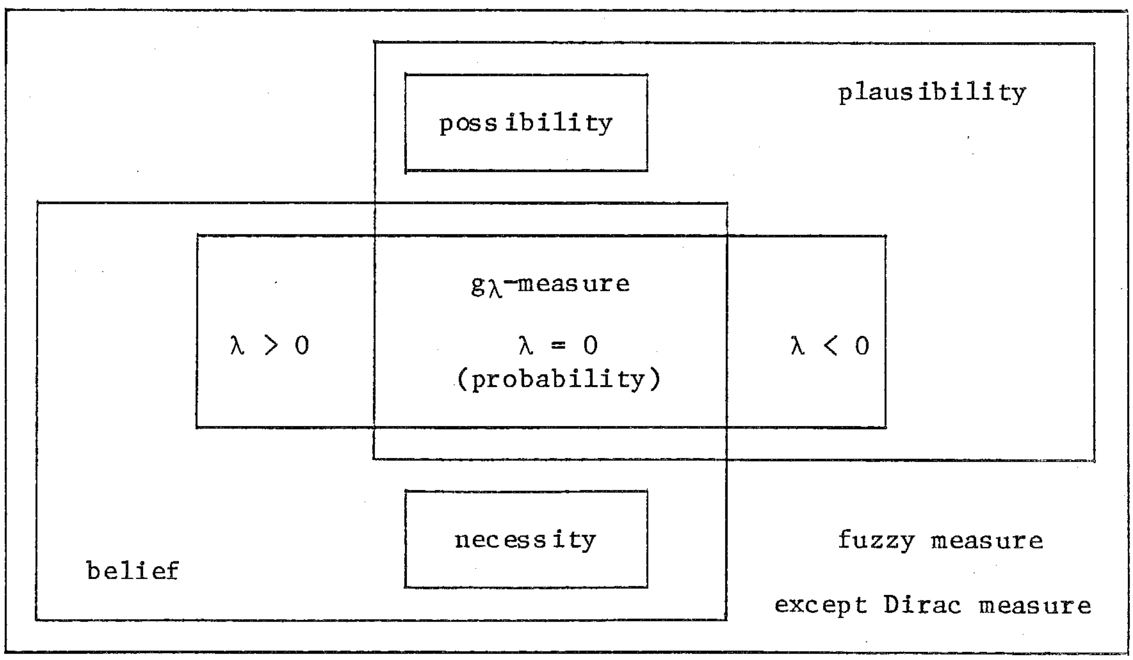


Fig. 1 The inclusion relationship between the various classes of fuzzy measures on a finite set X except Dirac measures.

Dubois and Prade [8] and Weber [24] investigated the fuzzy measures with the property (D):

$$\text{(D)} \qquad \mu(A \cup B) = \mu(A) \perp \mu(B) \quad \text{whenever} \quad A \cap B = \emptyset,$$

where $\perp$ is a t-conorm. A t-conorm is a binary operation on [0, 1] satisfying certain conditions (Schweizer and Sklar [18]); the λ-additions and the max operation $\vee$ are t-conorms. Weber [24] called a set function satisfiying (D) $\perp$-decomposable.

Dubois and Prade [8] further investigated the fuzzy measures with the property:

$$\mu(A \cap B) = \mu(A) \; T \; \mu(B) \quad \text{whenever} \quad A \cup B = X,$$

where T is a t-norm. A t-norm is a binary operation on [0, 1] such that $s \; T \; t = (1 - s) \perp (1 - t)$ for some t-conorm $\perp$ [18].

Now let us survey integration with respect to fuzzy measures. The integral with respect to probability measures is Lebesgue integral. Since fuzzy measures are non-additive, the ordinary definition of Lebesgue integral does not apply to fuzzy measures. Sugeno [21] proposed another integral, "the fuzzy integral," for an integral with respect to fuzzy measures. The fuzzy integral $⨍ f \circ \mu$ of a measurable function $f : X \to [0, 1]$ with respect to a fuzzy measure μ is defined by

$$⨍ f \circ \mu = \sup_{\alpha \varepsilon [0,1]} [\alpha \wedge \mu(\{x | f(x) \geq \alpha\})],$$

where $\wedge$ stands for minimum. This integral as an expectation of f has nice properties. But it is not an extension of Lebesgue integral while the fuzzy measure is an extension of the probability measure. That is, if P is a probability measure, the fuzzy integral with respect to P does not coincide with Lebesgue integral with respect to P;

$$⨍ f \circ P \neq \int f\, dP.$$

Considering good properties of Lebesgue integral, we want an extended Lebesgue integral for fuzzy measures.

Weber [24] defined an extended Lebesgue integral with respect to $\perp$-decomposable measures in case where $\perp$ is a continuous Archimedean t-conorm. If $\perp$ is a continuous Archimedean t-conorm, then $\perp$ is expressed as

$$s \perp t = g^{*}(g(s) + g(t)),$$

where g is a nondecreasing continuous function on [0, 1], which is called an additive generator, and g^{*} is a pseudo-inverse of g [18]. Weber's integral is defined as

$$\int f \perp \mu = g^{*}\left(\int f\, d\bar{\mu}\right),$$

where $\bar{\mu}$ is an ordinary measure for which $\mu = g^{*} \circ \bar{\mu}$. Kruse [12] defined an integral with respect to g_λ in a similar manner:

$$\int f\, dg_\lambda = g^{*}\left(\int g(f)\, d\bar{g}_\lambda\right),$$

where g is the additive generator of λ-addition and $\bar{g}_\lambda$ is an ordinary measure for which $g_\lambda = g^{*} \circ \bar{g}_\lambda$. If $\perp$ has no additive generator, that is, $\perp$ is non-Archimedean, then Weber's and Kruse's integrals cannot be defined.

Weber [24] pointed out that a functional defined by Choquet [3] can be regarded as an integral with respect to fuzzy measures. This functional is defined by

$$(C)\int f\, d\mu = \int_0^\infty \mu(\{x \mid f(x) \geq r\})\, dr.$$

Weber called it Choquet's integral. This integral is an extension of Lebesgue integral. Dempster's upper and lower expected values [5] of a nonnegative function are Choquet's integrals.

Höhle [10] defined another integral. His integral of a simple function

$$h = \sum_{i=1}^{n} b_i 1_{B_i} \qquad (0 \leq b_1 \leq b_2 \leq \cdots \leq b_n,\ B_i \cap B_j = \emptyset \text{ for } i \neq j)$$

is defined as

$$\sum_{m=1}^{n} b_m [\mu(\bigcup_{i=0}^{m} B_i) - \mu(\bigcup_{i=0}^{m-1} B_i)],$$

where $B_0 = X - \bigcup_{i=1}^{n} B_i$. We denote the characteristic function of a set B by 1_B, i.e.,

$$1_B = \begin{cases} 1 & x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Höhle's integral is also an extension of Lebesgue integral.

We have had many studies also concerning applications (Sugeno [21], Seif and Aguilar-Martin [19], Ishii and Sugeno [11], Onisawa, Sugeno, Nishiwaki, Kawai, and Harima [16]). The concept of fuzzy measures has been applied mainly to fuzzy modelling of subjective evaluation.

In Part I of this dissertation we consider a set function μ on a σ-algebra $\mathcal{X}$ of sets with the following properties:

(S1) $\mu(\emptyset) = 0$.

(S2) If $A \in \mathcal{X}$, $B \in \mathcal{X}$, and $A \subset B$, then $\mu(A) \leq \mu(B)$.

(S3) If $A \in \mathcal{X}$, $B \in \mathcal{X}$, and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) \hat{+} \mu(B)$.

(S4) If $\{A_n\} \subset \mathcal{X}$ and $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.

The condition $\mu(X) = 1$ is not essential. The condition (F4) does not fit (S3) (see Puri and Ralescu [17] and Section 1.2 in Part I of this dissertation). For instance, a possibility measure does not satisfy (S4) [17]. A pseudo-addition is defined as a binary operation on $[0, \infty]$ characterized by the above four conditions. It is a continuous "t-conorm" defined on $[0, \infty]$. A pseudo-additive measure is defined as a set function with the above properties if $\hat{+}$ is a pseudo-addition. The class of pseudo-additive measures is large (Fig. 2).

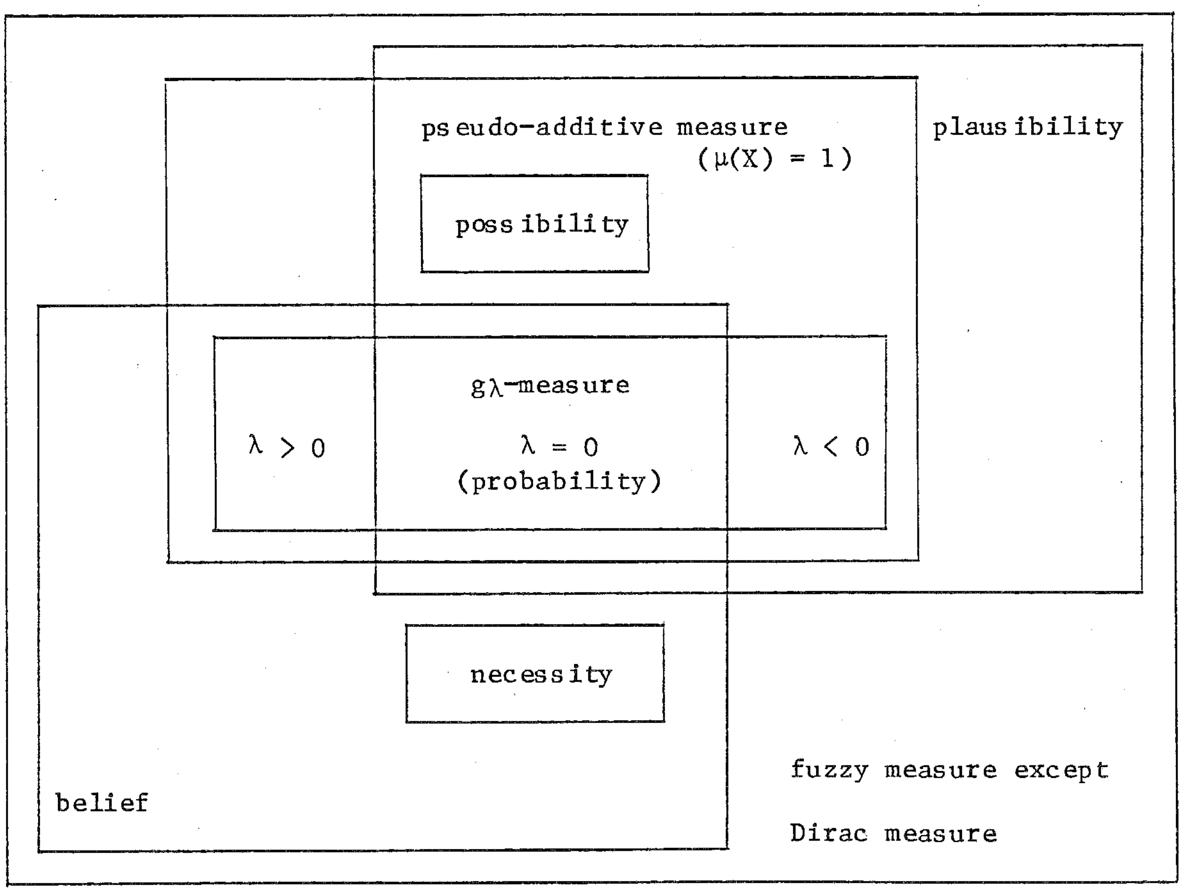


Fig. 2 The inclusion relationship between the class of pseudo-additive measures and the other classes of fuzzy measures.

As previously stated, Weber and Kruse defined integrals for continuous Archimedean t-conorms. We shall deal with non-Archimedean case; we define an extended Lebesgue integral with respect to pseudo-additive measures and a pseudo-addition is generally non-Archimedean. We consider a multiplication-like operation $\hat{\cdot}$ consistent with a pseudo-addition $\hat{+}$. And we define the integral by substituting the addition + and the multiplication • in the definition of Lebesgue integral for $\hat{+}$ and $\hat{\cdot}$, respectively. The resulting integrals include not only Lebesgue integral but also the fuzzy integral with respect to F-additive fuzzy measures.

Considering the original features of fuzzy measures and the argument at the begining of this Introduction, the condition (S3) is too strong. For a given $\hat{+}$, if $1 \hat{+} 1 = 3$, then one and one always makes three. But we can easily imagine a situation expressed by the following fuzzy measure μ:

$\mu(\{a\}) = \mu(\{b\}) = \mu(\{c\}) = 1$,

$\mu(\{a, b\}) = 3$,

$\mu(\{a, c\}) = 1.5$.

This fuzzy measure μ does not satisfy (S3). It is also necessary to study general fuzzy measures; Part II gives results of the study. In Part II, in addition to the condition $\mu(X) = 1$ and (F4), we remove (F3) from the definition of fuzzy measures; a necessity measure, which is a meaningful set function, does not satisfy (F3).

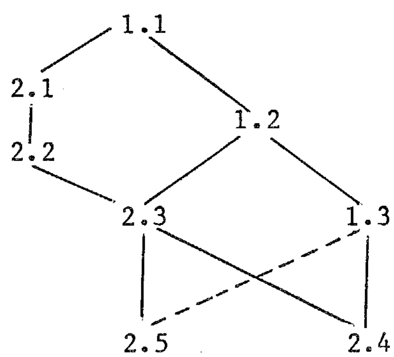
Part II discusses representation of fuzzy measures and Choquet's integral. It is suggested by Höhle [9]. Its essence is that a fuzzy measure μ on a σ-algebra $\mathscr{X}$ is expressed in terms of a measure m on a σ-algebra $\mathcal{Y}$ and a mapping $H : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\mu(A) = m \circ H(A) \quad \forall A \in \mathcal{X}.$$

Dempster's induction of upper and lower probabilities by a multivalued mapping is a particular case of this representation. Choquet's integral is not only definable for all fuzzy measures but also reasonable. In addition, this integral is closely related to the representation of fuzzy measures.

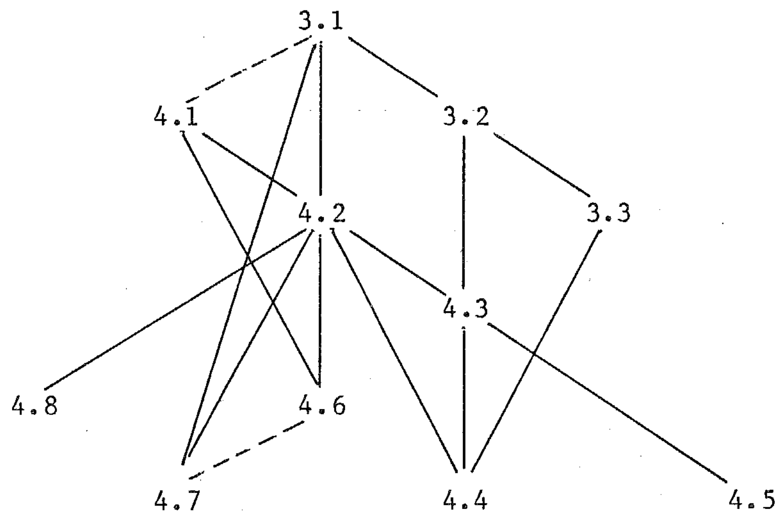
Lastly we state the constitution of this dissertation. As previously stated, it consists of two parts. Part I discusses pseudo-additive measures and integrals with respect to pseudo-additive measures. Part II discusses general fuzzy measures and Choquet's integral.

The following diagram expresses the interdependency of sections in Part I. (A dashed line indicates a minor dependence.)



In Chapter 1 pseudo-additions are characterized (§ 1.1) and pseudo-additive measures are defined by using pseudo-additions (§ 1.2). Section 1.3 shows that, for a pseudo-additive measure satisfying a certatin condition, the universal set can be partitioned by the values of the pseudo-additive measure. In Chapter 2 a multiplication-like operation $\hat{}$ consistent with a pseudo-addition $\hat{+}$ is characterized (§ 2.1 and §2.2) and the extended Lebesgue integral is defined using $\hat{}$ and $\hat{+}$ (§2.3). Section 2.4 and Section 2.5 show that Radon-Nikodym-like theorems hold.

The constitution of Part II is a little complicated. The following diagram expresses the interdependency of sections in Part II.



In Chapter 3 in Part II we discuss the representation of fuzzy measures by additive measures (§ 3.2) and give an interpretation to general fuzzy measures (§ 3.3). Chapter 4 discusses Choquet's integral as an integral with respect to fuzzy measures. Section 4.3 shows that Choquet's integral is closely related to the representation of fuzzy measures. In Section 4.4 concrete examples show Choquet's integral to be realistic. In Section 4.5 we derive Choquet's integral from the representation of fuzzy measures. Section 4.6 discusses "conformability," which is a binary relation between measurable functions. If there is this relation between two functions, they have interesting properties. In Section 4.7 the concept of null sets in fuzzy measure theory is proposed. Section

4.8 discusses some special fuzzy measures and Choquet's integral as an expectation.

PART I

PUSEUDO-ADDITIVE MEASURES AND INTEGRALS

CHAPTER 1 PSEUDO-ADDITIVE MEASURES

§ 1.1 PSEUDO-ADDITIONS

As mentioned in Introduction, in Part I we consider a set function with the properties:

(S1) $\mu(\emptyset) = 0$.

(S2) If $A \in \mathcal{X}$, $B \in \mathcal{X}$, and $A \subset B$, then $\mu(A) \leq \mu(B)$.

(S3) If $A \in \mathcal{X}$, $B \in \mathcal{X}$, and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) \hat{+} \mu(B)$.

(S4) If $\{A_n\} \subset \mathcal{X}$ and $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.

For the consistency with the conditions (S1) - (S4), it is necessary that a binary operation $\hat{+}$ satisfies the followings:

(P1) $s \hat{+} 0 = 0 \hat{+} s = s$,

(P2) $(s \hat{+} t) \hat{+} u = s \hat{+} (t \hat{+} u)$,

(P3) $s \leq s'$ and $t \leq t'$ $\Longrightarrow$ $s \hat{+} t \leq s' \hat{+} t'$,

(P'4) $s_n \uparrow s$ and $t_n \uparrow t$ $\Longrightarrow$ $s_n \hat{+} t_n \uparrow s \hat{+} t$.

For the sake of simplicity we put a stronger condition

(P4) $s_n \to s$ and $t_n \to t$ $\Longrightarrow$ $(s_n \hat{+} t_n) \to (s \hat{+} t)$

in place of (P'4). (P4) is very natural for an operation $\hat{+}$ on $[0, \infty]$.

1.1.1 DEFINITION. A binary operation $\hat{+}$ on $[0, \infty]$ satisfying (P1) - (P4) is called a pseudo-addition.

The ordinary addition + satisfies those conditions, that is, the pseudo-addition is an extension of the ordinary addition.

A pseudo-addition can be represented by a family of one-place functions.

1.1.2 DEFINITION. Let $\{(\alpha_k, \beta_k) : k \in K\}$ be a family of disjoint open intervals in $[0, \infty]$ indexed by a countable set K. For each $k \in K$, associate a continuous and strictly increasing function

$$g_k : [\alpha_k, \beta_k] \to [0,\infty].$$

We say that a binary operation $\hat{+}$ has a representation

$$\{\langle(\alpha_k, \beta_k), g_k\rangle : k \in K\}$$

iff

$$s \hat{+} t = \begin{cases} g_k^*(g_k(s) + g_k(t)) & (s, t) \in [\alpha_k, \beta_k]^2, \\ \max(s, t) & \text{otherwise,} \end{cases}$$

where g_k^* is the pseudo-inverse of g_k, which is defined by

$$g_k^*(s) = g_k^{-1}(\min(s, g_k(\beta_k))).$$

For example, the ordinary addition + has the representation $\{\langle(0,\infty), T\rangle\}$, where $T(s) = s$, $\forall s \in [0,\infty]$, and the binary operation $\vee$, i.e., max, has the representation $\emptyset$, that is, it has no (α_k, β_k).

The next theorem holds (Mostert and Shields [15] and Ling [13]).

1.1.3 THEOREM. A binary operation is a pseudo-addition iff it has a representation $\{\langle(\alpha_k, \beta_k), g_k\rangle : k \in K\}$.

As a corollary of this, we obtain that a pseudo-addition is commutative. Throughout the rest of Part I, $\hat{+}$ is used as a pseudo-addition and $\{\langle(\alpha_k, \beta_k), g_k\rangle : k \in K\}$ is a representation of $\hat{+}$. The set of all idempotent elements with respect to $\hat{+}$ is denoted by I, that is, $I = \{s \mid s \hat{+} s = s\}$. Obviously I is a closed set and

$$I = [0, \infty] - \bigcup_{k \in K}(\alpha_k, \beta_k).$$

We write

$$\hat{\sum_{i=1}^{n}} s_i = s_1 \hat{+} \cdots \hat{+} s_n,$$

and

$$\hat{\sum_{i=1}^{\infty}} x_i = \lim_{n\to\infty} \hat{\sum_{i=1}^{n}} x_i.$$

1.1.4 <u>DEFINITION</u>. A half open interval $(\alpha_k, \beta_k]$ is called <u>nilpotent</u> iff, for each $s \in (\alpha_k, \beta_k]$, there is a positive integer n such that $ns = \beta_k$, formally,

$$\hat{\sum_{i=1}^{n}} s = \beta_k.$$

It is easy to show that $(\alpha_k, \beta_k]$ is nilpotent iff $g_k(\beta_k) < \infty$.

Obviously $\{0\} \cup (\alpha_k, \beta_k]$ is a submonoid of $([0, \infty], \hat{+})$ for every $k \in K$.

1.1.5 <u>DEFINITION</u>. For each $k \in K$, we introduce a function

$$\bar{g}_k : \{0\} \cup (\alpha_k, \beta_k] \to [0, \infty]$$

defined by

$$\bar{g}_k(s) = \begin{cases} g_k(s) & s \in (\alpha_k, \beta_k], \\ 0 & s = 0. \end{cases}$$

Then the pseudo-addition on $\{0\} \cup (\alpha_k, \beta_k]$ is expressed by

$$s \hat{+} t = \bar{g}_k^*(\bar{g}_k(s) + \bar{g}_k(t)),$$

where $\bar{g}_k^*$ is defined by

$$\bar{g}_k^*(s) = \begin{cases} g_k^*(s) & s > 0, \\ 0 & s = 0. \end{cases}$$

These functions $\bar{g}_k$ and $\bar{g}_k^*$ will be used in the succeeding sections.

§ 1.2 PSEUDO-ADDITIVE MEASURES

Let $(X, \mathfrak{X})$ be a measurable space.

1.2.1 DEFINITION. A set function $\mu : \mathfrak{X} \to [0, \infty]$ is called a pseudo-additive measure (with respect to $\hat{+}$) iff μ satisfies the following conditions:

(1) $\mu(\emptyset) = 0$,

(2) $A, B \in \mathfrak{X}$ and $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) \hat{+} \mu(B)$,

(3) $\{A_n\} \subset \mathfrak{X}$ and $A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$.

We write a pseudo-additive measure with respect to $\hat{+}$ as a $\hat{+}$-measure for short and call the triplet $(X, \mathfrak{X}, \mu)$ a $\hat{+}$-measure space. Obviously the ordinary measure is the pseudo-additive measure with respect to the ordinary addition +. In the sequel we shall write

$\sum_n, \cup_n, \cap_n$, etc. in place of $\sum_{n=1}^{\infty}, \bigcup_{n=1}^{\infty}, \bigcap_{n=1}^{\infty}$, etc.

By the difinition, a $\hat{+}$-measure μ is monotone :

$$A, B \in \mathfrak{X} \text{ and } A \subset B \Rightarrow \mu(A) \leq \mu(B).$$

It is also σ-pseudo-additive :

$$\{A_n\} \text{ is a disjoint sequence of sets in } \mathfrak{X} \Rightarrow \mu(\bigcup_n A_n) = \hat{\sum_n} \mu(A_n).$$

It is easy to show that a $\hat{+}$-measure is not always continuous from above. However the next theorem holds.

1.2.2 THEOREM. If μ is a $\hat{+}$-measure on $\mathfrak{X}$ and if $\{A_n\}$ is a decreasing sequence of sets in $\mathfrak{X}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$ or $\lim_{n\to\infty} \mu(A_n) \notin I$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_n A_n).$$

<u>Proof</u>. If $\lim_{n\to\infty} \mu(A_n) = 0$, then, since $0 \leq \mu(\bigcap_n A_n) \leq \mu(A_n)$ for every n, $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_n A_n) = 0$. Assume that $a = \lim_{n\to\infty} \mu(A_n)$ is not idempotent. Then there exists $k \in K$ such that $a \in (\alpha_k, \beta_k)$, and there exists a positive integer N such that $\mu(A_N) \in (\alpha_k, \beta_k)$. Since $\{A_N - A_n\}$ is an increasing sequence, and since for $N \leq n$

$$\mu(A_N) = \mu(A_n) \hat{+} \mu(A_N - A_n),$$

it follows that

$$\mu(A_N) = a \hat{+} \mu(A_N - \bigcap_n A_n),$$

as $n \to \infty$. On the other hand, obviously

$$\mu(A_N) = \mu(\bigcap_n A_n) \hat{+} \mu(A_N - \bigcap_n A_n).$$

Therefore, if $\mu(A_N - \bigcap_n A_n) < \alpha_k$, then

$$a = \mu(A_N) = \mu(\bigcap_n A_n).$$

Furthermore, if $\mu(A_N - \bigcap_n A_n) \geq \alpha_k$, then

$$a = g_k^*(g_k(\mu(A_N)) - g_k(\mu(A_N - \bigcap_n A_n))) = \mu(\bigcap_n A_n).$$

§ 1.3 DECOMPOSITION THEOREM

We define a concept substitute for σ-finiteness.

1.3.1 DEFINITION. Let μ be a $\hat{+}$-measure on $(X, \mathcal{X})$. μ is said to be σ-decomposable iff it holds that if $\mathcal{P}$ is a class of mutually disjoint non-null sets in $\mathcal{X}$, then $\mathcal{P}$ is countable.

1.3.2 PROPOSITION. σ-finiteness of an ordinary measure μ implies its σ-decomposability.

Proof. Let $\mathcal{P}$ be a class of mutually disjoint non-null sets. Since μ is σ-finite, there exists a sequence $\{B_n\}$ such that $X = \bigcup_n B_n$ and

$\mu(B_n) < \infty$ for $n = 1, 2, \cdots$. We write, for $n, m = 1, 2, \cdots$,

$$\mathcal{P}_{n,m} = \{A \in \mathcal{P} : \mu(B_n)/2^m < \mu(A \cap B_n) \leq \mu(B_n)/2^{m-1}\}.$$

Since $\mathcal{P}_{n,m}$ has at most 2^m-1 elements, $\mathcal{P} = \bigcup_{n,m} \mathcal{P}_{n,m}$ is countable.

The converse of this proposition does not hold, because, if $\mathcal{X} = \{X, \emptyset\}$ and $\mu(X) = \infty$, then μ is not σ-finite but σ-decomposable. We shall show the precise relation between σ-finiteness of an ordinary measure and its σ-decomposability.

1.3.3 DEFINITION. For every $k \in K$ we define $\mathcal{W}_k(\mu)$ (or $\mathcal{W}_k$) to be a class of all those sets $A \in \mathcal{X}$ with the following properties:

(WK-1) $B \in \mathcal{X}$ and $B \subset A \Rightarrow \mu(B) \in \{0\} \cup (\alpha_k, \beta_k]$, (WK-2) there exists a sequence $\{B_n\} \subset \mathfrak{X}$ such that $A = \bigcup_n B_n$ and $\mu(B_n) < \beta_k$ for $n = 1, 2, \cdots$.

Similarly $\mathcal{W}_I(\mu)$ (or $\mathcal{W}_I$) is defined to be a class of all those sets $A \varepsilon \mathfrak{X}$ satisfying

(WI) $B \varepsilon \mathfrak{X}$ and $B \subset A \Rightarrow \mu(B) \varepsilon I$.

1.3.4 <u>DEFINITION</u>. For $A, B \varepsilon \mathfrak{X}$, we denote $\mu(B - A) = 0$ by $A \subset B\ [\mu]$, and $\mu((B - A) \cup (A - B)) = 0$ by $A = B\ [\mu]$. Let $\mathcal{C} \subset \mathfrak{X}$ and $M \varepsilon \mathcal{C}$. We say that M is <u>$\mu$-maximal in</u> $\mathcal{C}$ iff $C \subset M\ [\mu]$ for every $C \varepsilon \mathcal{C}$.

1.3.5 <u>LEMMA</u>. If μ is a σ-decomposable $\hat{+}$-measure on $\mathfrak{X}$ and if $\mathcal{C}$ is a subclass of $\mathfrak{X}$ satisfying the following conditions:

(L1) $\mathcal{C} \neq \emptyset$,

(L2) $C \varepsilon \mathcal{C}$ and $D \subset C\ [\mu] \Rightarrow D \varepsilon \mathcal{C}$,

(L3) $\{C_n\} \subset \mathcal{C} \Rightarrow \bigcup_n C_n \varepsilon \mathcal{C}$,

then $\mathcal{C}$ has a μ-maximal set M and

$$A \varepsilon \mathcal{C} \Longleftrightarrow A \subset M\ [\mu].$$

<u>Proof</u>. Let $\mathbb{D}$ be a set of all classes consisting of mutually disjoint non-null sets in $\mathcal{C}$. If $\mathbb{D} = \emptyset$, then any element $M \varepsilon \mathcal{C}$ is μ-maximal in $\mathcal{C}$. Let us assume $\mathbb{D} \neq \emptyset$. $\mathbb{D}$ is inductively ordered with respect to class inclusion. By Zorn's lemma there exists a maximal element $\mathcal{D}_0$ in $\mathbb{D}$. Since μ is σ-decomposable, $\mathcal{D}_0$ is countable, so if $M = \cup \mathcal{D}_0$, then $M \varepsilon \mathcal{C}$.

We show that M is μ-maximal in $\mathcal{C}$. Let us assume that there exists $C \varepsilon \mathcal{C}$ such that $\mu(C - M) > 0$. Since $C - M \varepsilon \mathcal{C}$, $\mathcal{D}_0 \cup \{C - M\}$ is greater than $\mathcal{D}_0$ in $\mathbb{D}$. This contradicts the fact that $\mathcal{D}_0$ is maximal in $\mathbb{D}$.

The second assertion is obvious.

1.3.6 <u>THEOREM</u> (DECOMPOSITION THEOREM). If μ is a σ-decomposable $\hat{+}$-measure on $(X, \mathcal{X})$, then the followings hold:

(1) For every $k \in K$ $\mathcal{W}_k$ has a μ-maximal set W_k. $\mathcal{W}_I$ also has a μ-maximal set W_I.

(2) $W_k \cap W_{k'} = \emptyset\ [\mu]$ for $k \neq k'$, and $W_I \cap W_k = \emptyset\ [\mu]$ $\forall\, k \in K$.

(3) $X = \bigcup_{k \in K} W_k \cup W_I\ [\mu]$.

<u>Proof</u>. (1) It is easy to show that $\mathcal{W}_I$ and $\mathcal{W}_k$ satisfy the conditions (L1) - (L3) in Lemma 1.3.5. Hence $\mathcal{W}_I$ and $\mathcal{W}_k$ have μ-maximal elements W_I and W_k, respectively.

(2) Let $A = W_k \cap W_I$. Since $A \in \mathcal{W}_k$, there exists $\{B_n\} \subset \mathcal{X}$ such that $\bigcup_n B_n = A$ and $\mu(B_n) \in \{0\} \cup (\alpha_k, \beta_k)$ for $n = 1, 2, \cdots$. On the other hand, the fact that $B_n \subset A \in \mathcal{W}_I$ implies $B_n \in \mathcal{W}_I$, that is, $\mu(B_n) \in I$. Therefore it follows that $\mu(B_n) = 0$ for $n = 1,2, \cdots$, and hence that $\mu(A) = 0$. Similary $W_k \cap W_{k'} = \emptyset\ [\mu]$ for $k \neq k'$.

(3) Let $E = X - [\bigcup_{k \in K} W_k \cup W_I]$. If $E \in \mathcal{W}_I$, then $E \subset W_I\ [\mu]$, and so $\mu(E) = 0$. We now assume that $E \notin \mathcal{W}_I$, then there exists a subset A of E such that $\mu(A) \in (\alpha_k, \beta_k)$ for some $k \in K$. We write

$$\mathcal{E} = \{B \in \mathcal{X} \mid B \subset A \text{ and } \mu(B) \leq \alpha_k\}.$$

Then it is easy to check that $\mathcal{E}$ satisfies (L1) - (L3) in Lemma 1.3.5. Let M be a μ-maximal element in $\mathcal{E}$ and let $B = A - M$. Obviously $\mu(B) = \mu(A) > 0$ and $B \in \mathcal{W}_k$. Since $B \subset E$, this contradicts the definition of E.

By this theorem the relation between σ-finiteness of an ordinary measure and its σ-decomposability is made clear.

1.3.7 PROPOSITION. If μ is an ordinary measure on $(X, \mathfrak{X})$, then μ is σ-finite iff μ is σ-decomposable and $W_I = \emptyset\ [\mu]$.

Proof. Suppose that μ is σ-finite. By Proposition 3.2, μ is σ-decomposable. Since $I = \{0, \infty\}$ in this case, obviously $W_I = \emptyset\ [\mu]$.

Conversely suppose that μ is σ-decomposable and $W_I = \emptyset\ [\mu]$. In the reprsentation of the ordinal addition, we have $K = \{0\}$ and $(\alpha_0, \beta_0) = (0, \infty)$. By the previous theorem, we have $X = W_0 \cup W_I = W_0\ [\mu]$, that is, $X \in \mathcal{W}_0$. It follows from the property (WK-2) of $\mathcal{W}_0$ that μ is σ-finite.

Lastly we show a correspondence of a $\hat{+}$-measure with an ordinary measure.

1.3.8 THEOREM. Let $(X, \mathfrak{X}, \mu)$ be a $\hat{+}$-measure space. If X has the property (WK-1) for some $k \in K$, there exists an ordinary measure $\bar{\mu}$ such that $\mu = \bar{g}_k^* \circ \bar{\mu}$. Moreover if X has the properties (WK-1) and (WK-2), i.e., $X \in \mathcal{W}_k$, then $\bar{\mu}$ is σ-finite and unique.

Proof. Let

$$\mathcal{A}_0 = \{A \in \mathfrak{X} : \mu(A) < \beta_k\},$$

$$\mathcal{A} = \{\bigcup_n A_n : A_n \in \mathcal{A}_0, \text{ for } n = 1, 2, \ldots\} \text{ and}$$

$$\mathcal{B} = \mathfrak{X} - \mathcal{A}.$$

If $\{A_n\}$ and $\{B_m\}$ are mutually disjoint sequences of sets in $\mathcal{A}_0$ and $\bigcup_n A_n = \bigcup_m B_m$, then

$$\sum_n \bar{g}_k \circ \mu(A_n) = \sum_n \bar{g}_k \circ \mu(\bigcup_m (A_n \cap B_m))$$

$$= \sum_n \bar{g}_k[\hat{\sum_m} \mu(A_n \cap B_m)]$$

$$= \sum_n \bar{g}_k \circ \bar{g}_k^*[\sum_m \bar{g}_k \circ \mu(A_n \cap B_m)]$$

$$= \sum_{n,m} \bar{g}_k \circ \mu(A_n \cap B_m)$$

$$= \sum_m \bar{g}_k \circ \mu(B_m).$$

Therefore we can define $\bar{\mu}$ by

$$\bar{\mu}(A) = \begin{cases} \sum_n \bar{g}_k \circ \mu(A_n), & \text{for } A = \bigcup_n A_n, \\ & \quad \text{where } \{A_n\} \subset \mathcal{A}_0 \text{ is mutually disjoint,} \\ \infty, & \text{for } A \in \mathcal{B}. \end{cases}$$

We show that $\bar{\mu}$ is an ordinary measure on $\mathcal{X}$. Obviously $\bar{\mu}(\emptyset) = 0$. Let $\{A_n\}$ be a disjoint sequence of the sets in $\mathcal{X}$. If $\bigcup_n A_n \in \mathcal{B}$, then there exists a positive integer m such that $A_m \in \mathcal{B}$. Thus $\bar{\mu}(\bigcup_n A_n) = \infty = \sum_n \bar{\mu}(A_n)$. Let us assume that $\bigcup_n A_n \in \mathcal{A}$. Then there exists a disjoint sequence $\{B_m\} \subset \mathcal{A}_0$ such that $\bigcup_n A_n = \bigcup_m B_m$. Since $A_n \cap B_m \in \mathcal{A}_0$ for n, m = 1, 2, ...,

$$\bar{\mu}(\bigcup_n A_n) = \bar{\mu}(\bigcup_m B_m)$$

$$= \sum_m \bar{g}_k \circ \mu(B_m)$$

$$= \sum_m \bar{g}_k \circ \mu(\bigcup_n A_n \cap B_m)$$

$$= \sum_{n,m} \bar{g}_k \circ \mu(A_n \cap B_m)$$

$$= \sum_n \bar{\mu}(A_n).$$

It follows from the definition of $\bar{\mu}$ that $\mu = \bar{g}_k^* \circ \bar{\mu}$. If $X \in \mathcal{W}_k$, then, since $\mathcal{X} = \mathcal{A}$, $\bar{\mu}$ is σ-finite and unique.

CHAPTER 2 INTEGRALS WITH RESPECT TO PSEUDO-ADDITIVE MEASURES

§ 2.1 MULTIPLICATIONS I

It is natural to assume that an integral of a function

$$f(x) = \begin{cases} a & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

with respect to a $\hat{+}$-measure μ depends only on a and $\mu(A)$. We introduce a binary operation $\hat{\cdot}$ called a multiplication and express the integral of f by $a \mathbin{\hat{\cdot}} \mu(A)$. Furthermore we require that the indefinite integral with respect to a $\hat{+}$-measure is also a $\hat{+}$-measure. So we set up the following conditions for $\hat{\cdot}$:

(M1) $a \mathbin{\hat{\cdot}} (s \mathbin{\hat{+}} t) = (a \mathbin{\hat{\cdot}} s) + (a \mathbin{\hat{\cdot}} t)$,

(M2) $a \leq b \Rightarrow a \mathbin{\hat{\cdot}} s \leq b \mathbin{\hat{\cdot}} s$,

(M3) $a \mathbin{\hat{\cdot}} s = 0 \Leftrightarrow a = 0$ or $s = 0$,

(M4) there exists a left identity element, that is, $\exists\, e \in [0, \infty]$, $\forall s \in [0, \infty]$, $e \mathbin{\hat{\cdot}} s = s$,

(M5) $0 < a < \infty$, $a_n \to a$ and $s_n \to s \Rightarrow (a_n \mathbin{\hat{\cdot}} s_n) \to (a \mathbin{\hat{\cdot}} s)$, and $(+\infty) \mathbin{\hat{\cdot}} s = \lim_{a\to\infty} a \mathbin{\hat{\cdot}} s$.

2.1.1 DEFINITION. We call a left operation $\hat{\cdot}$ on $([0, \infty], \hat{+})$ satisfying (M1) – (M5) a multiplication consistent with $\hat{+}$.

For example, the ordinary multiplication, denoted $\cdot$, is one of the multiplications consistent with the ordinary addition +, and both $\wedge$ (min) and $\cdot$ are multiplications consistent with $\vee$ (max).

The next theorem shows the structure of a multiplication on $[\alpha_k, \beta_k]$.

2.1.2 THEOREM. If $\hat{\cdot}$ is a multiplication consistent with $\hat{+}$, then there exists a family of nondecreasing continuous functions $\{h_k: k \in K\}$ satisfying

(1) $a \hat{\cdot} s = g_k^*(h_k(a)\, g_k(s)) \quad \forall\, a > 0, \forall\, s \in [\alpha_k, \beta_k]$,

(2) $h_k(e) = 1$,

(3) $0 < h_k(a) < \infty$ for $0 < a < \infty$,

(4) if $(\alpha_k, \beta_k]$ is nilpotent, then for $0 < a \leq e$ $h_k(a) = 1$.

We cannot characterize the structure of a multiplication on I, the set of all idempotent elements. But by the above theorem, if

$$\overline{\bigcup_{k \varepsilon K} [\alpha_k, \beta_k]} = [0, \infty],$$

then, for every idempotent element s,

$$a \hat{\cdot} s = \begin{cases} 0 & a = 0, \\ s & a > 0. \end{cases}$$

In the rest of this section we prove Theorem 2.1.2. First we prove a sequence of lemmas. Assume that a composition $(a, s) \mapsto (a \hat{\cdot} s)$ has the properties (M1) - (M5).

2.1.3 LEMMA. If s is idempotent, then so is $a \hat{\cdot} s$.

Proof. It follows from (M1) that

$$(a \hat{\cdot} s) \hat{+} (a \hat{\cdot} s) = a \hat{\cdot} (s \hat{+} s) = a \hat{\cdot} s.$$

2.1.4 LEMMA. $a \hat{\cdot} \alpha_k = \alpha_k$ for every $a \geq e$, where e is a left identity.

Proof. If $e = \infty$, it is trivial. So assume that $e < \infty$, and assume that there exists a number $a \in (e, \infty)$ such that $a \hat{\cdot} \alpha_k \neq \alpha_k$. By (M2) we have that $a \hat{\cdot} \alpha_k > e \hat{\cdot} \alpha_k = \alpha_k$. Then it follows from (M5) that there exists a number a_0 such that $e < a_0 < a$ and $\alpha_k < (a_0 \hat{\cdot} \alpha_k) < \beta_k$. This contradicts Lemma 2.1.3. Therefore $a \hat{\cdot} \alpha_k = \alpha_k$ for $e \leq a < \infty$, and $(+\infty) \hat{\cdot} \alpha_k = \lim_{a \to \infty} (a \hat{\cdot} \alpha_k) = \alpha_k$.

2.1.5 LEMMA. $a \hat{\cdot} \beta_k = \beta_k$ for $0 < a \leq e$.

Proof. Similar to the previous lemma.

2.1.6 LEMMA.

(1) $\hat{\sum}_{j=1}^{n} s = g_k^*(ng_k(s)), \quad \forall\, s \in (\alpha_k, \beta_k]$.

(2) $\hat{\sum}_{j} s = \beta_k, \quad \forall\, s \in (\alpha_k, \beta_k]$.

(3) $a \hat{\cdot} (\hat{\sum}_j s) = \hat{\sum}_j (a \hat{\cdot} s), \quad \forall\, a, s \in [0, \infty]$.

Proof. Trivial.

2.1.7 LEMMA. $a \hat{\cdot} \beta_k = \beta_k$ for $a \geq e$.

Proof. Let $e \leq a < \infty$. Since $a \hat{\cdot} \beta_k \geq e \hat{\cdot} \beta_k = \beta_k$ and $a \hat{\cdot} \alpha_k = \alpha_k$, there exists a number $s \in (\alpha_k, \beta_k]$ such that $a \hat{\cdot} x = \beta_k$. Hence

$$a \hat{\cdot} \beta_k = a \hat{\cdot} (\hat{\sum}_j s)$$

$$= \hat{\sum}_j (a \hat{\cdot} s)$$

$$= \hat{\sum_j} \beta_k$$

$$= \beta_k$$

and

$$(+\infty) \hat{\cdot} \beta_k = \lim_{a\to\infty} (a \hat{\cdot} \beta_k)$$

$$= \beta_k.$$

2.1.8 <u>LEMMA</u>. $a \hat{\cdot} \alpha_k = \alpha_k$ for $0 < a \leq e$.

<u>Proof</u>. Let us assume that there exists a number a such that $0 < a \leq e$ and $a \hat{\cdot} \alpha_k \neq \alpha_k$. It follows from the monotonicity of $\hat{\cdot}$ that $a \hat{\cdot} \alpha_k < \alpha_k$. Since $a \hat{\cdot} \beta_k = \beta_k$, there exists a number $s \in (\alpha_k, \beta_k)$ such that $a \hat{\cdot} s = \alpha_k$. Hence

$$a \hat{\cdot} \beta_k = a \hat{\cdot} (\hat{\sum_j} s)$$

$$= \hat{\sum_j} (a \hat{\cdot} s)$$

$$= \hat{\sum_j} \alpha_k$$

$$= \alpha_k.$$

This contradicts the fact that $a \hat{\cdot} \beta_k = \beta_k$.

<u>Proof of Theorem 2.1.2</u>. By previous lemmas we obtain the fact that, if $a \in (0, \infty)$ and $s \in [\alpha_k, \beta_k]$, then $a \hat{\cdot} s \in [\alpha_k, \beta_k]$. We define a function $f_k : (0, \infty) \to (\alpha_k, \beta_k]$ by

$$f_k(a) = \min\{s \mid a \hat{\cdot} s = \beta_k\}.$$

Let $a \in (0, \infty)$ and $s, t \in [\alpha_k, \beta_k]$. Suppose that $g_k(s) + g_k(t) < g_k(f_k(a))$. We obtain

$$g_k(s) + g_k(t) < g_k(f_k(a))$$
$$\Rightarrow \quad s \hat{+} t < f_k(a)$$
$$\Rightarrow \quad a \hat{\cdot} (s \hat{+} t) < \beta_k$$
$$\Rightarrow \quad (a \hat{\cdot} s) \hat{+} (a \hat{\cdot} t) < \beta_k$$
$$\Rightarrow \quad g_k(a \hat{\cdot} s) + g_k(a \hat{\cdot} t) < g_k(\beta_k).$$

Therefore it follows from (M1) that

$$g_k(a \hat{\cdot} g_k^*(g_k(s) + g_k(t))) = g_k(a \hat{\cdot} s) + g_k(a \hat{\cdot} t).$$

We introduce into this equation the notations

$$v = g_k(s), \quad w = g_k(t), \quad E_a(u) = g_k(a \hat{\cdot} g_k^*(u)).$$

We have

$$E_a(v + w) = E_a(v) + E_a(w).$$

Since the continuity of g_k and (M5) imply the continuity of E_a, there exists a function h_k such that

$$E_a(v) = h_k(a)v \qquad \text{for } 0 \leq v < g_k(f_k(a)),$$

so we have

$$a \hat{\cdot} s = g_k^*(h_k(a)g_k(s)) \qquad \text{for } \alpha_k \leq s < f_k(a).$$

By (M5) and the continuity and monotonicity of g_k, the above equation holds for $\alpha_k \leq s \leq \beta_k$. (M2) and (M5) imply that h_k is continuous and nondecreasing. So we define $h_k(\infty) = \lim_{a\to\infty} h_k(a)$, then we obtain

$$a \hat{\cdot} s = g_k^*(h_k(a)g_k(s))$$

for $a \in (0, \infty]$ and $s \in [\alpha_k, \beta_k]$.

We next show (3). If $h_k(a) = 0$, then

$$a \hat{\cdot} \beta_k = g_k^*(h_k(a)g_k(\beta_k)) = \alpha_k$$

and this contradicts that $a \hat{\cdot} \beta_k = \beta_k$. If $h_k(a) = \infty$, then, for every $s \in (\alpha_k, \beta_k]$,

$$a \hat{\cdot} s = g_k^*(h_k(a)g_k(s)) = g_k^*(\infty) = \beta_k,$$

hence $a \hat{\cdot} \alpha_k = \lim_{s \to \alpha_k + 0} a \hat{\cdot} s = \beta_k$, and this contradicts that $a \hat{\cdot} \alpha_k = \alpha_k$.

In addition, (2) follows from (M4).

Lastly we prove (4). Let $(\alpha_k, \beta_k]$ be nilpotent. It is sufficient to prove that, if $a \in (0, e]$ and $s \in [\alpha_k, \beta_k]$, then $a \hat{\cdot} s = s$. Assume that there are numbers $a \in (0, e]$ and $s \in [\alpha_k, \beta_k]$ such that $a \hat{\cdot} s \neq s$. Since $a \hat{\cdot} s \leq e \hat{\cdot} s = s$, we have $a \hat{\cdot} s < s$, that is, $g_k(a \hat{\cdot} s) < g_k(s)$. If we write $y = g_k{}^*(g_k(\beta_k) - g_k(s))$, then

$$\begin{aligned} g_k(a \hat{\cdot} s) + g_k(a \hat{\cdot} t) &\leq g_k(a \hat{\cdot} s) + g_k(t) \\ &= g_k(a \hat{\cdot} s) + g_k(\beta_k) - g_k(s) \\ &< g_k(\beta_k), \end{aligned}$$

therefore

$$\beta_k > (a \hat{\cdot} s) \hat{+} (a \hat{\cdot} t) = a \hat{\cdot} (s \hat{+} t) = a \hat{\cdot} \beta_k = \beta_k.$$

This is a contradiction. The proof is now complete.

§ 2.2 MULTIPLICATIONS II

If, for every idempotent element x,

$$a \mathbin{\hat{\cdot}} x = \begin{cases} 0 & \text{if } a = 0, \\ x & \text{if } a > 0. \end{cases}$$

then the converse of Theorem 2.1.2 holds.

2.2.1 <u>PROPOSITION</u>. Let e be a number in $(0, \infty]$ and let $\{h_k : k \in K\}$ be a family of nondecreasing continuous functions on $(0, \infty]$ satisfying

(1) $h_k(e) = 1$,

(2) $0 < h_k(a) < \infty$ for $0 < a < \infty$,

(3) if $(\alpha_k, \beta_k]$ is nilpotent, $h_k(a) = 1$ for $0 < a \leq e$.

If

$$a \mathbin{\hat{\cdot}} s = \begin{cases} 0 & \text{if } a = 0, \\ s & \text{if } a > 0 \text{ and } s \in I, \\ g_k^*(h_k(a)g_k(s)) & \text{if } a > 0 \text{ and } s \in [\alpha_k, \beta_k], \end{cases}$$

then $\hat{\cdot}$ is a multiplication consistent with $\hat{+}$.

<u>Proof</u>. It is easy to check that $\hat{\cdot}$ satisfies the conditions (M1) - (M5).

If $\hat{+}$ has no nilpotent interval, there is a multiplication with good properties.

2.2.2 PROPOSITION. If $\hat{+}$ has no nilpotent interval, and if

$$a \mathbin{\hat{\cdot}} s = \begin{cases} 0 & \text{if } a = 0, \\ s & \text{if } a > 0 \text{ and } s \in I, \\ g_k^*(a\, g_k(s)) & \text{if } a > 0 \text{ and } s \in [\alpha_k, \beta_k], \end{cases}$$

then $\hat{\cdot}$ is a multiplication consistent with $\hat{+}$ with the following properties :

$$(a + b) \mathbin{\hat{\cdot}} s = (a \mathbin{\hat{\cdot}} s) \mathbin{\hat{+}} (b \mathbin{\hat{\cdot}} s),$$

$$(ab) \mathbin{\hat{\cdot}} s = a \mathbin{\hat{\cdot}} (b \mathbin{\hat{\cdot}} s).$$

Proof. Trivial.

Throughout the rest of Part I, the symbol $\hat{\cdot}$ is used as a multiplication consistent with $\hat{+}$, and $\{h_k : k \in K\}$ is a family of functions satisfying the conditions (1) - (4) in Theorem 2.1.2. We further define $h_k(0) = 0 \;\; \forall\, k \in K$ for convenience.

The next proposition is used in the following sections.

2.2.3 PROPOSITION.

(1) If $a_j \in [0, \infty]$ and $s_j \in \{0\} \cup (\alpha_k, \beta_k]$ for $j = 1, 2, \cdots, n$, then

$$\hat{\sum_{j=1}^{n}} (a_j \mathbin{\hat{\cdot}} s_j) \in \{0\} \cup (\alpha_k, \beta_k]$$

and

$$\hat{\sum_{j=1}^{n}} (a_j \mathbin{\hat{\cdot}} s_j) = \bar{g}_k^*\Big(\sum_{j=1}^{n} h_k(a_j)\bar{g}_k(s_j)\Big).$$

(2) If $a_j \in [0, \infty]$ and $s_j \in I$ for $j = 1, 2, \cdots, n$, then

$$\hat{\sum_{j=1}^{n}} (a_j \hat{\cdot} s_j) \in I.$$

Proof. (1) The first assertion is trivial. We have

$$\hat{\sum_{j=1}^{n}} (a_j \hat{\cdot} s_j) = \bar{g}_k^*(\sum_{j=1}^{n} \bar{g}_k \circ \bar{g}_k^*(h_k(a_j)\bar{g}_k(s_j))).$$

Hence if $h_k(a_j)\bar{g}_k(s_j) < \bar{g}_k(\beta_k)$ for every j, then we obtain

$$\hat{\sum_{j=1}^{n}} (a_j \hat{\cdot} s_j) = \bar{g}_k^*(\sum_{j=1}^{n} h_k(a_j)\bar{g}_k(s_j)).$$

Let us assume that $h_k(a_{j_o})\bar{g}_k(s_{j_o}) \geq \bar{g}_k(\beta_k)$ for some j_o.

Then

$$\sum_{j=1}^{n} \bar{g}_k \circ \bar{g}_k^*(h_k(a_j)\bar{g}_k(s_j)) \geq \bar{g}_k(\beta_k)$$

and

$$\sum_{j=1}^{n} h_k(a_j)\bar{g}_k(s_j) \geq \bar{g}_k(\beta_k).$$

Therefore

$$\hat{\sum_{j=1}^{n}} (a_j \hat{\cdot} s_j) = \beta_k = \bar{g}_k^*(\sum_{j=1}^{n} h_k(a_j)\bar{g}_k(s_j)).$$

(2) Trivial.

§ 2.3 INTEGRALS

Now we define the integrals with respect to the $\hat{+}$-measures. We develop a theory in a similar manner with to the ordinary integral theory.

Let $(X, \mathfrak{X}, \mu)$ be a $\hat{+}$-measure space.

2.3.1 DEFINITION. For a nonnegative simple function

$$f(x) = \begin{cases} a_j & \text{if } x \in A_j \quad j = 1, 2, \cdots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $A_j \in \mathfrak{X}$, $0 \leq a_j < \infty$ for $j = 1, 2, \cdots, n$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, we define the integral of f over $B \in \mathfrak{X}$ as

$$\hat{\int}_B f \, d\mu = \hat{\sum_{j=1}^{n}} a_j \hat{\cdot} \mu(A_j \cap B).$$

For a nonnegative measurable function f on X, we define the integral of f over $B \in \mathfrak{X}$ as

$$\hat{\int}_B f d\mu = \lim_{n\to\infty} \hat{\int}_B f_n d\mu,$$

where $\{f_n\}$ is a sequence of nonnegative simple functions such that $f_n(x) \uparrow f(x)$ for every $x \in B$.

Obviously this integral is well-defined. Note that the definition depends on the choice of a multiplication $\hat{\cdot}$.

We define the characteristic function of $A \in \mathfrak{X}$ by χ_A :

$$\chi_A(x) = \begin{cases} e & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

where e is the left identity of the multiplication $\hat{\cdot}$; the characteristic function χ_A also depends on the choice of a multiplication $\hat{\cdot}$. This integral has the same properties as Lebesgue's one.

2.3.2 PROPOSITION. Let $A, B \in \mathcal{X}$ and let f and g be nonnegative measurable functions on X.

(1) $f \leq g$ a.e. $\Rightarrow \hat{\int} f\, d\mu \leq \hat{\int} g\, d\mu$.

(2) $A \cap B = \emptyset \Rightarrow \hat{\int}_{A \cup B} f\, d\mu = \hat{\int}_A f\, d\mu \,\hat{+}\, \hat{\int}_B f\, d\mu$.

(3) $\hat{\int} \chi_A\, d\mu = \mu(A)$.

(4) $\hat{\int} f\, d\mu = 0 \Rightarrow f = 0$ a.e.

(5) $\hat{\int}_A f\, d\mu = \hat{\int} (\chi_A \hat{\cdot} f)\, d\mu$.

(6) The monotone convergence theorem; if $\{f_n\}$ is a sequence of nonnegative measurable functions on X such that $f_n(x) \uparrow f(x)$ a.e., then

$$\hat{\int} f\, d\mu = \lim_{n\to\infty} \hat{\int} f_n\, d\mu.$$

(7) If $\nu(A) = \hat{\int}_A f\, d\mu$ for every $A \in \mathcal{X}$, then ν is a $\hat{+}$-measure on $(X, \mathcal{X})$.

We omit the proof. From Proposition 2.2.2, the next proposition follows.

2.3.3 PROPOSITION. Assume that $\hat{+}$ has no nilpotent interval, and further assume

$$a \mathbin{\hat{\cdot}} s = \begin{cases} 0 & \text{if } a = 0, \\ s & \text{if } a > 0 \text{ and } s \in I, \\ g_k^*(ag_k(s)) & \text{if } a > 0 \text{ and } s \in [\alpha_k, \beta_k]. \end{cases}$$

Let f and g be nonnegative measurable functions, and let $0 \leq a, b \leq \infty$, then

(1) $\hat{\int} (af + bg)\, d\mu = (a \mathbin{\hat{\cdot}} \hat{\int} f\, d\mu) \mathbin{\hat{+}} (b \mathbin{\hat{\cdot}} \hat{\int} g\, d\mu)$,

(2) if $\nu(A) = \hat{\int}_A f\, d\mu$ for every $A \in \mathfrak{X}$, then

$$\hat{\int} g\, d\nu = \hat{\int} (g \mathbin{\hat{\cdot}} f)\, d\mu.$$

Lastly we show a relation between an integral with respect to some $\hat{+}$-measure and a Lebesgue integral.

2.3.4 THEOREM. If X has the property (WK-1) for some $k \in K$, and if $\bar{\mu}$ is an ordinary measure such that $\mu = \bar{g}_k^* \circ \bar{\mu}$, then, for every nonnegative measurable function f on X,

$$\hat{\int} f d\mu = \bar{g}_k^*(\int h_k \circ f\, d\bar{\mu}).$$

Proof. By the left continuity of $\bar{g}_k^*$ and the monotone convergence theorem, it is sufficient to prove the theorem for a simple function f.

If $f(x) = 0$ for almost all $x \in X$, it is trivial. So let

$$f(x) = \begin{cases} a_j & \text{if } x \in A_j \text{ for } j = 1, 2, \cdots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu(A_j) \geq 0$, $0 < a_j < \infty$ for $j = 1, 2, \cdots, n$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. Then we have

$$\hat{\int} f\, d\mu = \hat{\sum_{j=1}^{n}} \left(a_j \mathbin{\hat{\cdot}} \mu(A_j)\right)$$

$$= \bar{g}_k^*\left(\sum_{j=1}^{n} h_k(a_j)\bar{g}_k \circ \mu(A_j)\right)$$

and

$$\bar{g}_k^*\left(\int h_k \circ f\, d\bar{\mu}\right) = \bar{g}_k^*\left(\sum_{j=1}^{n} h_k(a_j)\bar{\mu}(A_j)\right).$$

If $\bar{g}_k \circ \mu(A_j) = \bar{\mu}(A_j)$ for every j, then the theorem follows. So let us assume that

$$\bar{g}_k \circ \mu(A_{j_o}) \neq \bar{\mu}(A_{j_o}) \text{ for some } j_o.$$

In this case $(\alpha_k, \beta_k]$ is nilpotent. By the proof of Theorem 1.3.8, we have $\bar{g}_k \circ \mu \leq \bar{\mu}$ and $\mu(A_{j_o}) = \beta_k$. Thus

$$\sum_{j=1}^{n} h_k(a_j)\bar{\mu}(A_j) \geq \sum_{j=1}^{n} h_k(a_j)\bar{g}_k \circ \mu(A_j)$$

$$\geq h_k(a_{j_o})\bar{g}_k \circ \mu(A_{j_o})$$

$$\geq g_k(\beta_k).$$

Therefore

$$\hat{\int} f\, d\mu = \beta_k = \bar{g}_k^*\left(\int h_k \circ f\, d\bar{\mu}\right).$$

The proof is complete.

§ 2.4 RADON-NIKODYM-LIKE THEOREM I

We now prove a Radon-Nikodym-like theorem for the integral defined with a certain multiplication.

The next lemma follos from 1.3.5.

2.4.1 LEMMA. If μ and ν are $\hat{+}$-measures on $(X, \mathcal{X})$, and if μ is σ-decomposable, then the class of ν-null sets, $\{A \in \mathcal{X} \mid \nu(A) = 0\}$, has a μ-maximal set N.

We call the set N a μ-maximal ν-null set.

Throughout this section, we assume that a multiplication satisfies the following conditions:

(A1) $a \,\hat{\cdot}\, s = s \qquad \forall\, a > 0,\ \forall\, s \in I,$

(A2) if $(\alpha_k, \beta_k]$ is not nilpotent, $\lim_{a \to +0} a \,\hat{\cdot}\, s = \alpha_k \qquad \forall\, s \in (\alpha_k, \beta_k),$

(A3) $(+\infty) \,\hat{\cdot}\, s = \beta_k \qquad \forall\, s \in (\alpha_k, \beta_k),\ \forall\, k \in K.$

2.4.2 THEOREM. (Radon-Nikodym-like theorem I)

If μ and ν are $\hat{+}$-measures on $(X, \mathcal{X})$, and if μ is σ-decomposable, then there exists a function f such that, for every $A \in \mathcal{X}$,

$$\hat{\int}_A f \, d\mu = \nu(A)$$

iff the following conditions are satisfied:

(1) $\mu(A) = 0 \Rightarrow \nu(A) = 0,$

(2) for every $k \in K$, if $A \in \mathcal{W}_k(\mu)$, then $\nu(A) \in \{0\} \cup (\alpha_k, \beta_k],$

(3) if $(\alpha_k, \beta_k]$ is nilpotent and $A \in \mathcal{W}_k(\mu)$, then $\mu(A - N) \leq \nu(A - N),$

(4) if $A \in \mathcal{W}_I(\mu)$, then $\mu(A - N) = \nu(A - N),$

where N is a μ-maximal ν-null set.

In the rest of this section we prove this theorem. We first consider $\mathcal{W}_k(\mu)$.

2.4.3 LEMMA. Let $(\alpha_k, \beta_k]$ be not nilpotent. If μ and ν are $\hat{+}$-measures on $(X, \mathcal{X})$, if μ is σ-decomposable, and if $X \in \mathcal{W}_k(\mu)$, then there exists a function f such that, for every $A \in \mathcal{X}$

$$\hat{\int}_A f \, d\mu = \nu(A)$$

iff the following conditions are satisfied:

(1) $\mu(A) = 0 \implies \nu(A) = 0$

(2) $\nu(A) \in \{0\} \cup (\alpha_k, \beta_k]$ for every $A \in \mathcal{X}$.

Proof. The necessity is trivial, so we prove the sufficiency. By Theorem 1.3.8 there exist ordinary measures $\bar{\mu}$ and $\bar{\nu}$ such that $\mu = \bar{g}_k^* \circ \bar{\mu}$, $\nu = \bar{g}_k^* \circ \bar{\nu}$ and $\bar{\mu}$ is σ-finite. Since $\bar{\nu}$ is absolutely continuous with respect to $\bar{\mu}$, there exists a function $\bar{f}$ such that, for every $A \in \mathcal{X}$, $\int_A \bar{f} \, d\bar{\mu} = \bar{\nu}(A)$. By the assumption (A2) the range of h_k is $[0, \infty]$, hence there is a measurable function f such that $\bar{f} = h_k \circ f$. Therefore for every $A \in \mathcal{X}$

$$\hat{\int}_A f \, d\mu = \bar{g}_k^*[\int_A \bar{f} \, d\bar{\mu}]$$

$$= \bar{g}_k^*(\bar{\nu}(A))$$

$$= \nu(A).$$

2.4.4 LEMMA. Let $(\alpha_k, \beta_k]$ be nilpotent. If μ and ν are $\hat{+}$-measures on $(X, \mathcal{X})$, if μ is σ-decomposable, and if $X \in \mathcal{W}_k(\mu)$, then there exists a function f such that, for every $A \in \mathcal{X}$,

$$\hat{\int}_A f \, d\mu = \nu(A)$$

iff the following conditions are satisfied:

(1) $\mu(A) = 0 \Rightarrow \nu(A) = 0$,

(2) $\nu(A) \in \{0\} \cup (\alpha_k, \beta_k]$ for every $A \in \mathcal{X}$,

(3) if N is a μ-maximal ν-null set, then for every $A \in \mathcal{X}$

$$\nu(A - N) \geq \mu(A - N).$$

Proof. Let $\bar{\mu}$ and $\bar{\nu}$ be ordinary measures such that $\mu = \bar{g}_k^* \circ \bar{\mu}$ and $\nu = \bar{g}_k^* \circ \bar{\nu}$. First suppose that $\hat{\int}_A f \, d\mu = \nu(A)$ for every $A \in \mathcal{X}$. (1) and (2) are trivial. Since $h_k(a) \geq 1$ for $a > 0$, and since $f(x) > 0$ for almost all $x \in X - N$, we have

$$\begin{aligned}
\nu(A - N) &= \hat{\int}_{A-N} f \, d\mu \\
&= \bar{g}_k^*[\int_{A-N} h_k \circ f \, d\bar{\mu}] \\
&\geq \bar{g}_k^*[\int_{A-N} d\bar{\mu}] \\
&= \bar{g}_k^*(\bar{\mu}(A - N)) \\
&= \mu(A - N).
\end{aligned}$$

Conversely suppose (1) - (3). Similarly to the previous lemma we obtain a function $\bar{f}$ such that, for every $A \in \mathcal{X}$, $\int_A \bar{f} \, d\bar{\mu} = \bar{\nu}(A)$.

Since $\bar{\mu}(A - N) \leq \bar{\nu}(A - N)$ for every $A \in \mathcal{X}$, $\bar{f}(x) \geq 1$ for almost all $x \in X - N$. And obviously $\bar{f}(x) = 0$ for almost all $x \in N$. Therefore there is a measurable function f such that $\bar{f} = h_k \circ f$. Hence it follows that, for every $A \in \mathcal{X}$,

$$\hat{\int}_A f \, d\mu = \nu(A).$$

Next we consider $\mathcal{W}_I(\mu)$.

2.4.5 LEMMA. If $A \in \mathcal{W}_I(\mu)$ and if $f(x) > 0$ for almost all $x \in A$, then

$$\hat{\int}_A f \, d\mu = \mu(A).$$

Proof. By the assumption (A1), for every $a \in (0, \infty]$,

$$\hat{\int}_A a \, d\mu = a \hat{\cdot} \mu(A) = \mu(A).$$

Hence for every positive integer n,

$$\hat{\int}_A f \, d\mu = \hat{\int}_{A \cap \{f \geq 1/n\}} f \, d\mu \;\hat{+}\; \hat{\int}_{A \cap \{f < 1/n\}} f \, d\mu$$

$$\geq \hat{\int}_{A \cap \{f \geq 1/n\}} 1/n \, d\mu$$

$$= \mu(A \cap \{f \geq 1/n\}),$$

so we have

$$\hat{\int}_A f \, d\mu \geq \lim_{n\to\infty} \mu(A \cap \{f \geq 1/n\}) = \mu(A).$$

On the other hand,

$$\hat{\int}_A f\, d\mu \leq \hat{\int}_A (+\infty)\, d\mu = \mu(A).$$

The proof is complete.

2.4.6 LEMMA. If μ and ν are $\hat{+}$-measures on $(X, \mathcal{X})$, and if $X \varepsilon \mathcal{W}_I(\mu)$, then there exists a function f such that, for every A ε

$$\hat{\int}_A f\, d\mu = \nu(A)$$

iff $\nu(A - N) = \mu(A - N)$ for every $A \varepsilon \mathcal{X}$,

where N is a μ-maximal ν-null set.

Proof. If $\hat{\int}_A f\, d\mu = \nu(A)$ for every $A \varepsilon \mathcal{X}$, then, since $f(x) > 0$ for almost all $x \varepsilon X - N$,

$$\nu(A - N) = \hat{\int}_{A-N} f\, d\mu = \mu(A - N)$$

for every $A \varepsilon \mathcal{X}$.

On the other hand, if $\nu(A - N) = \mu(A - N)$ for every $A \varepsilon \mathcal{X}$, and if f is a measurable function such that $f(x) = 0$ for $x \varepsilon N$ and $f(x) > 0$ for $x \varepsilon X - N$, then

$$\hat{\int}_A f\, d\mu = \nu(A)$$

for every $A \varepsilon \mathcal{X}$.

Now Theorem 2.4.2. follows from the above lemmas and Theorem 1.3.6.

§ 2.5 RADON-NIKODYM-LIKE THEOREM II

If $\overline{\bigcup_{k \in K} [\alpha_k, \beta_k]} = [0, \infty]$, then the assumption (A1) in the previous section is satisfied. However, for example, if $\hat{+}$ is $\vee$, i.e., max, then (A1) means that, for every $s \in [0, \infty]$

$$a \mathbin{\hat{\cdot}} s = \begin{cases} 0 & \text{if } a = 0, \\ s & \text{if } a > 0. \end{cases}$$

and this multiplication is not natural. So in this section we prove a Radon-Nikodym-like theorem for $\vee$-measures; a $\vee$-measure is called F-additive in [21].

Let μ be a σ-decomposable $\vee$-measure on $(X, \mathcal{X})$ and let ν be a $\vee$-measure on $(X, \mathcal{X})$ such that $\nu(A) = 0$ whenever $\mu(A) = 0$. Let $a \in [0, \infty]$ and $\mathcal{E}(a)$ be the class of all those sets $A \in \mathcal{X}$ that $\nu(B) \geq a \mathbin{\hat{\cdot}} \mu(B)$ for every measurable subset B of A. It follows from Lemma 1.3.5 that $\mathcal{E}(a)$ has a μ-maximal set. We denote the μ-maximal set of $\mathcal{E}(a)$ by $[\nu/\mu](a)$.

2.5.1 LEMMA. If $0 \leq a \leq b \leq \infty$, and if $A \subset [\nu/\mu](a) - [\nu/\mu](b)$ and $A \in \mathcal{X}$, then $a \mathbin{\hat{\cdot}} \mu(A) \leq \nu(A) \leq b \mathbin{\hat{\cdot}} \mu(A)$.

Proof. Let $A \subset [\nu/\mu](a) - [\nu/\mu](b)$. It is sufficient to prove that $\nu(A) \leq b \mathbin{\hat{\cdot}} \mu(A)$. Let $\mathcal{C}$ be the class of all those sets $C \in \mathcal{E}$ that $C \subset A$ and $\nu(C) \leq b \mathbin{\hat{\cdot}} \mu(C)$. In the same way as the proof of Lemma 1.3.5, we obtain a set $M \in \mathcal{C}$ such that, if $C \in \mathcal{C}$ and $C \cap M = \emptyset$, then $\mu(C) = 0$. Therefore we have $A - M \subset [\nu/\mu](b)\ [\mu]$. It follows from the definition of A that $\mu(A - M) = 0$, and hence that $\nu(A) \leq b \mathbin{\hat{\cdot}} \mu(A)$.

2.5.2 DEFINITION. We say that a nonnegative real number s is multiplicatively finite, m-finite for short, if $\lim_{a \to +0} a \hat{\cdot} s = 0$. A measurable set A is called m-finite (with respect to μ) if μ(A) is m-finite, and a v-measure μ on $(X, \mathcal{X})$ is called σ-m-finite if X is a countable union of m-finite sets.

2.5.3 THEOREM. (Radon-Nikodym-like theorem II).
If μ is a σ-m-finite σ-decomposable v-measure on $(X, \mathcal{X})$ and if ν is a v-measure on $(X, \mathcal{X})$, then there exists a function f such that

$$\hat{\int}_A f \, d\mu = \nu(A) \qquad \text{for every } A \in \mathcal{X}$$

iff, for every m-finite set A with respect to μ

$$\nu(A) \leq (+\infty) \hat{\cdot} \mu(A).$$

Moreover if every positive m-finite number is right reducible (that is, if s is positive m-finite, then the equation $a \hat{\cdot} s = b \hat{\cdot} s$ implies that a = b), then the function f is unique in the sense of a.e.

Proof. If $A \in \mathcal{X}$ and if $\hat{\int}_A f \, d\mu = \nu(A)$, then

$$\nu(A) \leq \hat{\int}_A (+\infty) \, d\mu = (+\infty) \hat{\cdot} \mu(A).$$

Conversely suppose that $\nu(A) \leq (+\infty) \hat{\cdot} \mu(A)$ for every m-finite set A. We may assume that X is m-finite. We write

$$H_n^j = [\nu/\mu](j/2^n) - [\nu/\mu]((j+1)/2^n) \quad \text{for } n, j = 0, 1, 2, \cdots,$$

and

$$H_\infty = [\nu/\mu](+\infty) \; (= X - \bigcup_j H_n^j [\mu]).$$

For n = 0, 1, 2, ⋯, we define

$$f_n(x) = \begin{cases} j/2^n, & x \in H_n^j \quad \text{for } j = 0, 1, 2, \cdots \\ +\infty, & x \in H_\infty \end{cases}$$

and

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We shall prove that, $\hat{\int}_A f \, d\mu = \nu(A)$ for every $A \in \mathcal{X}$.

Let $A \in \mathcal{X}$.

$$\hat{\int}_A f \, d\mu = \lim_{n\to\infty} \hat{\int}_A f_n \, d\mu$$

$$= \lim_{n\to\infty} [\hat{\int}_{A\cap H_\infty} f_n \, d\mu \vee \bigvee_j \hat{\int}_{A\cap H_n^j} f_n \, d\mu]$$

$$= \lim_{n\to\infty} \{[(+\infty) \hat{\cdot} \mu(A \cap H_\infty)] \vee \bigvee_j [(j/2^n) \hat{\cdot} \mu(A \cap H_n^j)]\}$$

$$\leq \lim_{n\to\infty} [\nu(A \cap H_\infty) \vee \bigvee_j \nu(A \cap H_n^j)]$$

$$= \nu(A).$$

We show the converse inequation: $\hat{\int}_A f \, d\mu \geq \nu(A)$.

Obviously we have

$$\nu(A) = \nu(A \cap H_\infty) \vee \bigvee_j \nu(A \cap H_0^j).$$

Hence, if $\nu(A) = \nu(A \cap H_\infty)$, then

$$\hat{\int}_A f \, d\mu \geq \hat{\int}_{A\cap H_\infty} f \, d\mu$$

$$= (+\infty) \hat{\cdot} \mu(A \cap H_\infty)$$

$$= \nu(A \cap H_\infty)$$

$= \nu(A)$.

So let us assume that $\nu(A) > \nu(A \cap H_\infty)$ and let c be an arbitrary number such that $\nu(A) > c$. Then there is an integer j_0 such that

$\nu(A \cap H_0^{j_0}) > c$. Since

$$A \cap H_n^j = (A \cap H_{n+1}^{2j}) \cup (A \cap H_{n+1}^{2j+1}) \ [\mu]$$

and

$$\nu(A \cap H_n^j) = \nu(A \cap H_{n+1}^{2j}) \vee \nu(A \cap H_{n+1}^{2j+1}),$$

there exists a sequence $\{j_n\}$ such that

$$A \cap H_m^{j_m} \subset A \cap H_n^{j_n} \ [\mu] \quad \text{for } m \geq n$$

and

$$\nu(A \cap H_n^{j_n}) = \nu(A \cap H_0^{j_0}) \quad \text{for } n = 1, 2, \cdots .$$

Let $s_n = \mu(A \cap H_n^{j_n})$, $a_n = j_n/2^n$ and $b_n = (j_n + 1)/2^n$ for $n = 1, 2, \cdots$.

The sequence $\{s_n\}$ converges since it is nonnegative and nonincreasing. Obviously $\{a_n\}$ and $\{b_n\}$ converge to the same number. By Lemma 2.5.1, we have $a_n \hat{\cdot} s_n \leq \nu(A \cap H_n^{j_n}) \leq b_n \hat{\cdot} s_n$. In addition s_n is m-finite for $n = 1, 2, \cdots$. It follows from these facts that $\lim_{n\to\infty} (a_n \hat{\cdot} s_n) =$ $= \nu(A \cap H_0^{j_0})$, and hence, that there is an integer m such that $(j_m/2^m) \hat{\cdot} \mu(A \cap H_m^{j_m}) > c$. Therefore we have

$$\hat{\int}_A f \, d\mu \geq \hat{\int}_A f_m \, d\mu$$

$$\geq (j_m/2^m) \hat{\cdot} \mu(A \cap H_m^{j_m})$$

$$> c.$$

Concequently it follows that

$$\hat{\int}_A f\, d\mu \geq \nu(A).$$

We now prove the second assertion. Suppose that $\hat{\int}_A f\, d\mu = \hat{\int}_A g\, d\mu$ for every $A \in \mathcal{X}$. Let P be a set of all pair (r, s) of nonnegative rational numbers such that $r > s$, and let

$$A_{r,s} = \{x \mid f(x) > r > s > g(x)\}$$

for every $(r, s) \in P$. Since

$$\hat{\int}_{A_{r,s}} f\, d\mu \geq \hat{\int}_{A_{r,s}} r\, d\mu = r \hat{\cdot} \mu(A_{r,s})$$

and

$$\hat{\int}_{A_{r,s}} g\, d\mu \leq \hat{\int}_{A_{r,s}} s\, d\mu = s \hat{\cdot} \mu(A_{r,s}),$$

it follows from the assumption of right reducibility that $\mu(A_{r,s}) = 0$. Therefore

$$\mu(\{x \mid f(x) > g(x)\}) = 0,$$

and similarly

$$\mu(\{x \mid g(x) > f(x)\}) = 0,$$

hence $f = g$ a.e. The proof is complete.

If $\hat{}$ is $\wedge$, i.e., min, the integral with respect to a $\vee$-measure is the fuzzy integral in the sense of Sugeno [21] :

$$\hat{\int}_A f\, d\mu = ⨍_A f \circ \mu = \sup_{\alpha \in [0,\infty]} [\alpha \vee \mu(\{x \mid f(x) > \alpha\} \cap A)].$$

In this case every number in $[0, \infty]$ is m-finite. We have the following corollary.

2.5.4 <u>COROLLARY</u>. Let a multiplication $\hat{\cdot}$ be $\wedge$. If μ is a σ-decomposable $\vee$-measure on $(X, \mathcal{X})$ and if ν is a $\vee$-measure on $(X, \mathcal{X})$, then there exists a function f such that

$$\hat{\int}_A f \, d\mu = \nu(A) \qquad \text{for every } A \in \mathcal{X}$$

iff $\nu(A) \leq \mu(A)$ for every $A \in \mathcal{X}$.

If a multiplication is the ordinary one, then m-finiteness is equivalent to finiteness.

2.5.5 <u>COROLLARY</u>. Let a multiplication $\hat{\cdot}$ be the ordinary one. If μ is a σ-decomposable σ-finite $\vee$-measure on $(X, \mathcal{X})$ and if ν is a $\vee$-measure on $(X, \mathcal{X})$, there exists a function f such that

$$\hat{\int}_A f \, d\mu = \nu(A) \qquad \text{for every } A \in \mathcal{X}$$

iff $\nu(A) = 0$ whenever $\mu(A) = 0$. Moreover the function f is a.e. unique.

PART II

FUZZY MEASURES AND CHOQUET'S INTEGRAL

CHAPTER 3 FUZZY MEASURES

§ 3.1 FUZZY MEASURE SPACES

3.1.1 DEFINITION. A fuzzy measurable space is a pair $(X, \mathcal{X})$, where X is a non-empty set and $\mathcal{X}$ is a class of subsets of X containing the empty set $\emptyset$ and the whole set X. If $(X, \mathcal{X})$ is a fuzzy measurable space, an element of $\mathcal{X}$ is called a measurable set.

3.1.2 NOTATION. Phrases of the following form will be frequently encountered: the class of subsets $\mathcal{X}$ is closed under (•••), where the parentheses contain the symbols of set-theoritic operations, followed by the letters f, c, or m, which indicate respectively: finite, countable, and monotone. Two examples will suffice to make this usage clear: "$\mathcal{X}$ is closed under ($\cup$f, $\cap$c)" means that a finite union, or an countable intersection of elements of $\mathcal{X}$ belongs to $\mathcal{X}$. "$\mathcal{X}$ is closed under ($\cup$m, ^{c})" means that the union of a monotone sequence of elements of $\mathcal{X}$ belongs to $\mathcal{X}$, and that the complement of any set in $\mathcal{X}$ belongs to $\mathcal{X}$.

The closure of a class of subsets $\mathcal{X}$ under ($\cup$c) [resp. ($\cap$c)], $\{\bigcup_{n=1}^{\infty} A_n$ [resp. $\bigcap_{n=1}^{\infty} A_n] \mid A_n \in \mathcal{X}\}$, is denoted by $\mathcal{X}_\sigma$ [resp. $\mathcal{X}_\delta$].

3.1.3 DEFINITIONS. Let $(X, \mathcal{X})$ be a fuzzy measurable space. A fuzzy measure μ on $\mathcal{X}$ (or on $(X, \mathcal{X})$) is a set function $\mu : \mathcal{X} \to [0,\infty]$ with the properties:

(1) $\mu(\emptyset) = 0$,

(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$.

A triplet $(X, \mathcal{X}, \mu)$ is called a <u>fuzzy measure space</u>.

An ordinary measure space is a fuzzy measure space.

3.1.4 <u>DEFINITIONS</u>. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space.

(1) μ is said to be <u>continuous from below</u> if, for every increasing sequence $\{A_n\}$ of sets in $\mathcal{X}$ for which $A_n \uparrow A \in \mathcal{X}$, we have $\mu(A_n) \uparrow \mu(A)$.

(2) μ is said to be <u>conditionally continuous from above</u> if, for every decreasing sequence $\{A_n\}$ of sets in $\mathcal{X}$ for which $A_n \downarrow A \in \mathcal{X}$ and for which $\mu(A_m) < \infty$ for at least one value of m, we have $\mu(A_n) \downarrow \mu(A)$.

(3) μ is said to be <u>continuous from above</u> if, for every decreasing sequence $\{A_n\}$ of sets in $\mathcal{X}$ for which $A_n \downarrow A \in \mathcal{X}$, we have $\mu(A_n) \downarrow \mu(A)$.

The next result 3.1.5 is an extension theorem for fuzzy measures. Batle and Trillas [2] gave the first half of the next theorem and pointed out that its proof is very similar to that of a theorem in the theory of capacities (Choquet [4] and Meyer [14]). This applies to the proof of the second half.

3.1.5 <u>THEOREM</u>. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and be closed under $(\cup f, \cap f)$.

(1) ([2]) If μ is countinuous from below, then there exists a unique fuzzy measure $\bar{\mu}$ on $\mathcal{X}_\sigma$ such that $\bar{\mu}$ is continuous from below and $\bar{\mu}(A) = \mu(A)$ for $A \in \mathcal{X}$.

(2) If μ is countinuous from above, then there exists a unique fuzzy measure $\bar{\mu}$ on $\mathcal{X}_\delta$ such that $\bar{\mu}$ is continuous from above and $\bar{\mu}(A) = \mu(A)$ for $A \in \mathcal{X}$.

§ 3.2 REPRESENTATION OF FUZZY MEASURES

3.2.1 DEFINITIONS. Let $(X, \mathscr{X})$ be a fuzzy measurable space and $(Y, \mathscr{Y})$ an ordinary measurable space. A mapping $H : \mathscr{X} \to \mathscr{Y}$ is called an interpreter (for measurable sets) if H satisfies the following conditions:

(1) $H(\emptyset) = \emptyset$,

(2) $H(A) \subset H(B)$ whenever $A \subset B$,

(3) $H(X) = Y$.

A triplet $(Y, \mathscr{Y}, H)$ is called a frame of $(X, \mathscr{X})$ (for representation) if H is an interpreter from $\mathscr{X}$ to $\mathscr{Y}$.

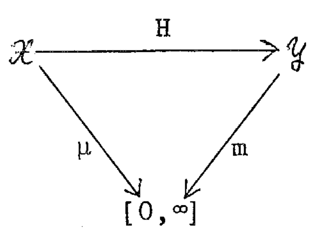
An interpreter H is said to be continuous from below [resp. continuous from above] if, for every increasing [resp. decreasing] sequence $\{A_n\}$ of for which $A_n \uparrow A \in \mathscr{X}$ [resp. $A_n \downarrow A \in \mathscr{X}$], we have $H(A_n) \uparrow H(A)$ [resp. $H(A_n) \downarrow H(A)$].

The following proposition is obvious.

3.2.2 PROPOSITION. Let $(X, \mathscr{X})$ be a fuzzy measurable space, $(Y, \mathscr{Y}, m)$ be an ordinary measure space, and $H : \mathscr{X} \to \mathscr{Y}$ be an interpreter. Let μ be a set function on $\mathscr{X}$ defined by

$$\mu(A) = m(H(A)) \qquad \forall\, A \in \mathscr{X},$$

that is, μ makes the following diagram commute:



Then μ is a fuzzy measure on $\mathcal{X}$. Moreover, if H is continuous from below [resp. continuous from above], then μ is continuous from below [resp. conditionally continuous from above].

3.2.3 DEFINITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space. A quadruplet $(Y, \mathcal{Y}, m, H)$ is called a representation of μ (or of $(X, \mathcal{X}, \mu)$) if $(Y, \mathcal{Y}, m)$ is an ordinary measure space and H is an interpreter from $\mathcal{X}$ to $\mathcal{Y}$ such that $\mu(A) = m(H(A)) \quad \forall A \in \mathcal{X}$.

The concept of the representation was proposed by Höhle [9]. The above definition is a generalization of his; in his definition $Y = \{0,1\}^{\mathcal{X}}$, $H(A) = \{y \in Y | y(A) = 1\}$, and $\mu(X) = 1$.

3.2.4 THEOREM. For every fuzzy measure μ, there exsists a representation of μ. Moreover, if μ is continuous from below [resp. continuous from above], then there exsists a representation of μ such that H is continuous from below [resp. continuous from above].

Proof. Let $(X, \mathcal{X}, \mu)$ be an arbitrary fuzzy measure space. Let Y be the open interval $(0, \mu(X))$ in the real line, $\mathcal{Y}$ be the class of all Borel subsets of Y, and m be Lebesgue measure on $\mathcal{Y}$. We define an interpreter $H : \mathcal{X} \to \mathcal{Y}$ by

$$H(A) = (0, \mu(A)) \quad \forall A \in \mathcal{X}.$$

Then $(Y, \mathcal{Y}, m, H)$ is a representation of μ. If μ is continuous from below, then obviously H is continuous from below.

Suppose that μ is continuous from above. If $\mu(X) < \infty$, let Y be the closed interval $(0, \mu(X)]$, and if $\mu(X) = \infty$, let Y be the interval $(0, \infty)$.

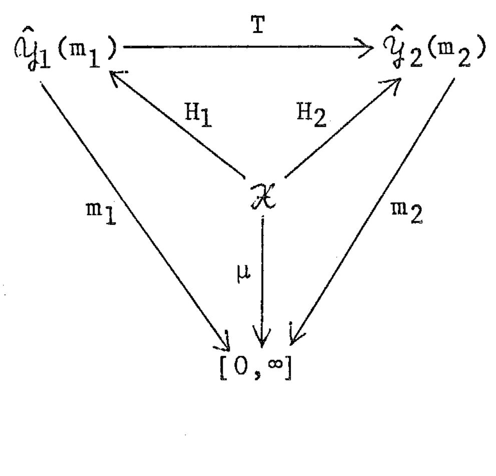
$\mathcal{Y}$ and m are defined in the same way as above. Now we define H by

$$H(A) = \begin{cases} (0,\ \mu(A)] & \text{if } \mu(A) < \infty, \\ (0,\ \infty) & \text{if } \mu(A) = \infty. \end{cases}$$

Then, since μ is continuous from above, H is also continuous from above.

3.2.5 NOTATION. If $(Y, \mathcal{Y}, m)$ is an ordinary measure space, we denote the measure algebra associated with $(Y, \mathcal{Y}, m)$ by $(\mathcal{Y}(m), m)$. If $(Y, \mathcal{Y}, m, H)$ is a representation of a fuzzy measure, we denote by $\hat{\mathcal{Y}}$ the σ-algebra generated by $\{H(A) | A \in \mathcal{X}\}$.

3.2.6 DEFINITION. Let $R_1 = (Y_1, \mathcal{Y}_1, m_1, H_1)$ and $R_2 = (Y_2, \mathcal{Y}_2, m_2, H_2)$ be representations of a fuzzy measure μ. R_1 and R_2 are said to be equivalent if there exists an isomorphism T from $(\hat{\mathcal{Y}}_1(m_1), m_1)$ to $(\hat{\mathcal{Y}}_2(m_2), m_2)$ such that $T \circ H_1 = H_2$. That is, the following diagram commutes:



A fuzzy measure generally has infinitely many and mutually non-equivalent representations. We give a simple example.

3.2.7 EXAMPLE. Let $X = \{x_1, x_2\}$, $\mathcal{X} = 2^X$, $Y = \{y_0, y_1, y_2, y_3\}$, and $\mathcal{Y} = 2^Y$. We define an interpreter $H : \mathcal{X} \to \mathcal{Y}$ as follows:

$$H(\emptyset) = \emptyset,$$
$$H(\{x_1\}) = \{y_0, y_1\},$$
$$H(\{x_2\}) = \{y_0, y_2\},$$
$$H(X) = Y.$$

Let μ be a fuzzy measure on 2^X such that $0 < \mu(\{x_1\}) < \mu(X)$ and $0 < \mu(\{x_2\}) < \mu(X)$. Then it is clear that the system of equalities

$$\begin{cases} m_0 + m_1 & = \mu(\{x_1\}) \\ m_0 \quad + m_2 & = \mu(\{x_2\}) \\ m_0 + m_1 + m_2 + m_3 & = \mu(X) \end{cases}$$

has infinitely many solutions (m_0, m_1, m_2, m_3) under the condition $m_0, m_1, m_2, m_3 \geq 0$. For any nonnegative solution (m_0, m_1, m_2, m_3), if m is the measure on 2^Y defined by $m(\{y_i\}) = m_i$ $i = 0, 1, 2, 3$, then $(Y, 2^Y, m, H)$ is a representation of μ. In addition, if (m_0, m_1, m_2, m_3) and (m_0', m_1', m_2', m_3') are different nonnegative solutions, then the representations $(Y, 2^Y, m, H)$ and $(Y, 2^Y, m', H)$ are not equivalent, where m and m' are the measures associated with (m_0, m_1, m_2, m_3) and (m_0', m_1', m_2', m_3'), respectively. Therefore μ has infinitely many and mutually nonequivalent representations.

We shall prove that, for any fuzzy measurable space $(X, \mathcal{X})$, there exist a frame $(S_X, \mathcal{S}_X, H_X)$ of $(X, \mathcal{X})$ such that, for any fuzzy measure μ on $\mathcal{X}$ and for any representation R of μ, there exists an ordinary measure m_R on $\mathcal{S}_X$ such that $(S_X, \mathcal{S}_X, m_R, H_X)$ is a representation of μ which is equivalent to R.

3.2.8 DEFINITIONS. Let $(X, \mathcal{X})$ be a fuzzy measurable space. A semi-filter in $(X, \mathcal{X})$ is a subclass θ of $\mathcal{X}$ with the properties:

(1) $\emptyset \notin \theta$, $X \in \theta$,

(2) $A \in \theta$, $A \subset B \in \mathcal{X} \Rightarrow B \in \theta$.

We denote the set of all the semi-filters in $(X, \mathcal{X})$ by S_X and define a mapping $H_X : \mathcal{X} \to 2^{S_X}$ by

$$H_X(A) = \{\theta | A \in \theta\} \qquad \forall A \in \mathcal{X}.$$

We denote by $\mathcal{S}_X$ the σ-algebra generated by $\{H_X(A) | A \in \mathcal{X}\}$. The triplet $(S_X, \mathcal{S}_X, H_X)$ is called the universal frame of $(X, \mathcal{X})$ for representation.

S_X is non-empty since the classes $\{A \in \mathcal{X} | A \neq \emptyset\}$ and $\{X\}$ belong to S_X. It is clear that H_X is an interpreter from $\mathcal{X}$ to $\mathcal{S}_X$.

3.2.9 THEOREM. Let $(X, \mathcal{X}, \mu)$ be an arbitrary fuzzy measure space. For every representation $R = (Y, \mathcal{Y}, m, H)$ of μ, there exists a measure m_R on $\mathcal{S}_X$ such that $(S_X, \mathcal{S}_X, m_R, H_X)$ is a representation of μ which is equivalent to R.

Proof. For each $y \in Y$, we define a subclass $\tau(y)$ of $\mathcal{X}$ by

$$\tau(y) = \{A \in \mathcal{X} | y \in H(A)\}.$$

By the definition of an interpretation H, $\tau(y)$ is a semi-filter in for every $y \in Y$. Hence we can regard τ as a mapping from Y to S_X. Since for every $A \in \mathcal{X}$ and for every $y \in Y$

$$\begin{aligned} y \in \tau^{-1} \circ H_X(A) &\Longleftrightarrow \tau(y) \in H_X(A) \\ &\Longleftrightarrow A \in \tau(y) \\ &\Longleftrightarrow y \in H(A), \end{aligned}$$

it follows that $H = \tau^{-1} \circ H_X$, and hence that τ is measurable. Let m_R be the image measure $m \circ \tau^{-1}$. Then it follows that $\mu = m \circ H = m \circ \tau^{-1} \circ H_X = m_R \circ H_X$, and hence that $R_X = (S_X, \mathcal{S}_X, m_R, H_X)$ is a representation of μ.

By showing that τ^{-1} can be regarded as an isomorphism from $(\mathcal{S}_X(m_R), m_R)$ to $(\hat{\mathcal{Y}}(m), m)$, we prove that R_X is equivalent to R. Since

$$\tau^{-1}(E \,\Delta\, F) = \tau^{-1}(E) \,\Delta\, \tau^{-1}(F) \quad \forall\, E, F \in \mathcal{S}_X,$$

it follows that, for $E, F \in \mathcal{S}_X$

$$m_R(E \,\Delta\, F) = 0 \iff m(\tau^{-1}(E) \,\Delta\, \tau^{-1}(F)) = 0.$$

Therefore τ^{-1} is a one-to-one homomorphism from $(\mathcal{S}_X(m_R), m_R)$ to $(\mathcal{Y}(m), m)$; note that this $\mathcal{Y}$ is not $\hat{\mathcal{Y}}$. Hence in order to prove that τ^{-1} is an isomorphism from $(\mathcal{S}_X(m_R), m_R)$ to $(\hat{\mathcal{Y}}(m), m)$, it is sufficient to prove that $\hat{\mathcal{Y}} = \mathcal{Y}_0$, where $\mathcal{Y}_0$ is the σ-subalgebra $\{\tau^{-1}(E) | E \in \mathcal{S}_X\}$ of $\mathcal{Y}$. Since $H = \tau^{-1} \circ H_X$, it follows that $\{H(A) | A \in \mathcal{X}\} \subset \mathcal{Y}_0$, and therefore, since $\hat{\mathcal{Y}}$ is generated by $\{H(A) | A \in \mathcal{X}\}$, that $\hat{\mathcal{Y}} \subset \mathcal{Y}_0$. Let $\mathcal{S}_0$ be the σ-subalgebra $\{E \in \mathcal{S}_X | \tau^{-1}(E) \in \hat{\mathcal{Y}}\}$ of $\mathcal{S}_X$. Since $H = \tau^{-1} \circ H_X$, it follows that $\{H_X(A) | A \in \mathcal{X}\} \subset \mathcal{S}_0$, and therefore, since $\mathcal{S}_X$ is generated by $\{H_X(A) | A \in \mathcal{X}\}$, that $\mathcal{S}_0 = \mathcal{S}_X$. This implies that $\tau^{-1}(E) \in \hat{\mathcal{Y}} \;\forall\, E \in \mathcal{S}_X$, i.e., $\mathcal{Y}_0 \subset \hat{\mathcal{Y}}$, and hence we have $\hat{\mathcal{Y}} = \mathcal{Y}_0$.

The next result follows from 3.2.4 and 3.2.9.

3.2.10 <u>COROLLARY</u>. ([9]) For every fuzzy measure μ on $(X, \mathcal{X})$, there exists a measure m on $\mathcal{S}_X$ such that $(S_X, \mathcal{S}_X, m, H_X)$ is a representation of μ.

Theorem 3.2.9 and Corollary 3.2.10 imply that the universal frame

$(S_X, \mathcal{S}_X, H_X)$ is sufficient for representation of fuzzy measures on $(X, \mathcal{X})$.

We next consider representation of continuous-from-below fuzzy measures.

3.2.11 DEFINITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space. An lower semi-filter in $\mathcal{X}$ is a semi-filter θ with the property:

$$\text{If } \{A_n\} \subset \mathcal{X} - \theta \text{ and } A_n \uparrow A, \text{ then } A \notin \theta.$$

We denote the set of all the lower semi-filters in $\mathcal{X}$ by $\underline{S}_X$ and define a mapping $\underline{H}_X$ by

$$\underline{H}_X(A) = \{\theta \in \underline{S}_X | A \in \theta\} \quad \forall A \in \mathcal{X}.$$

We denote by $\underline{\mathcal{S}}_X$ the σ-algebra generated by $\{\underline{H}_X(A) | A \in \mathcal{X}\}$.

$\underline{S}_X$ is non-empty since the class $\mathcal{X} - \{\emptyset\}$ belongs to $\underline{S}_X$. It is clear that $\underline{H}_X$ is a continuous-from-below interpreter form $\mathcal{X}$ into $\underline{\mathcal{S}}_X$. In the same way as 3.2.9, we have the next theorem.

3.2.12 THEOREM. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and μ be continuous from below. For every representation $R = (Y, \mathcal{Y}, m, H)$ of μ for which H is continuous forom below, there exists a measure m_R such that $(\underline{S}_X, \underline{\mathcal{S}}_X, m_R, \underline{H}_X)$ is a representation of μ which is equivalent to R.

3.2.13 COROLLARY. For every continuous-from-below fuzzy measure μ on $(X, \mathcal{X})$, there exists a measure m on $\underline{\mathcal{S}}_X$ such that $(\underline{S}_X, \underline{\mathcal{S}}_X, m, \underline{H}_X)$ is a representation of μ.

The same argument applies to continuous-from-above fuzzy measures.

3.2.14 <u>DEFINITION</u>. Let $(X, \mathcal{X})$ be a fuzzy measurable space. A <u>upper semi-filter</u> in $\mathcal{X}$ is a semi-filter θ in $\mathcal{X}$ with the property:

If $\{A_n\} \subset \theta$, $A \varepsilon \mathcal{X}$, and $A_n \downarrow A$, then $A \varepsilon \theta$.

We denote the set of all the upper semi-filters in $\mathcal{X}$ by $\bar{S}_X$ and define a mapping $\bar{H}_X$ by

$$\bar{H}_X(A) = \{\theta \varepsilon \bar{S}_X | A \varepsilon \theta\} \quad \forall A \varepsilon \mathcal{X}.$$

We denote by $\bar{\mathcal{S}}_X$ the σ-algebra generated by $\{\bar{H}_X(A) | A \varepsilon \mathcal{X}\}$.

$\bar{S}_X$ is non-empty since the class $\{X\}$ belongs to $\bar{S}_X$. It is clear that $\bar{H}_X$ is a continuous-from-above interpreter form $\mathcal{X}$ into $\bar{\mathcal{S}}_X$.

3.2.15 <u>THEOREM</u>. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and μ be continuous from above. For every representation $R = (Y, \mathcal{Y}, m, H)$ of μ for which H is continuous forom above, there exists a measure m_R such that $(\bar{S}_X, \bar{\mathcal{S}}_X, m_R, \bar{H}_X)$ is a representation of μ which is equivalent to R.

3.2.16 <u>COROLLARY</u>. For every continuous-from-above fuzzy measure μ on $(X, \mathcal{X})$, there exists a measure m on $\bar{\mathcal{S}}_X$ such that $(\bar{S}_X, \bar{\mathcal{S}}_X, m, \bar{H}_X)$ is a representation of μ.

§ 3.3 AN INTERPRETATION OF FUZZY MEASURES

A fuzzy measure is a sort of "measure." Therefore a fuzzy measure μ on $\mathcal{X}$ must measure a certain attribute of $\mathcal{X}$, and faithfully represent the properties of the attribute. In this section, we consider what situation a fuzzy measure expresses. We begin with examples.

3.3.1 EXAMPLE. "A WORKSHOP". Let X be a finite set of the workers in a workshop and suppose that they produce the same products. For each $A \in 2^X$, we consider the situation that the members of A work in the workshop. A group A may have various ways to work: various combinations of joint work and devided work. But suppose that a group A works in the most efficient way. Let $\mu(A)$ be the number of the products made by A in one hour. Then μ is a measure of the productivity of groups: the attribute of 2^X in question is the productivity. By the definition of μ, the following statements are natural:

(1) $\mu(\emptyset) = 0$,

(2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

That is, μ is a fuzzy measure on 2^X.

μ is not necessarily additive. Let A and B be disjoint subsets of X, and let us consider the productivity of the coupled group $A \cup B$. If A and B work separately, then $\mu(A \cup B) = \mu(A) + \mu(B)$. But, since they generally interact on each other, the equation may not always hold. The inequality $\mu(A \cup B) > \mu(A) + \mu(B)$ shows the effective cooperation of members of $A \cup B$. The converse inequality $\mu(A \cup B) < \mu(A) + \mu(B)$ shows the incompatibility between A's operations and B's, that is, the impossiblity of separate working. For example, the incompatibility is caused by insufficient equipments and/or insufficient workstations in the workshop: sufficient equipments and/or sufficient workstations make separate working possible. As a matter of fact, $A \cup B$ may have both effective cooperation and incompatible operation. Therefore if the degree of the effective cooperation is greater than that of the incompatible operation, the inequality $\mu(A \cup B) > \mu(A) + \mu(B)$ holds. If it is not the case, the converse inequality holds.

Now let $X = \{x_1, x_2\}$ for the sake of simplicity. The same argument is applicable to the case where $A = \{x_1\}$ and $B = \{x_2\}$. Let m_0, m_1, m_2, and m_3 be the measures of the productivity corresponding to the incompatible operation, x_1's compatible operation, x_2's compatible operation, and the effective cooperation, respectively. One's compatible operation means the operation not prevented by the other's operation. We can simply imagine the case where there are two workstations to do a certain job. While the incompatible operation means that there is only one workstation for two workers and they have to use it in series.

Then μ can be represented as follows:

$$\begin{aligned} \mu(\emptyset) &= 0, \\ \mu(\{x_1\}) &= m_0 + m_1, \\ \mu(\{x_2\}) &= m_0 + m_2, \\ \mu(\{x_1, x_2\}) &= m_0 + m_1 + m_2 + m_3. \end{aligned} \tag{3.1}$$

Fig. 3.1 illustrates Eq. (3.1), where the measures of the productivity are represented with area.

We denote the index set by Y, i.e., $Y = \{0, 1, 2, 3\}$, and an interpreter $H : 2^X \to 2^Y$ by

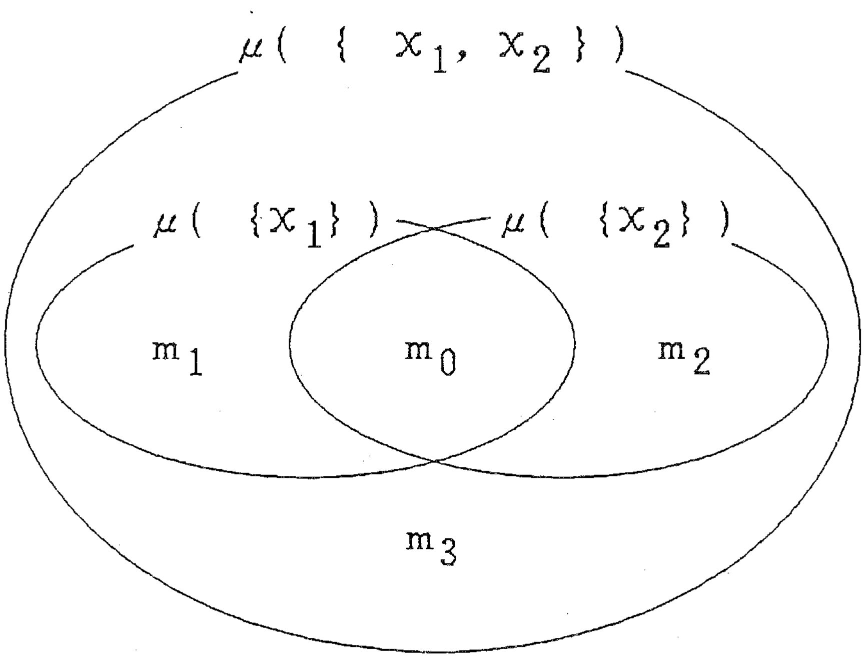


Fig. 3.1 Illustration of Eq. (3.1).

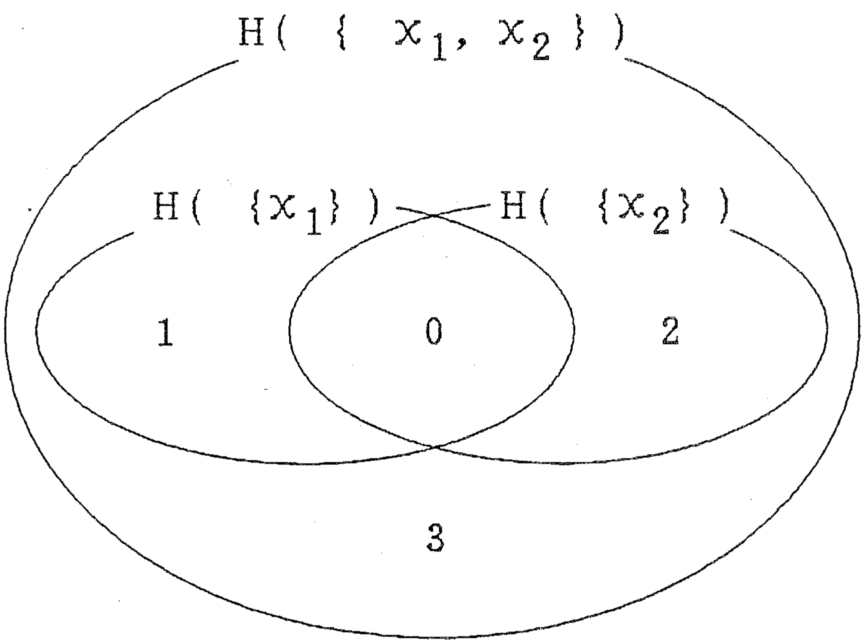


Fig. 3.2 The interpreter H (Venn diagram).

$$H(\emptyset) = \emptyset,$$
$$H(\{x_1\}) = \{0, 1\},$$
$$H(\{x_2\}) = \{0, 2\}, \qquad (3.2)$$
$$H(\{x_1, x_2\}) = \{0, 1, 2, 3\}.$$

Then μ is exressed as

$$\mu(A) = \sum_{i \varepsilon H(A)} m_i \qquad \forall A \varepsilon 2^X. \qquad (3.3)$$

If we define an ordinary measure m on 2^X by

$$m(E) = \sum_{i \varepsilon E} m_i \qquad \forall E \varepsilon 2^Y,$$

then $(Y, 2^Y, m, H)$ is a representation of μ.

If $(Y, \mathcal{Y}, m, H)$ is a representation of a fuzzy measure, we call an element of Y a feature in respect of the attribute (or a feature for short). In this example "a feature in respect of the attribute" means "a feature in respect of productivity", since the attribute in question is productivity. For example, $3 \varepsilon Y$ is the feature of the effective cooperation of x_1 and x_2. In general if $i \varepsilon H(A)$, we say that i is a feature of A, or A has the feature i. Then Eq. (3.3) expresses that the measure of A equals the sum of all the measures m_i's of the features of A. In addition, we call the set H(A) of the features of A "A in the attribute".

Note that the interpreter H from 2^X to 2^Y does not preserve the operations ∪ and ∩ but the inclusion ⊂. This is because of the interactions 0 and $3 \varepsilon Y$ between x_1 and x_2.

We give another example.

3.3.2 <u>EXAMPLE</u>. "A RARE BOOK". Suppose there is a rare book consisting of two volumes. We denote the first volume and the second volume by x_1

and x_2 respectively. Suppose that there is a secondhand bookseller who buys them at the prices:

$\mu(\{x_1\})$ dollars per first volume,

$\mu(\{x_2\})$ dollars per second volume,

$\mu(\{x_1, x_2\})$ dollars per set of two volumes.

Since he sets a high value on a complete set, there holds

$$\mu(\{x_1, x_2\}) > \mu(\{x_1\}) + \mu(\{x_2\}).$$

We define the set of features Y by $Y = \{y_1, y_2, y_3\}$, and an interpreter $H : 2^X \to 2^Y$ and a measure m on 2^Y as follows:

$H(\emptyset) = \emptyset$,

$H(\{x_1\}) = \{y_1\}$,

$H(\{x_2\}) = \{y_2\}$,

$H(X) = Y$,

and

$m(\{y_1\}) = \mu(\{x_1\})$,

$m(\{y_2\}) = \mu(\{x_2\})$,

$m(\{y_3\}) = \mu(X) - \mu(\{x_1\}) - \mu(\{x_2\})$.

Then $(Y, 2^Y, m, H)$ is a representation of μ. We can say that $m(\{y_1\})$ is the value of x_1, $m(\{y_2\})$ is the value of x_2, and $m(\{y_3\})$ is the value of being a complete set, in other words the increment of price caused by being a complete set.

The same argument applies when we discuss weight of objects.

3.3.3 <u>EXAMPLE</u>. "WEIGHT OF OBJECTS". Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set consisting of n objects, and let $\mu(A)$ be the weight of $A \in 2^X$. Obviously μ is an additive measure, which is a special fuzzy measure. Since the elements of X are not interactive, the weight of A is characterized only by each element contained in A. Therefore we define the set of the features by $Y = \{y_1, y_2, \cdots, y_n\}$, where $y_i \in Y$ means a feature "containing x_i." Since H(A) must satisfy that

$$x_i \in A \iff y_i \in H(A) \qquad \forall A \in 2^X,$$

H(A) is defined by $H(A) = \{y_i | x_i \in A\}$. If we define a measure m on 2^Y by $m(\{y_i\}) = \mu(\{x_i\}) \ \forall y_i \in Y$, where $m(\{y_i\})$ represents the increment of weight caused by the feature y_i, then we have

$$\mu(A) = m(H(A)) \qquad \forall A \in 2^X,$$

that is, $(Y, 2^Y, m, H)$ is a representation of μ. The mapping $H : 2^X \to 2^Y$ is a Boolean algebra isomorphism. That is, H(A) is not distinguished from A. In case of ordinary measures, the distinction between "2^X" and "2^X in the attribute", i.e. 2^Y, is unnecessary.

Now we describe a general case. We consider to measure an attribute of 2^X by a set function μ. We assume that the subsets of 2^X interact on each other, and we distinguish "2^X in the attribute" from "2^X." That is, we do not assume the mapping $H : 2^X \to 2^Y$ to be a Boolean algebra isomorphism.

We explain in detail. First suppose it is possible that $H(A) \cup H(B) \subsetneq H(A \cup B)$. This means that $A \cup B$ can have a feature which neither A nor B has. Such a feature can be called a cooperative action of A and B. The set of the cooperative actions is $H(A \cup B) - [H(A) \cup H(B)]$. Secondly suppose that the condition $A \cap B = \emptyset$ unnecessarily means that $H(A) \cap H(B) = \emptyset$. The set $H(A) \cap H(B)$ represents the overlap of A and B in the attribute, which is interpreted as an incompatibility between A and B or as common features between A and B.

Accordingly we assume that H satisfies only the conditions:

(1) $H(\emptyset) = \emptyset$,

(2) $A \subset B \Rightarrow H(A) \subset H(B)$,

(3) $H(X) = Y$,

that is, H is an interpreter. (1) means that the empty set has no feature in respect of the attribute. (2) means that, if $A \subset B$, then B has all the features of A. (3) means that only the features of the subsets of X are considered. It is not a very essential condition from a mathematical point of view.

For each feature $y \in Y$, let m_y be its measure, and suppose

$$\mu(A) = \sum_{y \in H(A)} m_y \quad \forall A \in 2^X,$$

i.e., the measure $\mu(A)$ of A is the sum of all the measures of the features of A. Let m be an ordinary measure on 2^Y defined by

$$m(E) = \sum_{y \in E} m_y \qquad \forall E \in 2^Y.$$

Then we have that

$$\mu(A) = m(H(A)) \qquad \forall A \in 2^X. \tag{3.4}$$

Obviously this set function μ is a fuzzy measure and $(Y, 2^Y, m, H)$ is a representation of μ.

Finally we state the difference between a fuzzy measure and an ordinary measure. The separateness (no interactions) of the subsets of X brings an ordinary measure, while their interaction brings a fuzzy measure. The identity of "2^X" and "2^X in the attribute" brings an ordinary measure, while these distinction brings a fuzzy measure.

3.3.4 <u>REMARK</u>. We can express these relations in other words. We need a preparation for that. Since

$$\mu(A \cup B) - \mu(B) = m(H(A \cup B)) - m(H(B))$$
$$= m(H(A \cup B) - H(B))$$
$$= \sum_{\substack{y \in H(A \cup B) \\ y \notin H(B)}} m_y,$$

$\mu(A \cup B) - \mu(B)$ is a meaningful quantity, which is the measure of the difference between $A \cup B$ and B in the attribute, or the sum of the measures of the features of $A \cup B$ which B does not have. This quantity is interpreted as the effect of A joining B. Let us fix a set A. If μ is an ordinary measure, then for every B for which $A \cap B = \emptyset$, $\mu(A \cup B) - \mu(B)$ is equal to the constant $\mu(A)$. But if μ is a fuzzy measure, this is not true; the value $\mu(A \cup B) - \mu(B)$ depends on B. Therefore we can state the difference of a fuzzy measure from an ordinary measure. For an ordinary measure, the effect of A joining B does not depend on B. For a fuzzy measure, the effect depends on B.

CHAPTER 4 CHOQUET'S INTEGRAL AS AN INTEGRAL WITH RESPECT TO FUZZY MEASURES

§ 4.1 MEASURABLE FUNCTIONS

4.1.1 DEFINITIONS. Let $(X, \mathcal{X})$ be a fuzzy measurable space. A c-measurable [resp. o-measurable] function f on X is a function $f : X \to [0,\infty]$ such that $\{x|f(x) \geq r\}$ [resp. $\{x|f(x) > r\}$] $\in \mathcal{X}$ $\forall\, r \in [0,\infty)$. A function f is said to be measurable if f is c-measurable or o-measurable. For the sake of convinience, we frequently denote $\{x|f(x) \geq r\}$ and $\{x|f(x) > r\}$ by $\{f \geq r\}$ and $\{f > r\}$, respectively.

Note that we deal only with nonnegative functions. It is obvious that the caracteristic function 1_A of a measurable set A is both o-measurable and c-measurable.

Since the following propositions are very elementary, we omit the proofs.

4.1.2 PROPOSITION. If $(X, \mathcal{X})$ is a fuzzy measurable space and if $\mathcal{X}$ is closed under ($\cup$m) [resp. ($\cap$m)], then c-measurability [resp. o-measurability] implies o-measurability [resp. c-measurability].

4.1.3 COROLLARY. Let $(X, \mathcal{X})$ be a fuzzy measurable space. If $\mathcal{X}$ is a monotone class, i.e., $\mathcal{X}$ is closed under ($\cup$m, $\cap$m), then c-measurability and o-measurability are equivalent.

4.1.4 <u>PROPOSITION</u>. Let $\phi : [0,\infty] \to [0,\infty]$ be a right [resp. left] continuous non-decreasing function. If f is a c-measurable [resp. o-measurable] function, then $\phi \circ f$ is c-measurable [resp. o-measurable].

4.1.5 <u>COROLLARY</u>. If f is a c-measurable [resp. o-measurable] function and $a \geq 0$, then af and (f + a) are c-measurable [resp. o-measurable].

4.1.6 <u>PROPOSITION</u>. Let $(X, \mathcal{X})$ be a fuzzy measurable space.

(1) Let $\mathcal{X}$ be closed under ($\cup$f). If f and g are both c-measurable [resp. o-measurable], then $f \vee g$ is c-measurable [resp. o-measurable].

(2) Let $\mathcal{X}$ be closed under ($\cap$f). If f and g are both c-measurable [resp. o-measurable], then $f \wedge g$ is c-measurable [resp. o-measurable].

4.1.7 <u>PROPOSITION</u>. Let $(X, \mathcal{X})$ be a fuzzy measurable space and $\mathcal{X}$ be closed under ($\cap$f, $\cup$c). If f and g are c-measurable [resp. o-measurable] functions on X, then (f + g) and fg are c-measurable [resp. o-measurable].

4.1.8 <u>PROPOSITION</u>. Let $(X, \mathcal{X})$ be a fuzzy measurable space and $\{f_n\}$ a sequence of c-measurable [resp. o-measurable] functions on X.

(1) If $\mathcal{X}$ is closed under ($\cap$m) [resp. ($\cup$m)] and if $f_n \downarrow f$ [resp. $f_n \uparrow f$], then f is c-measurable [resp. o-measurable].

(2) If $\mathcal{X}$ is closed under ($\cap$c) [resp. ($\cup$c)], then inf f_n is c-measurable [resp. sup f_n is o-measurable].

(3) If $\mathcal{X}$ is σ-lattice, i.e., $\mathcal{X}$ is closed under ($\cup$c, $\cap$c), then (lim inf f_n) and (lim sup f_n) are measurable.

§ 4.2 CHOQUET'S INTEGRAL

4.2.1 PROPOSITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and f a function on X which is both o-measurable and c-measurable. Then

$$\mu(\{x | f(x) > r\}) = \mu(\{x | f(x) \geq r\})$$

except at most countable values of r.

Proof. The monotonicity of μ implies that the functions $\mu(\{f > r\})$ and $\mu(\{f \geq r\})$ are non-increasing. In addition, we have

$$\mu(\{f > r\}) \leq \mu(\{f \geq r\}) \leq \lim_{s \uparrow r} \mu(\{f > s\}) \quad \forall\, r \in [0, \infty].$$

Therefore the assertion follows.

By virtue of 4.2.1 we can define Choquet's integral as follows:

4.2.2 DEFINITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space. Choquet's integral of measurable function f with respect to μ, denoted by $(C)\int f \, d\mu$, is defined by

$$(C)\int f \, d\mu = \begin{cases} \int_0^\infty \mu(\{x | f(x) > r\}) \, dr & \text{if f is o-measurable,} \\ \int_0^\infty \mu(\{x | f(x) \geq r\}) \, dr & \text{if f is c-measurable.} \end{cases}$$

4.2.3 PROPOSITION ([3]). Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and f a measurable function. If, for every $r \in [0, \infty)$

$$F(r) = \begin{cases} - \mu(\{f > r\}) & \text{if f is o-measurable,} \\ - \mu(\{f \geq r\}) & \text{if f is c-measurable,} \end{cases}$$

then

$$(C)\int f \, d\mu = \int_0^\infty r \, dF(r).$$

The integral in the right side is Stieltjes integral.

Since the following proposition is very elementary, we omit the proof.

4.2.4 PROPOSITION. ((1) - (3) are shown in [3].) Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space. Let f and g be measurable functions and $\{f_n\}$ a sequence of measurable functions.

(1) $f \leq g$ implies $(C)\int f \, d\mu \leq (C)\int g \, d\mu$.

(2) $(C)\int af \, d\mu = a \, (C)\int f \, d\mu \quad \forall a \geq 0$.

(3) If μ is an ordinary measure, then Choquet's integral coincides with Lebesgue integral; $(C)\int f \, d\mu = \int f \, d\mu$.

(4) $(C)\int 1_A \, d\mu = \mu(A) \quad \forall A \in \mathcal{X}$.

(5) If μ is continuous from below, then $f_n \uparrow f$ implies that

$$(C)\int f_n \, d\mu \uparrow (C)\int f \, d\mu.$$

(6) Let μ be conditionally continuous from above. If $f_n \downarrow f$ and if for at least one value of m f_m is bounded and $\mu(\{f_m > 0\}) < \infty$, then

$$(C)\int f_n \, d\mu \downarrow (C)\int f \, d\mu.$$

The next proposition shows an expression of Choquet's integral of a simple function. We omit the proof.

4.2.5 PROPOSITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measurable space. Every simple function f on X can be written as

$$f = \sum_{i=1}^{n} (a_i - a_{i-1}) \, 1_{A_i}, \tag{2.1}$$

where $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$. Choquet's integral of a simple function f written as (2.1) with respect to μ can be written as

$$(C)\int f\, d\mu = \sum_{i=1}^{n} (a_i - a_{i-1})\ \mu(A_i). \tag{2.2}$$

4.2.6 REMARK. If we assume a fuzzy measure to be continuous from below, then we can define the integral like Lebesgue integral: for a simple function f defined as (2.1), we define the integral of f as (2.2), and for a non-simple measurable function f, we define the integral of f as

$$(C)\int f\, d\mu = \lim_{n\to\infty} (C)\int f_n\, d\mu,$$

where $\{f_n\}$ is a sequence of simple functions such that for every $x \in X$ $f_n(x) \uparrow f(x)$.

Since fuzzy measure is generally non-additive, also Choquet's integral is generally non-additive. Concerning the additivity of Choquet's integral, the next theorem holds.

4.2.7 THEOREM ([3]). Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space.

(1) $$(C)\int (f + g)\, d\mu \le (C)\int f\, d\mu + (C)\int g\, d\mu$$

for every pair (f, g) of measurable functions if and only if

$$\mu(A \cup B) + \mu(A \cap B) \le \mu(A) + \mu(B) \quad \forall A \in \mathcal{X}, \forall B \in \mathcal{X}.$$

(2) $$(C)\int (f + g)\, d\mu \ge (C)\int f\, d\mu + (C)\int g\, d\mu$$

for every pair (f, g) of measurable functions if and only if

$$\mu(A \cup B) + \mu(A \cap B) \ge \mu(A) + \mu(B) \quad \forall A \in \mathcal{X}, \forall B \in \mathcal{X}.$$

In Section 4.6 we shall discuss the additivity again.

§ 4.3 REPRESENTATION OF CHOQUET'S INTEGRAL

4.3.1 PROPOSITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space and $(Y, \mathcal{Y}, H)$ a frame of $(X, \mathcal{X})$ for representation. For every function f on X which is both o-measurable and c-measurable,

$$\sup \{r | y \in H(\{f > r\})\} = \sup \{r | y \in H(\{f \geq r\})\} \quad \forall\, y \in Y.$$

Proof. This assertion is immediate from the monotonicity of the interpreter H.

4.3.2 DEFINITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space and $(Y, \mathcal{Y}, H)$ a frame of $(X, \mathcal{X})$ for representation. For every measurable function f on X, we define a function h(f) on Y by, for every $y \in Y$,

$$h(f)(y) = \begin{cases} \sup \{r | y \in H(\{f > r\})\} & \text{if } f \text{ is o-measurable,} \\ \sup \{r | y \in H(\{f \geq r\})\} & \text{if } f \text{ is c-measurable.} \end{cases}$$

We regard h as a mapping and call h the interpreter (for measurable functions) induced by H.

By virtue of 4.3.1, an interperter for measurable functions is well-defined. The interpreter h induced by an interpreter H can be regarded as an extension of H; we can easily obtain that $h(1_A) = 1_{H(A)} \quad \forall\, A \in \mathcal{X}$. This is the reason h has the same name as H.

4.3.3 PROPOSITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space, $(Y, \mathcal{Y}, H)$ a frame of $(X, \mathcal{X})$ for representation, and h the interpreter for measurable functions induced by H. For any measurable function f, h(f) is a measurable function on $(Y, \mathcal{Y})$.

Proof. Let f be an o-measurable function on $(X, \mathcal{X})$. It follows from 4.3.2 and the monotonicity of H that

$$\{y|h(f)(y) > r\} = \bigcup_{s>r} H(\{x|f(x) > s\}) \qquad \forall\ r \in [0,\infty).$$

Since $H(\{x|f(x) > s\}) \in \mathcal{Y}$ $\forall$ s $\in [0,\infty)$ and since $\mathcal{Y}$ is σ-algebra, it follows that $\{y|h(f)(y) > r\} \in \mathcal{Y}$ $\forall$ r $\in [0,\infty)$, and hence that h(f) is measurable. If f is c-measurable, similarly h(f) is measurable.

The following theorem is the main result in this section.

4.3.4 THEOREM. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, $(Y, \mathcal{Y}, m, H)$ a representation of μ, and h the interpreter for measurable functions induced by H. For any measurable function f, Choquet's integral of f with respect to μ is equal to Lebesgue integral of h(f) with respect to m:

$$(C)\int f\ d\mu = \int h(f)\ dm.$$

Proof. Let f be an o-measurable function on $(X, \mathcal{X})$. Since

$$\lim_{s \downarrow r} \mu(\{f > s\}) \leq m(\{h(f) > r\}) \leq \mu(\{f > r\}) \quad \forall\ r \in [0,\infty],$$

and since $\mu(\{f > r\})$ and $m(\{h(f) > r\})$ are non-increasing functions on $[0,\infty)$, it follows that

$$\mu(\{f > r\}) = m(\{h(f) > r\})$$

except at most countable values of r. Therefore we obtain

$$\int_0^\infty \mu(\{f > r\})\ dr = \int_0^\infty m(\{h(f) > r\})\ dr.$$

Since m is an ordinal measure, it follows that

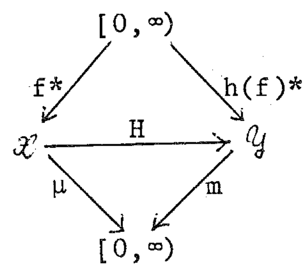
$$(C)\int f\ d\mu = \int h(f)\ dm.$$

4.3.5 REMARK. Let $(Z, \mathcal{Z})$ be a fuzzy measurable space. We denote by $\mathcal{F}(Z)$ the set of all the nonnegative o-measurable functions on Z, and by $\mathcal{L}(Z)$ the set of all the mappings $\phi : [0, \infty) \to \mathcal{Z}$ with the property:

$$\phi(r) = \bigcup_{s>r} \phi(s) \qquad \forall\, r \in [0, \infty).$$

It is easy to check the followings. The mapping $f \to f^*$, where $f \in \mathcal{F}(Z)$ and $f^*(r) = \{x \mid f(x) > r\}$ $\forall\, r \in [0, \infty)$, is a bijection from $\mathcal{F}(Z)$ onto $\mathcal{L}(Z)$, and its inverse mapping is $\phi \to {}^*\phi$, where $\phi \in \mathcal{L}(Z)$ and ${}^*\phi(z) = \sup\{r \mid z \in \phi(r)\}$.

Now let us recall the proof of the previous theorem. Then we can say that the following diagram is "almost" commutative.



Strictly speaking, since $H \circ f^* \notin \mathcal{L}(Y)$, the diagram is not commutative. But, if H is continuous from below, in this case μ is also continuous from below, then it is commutative, that is, the equation $h(f) = {}^*(H \circ f^*)$ holds.

For c-measurable functions, the similar argument applies.

The next result is concerning an interpreter for measurable function associated with a universal frame for representation.

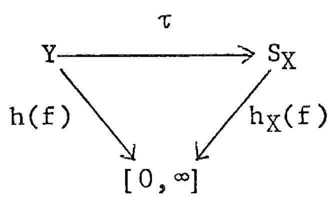
4.3.6 PROPOSITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space, $(Y, \mathcal{Y}, H)$ a frame of $(X, \mathcal{X})$, and $(S_X, \mathcal{S}_X, H_X)$ the universal frame of $(X, \mathcal{X})$. Let h and h_X be the interpreters induced by H and H_X, respectively. Let τ be a mapping from Y into S_X defined in the proof of Theorem 3.2.9:

$$\tau(y) = \{A \in \mathcal{X} \mid y \in H(A)\} \quad \forall\, y \in Y.$$

Then, for every measurable function f on X,

$$h(f)(y) = h_X(f)(\tau(y)),$$

that is, the following diagram commutes:



Proof. Trivial.

The next proposition shows expressions of h(f) for a simple function f. In 4.2.5, we pointed out that every simple function f can be written as

$$f = \sum_{i=1}^{n} (a_i - a_{i-1})\, 1_{A_i}, \tag{3.1}$$

where $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$. It is easy to show that a simple function f written as (3.1) is rewritten as

$$f = \bigvee_{i=1}^{n} a_i\, 1_{A_i}. \tag{3.2}$$

4.3.7 PROPOSITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space, $(Y, \mathcal{Y}, H)$ a frame of $(X, \mathcal{X})$ for representation, and h the interpreter for measurable functions induced by H. If f is a simple function defined as (3.1), then

$$h(f) = \sum_{i=1}^{n} (a_i - a_{i-1})\, 1_{H(A_i)}$$

$$= \bigvee_{i=1}^{n} a_i\, 1_{H(A_i)}.$$

Proof. Trivial.

An interpreter h has another expression.

4.3.8 PROPOSITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, $(Y, \mathcal{Y}, m, H)$ a representation of μ, and h the interpreter for measurable functions induced by H. For any measurable function f,

$$h(f)(y) = \sup_{A: y \in H(A)} \inf_{x \in A} f(x) \qquad \forall\, y \in Y.$$

Proof. Let y be an arbitrary element of Y and $a = \sup_{A: y \in H(A)} \inf_{x \in A} f(x)$. We may assume that f is o-measurable; the same proof applies to the case that f is c-measurable. Suppose that $a > s$. Then there exists $A \in \mathcal{X}$ such that $y \in H(A)$ and $\inf_{x \in A} f(x) > s$. Since $f(x) > s \quad \forall\, x \in A$, it follows that $y \in H(A) \subset H(\{f > s\})$, and hence that $h(f)(y) \geq s$. Therefore $h(f)(y) \geq a$. Now suppose that $h(f)(y) > t$ and let $B = \{f > t\}$. Then by the definition of h we have that $y \in H(B)$. Therefore it follows that

$$t \leq \inf_{x \in B} f(x) \leq a,$$

and hence that $a \geq h(f)(y)$. The proof is complete.

The above proposition will be used in Section 4.6.

§ 4.4 AN INTERPRETATION OF CHOQUET'S INTEGRAL

We again consider the examples "A WORKSHOP" (Example 3.3.1) and "A RARE BOOK" (Example 3.3.2). The two examples show that Choquet's integral is meaningful.

4.4.1 <u>EXAMPLE</u>. "A WORKSHOP" (continued from 3.3.1). Let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of the workers. Suppose that each worker x_i works $f(x_i)$ hours a day from the opening hour. Without loss of generality, we can assume that $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$.

Let us aggregate the working hours of all the workers in the following way. First the group X with n workers works $f(x_1)$ hours, next the group $X - \{x_1\} = \{x_2, x_3, \cdots, x_n\}$ works $f(x_2) - f(x_1)$ hours, then the group $X - \{x_1, x_2\} = \{x_3, x_4, \cdots, x_n\}$ works $f(x_3) - f(x_2)$ hours, $\cdots$, lastly one worker x_n works $f(x_n) - f(x_{n-1})$ hours. Therefore, since a group $A \subset X$ produces the amount $\mu(A)$ in one hour, the total number of the products produced by them is expressed by

$$\begin{aligned} & f(x_1)\mu(X) + [f(x_2) - f(x_1)]\mu(X - \{x_1\}) \\ & \quad + [f(x_3) - f(x_2)]\mu(X - \{x_1, x_2\}) \\ & \quad + \cdots \\ & \quad + [f(x_n) - f(x_{n-1})]\mu(\{x_n\}), \\ & = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\mu(\{x_i, x_{i+1}, \cdots, x_n\}), \end{aligned}$$

where $f(x_0) = 0$. This amount is nothing but Choquet's integral of f with respect to μ (see 4.2.5, an expression of Choquet's integral of simple functions).

4.4.2 EXAMPLE. "A RARE BOOK" (continued from 3.3.2). A certain person sells $f(x_1)$ first volumes and $f(x_2)$ second volumes to the secondhand bookseller. We may assume that $f(x_1) \leq f(x_2)$. Then, since he sells $f(x_1)$ complete sets and $[f(x_2) - f(x_1)]$ second volumes, and since $\mu(A)$ is the price of A, he gets

$$f(x_1)\mu(\{x_1, x_2\}) + [f(x_2) - f(x_1)]\mu(\{x_2\})$$

dollars. This is also Choquet's integral.

Choquet's integral reflects the interaction of subsets. Let us compare Choquet's integral of

$$f = \sum_{i=1}^{n} (a_i - a_{i-1}) 1_{A_i},$$

where $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$,

$$(C)\int f \, d\mu = \sum_{i=1}^{n} (a_i - a_{i-1}) \mu(A_i), \tag{4.1}$$

with the quantity

$$\sum_{i=1}^{n} a_i \mu(A_i - A_{i+1}), \tag{4.2}$$

where $A_{n+1} = \emptyset$. The quantity (4.2) is the classical integral which is based on another expression of f in the form:

$$f = \sum_{i=1}^{n} a_i 1_{B_i},$$

where $B_i = A_i - A_{i+1}$, $A_{n+1} = \emptyset$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Let us recall that a fuzzy measure reflects the interaction among subsets (see Section 3.3). Considering that B_i's are pairwise disjoint, this merit of μ is not utilized at the quantity (4.2). On the contrary, A_i's used in (4.1) are not disjoint: they form a monotone sequence $A_1 \supset A_2 \supset \cdots \supset A_n$. (If there are no interactions among the subsets, that is, μ is an additive measure, then the integral (4.1) coincides with the quantity (4.2). See 4.2.4 (3).)

We can interpret Choquet's integral (4.1) according to Remark 3.3.4. The integral (4.1) is rewritten as

$$(C)\int f \, d\mu = \sum_{i=1}^{n} a_i[\mu(A_i) - \mu(A_{i+1})], \qquad (4.3)$$

where $A_{n+1} = \emptyset$. Since A_i is the disjoint union of A_{i+1} and B_i, $\mu(A_i) - \mu(A_{i+1})$ means the effect of B_i joining to A_{i+1} (see 3.3.4). Then the integral (4.3) is the sum of all a_i unit's effect of B_i joining to A_{i+1}. We can consider that the expression (4.3) is based on the of f in the form

$$f = \sum_{i=1}^{n} a_i(1_{A_i} - 1_{A_{i+1}}).$$

Next we consider an interpreter h for measurable functions. Let us recall the example "A WORKSHOP" (Example 3.3.1 and Example 4.4.1). Let $X = \{x_1, x_2\}$ again and let $f(x_1) = a_1$, $f(x_2) = a_2$, and $a_1 \leq a_2$. Then, since $Y = \{0, 1, 2, 3\}$ (see 3.3.1), we have

$$\begin{cases} h(f)(0) = a_2, \\ h(f)(1) = a_1, \\ h(f)(2) = a_2, \\ h(f)(3) = a_1, \end{cases} \qquad (4.4)$$

and

$$\begin{aligned} \int h(f) \, dm &= a_1 \, m(\{1, 3\}) + a_2 \, m(\{0, 2\}) \\ &= a_1 \, m(Y) + [a_2 - a_1] \, m(\{0, 2\}) \\ &= (C)\int f \, d\mu. \end{aligned}$$

Considering that

$0 \in Y$ is the feature "the imcompatible operations between x_1 and x_2",

$1 \in Y$ is the feature "the compatible operations of x_1",

$2 \in Y$ is the feature "the compatible operations of x_2",

$3 \in Y$ is the feature "the effective cooperation between x_1 and x_2",

Eq. (4.4) is convincing.

In the case of the example "A RARE BOOK" (Example 3.3.2 and Example 4.4.2), since $f(x_1) \leq f(x_2)$, we have

$$\begin{cases} h(f)(y_1) = f(x_1), \\ h(f)(y_2) = f(x_2), \\ h(f)(y_3) = f(x_1), \end{cases}$$

and

$$\begin{aligned} \int h(f)\ dm &= f(x_1)\ m(\{y_1,\ y_3\}) + f(x_2)\ m(\{y_2\}) \\ &= f(x_1)\ m(Y) + [f(x_2) - f(x_1)]\ m(\{y_2\}) \\ &= (C)\int f\ d\mu. \end{aligned}$$

Therefore, considering the interpretation of Y and m (see 3.3.2), the values of h(f) are also convincing.

Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, $(Y, \mathcal{Y}, m, H)$ a representation of μ, and f an o-measurable function on X. (The same argument applies if f is c-measurable.) Let y_0 be an arbitrary element of Y. Since $h(f)(y_0) = \sup\ \{r | y_0 \in H(\{f > r\})\}$, we have

$$y_0 \in H(\{f > r\}) \qquad \forall\ r < h(f)(y_0)$$

and

$$y_0 \notin H(\{f > r\}) \qquad \forall\ r > h(f)(y_0).$$

That is, $\{f > r\}$ has the feature y_0 for $r < h(f)(y_0)$, and $\{f > r\}$ does not have the feature y_0 for $r > h(f)(y_0)$. Hence the value $h(f)(y_0)$ is just the value the function f brings on the feature y_0.

§ 4.5 DERIVATION OF CHOQUET'S INTEGRAL

In Section 4.2 we defined Choquet's integral and then in Section 4.3 we considered the relation between Choquet's integral and the representation of fuzzy measures. By contrast, in this section we derive Choquet's integral from the representation. We deal only with a finite set X and the power set 2^X of X as a fuzzy measurable space.

First we define the notation used here as follows: let Z be a non-empty finite set,

M(Z) : the set of all the ordinary measures on 2^Z,

FM(Z) : the set of all the fuzzy measures on 2^Z,

F(Z) : the set of all the nonnegative functions on Z.

Let X be a non-empty finite set. We consider a mapping $I : F(X) \times FM(X) \to [0,\infty]$ and deduce that

$$I(f,\mu) = (C)\int f\, d\mu \quad \forall\, f \in F(X) \quad \forall\, \mu \in FM(X)$$

from certain conditions.

The following condition (C0) means thet $I(f,\mu)$ is expressed by a representation of μ.

(C0) For every frame $(Y, 2^Y, H)$ for which Y is finite, there exists a mapping $\eta : F(X) \to F(Y)$ such that

$$I(f, m\circ H) = \int \eta(f)\, dm \qquad \forall\, f \in F(X) \quad \forall\, m \in M(Y).$$

In order that $I(f,\mu)$ is regarded as an extended Lebesgue integral of f with respect to μ, the following three conditions are necessary.

(C1) For a given $f = a1_A \in F(X)$, and for given $\mu_1, \mu_2 \in FM(X)$, if $\mu_1(A) = \mu_2(A)$, then $I(f,\mu_1) = I(f,\mu_2)$.

(C2) For a given $\mu \in FM(X)$, if μ is an additive measure, then

$$I(f,\mu) = \int f \, d\mu \qquad \forall \; f \in F(X).$$

(C3) For given f, g $\in F(X)$, if $f(x) \leq g(x) \quad \forall \; x \in X$, then

$$I(f,\mu) \leq I(g,\mu).$$

(C1) means that the value $I(a1_A, \mu)$ depends on $\mu(A)$, and not on $\{\mu(B) | B \neq A\}$. (C2) means that the mapping I is an extention of Lebesgue integral. (C3) means the monotonicity of the mapping I.

4.5.1 THEOREM. The mapping $I : F(X) \times FM(X) \to [0,\infty]$ satisfies the conditions (C0) - (C3) iff

$$I(f,\mu) = (C)\int f \, d\mu \qquad \forall \; f \in F(X) \quad \forall \; \mu \in FM(X).$$

In the rest of this section we prove the above theorem. It is obvious that, if

$$I(f,\mu) = (C)\int f \, d\mu \qquad \forall \; f \in F(X) \quad \forall \; \mu \in FM(X),$$

then the mapping I satisfies (C0), (C1), (C2), and (C3); we have shown in Section 4.3 that Choquet's integral satisfies these conditions. We shall prove the converse.

Corollary 4.2.10 implies that, for a given finite set X, there exist a frame $(Y, 2^Y, H)$ of $(X, 2^X)$ such that Y is a finite set and $(Y, 2^Y, H)$ has the following property (U).

(U) For every $\mu \in FM(X)$, there is $m \in M(Y)$ such that $\mu = m \circ H$.

Note that, if X is finite, then S_X, the set of all semi-filters in 2^X, is also finite and ${}_X = 2^{S_X}$; $(S_X, 2^{S_X}, H_X)$ satisfies the above condition (see Section 4.2). That is, we can use the above property (U).

(C0) implies that there exists a mapping $\eta : F(X) \to F(Y)$

$$I(f,\ m \circ H) = \int \eta(f)\ dm \quad \forall\ f\ \varepsilon\ F(X)\ \ \forall\ m\ \varepsilon\ M(Y).$$

By the property (U), in order to show that

$$I(f,\mu) = (C)\int f\ d\mu \quad \forall\ f\ \varepsilon\ F(X)\ \ \forall\ \mu\ \varepsilon\ FM(X),$$

it is sufficient to show that

$$\int \eta(f)\ dm = (C)\int f\ d(m \circ H) \quad \forall\ f\ \varepsilon\ F(X)\ \ \forall\ m\ \varepsilon\ M(Y).$$

The conditions (C1), (C2), and (C3) can be rewritten as (D1), (D2), and (D3), respectively:

(D1) For a given $f = a1_A\ \varepsilon\ F(X)$ and for given $m_1, m_2\ \varepsilon\ M(Y)$, if $m_1(A) = m_2(A)$, then

$$\int \eta(f)\ dm_1 = \int \eta(f)\ dm_2.$$

(D2) For a given $\hat{m}\ \varepsilon\ M(Y)$, if $\hat{m} \circ H$ is an additive measure on 2^X, then

$$\int \eta(f)\ d\hat{m} = \int f\ d(\hat{m} \circ H) \qquad \forall\ f\ \varepsilon\ F(X).$$

(D3) For a given $f, g\ \varepsilon\ F(X)$, if $f(x) \leq g(x)\ \ \forall\ x\ \varepsilon\ X$, then

$$\int \eta(f)\ dm \leq \int \eta(g)\ dm \qquad \forall\ m\ \varepsilon\ M(Y).$$

By Theorem 4.3.4, in order to prove this theorem, it is sufficient to deduce from (D1) – (D3) that η is the interpreter induced by H. That is, by Propositon 4.3.7, it is sufficient to prove that

$$\eta(\bigvee_{i=1}^{n} a_i 1_{A_i}) = \bigvee_{i=1}^{n} a_i 1_{H(A_i)},$$

where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$. We do it by proving a sequence of lemmas.

The most important tool we use here is a Dirac measure. Let Z be a non-empty finite set and let z_0 be an arbitrary element of Z. The

Dirac measure δ_{z_0} focused on z_0 is an ordinary measure on 2^Z defined by, for every $A \varepsilon 2^Z$,

$$\delta_{z_0}(A) = \begin{cases} 1 & z_0 \varepsilon A, \\ 0 & z_0 \not\varepsilon A. \end{cases}$$

δ_{z_0} has the following important property:

$$\int f \, d\delta_{z_0} = f(z_0) \qquad \forall f \varepsilon F(Z).$$

4.5.2 LEMMA. Let us assume (D2). If $f = \bigvee_{i=1}^{n} a_i 1_{A_i} \varepsilon F(X)$, where $a_1 \leq a_2 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$, if $\hat{m} \varepsilon M(Y)$, and if $\hat{m} \circ H$ is an additive measure on 2^X, then

$$\int \eta(f) \, d\hat{m} = \int \bigvee_{i=1}^{n} a_i 1_{H(A_i)} \, d\hat{m}.$$

Proof. By an easy calculation, we obtain

$$\int \bigvee_{i=1}^{n} a_i 1_{H(A_i)} \, d\hat{m} = \sum_{i=1}^{n} (a_i - a_{i-1}) \, \hat{m}(H(A_i)),$$

and

$$\int f \, d(\hat{m} \circ H) = \sum_{i=1}^{n} (a_i - a_{i-1}) \, \hat{m}(H(A_i)),$$

where $a_0 = 0$. It follows from (D2) that

$$\int \eta(f) \, d\hat{m} = \int f \, d(\hat{m} \circ H).$$

The proof is complete.

4.5.3 LEMMA. For every $y_0 \varepsilon Y$ and for every $A_0 \varepsilon 2^X$, there exists $\hat{m} \varepsilon M(Y)$ such that $\hat{m} \circ H$ is an additive measure on 2^X and

$$\hat{m}(H(A_0)) = \delta_{y_0}(H(A_0)),$$

where δ_{y_0} is the Dirac measure focused on y_0.

Proof. Let $y_0 \in Y$ and let $A_0 \in 2^X$. If $y_0 \in H(A_0)$, let $x_0 \in A_0$, and if $y_0 \notin H(A_0)$, let $x_0 \in X - A_0$. Let δ_{x_0} be the Dirac measure focused on x_0. By the property (U) there exists $\hat{m} \in M(Y)$ such that $\delta_{x_0} = \hat{m} \circ H$. Then we have

$$\hat{m}(H(A_0)) = \delta_{x_0}(A_0) = \delta_{y_0}(H(A_0)).$$

Since $\delta_{x_0} = \hat{m} \circ H$ is an additive measure, the proof is complete.

4.5.4 LEMMA. Let us assume (D1) and (D2). If $f = a1_A \in F(X)$, then

$$\eta(f) = a1_{H(A)}.$$

Proof. Let y_0 be an arbitrary element of Y. By the previous lemma, there is $\hat{m} \in M(Y)$ such that $\hat{m} \circ H$ is an additive measure on 2^X and

$$\hat{m}(H(A)) = \delta_{y_0}(H(A)). \tag{5.1}$$

Then it follows from (D1) that

$$\int \eta(f)\, d\delta_{y_0} = \int \eta(f)\, d\hat{m}. \tag{5.2}$$

By 4.5.2 we have

$$\int \eta(f)\, d\hat{m} = \int a1_{H(A)}\, d\hat{m}. \tag{5.3}$$

Eq.(5.1) implies that

$$\int a1_{H(A)}\, d\hat{m} = \int a1_{H(A)}\, d\delta_{y_0} \tag{5.4}$$

By (5.2), (5.3), and (5.4) we obtain

$$\int \eta(f)\, d\delta_{y_0} = \int a1_{H(A)}\, d\delta_{y_0},$$

hence

$$\eta(f)(y_0) = a1_{H(A)}(y_0).$$

4.5.5 LEMMA. The condition (D3) is equivalent to the following condition:

$$\text{if } f(x) \leq g(x) \quad \forall\, x \in X, \text{ then}$$

$$\eta(f)(y) \leq \eta(g)(y) \quad \forall\, y \in Y.$$

Proof. If $y_0 \in Y$ and if δ_{y_0} is the Dirac measure focused on y_0, then

$$\int \eta(f)\, d\delta_{y_0} = \eta(f)(y_0).$$

Therefore the condition

$$\int \eta(f)\, dm \leq \int \eta(g)\, dm \quad \forall\, m \in M(Y)$$

is equivalent to the condition

$$\eta(f)(y) \leq \eta(g)(y) \quad \forall\, y \in Y.$$

Thus the lemma follows.

In the rest of this section we assume (D1) - (D3).

4.5.6 LEMMA. If $f = \bigvee_{i=1}^{n} a_i 1_{A_i} \in F(X)$, where $a_1 \leq a_2 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$, then $\eta(f) \geq \bigvee_{i=1}^{n} a_i 1_{H(A_i)}$.

Proof. Since $f \geq a_i 1_{A_i}$ $i = 1, 2, \cdots, n$, it follows from Lemma 4.5.4 and Lemma 4.5.5 that

$$\eta(f) \geq \eta(a_i 1_{A_i}) = a_i 1_{H(A_i)} \quad \text{for } i = 1, 2, \cdots, n.$$

Thus

$$\eta(f) \geq \bigvee_{i=1}^{n} a_i 1_{H(A_i)}.$$

For any two set functions ν_1 and ν_2 with the same domain, we define $\nu_1 + \nu_2$ and $\nu_1 - \nu_2$ by

$$(\nu_1 + \nu_2)(A) = \nu_1(A) + \nu_2(A)$$

and

$$(\nu_1 - \nu_2)(A) = \nu_1(A) - \nu_2(A)$$

in an ordinary way.

4.5.7 LEMMA. For every $y_0 \varepsilon Y$, there exists $m \varepsilon M(Y)$ such that $(m + \delta_{y_0})\circ H$ is an additive measure on 2^X, where δ_{y_0} is the Dirac measure focused on y_0.

Proof. Let μ_c be the counting measure on 2^X, that is, $\mu_c(A)$ is the number of the elements of the subset A. Note that μ_c is an additive measure on 2^X. Let $y_0 \varepsilon Y$, δ_{y_0} be the Dirac measure focused on y_0, and $\mu = \mu_c - (\delta_{y_0}\circ H)$.

μ is a fuzzy measure on 2^X. Since

$$\mu(A) = \begin{cases} \mu_c(A) - 1 & y_0 \varepsilon H(A), \\ \mu_c(A) & \text{otherwise}, \end{cases}$$

we have

$$\mu_c(A) - 1 \leq \mu(A) \leq \mu_c(A) \quad \forall A \varepsilon 2^X.$$

Since μ_c is the counting measure,

$$\mu_c(A) \leq \mu_c(B) - 1 \quad \text{whenever} \quad A \subsetneq B.$$

Therefore

$$\mu(A) \leq \mu(B) \quad \text{whenever} \quad A \subsetneq B.$$

In addition, $\mu(\emptyset) = \mu_c(\emptyset) - \delta_{y_0}(H(\emptyset)) = 0 - 0 = 0.$

By the property (U), there is $m \varepsilon M(Y)$ such that $\mu = m\circ H$.

Since

$$\begin{aligned} \mu_c &= \mu + (\delta_{y_0}\circ H) \\ &=(m\circ H) + (\delta_{y_0}\circ H) \\ &=(m + \delta_{y_0})\circ H, \end{aligned}$$

the proof is complete.

Proof of Theorem 4.5.1. It is sufficient to prove that, if

$f = \bigvee_{i=1}^{n} a_i 1_{A_i} \in F(X)$, where $a_1 \leq a_2 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$,

then

$$\eta(f) = \bigvee_{i=1}^{n} a_i 1_{H(A_i)}.$$

Let y_0 be an arbitrary element of Y and let δ_{y_0} be the Dirac measure focused on y_0. By the previous lemma, there is $m \in M(Y)$ such that $(m + \delta_{y_0})\circ H$ is an additive measure on 2^X. Then it follows from Lemma 4.5.2 that

$$\int \eta(f)\, d(m + \delta_{y_0}) = \int \bigvee_{i=1}^{n} a_i 1_{H(A_i)}\, d(m + \delta_{y_0}).$$

Hence

$$\int \eta(f)\, dm + \int \eta(f)\, d\delta_{y_0}$$

$$= \int \bigvee_{i=1}^{n} a_i 1_{H(A_i)}\, dm + \int \bigvee_{i=1}^{n} a_i 1_{H(A_i)}\, d\delta_{y_0}. \qquad (5.5)$$

On the other hand, it follows from Lemma 4.5.6 that

$$\int \eta(f)\, dm \geq \int \bigvee_{i=1}^{n} a_i 1_{H(A_i)}\, dm \qquad (5.6)$$

and

$$\int \eta(f)\, d\delta_{y_0} \geq \int \bigvee_{i=1}^{n} a_i 1_{H(A_i)}\, d\delta_{y_0}. \qquad (5.7)$$

By (5.5), (5.6), and (5.7) we obtain

$$\int \eta(f)\, d\delta_{y_0} = \int \bigvee_{i=1}^{n} a_i 1_{H(A_i)}\, d\delta_{y_0}.$$

Therefore

$$\eta(f)(y_0) = \bigvee_{i=1}^{n} a_i 1_{H(A_i)}(y_0).$$

§ 4.6 CONFORMABILITY

4.6.1 DEFINITION. Let f and g be fuctions defined on a set X. f is said to be conformable to g if

$$f(x_1) < f(x_2) \implies g(x_1) \leq g(x_2) \quad \forall\, x_1, x_2 \in X.$$

We write $f \sim g$ when f is conformable to g.

Obviously the relation $\sim$ is reflexive, symmetric, and not transitive.

4.6.2 PROPOSITION. Let ϕ be a nondecreasing function defined on the real line. Then $f \sim \phi \circ f$ for every function f.

Proof. Trivial.

4.6.3 COROLLARY. Let f be a function and a be a real number.

(1) $f \sim a$.

(2) $f \sim af$ for $a \geq 0$.

(3) $f \sim (f + a)$.

4.6.4 PROPOSITION. Let f, g, and h be functions with the same domain. Let $S(\cdot, \cdot)$ be a two-place function which is nondecreasing in each place. If $f \sim g$ and $f \sim h$, then $f \sim S(g, h)$.

Proof. Trivial.

4.6.5 COROLLARY. Let f, g, and h be functions with the same domain. If $f \sim g$ and $f \sim h$, then $f \sim (g + h)$, $f \sim (g \vee h)$, and $f \sim (g \wedge h)$. Moreover if g and h are nonnegative, then $f \sim (gh)$.

4.6.6 LEMMA. Let f and g be nonnegative functions with the same domain. The following conditions (1) - (3) are equivalent.

(1) $f \sim g$.

(2) For $s, t \in [0, \infty)$

$$\{x \mid f(x) > s\} \subset \{x \mid g(x) > t\} \text{ or } \{x \mid g(x) > t\} \subset \{x \mid f(x) > s\}.$$

(3) For $s, t \in [0, \infty)$

$$\{x \mid f(x) \geq s\} \subset \{x \mid g(x) \geq t\} \text{ or } \{x \mid g(x) \geq t\} \subset \{x \mid f(x) \geq s\}.$$

Proof. (1) $\Rightarrow$ (2). Let s and t be arbitrary nonnegative real numbers. Let us assume that $\{f > s\} \not\subset \{g > t\}$. Then there exists $x_0 \in \{f > s\}$ such that $x_0 \notin \{g > t\}$. Suppose that $x \in \{g > t\}$. Since $g(x_0) < g(x)$ and $f \sim g$, it follows that $f(x_0) \leq f(x)$, and hence that $x \in \{f > s\}$. Therefore $\{g > t\} \subset \{f > s\}$.

(2) $\Rightarrow$ (1). Let us assume that there exist $x, x' \in X$ such that $f(x) < f(x')$ and $g(x') < g(x)$. Then there exist real numbers s and t such that $f(x) < s < f(x')$ and $g(x') < t < g(x)$. Since $x \in \{g > t\}$ and $x \notin \{f > s\}$, it follows that $\{f > s\} \not\subset \{g > t\}$. Similarly it follows that $\{f > s\} \not\supset \{g > t\}$. This is a contradiction.

Similarly (1) is equivalent to (3).

4.6.7 THEOREM. Let f and g be measurable functions on a fuzzy measurable space, and $S : [0,\infty] \times [0,\infty] \to [0,\infty]$ be a continuous two-place function which is nondecreasing in each place. If f and g are both o-measurable [resp. c-measurable], and if $f \sim g$, then $S(f, g)$ is o-measurable [resp. c-measurable].

Proof. We prove only the case that f and g are both o-measurable. The proof of the other case is similar. For convenience we denote the sets $\{f > s\}$, $\{g > t\}$, and $\{S(f, g) > r\}$ by F_s, G_t, and H_r, respectively.

Let r be an arbitrary nonnegative real number. Let

$$s = \inf \{\alpha | F_\alpha \subset H_r\},$$

$$t = \inf \{\beta | G_\beta \subset H_r\}.$$

By the previous lemma, we can assume that $G_t \subset F_s$ without loss of generality. Since F_s is a measurable set, in order to prove that S(f, g) is o-measurable, it is sufficient to prove that $H_r = F_s$. Since $F_s = \bigcup_{\alpha > s} F_\alpha$ and $F_\alpha \subset H_r$ for $\alpha > s$, it follows that $F_s \subset H_r$.

Suppose that $\alpha < s$ and $\beta < t$. Then $F_\alpha \supsetneq H_r$ and $G_\beta \supsetneq H_r$. Without loss of generality, we can assume $F_\alpha \subset G_\beta$. Then there exists $x_0 \in F_\alpha - H_r$ and we have $S(\alpha, \beta) \leq S(f(x_0), g(x_0)) \leq r$. It follows from the continuity of S that $S(s, t) \leq r$.

Suppose that $x \notin F_s$. Then, by the assumption $G_t \subset F_s$, we have $f(x) \leq s$ and $g(x) \leq t$. It follows that

$$S(f(x), g(x)) \leq S(s, t) \leq r,$$

and hence that $x \notin H_r$. This implies that $H_r \subset F_s$, and hence we have $H_r = F_s$. The proof is complete.

4.6.8 COROLLARY. Let f and g be measurable functions on a fuzzy measurable space. If f and g are both o-measurable [resp. c-measurable], and if $f \sim g$, then $f + g$, fg, $f \vee g$, and $f \wedge g$ are o-measurable [resp. c-measurable].

4.6.9 LEMMA. Let f and g be functions defined on a set X. If $f \sim g$, then, for every $A \subset X$,

$$\inf_{x \in A} [f(x) + g(x)] = \inf_{x \in A} f(x) + \inf_{x \in A} g(x)$$

and

$$\sup_{x \varepsilon A} [f(x) + g(x)] = \sup_{x \varepsilon A} f(x) + \sup_{x \varepsilon A} g(x).$$

Proof. Let $s = \inf_{x \varepsilon A} f(x)$ and $t = \inf_{x \varepsilon A} g(x)$. Obviously $\inf_{x \varepsilon A} [f(x) + g(x)] \geq s + t$. For an arbitrary positive number ε, if we write $B = \{x \varepsilon A | f(x) < s + \varepsilon/2\}$ and $C = \{x \varepsilon A | g(x) < t + \varepsilon/2\}$, then, since $f \sim g$, it follows that $B \cap C \neq \emptyset$, and hence that $\inf_{x \varepsilon A} [f(x) + g(x)] \leq s + t + \varepsilon$. Therefore $\inf_{x \varepsilon A} [f(x) + g(x)] = s + t$. Similarly the other equality follows.

4.6.10 THEOREM. Let $(X, \mathcal{X})$ be a fuzzy measure space, $(Y, \mathcal{Y}, H)$ a frame of $(X, \mathcal{X})$ for representation, and h the interpreter for measurable functions induced by H. If $f \sim g$, then

$$h(f + g) = h(f) + h(g).$$

Proof. We use Lemma 4.3.8:

$$h(f)(y) = \sup_{A: y \varepsilon H(A)} \inf_{x \varepsilon A} f(x).$$

For each $A \varepsilon \mathcal{X}$, let $F(A) = \inf_{x \varepsilon A} f(x)$ and $G(A) = \inf_{x \varepsilon A} g(x)$. The fact that $f \sim g$ implies that $F \sim G$, i.e.,

$$G(A) \leq G(B) \quad \text{whenever} \quad F(A) < F(B).$$

By the previous lemma we obtain that for every $y \varepsilon Y$

$$h(f + g)(y) = \sup_{A: y \varepsilon H(A)} \inf_{x \varepsilon A} [f(x) + g(x)]$$

$$= \sup_{A: y \varepsilon H(A)} [\inf_{x \varepsilon A} f(x) + \inf_{x \varepsilon A} g(x)]$$

$$= \sup_{A: y \varepsilon H(A)} [F(A) + G(A)]$$

$$= \sup_{A: y \in H(A)} F(A) + \sup_{A: y \in H(A)} G(A)$$

$$= h(f)(y) + h(g)(y).$$

4.6.11 <u>THEOREM</u>. Let $(X, \mathcal{X})$ be a fuzzy measurable space, $(S_X, \mathcal{S}_X, H_X)$ a universal frame of $(X, \mathcal{X})$ for representation, and h_X the interpreter for measurable functions induced by H_X. For given measurable functions f and g, if they are both o-measurable or both c-measurable, then the following conditions are equivalent to each other.

(1) $f \sim g$.

(2) $h_X(f + g) = h_X(f) + h_X(g)$.

(3) For every fuzzy measure μ on $(X, \mathcal{X})$,

$$(C)\int (f + g)\, d\mu = (C)\int f\, d\mu + (C)\int g\, d\mu.$$

<u>Proof</u>. (1) $\Rightarrow$ (2) follows from 4.6.10. (2) $\Rightarrow$ (3) follows from 3.2.10 and 4.3.4. Hence it is sufficient to prove that (3) $\Rightarrow$ (1). Suppose that $f \not\sim g$. Then there exist two points x_1 and x_2 in X such that $f(x_1) < f(x_2)$ and $g(x_1) > g(x_2)$. We define a set function μ on $\mathcal{X}$ by

$$\mu(A) = \begin{cases} 1 & \text{if } x_1 \in A \text{ and } x_2 \in A \\ 0 & \text{otherwise} \end{cases} \qquad \forall A \in \mathcal{X}.$$

Then μ is a fuzzy measure. By an easy calculation we obtain that

$$(C)\int (f + g)\, d\mu = \min\,[f(x_1) + g(x_1),\ f(x_2) + f(x_2)],$$

$$(C)\int f\, d\mu = f(x_1),$$

$$(C)\int g\, d\mu = g(x_2),$$

and hence that

$$(C)\int (f + g)\, d\mu > (C)\int f\, d\mu + (C)\int g\, d\mu.$$

The proof is complete.

§ 4.7 NULL SETS

The concept of "almost everywhere" is one of the most important concepts in the ordinary measure theory. It is defined by the concept of "null set." A null set is defined as a measurable set of measure zero.

But this definition of null sets is unsuited for fuzzy measures. If μ is a fuzzy measure, then it is possible that there exist measurable sets A and B such that $\mu(A) = \mu(B) = 0$ but $\mu(A \cup B) > 0$. Are these sets suited to be called null sets? Furthermore the following example shows that there exist measurable functions f and g such that f = g a.e. in the sense of ordinary null sets but $(C)\int f\, d\mu \neq (C)\int g\, d\mu$.

4.7.1 EXAMPLE. Let $X = \{x_1, x_2\}$ and μ a fuzzy measure on 2^X defined as $\mu(\{x_1\}) = 0$, $\mu(\{x_2\}) = 0$, and $\mu(X) = 1$. Let $f(x_1) = 0$, $f(x_2) = 1$, $g(x_1) = 2$, and $g(x_2) = 1$. Then f = g a.e. in the ordinary sense, i.e., $\mu(\{f \neq g\}) = \mu(\{x_1\}) = 0$. But we have $(C)\int f\, d\mu = 0$ and $(C)\int g\, d\mu = 1$.

We should generalize the definition of null sets.

4.7.2 DEFINITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space. A subset N of X is called a null set if, for every pair (A, B) of measurable sets,

$$A \subset B \cup N \quad \text{implies} \quad \mu(A) \leq \mu(B).$$

A proposition P(x) concerning the points of X is said to be true almost everywhere (or a.e. for short) if there exsists a null set N such that $x \in X - N$ implies that P(x) is true.

Note that a null set is not always measurable.

4.7.3 PROPOSITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space.

(1) The empty set is a null set.

(2) If $N \in \mathcal{X}$ is a null set, then $\mu(N) = 0$.

(3) A subset of a null set is a null set.

The following proposition makes the definition of null sets comprehensible.

4.7.4 PROPOSITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, $\mathcal{X}$ be closed under $(\cup f)$, and $N \in \mathcal{X}$. Then the following conditions (1) and (2) are equivalent.

(1) N is a null set.

(2) $\mu(A \cup N) = \mu(A)$ $\forall A \in \mathcal{X}$.

Proof. (1) $\Rightarrow$ (2). Since $(A \cup N) \subset A \cup N$, by the definition of null sets we have $\mu(A \cup N) \leq \mu(A)$. Hence it follows from the monotonicity of μ that $\mu(A \cup N) = \mu(A)$.

(2) $\Rightarrow$ (1). Let $A, B \in \mathcal{X}$. If $A \subset B \cup N$, then $\mu(A) \leq \mu(B \cup N) = \mu(B)$. Hence N is a null set.

4.7.5 COROLLARY. Let $(X, \mathcal{X}, \mu)$ be an ordinary measure space and $N \in \mathcal{X}$. Then N is a null set if and only if $\mu(N) = 0$.

4.7.6 COROLLARY. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space. If $\mathcal{X}$ is closed under $(\cup f)$, then a finite union of measurable null sets is a null set. Moreover, if $\mathcal{X}$ is closed under $(\cup c)$ and if μ is continuous from below, then a countable union of measurable null sets is a null set.

4.7.7 PROPOSITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and $N \subset X$. N is a null set if and only if, for every pair (A, B) of measurable sets

$$A \cap N^c \subset B \text{ implies } \mu(A) \leq \mu(B).$$

Proof. Since $A \subset B \cup N \iff A \cap N^c \subset B$, our assertion is immediate from 4.7.2.

Proposition 4.7.8 and Corollary 4.7.9 are dual of 4.7.4 and 4.7.6, respectively.

4.7.8 PROPOSITION. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, $\mathcal{X}$ be closed under ($\cap$f), and $N^c \in \mathcal{X}$. Then the following conditions (1) and (2) are equivalent.

(1) N is a null set.

(2) $\mu(A \cap N^c) = \mu(A) \quad \forall A \in \mathcal{X}$.

4.7.9 COROLLARY. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space.

(1) Let $\mathcal{X}$ be closed under ($\cap$f). If $\{N_i | i = 1, \cdots, n\}$ is a finite sequence of null sets such that $N_i^c \in \mathcal{X}$ $i = 1, \cdots, n$, then $\bigcup_{i=1}^{n} N_i$ is a null set.

(2) Let $\mathcal{X}$ be closed under ($\cap$c) and μ be continuous from above. If $\{N_n\}$ is a sequence of null sets such that $N_n^c \in \mathcal{X}$ $n = 1, 2, \cdots$, then $\bigcup_{n=1}^{\infty} N_n$ is a null set.

4.7.10 <u>LEMMA</u>. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, N a null set, and $\mathcal{X}' = \{A \cap N^c | A \in \mathcal{X}\}$. Then $(N^c, \mathcal{X}')$ is a fuzzy measurable space and there exists a fuzzy measure μ' on $\mathcal{X}'$ such that

$$\mu'(A \cap N^c) = \mu(A) \quad \forall A \in \mathcal{X}.$$

<u>Proof</u>. It is obvious that $(N^c, \mathcal{X}')$ is a fuzzy measurable space. It follows from 4.7.7 that

$$\mu(A) = \mu(B) \quad \text{whenever} \quad A \cap N^c = B \cap N^c.$$

Therefore we can define a set function μ' on $\mathcal{X}'$ by

$$\mu'(A \cap N^c) = \mu(A) \quad \forall A \in \mathcal{X},$$

and obviously μ' is a fuzzy measure.

4.7.11 <u>DEFINITION</u>. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, N a null set, and $\mathcal{X}' = \{A \cap N^c | A \in \mathcal{X}\}$. The fuzzy measure μ' on $(N^c, \mathcal{X}')$ defined by

$$\mu'(A \cap N^c) = \mu(A) \quad \forall A \in \mathcal{X}$$

is called the <u>restriction</u> of μ to $(N^c, \mathcal{X}')$ (or to N^c).

4.7.12 <u>LEMMA</u>. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space, N a null set, and μ' the restriction of μ to N^c. For any measurable function f,

$$(C)\int f \, d\mu = (C)\int f|N^c \, d\mu',$$

where $f|N^c$ is the restriction of f to N^c.

<u>Proof</u>. Let f be an o-measurable function. The same proof applies to

the case that f is c-measurable.

$$(C)\int f\, d\mu = \int_0^\infty \mu(\{f > r\})\, dr$$

$$= \int_0^\infty \mu'(\{f > r\} \cap N^c)\, dr$$

$$= \int_0^\infty \mu'(\{(f|N^c) > r\})\, dr$$

$$= (C)\int f|N^c\, d\mu'.$$

4.7.13 THEOREM. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space. If f and g are measurable functions such that f = g a.e., then

$$(C)\int f\, d\mu = (C)\int g\, d\mu.$$

Proof. Let N be a null set such that $f(x) = g(x)$ $\forall\, x \in N^c$. Let μ' be the restriction of μ to N^c. Then it follows from 4.7.12 that

$$(C)\int f\, d\mu = (C)\int f|N^c\, d\mu'$$

$$= (C)\int g|N^c\, d\mu'$$

$$= (C)\int g\, d\mu\ .$$

The next result follows from 4.6.11 and 4.7.12.

4.7.14 THEOREM. Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space. If f and g are measurable functions such that $f \sim g$ a.e. and $f + g$ is measurable, then

$$(C)\int (f + g)\, d\mu = (C)\int f\, d\mu + (C)\int g\, d\mu,$$

where $f \sim g$ a.e. means that there exists a null set N such that

$$g(x_1) \leq g(x_2) \quad \text{whenever} \quad f(x_1) < f(x_2) \quad \forall\, x_1, \forall\, x_2 \in N^c.$$

§ 4.8 EXPECTATION

4.8.1 DEFINITION. A fuzzy measure μ is said to be normalized if $\mu(X) = 1$. Choquet's integral of a measurable function f with respect to a normarized fuzzy measure is called the expectation of f.

4.8.2 DEFINITIONS (Tsukamoto [23]). Let $(X, \mathcal{X})$ be a fuzzy measurable space. Let $B \subset X$ and $B \neq \emptyset$. The 0-1 possibility measure focused on B, denoted by Π_B, is a set function on $\mathcal{X}$ defined by, for every $A \in \mathcal{X}$

$$\Pi_B(A) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The 0-1 necessity measure focused on B, denoted by N_B, is a set function on $\mathcal{X}$ defined by, for every $A \in \mathcal{X}$

$$N_B(A) = \begin{cases} 1 & \text{if } B \subset A, \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition is obvious.

4.8.3 PROPOSITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space and B a non-empty subset of X.

(1) Π_B and N_B are normalized fuzzy measures on $\mathcal{X}$.

(2) If $A \in \mathcal{X}$ and $A^c \in \mathcal{X}$, then $\Pi_B(A) + N_B(A^c) = 1$.

(3) If $\{A_\lambda | \lambda \in \Lambda\} \subset \mathcal{X}$ and $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathcal{X}$, then $\Pi_B(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigvee_{\lambda\in\Lambda} \Pi_B(A_\lambda)$.

(4) If $\{A_\lambda | \lambda \in \Lambda\} \subset \mathcal{X}$ and $\bigcap_{\lambda\in\Lambda} A_\lambda \in \mathcal{X}$, then $N_B(\bigcap_{\lambda\in\Lambda} A_\lambda) = \bigwedge_{\lambda\in\Lambda} N_B(A_\lambda)$.

(5) For every fuzzy measure μ on $\mathcal{X}$,

$$\mu \text{ is normalized} \iff N_X \leq \mu \leq \Pi_X.$$

4.8.4 PROPOSITION. Let $(X, \mathcal{X})$ be a fuzzy measurable space, B a non-empty subset of X, and f a measurable function.

(1) $(C)\int f \, d\Pi_B = \sup_{x \in B} f(x)$.

(2) $(C)\int f \, dN_B = \inf_{x \in B} f(x)$.

(3) For every normalized fuzzy measure μ,

$$\inf_{x \in X} f(x) \leq (C)\int f \, d\mu \leq \sup_{x \in X} f(x).$$

Proof. (1) Let $a = \sup_{x \in B} f(x)$. Then it follows that

$$(C)\int f \, d\Pi_B = \int_0^\infty \Pi_B(\{x | f(x) > r\}) \, dr = \int_0^a dr = a.$$

(2) Similar to (1).

(3) The assertion is immediate from (1), (2), and the fact that $N_X \leq \mu \leq \Pi_X$.

The assertion (3) shows that Choquet's integral with respect to a normalized fuzzy measure is suited to be called expectation.

A similar argument applies to a probability space.

4.8.5 DEFINITIONS. Let $(X, \mathcal{X}, P)$ be a probability space and B a non-null measurable set. The essential 0-1 possibility measure focused on B, denoted by $\text{ess}.\Pi_B$, is a set function on $\mathcal{X}$ defined by, for every $A \in \mathcal{X}$

$$\text{ess}.\Pi_B(A) = \begin{cases} 1 & \text{if } P(A \cap B) > 0, \\ 0 & \text{if } P(A \cap B) = 0. \end{cases}$$

The essential 0-1 necessity measure focused on B, denoted by $\text{ess}.N_B$, is a set function on $\mathcal{X}$ defined by, for every $A \in \mathcal{X}$

$$\text{ess}.N_B(A) = \begin{cases} 1 & \text{if } P(B - A) = 0, \\ 0 & \text{if } P(B - A) > 0. \end{cases}$$

Let μ be a fuzzy measure on $\mathcal{X}$, we write $\mu << P$ if $\mu(A) = 0$ for every measurable set A for which $P(A) = 0$.

Tsukamoto's G_0-measure and G_∞-measure [23] are the essential 0-1 possibility measure focused on X and the essential 0-1 necessity measure focused on X, respectively.

The following proposition is obvious.

4.8.6 PROPOSITION. Let $(X, \mathcal{X}, P)$ be a probability measure space and B a non-null measurable set.

(1) $\text{ess}.\Pi_B$ and $\text{ess}.N_B$ are normalized fuzzy measures on $\mathcal{X}$.

(2) $\text{ess}.\Pi_B << P$ and $\text{ess}.N_B << P$.

(3) $\text{ess}.\Pi_B(A) + \text{ess}.N_B(A^c) = 1$.

(4) $\text{ess}.\Pi_B(\bigcup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} \text{ess}.\Pi_B(A_n)$.

(5) $\text{ess}.N_B(\bigcap_{n=1}^{\infty} A_n) = \bigwedge_{n=1}^{\infty} \text{ess}.N_B(A_n)$

(6) For every fuzzy measure μ on $\mathcal{X}$,

$$(\mu(X) = 1 \text{ and } \mu << P) \iff \text{ess}.N_X \leq \mu \leq \text{ess}.\Pi_X.$$

4.8.7 PROPOSITON. Let $(X, \mathcal{X}, P)$ be a probability space, B a non-null measurable set, f a nonnegative measurable function on X.

(1) $(C)\int f \, d(\text{ess}.\Pi_B) = \text{ess} \sup_{x \in B} f(x)$.

(2) $(C)\int f \, d(\mathrm{ess}.N_B) = \operatorname*{ess\,inf}_{x \varepsilon B} f(x)$,

where $\operatorname*{ess\,inf}_{x \varepsilon B} f(x)$ is defined as the dual of $\operatorname*{ess\,sup}_{x \varepsilon B} f(x)$, that is,

$$\operatorname*{ess\,inf}_{x \varepsilon B} f(x) = \sup \{r \mid P(\{f < r\}) = 0\}.$$

(3) For every normalized fuzzy measure μ on $\mathcal{X}$ for which $\mu << P$,

$$\operatorname*{ess\,inf}_{x \varepsilon B} f(x) \leq (C)\int f \, d\mu \leq \operatorname*{ess\,sup}_{x \varepsilon B} f(x).$$

Proof. Similar to 4.8.4.

CONCLUSION

We gave two interpretations to the non-additivity and monotonicity of a fuzzy measure; one is that the fuzzy measure is characterized by a pseudo-additivity (Part I) and the other is that the fuzzy measure is characterized by an interpreter H, which reflects the interaction of subsets (Part II). On the basis of the interpretations we defined two different fuzzy integrals. We expect that these integrals will be applied to various problems.

We should continue this research. There are problems for solution; how to integrate a function which is not nonnegative, Radon-Nikodym-like theorem for Choquet's integral, etc.

Choquet's integral is suitable for the interpretation that a fuzzy measure expresses the interaction of subsets (Section 4.4 in Part II). On the basis of this the author thinks the following application. Let us consider data $\{(f_i, r_i) | i = 1, 2, \cdots, n\}$, where f_i is a nonnegative function on a finite set X and r_i is a nonnegative real number which can be regarded as a representative value of f_i. Let us assume that Choquet's integral model expresses a relation between f_i and r_i such that

$$r_i = (C)\int f_i \, d\mu \qquad i = 1, 2, \cdots, n.$$

Then, by identifying μ, we can analyze the interaction among subsets of X.

There are many sides to the set $\mathbb{R}$ of all real numbers: a totally ordered set $(\mathbb{R}, \leq)$, an additive group $(\mathbb{R}, +, 0)$, a field $(\mathbb{R}, +, \times, 0, 1)$, a one-dimensional vector space over the field $\mathbb{R}$, a topological space, a metric space, etc. In addition we can induce other mathematical structures to $\mathbb{R}$. The set $\mathbb{R}^+$ of all nonnegative real numbers is a subset of the set $\mathbb{R}$.

An ordinary measure is nonnegative real-valued and defined by addition. $\mathbb{R}^+$ as its range is generally regarded as the positive cone of the one-dimensional ordered vector space $\mathbb{R}$. We do not regard $\mathbb{R}^+$ as a subset of the field $\mathbb{R}$ since the multiplication $\times$ as a binary operation on $\mathbb{R}^+$ is meaningless; for instance, a product of length and length is not length. (The multiplication $\times$ as a bilinear mapping is meaningful; a product of length and length is area.)

A fuzzy measure is nonnegative real-valued and defined only by order relation. Then a question arises: what is the structure of $\mathbb{R}^+$ as the range of the fuzzy measure? By the monotonicity of the fuzzy measure, $\mathbb{R}^+$ is at least an ordered set. However, we can consider richer structure.

In Part I we gave an answer to the above question: $\mathbb{R}^+$ is an ordered monoid $(\mathbb{R}^+, \hat{+}, 0)$ with a left operation $\hat{\cdot}$. In Part II $\mathbb{R}^+$ is the positive cone of the one-dimensional ordered vector space $\mathbb{R}$. This is a particular case of the answer given in Part I; $\hat{+}$ and $\hat{\cdot}$ are the ordinary addition $+$ and the ordinary multiplication $\cdot$, respectively.

There may be other answers. If the structure of $\mathbb{R}^+$ is fixed, there is the problem: how to integrate a function. There is probably more than one solution. Therefore the author thinks that the two approaches reported here are natural but they are two out of many.

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