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PAPER *Special Section on Discrete Mathematics and Its Applications*

Universal Graphs for Graphs with Bounded Path-Width

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SUMMARY A graph G is said to be universal for a family $\mathcal{F}$ of graphs if G contains every graph in $\mathcal{F}$ as a subgraph. A minimum universal graph for $\mathcal{F}$ is a universal graph for $\mathcal{F}$ with the minimum number of edges. This paper considers a minimum universal graph for the family $\mathcal{F}_n^k$ of graphs on n vertices with path-width at most k. We first show that the number of edges in a universal graph for $\mathcal{F}_n^k$ is at least $\Omega(kn\log(n/k))$. Next, we construct a universal graph for $\mathcal{F}_n^k$ with $O(kn\log(n/k))$ edges, and show that the number of edges in a minimum universal graph for $\mathcal{F}_n^k$ is $\Theta(kn\log(n/k))$.

key words: *universal graph, path-width, k-path, parallel computing*

1. Introduction

Given a family $\mathcal{F}$ of graphs, a graph G is said to be *universal* for $\mathcal{F}$ if G contains every graph in $\mathcal{F}$ as a subgraph. A *minimum universal graph* for $\mathcal{F}$ is a universal graph for $\mathcal{F}$ with the minimum number of edges. We denote the number of edges in a minimum universal graph for $\mathcal{F}$ by $f(\mathcal{F})$. $f(\mathcal{F})$ is $O(n^2)$ for any family $\mathcal{F}$ of graphs on n vertices, since a complete graph on n vertices is trivially a universal graph for $\mathcal{F}$. Determining $f(\mathcal{F})$ has been known to have applications to the circuit design, data representation, and parallel computing [2], [3], [10], [12], [14]. Bhatt, Chung, Leighton, and Rosenberg showed a general upper bound for $f(\mathcal{F})$ for a family $\mathcal{F}$ of bounded-degree graphs by means of the size of separators [3]. For general families of (unbounded-degree) graphs, the following three results have been known:

(I) If $\mathcal{F}$ is the family of all planar graphs on n vertices, $f(\mathcal{F})$ is $\Omega(n\log n)$ and $O(n\sqrt{n})$ [1];

(II) If $\mathcal{F}$ is the family of all trees on n vertices, $f(\mathcal{F})$ is $\Theta(n\log n)$ [6];

(III) If $\mathcal{F}$ is the family of all 2-paths on n vertices, $f(\mathcal{F})$ is $\Theta(n\log n)$ [13]. (A 2-path is a special kind of outerplanar graph.)

This paper shows a generalization of (III).

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We consider finite undirected graphs without loops or multiple edges. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a sequence of subsets of $V(G)$. The *width* of $\mathcal{X}$ is $\max_{1\leq i\leq r}|X_i| - 1$. $\mathcal{X}$ is called a *path-decomposition* of G if the following conditions are satisfied: (i) For any distinct i and j, $X_i \nsubseteq X_j$; (ii) $\bigcup_{i=1}^r X_i = V(G)$; (iii) For any edge $(u,v) \in E(G)$, there exists an i such that $u, v \in X_i$; (iv) For all a, b, and c with $1 \leq a \leq b \leq c \leq r$, $X_a \cap X_c \subseteq X_b$. The *path-width* of G, denoted by $pw(G)$, is the minimum width over all path-decompositions of G [11]. We denote the family of all graphs on n vertices with path-width at most k by $\mathcal{F}_n^k$.

The purpose of this paper is to prove the following:

Theorem 1: For any integer k ($k \geq 1$) and n ($n \geq 12k$), $f(\mathcal{F}_n^k)$ is $\Theta(kn\log(n/k))$. □

We will prove this theorem by showing that $f(\mathcal{F}_n^k)$ is $\Omega(kn\log(n/k))$ in Sect. 3, and $f(\mathcal{F}_n^k)$ is $O(kn\log(n/k))$ in Sect. 4. It follows from Theorem 1 that if $\mathcal{F}$ is the family of all planar graphs on n vertices with bounded path-width then $f(\mathcal{F})$ is $\Theta(n\log n)$.

Many related results can be found in the literature [1]–[10], [12]–[14].

2. Preliminaries

A *k-clique* of a graph G is a complete subgraph of G on k vertices. For a positive integer k, *k-trees* are defined recursively as follows: (1) The complete graph on k vertices is a k-tree; (2) Given a k-tree Q on n vertices ($n \geq k$), a graph obtained from Q by adding a new vertex adjacent to the vertices of a k-clique of Q is a k-tree on $n+1$ vertices. A k-tree Q is called a *k-path* if either $|V(Q)| \leq k+1$ or Q has exactly two vertices of degree k. A *k-separator* S of a connected graph G is an induced subgraph of G on k vertices such that $G\backslash V(S)$ has at least two connected components where $G\backslash V(S)$ is the graph obtained from G by deleting $V(S)$. It is well-known that a k-separator of a k-tree Q is a k-clique of Q. For a positive integer k, *k-intercats (interior k-caterpillars)* are defined as follows: (1) A k-path is a k-intercat; (2) Given a k-intercat Q on n vertices ($n \geq k+2$), a graph obtained from Q by adding a new vertex adjacent to the vertices of a k-separator of Q is also a k-intercat on $n+1$ vertices.

A 1-path, 1-intercat, and 1-tree are an ordinary path, caterpillar, and tree, respectively. A subgraph of a k-path, k-intercat, and k-tree are called a *partial k-path*, *partial k-intercat*, and *partial k-tree*, respectively.

It is well-known that any k-intercat R on n vertices $(n \geqq k)$ can be obtained as follows: (1) Define that Q_k is the complete graph on k vertices C_k; (2) Given Q_i and C_i $(k \leqq i \leqq n-1)$, define that Q_{i+1} is the k-intercat obtained from Q_i by adding vertex $v_{i+1} \notin V(Q_i)$ adjacent to the vertices in C_i, and let $C_{i+1} = (C_i \cup \{v_{i+1}\}) - \{w_i\}$ where $w_i \in C_i \cup \{v_{i+1}\}$; (3) Define $R = Q_n$.

A path-decomposition with width k is called a k-path-decomposition. A k-path-decomposition $(X_1, X_2, \ldots, X_r)$ is said to be *full* if $|X_i| = k+1$ $(1 \leqq i \leqq r)$ and $|X_j \cap X_{j+1}| = k$ $(1 \leqq j \leqq r-1)$.

Lemma 1: For any graph G with path-width k, there exists a full k-path-decomposition of G.

Proof: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a k-path-decomposition of G such that $\sum_{i=1}^{r}(|X_i| - k)$ is maximum. We shall show that $\mathcal{X}$ is a full k-path-decomposition of G. If $r = 1$ then $\mathcal{X}$ is trivially a full k-path-decomposition of G. Thus we assume that $r \geqq 2$.

Suppose that $|X_i| \leqq k$ for some i $(2 \leqq i \leqq r)$. Let $v \in X_{i-1} - X_i$. The sequence $\mathcal{X}' = (X_1, X_2, \ldots, X_{i-1}, X_i \cup \{v\}, X_{i+1}, \ldots, X_r)$ satisfies conditions (ii), (iii), and (iv) in the definition of path-decomposition. Assume that $X_j \subseteqq X_i \cup \{v\}$ for some $j (\neq i)$. If $j > i$ then $X_j \subseteqq X_i$ since $v \notin X_j$, contradicting the condition (i) in the definition of path-decomposition. Thus $j = i-1$ since $X_j = X_j \cap (X_i \cup \{v\}) \subseteqq X_{i-1}$. Therefore, $(X_1, X_2, \ldots, X_{i-2}, X_i \cup \{v\}, X_{i+1}, \ldots, X_r)$ is a k-path-decomposition of G. But this is contradicting the choice of $\mathcal{X}$ since $|X_{i-1}| \leqq k$, for otherwise $X_i \subseteqq X_{i-1}$. Thus $\mathcal{X}'$ satisfies condition (i) in the definition of path-decomposition, and $\mathcal{X}'$ is a k-path-decomposition of G. But again this is contradicting the choice of $\mathcal{X}$. Thus $|X_i| = k+1$ for any i $(2 \leqq i \leqq r)$. Since $(X_r, \ldots, X_1)$ is also a path-decomposition of G, $|X_i| = k+1$ for any i $(1 \leqq i \leqq r)$.

Suppose next that $|X_i \cap X_{i+1}| \leqq k-1$ for some i $(1 \leqq i \leqq r-1)$. Let $v \in X_{i+1} - X_i$ and $w \in X_i - X_{i+1}$. The sequence $\mathcal{X}' = (X_1, \ldots, X_i, (X_i \cup \{v\}) - \{w\}, X_{i+1}, \ldots, X_r)$ satisfies conditions (ii), (iii), and (iv) in the definition of path-decomposition. Assume that $X_j \subseteqq (X_i \cup \{v\}) - \{w\}$ or $(X_i \cup \{v\}) - \{w\} \subseteqq X_j$ for some j $(1 \leqq j \leqq r)$. Since $|(X_i \cup \{v\}) - \{w\}| = |X_j| = k+1$, $X_j = (X_i \cup \{v\}) - \{w\}$. Then $j = i$ or $j = i+1$ since $X_j = X_j \cap ((X_i \cup \{v\}) - \{w\}) \subseteqq X_i$ if $j \leqq i$, $X_j = ((X_i \cup \{v\}) - \{w\}) \cap X_j \subseteqq X_{i+1}$ otherwise. But this is contradicting the assumption that $|X_i \cap X_{i+1}| \leqq k-1$. Thus $\mathcal{X}'$ satisfies condition (i) in the definition of path-decomposition, and $\mathcal{X}'$ is a k-path-decomposition of G. But this is contradicting the choice of $\mathcal{X}$ since $|(X_i \cup \{v\}) - \{w\}| = k+1$. Thus $|X_i \cap X_{i+1}| = k$ for any i $(1 \leqq i \leqq r-1)$.

Therefore $\mathcal{X}$ is a full k-path-decomposition of G. □

Theorem 2: For any graph G and an integer k $(k \geqq 1)$, $pw(G) \leqq k$ if and only if G is a partial k-intercat.

Proof: Suppose that $pw(G) = h \leqq k$. There exists a full h-path-decomposition $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of G by Lemma 1. If $r = 1$ then G is a subgraph of a complete graph on $h+1$ vertices, and so we conclude that G is a partial h-intercat. Thus we assume that $r \geqq 2$. We construct an h-intercat R from $\mathcal{X}$ as follows:

1. Let v_1 be a vertex in $X_1 \cap X_2$. Define that Q_0 is the complete graph on $X_1 - \{v_1\}$;
2. Define that Q_1 is the h-intercat obtained from Q_0 by adding v_1 and the edges connecting v_1 and the vertices in $X_1 - \{v_1\}$;
3. Given Q_i $(1 \leqq i \leqq r-1)$, define that Q_{i+1} is the h-intercat obtained from Q_i by adding $v_{i+1} \in X_{i+1} - X_i$ and the edges connecting v_{i+1} and the vertices in $X_{i+1} - \{v_{i+1}\}$;
4. Define $R = Q_r$.

Since $|X_{i+1} - X_i| = 1$ from the definition of full h-path-decomposition, v_{i+1} is uniquely determined $(1 \leqq i \leqq r-1)$. Since $X_{i+1} - \{v_{i+1}\} = ((X_i - \{v_i\}) \cup \{v_i\}) - \{w_i\}$ where $w_i \in X_i - X_{i+1}$ $(1 \leqq i \leqq r-1)$, R is an h-intercat. Furthermore, we have $V(R) = V(G)$ and $E(R) \supseteqq E(G)$ from the definitions of path-decomposition and Q_i. Thus G is a partial h-intercat, and so a partial k-intercat.

Conversely, suppose, without loss of generality, that G is a partial h-intercat $(h \leqq k)$ with n' $(n' > h)$ vertices and R is an h-intercat such that $V(R) \supseteqq V(G)$ and $E(R) \supseteqq E(G)$. Let $n = |V(R)|$. As we mentioned before, we can assume that R can be obtained as follows:

1. Define that Q_h is the complete graph on h vertices C_h;
2. Given Q_i and C_i $(h \leqq i \leqq n-1)$, define that Q_{i+1} is the h-intercat obtained from Q_i by adding vertex $v_{i+1} \notin V(Q_i)$ adjacent to the vertices in C_i, and let $C_{i+1} = (C_i \cup \{v_{i+1}\}) - \{w_i\}$ where $w_i \in C_i \cup \{v_{i+1}\}$;
3. Define $R = Q_n$.

We define that $X_i = C_i \cup \{v_{i+1}\}$ $(h \leqq i \leqq n-1)$ and $\mathcal{X} = (X_h, X_{h+1}, \ldots, X_{n-1})$. It is easy to see that $\bigcup_{i=h}^{n-1} X_i = V(R)$ and each vertex appears in consecutive X_i's. Thus $\mathcal{X}$ satisfies conditions (ii) and (iv) in the definition of path-decomposition. Since $w_i \in X_i - X_{i+1}$ and $v_{i+2} \in X_{i+1} - X_i$, $X_i \nsubseteqq X_{i+1}$ and $X_{i+1} \nsubseteqq X_i$ $(h \leqq i \leqq n-2)$. Thus $X_i \nsubseteqq X_j$ for any distinct i and j, for otherwise $X_i = X_i \cap X_j \subseteqq X_{i+1}$ $(i < j)$ or $X_i = X_i \cap X_j \subseteqq X_{i-1}$ $(i > j)$. Hence $\mathcal{X}$ satisfies condition (i) in the definition of path-decomposition. Since each edge of R either connects v_{i+1} and a vertex in C_i for some i ($h \leqq i \leqq n-1$) or connects vertices in C_h, both ends of each edge of R are contained in some X_i. Thus $\mathcal{X}$ satisfies condition (iii) in the definition of path-decomposition. It is easy to see that $|X_i| = h+1$ ($h \leqq i \leqq n-1$) and $|X_i \cap X_{i+1}| = |C_{i+1}| = h$ ($h \leqq i \leqq n-2$). Thus the sequence $\mathcal{X}$ is a full h-path-decomposition of R. Therefore, we have that $pw(G) \leqq pw(R) \leqq h \leqq k$. □

3. Lower Bound

Let $d_G(v)$ be the degree of a vertex v in G. Let $D(G) = (\delta_G^1, \delta_G^2, \ldots, \delta_G^n)$ be the degree sequence for a graph G with n vertices, where $\delta_G^1 \geqq \delta_G^2 \geqq \cdots \geqq \delta_G^n$. For graphs G and H with m and n vertices, respectively, we define $D(G) \geqq D(H)$ if and only if $m \geqq n$ and $\delta_G^i \geqq \delta_H^i$ for any i ($1 \leqq i \leqq n$).

Lemma 2: If a graph G is a universal graph for a family $\mathcal{F}$ of graphs, $D(G) \geqq D(H)$ for any graph H in $\mathcal{F}$.

Proof: For otherwise, G cannot contain H as a subgraph. □

Lemma 3: For any integer k ($k \geq 1$) and s ($1 \leqq s \leqq \lfloor (n-2k)/k \rfloor$), there exists a k-intercat $R(k,s)$ on n vertices such that $\delta_{R(k,s)}^{ks} \geqq \lfloor (n-2k)/s \rfloor + k$.

Proof: Let $r = \lfloor (n-2k)/s \rfloor$. $R(k,s)$ can be constructed as follows:

1. Define that $Q(k,k)$ is the complete graph on the vertices $C_k = \{v_1, v_2, \ldots, v_k\}$;
2. Given $Q(k,i)$ and C_i ($k \leqq i < 2k$), define that $Q(k,i+1)$ is the k-intercat obtained from $Q(k,i)$ by adding vertex v_{i+1} adjacent to the vertices in C_i, and let $C_{i+1} = (C_i \cup \{v_{i+1}\}) - \{v_{i+1-k}\}$;
3. Given $Q(k,i)$ and C_i ($2k + jr \leqq i < r + k + jr, 0 \leqq j \leqq s-2$), define that $Q(k,i+1)$ is the k-intercat obtained from $Q(k,i)$ by adding vertex v_{i+1} adjacent to the vertices in C_i, and let $C_{i+1} = C_i$;
4. Given $Q(k,i)$ and C_i ($r + k + jr \leqq i < r + 2k + jr, 0 \leqq j \leqq s-2$), define that $Q(k,i+1)$ is the k-intercat obtained from $Q(k,i)$ by adding vertex v_{i+1} adjacent to the vertices in C_i, and let $C_{i+1} = (C_i \cup \{v_{i+1}\}) - \{v_{i+1-r}\}$;
5. Given $Q(k,i)$ and C_i ($2k + (s-1)r \leqq i \leqq n-1$), define that $Q(k,i+1)$ is the k-intercat obtained from $Q(k,i)$ by adding vertex v_{i+1} adjacent to the vertices in C_i, and let $C_{i+1} = C_i$;
6. Define $R(k,s) = Q(k,n)$.

It is easy to see that $|C_i| = k$ and $Q(k,i)$ is a k-intercat for any i ($k \leqq i \leqq n$). It is also easy to see that $d_{R(k,s)}(v_{k+i+jr}) = r + k$ ($1 \leqq i \leqq k, 0 \leqq j \leqq s-2$), and $d_{R(k,s)}(v_{k+i+(s-1)r}) \geqq r + k$ ($1 \leqq i \leqq k$). Thus we have $\delta_{R(k,s)}^{ks} \geqq r + k$. □

Theorem 3: For any integer k ($k \geq 1$) and n ($n \geq 3k$), $f(\mathcal{F}_n^k)$ is $\Omega(kn \log(n/k))$.

Proof: Let G be a universal graph for $\mathcal{F}_n^k$ and $t = \lfloor (n-2k)/k \rfloor$. Notice that $2|E(G)| = \sum_{v \in V(G)} d_G(v) \geqq \sum_{i=1}^n \delta_G^i > \sum_{i=1}^{tk} \delta_G^i \geqq k \sum_{i=1}^t \delta_G^{ki}$. By Lemmas 2, 3, and Theorem 2,

$$\begin{aligned} k\sum_{i=1}^{t}\delta_G^{ki} &= k\sum_{i=1}^{t}\left(\left\lfloor\frac{n-2k}{i}\right\rfloor + k\right) \\ &\geqq k\sum_{i=1}^{t}\left(\frac{n-2k}{i} + k - 1\right) \\ &> k(n-2k)\log_e\left(\frac{n-2k}{k}\right) \\ &\quad +(k-1)(n-3k). \end{aligned}$$

Thus $|E(G)|$ is $\Omega(kn \log(n/k))$. □

4. Upper Bound

We show an upper bound by constructing the graph G_n^k with n vertices and $O(kn \log(n/k))$ edges, and proving that G_n^k is a universal graph for $\mathcal{F}_n^k$.

Let $k^* = 2^{\lceil \log k \rceil}$, b_i be the maximum power of 2 such that $b_i | i$, and $b_{i,j} = \max(b_i, b_j)$. Notice that $k \leqq k^* < 2k$. Let G_n^k ($k \geqq 1, n \geqq 1$) be the graph obtained by the following construction procedure:

(1) Let $u_1, u_2, \ldots, u_n$ be n vertices;

(2) For any distinct i and j, join u_i and u_j by an edge if $|j-i| \leqq 3k^* b_{i,j} + k - 1$.

Theorem 4: For any integer k ($k \geqq 1$) and n ($n \geqq 12k$), $|E(G_n^k)| = O(kn \log(n/k))$.

Proof: Let E_i ($1 \leqq i \leqq n$) be the set of edges $(u_i, u_j) \in G_n^k$ such that $|j-i| \leqq 3k^* b_i + k - 1$. It is easy to see that $|E_i| \leqq \min(2(3k^* b_i + k - 1), n-1)$ for any i ($1 \leqq i \leqq n$), and $\bigcup_{i=1}^n E_i = E(G_n^k)$. Notice that $|\{i \mid b_i = 2^h, 1 \leqq i \leqq n\}| = \lfloor (n+2^h)/2^{h+1} \rfloor$ and $|\{i \mid b_i \geqq 2^h, 1 \leqq i \leqq n\}| = \lfloor n/2^h \rfloor$ for any integer h ($h \geqq 0$). Since $2(3k^* 2^{\log(n/(6k^*))} + k - 1) \geqq n$, the total number of edges added in (2) is at most

$$\begin{aligned} \sum_{i=1}^{n}|E_i| &< \sum_{h=0}^{\lfloor \log \frac{n}{6k^*} \rfloor} 2(3k^*2^h + k - 1)\left\lfloor\frac{n+2^h}{2^{h+1}}\right\rfloor \\ &\quad +(n-1)\left\lfloor\frac{n}{2^{\lfloor \log \frac{n}{6k^*}\rfloor + 1}}\right\rfloor \\ &< \sum_{h=0}^{\lfloor \log \frac{n}{6k^*} \rfloor} (3k^*2^h + k - 1)\left(\frac{n}{2^h} + 1\right) \\ &\quad +6k^*(n-1) \end{aligned}$$

$$< (6kn+k-1)\log\frac{n}{6k}+(20k-1)n$$
$$-(6k^2+8k+1).$$

Thus $|E(G_n^k)| = O(kn\log(n/k))$. □

Theorem 5: For any integer k ($k \geq 1$) and n ($n \geqq 1$), G_n^k is a universal graph for $\mathcal{F}_n^k$.

Proof: By Theorem 2, it is sufficient to show that any k-intercat is a subgraph of G_n^k. Let R be a k-intercat in $\mathcal{F}_n^k$. We shall show that R is a subgraph of G_n^k. If $n \leqq 4k$, R is a subgraph of G_n^k since G_n^k is the complete graph on n vertices. Thus we assume that $n \geqq 4k+1$. As we mentioned before, we can assume that R can be obtained as follows:

1. Define that Q_k is the complete graph on the vertices $C_k = \{v_1, v_2, \ldots, v_k\}$;
2. Given Q_i and C_i ($k \leqq i \leqq n-1$), define that Q_{i+1} is the k-intercat obtained from Q_i by adding vertex $v_{i+1} \notin V(Q_i)$ adjacent to the vertices in C_i, and let $C_{i+1} = (C_i \cup \{v_{i+1}\}) - \{w_i\}$ where $w_i \in C_i \cup \{v_{i+1}\}$;
3. Define $R = Q_n$.

For the construction above, we have the following two lemmas.

Lemma 4: If $(v_a, v_c) \in E(R)$ then $(v_a, v_b) \in E(R)$ for any distinct a, b, and c ($1 \leqq a < b < c \leqq n$).

Proof: Assume contrary that $(v_a, v_b) \notin E(R)$. Since $v_a \notin C_{b-1}$ and $v_a \in C_{c-1}$, $v_a = v_{i+1}$ for some i ($b-1 \leqq i \leqq c-2$), contradicting that $v_{i+1} \notin V(Q_i)$. □

Define $l_i = \max(d \mid (v_i, v_{i+d}) \in E(R) \vee d = 0)$ for any i ($1 \leqq i \leqq n$).

Lemma 5: For any integer i ($1 \leqq i \leqq n-1$), $l_i = 0$ if and only if $|\{v_j \mid (v_j, v_{i+1}) \in E(R), j < i\}| = k$.

Proof: First, assume that $1 \leqq i \leqq k$. Since $(v_i, v_{k+1}) \in E(R)$, $l_i > 0$. Since Q_k is the complete graph on the vertices $v_1, v_2, \ldots,$ and v_k, $|\{v_j \mid (v_j, v_{i+1}) \in E(R), j < i\}| = i-1 < k$.

Next, assume that $k+1 \leqq i \leqq n-1$. Notice that v_{i+1} is adjacent to the vertices in C_i in Q_{i+1}, $\{v_j \mid (v_j, v_{i+1}) \in E(R), j < i\} = C_i - \{v_i\}$, and $|C_i| = k$. Suppose that $l_i = 0$. By the definition of l_i, we have $(v_i, v_{i+1}) \notin E(R)$, and $v_i \notin C_i$. Thus $|\{v_j \mid (v_j, v_{i+1}) \in E(R), j < i\}| = |C_i| = k$. Conversely, suppose that $|\{v_j \mid (v_j, v_{i+1}) \in E(R), j < i\}| = k$. Since $|C_i| = k$, we have $v_i \notin C_i$, and $(v_i, v_{i+1}) \notin E(R)$. Thus $l_i = 0$ by Lemma 4. □

Let $l_i^* = 2^{\lceil \log l_i \rceil}$ if $l_i \geqq 1$, $l_i^* = 1$ otherwise. Let $m_i = \lceil l_i^*/(2k^*) \rceil$. Now we define mapping ϕ:$\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ as follows:

Step 1: Let $D_0 = \emptyset$, $U_0 = \{1, 2, \ldots, n\}$, and $i = 1$.

Step 2: Define that $\phi(i)$ is the minimum integer such that $\phi(i) \in U_{i-1}$ and $m_i | \phi(i)$.

Step 3: Let $D_i = D_{i-1} \cup \{\phi(i)\}$ and $U_i = U_{i-1} - \{\phi(i)\}$.

Step 4: If $i = n$, halt. Otherwise, set $i = i+1$, and return to Step 2.

Notice that $m_i \leqq b_{\phi(i)}$ for any i ($1 \leqq i \leqq n$) since both m_i and $b_{\phi(i)}$ are power of 2 that divide $\phi(i)$. Notice that $l_i \leqq l_i^* < 2l_i$ if $l_i \geqq 1$.

Lemma 6: ϕ is a 1–1 mapping satisfying that $-k \leqq \phi(i) - i \leqq \lceil l_i^*/2 \rceil - 1$ for any i ($1 \leqq i \leqq n$).

Proof: By induction on i, we show that

$$(*)\ -k \leqq \phi(i) - i \leqq \left\lceil \frac{l_i^*}{2} \right\rceil - 1,$$

and

$$(\star)\ \phi(i) - i \leqq l_i - k - 1 \text{ if } m_i \geqq 2.$$

Assume that the algorithm have determined $\phi(1)$, $\phi(2), \ldots, \phi(i-1)$ satisfying conditions $(*)$ and $(\star)$, and $\{1, 2, \ldots, i-h-1\} \subseteqq D_{i-1}$ and $i-h \in U_{i-1}$ ($0 \leqq h \leqq k, h < i$). Notice h depends on i and that these assumptions are trivially true if $i = 1$, since $D_0 = \emptyset$ and $1 \in U_0$. We show that the conditions $(*)$ and $(\star)$ hold also for $\phi(i)$ ($i \geqq 1$), and there exists h' ($0 \leqq h' \leqq k, h' < i+1$) such that $\{1, 2, \ldots, i-h'\} \subseteqq D_i$ and $i - h' + 1 \in U_i$.

First, suppose that $0 \leqq h \leqq k-1$. It is easy to see that

$$\begin{aligned} -h \leqq \phi(i) - i &\leqq -h + (h+1)m_i - 1 \\ &= (h+1)(m_i - 1) \\ &< (h+1)\frac{l_i^*}{2k} \leqq \frac{l_i^*}{2} \leqq \left\lceil \frac{l_i^*}{2} \right\rceil. \end{aligned}$$

Notice that $\phi(i) \leqq i + \lceil l_i^*/2 \rceil - 1 \leqq i + l_i - 1 < n$ if $l_i \geqq 1$, and $\phi(i) \leqq i + \lceil l_i^*/2 \rceil - 1 = i$ otherwise. Thus $\phi(i)$ is uniquely determined in Step 2 in the algorithm. If $m_i \geqq 2$ then

$$\begin{aligned} \phi(i) - i &\leqq (h+1)\left(\frac{l_i^*}{2k^*} - 1\right) \\ &\leqq (h+1)\frac{l_i - k - 1}{k} \\ &\leqq l_i - k - 1. \end{aligned}$$

Thus the conditions $(*)$ and $(\star)$ hold also for $\phi(i)$. Since $h \leqq k-1$, there exists h' ($0 \leqq h' \leqq h+1 \leqq k, h' < i+1$) such that $\{1, 2, \ldots, i-h'\} \subseteqq D_i$ and $i - h' + 1 \in U_i$.

Next, suppose that $h = k$. We will show that $m_i = 1$ and $\phi(i) - i = -k$. Let $W = \{j \mid \phi(j) \geqq i-k+1, j < i\}$. Since $i - k \in U_{i-1}$, $|W| = k$ and $m_j \geqq 2$ for any $j \in W$. Notice that $j < i+1 < \phi(j)+k+1 \leqq j + l_j$ for any $j \in W$ by the definition of W and the condition $(\star)$. Since $(v_j, v_{j+l_j}) \in E(R)$ for any $j \in W$ by the definition of l_j, $(v_j, v_{i+1}) \in E(R)$ by Lemma 4. Thus $l_i = 0$ by Lemma 5, and we have $m_i = 1$ and $\phi(i) - i = -k$. Therefore the conditions $(*)$ and $(\star)$ hold also for $\phi(i)$. Since $\phi(i) = i - k$, there exists h'

$(0 \leqq h' \leqq k, h' < i+1)$ such that $\{1, 2, \ldots, i-h'\} \subseteqq D_i$ and $i - h' + 1 \in U_i$.

Thus ϕ is a 1–1 mapping satisfying $(*)$ for any $\phi(i)$. □

Lemma 7: If $(v_i, v_j) \in E(R)$ then $(u_{\phi(i)}, u_{\phi(j)}) \in E(G_n^k)$.

Proof: Without loss of generality, we assume that $i < j$. Notice that $1 \leqq j - i \leqq l_i \leqq l_i^*$. Since ϕ is a 1–1 mapping, $\phi(i) \neq \phi(j)$. From Lemma 6, we have $-k \leqq \phi(i) - i \leqq \lceil l_i^*/2 \rceil - 1$ and $-k \leqq \phi(j) - j \leqq \lceil l_j^*/2 \rceil - 1$. Thus $-(\lceil l_i^*/2 \rceil + k - 2) \leqq \phi(j) - \phi(i) \leqq l_i^* + \lceil l_j^*/2 \rceil + k - 1$ and $|\phi(j) - \phi(i)| \leqq l_i^* + \lceil l_j^*/2 \rceil + k - 1$.

If $l_j^* > l_i^*$ then $|\phi(j) - \phi(i)| < \lceil 3l_j^*/2 \rceil + k - 1 \leqq 3k^* m_j + k - 1 \leqq 3k^* b_{\phi(i),\phi(j)} + k - 1$. Notice that $m_j \leqq b_{\phi(j)} \leqq b_{\phi(i),\phi(j)}$. Thus $(u_{\phi(i)}, u_{\phi(j)}) \in E(G_n^k)$ by the definition of G_n^k. The same type of argument applies when $l_j^* \leqq l_i^*$. □

By Lemmas 6 and 7, we conclude that R is a subgraph of G_n^k. This completes the proof of Theorem 5. □

Theorem 1 follows from Theorems 3, 4, and 5.

We conclude with the following open problems.

1. Close up the gap between upper and lower bounds in (I).
2. Generalize (II) to k-trees ($k \geqq 2$).

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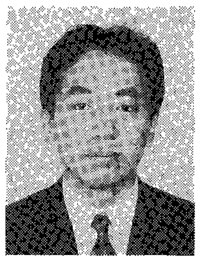

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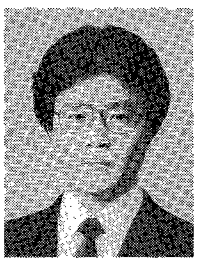

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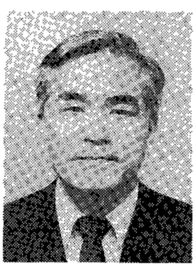

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