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## MULTI-DIMENSIONAL TRAVELING FRONTS IN BISTABLE REACTION-DIFFUSION EQUATIONS

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**ABSTRACT.** This paper studies traveling front solutions of convex polyhedral shapes in bistable reaction-diffusion equations including the Allen-Cahn equations or the Nagumo equations. By taking the limits of such solutions as the lateral faces go to infinity, we construct a three-dimensional traveling front solution for any given  $g \in C^\infty(S^1)$  with  $\min_{0 \leq \theta \leq 2\pi} g(\theta) = 0$ .

**1. Introduction.** In this paper we consider the following equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u) && \text{in } \mathbb{R}^3, t > 0, \\ u|_{t=0} &= u_0 && \text{in } \mathbb{R}^3. \end{aligned}$$

A given function  $u_0$  belongs to  $BU(\mathbb{R}^3)$ . Here  $BU(\mathbb{R}^3)$  is the space of bounded uniformly continuous functions from  $\mathbb{R}^3$  to  $\mathbb{R}$  with the supremum norm. The Laplacian  $\Delta$  stands for  $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ . Here  $f$  is a bistable or multi-stable nonlinear term. Here we write a typical example:  $f(u) = -(u+1)(u+a)(u-1)$  with  $a \in (0, 1)$ . This equation is called the Allen-Cahn equation, the Nagumo equation or the scalar Ginzburg-Landau equation. It appears in various fields, say, in population genetics [10], ecology [35, 34], bistable transmission in electronic circuits [25], phase transitions in metallurgy, van der Waals theory and Landau theory [1, 21] and chemical reactions [40]. Traveling fronts are often observed in these fields. For the literature for one-dimensional traveling fronts, one can refer to Kanel' [18, 19], Nagumo, Yoshizawa and Arimoto [25], Aronson and Weinberger [2, 3], Fife and McLeod [9] and X. Chen [6] and many other works.

In the one-dimensional space, let  $\Phi(x - kt)$  be a traveling wave that connects two stable stationary states  $\pm 1$  with speed  $k$ . By putting  $\mu = x - kt$ ,  $\Phi$  satisfies

$$\begin{aligned} -\Phi''(\mu) - k\Phi'(\mu) - f(\Phi(\mu)) &= 0 && -\infty < \mu < \infty, \\ \Phi(-\infty) &= 1, \quad \Phi(\infty) = -1. \end{aligned} \tag{1}$$

To fix the phase we set  $\Phi(0) = 0$ . Especially one has  $k = \sqrt{2}a$  and  $\Phi(\mu) = -\tanh(\mu/\sqrt{2})$  for  $f(u) = -(u+1)(u+a)(u-1)$  with  $a \in (0, 1)$ .

The following is the assumptions on  $f$  in this paper.

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(A1)  $f \in C^1(\mathbb{R})$  satisfies  $f(1) = 0$ ,  $f(-1) = 0$ ,  $f'(1) < 0$ ,  $f'(-1) < 0$ ,  $\int_{-1}^1 f > 0$  and

$$\begin{aligned} f(u) &< 0 & \text{if } u > 1, \\ f(u) &> 0 & \text{if } u < -1. \end{aligned}$$

(A2) There exists  $\Phi(\mu)$  that satisfies (1) for some  $k > 0$ .

(A3) For some constant  $\alpha$  with

$$\max\{u \mid f(u) = 0, -1 < u < 1\} < \alpha < 1$$

and some  $\tilde{f} \in C^\infty[-1, 1]$  with  $\tilde{f}(-1) = 0$ ,  $\tilde{f}(1) = 0$ ,  $\tilde{f}'(-1) < 0$ ,  $\tilde{f}'(1) < 0$ ,  $\int_{-1}^1 \tilde{f} < 0$  and

$$f(u) \leq \tilde{f}(u) \quad \text{for all } u \in [-1, \alpha],$$

there exists  $s_* > 0$  and  $\psi(\xi)$  satisfying

$$\begin{aligned} -\psi''(\xi) - s_*\psi'(\xi) - \tilde{f}(\psi(\xi)) &= 0 & -\infty < \xi < +\infty, \\ \psi(-\infty) &= -1, & \psi(+\infty) = 1. \end{aligned}$$

We put  $F(u) \stackrel{\text{def}}{=} \int_{-1}^u f$ . Then  $-F(1) < 0$  follows from  $\int_{-1}^1 f > 0$ . Now we present  $f$  that satisfies the assumptions stated above.

**Example 1.** In addition to (A1) assume that there exists  $a \in (-1, 1)$  with

$$\begin{aligned} f(u) &< 0 & \text{if } -1 < u < -a, \\ f(u) &> 0 & \text{if } -a < u < 1. \end{aligned}$$

Then assumptions (A2) and (A3) follow. See Figure 1.

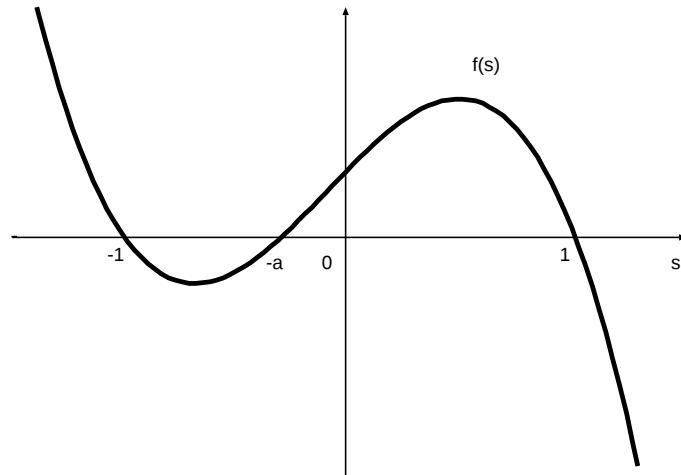


FIGURE 1. The graph of a bistable nonlinear term  $f$ .

The following an example of multistable nonlinearity  $f$ .

**Example 2.** For constants  $a_1, a_2, a$  with  $-1 < a_1 < a_2 < -a < 1$  assume  $f$  satisfies (A1),  $F'(a_1) = 0$ ,  $F'(a_2) = 0$ ,  $-F''(a_2) > 0$ ,  $-F(a_2) > 0$ ,  $F'(-a) = 0$  and

$$\begin{aligned} -F'(u) &> 0 & \text{if } u &\in (-1, a_1) \cup (a_2, -a), \\ -F'(u) &< 0 & \text{if } u &\in (a_1, a_2) \cup (-a, 1). \end{aligned}$$

Then  $f = F'$  satisfies (A2) and (A3).

We show  $f$  in Example 1 or Example 2 satisfies (A1), (A2) and (A3) in Lemma 2.1. We derive the profile equation for a traveling wave with speed  $c$ . We adopt the moving coordinate of speed  $c$  toward the  $z$ -axis without loss of generality. We put  $s = z - ct$  and  $u(x, y, z, t) = w(x, y, s, t)$ . We denote  $w(x, y, s, t)$  by  $w(x, y, z, t)$  for simplicity. Then we obtain

$$\begin{aligned} w_t - w_{xx} - w_{yy} - w_{zz} - cw_z - f(w) &= 0 & \text{in } \mathbb{R}^3, t > 0, \\ w|_{t=0} &= u_0 & \text{in } \mathbb{R}^3. \end{aligned} \quad (2)$$

Here  $w_t$  stands for  $\partial w / \partial t$  and so on. We write the solution as  $w(x, y, z, t; u_0)$ . If  $v$  is a traveling wave with speed  $c$ , it satisfies

$$\mathcal{L}[v] \stackrel{\text{def}}{=} -v_{xx} - v_{yy} - v_{zz} - cv_z - f(v) = 0 \quad \text{in } \mathbb{R}^3. \quad (3)$$

Equation (3) is the profile equation for traveling waves.

Here we briefly explain well-known traveling waves for bistable reaction-diffusion equations. For stationary ball solutions, see [4] and [29] and references therein. Recently del Pino, Kowalczyk and Wei [8] studied stationary solutions related to de Giorgi conjecture.

Planar traveling fronts are studied by Xin [41], Kapitula [20], Levermore and Xin [23], Matano, Nara and myself [24], Roquejoffre and Roussier-Michon [32] and so on.

Multi-dimensional traveling front solutions have been studied recently. Two-dimensional V-form fronts are studied by Ninomiya and myself [27, 28] and Hamel, Monneau and Roquejoffre [13, 14]. See also Haragus and Scheel [17]. Traveling fronts with cylindrical symmetry are studied by [13, 14, 7]. Pyramidal traveling fronts are studied by [37, 38] in the three-dimensional space and by Kurokawa and myself [22] in the  $N$ -dimensional space for  $N \geq 4$ .

The aim of this paper is as follows.

1. Traveling front solutions of convex polyhedral shapes are constructed in Theorem 3.2 of §3. We call them generalized pyramidal traveling front solutions.
2. For any given  $g \in C^\infty(S^1)$  with  $\min_{0 \leq \theta \leq 2\pi} g(\theta) = 0$ , we construct a three-dimensional traveling front solution in Theorem 6.1 by sending the number of lateral faces of a convex polyhedron to infinity in §5.

Let  $n$  be as in (7). The cross section of a convex polyhedron  $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq h(x, y)\}$  for sufficiently large  $z > 0$  is an  $n$ -polygon. Here  $h$  is given by (7). To guarantee the convergence of generalized pyramidal traveling front solutions in Theorem 3.2 to three-dimensional traveling front solutions in Theorem 6.1, we need a uniform estimate on the width of generalized pyramidal traveling front solutions with respect to  $n$ . We discuss this uniform estimate in §4 and prove Theorem 6.1 in §5.

**2. Preliminaries.** Hereafter we assume

$$c > k.$$

Since the curvature often accelerates the speed, we have many traveling front solutions when  $c > k$ . Though it is an interesting problem to study the traveling front solutions when  $c \leq k$ , we just study  $c > k$  in this paper.

Under  $f(-1) = 0$  and  $f(1) = 0$ ,  $\Phi(\mu)$  in (1) satisfies

$$\Phi'(\mu) < 0 \quad \text{for all } \mu \in \mathbb{R}, \quad (4)$$

$$\sup_{\mu \in \mathbb{R}} e^{\lambda_0 |\mu|} |\Phi'(\mu)| < \infty, \quad \sup_{\mu \in \mathbb{R}} e^{\lambda_0 |\mu|} |\Phi''(\mu)| < \infty. \quad (5)$$

Here  $\lambda_0 > 0$  is a constant. See [9] for more details.

We set

$$\beta \stackrel{\text{def}}{=} \frac{1}{2} \min \{-f'(-1), -f'(1)\} > 0,$$

and

$$M \stackrel{\text{def}}{=} \max_{-2 \leq u \leq 2} |f'(u)| > 0.$$

By the assumptions on  $f$  we take  $\delta_* \in (0, 1/4)$  small enough such that we have

$$-f'(s) > \beta \quad \text{if } |s+1| \leq 2\delta_* \text{ or } |s-1| \leq 2\delta_*.$$

Now we show the following lemma.

**Lemma 2.1.** *Assumptions (A2) and (A3) holds true in Example 1 and Example 2.*

*Proof.* For Example 1 the existence of  $(k, \Phi)$  and  $k > 0$  follow from [9, 2, 3]. We take  $G \in C^\infty[-1, 1]$  with  $G(1) > 0$ ,  $G'(1) = 0$ ,  $G''(1) > 0$  and

$$\begin{aligned} G(u) &= -F(u) & \text{if } -1 \leq u \leq -a, \\ -F(u) &\leq G(u) & \text{if } u > -a, \\ G'(u) &< 0 & \text{if } -a < u < 1. \end{aligned}$$

We put  $\tilde{f} \stackrel{\text{def}}{=} -G'$ . Then the existence of  $(s_*, \psi)$  and  $s_* > 0$  follow from [9, 2, 3].

For Example 2 there exists a traveling wave connecting 1 and  $a_2$  with positive speed along the  $x$ -axis, and there exists a traveling wave connecting  $a_2$  and  $-1$  with negative speed. Now [9, Theorem 2.7] gives  $(k, \Phi)$  and  $F(-1) > F(1)$  gives  $k > 0$ . To show (A3) for Example 2, we use  $G$  stated above with  $G(1) < G(a_2)$ . Then there exists a traveling wave connecting  $-1$  and  $a_2$  with positive speed and a traveling wave connecting  $a_2$  and 1 with negative speed along the  $x$ -axis. Then [9, Theorem 2.7] gives  $(s_*, \psi)$  and  $s_* > 0$  follow from  $G(-1) < G(1)$ .  $\square$

Let  $n \geq 3$  be a given integer. Let  $\{\theta_j\}_{1 \leq j \leq n}$  satisfy

$$0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$$

and

$$\max_{1 \leq j \leq n} (\theta_{j+1} - \theta_j) < \pi,$$

where  $\theta_{n+1} \stackrel{\text{def}}{=} \theta_1 + 2\pi$ . Let  $c_j \geq 0$  be given for  $1 \leq j \leq n$  with

$$\min_{1 \leq j \leq n} c_j \geq 0.$$

We put

$$\tau \stackrel{\text{def}}{=} \frac{\sqrt{c^2 - k^2}}{k} > 0. \quad (6)$$

Now

$$\nu_j \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + \tau^2}} \begin{pmatrix} -\tau \cos \theta_j \\ -\tau \sin \theta_j \\ 1 \end{pmatrix}$$

is the unit normal vector of a plane  $\{z = \tau(x \cos \theta_j + y \sin \theta_j)\}$ . We put

$$\begin{aligned} h_j(x, y) &\stackrel{\text{def}}{=} \tau(x \cos \theta_j + y \sin \theta_j - c_j), \\ h(x, y) &\stackrel{\text{def}}{=} \max_{1 \leq j \leq n} h_j(x, y) = \tau \max_{1 \leq j \leq n} (x \cos \theta_j + y \sin \theta_j - c_j). \end{aligned} \quad (7)$$

A set  $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq h(x, y)\}$  is a convex polyhedron. If  $(c_1, \dots, c_n) = (0, \dots, 0)$ , the polyhedron is a pyramid in  $\mathbb{R}^3$ .

We set

$$\Omega_j = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) = h_j(x, y), h(x, y) \geq \tau\Theta\},$$

where  $\Theta$  is a number given by (17). We note that  $\Omega_j \neq \emptyset$  for all  $1 \leq j \leq n$ . Here  $\Omega_1, \Omega_2, \dots, \Omega_n$  are located counterclockwise.

Now we set

$$S_j \stackrel{\text{def}}{=} \{(x, y, h_j(x, y)) \in \mathbb{R}^3 \mid (x, y) \in \Omega_j\}$$

for  $j = 1, \dots, n$ . We put

$$\Gamma_j \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid z = h_j(x, y) = h_{j+1}(x, y) \geq \tau\Theta, (x, y) \in \mathbb{R}^2\},$$

which is a part of an edge of a polyhedron  $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq h(x, y)\}$ . If  $(c_1, \dots, c_n) = (0, \dots, 0)$ ,  $\Theta = 0$  is valid and  $\Gamma_j$  and  $\cup_{j=1}^n \Gamma_j$  represent an edge and the set of all edges for a pyramid, respectively.

Equation (3) has a solution  $\Phi((k/c)(z - h_j(x, y)))$ . It is called a planar traveling front associated with the lateral face  $S_j$ . Now we put

$$\underline{v}(x, y, z) \stackrel{\text{def}}{=} \Phi\left(\frac{k}{c}(z - h(x, y))\right) = \max_{1 \leq j \leq n} \Phi\left(\frac{k}{c}(z - h_j(x, y))\right). \quad (8)$$

We define

$$D(\gamma) \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid \text{dist}((x, y, z), \cup_{j=1}^n \Gamma_j) \geq \gamma\} \quad (9)$$

for  $\gamma \geq 0$ .

We state the uniqueness and stability of a two-dimensional V-form front in the two-dimensional plane.

**Theorem 2.2** (Two-dimensional V-form fronts [27, 28]). *Assume (A1) and (A2). For any given  $c \in (k, +\infty)$ , there exists unique  $v_*(x, y; c)$  that satisfies*

$$\begin{aligned} -(v_*)_{xx} - (v_*)_{yy} - c(v_*)_y - f(v_*) &= 0 \quad \text{for } (x, y) \in \mathbb{R}^2, \\ \lim_{R \rightarrow \infty} \sup_{x^2 + y^2 > R^2} \left| v_*(x, y) - \Phi\left(\frac{k}{c}(y - \tau|x|)\right) \right| &= 0. \end{aligned} \quad (10)$$

One has

$$\Phi\left(\frac{k}{c}(y - \tau|x|)\right) < v_*(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2, \quad (11)$$

$$\inf_{-1 + \delta \leq v_*(x, y) \leq 1 - \delta} (-(v_*)_y(x, y)) > 0 \quad \text{for any } \delta \in (0, \delta_*]. \quad (12)$$

Moreover, for a positive constant  $\sigma$  with

$$\frac{\beta + M}{\beta \inf_{-1+\delta_* \leq v_*(x,y) \leq 1-\delta_*} (-(v_*)_y(x,y))} < \sigma,$$

$v_*(x, y - \sigma\delta_*(1 - e^{-\beta t}); c) + \delta_*e^{-\beta t}$  is a supersolution of (2).

See also Hamel, Monneau and Roquejoffre [13, 14].

We define

$$p(x, y) \stackrel{\text{def}}{=} \tau \max_{1 \leq j \leq n} (x \cos \theta_j + y \sin \theta_j) \quad (13)$$

We state the existence and stability of pyramidal traveling fronts in  $\mathbb{R}^3$ . We set  $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ .

**Theorem 2.3** (Pyramidal traveling fronts [37, 38]). *Let  $c > k$  and let  $p(x, y)$  be given by (13). Under the assumptions (A1) and (A2), there exists a solution  $V(x, y, z)$  to (3) with*

$$\lim_{\gamma \rightarrow +\infty} \sup_{(x,y,z) \in D(\gamma)} \left| V(x, y, z) - \Phi \left( \frac{k}{c}(z - p(x, y)) \right) \right| = 0. \quad (14)$$

Moreover one has

$$\begin{aligned} V_z(x, y, z) < 0, \quad \Phi \left( \frac{k}{c}(z - p(x, y)) \right) < V(x, y, z) < 1 \quad \text{for all } (x, y, z) \in \mathbb{R}^3, \\ \inf_{-1+\delta \leq V(\mathbf{x}) \leq 1-\delta} (-V_z(\mathbf{x})) > 0 \quad \text{for any } \delta \in (0, \delta_*], \\ \lim_{R \rightarrow \infty} \sup_{|z-p(x,y)| \geq R} |V_z(x, y, z)| = 0. \end{aligned} \quad (15)$$

Suppose

$$\lim_{\gamma \rightarrow +\infty} \sup_{\mathbf{x} \in D(\gamma)} |u_0(\mathbf{x}) - V(\mathbf{x})| = 0. \quad (16)$$

Then

$$\lim_{t \rightarrow +\infty} \sup_{(x,y,z) \in \mathbb{R}^3} |u(x, y, z - ct, t) - V(x, y, z)| = 0$$

holds true. Especially  $V(x, y, z)$  is uniquely determined by (3) and (14).

Thus a pyramidal traveling front solution  $V$  is asymptotically stable globally in space if a given fluctuation decays at infinity. Now we call  $V$  as in Theorem 2.3 the pyramidal traveling wave  $V(\mathbf{x}; p)$  associated with a pyramid  $z = p(x, y)$ .

We set

$$\kappa_j \stackrel{\text{def}}{=} \cos \left( \frac{\theta_{j+1} - \theta_j}{2} \right) > 0, \quad \phi_j \stackrel{\text{def}}{=} \frac{\theta_j + \theta_{j+1}}{2} \quad \text{for all } 1 \leq j \leq n,$$

and define

$$E_j(\mathbf{x}) = v_* \left( x \sin \phi_j - y \cos \phi_j, \frac{z - \tau \kappa_j (x \cos \phi_j + y \sin \phi_j)}{\sqrt{1 + \tau^2 \kappa_j^2}}, \frac{c}{\sqrt{1 + \tau^2 \kappa_j^2}} \right).$$

A pyramidal traveling front is uniquely determined as a combination of two-dimensional V-form fronts.

**Theorem 2.4** ([38]). Assume (A1) and (A2). Let  $V$  be as in Theorem 2.3 for  $z = p(x, y)$ . Then it satisfies

$$\lim_{R \rightarrow \infty} \sup_{|\mathbf{x}| \geq R} \left| V(\mathbf{x}) - \max_{1 \leq j \leq n} E_j(\mathbf{x}) \right| = 0,$$

$$\max_{1 \leq j \leq n} E_j(\mathbf{x}) < V(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

Conversely, if (3) has a solution  $v$  with

$$\lim_{R \rightarrow \infty} \sup_{|\mathbf{x}| \geq R} \left| v(\mathbf{x}) - \max_{1 \leq j \leq n} E_j(\mathbf{x}) \right| = 0,$$

then one has  $v \equiv V$ .

**Lemma 2.5.** Let  $p(x, y)$  be given by (13) and let  $V$  be as in Theorem 2.3. If  $(X_0, Y_0, Z_0) \in \mathbb{R}^3$  satisfies  $Z_0 \geq p(X_0, Y_0)$ , one has

$$V(x, y, z) \leq V(x - X_0, y - Y_0, z - Z_0) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

*Proof.* We have

$$p(x, y) \leq Z_0 + p(x - X_0, y - Y_0) \quad \text{for all } (x, y) \in \mathbb{R}^2$$

and thus

$$\Phi\left(\frac{k}{c}(z - p(x, y))\right) \leq \Phi\left(\frac{k}{c}(z - Z_0 - p(x - X_0, y - Y_0))\right) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

Using both sides as initial values of (2) and taking the limit of  $t = +\infty$ , we obtain

$$V(x, y, z) \leq V(x - X_0, y - Y_0, z - Z_0) \quad \text{for all } (x, y, z) \in \mathbb{R}^3$$

due to Theorem 2.3. This completes the proof.  $\square$

**3. Generalized pyramidal traveling fronts.** In this section we study traveling front solutions of convex polyhedral shapes. In this section we assume only (A1) and (A2).

We first note that  $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq h(x, y)\}$  is a convex polyhedron. Indeed, for  $i = 1, 2$  let  $(x_i, y_i, z_i)$  satisfy  $z_i \geq h(x_i, y_i)$ . Then we have  $z_i \geq h_j(x_i, y_i)$  for all  $1 \leq j \leq n$  and  $i = 1, 2$ . Since  $h_j$  is affine we have

$$\theta z_1 + (1 - \theta) z_2 \geq \theta h_j(x_1, y_1) + (1 - \theta) h_j(x_2, y_2) = h_j(\theta x_1 + (1 - \theta) x_2)$$

for all  $1 \leq j \leq n$  and any  $\theta \in (0, 1)$ . Thus we have

$$\theta z_1 + (1 - \theta) z_2 \geq \max_{1 \leq j \leq n} h_j(\theta x_1 + (1 - \theta) x_2) = h(\theta x_1 + (1 - \theta) x_2).$$

Thus  $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq h(x, y)\}$  is a convex polyhedron.

For any  $\zeta \in \mathbb{R}$  and  $1 \leq j \leq n$ , let  $(X_j(\zeta), Y_j(\zeta))$  be defined by

$$h_j(X_j(\zeta), Y_j(\zeta)) = h_{j+1}(X_j(\zeta), Y_j(\zeta)) = \tau \zeta,$$

that is,

$$\begin{pmatrix} X_j(\zeta) \\ Y_j(\zeta) \end{pmatrix} = \frac{1}{\sin(\theta_{j+1} - \theta_j)} \begin{pmatrix} (\zeta + c_j) \sin \theta_{j+1} - (\zeta + c_{j+1}) \sin \theta_j \\ -(\zeta + c_j) \cos \theta_{j+1} + (\zeta + c_{j+1}) \cos \theta_j \end{pmatrix}.$$

For every  $\zeta \in \mathbb{R}$  a set  $\{(x, y) \in \mathbb{R}^2 \mid h(x, y) \leq \zeta\}$  is the empty set or a non-empty convex closed set in  $\mathbb{R}^2$ . Indeed, for  $i = 1, 2$  let  $(x_i, y_i)$  satisfy  $h(x_i, y_i) \leq \zeta$ . Then we have  $h_j(x_i, y_i) \leq \zeta$  for all  $1 \leq j \leq n$  and  $i = 1, 2$ . Since  $h_j$  is affine, we have

$$h_j(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_2) = \theta h_j(x_1, y_1) + (1 - \theta)h_j(x_2, y_2) \leq \zeta$$

for all  $1 \leq j \leq n$ . Thus we get

$$h(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_2) = \max_{1 \leq j \leq n} h_j(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_2) \leq \zeta.$$

Thus  $\{(x, y) \in \mathbb{R}^2 \mid h(x, y) \leq \zeta\}$  is the empty set or a non-empty convex closed set in  $\mathbb{R}^2$ .

If  $\zeta < -\max_{1 \leq j \leq n} c_j$ , then  $\{(x, y) \in \mathbb{R}^2 \mid h(x, y) \leq \zeta\}$  is the empty set. On the other hand, it becomes a convex  $n$ -polygon if  $\zeta > 0$  is large enough. We find and fix such a large number. For this purpose we set

$$\Theta \stackrel{\text{def}}{=} \max_{1 \leq j \leq n} \frac{c_j \sin(\theta_{j+1} - \theta_{j-1}) - c_{j-1} \sin(\theta_{j+1} - \theta_j) - c_{j+1} \sin(\theta_j - \theta_{j-1})}{\sin(\theta_{j+1} - \theta_j) + \sin(\theta_j - \theta_{j-1}) - \sin(\theta_{j+1} - \theta_{j-1})}, \quad (17)$$

and fix any positive number  $\rho \in (\Theta, \infty)$ .

**Lemma 3.1.** *A set  $\{(x, y) \in \mathbb{R}^2 \mid h(x, y) \leq \tau\rho\}$  is a convex  $n$ -polygon in the  $x$ - $y$  plane with vertices  $\{(X_j(\rho), Y_j(\rho))\}_{1 \leq j \leq n}$ .*

*Proof.* For any  $j$  with  $0 < \theta_{j+1} - \theta_{j-1} < \pi$ , we require that the intersection point of  $h_{j-1}(x, y) = \tau\rho$  and  $h_{j+1}(x, y) = \tau\rho$  in the  $x$ - $y$  plane should satisfy  $h_j(x, y) > \rho$ . Then  $(X_1(\rho), Y_1(\rho)), \dots, (X_n(\rho), Y_n(\rho))$  are located counterclockwise and a set  $\{(x, y) \mid h(x, y) \leq \tau\rho\}$  becomes an  $n$ -polygon. The intersection point of the two lines is

$$\frac{1}{\sin(\theta_{j+1} - \theta_{j-1})} \left( \begin{aligned} &(\rho + c_{j-1}) \sin \theta_{j+1} - (\rho + c_{j+1}) \sin \theta_{j-1} \\ &(\rho + c_{j+1}) \cos \theta_{j-1} - (\rho + c_{j-1}) \cos \theta_{j+1} \end{aligned} \right).$$

Thus, if  $0 < \theta_{j+1} - \theta_{j-1} < \pi$ ,  $\rho$  should satisfy

$$\begin{aligned} &((\rho + c_{j-1}) \sin \theta_{j+1} - (\rho + c_{j+1}) \sin \theta_{j-1}) \cos \theta_j \\ &+ ((\rho + c_{j+1}) \cos \theta_{j-1} - (\rho + c_{j-1}) \cos \theta_{j+1}) \sin \theta_j > (\rho + c_j) \sin(\theta_{j+1} - \theta_{j-1}), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &c_j \sin(\theta_{j+1} - \theta_{j-1}) - c_{j-1} \sin(\theta_{j+1} - \theta_j) - c_{j+1} \sin(\theta_j - \theta_{j-1}) \\ &< (\sin(\theta_{j+1} - \theta_j) + \sin(\theta_j - \theta_{j-1}) - \sin(\theta_{j+1} - \theta_{j-1})) \rho \end{aligned} \quad (18)$$

We have

$$\begin{aligned} &\sin(\theta_{j+1} - \theta_j) + \sin(\theta_j - \theta_{j-1}) - \sin(\theta_{j+1} - \theta_{j-1}) \\ &= 4 \sin \frac{\theta_{j+1} - \theta_j}{2} \sin \frac{\theta_j - \theta_{j-1}}{2} \sin \frac{\theta_{j+1} - \theta_{j-1}}{2} > 0. \end{aligned}$$

Now we can see that  $\rho > \Theta$  gives (18). This completes the proof.  $\square$

We have  $h(X_j(\rho), Y_j(\rho)) = \tau\rho$  for all  $1 \leq j \leq n$ . Thus we obtain

$$h(x, y) \leq \tau\rho + p(x - X_j(\rho), y - Y_j(\rho)) \quad \text{for all } (x, y) \in \mathbb{R}^2, 1 \leq j \leq n.$$

Let  $V_0(\mathbf{x})$  denote the pyramidal traveling front  $V(\mathbf{x}; p)$  as in Theorem 2.3. Let  $\underline{v}$  be as in (8). Then  $\underline{v}$  is a subsolution to (3) and satisfies

$$\underline{v}(x, y, z) < V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho)$$

for all  $(x, y, z) \in \mathbb{R}^3$  and  $1 \leq j \leq n$ . Now

$$\min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho)$$

is a supersolution to (3). We define

$$V(\mathbf{x}) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} w(\mathbf{x}, t; \underline{v})$$

for any  $\mathbf{x} \in \mathbb{R}^3$ . Then  $V$  is a solution to (3) with

$$\underline{v}(x, y, z) < V(x, y, z) \leq \min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho)$$

for all  $(x, y, z) \in \mathbb{R}^3$ . See Sattinger [33, Theorem 3.6] for general arguments.

Using (12), (15) and  $\lim_{R \rightarrow \infty} \sup_{|\mathbf{x}| \geq R} |V(\mathbf{x}) - \max_{1 \leq j \leq n} E_j(\mathbf{x})| = 0$  and applying the Schauder interior estimate [11, Theorem 9.11] to

$$\left( -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - c \frac{\partial}{\partial z} \right) (V - E_j) = f(V) - f(E_j),$$

we obtain

$$\inf_{-1+\delta \leq V(\mathbf{x}) \leq 1-\delta} (-V_z(\mathbf{x})) > 0 \quad \text{for any } \delta \in (0, \delta_*].$$

For all  $1 \leq j \leq n$  we have

$$\max\{h_j(x, y), h_{j+1}(x, y)\} \leq h(x, y) \quad \text{in } \mathbb{R}^2.$$

Then we get

$$\Phi\left(\frac{k}{c}(z - \max\{h_j(x, y), h_{j+1}(x, y)\})\right) \leq \underline{v}(x, y, z) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

We consider the left-hand side or the right-hand side as an initial value of (2), respectively. Using Theorem 2.2 and sending  $t \rightarrow +\infty$ , we obtain

$$E_j(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho) \leq V(x, y, z) \quad \text{for all } (x, y, z) \in \mathbb{R}^3,$$

and thus

$$\begin{aligned} \max_{1 \leq j \leq n} E_j(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho) &\leq V(x, y, z) \\ &\leq \min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho) \end{aligned} \quad (19)$$

for all  $(x, y, z) \in \mathbb{R}^3$ . We set

$$\widehat{c} \stackrel{\text{def}}{=} \max_{1 \leq j \leq n} c_j \geq 0.$$

Using

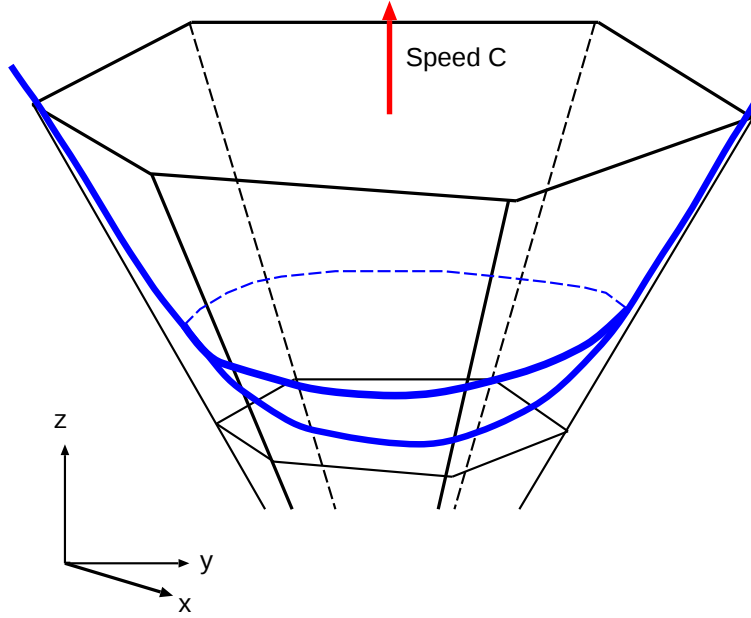
$$-\tau\widehat{c} + p(x - X_j(-\widehat{c}), y - Y_j(-\widehat{c})) \leq h(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2,$$

we obtain

$$\Phi\left(\frac{k}{c}(z + \tau\widehat{c} - p(x - X_j(-\widehat{c}), y - Y_j(-\widehat{c})))\right) \leq \Phi\left(\frac{k}{c}(z - h(x, y))\right).$$

Considering the left-hand side or the right-hand side as an initial condition of (2) and sending  $t \rightarrow \infty$ , we obtain

$$V_0(x - X_j(-\widehat{c}), y - Y_j(-\widehat{c}), z + \tau\widehat{c}) \leq V(x, y, z),$$

FIGURE 2. A generalized pyramidal traveling front  $V(\mathbf{x}; h)$ .

and thus

$$\begin{aligned} \max_{1 \leq j \leq n} V_0(x - X_j(-\hat{c}), y - Y_j(-\hat{c}), z + \tau \hat{c}) &\leq V(x, y, z) \\ &\leq \min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau \rho) \end{aligned} \quad (20)$$

for all  $(x, y, z) \in \mathbb{R}^3$ .

**Theorem 3.2** (Generalized pyramidal traveling front solutions). *Let  $c > k$  and let  $h(x, y)$  be given by (7). Under the assumptions (A1) and (A2) there exists a solution  $V(x, y, z)$  to (3) with*

$$\lim_{\gamma \rightarrow +\infty} \sup_{(x, y, z) \in D(\gamma)} \left| V(x, y, z) - \Phi \left( \frac{k}{c}(z - h(x, y)) \right) \right| = 0. \quad (21)$$

Moreover one has

$$V_z(x, y, z) < 0, \quad -1 < \Phi \left( \frac{k}{c}(z - h(x, y)) \right) < V(x, y, z) < 1 \quad \text{for all } (x, y, z) \in \mathbb{R}^3,$$

$$\lim_{R \rightarrow \infty} \sup_{|z - h(x, y)| \geq R} |V_z(x, y, z)| = 0,$$

$$\lim_{R \rightarrow \infty} \sup_{|x| \geq R} |V(x, y, z) - \max_{1 \leq j \leq n} E_j(x - X_j(\rho), y - Y_j(\rho), z - \tau \rho)| = 0.$$

Suppose

$$\lim_{\gamma \rightarrow +\infty} \sup_{(x, y, z) \in D(\gamma)} |u_0(x, y, z) - V(x, y, z)| = 0. \quad (22)$$

Then

$$\lim_{t \rightarrow +\infty} \sup_{(x, y, z) \in \mathbb{R}^3} |u(x, y, z - ct, t) - V(x, y, z)| = 0$$

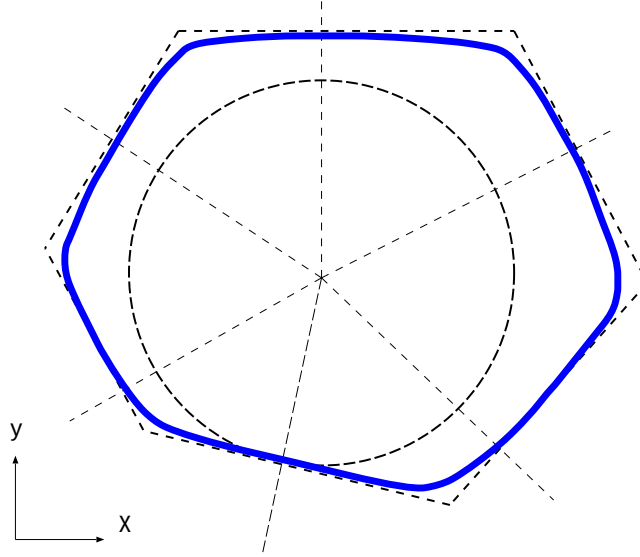


FIGURE 3. The cross section of a generalized pyramidal traveling front  $V(\mathbf{x}; h)$ .

holds true. Especially  $V(x, y, z)$  is uniquely determined by (3) and (21).

*Proof.* It suffices to prove the uniqueness and stability of  $V$ . We have

$$V(x, y, z) \leq \min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho).$$

and define

$$V^*(\mathbf{x}) \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} w(\mathbf{x}, t; \min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho))$$

for every  $\mathbf{x} \in \mathbb{R}^3$ . Then  $V^*$  satisfies

$$V(x, y, z) \leq V^*(x, y, z) \leq \min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho)$$

for all  $(x, y, z) \in \mathbb{R}^3$ . This implies

$$\lim_{R \rightarrow \infty} \sup_{|\mathbf{x}| \geq R} |V^*(\mathbf{x}) - V(\mathbf{x})| = 0.$$

We choose a positive constant  $\sigma$  with

$$\beta\sigma \inf_{-1+\delta_* \leq V(\mathbf{x}) \leq 1-\delta_*} (-V_z(\mathbf{x})) > \beta + \sup_{|s| \leq 2} |f'(s)|.$$

Similarly as in [27, Lemma 4.4] we find that

$$V(x, y, z - \sigma\delta(1 - e^{-\beta t})) + \delta e^{-\beta t}$$

is a supersolution to (2) for any  $\delta \in (0, \delta_*)$ .

For any given  $\delta \in (0, \delta_*)$ , we take  $\lambda > 0$  large enough and get

$$V^*(x, y, z) \leq V(x, y, z - \lambda) + \delta.$$

Then we have

$$V^*(x, y, z) \leq V(x, y, z - \lambda - \sigma\delta(1 - e^{-\beta t})) + \delta e^{-\beta t}.$$

Sending  $t \rightarrow \infty$ , we obtain

$$V(x, y, z) \leq V^*(x, y, z) \leq V(x, y, z - \lambda - \sigma\delta) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

We define

$$\Lambda \stackrel{\text{def}}{=} \inf\{\lambda \in (-\infty, \infty) \mid V^*(x, y, z) \leq V(x, y, z - \lambda) \quad \text{for all } (x, y, z) \in \mathbb{R}^3\}.$$

We have  $\Lambda \geq 0$ . If  $\Lambda = 0$ ,  $V^* \equiv V$  follows. We prove  $\Lambda = 0$  by contradiction. Assume  $\Lambda > 0$ . Then the comparison principle gives

$$V^*(x, y, z) < V(x, y, z - \Lambda) \quad \text{for all } (x, y, z) \in \mathbb{R}^3$$

Now we fix  $R_0 > 0$  large enough and get

$$2\sigma \sup_{|z-h(x,y)| \geq R_0 - \Lambda - 1} |V_z(x, y, z)| < 1.$$

Taking

$$k_1 \in \left(0, \min\left\{\delta_*, \frac{1}{2\sigma}, \frac{\Lambda}{2\sigma}\right\}\right)$$

small enough and using

$$\lim_{R \rightarrow \infty} \sup_{|\mathbf{x}| \geq R} |V^*(\mathbf{x}) - V(\mathbf{x})| = 0,$$

we have

$$V^*(x, y, z) < V(x, y, z - \Lambda + 2\sigma k_1)$$

if  $|z - h(x, y)| \leq R_0 - \Lambda - 1$ . If  $|z - h(x, y)| > R_0 - \Lambda - 1$ , we have

$$V(x, y, z - \Lambda + 2\sigma k_1) - V(x, y, z - \Lambda) = 2\sigma k_1 \int_0^1 V_z(x, y, z - \Lambda + 2\sigma k_1 \theta) d\theta \geq -k_1.$$

Combining these two estimates together, we obtain

$$V^*(x, y, z) \leq V(x, y, z - \Lambda + 2\sigma k_1) + k_1 \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

Then we get

$$V^*(x, y, z) \leq V(x, y, z - \Lambda + 2\sigma k_1 - \sigma k_1(1 - e^{-\beta t})) + k_1 e^{-\beta t}.$$

Sending  $t \rightarrow +\infty$ , we obtain

$$V^*(x, y, z) \leq V(x, y, z - \Lambda + \sigma k_1) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

This contradicts the definition of  $\Lambda$ . Thus  $V^* \equiv V$  follows. This implies

$$\lim_{t \rightarrow \infty} \left\| w(\mathbf{x}, t; \min_{1 \leq j \leq n} V_0(x - X_j(\rho), y - Y_j(\rho), z - \tau\rho)) - V(\mathbf{x}) \right\|_{L^\infty(\mathbb{R}^3)} = 0.$$

Applying a similar argument to

$$w(\mathbf{x}, t; \max_{1 \leq j \leq n} V_0(x - X_j(-\hat{c}), y - Y_j(-\hat{c}), z + \tau\hat{c})),$$

we obtain

$$\lim_{t \rightarrow \infty} \|w(\mathbf{x}, t; V_0(x - X_j(-\hat{c}), y - Y_j(-\hat{c}), z + \tau\hat{c})) - V(\mathbf{x})\|_{L^\infty(\mathbb{R}^3)} = 0.$$

For any fixed  $t \geq 0$ ,  $w(\mathbf{x}, t; \cdot)$  is a continuous mapping in  $BU(\mathbb{R}^3)$ . Using this continuity, Theorem 2.3 and the comparison principle, we obtain

$$\lim_{t \rightarrow +\infty} \|w(\mathbf{x}, t; u_0) - V(\mathbf{x})\|_{L^\infty(\mathbb{R}^3)} = 0.$$

This completes the proof.  $\square$

A set  $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq h(x, y)\}$  is a convex polyhedron for any  $h$  in (7). We call it a generalized pyramid and call  $V$  in Theorem 3.2 the generalized pyramidal traveling front solution associated with  $z = h(x, y)$ , and denote it by  $V(\mathbf{x}; h)$ . See Figure 2. For the cross section of  $V(\mathbf{x}; h)$  see Figure 3. When  $h = p$ ,  $V(\mathbf{x}; h)$  in Theorem 3.2 equals  $V(\mathbf{x}; p)$  in Theorem 2.3.

**Lemma 3.3.** *Let  $h(x, y)$  be as in (7) and let  $\tilde{h}(x, y)$  be*

$$\tilde{h}(x, y) \stackrel{\text{def}}{=} \tau \max_{1 \leq j \leq n} (x \cos \theta_j + y \sin \theta_j - \tilde{c}_j),$$

where  $\{\tilde{c}_j\}$  satisfies

$$\min_{1 \leq j \leq n} \tilde{c}_j \geq 0.$$

If  $h(x, y) \leq \tilde{h}(x, y)$  for all  $(x, y) \in \mathbb{R}^2$ , one has  $V(\mathbf{x}; h) \leq V(\mathbf{x}; \tilde{h})$  for all  $\mathbf{x} \in \mathbb{R}^3$ .

*Proof.* We have

$$\Phi\left(\frac{k}{c}(z - h(x, y))\right) \leq \Phi\left(\frac{k}{c}(z - \tilde{h}(x, y))\right) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

Using each side as an initial value of (2) and taking the limit of  $t = +\infty$ , we obtain  $V(\mathbf{x}; h) \leq V(\mathbf{x}; \tilde{h})$ .  $\square$

As in Theorem 2.3,  $V$  is strictly monotone decreasing in  $z$ . This suggests that  $V$  is strictly monotone decreasing along a direction near the  $z$ -axis. Indeed we have the following lemma.

**Lemma 3.4.** *Let  $V$  be as in Theorem 3.2. Let*

$$\mathbf{t} = \frac{1}{\sqrt{1 + t_1^2 + t_2^2}} \begin{pmatrix} t_1 \\ t_2 \\ 1 \end{pmatrix}$$

be a given constant vector with

$$\sqrt{t_1^2 + t_2^2} \leq \frac{1}{\tau}.$$

Then one has

$$\frac{\partial V}{\partial \mathbf{t}} < 0 \quad \text{in } \mathbb{R}^3.$$

*Proof.* For  $1 \leq j \leq n$  we put

$$P_j(x, y, z) \stackrel{\text{def}}{=} \Phi\left(\frac{k}{c}(z - h_j(x, y))\right) \quad \text{in } \mathbb{R}^3,$$

and have

$$\nabla P_j = \Phi'\left(\frac{k}{c}(z - h_j(x, y))\right) \boldsymbol{\nu}_j.$$

For  $1 \leq j \leq n$  we have  $(\mathbf{t}, \boldsymbol{\nu}_j) \geq 0$ . Indeed, we have

$$\begin{aligned} (\mathbf{t}, \boldsymbol{\nu}_j) &= \frac{k}{c} \frac{1}{\sqrt{1+t_1^2+t_2^2}} (1 - \tau(t_1 \cos \theta_j + t_2 \sin \theta_j)) \\ &\geq \frac{k}{c} \frac{1}{\sqrt{1+t_1^2+t_2^2}} \left(1 - \tau \sqrt{t_1^2+t_2^2}\right) \geq 0. \end{aligned}$$

Then we get

$$\frac{\partial P_j}{\partial \mathbf{t}} = (\nabla P_j, \mathbf{t}) = \Phi' \left( \frac{k}{c} (z - h_j(x, y)) \right) (\mathbf{t}, \boldsymbol{\nu}_j) \leq 0.$$

Thus we get

$$P_j(\mathbf{x} + \varepsilon \mathbf{t}) \leq P_j(\mathbf{x})$$

for all  $\mathbf{x} \in \mathbb{R}^3$ ,  $\varepsilon \in (0, 1)$  and  $1 \leq j \leq n$ . Thus we have

$$\underline{v}(\mathbf{x} + \varepsilon \mathbf{t}) \leq \underline{v}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3, \varepsilon \in (0, 1).$$

Using the comparison principle, we have

$$w(\mathbf{x}, t; \underline{v}(\cdot + \varepsilon \mathbf{t})) \leq w(\mathbf{x}, t; \underline{v}).$$

By Theorem 3.2 we have

$$V = \lim_{t \rightarrow +\infty} w(\mathbf{x}, t; \underline{v}) \quad \text{in } L^\infty(\mathbb{R}^3).$$

Sending  $t \rightarrow +\infty$ , we have

$$V(\mathbf{x} + \varepsilon \mathbf{t}) \leq V(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3, \varepsilon \in (0, 1)$$

and thus

$$\frac{\partial V}{\partial \mathbf{t}} = \lim_{\varepsilon \rightarrow +0} \frac{V(\mathbf{x} + \varepsilon \mathbf{t}) - V(\mathbf{x})}{\varepsilon} \leq 0.$$

Applying the strong maximum principle to

$$\left( -\Delta - c \frac{\partial}{\partial z} - f'(V) \right) \frac{\partial V}{\partial \mathbf{t}} = 0 \quad \text{in } \mathbb{R}^3,$$

we obtain

$$\frac{\partial V}{\partial \mathbf{t}} < 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

This completes the proof.  $\square$

We now consider the case where  $h$  is symmetric with respect to the  $x$ -axis and the  $y$ -axis, respectively.

**Lemma 3.5.** *Let  $V$  be as in Theorem 3.2. Assume*

$$h(x, y) = h(|x|, |y|) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

*Then one has*

$$V(x, y, z) = V(|x|, |y|, z) \quad \text{for all } (x, y, z) \in \mathbb{R}^3$$

*and*

$$\begin{aligned} V_x(x, y, z) &> 0 \quad \text{if } x > 0, (y, z) \in \mathbb{R}^2, \\ V_x(0, y, z) &= 0 \quad \text{if } (y, z) \in \mathbb{R}^2, \\ V_y(x, y, z) &> 0 \quad \text{if } y > 0, (x, z) \in \mathbb{R}^2, \\ V_y(x, 0, z) &= 0 \quad \text{if } (x, z) \in \mathbb{R}^2. \end{aligned}$$

*Proof.* First  $V(x, y, z) = V(|x|, |y|, z)$  follows from the uniqueness in Theorem 3.2. We have

$$h(x, y) = \tau \max \{x \cos \theta_j + y \sin \theta_j - c_j \mid 1 \leq j \leq n, \cos \theta_j \geq 0\}$$

in  $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$ . Thus we get

$$\begin{aligned} h_x(x, y) &\geq 0 & \text{in } \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}, \\ h_x(x, y) &= 0 & \text{on } \{(x, y) \in \mathbb{R}^2 \mid x = 0\}. \end{aligned}$$

Consequently

$$(\underline{v})_x = -\frac{k}{c} \Phi' \left( \frac{k}{c} (z - h(x, y)) \right) h_x(x, y)$$

satisfies

$$\begin{aligned} (\underline{v})_x(x, y, z) &\geq 0 & \text{in } \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0\}, \\ (\underline{v})_x(x, y, z) &= 0 & \text{on } \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}. \end{aligned}$$

For every  $t \geq 0$ ,  $w(\mathbf{x}, t; \underline{v})$  is symmetric with respect to  $\{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$  and satisfies

$$\begin{aligned} \left( \frac{\partial}{\partial t} - \Delta - c \frac{\partial}{\partial z} - f'(w(\mathbf{x}, t; \underline{v})) \right) w_x &= 0 & \text{in } \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}, t > 0, \\ w_x &= 0 & \text{on } \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}, t > 0, \\ w_x(\mathbf{x}, 0; \underline{v}) &= (\underline{v})_x(\mathbf{x}) & \text{in } \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}. \end{aligned}$$

The comparison principle gives

$$w_x(\mathbf{x}, t; \underline{v}) \geq 0 \quad \text{in } \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}, t > 0.$$

For any  $\varepsilon > 0$  we have

$$w(x, y, z, t; \underline{v}) \leq w(x + \varepsilon, y, z, t; \underline{v}) \quad \text{in } \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}, t > 0.$$

Sending  $t \rightarrow +\infty$ , we have

$$V(x, y, z) \leq V(x + \varepsilon, y, z) \quad \text{for all } x > 0, (y, z) \in \mathbb{R}^2.$$

Sending  $\varepsilon \rightarrow 0$ , we have  $V_x \geq 0$  if  $x > 0$ . Then the strong maximum principle gives

$$\begin{aligned} V_x(\mathbf{x}) &> 0 & \text{if } x > 0, \\ V_x(\mathbf{x}) &= 0 & \text{if } x = 0. \end{aligned}$$

Similarly we have

$$\begin{aligned} V_y(\mathbf{x}) &> 0 & \text{if } y > 0, \\ V_y(\mathbf{x}) &= 0 & \text{if } y = 0. \end{aligned}$$

This completes the proof.  $\square$

**4. Uniform estimates on widths of generalized pyramidal traveling front solutions.** In this section we deal with generalized pyramidal traveling front solutions in Theorem 3.2, and give estimates of widths of their internal transition layers uniformly in  $n$ , where  $n$  is the number of lateral faces. For this purpose we use expanding balls, that is, developing fronts with spherical symmetry, and give upper and lower estimates of  $V$  by the comparison principles. Hereafter we assume (A1), (A2) and (A3). First we give an upper bound of  $V$ . Let  $\psi(0) = \alpha$  in (A3). Let  $\mathbf{x}_0 = (x_0, y_0, z_0)$  satisfy  $V(\mathbf{x}_0) = \alpha$ . We set

$$\Omega_0 \stackrel{\text{def}}{=} \left\{ (x, y, z) \mid z - z_0 \geq \tau \sqrt{(x - x_0)^2 + (y - y_0)^2} \right\}.$$

From Theorem 3.2 and Lemma 3.4 we have

$$\begin{aligned} -\Delta V - f(V) &= c(V)_z < 0 && \text{in } \mathbb{R}^3, \\ V(\mathbf{x}) &\leq \alpha && \text{for all } \mathbf{x} \in \Omega_0. \end{aligned}$$

Especially we have

$$\begin{aligned} -\Delta V - \tilde{f}(V) &< 0 && \text{in } \Omega_0, \\ V(\mathbf{x}) &\leq \alpha && \text{for all } \mathbf{x} \in \Omega_0. \end{aligned}$$

We consider the following boundary value problem:

$$\begin{aligned} \hat{w}_t - \Delta \hat{w} - \tilde{f}(\hat{w}) &= 0 && \mathbf{x} \in \Omega_0, t > 0, \\ \hat{w}(\mathbf{x}, t) &= \alpha && \mathbf{x} \in \partial\Omega_0, t > 0, \\ \hat{w}(\mathbf{x}, 0) &= V(\mathbf{x}) && \mathbf{x} \in \Omega_0. \end{aligned} \quad (23)$$

Then the maximum principle gives

$$V(\mathbf{x}) \leq \hat{w}(\mathbf{x}, t) \quad \text{for all } \mathbf{x} \in \Omega_0, t > 0.$$

Now we construct a supersolution that gives an upper estimate on  $V$ . For given  $Q > 0$ ,  $\delta > 0$  and  $D > 0$ , let  $q(t)$  be given by

$$\begin{aligned} q'(t) &= s_* - \frac{2}{q(t)} - D \left( \frac{1}{q(t)^2} + \delta e^{-\beta t} \right) && \text{for } t > 0, \\ q(0) &= Q. \end{aligned} \quad (24)$$

Taking  $Q > 0$  larger and taking  $\delta > 0$  smaller if necessary, we can assume

$$\begin{aligned} \frac{2}{Q} + D \left( \frac{1}{Q^2} + \delta \right) &< \frac{s_*}{2}, \\ Q + \frac{s_*}{2}t &\leq q(t) \leq Q + s_*t && \text{for all } t \geq 0. \end{aligned}$$

Then we have  $q'(t) > 0$  for all  $t > 0$ . We fix a constant  $\lambda_1 > 0$  to satisfy

$$\sup_{\mu \in \mathbb{R}} e^{\lambda_1 |\mu|} |\psi'(\mu)| < \infty,$$

and we put

$$\bar{w}(r, t) \stackrel{\text{def}}{=} \psi(r - q(t)) + \exp \left( -r - \frac{\lambda_1}{2} q(t) \right) + \frac{1}{q(t)^2} + \delta e^{-\beta t}. \quad (25)$$

**Lemma 4.1** (Yagisita [42, Proposition 3.1]). *For positive constants  $L$  and  $D$  depending only on  $f$  the following holds true. If  $Q > 0$  is large enough and  $\delta > 0$  is small enough,  $\bar{w}(r, t)$  given by (25) satisfies*

$$\begin{aligned} \bar{w}_t &\geq \bar{w}_{rr} + \frac{2}{r}\bar{w}_r + \tilde{f}(\bar{w}) && \text{for } r > 0, t > 0, \\ \bar{w}_r(0, t) &< 0 && \text{for } 0 \leq r \leq L, t > 0. \end{aligned}$$

*Proof.* The same proof in [42, Propositions 3.1] can be carried out under the condition (A1), (A2) and (A3). If  $L > 0$  is small enough, we have

$$\begin{aligned} -\bar{w}_r(r, t) &= -\psi'(r - q(t)) + \exp\left(-r - \frac{\lambda_1}{2}q(t)\right) \\ &\geq \exp\left(-r - \frac{\lambda_1}{2}q(t)\right) \left[1 - \left(\sup_{\mu \in \mathbb{R}} e^{\lambda_1|\mu|} |\psi'(\mu)|\right) \exp\left(\frac{\lambda_1}{2}q(t) - \lambda_1|q(t) - r| + r\right)\right] \\ &> 0, \end{aligned}$$

for all  $0 \leq r \leq L$  and  $t > 0$ , since  $Q$  is large enough. This gives the last inequality.  $\square$

If

$$u_0(\mathbf{x}) \leq \bar{w}(|\mathbf{x}|, 0) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3,$$

then the maximum principle [31, Theorem 10, Chap. 3, Sec. 6] gives

$$w(\mathbf{x}, t; u_0) \leq \bar{w}(|\mathbf{x}|, t) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3, t > 0.$$

Now we give an upper bound of a generalized pyramidal traveling front solution  $V$  as follows.

**Lemma 4.2** (Upper estimate of  $V$ ). *Let  $V$  be as in Theorem 3.2. Let  $\mathbf{x}_0 = (x_0, y_0, z_0)$  satisfy  $V(\mathbf{x}_0) = \alpha$ . Then one has*

$$\begin{aligned} -1 &< V(x_0, y_0, z_0 + z) \\ &\leq \psi\left(-\frac{k}{c}z\right) + \exp\left(-\frac{\lambda_1 k}{2c}z\right) + \frac{c^2}{k^2 z^2} + \delta \exp\left(\frac{\beta Q}{s_*}\right) \exp\left(-\frac{\beta k}{cs_*}z\right) \end{aligned}$$

for all

$$z \geq \frac{3cQ}{k}.$$

*Epecially the convergence  $\lim_{z \rightarrow +\infty} V(x_0, y_0, z_0 + z) = -1$  is uniform in  $n$  and  $h$  and  $\mathbf{x}_0$ .*

*Proof.* It suffices to prove the lemma for  $\mathbf{x}_0 = (0, 0, z_0)$  and  $V(0, 0, z_0) = \alpha$ . We take any  $T > 0$  with  $q(T) > 3Q$ . For every

$$z \geq z_0 + \frac{c}{k}q(T)$$

we have  $\overline{B((0, 0, z); q(T))} \subset \Omega_0$ . We set

$$z^{(0)} \stackrel{\text{def}}{=} z_0 + \frac{c}{k}q(T)$$

and

$$z^{(j)} \stackrel{\text{def}}{=} z^{(j-1)} + 2Q \quad \text{for all } j \in \mathbb{N}.$$

We define  $\mathbf{x}^{(j)} \stackrel{\text{def}}{=} (0, 0, z^{(j)})$  for all  $j \in \mathbb{N}$ .

Assume that, for some  $j \in \mathbb{N}$ , one has

$$V(\mathbf{x}) \leq \psi(|\mathbf{x} - \mathbf{x}^{(j)}| - Q) + \exp\left(-|\mathbf{x} - \mathbf{x}^{(j)}| - \frac{\lambda_1}{2}Q\right) + \frac{1}{Q^2} + \delta \quad (26)$$

for all  $\mathbf{x} \in \overline{B(\mathbf{x}^{(j)}; Q)}$ . From Theorem 3.2 this inequality holds true if  $j$  is large enough.

Using  $\psi(0) = \alpha$ , we have

$$V(\mathbf{x}) \leq \alpha \leq \psi(|\mathbf{x} - \mathbf{x}^{(j)}| - Q) + \exp\left(-|\mathbf{x} - \mathbf{x}^{(j)}| - \frac{\lambda_1}{2}Q\right) + \frac{1}{Q^2} + \delta$$

for all  $\mathbf{x} \in \Omega_0 \setminus \overline{B(\mathbf{x}^{(j)}; Q)}$ . Thus we have

$$V(\mathbf{x}) \leq \psi(|\mathbf{x} - \mathbf{x}^{(j)}| - Q) + \exp\left(-|\mathbf{x} - \mathbf{x}^{(j)}| - \frac{\lambda_1}{2}Q\right) + \frac{1}{Q^2} + \delta \quad \text{for all } \mathbf{x} \in \Omega_0.$$

We have

$$|\mathbf{x} - \mathbf{x}^{(j)}| \geq q(T) \geq q(t) \quad \text{for } \mathbf{x} \in \partial\Omega_0, t \in [0, T].$$

Now

$$\overline{w}(|\mathbf{x} - \mathbf{x}^{(j)}|, t) = \psi(|\mathbf{x} - \mathbf{x}^{(j)}| - q(t)) + \exp\left(-|\mathbf{x} - \mathbf{x}^{(j)}| - \frac{\lambda_1}{2}q(t)\right) + \frac{1}{q(t)^2} + \delta e^{-\beta t}$$

satisfies

$$\overline{w}(|\mathbf{x} - \mathbf{x}^{(j)}|, t) \geq \alpha \quad \text{for all } \mathbf{x} \in \partial\Omega_0, 0 \leq t \leq T.$$

By the comparison principle we obtain

$$\widehat{w}(\mathbf{x}, t) \leq \overline{w}(|\mathbf{x} - \mathbf{x}^{(j)}|, t) \quad \text{for } \mathbf{x} \in \Omega_0, 0 \leq t \leq T.$$

Taking  $t = T$ , we obtain

$$V(\mathbf{x}) \leq \psi(|\mathbf{x} - \mathbf{x}^{(j)}| - q(T)) + \exp\left(-|\mathbf{x} - \mathbf{x}^{(j)}| - \frac{\lambda_1}{2}q(T)\right) + \frac{1}{q(T)^2} + \delta e^{-\beta T} \quad (27)$$

for all  $\mathbf{x} \in \Omega_0$ .

For every  $\mathbf{x} \in \overline{B(\mathbf{x}^{(j-1)}; Q)}$  we have

$$0 \leq |\mathbf{x} - \mathbf{x}^{(j)}| - |\mathbf{x} - \mathbf{x}^{(j-1)}| \leq 2Q.$$

Combining this inequality and  $q(T) > 3Q$ , we get

$$|\mathbf{x} - \mathbf{x}^{(j)}| - q(T) \leq |\mathbf{x} - \mathbf{x}^{(j-1)}| - Q.$$

Thus we obtain

$$V(\mathbf{x}) \leq \psi(|\mathbf{x} - \mathbf{x}^{(j-1)}| - Q) + \exp\left(-|\mathbf{x} - \mathbf{x}^{(j-1)}| - \frac{\lambda_1}{2}Q\right) + \frac{1}{Q^2} + \delta$$

for all  $\mathbf{x} \in \overline{B(\mathbf{x}^{(j-1)}; Q)}$ . Now we see that (26) is valid for  $j - 1$ . By iteration we can see that (26) is valid for  $j = 0$  and that (27) is also valid for  $j = 0$ . Substituting  $\mathbf{x} = \mathbf{x}^{(0)}$  in (27) for  $j = 0$ , we obtain

$$V\left(0, 0, z_0 + \frac{c}{k}q(T)\right) \leq \psi(-q(T)) + \exp\left(-\frac{\lambda_1 q(T)}{2}\right) + \frac{1}{q(T)^2} + \delta e^{-\beta T}$$

for any  $T > 0$  with  $q(T) > 3Q$ . Now we complete the proof by putting  $q(T) = kz/c$  and using  $q(T) \leq Q + s_*T$ .  $\square$

Next we give lower estimates on  $V$  uniformly in  $n$ . Let  $V$  be as in Theorem 3.2 and let  $\mathbf{x}_0 = (x_0, y_0, z_0)$  satisfy  $V(x_0, y_0, z_0) = \alpha$ . We put

$$\mathcal{D}_1 \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid z < z_0 - \tau \sqrt{(x - x_0)^2 + (y - y_0)^2}\},$$

and have

$$\alpha \leq V(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{D}_1.$$

We denote  $w(\mathbf{x}, t; \alpha)$  simply by  $w(t; \alpha)$ , since it depends only on  $t$ . We have

$$\begin{aligned} w_t(t; \alpha) &= f(w(t; \alpha)) & t > 0, \\ w(0; \alpha) &= \alpha. \end{aligned}$$

Now  $w(t; \alpha)$  is strictly monotone increasing in  $t > 0$  with  $w(+\infty; \alpha) = 1$ . We put

$$v_1(\mathbf{x}, t) \stackrel{\text{def}}{=} w(t; \alpha) - V(\mathbf{x}) = w(t; \alpha) - w(\mathbf{x}, t; V).$$

Then we obtain

$$\left( \frac{\partial}{\partial t} - \Delta - c \frac{\partial}{\partial z} - \int_0^1 f'(\theta w(t; \alpha) + (1 - \theta)w(\mathbf{x}, t; V)) d\theta \right) v_1 \leq 0$$

for all  $\mathbf{x} \in \mathbb{R}^3$ ,  $t > 0$ , and

$$v_1(\mathbf{x}, 0) \leq \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{D}_1, \\ 2 & \text{if } \mathbf{x} \in \mathbb{R}^3 \setminus \mathcal{D}_1. \end{cases}$$

We set  $\widehat{v}_1(\mathbf{x}, t)$  by

$$\begin{aligned} \left( \frac{\partial}{\partial t} - \Delta - c \frac{\partial}{\partial z} - M \right) \widehat{v}_1 &= 0 & \text{for all } \mathbf{x} \in \mathbb{R}^3, t > 0. \\ \widehat{v}_1(\mathbf{x}, 0) &= \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{D}_1, \\ 2 & \text{if } \mathbf{x} \in \mathbb{R}^3 \setminus \mathcal{D}_1, \end{cases} \end{aligned}$$

and have

$$v_1(\mathbf{x}, t) \leq \widehat{v}_1(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in \mathbb{R}^3, t > 0.$$

Now we have

$$\begin{aligned} &\widehat{v}_1(\mathbf{x}, t) \\ &= 2e^{Mt} \int_{\mathbb{R}^3 \setminus \mathcal{D}_1} \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp \left( -\frac{(x - x')^2 + (y - y')^2 + (z + ct - z')^2}{4t} \right) dx' dy' dz'. \end{aligned}$$

Here recall that

$$\operatorname{erfc} x \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad \text{for } x \in \mathbb{R}$$

satisfies

$$0 < \operatorname{erfc} x < \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-xt} dt = \frac{2}{\sqrt{\pi}} \frac{1}{x} e^{-x^2} \quad \text{for all } x > 0.$$

We define a positive constant as

$$b \stackrel{\text{def}}{=} \min \left\{ \frac{k}{2\sqrt{3}c^2}, \frac{\sqrt{5}}{4\sqrt{6}} \frac{k}{c\sqrt{M}} \right\} > 0.$$

**Lemma 4.3** (Lower estimate of  $V$ ). *Let  $V$  be as in Theorem 3.2. If  $V(x_0, y_0, z_0) = \alpha$ , one has*

$$\sup_{t>0} (w(t; \alpha) - \widehat{v}_1(\mathbf{x}, t)) \leq V(\mathbf{x}) < 1.$$

*Epecially one has*

$$w(bz; \alpha) - \frac{768\sqrt{3}c^3b^{\frac{3}{2}}}{k^3(\pi z)^{\frac{3}{2}}} \exp(-Mbz) \leq V(x_0, y_0, z_0 - z) < 1$$

*for all  $z > 0$ .*

*Proof.* The argument stated above gave the first inequality. It suffices to prove the second inequality. Since we have

$$\overline{B((x_0, y_0, z_0 - z); (k/c)z)} \subset \overline{\mathcal{D}_1},$$

we obtain

$$\begin{aligned} 0 < \widehat{v}_1(x_0, y_0, z_0 - z, t) &\leq 2e^{Mt} \int_{|x| \geq \frac{kz}{\sqrt{3c}}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} dx \int_{|y| \geq \frac{kz}{\sqrt{3c}}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} dy \\ &\quad \times \left( \int_{-\infty}^{ct - \frac{kz}{\sqrt{3c}}} + \int_{ct + \frac{kz}{\sqrt{3c}}}^{\infty} \right) \frac{1}{\sqrt{4\pi t}} e^{-\frac{\zeta^2}{4t}} d\zeta \\ &\leq 2e^{Mt} \left( \operatorname{erfc} \left( \frac{kz}{\sqrt{3c}\sqrt{4t}} \right) \right)^2 \operatorname{erfc} \left( \frac{\frac{kz}{\sqrt{3c}} - ct}{\sqrt{4t}} \right). \end{aligned}$$

Putting  $t = bz$ , we get

$$0 < \widehat{v}_1(x_0, y_0, z_0 - z, bz) \leq 2e^{Mbz} \left( \operatorname{erfc} \left( \frac{k\sqrt{z}}{2\sqrt{3c}\sqrt{b}} \right) \right)^2 \operatorname{erfc} \left( \frac{kz}{4\sqrt{3c}\sqrt{b}} \right),$$

which gives

$$0 < \widehat{v}_1(x_0, y_0, z_0 - z, bz) \leq \frac{768\sqrt{3}c^3b^{\frac{3}{2}}}{k^3(\pi z)^{\frac{3}{2}}} \exp \left( - \left( \frac{5k^2}{48c^2b} - Mb \right) z \right).$$

This completes the proof.  $\square$

By using

$$\lim_{z \rightarrow +\infty} \left( w(bz; \alpha) - \frac{768\sqrt{3}c^3b^{\frac{3}{2}}}{k^3(\pi z)^{\frac{3}{2}}} \exp(-Mbz) \right) = 1,$$

this lemma implies

$$\lim_{z \rightarrow +\infty} V(x_0, y_0, z_0 - z) = 1$$

uniformly in  $n$ ,  $h$  and  $\mathbf{x}_0$ .

Let  $V$  be as in Lemma 3.5 and let  $\mathbf{x}_0 = (x_0, x_0, z_0)$  satisfy  $V(\mathbf{x}_0) = \alpha$  with  $x_0 > 0$ . Then we have

$$V(-x_0, x_0, z_0) = V(x_0, -x_0, z_0) = V(-x_0, -x_0, z_0) = \alpha$$

and

$$\alpha \leq V(x, y, z) \quad \text{if } x \geq x_0 \text{ and } y \geq x_0 \text{ and } z \leq z_0$$

from Lemma 3.5 and Theorem 3.2. We put

$$\mathcal{D}_2 \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 \mid x > x_0 \text{ and } y > x_0 \text{ and } z < z_0\}.$$

We set

$$\widehat{v}_2(\mathbf{x}, t) \stackrel{\text{def}}{=} 2e^{Mt} \int_{\mathbb{R}^3 \setminus \mathcal{D}_2} \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp\left(-\frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{4t}\right) dx' dy' dz'.$$

**Lemma 4.4** (Lower estimate of  $V$ ). *Let  $V$  be as in Lemma 3.5 and  $\mathbf{x}_0 = (x_0, x_0, z_0)$  satisfy  $V(\mathbf{x}_0) = \alpha$  with  $x_0 > 0$ . Then one has*

$$\sup_{t>0} \left( w(t; \alpha) - \frac{1}{4} e^{Mt} \operatorname{erfc}\left(\frac{x}{\sqrt{4t}}\right) \operatorname{erfc}\left(\frac{y}{\sqrt{4t}}\right) \operatorname{erfc}\left(\frac{z}{\sqrt{4t}}\right) \right) \leq V(x_0 + x, x_0 + y, z_0 - z) < 1$$

for all  $x \geq 0$ ,  $y \geq 0$  and  $z \geq 0$ .

*Proof.* We have

$$0 < \widehat{v}_2(x, y, z, t) \leq \frac{1}{4} e^{Mt} \operatorname{erfc}\left(\frac{x-x_0}{\sqrt{4t}}\right) \operatorname{erfc}\left(\frac{y-x_0}{\sqrt{4t}}\right) \operatorname{erfc}\left(\frac{z_0-z}{\sqrt{4t}}\right).$$

Then we complete the proof by the argument stated above.  $\square$

Combining an upper estimate and a lower estimate, we obtain the following estimate on  $V$ .

**Proposition 1.** *Under assumptions (A1), (A2) and (A3), let  $V$  be as in Theorem 3.2. Let  $\mathbf{x}_0 = (x_0, y_0, z_0)$  satisfy  $V(\mathbf{x}_0) = \alpha$ . Then one has*

$$\begin{aligned} -1 &< V(x_0, y_0, z_0 + z) \\ &\leq \psi\left(-\frac{k}{c}z\right) + \exp\left(-\frac{\lambda_1 k}{2c}z\right) + \frac{c^2}{k^2 z^2} + \delta \exp\left(\frac{\beta Q}{s_*}\right) \exp\left(-\frac{\beta k}{cs_*}z\right) \end{aligned}$$

for all  $z \geq 3cQ/k$ . One has

$$w(bz; \alpha) - \frac{768\sqrt{3}c^3 b^{\frac{3}{2}}}{k^3(\pi z)^{\frac{3}{2}}} \exp(-Mbz) \leq V(x_0, y_0, z_0 - z) < 1$$

for all  $z > 0$ . Here constants  $Q > 0$  and  $\delta > 0$  are as in Lemma 4.1.

*Proof.* This proposition follows from Lemma 4.2 and Lemma 4.3.  $\square$

Let  $\mathbf{x}_0 = (x_0, y_0, z_0)$  satisfy  $V(\mathbf{x}_0) = \alpha$ . This proposition implies that the convergence

$$\lim_{z \rightarrow +\infty} V(x_0, y_0, z_0 + z) = -1, \quad \lim_{z \rightarrow +\infty} V(x_0, y_0, z_0 - z) = 1$$

is uniform in  $n$  and  $h$  and  $\mathbf{x}_0$ .

**5. The limits of pyramidal traveling fronts as the number of lateral faces goes to infinity.** Traveling fronts with cylindrical symmetry are studied in [13] and [14] for bistable nonlinearity.

Here we construct cylindrically symmetric traveling front solutions by another method under the assumptions (A1), (A2) and (A3). We consider pyramids whose cross sections are regular  $2^n$ -polygons. In the limit of  $n \rightarrow \infty$  we study traveling front solutions with cylindrical symmetry. For every  $n \geq 2$  we set

$$\theta_j = \frac{2\pi j}{2^n} \quad (0 \leq j \leq 2^n - 1)$$

and put

$$p^{(n)}(x, y) = \tau \max_{0 \leq j \leq 2^n - 1} \left( x \cos \frac{2\pi j}{2^n} + y \sin \frac{2\pi j}{2^n} \right). \quad (28)$$

Let  $V^{(n)}$  be the pyramidal traveling front solution associated with  $z = p^{(n)}(x, y)$  as in Theorem 2.3. We define  $z_0^{(n)} \in (-\infty, \infty)$  by

$$V^{(n)}(0, 0, z_0^{(n)}) = 0,$$

and we set

$$U^{(n)}(x, y, z) \stackrel{\text{def}}{=} V^{(n)}(x, y, z + z_0^{(n)}) \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

Then we have

$$\begin{aligned} U^{(n)}(0, 0, 0) &= 0, \\ -1 &< U^{(n)}(\mathbf{x}) < 1 \quad \text{for all } \mathbf{x} \in \mathbb{R}^3. \end{aligned}$$

For any compact set  $K_1 \subset \mathbb{R}^3$  and  $p_1 > 3$ ,  $(U^{(n)})_{n \in \mathbb{N}}$  is bounded in  $L^{p_1}(K_1)$ . The Schauder interior estimates implies that  $(U^{(n)})_{n \in \mathbb{N}}$  is bounded in  $W^{2,p_1}(K_1)$ . The compact imbedding  $W^{2,p_1}(K_1) \subset C^{1,\alpha_1}(\overline{K_1})$  for  $\alpha_1 < 1 - 3/p_1$  implies that  $(U^{(n)})_{n \in \mathbb{N}}$  has a subsequence that converges in  $C^{1,\alpha_1}(\overline{K_1})$ . Let  $U$  be the limit of the subsequence of  $(U^{(n)})_{n \in \mathbb{N}}$ . Then using the Schauder interior again, a subsequence of  $(U^{(n)})_{n \in \mathbb{N}}$  converges to  $U$  in  $C^2(K_1)$ . By the diagonal argument we see that a subsequence of  $(U^{(n)})_{n \in \mathbb{N}}$  converges to  $U$  in  $C_{\text{loc}}^2(\mathbb{R}^3)$ . We have

$$U = \lim_{n \rightarrow \infty} U^{(n)} \quad \text{in } C_{\text{loc}}^2(\mathbb{R}^3). \quad (29)$$

Now  $V^{(n)}$  and  $U^{(n)}$  are symmetric with respect to a plane

$$\left\{ (x, y, z) \mid y \cos \frac{\pi j}{2^n} = x \sin \frac{\pi j}{2^n} \right\}$$

for each  $0 \leq j \leq 2^n - 1$ . Thus  $U$  has the same symmetry, which implies that  $U$  has cylindrical symmetry with respect to the  $z$ -axis, that is, a function of  $r = \sqrt{x^2 + y^2}$  and  $z$ . We write  $U = U(r, z)$  for  $r \geq 0$  and  $z \in \mathbb{R}$ . Then it satisfies

$$\begin{aligned} \left( -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} - c \frac{\partial}{\partial z} \right) U - f(U(r, z)) &= 0 \quad \text{for } r > 0, z \in \mathbb{R}, \\ U_r(0, z) &= 0 \quad \text{for } z \in \mathbb{R}, \\ U(0, 0) &= 0. \end{aligned}$$

**Lemma 5.1.** *Let  $U$  be as in (29). Let  $r_0 \geq 0$  and  $z_0 \in \mathbb{R}$  satisfy  $U(r_0, z_0) = \alpha$ . Then one has*

$$-1 < U(r_0, z_0 + z) \leq \psi\left(-\frac{k}{c}z\right) + \exp\left(-\frac{\lambda_1 k}{2c}z\right) + \frac{c^2}{k^2 z^2} + \delta \exp\left(\frac{\beta Q}{s_*}\right) \exp\left(-\frac{\beta k}{cs_*}z\right)$$

for all  $z \geq 3cQ/k$ . One has

$$w(bz; \alpha) - \frac{768\sqrt{3}c^3 b^{\frac{3}{2}}}{k^3(\pi z)^{\frac{3}{2}}} \exp(-Mbz) \leq U(r_0, z_0 - z) < 1$$

for all  $z > 0$ . Here constants  $Q > 0$  and  $\delta > 0$  are as in Lemma 4.1.

*Proof.* Since  $U^{(n)}$  satisfies Proposition 1, we have estimates on the limiting function  $U$  by sending  $n \rightarrow \infty$ . This completes the proof.  $\square$

This lemma implies that the convergence

$$\lim_{z \rightarrow +\infty} U(r_0, z_0 + z) = -1, \quad \lim_{z \rightarrow +\infty} V(r_0, z_0 - z) = 1$$

is uniform in  $(r_0, z_0)$ .

**Lemma 5.2.** *Let  $U$  be as in (29). Then one has*

$$\begin{aligned} -1 < U(r, z) < 1 & \quad \text{for all } r \geq 0, z \in \mathbb{R}, \\ U_r(r, z) > 0 & \quad \text{for all } r > 0, z \in \mathbb{R}, \\ U_z(r, z) < 0 & \quad \text{for all } r \geq 0, z \in \mathbb{R} \end{aligned}$$

and

$$\frac{\partial U}{\partial \mathbf{a}} < 0$$

for all  $\mathbf{a} \stackrel{\text{def}}{=} (a_1, 1)$  with  $|a_1| \leq \tau^{-1}$ .

*Proof.* Since  $U^{(n)}$  satisfies Lemma 3.5 and Theorem 2.3, we have  $U_r \geq 0$  and  $U_z \leq 0$  by sending  $n \rightarrow \infty$ . Since  $U$  is not identically a constant by Lemma 5.1, we obtain

$$\begin{aligned} U_r(r, z) > 0 & \quad \text{for all } r > 0, z \in \mathbb{R}, \\ U_z(r, z) < 0 & \quad \text{for all } r \geq 0, z \in \mathbb{R} \end{aligned}$$

by the strong maximum principle. Finally we have

$$\frac{\partial U}{\partial \mathbf{a}} < 0$$

from Lemma 3.4.  $\square$

Then Lemma 5.1 gives

$$U(0, -\infty) = 1, \quad U(0, \infty) = -1.$$

For any  $r \geq 0$  we define  $\phi(r) \in \mathbb{R}$  by  $U(r, \phi(r)) = 0$ . Then we have

$$\phi'(r) = \frac{U_r(r, \phi(r))}{-U_z(r, \phi(r))} > 0,$$

and

$$\begin{aligned} 0 < \phi'(r) \leq \tau & \quad \text{for all } r > 0, \\ \phi(0) = 0, \quad \phi'(0) = 0. \end{aligned}$$

The Schauder estimate [11, Theorem 9.11] gives

$$\sup_{\mathbb{R}^3} |\nabla U| < \infty, \quad \|U\|_{W^{2,\infty}(\mathbb{R}^3)} < \infty.$$

**Lemma 5.3** (Estimate on the cross section of  $U$ ). *One has*

$$0 < \liminf_{r \rightarrow \infty} \phi'(r).$$

For any given  $\varepsilon > 0$ , there exists  $R_\varepsilon > 0$  such that, if  $r \geq 0$  satisfies  $-1 + \varepsilon < U(r, \phi(r_0)) < 1 - \varepsilon$  for any  $r_0 > 0$ , one has  $|r - r_0| \leq R_\varepsilon$ . Here  $R_\varepsilon$  is independent of  $r_0$ .

*Proof.* It suffices to prove  $\liminf_{r \rightarrow \infty} \phi'(r) > 0$ . Then the latter statement follows from Lemma 5.1 and Lemma 5.2.

Assume the contrary. Then there exists  $(r_i)_{i \in \mathbb{N}} \subset (0, \infty)$  with  $\lim_{i \rightarrow \infty} r_i = \infty$  and  $\lim_{i \rightarrow \infty} \phi'(r_i) = 0$ . Then we have  $\lim_{i \rightarrow \infty} U_r(r_i, \phi(r_i)) = 0$ . For any given  $a > 0$  we take  $i$  with  $r_i > 2a$  and have

$$\begin{aligned} \left( -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} - c \frac{\partial}{\partial z} \right) U_r &= 0 \quad \text{in } B((r_i, \phi(r_i)); a), \\ U_r &> 0 \quad \text{in } \overline{B((r_i, \phi(r_i)); a)}. \end{aligned}$$

Then the Harnack inequality [11, Corollary 9.25] gives

$$0 \leq \sup_{B((r_i, \phi(r_i)); a)} U_r \leq C U_r(r_i, \phi(r_i)),$$

where a constant  $C$  is independent of  $i$ . Taking a subsequence of  $r_i$  and call the subsequence  $(r_i)$  if necessary, we have  $\psi_0(z) \stackrel{\text{def}}{=} \lim_{i \rightarrow \infty} U(r + r_i, z + \phi(r_i))$ . We note that the right-hand side is independent of  $r$ .

We have

$$\begin{aligned} \int_{\mathbb{R}^3} (-U(r + r_i, z + \phi(r_i)) \Delta \varphi + c U(r + r_i, z + \phi(r_i)) \varphi_z \\ - f(U(r + r_i, z + \phi(r_i)) \varphi) = 0 \end{aligned}$$

for all  $\varphi \in C_0^\infty(\mathbb{R}^3)$ . Sending  $i \rightarrow \infty$ , we have

$$-\psi_0 \Delta \varphi + c \psi_0 \varphi_z - f(\psi_0) \varphi = 0 \quad \text{in } \mathbb{R}^3.$$

Combining this equality and Lemma 5.1, we have

$$\begin{aligned} (-\partial_z^2 - c \partial_z) \psi_0 - f(\psi_0(z)) &= 0 \quad z \in \mathbb{R}, \\ \psi_0(-\infty) &= 1, \quad \psi_0(+\infty) = -1, \end{aligned}$$

which contradicts the uniqueness of a one-dimensional traveling front solution in [6] or [9], because we have  $c > k$  and we already have a one-dimensional traveling front solution  $(k, \Phi)$ . This completes the proof.  $\square$

**Lemma 5.4.** *One has*

$$\limsup_{r \rightarrow \infty} \frac{\phi(r)}{r} = \tau.$$

*Proof.* For any  $r > 1$  we set

$$\tilde{v}(x, z; r) \stackrel{\text{def}}{=} U(r + x, \phi(r) + z)$$

and have

$$\left( -\frac{\partial^2}{\partial x^2} - \frac{1}{r+x} \frac{\partial}{\partial x} - \frac{\partial^2}{\partial z^2} - c \frac{\partial}{\partial z} \right) \tilde{v} - f(\tilde{v}(x, z; r)) = 0 \quad -r < x < \infty, z \in \mathbb{R},$$

$$\tilde{v}(0, 0; r) = 0.$$

Sending  $r \rightarrow +\infty$  and using Lemma 5.1, we have a limit function

$$\tilde{v}_0(x, z) \stackrel{\text{def}}{=} \lim_{r \rightarrow \infty} \tilde{v}(x, z; r) \quad \text{for all } (x, z) \in \mathbb{R}^2$$

with

$$\left( -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} - c \frac{\partial}{\partial z} \right) \tilde{v}_0 - f(\tilde{v}_0(x, z)) = 0 \quad (x, z) \in \mathbb{R}^2,$$

$$\tilde{v}_0(0, 0) = 0,$$

and

$$\lim_{z \rightarrow +\infty} \tilde{v}_0(x, z) = -1, \quad \lim_{z \rightarrow -\infty} \tilde{v}_0(x, z) = +1 \quad \text{for any fixed } x \in \mathbb{R}. \quad (30)$$

We have

$$0 \leq \frac{\phi(r)}{r} = \frac{1}{r} \int_0^r \phi'(s) ds \leq \tau.$$

We put

$$\tau' \stackrel{\text{def}}{=} \limsup_{r \rightarrow \infty} \frac{\phi(r)}{r} \in (0, \tau].$$

We obtain a contradiction by assuming  $\tau' < \tau$ . We set  $c' \in (0, c)$  by

$$\frac{\sqrt{(c')^2 - k^2}}{k} = \tau'.$$

We have

$$\phi(|x|) \leq c'|x| \quad \text{for all } r \geq 0.$$

Since the absolute value of the gradient of  $\{(x, z) \mid \tilde{v}_0(x, z) = 0\}$  is no more than  $\tau'$ , we obtain

$$\tilde{v}_0(x, z) < v_*(x, z - z_2; c') + \delta_* \quad \text{for all } (x, z) \in \mathbb{R}^2$$

by taking  $z_2 > 0$  large enough. Using Theorem 2.2, we have

$$\tilde{v}_0(x, z) \leq v_*(x, z + (c - c')t - z_2 - \sigma\delta_*(1 - e^{-\beta t})) + \delta_* e^{-\beta t} \quad \text{for all } (x, z) \in \mathbb{R}^2, t > 0.$$

This contradicts  $\tilde{v}_0(0, 0) = 0$  by sending  $t \rightarrow +\infty$ .  $\square$

We have  $0 \leq \phi' \leq \tau$  and

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \int_0^r \phi'(s) ds = \tau. \quad (31)$$

If we have  $\varepsilon_0 > 0$ ,  $\delta_0 > 0$  and  $(\sigma_i)_{i \geq 2} \subset (0, \infty)$  with

$$\begin{aligned} 0 < \sigma_2 < \sigma_3 < \cdots < \sigma_i < \cdots \rightarrow \infty, \\ \sup_{i \geq 2} |\sigma_{i+1} - \sigma_i| < \infty \\ \sup_{s \in [\sigma, \sigma_i + \delta_0]} \phi'(s) \leq \tau - \varepsilon_0 \quad \text{for all } i \geq 2, \end{aligned}$$

then we have  $\limsup_{r \rightarrow \infty} (1/r) \int_0^r \phi'(s) ds < \tau$ , which contradicts (31). Then there exists a sequence  $(s_i) \subset (0, \infty)$  with

$$\lim_{i \rightarrow \infty} s_i = \infty, \quad \lim_{i \rightarrow \infty} \phi'(s_i) = \tau, \quad \sup_i |s_{i+1} - s_i| < \infty. \quad (32)$$

Setting

$$\mathbf{a}_0 \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + \tau^2}} \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \frac{k}{c} \begin{pmatrix} 1 \\ \tau \end{pmatrix},$$

we obtain

$$\lim_{i \rightarrow \infty} \frac{\partial U}{\partial \mathbf{a}_0} \Big|_{(s_i, \phi(s_i))} = 0.$$

From Lemma 5.2 we have

$$-\frac{\partial U}{\partial \mathbf{a}_0} > 0 \quad \text{for all } r \geq 0, z \in \mathbb{R}.$$

Then applying the Harnack inequality to  $-\partial U / \partial \mathbf{a}_0$ , we obtain

$$\lim_{i \rightarrow \infty} \sup_{B((s_i, \phi(s_i)); \ell)} \left( -\frac{\partial U}{\partial \mathbf{a}_0} \right) = 0,$$

and thus

$$\lim_{i \rightarrow \infty} \sup_{B((s_i, \phi(s_i)); \ell)} \left| \frac{\partial U}{\partial \mathbf{a}_0} \right| = 0$$

for any given  $\ell > 0$ . Using  $\sup_i |s_{i+1} - s_i| < \infty$  and taking  $\ell > 0$  large enough, we get

$$\lim_{r \rightarrow \infty} \sup_{B((r, \phi(r)); \ell)} \left| \frac{\partial U}{\partial \mathbf{a}_0} \right| = 0. \quad (33)$$

Then the Schauder estimate gives

$$\lim_{r \rightarrow \infty} \left\| \frac{\partial U}{\partial \mathbf{a}_0} \right\|_{C^1(B((r, \phi(r)); \ell))} = 0. \quad (34)$$

**Lemma 5.5.** *One has  $\lim_{r \rightarrow \infty} \phi'(r) = \tau$  and*

$$\lim_{r \rightarrow \infty} U(r + x, \phi(r) + z) = \Phi \left( \frac{k}{c} (z - \tau x) \right) \quad \text{in } C_{\text{loc}}^2(\mathbb{R}^2).$$

*Proof.* For  $r_1 > 0$  we set  $(\xi, \eta) \in \mathbb{R}^2$  by

$$\begin{pmatrix} r - r_1 \\ z - \phi(r_1) \end{pmatrix} = \xi \begin{pmatrix} 1 \\ \frac{\tau}{\sqrt{1 + \tau^2}} \end{pmatrix} + \eta \begin{pmatrix} -\frac{\tau}{\sqrt{1 + \tau^2}} \\ 1 \end{pmatrix}.$$

We put

$$\tilde{w}(\xi, \eta) \stackrel{\text{def}}{=} \lim_{r_1 \rightarrow \infty} U \left( r_1 + \frac{\xi - \tau\eta}{\sqrt{1 + \tau^2}}, \phi(r_1) + \frac{\tau\xi + \eta}{\sqrt{1 + \tau^2}} \right),$$

and have

$$-\tilde{w}_{\eta\eta} - k\tilde{w}_\eta - f(\tilde{w}(\xi, \eta)) - \tilde{w}_{\xi\xi} - k\tau\tilde{w}_\xi = 0$$

for  $(\xi, \eta) \in \mathbb{R}^2$ . From (33) we have  $\tilde{w}_\xi \equiv 0$ . This implies that  $\tilde{w}$  is a function of  $\eta$  and is independent of  $\xi$ . We have

$$\begin{aligned} \left( -\frac{\partial^2}{\partial \eta^2} - k\frac{\partial}{\partial \eta} \right) \tilde{w} - f(\tilde{w}(\eta)) &= 0, \\ \tilde{w}(-\infty) &= 1, \quad \tilde{w}(0) = 0, \quad \tilde{w}(\infty) = -1 \end{aligned}$$

by using Lemma 5.1. The uniqueness of one-dimensional traveling front gives  $\tilde{w} \equiv \Phi$ . By using the Schauder interior estimate, we obtain the convergence in  $C_{\text{loc}}^2(\mathbb{R}^3)$ . We get  $\lim_{r \rightarrow \infty} \phi'(r) = \tau$  from (34). This completes the proof.  $\square$

If  $f$  is of a bistable type, a cylindrically symmetric traveling front  $U$  coincides to that of [13] and [14] due to [14, Theorem 3]. For the uniqueness of  $U$  for a bistable nonlinear term  $f$ , see [14]. The uniqueness of  $U$  for a multistable nonlinear term  $f$  is yet to be studied.

## 6. Limits of generalized pyramidal traveling fronts. We set

$$C^\infty(S^1) \stackrel{\text{def}}{=} \{g \in C^\infty(\mathbb{R}) \mid g(\theta + 2\pi) = g(\theta) \text{ for all } \theta \in \mathbb{R}\}.$$

We identify  $S^1$  with  $\mathbb{R}/2\pi\mathbb{Z}$ . Let  $g \in C^\infty(S^1)$  be any given function with

$$\min_{0 \leq \theta \leq 2\pi} g(\theta) = 0.$$

Let  $r_* \geq 1$  be large enough to satisfy

$$(r_*)^2 + r_* \min_{0 \leq \theta \leq 2\pi} (2g(\theta) - g''(\theta)) + \min_{0 \leq \theta \leq 2\pi} (g(\theta)^2 + 2g'(\theta)^2 - g(\theta)g''(\theta)) > 0. \quad (35)$$

We put

$$R(\theta) \stackrel{\text{def}}{=} r_* + g(\theta) \quad \text{for } 0 \leq \theta \leq 2\pi. \quad (36)$$

Then  $\mathcal{C} \stackrel{\text{def}}{=} \{(R(\theta) \cos \theta, R(\theta) \sin \theta) \mid 0 \leq \theta \leq 2\pi\}$  becomes a smooth convex closed curve with

$$R(0) = R(2\pi), \quad \min_{0 \leq \theta \leq 2\pi} R(\theta) = r_*, \quad 0 < \min_{0 \leq \theta \leq 2\pi} \kappa(\theta) \leq \max_{0 \leq \theta \leq 2\pi} \kappa(\theta) < +\infty. \quad (37)$$

Here  $\kappa(\theta)$  is the curvature of  $\mathcal{C}$  given by

$$\kappa(\theta) = \frac{R(\theta)^2 + 2R'(\theta)^2 - R(\theta)R''(\theta)}{(R(\theta)^2 + R'(\theta)^2)^{\frac{3}{2}}} > 0 \quad \text{for } \theta \in [0, 2\pi].$$

The boundary of

$$\mathcal{D} \stackrel{\text{def}}{=} \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2 \mid 0 \leq r < R(\theta), 0 \leq \theta \leq 2\pi\}$$

is  $\mathcal{C}$ . See Figure 4. We put

$$\kappa_{\min} = \min_{0 \leq \theta \leq 2\pi} \kappa(\theta), \quad \kappa_{\max} = \max_{0 \leq \theta \leq 2\pi} \kappa(\theta)$$

and have  $0 < \kappa_{\min} \leq \kappa_{\max} < \infty$ . We set

$$R_{\max} \stackrel{\text{def}}{=} \max_{0 \leq \theta \leq 2\pi} R(\theta) \in [1, \infty),$$

and have  $r_* \leq R_{\max} < \infty$ .

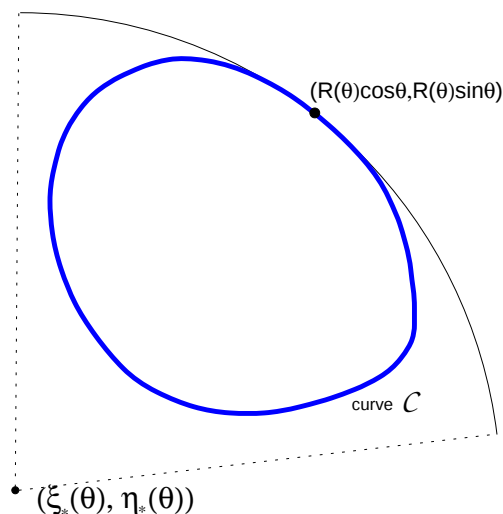


FIGURE 4. A smooth convex closed curve  $\mathcal{C}$ .

Let

$$R_* \in (\kappa_{\min}^{-1}, \infty)$$

be large enough such that, for each  $\theta \in [0, 2\pi)$ , one has a circle of radius  $R_*$  that circumscribes  $\mathcal{C}$  at  $(R(\theta) \cos \theta, R(\theta) \sin \theta)$ . Let  $(\xi_*(\theta), \eta_*(\theta))$  and  $\mathcal{B}(\theta)$  be the center and the interior of this circle for each  $\theta \in [0, 2\pi)$ . We have  $\overline{\mathcal{D}} \subset \overline{\mathcal{B}(\theta)}$  for all  $\theta \in [0, 2\pi)$ . We put

$$\begin{aligned} \mathcal{A}_0 &\stackrel{\text{def}}{=} \left\{ (\xi, \eta) \mid \overline{\mathcal{D}} \subset \overline{B((\xi, \eta); R_*)} \right\} \\ &= \left\{ (\xi, \eta) \in \mathbb{R}^2 \mid \sqrt{(x - \xi)^2 + (y - \eta)^2} \leq R_* \text{ if } (x, y) \in \mathcal{C} \right\}, \end{aligned} \quad (38)$$

and

$$\begin{aligned} \mathcal{A}_n &\stackrel{\text{def}}{=} \left\{ (\xi, \eta) \mid \overline{\mathcal{D}} \subset \{(x, y) \mid p^{(n)}(x - \xi, y - \eta) \leq \tau R_*\} \right\} \\ &= \left\{ (\xi, \eta) \in \mathbb{R}^2 \mid p^{(n)}(x - \xi, y - \eta) \leq \tau R_* \text{ if } (x, y) \in \mathcal{C} \right\} \end{aligned}$$

for every  $n \geq 2$ . We have  $(\xi_*(\theta), \eta_*(\theta)) \in \mathcal{A}_0$  for all  $\theta \in [0, 2\pi)$ . We note that  $\partial B(0; R_*)$  is the inscribed circle of a polygon  $\{(x, y) \in \mathbb{R}^2 \mid p^{(n)}(x, y) = \tau R_*\}$ .

Then we have

$$\lim_{n \rightarrow \infty} \text{dist}(\mathcal{A}_n, \mathcal{A}_0) = 0.$$

For any  $\varepsilon > 0$  we have

$$\text{dist}((\xi, \eta), \mathcal{C}) \leq R_* + \varepsilon \quad \text{for all } (\xi, \eta) \in \mathcal{A}_n,$$

that is,

$$\text{dist}(\mathcal{A}_n, \mathcal{C}) \leq R_* + \varepsilon$$

if  $n$  is large enough. Thus we get

$$\sqrt{\xi^2 + \eta^2} \leq R_{\max} + R_* + \varepsilon \quad \text{for all } (\xi, \eta) \in \mathcal{A}_n$$

if  $n$  is large enough. Now we study

$$h^{(n)}(x, y) \stackrel{\text{def}}{=} \sup_{(\xi, \eta) \in \mathcal{A}_n} p^{(n)}(x - \xi, y - \eta) - \tau R_*.$$

Since  $\partial B(0; R_*)$  is the inscribed circle of a polygon  $\{(x, y) \in \mathbb{R}^2 \mid p^{(n)}(x, y) = \tau R_*\}$ , we have

$$h^{(n)}(x, y) = \tau \max_{0 \leq j \leq 2^n - 1} \left( x \cos \frac{2\pi j}{2^n} + y \sin \frac{2\pi j}{2^n} - R \left( \frac{2\pi j}{2^n} \right) - R_* \right).$$

We have

$$p^{(n)}(x - \xi, y - \eta) - \tau R_* \leq p^{(n)}(x - R(\theta) \cos \theta, y - R(\theta) \sin \theta)$$

for all  $(\xi, \eta) \in \mathcal{A}_n$  and  $\theta \in [0, 2\pi]$ , that is,

$$\sup_{(\xi, \eta) \in \mathcal{A}_n} p^{(n)}(x - \xi, y - \eta) - \tau R_* \leq \inf_{0 \leq \theta \leq 2\pi} p^{(n)}(x - R(\theta) \cos \theta, y - R(\theta) \sin \theta)$$

for all  $(x, y) \in \mathbb{R}^2$ .

Since we have

$$p^{(n)}(x - \xi, y - \eta) - \tau R_* \leq h^{(n)}(x, y) \leq p^{(n)}(x - R(\theta) \cos \theta, y - R(\theta) \sin \theta),$$

for all  $(\xi, \eta) \in \mathcal{A}_n$  and  $(x, y) \in \mathbb{R}^2$ , we obtain

$$\begin{aligned} V(x - \xi, y - \eta, z + z_0^{(n)} + \tau R_*; p^{(n)}) &\leq V(x, y, z + z_0^{(n)}; h^{(n)}) \\ &\leq V(x - R(\theta) \cos \theta, y - R(\theta) \sin \theta, z + z_0^{(n)}; p^{(n)}) \end{aligned}$$

for all  $(\xi, \eta) \in \mathcal{A}_n$  and  $(x, y) \in \mathbb{R}^2$ . Using this inequality and the Schauder interior estimate and the Sobolev imbedding theorem, we can define

$$\tilde{U}(x, y, z) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} V(x, y, z + z_0^{(n)}; h^{(n)}) \quad \text{in } C_{\text{loc}}^2(\mathbb{R}^3).$$

Then  $\tilde{U}$  satisfies (3), that is,  $\mathcal{L}[\tilde{U}] = 0$  in  $\mathbb{R}^3$ . We have

$$\begin{aligned} U(\sqrt{(x - \xi)^2 + (y - \eta)^2}, z + \tau R_*) &\leq \tilde{U}(x, y, z) \\ &\leq U(\sqrt{(x - R(\theta) \cos \theta)^2 + (y - R(\theta) \sin \theta)^2}, z) \end{aligned}$$

for all  $(\xi, \eta) \in \mathcal{A}_0$  and  $0 \leq \theta \leq 2\pi$ . Especially we get (39) using  $(\xi_*(\theta), \eta_*(\theta)) \in \mathcal{A}_0$  for all  $\theta \in [0, 2\pi]$ .

We have

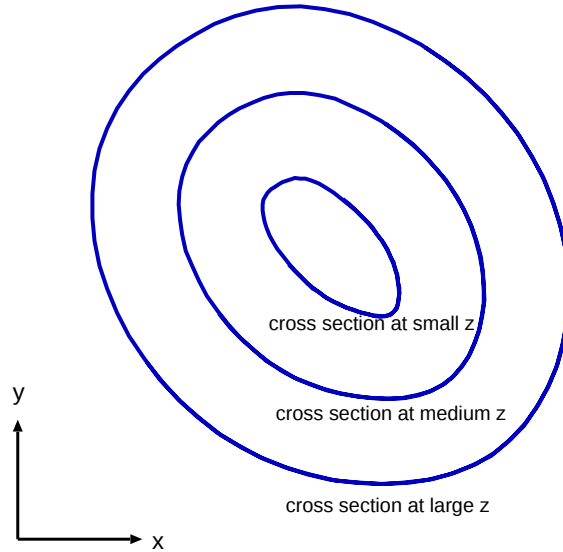
$$\tilde{U}_z(x, y, z) < 0 \quad \text{for all } (x, y, z) \in \mathbb{R}^3,$$

and

$$\frac{\partial \tilde{U}}{\partial \mathbf{a}} < 0.$$

for  $\mathbf{a} \stackrel{\text{def}}{=} (a_1, 1)$  with  $|a_1| \leq \tau^{-1}$ .

The following is the main assertion in this paper.

FIGURE 5. The cross section of zero level surface of  $\tilde{U}$ .

**Theorem 6.1.** Assume (A1), (A2) and (A3). Let  $g \in C^\infty(S^1)$  be any given function with  $\min_{0 \leq \theta \leq 2\pi} g(\theta) = 0$  and let  $R(\theta) = r_* + g(\theta)$ . Here  $r_*$  is as in (35). Then  $\tilde{U}$  is a solution of (3) with

$$\begin{aligned} \max_{0 \leq \theta < 2\pi} U(\sqrt{(x - \xi_*(\theta))^2 + (y - \eta_*(\theta))^2}, z + \tau R_*) &\leq \tilde{U}(x, y, z) \\ &\leq \min_{0 \leq \theta < 2\pi} U(\sqrt{(x - R(\theta) \cos \theta)^2 + (y - R(\theta) \sin \theta)^2}, z) \end{aligned} \quad (39)$$

for all  $(x, y, z) \in \mathbb{R}^3$ . Moreover one has

$$\begin{aligned} \lim_{s \rightarrow \infty} \sup_{x^2 + y^2 + z^2 \geq s^2} (\tilde{U}(x, y, z) - \min_{0 \leq \theta < 2\pi} U(\sqrt{(x - R(\theta) \cos \theta)^2 + (y - R(\theta) \sin \theta)^2}, z)) \\ = 0. \end{aligned} \quad (40)$$

Under (3) and (39),  $\tilde{U}$  is uniquely determined.

*Proof.* Since we have  $\lim_{r \rightarrow \infty} \phi'(r) = \tau$  in Lemma 5.5, we obtain

$$\begin{aligned} \lim_{\sqrt{r^2 + z^2} \rightarrow \infty} \left( U(\sqrt{(r \cos \theta - R(\theta) \cos \theta)^2 + (r \sin \theta - R(\theta) \sin \theta)^2}, z) \right. \\ \left. - U(\sqrt{(r \cos \theta - \xi_*(\theta))^2 + (r \sin \theta - \eta_*(\theta))^2}, z + \tau R_*) \right) = 0 \end{aligned}$$

for all  $\theta \in [0, 2\pi)$ . Thus we get

$$\lim_{s \rightarrow \infty} \sup_{x^2+y^2+z^2 \geq s^2} \left| \min_{0 \leq \theta < 2\pi} U(\sqrt{(x - R(\theta) \cos \theta)^2 + (y - R(\theta) \sin \theta)^2}, z) - \max_{0 \leq \theta < 2\pi} U(\sqrt{(x - \xi_*(\theta))^2 + (y - \eta_*(\theta))^2}, z + \tau R_*) \right| = 0$$

and obtain (40).

We show the uniqueness of  $U$ . From Lemma 5.5 and (40), we obtain

$$\inf_{-1+\delta_* \leq \tilde{U}(\mathbf{x}) \leq 1-\delta_*} \left( -\tilde{U}_z(\mathbf{x}) \right) > 0.$$

We choose a positive constant  $\sigma_1$  with

$$\beta \sigma_1 \inf_{-1+\delta_* \leq \tilde{U}(\mathbf{x}) \leq 1-\delta_*} (-\tilde{U}_z(\mathbf{x})) > \beta + \sup_{|s| \leq 2} |f'(s)|.$$

Then

$$\tilde{U}(x, y, z - \sigma_1 \delta(1 - e^{-\beta t})) + \delta e^{-\beta t}$$

is a supersolution to (2) for any  $\delta \in (0, \delta_*)$ .

Assume  $\hat{U}$  satisfies (3) and (39). We prove  $\hat{U} \equiv \tilde{U}$  by contradiction.

We take  $\lambda > 0$  large enough and get

$$\hat{U}(x, y, z) \leq V(x, y, z - \lambda) + \delta_*$$

by using (40). Then we have

$$\hat{U}(x, y, z) \leq \tilde{U}(x, y, z - \lambda - \sigma_1 \delta_*(1 - e^{-\beta t})) + \delta_* e^{-\beta t}.$$

Sending  $t \rightarrow \infty$ , we obtain

$$\hat{U}(x, y, z) \leq \tilde{U}(x, y, z - \lambda - \sigma_1 \delta_*) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

We define

$$\Lambda_1 \stackrel{\text{def}}{=} \inf \{ \lambda \in (-\infty, \infty) \mid \hat{U}(x, y, z) \leq \tilde{U}(x, y, z - \lambda) \text{ for all } (x, y, z) \in \mathbb{R}^3 \}.$$

We have  $\Lambda_1 \geq 0$ . Since  $\tilde{U}$  and  $\hat{U}$  satisfy (40), we have  $\Lambda_1 \geq 0$ . If  $\Lambda_1 = 0$ , we have  $\hat{U} \leq \tilde{U}$  and also have  $\tilde{U} \leq \hat{U}$  by the same argument, which gives  $\tilde{U} \equiv \hat{U}$ .

We prove  $\Lambda_1 = 0$  by contradiction. Assume  $\Lambda_1 > 0$ . Then the comparison principle gives

$$\hat{U}(x, y, z) < \tilde{U}(x, y, z - \Lambda_1) \quad \text{for all } (x, y, z) \in \mathbb{R}^3$$

Now we fix  $R_1 > 0$  large enough and get

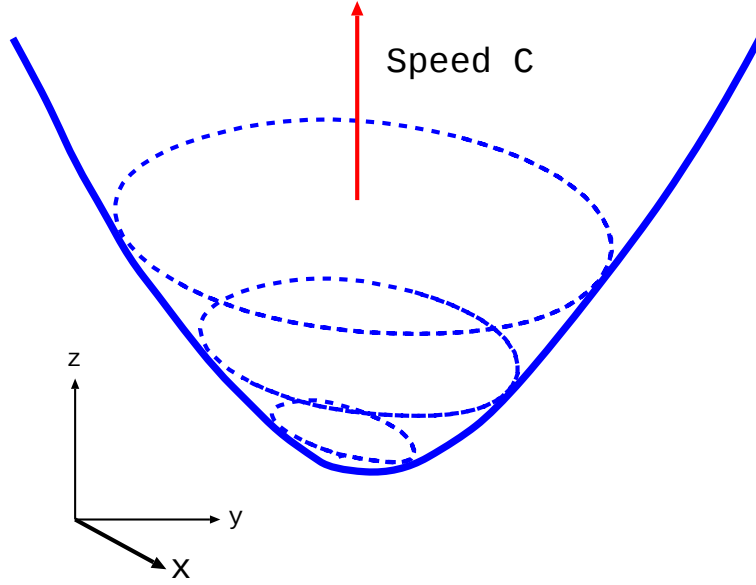
$$2\sigma_1 \sup_{|z - \phi(\sqrt{x^2 + y^2})| \geq R_1 - \Lambda_1 - 1} |\tilde{U}_z(x, y, z)| < 1.$$

Taking  $k_2 > 0$  small enough and using (40), we have

$$\hat{U}(x, y, z) < \tilde{U}(x, y, z - \Lambda_1 + 2\sigma_1 k_2)$$

if  $|z - \phi(\sqrt{x^2 + y^2})| \leq R_1 - \Lambda_1 - 1$ . If  $|z - \phi(\sqrt{x^2 + y^2})| > R_1 - \Lambda_1 - 1$ , we have

$$\begin{aligned} \tilde{U}(x, y, z - \Lambda_1 + 2\sigma_1 k_2) - \tilde{U}(x, y, z - \Lambda_1) &= 2\sigma_1 k_2 \int_0^1 \tilde{U}_z(x, y, z - \Lambda_1 + 2\sigma_1 k_2 \theta) d\theta \\ &\geq -k_2. \end{aligned}$$

FIGURE 6. The graph of  $\tilde{U}$ .

Combining these two estimates together, we obtain

$$\hat{U}(x, y, z) \leq \tilde{U}(x, y, z - \Lambda_1 + 2\sigma_1 k_2) + k_2 \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

Then we get

$$\hat{U}(x, y, z) \leq \tilde{U}(x, y, z - \Lambda_1 + 2\sigma_1 k_2 - \sigma k_2(1 - e^{-\beta t})) + k_2 e^{-\beta t}.$$

Sending  $t \rightarrow +\infty$ , we obtain

$$\hat{U}(x, y, z) \leq \tilde{U}(x, y, z - \Lambda_1 + \sigma_1 k_2) \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

This contradicts the definition of  $\Lambda_1$ . Thus  $\hat{U} \equiv \tilde{U}$  follows. This completes the proof.  $\square$

**Corollary 1** (Stability of  $\tilde{U}$ ). *If*

$$\begin{aligned} \max_{0 \leq \theta < 2\pi} U(\sqrt{(x - \xi_*(\theta))^2 + (y - \eta_*(\theta))^2}, z + \tau R_*) &\leq u_0(x, y, z) \\ &\leq \min_{0 \leq \theta < 2\pi} U(\sqrt{(x - R(\theta) \cos \theta)^2 + (y - R(\theta) \sin \theta)^2}, z) \end{aligned}$$

for  $(x, y, z) \in \mathbb{R}^3$ , one has

$$\lim_{t \rightarrow \infty} \sup_{(x, y, z) \in \mathbb{R}^3} |w(x, y, z, t; u_0) - \tilde{U}(x, y, z)| = 0.$$

*Proof.* Let

$$\begin{aligned} \underline{U}(x, y, z) &\stackrel{\text{def}}{=} \max_{0 \leq \theta < 2\pi} U(\sqrt{(x - \xi_*(\theta))^2 + (y - \eta_*(\theta))^2}, z + \tau R_*), \\ \overline{U}(x, y, z) &\stackrel{\text{def}}{=} \min_{0 \leq \theta < 2\pi} U(\sqrt{(x - R(\theta) \cos \theta)^2 + (y - R(\theta) \sin \theta)^2}, z) \end{aligned}$$

for  $(x, y, z) \in \mathbb{R}^3$ . Then  $\underline{U}$  is a subsolution and  $\overline{U}$  is a supersolution of (3), respectively. From Sattinger [33, Theorem 3.6] we see that  $\lim_{t \rightarrow \infty} w(\mathbf{x}, t; \underline{U})$  and  $\lim_{t \rightarrow \infty} w(\mathbf{x}, t; \overline{U})$  are solutions of (3) between  $\underline{U}$  and  $\overline{U}$ . Such solution is unique and is given by  $\tilde{U}$  by Theorem 6.1. This completes the proof.  $\square$

Thus a convex and bounded domain  $\mathcal{D}$  with a smooth boundary  $\mathcal{C}$  on a two-dimensional plane gives a three-dimensional traveling front  $\tilde{U}$  in the Allen-Cahn equation. If another smooth convex closed curve  $\mathcal{C}'$  satisfying the conditions stated above is given, we have  $\tilde{U}(\mathbf{x}; \mathcal{C}) \neq \tilde{U}(\mathbf{x}; \mathcal{C}')$  from the theorem. See Figure 5. The cross section of  $\{\mathbf{x} \in \mathbb{R}^3 \mid \tilde{U}(\mathbf{x}) = 0\}$  at  $z$  converges to that of a graph  $z = \phi(r - R(\theta))$  ( $0 \leq \theta \leq 2\pi$ ) as  $z \rightarrow +\infty$ , that is, we have

$$\lim_{z \rightarrow +\infty} \text{dist}(\{(x, y) \in \mathbb{R}^2 \mid \tilde{U}(x, y, z) = 0\}, \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2 \mid z = \phi(r - R(\theta))\}) = 0.$$

The cross section of  $\{\mathbf{x} \in \mathbb{R}^3 \mid \tilde{U}(\mathbf{x}) = 0\}$  at  $z$  is almost a circle after rescaling if  $z > 0$  is large enough, that is,

$$\left\{ \left( \frac{x}{\phi^{-1}(z)}, \frac{y}{\phi^{-1}(z)} \right) \in \mathbb{R}^2 \mid \tilde{U}(x, y, z) = 0 \right\}$$

converges to a unit circle as  $z \rightarrow +\infty$ .

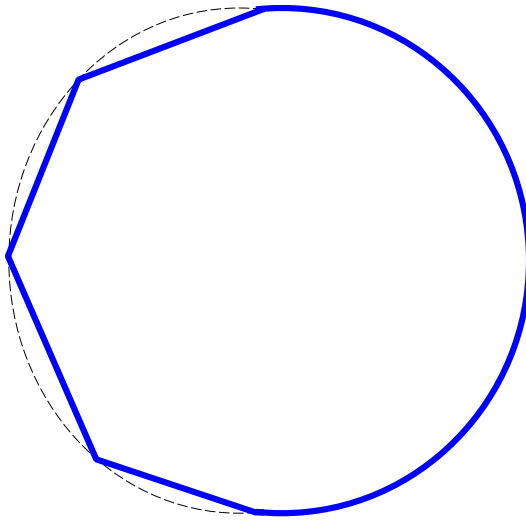


FIGURE 7. Does a mixed traveling front solution exist?

**7. Conjecture.** If  $f$  is of bistable type and satisfies the assumptions of [13, 14],  $U$  in § 5 coincides with the cylindrically symmetric traveling front solution in [14, Theorem 2] due to [14, Theorem 3]. See also [26]. Then [14] gives

$$\phi(r) = \tau r - k_0 \log r + O(1) \quad \text{as } r \rightarrow \infty$$

with a constant  $k_0 > 0$ . Then we have

$$\phi^{-1}(z) = \frac{1}{\tau}(z + k_0 \log z) + O(1) \quad \text{as } z \rightarrow +\infty.$$

This implies that  $z_0^n$  in § 5 satisfies  $\lim_{n \rightarrow \infty} z_0^n = \infty$  and that  $V^{(n)}$  converges to  $-1$  in  $C_{\text{loc}}^2(\mathbb{R}^3)$  as  $n \rightarrow \infty$ . This is because the curvature effect makes  $V^{(n)}$  go upwards as  $n \rightarrow \infty$ .

The zero level of  $U$  is given by  $z = \phi(r)$ . The radius of the cross section of  $z = \phi(r)$  at  $z \in \mathbb{R}$  is given by  $r = \phi^{-1}(z)$ . We can see that  $\phi^{-1}(z)$  is faster than linear growth. On the other hand, the zero level surface of a pyramidal traveling front converges to a pyramid. Thus the cross section at  $z \in \mathbb{R}$  is a polygon and the scale is proportional to  $z$  as  $z \rightarrow \infty$ . One might suspect that the difference of the growth rate of cross sections implies that there exists no traveling front whose cross section is the mixture of a circle and a polygon as in See Figure 7. The existence or non-existence of such mixed traveling front is yet to be studied.

**8. Discussion.** Generalized pyramidal traveling fronts or traveling fronts of various kinds of smooth shapes are constructed in this paper for a bistable or multistable reaction-diffusion with unbalanced energy potentials. For balanced bistable or multistable  $f$ , that is,  $\int_{-1}^1 f(s)ds = 0$ , can one construct a cylindrically non-symmetric traveling front solution with speed  $c \neq 0$ ? This question is an interesting open question as far as the author knows.

For multi-dimensional traveling fronts for reaction-diffusion equations of combustion types or Fisher-KPP types, one can see [12] and [15] for instance. In Smith and Pickering [36] or Buckmaster [5], flames of polyhedral shapes or flames of various kinds of smooth shapes are observed for burning gas in Bunsen burners. From these experiments one can suspects that reaction-diffusion equations of combustion types or Fisher-KPP types or reaction-diffusion systems might have traveling fronts of convex polyhedral shapes or various kinds of smooth shapes. This problem is also interesting and should be studied in future.

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