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PAPER *Special Section on Information Theory and Its Applications*

A General Formula of the Capacity Region for Multiple-Access Channels with Deterministic Feedback

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SUMMARY The multiple-access channel (MAC) becomes very popular in various communication systems, because multi-terminal communication systems have been widely used in practical systems, e.g., mobile phones and P2P, etc. For some MACs, it is known that feedback can enlarge the capacity region, where the capacity region is the set of rate pairs such that the error probability can be made arbitrarily small for sufficiently large block length. The capacity region for general MACs, which are not required to satisfy ergodicity and stationarity with perfect feedback was first shown by Tatikonda and Mitter without the proof, where perfect feedback means that the channel output is perfectly fed back to senders. In this paper, we generalize Tatikonda and Mitter's result to the case of deterministic feedback, where the values of deterministic functions of past channel outputs is fed back to senders. We show that the capacity region for general MACs with deterministic feedback can be represented by the information-spectrum formula introduced by Han and Verdú, and directed information introduced by Massey. We also investigate the compound MAC problem, the ε-coding problem, the strong converse property and the cost constraint problem for general MACs with deterministic feedback.

key words: *capacity region, directed information, feedback, information spectrum, multiple-access channel*

1. Introduction

Multi-terminal communication systems have been widely used in practical systems, e.g., mobile phones, P2P, etc., and this makes the multi-terminal information theory more important than ever. The multiple-access channel (MAC) is one of basic models in the multi-terminal information theory. For some MACs, it is known that feedback can enlarge the capacity region, where the capacity region is the set of rate pairs such that the error probability can be made arbitrarily small for sufficiently large block length. This result was first shown by Gaarder and Wolf [1]. They showed an example of a stationary memoryless MAC of which capacity region is enlarged by using perfect feedback, where perfect feedback means that the channel output is perfectly fed back to the senders. Later, Cover and Leung [2] derived an *single-letterized* inner region of the capacity region for stationary memoryless MACs with perfect feedback. Although this inner region is optimal for some MACs, it is not optimal in general.

The single-letterized capacity region for stationary memoryless MACs with feedback has not yet been clarified, although the single-letterized inner region and outer region have been clarified. On the other hand, Kramer [3] showed a *multi-letterized* capacity region for stationary memoryless MACs with perfect feedback by using *directed information*. Directed information is a causal version of mutual information, and was introduced by Massey [4]. In general, the multi-letterized capacity region is not computable. However, by using the multi-letterized capacity region, we can find the point which lies outside of the Cover-Leung inner region for some stationary memoryless MACs [5]. Recently, by using directed information, Permuter et al. [6] derived the capacity region for some finite-state MACs with time-invariant deterministic feedback, where time-invariant deterministic feedback means that the values of time-invariant deterministic functions of the channel output is fed back to the senders.

The capacity region for general MACs, which are not required to satisfy ergodicity and stationarity, without feedback was shown by Han [7], and the capacity region for general MACs with perfect feedback was shown by Tatikonda and Mitter [8] although they did not show the proof. In this paper, we generalize the result of Tatikonda and Mitter to the case of deterministic feedback, where deterministic feedback means that the values of deterministic functions of past channel outputs is fed back to the senders. The deterministic feedback includes the case of perfect feedback, time-invariant deterministic feedback, quantized feedback, arbitrarily delayed feedback, etc. Hence, the case we consider may be reviewed as a generalization of Permuter et al.'s case [6] we mentioned earlier.

One of contributions of this paper is clarifying the capacity region for general MACs with deterministic feedback. Then, we show that the information-spectrum formula introduced by Han and Verdú [9] and directed-information play an important role to characterize the capacity region. The proof of this result is based on Tatikonda and Mitter's method [8], [10], [11] which was used to clarify the channel capacity for general one-to-one channels with feedback. The key idea of their method is to consider a channel from code-functions to a receiver instead of a channel from codewords to a receiver, where the code-function is a function that maps a message and feedback into codewords. In order to demonstrate the availability of our capacity region for general MACs with deterministic feedback, we investigate some important classes of MACs, and clarify the capacity regions for these MACs when there exists deterministic feedback. For the class of binary additive noise MACs, we

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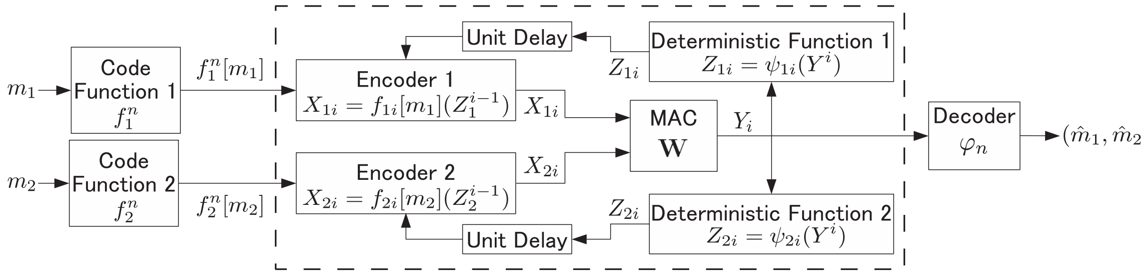

Fig. 1 MAC with deterministic feedback.

show that the capacity region cannot be enlarged by using deterministic feedback. For the class of stationary memoryless MACs, our capacity region is equal to the region obtained by Kramer [3], [5] when perfect feedback is available at the senders. For classes of mixed MACs and compound MACs, we clarify a relation between these two capacity regions. We also clarify the capacity region for the ε-coding problem and the cost constraint problem of general MACs with deterministic feedback. Furthermore, we investigate the strong converse property for general MACs with deterministic feedback, and show the relation between the strong converse property and directed information.

This paper is organized as follows. In Sect. 2, we provide a precise formulation of the deterministic feedback MAC problem. In Sect. 3, we define directed information and information-spectrum formula of directed information. In Sect. 4, we show the capacity region for general MACs with deterministic feedback. In Sect. 5, we give the proof of the capacity region for general MACs with deterministic feedback. In Sect. 6, we show the capacity region for some important classes of MACs with deterministic feedback. In Sect. 7, we investigate some coding problems related to MACs with deterministic feedback. In Sect. 8, we conclude the paper.

2. General MACs with Deterministic Feedback

In this chapter, we provide some notations and a precise formulation of the deterministic feedback MAC coding problem.

We will denote random variables by capital letters $X, Y, Z, \cdots$, the values they can take by lowercase letters $x, y, z, \cdots$, and the set of these values by calligraphic letters $\mathscr{X}, \mathscr{Y}, \mathscr{Z}, \cdots$. The probability distribution of a random variable X taking values in $\mathscr{X}$ will be denoted by P_X. In a similar manner, the probability distribution of a pair of random variables (X, Y) taking values in $\mathscr{X} \times \mathscr{Y}$ will be denoted by P_{XY}. The conditional distribution for X given Y will be denoted by $P_{X|Y}$. In what follows, all logarithms and exponentials are taken to the base of natural logarithm.

Let $\mathscr{X}_1$, $\mathscr{X}_2$, $\mathscr{Y}$, $\mathscr{Z}_1$ and $\mathscr{Z}_2$ be arbitrary finite sets. Let $\mathscr{F}_{1i}$ and $\mathscr{F}_{2i}$ be a set of all maps $f_{1i} : \mathscr{Z}_1^{i-1} \to \mathscr{X}_1$ and $f_{2i} : \mathscr{Z}_2^{i-1} \to \mathscr{X}_2$, respectively. We call these maps *code functions*. Let $\mathscr{F}_1^n \triangleq \bigtimes_{i=1}^n \mathscr{F}_{1i}$ and $\mathscr{F}_2^n \triangleq \bigtimes_{i=1}^n \mathscr{F}_{2i}$ denote the Cartesian product of $\mathscr{F}_{1i}$ and $\mathscr{F}_{2i}$, respectively, where $\mathscr{F}_{11} \triangleq \mathscr{X}_1$ and $\mathscr{F}_{21} \triangleq \mathscr{X}_2$. We will use notations $f_1^i(z_1^{i-1}) = (f_{11}, f_{12}(z_{11}), f_{13}(z_1^2), \cdots, f_{1i}(z_1^{i-1}))$ and $f_2^i(z_2^{i-1}) = (f_{21}, f_{22}(z_{21}), f_{23}(z_2^2), \cdots, f_{2i}(z_2^{i-1}))$, where $z_1^{i-1} = (z_{11}, z_{12}, \cdots, z_{1,i-1})$, $z_2^{i-1} = (z_{21}, z_{22}, \cdots, z_{2,i-1})$, and f_{11} and f_{21} are fixed elements of $\mathscr{X}_1$ and $\mathscr{X}_2$, respectively.

Now, we formulate the MAC with deterministic feedback. We illustrate the MAC with deterministic feedback in Fig. 1. We are given a pair of deterministic functions $\Psi = \{(\psi_1^n, \psi_2^n)\}_{n=1}^\infty$, where $\psi_1^n = (\psi_{11}, \psi_{12}, \cdots, \psi_{1n})$, $\psi_2^n = (\psi_{21}, \psi_{22}, \cdots, \psi_{2n})$, $\psi_{1i} : \mathscr{Y}^i \to \mathscr{Z}_1$ and $\psi_{2i} : \mathscr{Y}^i \to \mathscr{Z}_2$. We consider the situation that sender 1 and sender 2 independently send a message $m_1 \in \mathscr{M}_n^{(1)}$ and a message $m_2 \in \mathscr{M}_n^{(2)}$ to a receiver, where the massage sets are defined as $\mathscr{M}_n^{(1)} \triangleq \{1, \cdots, M_n^{(1)}\}$ and $\mathscr{M}_n^{(2)} \triangleq \{1, \cdots, M_n^{(2)}\}$. Sender 1 has encoder 1 and sender 2 has encoder 2. Encoder 1 outputs an n-length codeword $X_{11}, X_{12}, \cdots, X_{1n}$ ($X_{1i} \in \mathscr{X}_1$) corresponding to the message m_1 to the MAC. Similarly, encoder 2 outputs an n-length codeword $X_{21}, X_{22}, \cdots, X_{2n}$ ($X_{2i} \in \mathscr{X}_2$) corresponding to the message m_2 to the MAC. Then, for each time i ($1 \le i \le n$), the MAC outputs $Y_i \in \mathscr{Y}$, and the value of a deterministic function of channel outputs $Z_{1i} = \psi_{1i}(Y^i)$ is fed back to encoder 1 with unit delay. Similarly, $Z_{2i} = \psi_{2i}(Y^i)$ is fed back to encoder 2 with unit delay. Hence, past values of deterministic functions $Z_1^{i-1} = (Z_{11}, Z_{12}, \cdots, Z_{1,i-1})$ and $Z_2^{i-1} = (Z_{21}, Z_{22}, \cdots, Z_{2,i-1})$ are available at encoders. Thus, encoder 1 is defined by a set of $M_n^{(1)}$ code functions $\{f_1^n[m_1] \in \mathscr{F}_1^n\}_{m_1 \in \mathscr{M}_n^{(1)}}$, and encoder 2 is defined by a set of $M_n^{(2)}$ code functions $\{f_2^n[m_2] \in \mathscr{F}_2^n\}_{m_2 \in \mathscr{M}_n^{(2)}}$. We define their rates as

$$R_n^{(1)} \triangleq \frac{1}{n} \log M_n^{(1)} \text{ and } R_n^{(2)} \triangleq \frac{1}{n} \log M_n^{(2)}.$$

When the receiver observes all the channel outputs Y^n, one reconstructs the messages by using a decoder. The decoder decides that $(m_1, m_2) \in \mathscr{M}_n^{(1)} \times \mathscr{M}_n^{(2)}$ is transmitted from senders 1 and 2 if $Y^n \in \mathscr{D}_{m_1, m_2}$, where $\mathscr{D}_{m_1, m_2}$ is a disjoint partition of $\mathscr{Y}^n$ determined in advance. Thus, the decoder is defined by a map $\varphi_n : \mathscr{Y}^n \to \mathscr{M}_n^{(1)} \times \mathscr{M}_n^{(2)}$.

Next, we provide a precise formulation of general MACs. When there is feedback, the time ordering of channel inputs and channel outputs is

$$X_{11}, X_{21}, Y_1, X_{12}, X_{22}, Y_2, \cdots, X_{1n}, X_{2n}, Y_n.$$

This is because encoders must wait till values of deterministic functions are fed back. Hence, at time i, the channel output Y_i only depends on past channel inputs X_1^{i-1} and X_2^{i-1}, past channel outputs Y^{i-1}, and current channel inputs X_{1i} and X_{2i}. Thus, a MAC at each time i is characterized by a conditional distribution $W_{Y_i|X_1^iX_2^iY^{i-1}} : \mathscr{X}_1^i \times \mathscr{X}_2^i \times \mathscr{Y}^{i-1} \to \mathscr{Y}$. Then, a MAC until the time n is characterized by $W_{Y^n\|X_1^nX_2^n} \triangleq \prod_{i=1}^n W_{Y_i|X_1^iX_2^iY^{i-1}}$, where $W_{Y^n\|X_1^nX_2^n}$ is called the *causally conditioned distribution* of which we will give a precise definition in the next chapter. This represents the conditional distribution for channel outputs Y^n given channel inputs X_1^n and X_2^n when there exists feedback. Thus, we define a sequence $\{W_{Y^n\|X_1^nX_2^n}\}_{n=1}^\infty$ as a general MAC $\mathbf{W}$. Note that when a MAC is stationary memoryless, we can simply express a MAC $\mathbf{W}$ as a conditional distribution $W : \mathscr{X}_1 \times \mathscr{X}_2 \to \mathscr{Y}$.

We define an $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$-code as sets of $M_n^{(1)}$ and $M_n^{(2)}$ code functions, a decoder φ_n, and the error probability

$$\varepsilon_n = \frac{1}{M_n^{(1)}M_n^{(2)}} \sum_{m_1 \in \mathscr{M}_n^{(1)}} \sum_{m_2 \in \mathscr{M}_n^{(2)}} \cdot \Pr\left((m_1,m_2) \neq \varphi_n(Y^n)|m_1,m_2\right).$$

According to the definition of a MAC $\mathbf{W}$, we have

$$\Pr\left((m_1,m_2) \neq \varphi_n(Y^n)|m_1,m_2\right) = \sum_{y^n \in \mathscr{D}_{m_1,m_2}^c} \cdot W_{Y^n\|X_1^nX_2^n}(y^n\|f_1^n[m_1](\psi_1^{n-1}(y^{n-1})), f_2^n[m_2](\psi_2^{n-1}(y^{n-1}))), \tag{1}$$

where $\mathscr{D}_{m_1,m_2}^c$ denotes the complement of $\mathscr{D}_{m_1,m_2}$. Sequences $f_1^n[m_1](\psi_1^{n-1}(y^{n-1}))$ and $f_2^n[m_2](\psi_2^{n-1}(y^{n-1}))$ represent n-length codewords determined by massages m_1 and m_2, and deterministic feedbacks $\psi_1^{n-1}(y^{n-1})$ and $\psi_2^{n-1}(y^{n-1})$.

Now, define *achievability* and *capacity region* for a MAC $\mathbf{W}$ with deterministic feedback as follows:

Definition 1. A pair (R_1, R_2) is called *achievable* if there exists a sequence of $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$-codes such that $\lim_{n\to\infty} \varepsilon_n = 0$,

$$\liminf_{n\to\infty} R_n^{(1)} \geq R_1 \text{ and } \liminf_{n\to\infty} R_n^{(2)} \geq R_2.$$

The set of all achievable rates is called the *capacity region* for a pair of deterministic functions Ψ and a MAC $\mathbf{W}$ with deterministic feedback, and denoted by $\mathscr{C}^\Psi(\mathbf{W})$.

3. Causality and Directed Information

To represent the capacity region for general MACs with deterministic feedback, we use *directed information* introduced by Massey [4]. Here, we generalize Massey's notion [4] of directed information, and Kramer's notion [3], [5] of *causally conditional directed information* to the information-spectrum formula.

First, we introduce the *causally conditioned distribution* (cf. [3], [6], [12]).

Definition 2 (Causally conditioned distribution)**.** For a non-negative integer $d_Y \in \{0,1,2,\cdots\}$, we denote the causally conditioned distribution for X^n given Y^{n-d_Y} as $P_{X^n\|Y^{n-d_Y}}$ which is defined as

$$P_{X^n\|Y^{n-d_Y}}(x^n\|y^{n-d_Y}) \triangleq \prod_{i=1}^n P_{X_i|X^{i-1}Y^{i-d_Y}}(x_i|x^{i-1}, y^{i-d_Y}) \quad ({}^\forall x^n \in \mathscr{X}^n \text{ and } {}^\forall y^{n-d_Y} \in \mathscr{Y}^{n-d_Y}), \tag{2}$$

where we use the convention $P_{X_i|X^{i-1}Y^{i-d_Y}} = P_{X_i|X^{i-1}}$ when $i - d_Y \leq 0$.

According to this definition, we have

$$P_{X^nY^n}(x^n, y^n) = P_{X^n\|Y^{n-1}}(x^n\|y^{n-1})P_{Y^n\|X^n}(y^n\|x^n). \tag{3}$$

Since we define the causally conditioned distribution as a product of conditional distributions, whenever we use the notation $P_{X^n\|Y^{n-d_Y}}$, we implicitly assume that there exists a sequence of conditional distribution $\{P_{X_i|X^{i-1}Y^{i-d_Y}}\}_{i=1}^n$ that satisfies (2). Note that a causally conditioned distribution $P_{X^n\|Y^{n-d_Y}}$ determines a sequence of conditional distributions $\{P_{X_i|X^{i-1}Y^{i-d_Y}}\}_{i=1}^n$ such that $P_{X^n\|Y^{n-d_Y}} = \prod_{i=1}^n P_{X_i|X^{i-1}Y^{i-d_Y}}$ (see [12, Lemma 3]). Thus, we can identify a causally conditioned distribution $P_{X^n\|Y^{n-d_Y}}$ as a sequence of conditional distributions $\{P_{X_i|X^{i-1}Y^{i-d_Y}}\}_{i=1}^n$.

By using the notion of causally conditioning, we can define directed information and causally conditioned directed information as follows:

Definition 3 (Directed information)**.**

$$I(X^n \to Y^n) \triangleq E\left[\log \frac{P_{Y^n\|X^n}(Y^n\|X^n)}{P_{Y^n}(Y^n)}\right] = \sum_{i=1}^n I(X^i; Y_i|Y^{i-1}).$$

Definition 4 (Causally conditioned directed information)**.**

$$I(X^n \to Y^n\|Z^n) \triangleq E\left[\log \frac{P_{Y^n\|X^nZ^n}(Y^n\|X^n, Z^n)}{P_{Y^n\|Z^n}(Y^n\|Z^n)}\right] = \sum_{i=1}^n I(X^i; Y_i|Y^{i-1}Z^i).$$

Next, we introduce the *limit superior in probability* and *limit inferior in probability* [13]. For an arbitrary sequence of real-valued random variables $\{Z_n\}_{n=1}^\infty$, we define the following notion.

Definition 5 (Limit superior in probability)**.**

$$\text{p-}\limsup_{n\to\infty} Z_n \triangleq \inf\left\{\alpha : \lim_{n\to\infty} \Pr\{Z_n > \alpha\} = 0\right\}.$$

Definition 6 (Limit inferior in probability)**.**

$$\text{p-}\liminf_{n\to\infty} Z_n \triangleq \sup\left\{\beta : \lim_{n\to\infty}\Pr\{Z_n < \beta\} = 0\right\}.$$

Now, we define the information-spectrum formula of directed information and causally conditioned directed information [8].

Definition 7. For a given sequence of random variables $(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \{(X^n,Y^n,Z^n)\}_{n=1}^{\infty}$, we define

$$\overline{I}(\mathbf{X}\to\mathbf{Y}) \triangleq \text{p-}\limsup_{n\to\infty}\frac{1}{n}\log\frac{P_{Y^n\|X^n}(Y^n\|X^n)}{P_{Y^n}(Y^n)},$$

$$\underline{I}(\mathbf{X}\to\mathbf{Y}) \triangleq \text{p-}\liminf_{n\to\infty}\frac{1}{n}\log\frac{P_{Y^n\|X^n}(Y^n\|X^n)}{P_{Y^n}(Y^n)},$$

$$\overline{I}(\mathbf{X}\to\mathbf{Y}\|\mathbf{Z}) \triangleq \text{p-}\limsup_{n\to\infty}\frac{1}{n}\log\frac{P_{Y^n\|X^nZ^n}(Y^n\|X^n,Z^n)}{P_{Y^n\|Z^n}(Y^n\|Z^n)},$$

$$\underline{I}(\mathbf{X}\to\mathbf{Y}\|\mathbf{Z}) \triangleq \text{p-}\liminf_{n\to\infty}\frac{1}{n}\log\frac{P_{Y^n\|X^nZ^n}(Y^n\|X^n,Z^n)}{P_{Y^n\|Z^n}(Y^n\|Z^n)}.$$

4. The Capacity Region for General MACs with Deterministic Feedback

In this chapter, we show a general formula of the capacity region for general MACs with deterministic feedback.

Let $\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}} \triangleq \{P_{X_1^n\|Y^{n-1}}\}_{n=1}^{\infty}$ and $\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}} \triangleq \{P_{X_2^n\|Y^{n-1}}\}_{n=1}^{\infty}$ be a sequence of causally conditioned distributions. These represent channel input distributions. The set of all pairs $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})$ is denoted by $\mathscr{S}$. Let $\mathscr{S}^{\Psi}\subseteq\mathscr{S}$ be the set of all pairs $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})$ such that

$$P_{X_k^n\|Y^{n-1}}(x_k^n\|y^{n-1}) = P_{X_k^n\|Z_k^{n-1}}(x_k^n\|\psi_k^{n-1}(y^{n-1}))\quad (k=1,2), \tag{4}$$

where $\psi_k^{n-1}(y^{n-1}) = (\psi_{k1}(y_1),\psi_{k2}(y^2),\cdots,\psi_{k,n-1}(y^{n-1}))$. The condition (4) means, for a given $z_k^{n-1}\in\mathscr{Z}_k^{n-1}$, all channel input probabilities causally conditioned on $y^{n-1}\in\{y^{n-1}\in\mathscr{Y}^{n-1} : \psi^{n-1}(y^{n-1}) = z_k^{n-1}\}$ are the same probability. Then, we have the next theorem.

Theorem 1. For a general MAC $\mathbf{W}$ and a pair of deterministic functions Ψ, we have

$$\mathscr{C}^{\Psi}(\mathbf{W}) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^{\Psi}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}), \tag{5}$$

where

$$\mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \triangleq \{(R_1,R_2) : 0\le R_1\le \underline{I}(\mathbf{X}_1\to\mathbf{Y}\|\mathbf{X}_2),\ 0\le R_2\le\underline{I}(\mathbf{X}_2\to\mathbf{Y}\|\mathbf{X}_1),\ R_1+R_2\le\underline{I}(\mathbf{X}_1,\mathbf{X}_2\to\mathbf{Y})\}.$$

Here, $\underline{I}(\mathbf{X}_1\to\mathbf{Y}\|\mathbf{X}_2)$, $\underline{I}(\mathbf{X}_2\to\mathbf{Y}\|\mathbf{X}_1)$ and $\underline{I}(\mathbf{X}_1,\mathbf{X}_2\to\mathbf{Y})$ are calculated by $(\mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}) = \{(X_1^n,X_2^n,Y^n)\}_{n=1}^{\infty}$ subject to the joint probability distribution $\{P_{X_1^nX_2^nY^n}\}_{n=1}^{\infty}$ such that, for all $n=1,2,\cdots$,

$$P_{X_1^nX_2^nY^n}(x_1^n,x_2^n,y^n) = P_{X_1^n\|Y^{n-1}}(x_1^n\|y^{n-1})P_{X_2^n\|Y^{n-1}}(x_2^n\|y^{n-1}) \cdot W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n). \tag{6}$$

Note that the right-hand side of (5) is a closed set (see [13, Remark 7.7.1]).

When a pair of deterministic functions Ψ satisfies $\psi_{1i}(y^i) = y_{i-d_1+1}$ and $\psi_{2i}(y^i) = y_{i-d_2+1}$ for all $i=1,2,\cdots$, we call this case (d_1,d_2)-delayed feedback, where d_1 and d_2 are positive integers which represent delays. We use the convention that $\psi_{ki}(y^i) = y_0$ for some $y_0\in\mathscr{Y}$ when $i-d_k+1\le 0$ $(k=1,2)$. Then, we have the next corollary.

Corollary 1. For a general MAC $\mathbf{W}$ with (d_1,d_2)-delayed feedback, we have

$$\mathscr{C}^{\Psi}(\mathbf{W}) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}^{-d_1}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}^{-d_2}})\in\mathscr{S}^{d_1,d_2}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}^{-d_1}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}^{-d_2}}),$$

where $\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}^{-d_1}} \triangleq \{P_{X_1^n\|Y^{n-d_1}}\}$, $\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}^{-d_2}} \triangleq \{P_{X_2^n\|Y^{n-d_2}}\}$ and $\mathscr{S}^{d_1,d_2}$ is the set of all pairs $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}^{-d_1}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}^{-d_2}})$.

Note that when $d_1=d_2=1$, i.e., perfect feedback is available at encoders, $\mathscr{S}^{d_1,d_2}=\mathscr{S}$. Hence, in the case of perfect feedback, we denote $\mathscr{C}^{\Psi}(\mathbf{W})$ by $\mathscr{C}(\mathbf{W})$ for the sake of convenience. We also note that the capacity region for the perfect feedback case is shown by Tatikonda and Mitter without the proof [8].

5. Proof of Theorem 1

In this chapter, we prove Theorem 1. To this end, we follow the Tatikonda and Mitter's [8], [10], [11] method which was used to clarify the channel capacity for general one-to-one channels with feedback.

5.1 Interconnection of Code Functions to the MAC

In a general MAC with deterministic feedback, the channel from $\mathscr{F}_1^n$ and $\mathscr{F}_2^n$ to $\mathscr{Y}^n$ can be considered as a general MAC *without* deterministic feedback (this channel is framed by a dotted line in Fig. 1). Hence, any coding problem for the general MAC with deterministic feedback can be reduced to the coding problem for the general MAC without deterministic feedback. This is a key idea to prove Theorem 1, because this idea allows us to use Han's method [7] which was used to clarify the capacity region for general MACs without feedback. This key idea is an extension of Tatikonda and Mitter's idea [11]. In order to clarify the capacity region for general MACs without feedback, Han used the random-coding method, and introduced a probability distribution on the set of channel inputs. For the channel from $\mathscr{F}_1^n$ and $\mathscr{F}_2^n$ to $\mathscr{Y}^n$, i.e., the channel framed by the dotted line in Fig. 1, the channel inputs are code functions. Hence, to apply the random-coding method for the system of the MAC with deterministic feedback, we have to introduce the *code-function distribution*, i.e., a probability distribution on the set of code functions. We will denote this distributions on $\mathscr{F}_1^n$ and $\mathscr{F}_2^n$ as $P_{F_1^n}$ and $P_{F_2^n}$, respectively. Then, we consider the following situation. Code functions F_1^n and F_2^n are independently selected with the code-function distributions $P_{F_1^n}$ and $P_{F_2^n}$. For each time i, X_{1i} and X_{2i}, which are determined by functions F_1^n and F_2^n, and past values of deterministic functions of channel outputs $Z_1^{i-1} = \psi_1^{i-1}(Y^{i-1})$ and $Z_2^{i-1} = \psi_2^{i-1}(Y^{i-1})$, are fed into a MAC $\mathbf{W}$. Under this situation, let $Q_{F_1^nF_2^nX_1^nX_2^nY^nZ_1^nZ_2^n}$ be a joint probability distribution of code functions F_1^n and F_2^n, channel inputs X_1^n and X_2^n, channel outputs Y^n, and values of deterministic functions Z_1^n and Z_2^n, i.e., $Q_{F_1^nF_2^nX_1^nX_2^nY^nZ_1^nZ_2^n}$ is the probability distribution of the overall system of the MAC with deterministic feedback when we use the random code. Then, the distribution $Q_{F_1^nF_2^nX_1^nX_2^nY^nZ_1^nZ_2^n}$ can be represented by

$$\begin{aligned} &Q_{F_1^nF_2^nX_1^nX_2^nY^nZ_1^nZ_2^n}(f_1^n,f_2^n,x_1^n,x_2^n,y^n,z_1^n,z_2^n) \\ &= P_{F_1^n}(f_1^n)P_{F_2^n}(f_2^n)1_{\{f_1^n(z_1^{n-1})\}}(x_1^n)1_{\{f_2^n(z_2^{n-1})\}}(x_2^n) \\ &\quad \cdot W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n)1_{\{\psi_1^n(y^n)\}}(z_1^n)1_{\{\psi_2^n(y^n)\}}(z_2^n), \quad (7) \end{aligned}$$

where $1_{\mathscr{X}}(x)$ denotes the indicator function defined as

$$1_{\mathscr{X}}(x) \triangleq \begin{cases} 1 & \text{if } x \in \mathscr{X}, \\ 0 & \text{if } x \notin \mathscr{X}. \end{cases}$$

We call $Q_{F_1^nF_2^nX_1^nX_2^nY^nZ_1^nZ_2^n}$ the *consistent distribution*. When there is no confusion, we shall simply write Q_n for $Q_{F_1^nF_2^nX_1^nX_2^nY^nZ_1^nZ_2^n}$.

The consistent distribution Q_n is useful to deal with input-output relations in the system of the MAC with deterministic feedback. By using consistent distribution Q_n, the channel from $\mathscr{F}_1^n$ and $\mathscr{F}_2^n$ to $\mathscr{Y}^n$ in the system of the general MAC with deterministic feedback can be denoted as $\{Q_{Y^n|F_1^nF_2^n}\}_{n=1}^\infty$, where $Q_{Y^n|F_1^nF_2^n}$ is written as

$$\begin{aligned} &Q_{Y^n|F_1^nF_2^n}(y^n|f_1^n,f_2^n) \\ &= W_{Y^n\|X_1^nX_2^n}(y^n\|f_1^n(\psi_1^{n-1}(y^{n-1})),f_2^n(\psi_2^{n-1}(y^{n-1}))) \end{aligned} \quad (8)$$

according to (7). Since the channel from $\mathscr{F}_1^n$ and $\mathscr{F}_2^n$ to $\mathscr{Y}^n$ is independent of code-function distributions, it is natural that $Q_{Y^n|F_1^nF_2^n}$ is independent of code-function distributions. Hence, we use Han's random-coding method to the general MAC $\{Q_{Y^n|F_1^nF_2^n}\}_{n=1}^\infty$ without deterministic feedback, and then prove Theorem 1. To this end, we use some properties of the consistent distribution which will be shown in the next section.

5.2 Some Properties of Consistent Distribution

In this section, we show some useful properties of the consistent distribution.

In order to apply the random-coding method, we use a code-function distribution, and we assume that a codeword is randomly selected depending on a code-function distribution even if channel outputs are given. In other words, a channel input is induced by a given code-function distribution. By using the consistent distribution Q_n, for given channel outputs, the probability distribution of channel inputs can be represented by $Q_{X_k^n\|Y^{n-1}}$ $(k=1,2)$. The reason that this probability distribution has causality is that a channel input for each time depends on past channel outputs. In what follows, we call $Q_{X_k^n\|Y^{n-1}}$ $(k=1,2)$ *induced channel input distributions*. In the following Lemma 1 - Lemma 4, we discuss some properties of induced channel input distribution. To this end, we introduce following definitions. For each $k=1,2$, define $\Upsilon_{ki}(z_k^{i-1},x_{ki})$ and $\Upsilon_k^i(z_k^{i-1},x_k^i)$ as

$$\Upsilon_{ki}(z_k^{i-1},x_{ki}) \triangleq \{f_{ki} \in \mathscr{F}_{ki} : f_{ki}(z_k^{i-1}) = x_{ki}\} \quad (i \geq 2),$$

$$\Upsilon_k^i(z_k^{i-1},x_k^i) \triangleq \Upsilon_{k1}(x_{k1}) \bigtimes_{j=2}^{i} \Upsilon_{kj}(z_k^{j-1},x_{kj}) \subseteq \mathscr{F}_k^i \quad (i \geq 2),$$

while $\Upsilon_{k1}(x_{k1}) \triangleq \{x_{k1}\}$ and $\Upsilon_k^1(x_{k1}) \triangleq \Upsilon_{k1}(x_{k1})$.

We show that induced channel input distributions only depend on code-function distributions, and show a property of the product of induced channel input distributions.

Lemma 1. For a given consistent distribution Q_n, we have

$$\begin{aligned} &Q_{X_k^n\|Y^{n-1}}(x_k^n\|y^{n-1}) = P_{F_k^n}(\Upsilon_k^n(\psi_k^{n-1}(y^{n-1}),x_k^n)) \quad (k=1,2), \\ &Q_{X_1^nX_2^n\|Y^{n-1}}(x_1^n,x_2^n\|y^{n-1}) \\ &= Q_{X_1^n\|Y^{n-1}}(x_1^n\|y^{n-1})Q_{X_2^n\|Y^{n-1}}(x_2^n\|y^{n-1}). \end{aligned} \quad (9)$$

Proof. By following the proof of [11, Lemma 5.1], for all $i = 1,2\cdots$, the marginal distribution $Q_{X_1^iY^{i-1}}$ is given by

$$\begin{aligned} &Q_{X_1^iY^{i-1}}(x_1^i,y^{i-1}) \\ &= P_{F_{1i}|F_1^{i-1}}(\Upsilon_{1i}(\psi_1^{i-1}(y^{i-1}),x_{1i})|\Upsilon_1^{i-1}(\psi_1^{i-2}(y^{i-2}),x_1^{i-1})) \\ &\quad \cdot Q_{X_1^{i-1}Y^{i-1}}(x_1^{i-1},y^{i-1}). \end{aligned}$$

Thus, according to the definition (2) of the causally conditioned distribution, we have

$$\begin{aligned} Q_{X_1^n\|Y^{n-1}}(x_1^n\|y^{n-1}) &= \prod_{i=1}^n \frac{Q_{X_1^iY^{i-1}}(x_1^i,y^{i-1})}{Q_{X_1^{i-1}Y^{i-1}}(x_1^{i-1},y^{i-1})} \\ &= P_{F_1^n}(\Upsilon_1^n(\psi_1^{n-1}(y^{n-1}),x_1^n)). \end{aligned}$$

Similarly, we also have

$$Q_{X_2^n\|Y^{n-1}}(x_2^n\|y^{n-1}) = P_{F_2^n}(\Upsilon_2^n(\psi_2^{n-1}(y^{n-1}),x_2^n)),$$

and

$$\begin{aligned} Q_{X_1^nX_2^n\|Y^{n-1}}(x_1^n,x_2^n\|y^{n-1}) &= P_{F_1^n}(\Upsilon_1^n(\psi_1^{n-1}(y^{n-1}),x_1^n)) \\ &\quad \cdot P_{F_2^n}(\Upsilon_2^n(\psi_2^{n-1}(y^{n-1}),x_2^n)). \end{aligned}$$

These identities imply the lemma. □

The following lemma shows that induced channel input distributions always satisfy the condition (4). Hence, in Theorem 1, we can restrict a channel input distribution to an element of $\mathscr{S}^\Psi$.

Lemma 2. For a given consistent distribution Q_n, we have

$$Q_{X_k^n\|Y^{n-1}}(x_k^n\|y^{n-1}) = Q_{X_k^n\|Z_k^{n-1}}(x_k^n\|\psi_k^{n-1}(y^{n-1})) \quad (k=1,2).$$

Proof. The proof can be done in a similar manner to the proof of Lemma 1. For all $i = 1, 2, \cdots$, we have

$$\begin{aligned}&Q_{X_1^i Z_1^{i-1}}(x_1^i, \psi_1^{i-1}(y^{i-1}))\\&= P_{F_{1i}|F_1^{i-1}}(\Upsilon_{1i}(\psi_1^{i-1}(y^{i-1}), x_{1i})|\Upsilon_1^{i-1}(\psi_1^{i-2}(y^{i-2}), x_1^{i-1}))\\&\quad \cdot Q_{X_1^{i-1} Z_1^{i-1}}(x_1^{i-1}, \psi_1^{i-1}(y^{i-1})).\end{aligned}$$

Thus, we have

$$\begin{aligned}Q_{X_k^n \| Z_k^{n-1}}(x_k^n \| \psi_k^{n-1}(y^{n-1})) &= P_{F_k^n}(\Upsilon_k^n(\psi_k^{n-1}(y^{n-1}), x_k^n))\\&= Q_{X_k^n \| Y^{n-1}}(x_k^n \| y^n),\end{aligned}$$

where the last equality comes from Lemma 1. □

Now, we introduce the following notion. This notion was first introduced by Tatikonda [10].

Definition 8. For each $k = 1, 2$, we call a code-function distribution $P_{F_k^n}$ *good with respect to the channel input distribution* $P_{X_k^n \| Y^{n-1}}$ if

$$P_{F_k^n}(\Upsilon_k^n(\psi_k^{n-1}(y^{n-1}), x_k^n)) = P_{X_k^n \| Y^{n-1}}(x_k^n \| y^{n-1})$$

for all $x_k^n \in \mathscr{X}_k^n$ and $y^{n-1} \in \mathscr{Y}^{n-1}$.

Then, we have the following two lemmas similar to [11, Lemma 5.3 and 5.4]. Note that our lemmas use a causally conditioned distribution instead of a sequence of conditional distributions used in [11, Lemma 5.3 and 5.4]. However, since a causally conditioned distribution can be identified as a sequence of conditional distributions as we mentioned in Chapter 3, we can prove following lemmas in a similar way to the proofs of [11, Lemma 5.3 and 5.4].

Lemma 3. For each $k = 1, 2$, and a given channel input distribution $P_{X_k^n \| Y^{n-1}}$, the induced channel input distribution satisfies

$$Q_{X_k^n \| Y^{n-1}}(x_k^n \| y^{n-1}) = P_{X_k^n \| Y^{n-1}}(x_k^n \| y^{n-1}) \tag{10}$$

if and only if the code-function distribution $P_{F_k^n}$ is good with respect to $P_{X_k^n \| Y^{n-1}}$.

Lemma 4. For each $k = 1, 2$, given a deterministic function ψ_k^{n-1}, and any channel input distribution $P_{X_k^n \| Y^{n-1}}$ such that

$$P_{X_k^n \| Y^{n-1}}(x_k^n \| y^{n-1}) = P_{X_k^n \| Z_k^{n-1}}(x_k^n \| \psi_k^{n-1}(y^{n-1})), \tag{11}$$

there exists a code-function distribution $P_{F_k^n}$ that is good with respect to $P_{X_k^n \| Y^{n-1}}$.

According to Lemma 3 and Lemma 4, for any channel input distribution $P_{X_k^n \| Y^{n-1}}$ satisfying the condition (4), we have an induced channel input distribution $Q_{X_k^n \| Y^{n-1}}$ which is equal to $P_{X_k^n \| Y^{n-1}}$ by a proper choice of a code-function distribution $P_{F_k^n}$. This result of the induced channel input distribution plays an important role in order to prove the direct part of Theorem 1.

The next lemma is fundamental in this paper.

Lemma 5. For any $(f_1^n, f_2^n, x_1^n, x_2^n, y^n, z_1^n, z_2^n)$ satisfying $Q_n(f_1^n, f_2^n, x_1^n, x_2^n, y^n, z_1^n, z_2^n) > 0$, we have

$$\frac{Q_{Y^n|F_1^n F_2^n}(y^n|f_1^n, f_2^n)}{Q_{Y^n|F_2^n}(y^n|f_2^n)} = \frac{Q_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}{Q_{Y^n \| X_2^n}(y^n \| x_2^n)}, \tag{12}$$

$$\frac{Q_{Y^n|F_1^n F_2^n}(y^n|f_2^n, f_1^n)}{Q_{Y^n|F_1^n}(y^n|f_1^n)} = \frac{Q_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}{Q_{Y^n \| X_1^n}(y^n \| x_1^n)}, \tag{13}$$

and

$$\frac{Q_{Y^n|F_1^n F_2^n}(y^n|f_1^n, f_2^n)}{Q_{Y^n}(y^n)} = \frac{Q_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}{Q_{Y^n}(y^n)}. \tag{14}$$

The relation (14) in this lemma is analogous to [11, Lemma 5.2], and can be proved in a similar way to the proof of [11, Lemma 5.2]. However, relations (12) and (13) do not appear in the one-to-one channel, and hence we give the proof of these relations. To this end, we use the next lemma.

Lemma 6. For a given consistent distribution Q_n, we have

$$\begin{aligned}Q_{Y^n \| X_1^n}(y^n \| x_1^n) &= \sum_{f_2^n} \sum_{x_2^n} P_{F_2^n}(f_2^n) 1_{\{f_2^n(\psi_2^{n-1}(y^{n-1}))\}}(x_2^n)\\&\quad \cdot W_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n),\\Q_{Y^n \| X_2^n}(y^n \| x_2^n) &= \sum_{f_1^n} \sum_{x_1^n} P_{F_1^n}(f_1^n) 1_{\{f_1^n(\psi_1^{n-1}(y^{n-1}))\}}(x_1^n)\\&\quad \cdot W_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n),\end{aligned}$$

and

$$Q_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n) = W_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n). \tag{15}$$

Proof. We have

$$\begin{aligned}&Q_{X_1^n Y^n}(x_1^n, y^n)\\&= \sum_{f_1^n} \sum_{f_2^n} \sum_{x_2^n} P_{F_1^n}(f_1^n) P_{F_2^n}(f_2^n) 1_{\{f_1^n(\psi_1^{n-1}(y^{n-1}))\}}(x_1^n)\\&\quad \cdot 1_{\{f_2^n(\psi_2^{n-1}(y^{n-1}))\}}(x_2^n) W_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)\\&= P_{F_1^n}(\Upsilon_1^n(\psi_1^{n-1}(y^{n-1}), x_1^n))\\&\quad \cdot \sum_{f_2^n} \sum_{x_2^n} P_{F_2^n}(f_2^n) 1_{\{f_2^n(\psi_2^{n-1}(y^{n-1}))\}}(x_2^n)\\&\quad \cdot W_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n),\end{aligned} \tag{16}$$

$$\begin{aligned}&Q_{X_2^n Y^n}(x_2^n, y^n)\\&= P_{F_2^n}(\Upsilon_2^n(\psi_2^{n-1}(y^{n-1}), x_2^n))\\&\quad \cdot \sum_{f_1^n} \sum_{x_1^n} P_{F_1^n}(f_1^n) 1_{\{f_1^n(\psi_1^{n-1}(y^{n-1}))\}}(x_1^n)\\&\quad \cdot W_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n),\end{aligned} \tag{17}$$

and

$$\begin{aligned}&Q_{X_1^n X_2^n Y^n}(x_1^n, x_2^n, y^n)\\&= \sum_{f_1^n} \sum_{f_2^n} P_{F_1^n}(f_1^n) P_{F_2^n}(f_2^n) 1_{\{f_1^n(\psi_1^{n-1}(y^{n-1}))\}}(x_1^n)\end{aligned}$$

$$\cdot 1_{\{f_2^n(\psi_2^{n-1}(y^{n-1}))\}}(x_2^n) W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n)$$
$$= P_{F_1^n}(\Upsilon_1^n(\psi_1^{n-1}(y^{n-1}),x_1^n))P_{F_2^n}(\Upsilon_2^n(\psi_2^{n-1}(y^{n-1}),x_2^n))$$
$$\cdot W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n). \quad (18)$$

Thus, by combining (3), Lemma 1 and (16)–(18), we have the lemma. □

According to (3), (9) and (15), we have

$$\begin{aligned} &Q_{X_1^nX_2^nY^n}(x_1^n,x_2^n,y^n) \\ &= Q_{X_1^nX_2^n\|Y^{n-1}}(x_1^n,x_2^n\|y^{n-1})Q_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n) \\ &= Q_{X_1^n\|Y^{n-1}}(x_1^n\|y^{n-1})Q_{X_2^n\|Y^{n-1}}(x_2^n\|y^{n-1}) \\ &\quad \cdot W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n). \end{aligned} \quad (19)$$

This justifies that the joint probability distribution of channel inputs and outputs can be denoted as (6) in Theorem 1.

Now, we prove Lemma 5.

Proof of Lemma 5. $Q_n(f_1^n,f_2^n,x_1^n,x_2^n,y^n,z_1^n,z_2^n)>0$ implies

$$f_1^n(\psi_1^{n-1}(y^{n-1})) = x_1^n \text{ and } f_2^n(\psi_2^{n-1}(y^{n-1})) = x_2^n. \quad (20)$$

Hence, we have

$$\begin{aligned} Q_{Y^n|F_1^nF_2^n}(y^n|f_1^n,f_2^n) &\overset{\text{(a)}}{=} W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n) \\ &\overset{\text{(b)}}{=} Q_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n), \end{aligned} \quad (21)$$

where (a) comes from (8), and (b) comes from Lemma 6. On the other hand, we have

$$\begin{aligned} &Q_{F_2^nY^n}(f_2^n,y^n) \\ &= \sum_{\tilde{f}_1^n}\sum_{\tilde{x}_1^n}\sum_{\tilde{x}_2^n}\sum_{\tilde{z}_1^n}\sum_{\tilde{z}_2^n} P_{F_1^n}(\tilde{f}_1^n)P_{F_2^n}(f_2^n)1_{\{\tilde{f}_1^n(\tilde{z}_1^{n-1})\}}(\tilde{x}_1^n) \\ &\quad \cdot 1_{\{f_2^n(\tilde{z}^{n-1})\}}(\tilde{x}_2^n)W_{Y^n\|X_1^nX_2^n}(y^n\|\tilde{x}_1^n,\tilde{x}_2^n) \\ &\quad \cdot 1_{\{\psi_1^n(y^n)\}}(\tilde{z}_1^n)1_{\{\psi_2^n(y^n)\}}(\tilde{z}_2^n) \\ &= P_{F_2^n}(f_2^n)\sum_{\tilde{f}_1^n}\sum_{\tilde{x}_1^n} P_{F_1^n}(\tilde{f}_1^n)1_{\{\tilde{f}_1^n(\psi_1^{n-1}(y^{n-1}))\}}(\tilde{x}_1^n) \\ &\quad \cdot \sum_{\tilde{x}_2^n} 1_{\{f_2^n(\psi_2^{n-1}(y^{n-1}))\}}(\tilde{x}_2^n)W_{Y^n\|X_1^nX_2^n}(y^n\|\tilde{x}_1^n,\tilde{x}_2^n) \\ &\overset{\text{(c)}}{=} P_{F_2^n}(f_2^n)\sum_{\tilde{f}_1^n}\sum_{\tilde{x}_1^n} P_{F_1^n}(\tilde{f}_1^n)1_{\{\tilde{f}_1^n(\psi_1^{n-1}(y^{n-1}))\}}(\tilde{x}_1^n) \\ &\quad \cdot W_{Y^n\|X_1^nX_2^n}(y^n\|\tilde{x}_1^n,x_2^n), \end{aligned}$$

where (c) follows from (20). Hence, we obtain

$$\begin{aligned} Q_{Y^n|F_2^n}(y^n|f_2^n) &\overset{\text{(d)}}{=} \sum_{\tilde{f}_1^n}\sum_{\tilde{x}_1^n} P_{F_1^n}(\tilde{f}_1^n)1_{\{\tilde{f}_1^n(\psi^{n-1}(y^{n-1}))\}}(\tilde{x}_1^n) \\ &\qquad \cdot W_{Y^n\|X_1^nX_2^n}(y^n\|\tilde{x}_1^n,x_2^n) \\ &\overset{\text{(e)}}{=} Q_{Y^n\|X_2^n}(y^n\|x_2^n), \end{aligned} \quad (22)$$

where (d) comes from the fact that $Q_{F_2^n}=P_{F_2^n}$ which can be obtained by (7), and (e) comes from Lemma 6. Similarly, we also have

$$Q_{Y^n|F_1^n}(y^n|f_1^n) = Q_{Y^n\|X_1^n}(y^n\|x_1^n). \quad (23)$$

By combining (21), (22) and (23), we obtain the lemma. □

5.3 The Proof of Theorem 1

First, we prove the direct part. To this end, we employ the following lemma.

Lemma 7. We are given a pair of deterministic functions Ψ. For each $k=1,2$, let $P_{X_k^n\|Y^{n-1}}$ satisfy

$$P_{X_k^n\|Y^{n-1}}(x_k^n\|y^{n-1}) = P_{X_k^n\|Z_k^{n-1}}(x_k^n\|\psi_k^{n-1}(y^{n-1})),$$

for some $P_{X_k^n\|Z_k^{n-1}}$. Then, for a general MAC $\mathbf{W}$, any $\gamma>0$, $n=1,2,\cdots$ and any positive integers $M_n^{(1)}$ and $M_n^{(2)}$, there exists an $(n,M_n^{(1)},M_n^{(2)},\varepsilon_n)$-code satisfying

$$\begin{aligned} \varepsilon_n \le\, & P_{X_1^nX_2^nY^n}\Bigg(\frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{P_{Y^n\|X_2^n}(Y^n\|X_2^n)} \le R_n^{(1)}+\gamma \\ &\text{or } \frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{P_{Y^n\|X_1^n}(Y^n\|X_1^n)} \le R_n^{(2)}+\gamma \\ &\text{or } \frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{P_{Y^n}(Y^n)} \le R_n^{(1)}+R_n^{(2)}+\gamma\Bigg) \\ &+3e^{-n\gamma}, \end{aligned}$$

where $P_{X_1^nX_2^nY^n} = P_{X_1^n\|Y^{n-1}}\cdot P_{X_2^n\|Y^{n-1}}\cdot W_{Y^n\|X_1^nX_2^n}$.

Proof. Let $P_{F_1^n}$ and $P_{F_2^n}$ be code-function distributions good with respect to the channel input distribution $P_{X_1^n\|Y^{n-1}}$ and $P_{X_2^n\|Y^{n-1}}$, respectively. The existence of such code-function distributions $P_{F_1^n}$ and $P_{F_2^n}$ are guaranteed by Lemma 4. Let Q_n be the consistent distribution determined by $P_{F_1^n}$, $P_{F_2^n}$ and the MAC $\mathbf{W}$. As we mentioned in Sect. 5.1, the sequence of conditional distributions of code functions to channel outputs $\{Q_{Y^n|F_1^nF_2^n}\}_{n=1}^\infty$ can be considered as a MAC without feedback. Then, by following the proof of [7, Lemma 3], for the MAC without feedback $\{Q_{Y^n|F_1^nF_2^n}\}_{n=1}^\infty$ and any $\gamma>0$, we can show the existence of an $(n,M_n^{(1)},M_n^{(2)},\varepsilon_n)$-code that satisfies

$$\begin{aligned} \varepsilon_n \le\, & Q_{F_1^nF_2^nY^n}\Bigg(\frac{1}{n}\log\frac{Q_{Y^n|F_1^nF_2^n}(Y^n|F_1^n,F_2^n)}{Q_{Y^n|F_2^n}(Y^n|F_2^n)} \le R_n^{(1)}+\gamma \\ &\text{or } \frac{1}{n}\log\frac{Q_{Y^n|F_1^nF_2^n}(Y^n|F_1^n,F_2^n)}{Q_{Y^n|F_1^n}(Y^n|F_1^n)} \le R_n^{(2)}+\gamma \\ &\text{or } \frac{1}{n}\log\frac{Q_{Y^n|F_1^nF_2^n}(Y^n|F_1^n,F_2^n)}{Q_{Y^n}(Y^n)} \le R_n^{(1)}+R_n^{(2)}+\gamma\Bigg) \\ &+3e^{-n\gamma}. \end{aligned} \quad (24)$$

According to Lemma 5, (24) implies

$$\varepsilon_n \le Q_{X_1^nX_2^nY^n}\Bigg(\frac{1}{n}\log\frac{Q_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{Q_{Y^n\|X_2^n}(Y^n\|X_2^n)} \le R_n^{(1)}+\gamma$$

$$\text{or } \frac{1}{n}\log\frac{Q_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{Q_{Y^n\|X_1^n}(Y^n\|X_1^n)} \le R_n^{(2)}+\gamma$$
$$\left.\text{or } \frac{1}{n}\log\frac{Q_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{Q_{Y^n}(Y^n)} \le R_n^{(1)}+R_n^{(2)}+\gamma\right)$$
$$+3e^{-n\gamma}.$$

On the other hand, according to the assumption of the lemma, Lemma 3 shows $Q_{X_1^n\|Y^{n-1}} = P_{X_1^n\|Y^{n-1}}$ and $Q_{X_2^n\|Y^{n-1}} = P_{X_2^n\|Y^{n-1}}$. Hence, by noting (19), we have $Q_{1}^{n}{}_{X_2^nY^n} = P_{X_1^nX_2^nY^n}$. This completes the proof. □

Theorem 2.

$$\mathscr{C}^{\Psi}(\mathbf{W}) \supseteq \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^{\Psi}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}).$$

Proof. By using Lemma 7, we can prove the theorem in a similar way to the proof of the direct part of [7, Theorem 1]. □

Next, we prove the converse part.

Lemma 8. For any $\gamma > 0$ and all $n = 1,2,\cdots$, every $(n,M_n^{(1)},M_n^{(2)},\varepsilon_n)$-code satisfies

$$\varepsilon_n \ge Q_{X_1^nX_2^nY^n}\left(\frac{1}{n}\log\frac{Q_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{Q_{Y^n\|X_2^n}(Y^n\|X_2^n)} \le R_n^{(1)}-\gamma\right.$$
$$\text{or } \frac{1}{n}\log\frac{Q_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{Q_{Y^n\|X_1^n}(Y^n\|X_1^n)} \le R_n^{(2)}-\gamma$$
$$\left.\text{or } \frac{1}{n}\log\frac{Q_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{Q_{Y^n}(Y^n)} \le R_n^{(1)}+R_n^{(2)}-\gamma\right)$$
$$-3e^{-n\gamma}, \tag{25}$$

where Q_n is the consistent distribution defined by a general MAC $\mathbf{W}$, and uniform distributions $P_{F_1^n}$ and $P_{F_2^n}$ over $M_n^{(1)}$ code functions $\{f_1^n[m_1]\}_{m_1\in\mathscr{M}_n^{(1)}}$ and $M_n^{(2)}$ code functions $\{f_2^n[m_2]\}_{m_2\in\mathscr{M}_n^{(2)}}$, respectively.

Proof. By following the proof of [7, Lemma 4], for the MAC without feedback $\{Q_{Y^n|F_1^nF_2^n}\}_{n=1}^{\infty}$, we can show that any $(n,M_n^{(1)},M_n^{(2)},\varepsilon_n)$-code satisfies

$$\varepsilon_n \ge Q_{F_1^nF_2^nY^n}\left(\frac{1}{n}\log\frac{Q_{Y^n|F_1^nF_2^n}(Y^n|F_1^n,F_2^n)}{Q_{Y^n|F_2^n}(Y^n|F_2^n)} \le R_n^{(1)}-\gamma\right.$$
$$\text{or } \frac{1}{n}\log\frac{Q_{Y^n|F_1^nF_2^n}(Y^n|F_1^n,F_2^n)}{Q_{Y^n|F_1^n}(Y^n|F_1^n)} \le R_n^{(2)}-\gamma$$
$$\left.\text{or } \frac{1}{n}\log\frac{Q_{Y^n|F_1^nF_2^n}(Y^n|F_1^n,F_2^n)}{Q_{Y^n}(Y^n)} \le R_n^{(1)}+R_n^{(2)}-\gamma\right)$$
$$-3e^{-n\gamma}.$$

Thus, by using Lemma 5, we have (25). □

Theorem 3.

$$\mathscr{C}^{\Psi}(\mathbf{W}) \subseteq \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^{\Psi}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}).$$

Proof. Suppose that a rate pair (R_1,R_2) is achievable. According to the definition of the achievability, there exists a sequence of $(n,M_n^{(1)},M_n^{(2)},\varepsilon_n)$-codes such that $\lim_{n\to\infty}\varepsilon_n = 0$, $\liminf_{n\to\infty} R_n^{(1)} \ge R_1$ and $\liminf_{n\to\infty} R_n^{(2)} \ge R_2$. For all $n = 1,2,\cdots$, and the sequence of $(n,M_n^{(1)},M_n^{(2)},\varepsilon_n)$-codes, let Q_n be the consistent distribution defined by $\mathbf{W}$ and uniform distributions $P_{F_1^n}$ and $P_{F_2^n}$ over $M_n^{(1)}$ code functions $\{f_1^n[m_1]\}_{m_1\in\mathscr{M}_n^{(1)}}$ and $M_n^{(2)}$ code functions $\{f_2^n[m_2]\}_{m_2\in\mathscr{M}_n^{(2)}}$, respectively. Then, by using Lemma 8, we can show that $R_1 \le \underline{I}(\mathbf{X}_1\to\mathbf{Y}\|\mathbf{X}_2)$, $R_2 \le \underline{I}(\mathbf{X}_2\to\mathbf{Y}\|\mathbf{X}_1)$, and $R_1+R_2 \le \underline{I}(\mathbf{X}_1,\mathbf{X}_2\to\mathbf{Y})$, in a similar way to the proof of the converse part of [7, Theorem 1], where $\underline{I}(\mathbf{X}_1\to\mathbf{Y}\|\mathbf{X}_2)$, $\underline{I}(\mathbf{X}_2\to\mathbf{Y}\|\mathbf{X}_1)$ and $\underline{I}(\mathbf{X}_1,\mathbf{X}_2\to\mathbf{Y})$ are calculated by $(\mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}) = \{(X_1^n,X_2^n,Y^n)\}_{n=1}^{\infty}$ subject to the sequence of consistent distributions $\{Q_{X_1^nX_2^nY^n}\}_{n=1}^{\infty}$. Note that

$$\begin{aligned} &Q_{X_1^nX_2^nY^n}(x_1^n,x_2^n,y^n) \\ &= Q_{X_1^n\|Y^{n-1}}(x_1^n\|y^{n-1})Q_{X_1^n\|Y^{n-1}}(x_2^n\|y^{n-1}) \\ &\quad \cdot W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n), \end{aligned}$$

where the equality comes from Lemma 1 and Lemma 6. Since, according to Lemma 2,

$$Q_{X_k^n\|Y^{n-1}}(x_k^n\|y^{n-1}) = Q_{X_k^n\|Z_k^{n-1}}(x_k^n\|\psi_k^{n-1}(y^{n-1})),$$

we have $(\{Q_{X_1^n\|Y^{n-1}}\}_{n=1}^{\infty},\{Q_{X_2^n\|Y^{n-1}}\}_{n=1}^{\infty}) \in \mathscr{S}^{\Psi}$. Thus, the achievable rate pair (R_1,R_2) must satisfies

$$\begin{aligned} (R_1,R_2) &\in \mathscr{R}_{\mathbf{W}}(\{Q_{X_1^n\|Y^{n-1}}\}_{n=1}^{\infty},\{Q_{X_2^n\|Y^{n-1}}\}_{n=1}^{\infty}) \\ &\subseteq \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^{\Psi}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}). \end{aligned}$$ □

6. Some Examples of Capacity Regions for Some MACs

In this chapter, we show the capacity regions for some interesting MACs.

6.1 Binary Additive Noise MACs

In this section, we show a special case of general MACs for which the capacity region cannot be enlarged by using deterministic feedback.

Let $\mathscr{X}_1 = \mathscr{X}_2 = \mathscr{Y} = \{0,1\}$, and let $\mathbf{V} = \{V^n = (V_1^{(n)},V_2^{(n)},\cdots,V_n^{(n)})\}_{n=1}^{\infty}$ be an arbitrary nonstationary and nonergodic noise, where V^n is subject to the probability distribution P_{V^n} over $\mathscr{V}^n = \{0,1\}^n$. For each time i, the output of the MAC $Y_i^{(n)}$ is given by $Y_i^{(n)} = X_{1i}^{(n)} \oplus X_{2i}^{(n)} \oplus V_i^{(n)}$, where $X_{1i}^{(n)}$ and $X_{2i}^{(n)}$ are channel inputs and $V_i^{(n)}$ is independent from $X_{1i}^{(n)}$ and $X_{2i}^{(n)}$. Then, the MAC $\mathbf{W}$ can be represented by $\{W_{Y^n\|X_1^nX_2^n}=\prod_{i=1}^n W_{Y_i|X_1^iX_2^iY^{i-1}}\}_{n=1}^\infty$ such that

$$W_{Y_i|X_1^iX_2^iY^{i-1}}(y_i|x_1^i,x_2^i,y^{i-1}) = P_{V_i|V^{i-1}}(x_{1i}\oplus x_{2i}\oplus y_i|x_1^{i-1}\oplus x_2^{i-1}\oplus y^{i-1}), \quad (26)$$

for each $i=1,2,\cdots,n$ and $n=1,2,\cdots$, where

$$x_1^{i-1}\oplus x_2^{i-1}\oplus y^{i-1}=(x_{11}\oplus x_{21}\oplus y_1, x_{12}\oplus x_{22}\oplus y_2,\cdots, x_{1,i-1}\oplus x_{2,i-1}\oplus y_{i-1}).$$

We call this type of MAC the binary additive noise MAC. We can show the following result in a similar way to the proof of [7, Example 1].

$$\mathscr{C}^\Psi(\mathbf{W})=\{(R_1,R_2):0\le R_1, 0\le R_2, R_1+R_2\le \log 2-\overline{H}(\mathbf{V})\}. \quad (27)$$

where $\overline{H}(\mathbf{V})=\text{p-}\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{P_{V^n}(V^n)}$.

Note that the capacity region for the binary additive noise MAC without feedback is equal to (27) (see [7, Example 1]). Thus, this example shows that the feedback cannot enlarge the capacity region for binary additive noise MACs. We also mention that the capacity region for the additive noise MAC with time-invariant feedback was shown in [6, p.2462] with the entropy rate of the noise $\mathbf{V}$. Their result relies on the time-sharing principle. However, since the time-sharing may not always work when the noise $\mathbf{V}$ is non-stationary, their argument cannot directly be applied to the general noise $\mathbf{V}$.

6.2 Stationary Memoryless MACs

In this section, we show that for stationary memoryless MACs with perfect feedback, the capacity region in Theorem 1 is equal to the region derived by Kramer [3], [5].

We begin with giving some definitions and notations for the capacity region of stationary memoryless MACs with deterministic feedback. We denote a pair of deterministic function (ψ_1^n,ψ_2^n) by ψ^n. The set of all pairs $(P_{X_1^n\|Y^{n-1}},P_{X_2^n\|Y^{n-1}})$ is denoted by $\mathscr{S}_n$. Let $\mathscr{S}^{\psi^n}\subseteq\mathscr{S}_n$ be the set of all pairs $(P_{X_1^n\|Y^{n-1}},P_{X_2^n\|Y^{n-1}})$ such that

$$P_{X_k^n\|Y^{n-1}}(x_k^n\|y^{n-1})=P_{X_k^n\|Z_k^{n-1}}(x_k^n\|\psi_k^{n-1}(y^{n-1})) \quad (k=1,2).$$

For a stationary memoryless MAC W and a deterministic function ψ^n, we define the set $\mathscr{R}_n^{\psi^n}(W)$ by

$$\mathscr{R}_n^{\psi^n}(W)\triangleq\bigcup_{(P_{X_1^n\|Y^{n-1}},P_{X_2^n\|Y^{n-1}})\in\mathscr{S}^{\psi^n}}\mathscr{R}_{W,n}(P_{X_1^n\|Y^{n-1}},P_{X_2^n\|Y^{n-1}}),$$

where

$$\mathscr{R}_{W,n}(P_{X_1^n\|Y^{n-1}},P_{X_2^n\|Y^{n-1}})\triangleq\{(R_1,R_2): 0\le R_1\le\frac{1}{n}I(X_1^n\to Y^n\|X_2^n), 0\le R_2\le\frac{1}{n}I(X_2^n\to Y^n\|X_1^n), R_1+R_2\le\frac{1}{n}I(X_1^n,X_2^n\to Y^n)\},$$

and the triple of random variable (X_1^n,X_2^n,Y^n) is drawn according to the probability distribution

$$P_{X_1^nX_2^nY^n}(x_1^n,x_2^n,y^n)=P_{X_1^n\|Y^{n-1}}(x_1^n\|y^{n-1})P_{X_2^n\|Y^{n-1}}(x_2^n\|y^{n-1})\cdot W^n(y^n|x_1^n,x_2^n).$$

Here, $W^n(y^n|x_1^n,x_2^n)=\prod_{i=1}^n W(y_i|x_{1i},x_{2i})$. For the sake of convenience, in the case of perfect feedback, we denote $\mathscr{R}_n^{\psi^n}(W)$ by $\mathscr{R}_n(W)$, where

$$\mathscr{R}_n(W)=\bigcup_{(P_{X_1^n\|Y^{n-1}},P_{X_2^n\|Y^{n-1}})\in\mathscr{S}_n}\mathscr{R}_{W,n}(P_{X_1^n\|Y^{n-1}},P_{X_2^n\|Y^{n-1}}).$$

We now show the Kramer's capacity region.

Theorem 4 ([3, Theorem 3],[5, Theorem 5.1])**.** We are given a memoryless MAC W and a pair of deterministic functions Ψ such that $\psi_{ki}(y^i)=y_i$ $(k=1,2)$ for all $i=1,2,\cdots$, i.e., perfect feedback is available at encoders. Let $\lim_{n\to\infty}\text{conv}(\mathscr{R}_n(W))$ be the set of limit points of convergent sequences whose n-th term is in $\text{conv}(\mathscr{R}_n(W))$, where $\text{conv}(\cdot)$ denotes the convex hull of the region. Then, we have

$$\mathscr{C}(W)=\lim_{n\to\infty}\text{conv}(\mathscr{R}_n(W)).$$

The next lemma implies another expression of the channel capacity.

Lemma 9. For a stationary memoryless MAC W, we have

$$\lim_{n\to\infty}\text{conv}(\mathscr{R}_n(W))=\text{cl}\left(\bigcup_{n\in\mathbb{N}}\mathscr{R}_n(W)\right), \quad (28)$$

where $\mathbb{N}$ denotes the set of all natural numbers and $\text{cl}(\cdot)$ denotes closure of the region.

Proof. The proof appeared in the proof of [3, Theorem 3]. This lemma can also be proved by the *sup-additivity* of $\text{conv}(\mathscr{R}_n(W))$ and $\mathscr{R}_n(W)$ (see the proof of [6, Lemma 27]). □

We show that for stationary memoryless MACs with perfect feedback, the region in Theorem 1 is equal to the right-hand side of (28). First, the next theorem shows the region in Theorem 1 includes the right-hand side of (28).

Theorem 5. For any stationary memoryless channel W and any pair of deterministic functions Ψ,

$$\bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^\Psi}\mathscr{R}_W(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\supseteq\text{cl}\left(\bigcup_{n\in\mathbb{N}}\mathscr{R}_n^{\psi^n}(W)\right).$$

Proof. For any fixed $m \in \mathbb{N}$ and causally conditioned distributions $P_{X_1^m \| Y^{m-1}}$ and $P_{X_2^m \| Y^{m-1}}$ such that

$$P_{X_1^m \| Y^{m-1}}(x_1^m \| y^{m-1}) = P_{X_1^m \| Z_1^{m-1}}(x_1^m \| \psi_1^{m-1}(y^{m-1})), \quad (29)$$

$$P_{X_2^m \| Y^{m-1}}(x_2^m \| y^{m-1}) = P_{X_2^m \| Z_2^{m-1}}(x_2^m \| \psi_2^{m-1}(y^{m-1})), \quad (30)$$

we define the channel input distribution $\mathbf{P}_{\mathbf{X}_1 \| \mathbf{Y}} = \{P_{X_1^n \| Y^{n-1}}\}_{n=1}^{\infty}$ as

$$P_{X_1^n \| Y^{n-1}}(x_1^n \| y^{n-1}) = \prod_{i=1}^{\lfloor \frac{n}{m} \rfloor} P_{X_1^m \| Y^{m-1}}(x_{1,(i-1)m+1}^{im} \| y_{(i-1)m+1}^{im-1}) \cdot P_{X_1^{m'} \| Y^{m'-1}}(x_{1,\lfloor \frac{n}{m} \rfloor m+1}^{n} \| y_{\lfloor \frac{n}{m} \rfloor m+1}^{n-1}), \quad (31)$$

where $x_{1i}^j = (x_{1i}, x_{1,i+1}, \cdots, x_{1j})$ and $m' \triangleq n - \lfloor \frac{n}{m} \rfloor m$. $\mathbf{P}_{\mathbf{X}_2 \| \mathbf{Y}} = \{P_{X_2^n \| Y^{n-1}}\}_{n=1}^{\infty}$ is defined in a similar way. According to (29), (30) and (31), we have

$$\begin{aligned} &P_{X_k^n \| Y^{n-1}}(x_k^n \| y^{n-1}) \\ &= \prod_{i=1}^{\lfloor \frac{n}{m} \rfloor} P_{X_k^m \| Z_k^{m-1}}(x_{k,(i-1)m+1}^{im} \| \psi_{k,(i-1)m+1}^{im-1}(y_{(i-1)m+1}^{im-1})) \\ &\quad \cdot P_{X_k^{m'} \| Z_k^{m'-1}}(x_{k,\lfloor \frac{n}{m} \rfloor m+1}^{n} \| \psi_{k,\lfloor \frac{n}{m} \rfloor m+1}^{n-1}(y_{\lfloor \frac{n}{m} \rfloor m+1}^{n-1})), \end{aligned}$$

for each $k = 1, 2$. This implies $(\mathbf{P}_{\mathbf{X}_1 \| \mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2 \| \mathbf{Y}}) \in \mathscr{S}^{\Psi}$. By using this pair of channel input distributions $(\mathbf{P}_{\mathbf{X}_1 \| \mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2 \| \mathbf{Y}})$, $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y})$ in Theorem 1 can be represented as the sequence of random variables $\{(X_1^n, X_2^n, Y^n)\}_{n=1}^{\infty}$ subject to

$$\begin{aligned} &P_{X_1^n X_2^n Y^n}(x_1^n, x_2^n, y^n) \\ &= P_{X_1 \| Y^{n-1}}(x_1^n \| y^{n-1}) P_{X_2 \| Y^{n-1}}(x_2^n \| y^{n-1}) W^n(y^n | x_1^n, x_2^n) \\ &= \prod_{i=1}^{\lfloor \frac{n}{m} \rfloor} \Big(P_{X_1^m \| Y^{m-1}}(x_{1,(i-1)m+1}^{im} \| y_{(i-1)m+1}^{im-1}) \\ &\quad \cdot P_{X_2^m \| Y^{m-1}}(x_{2,(i-1)m+1}^{im} \| y_{(i-1)m+1}^{im-1}) \\ &\quad \cdot W^m(y_{(i-1)m+1}^{im} | x_{1,(i-1)m+1}^{im}, x_{2,(i-1)m+1}^{im}) \Big) \\ &\quad \cdot P_{X_1^{m'} \| Y^{m'-1}}(x_{1,\lfloor \frac{n}{m} \rfloor m+1}^{n} \| y_{\lfloor \frac{n}{m} \rfloor m+1}^{n-1}) \\ &\quad \cdot P_{X_2^{m'} \| Y^{m'-1}}(x_{2,\lfloor \frac{n}{m} \rfloor m+1}^{n} \| y_{\lfloor \frac{n}{m} \rfloor m+1}^{n-1}) \\ &\quad \cdot W^{m'}(y_{\lfloor \frac{n}{m} \rfloor m+1}^{n} | x_{1,\lfloor \frac{n}{m} \rfloor m+1}^{n}, x_{2,\lfloor \frac{n}{m} \rfloor m+1}^{n}). \end{aligned}$$

This implies that $\{(X_{1,(i-1)m+1}^{im}, X_{2,(i-1)m+1}^{im}, Y_{(i-1)m+1}^{im})\}_{i=1}^{\lfloor \frac{n}{m} \rfloor}$ are mutually independent, and are subject to the identical distribution. Thus, by noting that $P_{X_1^n X_2^n \| Y^{n-1}} = P_{X_1^n \| Y^{n-1}} \cdot P_{X_2^n \| Y^{n-1}}$, we have

$$P_{Y^n}(y^n) = \prod_{i=1}^{\lfloor \frac{n}{m} \rfloor} P_{Y^m}(y_{(i-1)m+1}^{im}) P_{Y^{m'}}(y_{\lfloor \frac{n}{m} \rfloor m+1}^{n}), \quad (32)$$

$$P_{Y^n \| X_1^n}(y^n \| x_1^n) = \prod_{i=1}^{\lfloor \frac{n}{m} \rfloor} P_{Y^m \| X_1^m}(y_{(i-1)m+1}^{im} \| x_{1,(i-1)m+1}^{im}) \cdot P_{Y^{m'} \| X_1^{m'}}(y_{\lfloor \frac{n}{m} \rfloor m+1}^{n} \| x_{1,\lfloor \frac{n}{m} \rfloor m+1}^{n}), \quad (33)$$

$$P_{Y^n \| X_2^n}(y^n \| x_1^n) = \prod_{i=1}^{\lfloor \frac{n}{m} \rfloor} P_{Y^m \| X_2^m}(y_{(i-1)m+1}^{im} \| x_{2,(i-1)m+1}^{im}) \cdot P_{Y^{m'} \| X_2^{m'}}(y_{\lfloor \frac{n}{m} \rfloor m+1}^{n} \| x_{2,\lfloor \frac{n}{m} \rfloor m+1}^{n}). \quad (34)$$

We define

$$\begin{aligned} i_n(x_1^n, x_2^n \to y^n) &\triangleq \log \frac{P_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}{P_{Y^n}(y^n)}, \\ i_n(x_1^n \to y^n \| x_2^n) &\triangleq \log \frac{P_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}{P_{Y^n \| X_2^n}(y^n \| x_2^n)}, \\ i_n(x_2^n \to y^n \| x_1^n) &\triangleq \log \frac{P_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}{P_{Y^n \| X_1^n}(y^n \| x_1^n)}. \end{aligned}$$

Then, according to (32)-(34), we have

$$E[i_n(X_1^n, X_2^n \to Y^n)] = \left\lfloor \frac{n}{m} \right\rfloor I(X_1^m, X_2^m \to Y^m) + I(X_1^{m'}, X_2^{m'} \to Y^{m'}), \quad (35)$$

$$E[i_n(X_1^n \to Y^n \| X_2^n)] = \left\lfloor \frac{n}{m} \right\rfloor I(X_1^m \to Y^m \| X_2^m) + I(X_1^{m'} \to Y^{m'} \| X_2^{m'}), \quad (36)$$

$$E[i_n(X_2^n \to Y^n \| X_1^n)] = \left\lfloor \frac{n}{m} \right\rfloor I(X_2^m \to Y^m \| X_1^m) + I(X_2^{m'} \to Y^{m'} \| X_1^{m'}). \quad (37)$$

On the other hand, according to Chebyshev's inequality, for any $\varepsilon > 0$, we have

$$\begin{aligned} &\Pr\left\{ \left| \frac{1}{n} i_n(X_1^n, X_2^n \to Y^n) - \frac{1}{n} E[i_n(X_1^n, X_2^n \to Y^n)] \right| \geq \varepsilon \right\} \\ &\leq \frac{1}{\varepsilon^2} V\left[\frac{1}{n} i_n(X_1^n, X_2^n \to Y^n) \right] \\ &= \frac{1}{\varepsilon^2 n^2} \left(\sum_{i=1}^{\lfloor \frac{n}{m} \rfloor} V\left[i_m(X_1^m, X_2^m \to Y^m) \right] + V\left[i_{m'}(X_1^{m'}, X_2^{m'} \to Y^{m'}) \right] \right) \\ &= \frac{1}{\varepsilon^2 n^2} \left(\left\lfloor \frac{n}{m} \right\rfloor \sigma_m^2 + \sigma_{m'}^2 \right), \end{aligned} \quad (38)$$

where σ_m^2 and $\sigma_{m'}^2$ is the variance of $i_m(X_1^m, X_2^m \to Y^m)$ and $i_{m'}(X_1^{m'}, X_2^{m'} \to Y^{m'})$, respectively, and we have σ_m^2, $\sigma_{m'}^2 < 8|\mathscr{Y}|^m/e^2$ in the similar way to [13, Remark 3.1.1]. Similarly, we also obtain

$$\begin{aligned} &\Pr\left\{ \left| \frac{1}{n} i_n(X_1^n \to Y^n \| X_2^n) - \frac{1}{n} E[i_n(X_1^n \to Y^n \| X_2^n)] \right| \geq \varepsilon \right\} \\ &\leq \frac{1}{\varepsilon^2 n^2} \left(\left\lfloor \frac{n}{m} \right\rfloor \frac{8|\mathscr{Y}|^m}{e^2} + \frac{8|\mathscr{Y}|^{m'}}{e^2} \right), \end{aligned} \quad (39)$$

$$\Pr\left\{\left|\frac{1}{n}i_n(X_2^n \to Y^n\|X_1^n) - \frac{1}{n}E[i_n(X_2^n \to Y^n\|X_1^n)]\right| \geq \varepsilon\right\}$$
$$\leq \frac{1}{\varepsilon^2 n^2}\left(\left\lfloor\frac{n}{m}\right\rfloor \frac{8|\mathscr{Y}|^m}{e^2} + \frac{8|\mathscr{Y}|^{m'}}{e^2}\right). \quad (40)$$

According to (35)-(37), we have

$$\lim_{n\to\infty}\frac{1}{n}E[i_n(X_1^n,X_2^n \to Y^n)] = \frac{1}{m}I(X_1^m,X_2^m \to Y^m),$$
$$\lim_{n\to\infty}\frac{1}{n}E[i_n(X_1^n \to Y^n\|X_2^n)] = \frac{1}{m}I(X_1^m \to Y^m\|X_2^m),$$
$$\lim_{n\to\infty}\frac{1}{n}E[i_n(X_2^n \to Y^n\|X_1^n)] = \frac{1}{m}I(X_2^m \to Y^m\|X_1^m),$$

and by using (38)-(40), sequences $\{\frac{1}{n}i_n(X_1^n,X_2^n \to Y^n)\}_{n=1}^\infty$, $\{\frac{1}{n}i_n(X_1^n \to Y^n\|X_2^n)\}_{n=1}^\infty$ and $\{\frac{1}{n}i_n(X_2^n \to Y^n\|X_1^n)\}_{n=1}^\infty$ converge to $\frac{1}{m}I(X_1^m,X_2^m \to Y^m)$, $\frac{1}{m}I(X_1^m \to Y^m\|X_2^m)$ and $\frac{1}{m}I(X_2^m \to Y^m\|X_1^m)$ in probability, respectively. Consequently, we have $\underline{I}(\mathbf{X}_1,\mathbf{X}_2 \to \mathbf{Y}) = \frac{1}{m}I(X_1^m,X_2^m \to Y^m)$, $\underline{I}(\mathbf{X}_1 \to \mathbf{Y}\|\mathbf{X}_2) = \frac{1}{m}I(X_1^m \to Y^m\|X_2^m)$ and $\underline{I}(\mathbf{X}_2 \to \mathbf{Y}\|\mathbf{X}_1) = \frac{1}{m}I(X_2^m \to Y^m\|X_1^m)$. This implies that for any $m \in \mathbb{N}$ and $(P_{X_1^m\|Y^{m-1}}, P_{X_2^m\|Y^{m-1}}) \in \mathscr{S}^{\psi^m}$, there exists a pair $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^\Psi$ satisfying

$$\mathscr{R}_W(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) = \mathscr{R}_{W,m}(P_{X_1^m\|Y^{m-1}}, P_{X_2^m\|Y^{m-1}}).$$

By noting that the right-hand side of (5) is closed set, we have the lemma. □

In order to show the opposite relation, we need the next lemma which can be proved in a similar way to [7, Lemma 1].

Lemma 10. For any sequence of random variables $(\mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}) = \{(X_1^n,X_2^n,Y^n)\}_{n=1}^\infty$, we have

$$\underline{I}(\mathbf{X}_1,\mathbf{X}_2 \to \mathbf{Y}) \leq \liminf_{n\to\infty}\frac{1}{n}I(X_1^n,X_2^n \to Y^n),$$
$$\underline{I}(\mathbf{X}_1 \to \mathbf{Y}\|\mathbf{X}_2) \leq \liminf_{n\to\infty}\frac{1}{n}I(X_1^n \to Y^n\|X_2^n),$$
$$\underline{I}(\mathbf{X}_2 \to \mathbf{Y}\|\mathbf{X}_1) \leq \liminf_{n\to\infty}\frac{1}{n}I(X_2^n \to Y^n\|X_1^n).$$

By using this lemma, we have the next theorem.

Theorem 6. For any stationary memoryless MAC W and any pair of deterministic functions Ψ,

$$\bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^\Psi} \mathscr{R}_\mathbf{W}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \subseteq \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}\mathscr{R}_n^{\psi^n}(W)\right).$$

Proof. According to Lemma 10, for any triple of random variables $(\mathbf{X}_1,\mathbf{X}_2,\mathbf{Y})$ defined by a stationary memoryless MAC W and a pair of channel input distribution $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^\Psi$, we have

$$\underline{I}(\mathbf{X}_1,\mathbf{X}_2 \to \mathbf{Y}) \leq \liminf_{n\to\infty}\frac{1}{n}I(X_1^n,X_2^n \to Y^n),$$
$$\underline{I}(\mathbf{X}_1 \to \mathbf{Y}\|\mathbf{X}_2) \leq \liminf_{n\to\infty}\frac{1}{n}I(X_1^n \to Y^n\|X_2^n),$$
$$\underline{I}(\mathbf{X}_2 \to \mathbf{Y}\|\mathbf{X}_1) \leq \liminf_{n\to\infty}\frac{1}{n}I(X_2^n \to Y^n\|X_1^n).$$

Thus, for any $\gamma > 0$ and all sufficiently large n, we obtain

$$\underline{I}(\mathbf{X}_1,\mathbf{X}_2 \to \mathbf{Y}) \leq \frac{1}{n}I(X_1^n,X_2^n \to Y^n) + \gamma,$$
$$\underline{I}(\mathbf{X}_1 \to \mathbf{Y}\|\mathbf{X}_2) \leq \frac{1}{n}I(X_1^n \to Y^n\|X_2^n) + \gamma,$$
$$\underline{I}(\mathbf{X}_2 \to \mathbf{Y}\|\mathbf{X}_1) \leq \frac{1}{n}I(X_2^n \to Y^n\|X_1^n) + \gamma.$$

Since $\gamma > 0$ can be arbitrary small and

$$P_{X_k^n\|Y^{n-1}}(x_k^n\|y^{n-1}) = P_{X_k^n\|Z^{n-1}}(x_k^n\|\psi^{n-1}(y^{n-1})) \quad (k=1,2),$$

we have

$$\mathscr{R}_\mathbf{W}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \subseteq \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}\mathscr{R}_{\mathbf{W},n}\right)$$

for any $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^\Psi$. □

By combining the Theorem 5 and Theorem 6, we have the following theorem and corollary.

Theorem 7. For any stationary memoryless channel W and any pair of deterministic functions Ψ,

$$\bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^\Psi} \mathscr{R}_W(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}\mathscr{R}_n^{\psi^n}(W)\right).$$

Corollary 2. For any stationary memoryless channel W with perfect feedback, we have

$$\bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}},\mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}} \mathscr{R}_W(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}\mathscr{R}_n(W)\right).$$

6.3 Mixed MACs

In this section, we characterize the capacity region for mixed MACs with deterministic feedback.

For a given m MACs (m may be infinite) $\mathbf{W}^{(i)} = \{W^{(i)}_{Y^n\|X_1^nX_2^n}\}_{n=1}^\infty$ $(i = 1,2,\cdots,m)$, we define a MAC $\mathbf{W} = \{W_{Y^n\|X_1^nX_2^n}\}_{n=1}^\infty$ as

$$W_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n) = \sum_{i=1}^m \alpha_i W^{(i)}_{Y^n\|X_1^nX_2^n}(y^n\|x_1^n,x_2^n), \quad (41)$$

where $\alpha_1,\alpha_2,\cdots$ are nonnegative constants satisfying $\sum_{i=1}^m \alpha_i = 1$. We call this MAC as the *mixed MAC* of a MAC family $\{\alpha_i,\mathbf{W}^{(i)}\}_{i=1}^m$. We can verify that $W_{Y^n\|X_1^nX_2^n}$ is causally conditioned distribution by using a sequence of conditional distributions $\{W_{Y_l|X_1^jX_2^jY^{j-1}}\}_{j=1}^n$ such that

$$W_{Y_j|X_1^jX_2^jY^{j-1}}(y_j|x_1^j,x_2^j,y^{j-1})$$

$$= \frac{\sum_{y_{j+1}^n \in \mathscr{Y}^{n-j}} \sum_{i=1}^m \alpha_i W^{(i)}_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}{\sum_{y_j^n \in \mathscr{Y}^{n-j+1}} \sum_{i=1}^m \alpha_i W^{(i)}_{Y^n \| X_1^n X_2^n}(y^n \| x_1^n, x_2^n)}.$$

This type of MAC is useful to characterize the capacity region of the compound MAC problem which we will discuss in the next section. To characterize the channel capacity region for mixed MACs, we use the following lemma.

Lemma 11. Let $\mathbf{W}$ be the mixed MAC of a MAC family $\{\alpha_i, \mathbf{W}^{(i)}\}_{i=1}^m$. Then, for any pair of deterministic function Ψ and any pair of channel input distribution $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}) \in \mathscr{S}^\Psi$, we have

$$\underline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y}) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y}^{(i)}),$$
$$\underline{I}(\mathbf{X}_1 \to \mathbf{Y} \| \mathbf{X}_2) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_1 \to \mathbf{Y}^{(i)} \| \mathbf{X}_2),$$
$$\underline{I}(\mathbf{X}_2 \to \mathbf{Y} \| \mathbf{X}_1) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_2 \to \mathbf{Y}^{(i)} \| \mathbf{X}_1).$$

where $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}) = \{(X_1^n, X_2^n, Y^n)\}_{n=1}^\infty$ and $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}^{(i)}) = \{(X_1^n, X_2^n, Y^{(i)n})\}_{n=1}^\infty$ are defined by a random variables subject to the probability distribution $P_{X_1^n X_2^n Y^n} = P_{X_1^n\|Y^{n-1}} \cdot P_{X_2^n\|Y^{n-1}} \cdot W_{Y^n\|X_1^n X_2^n}$ and $P_{X_1^n X_2^n Y^{(i)n}} = P_{X_1^n\|Y^{n-1}} \cdot P_{X_2^n\|Y^{n-1}} \cdot W^{(i)}_{Y^n\|X_1^n X_2^n}$, respectively.

Proof. We use some properties of the consistent distribution. Let $P_{F_1^n}$ and $P_{F_2^n}$ be a code-function distribution good with respect to the channel input distribution $P_{X_1^n\|Y^{n-1}}$ and $P_{X_2^n\|Y^{n-1}}$, respectively. Let Q_n be the consistent distribution determined by $P_{F_1^n}$, $P_{F_2^n}$ and the mixed MAC $\mathbf{W}$. Then, for all $n = 1, 2, \cdots$, we have

$$Q_n(f_1^n, f_2^n, x_1^n, x_2^n, y^n, z_1^n, z_2^n) = \sum_{i=1}^m \alpha_i Q_n^{(i)}(f_1^n, f_2^n, x_1^n, x_2^n, y^n, z_1^n, z_2^n). \quad (42)$$

where

$$Q_n^{(i)}(f_1^n, f_2^n, x_1^n, x_2^n, y^n, z_1^n, z_2^n) \triangleq P_{F_1^n}(f_1^n) P_{F_2^n}(f_2^n) 1_{\{f_1^n(y^{n-1})\}}(x_1^n) 1_{\{f_2^n(y^{n-1})\}}(x_2^n) \cdot W^{(i)}_{Y^n\|X_1^n X_2^n}(y^n \| x_1^n, x_2^n) 1_{\{\psi_1^n(y^n)\}}(z_1^n) 1_{\{\psi_2^n(y^n)\}}(z_2^n).$$

Thus, $Q_n^{(i)}$ is consistent distribution for $P_{F_1^n}$, $P_{F_2^n}$ and $\mathbf{W}^{(i)}$. According to (42), for each n,

$$Q_{F_1^n F_2^n Y^n}(f_1^n, f_2^n, y^n) = \sum_{i=1}^m \alpha_i Q^{(i)}_{F_1^n F_2^n Y^n}(f_1^n, f_2^n, y^n). \quad (43)$$

We define $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{Y})$ and $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{Y}^{(i)})$ as a sequence of random variables subject to the sequence of probability distributions $\{Q_{F_1^n F_2^n Y^n}\}_{n=1}^\infty$ and $\{Q_{F_1^n F_2^n Y^{(i)n}}\}_{n=1}^\infty$, respectively. Then, according to (43) and [13, Lemma 7.9.2], we have

$$\underline{I}(\mathbf{F}_1, \mathbf{F}_2; \mathbf{Y}) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{F}_1, \mathbf{F}_2; \mathbf{Y}^{(i)}),$$
$$\underline{I}(\mathbf{F}_1; \mathbf{Y} | \mathbf{F}_2) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{F}_1; \mathbf{Y}^{(i)} | \mathbf{F}_2),$$
$$\underline{I}(\mathbf{F}_2; \mathbf{Y} | \mathbf{F}_1) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{F}_2; \mathbf{Y}^{(i)} | \mathbf{F}_1),$$

where $\underline{I}(\mathbf{F}_1, \mathbf{F}_2; \mathbf{Y})$, $\underline{I}(\mathbf{F}_1; \mathbf{Y}|\mathbf{F}_2)$ and $\underline{I}(\mathbf{F}_2; \mathbf{Y}|\mathbf{F}_1)$ are *spectral (conditional) inf-mutual information rates* [13, (7.7.1)-(7.7.3)] of $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{Y})$, and $\underline{I}(\mathbf{F}_1, \mathbf{F}_2; \mathbf{Y}^{(i)})$, $\underline{I}(\mathbf{F}_1; \mathbf{Y}^{(i)}|\mathbf{F}_2)$ and $\underline{I}(\mathbf{F}_2; \mathbf{Y}^{(i)}|\mathbf{F}_1)$ are spectral (conditional) inf-mutual information rates of $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{Y}^{(i)})$. On the other hand, according to Lemma 5, we have $\underline{I}(\mathbf{F}_1, \mathbf{F}_2; \mathbf{Y}) = \underline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y})$, $\underline{I}(\mathbf{F}_1; \mathbf{Y}|\mathbf{F}_2) = \underline{I}(\mathbf{X}_1 \to \mathbf{Y}\|\mathbf{X}_2)$, $\underline{I}(\mathbf{F}_2; \mathbf{Y}|\mathbf{F}_1) = \underline{I}(\mathbf{X}_2 \to \mathbf{Y}\|\mathbf{X}_1)$, $\underline{I}(\mathbf{F}_1, \mathbf{F}_2; \mathbf{Y}^{(i)}) = \underline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y}^{(i)})$, $\underline{I}(\mathbf{F}_1; \mathbf{Y}^{(i)}|\mathbf{F}_2) = \underline{I}(\mathbf{X}_1 \to \mathbf{Y}^{(i)}\|\mathbf{X}_2)$ and $\underline{I}(\mathbf{F}_2; \mathbf{Y}^{(i)}|\mathbf{F}_1) = \underline{I}(\mathbf{X}_2 \to \mathbf{Y}^{(i)}\|\mathbf{X}_1)$. Hence, we obtain

$$\underline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y}) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y}^{(i)}),$$
$$\underline{I}(\mathbf{X}_1 \to \mathbf{Y} \| \mathbf{X}_2) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_1 \to \mathbf{Y}^{(i)} \| \mathbf{X}_2),$$
$$\underline{I}(\mathbf{X}_2 \to \mathbf{Y} \| \mathbf{X}_1) = \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_2 \to \mathbf{Y}^{(i)} \| \mathbf{X}_1).$$

Since $P_{F_1^n}$ and $P_{F_2^n}$ is good with respect to the channel input distribution $P_{X_1^n\|Y^{n-1}}$ and $P_{X_2^n\|Y^{n-1}}$, respectively, we have $Q_{X_1^n X_2^n Y^n} = P_{X_1^n X_2^n Y^n}$ and $Q_{X_1^n X_2^n Y^{(i)n}} = P_{X_1^n X_2^n Y^{(i)}}$. This completes the proof. □

The next theorem shows the capacity region for mixed MACs.

Theorem 8. For the mixed MAC $\mathbf{W}$ of a MAC family $\{\alpha_i, \mathbf{W}^{(i)}\}_{i=1}^m$ and a pair of deterministic functions Ψ, we have

$$\mathscr{C}^\Psi(\mathbf{W}) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^\Psi} \mathscr{R}_{\{\mathbf{W}^{(i)}\}_{i=1}^m}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}),$$

where

$$\mathscr{R}_{\{\mathbf{W}^{(i)}\}_{i=1}^m}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \triangleq \Big\{(R_1, R_2) : 0 \le R_1 \le \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_1 \to \mathbf{Y}^{(i)} \| \mathbf{X}_2),$$
$$0 \le R_2 \le \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_2 \to \mathbf{Y}^{(i)} \| \mathbf{X}_1),$$
$$R_1 + R_2 \le \inf_{i:\alpha_i>0} \underline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y}^{(i)})\Big\}.$$

Proof. We can easily prove the theorem by combining Theorem 1 and Lemma 11. □

6.4 Compound MACs with Deterministic Feedback

In this section, we characterize the capacity region for the compound MAC problem when there exists deterministic feedback.

First, we formulate the compound MAC problem. Suppose that general MACs $\mathbf{W}^{(i)} = \{W^{(i)}_{Y^n\|X_1^n X_2^n}\}$ $(i = 1, 2, \cdots, m)$ are given. Let $\mathscr{M}_n^{(1)} = \{1, 2, \cdots, M_n^{(1)}\}$ and

$\mathscr{M}_n^{(2)} = \{1,2,\cdots,M_n^{(2)}\}$ be a message set and fix encoders $\{f_n^{(1)}[m_1]\}_{m_1\in\mathscr{M}_n^{(1)}}$ and $\{f_n^{(2)}[m_2]\}_{m_2\in\mathscr{M}_n^{(2)}}$, and a decoder $\varphi_n : \mathscr{Y}^n \to \mathscr{M}_1^n \times \mathscr{M}_2^n$. Note that encoders may use feedback. For each MAC $\mathbf{W}^{(i)}$ the error probability is represented by

$$\varepsilon_n^{(i)} = \frac{1}{M_n^{(1)}M_n^{(2)}} \sum_{m_1\in\mathscr{M}_n^{(1)}} \sum_{m_2\in\mathscr{M}_n^{(2)}} \sum_{y^n\in\mathscr{D}_{m_1,m_2}^c} \cdot W^{(i)}_{Y^n\|X_1^nX_2^n}(y^n\|f_1^n[m_1](\psi_1^{n-1} \cdot (y^{n-1})), f_2^n[m_2](\psi_2^{n-1}(y^{n-1}))). \tag{44}$$

We define an $(n, M_n^{(1)}, M_n^{(2)}, (\varepsilon_n^{(i)})_{i=1}^m)$-code as sets of $M_n^{(1)}$ and $M_n^{(2)}$ code functions, a decoder φ_n and the error probabilities $\varepsilon_n^{(i)}$ $(i = 1,2,\cdots,m)$. Now, we define achievability and capacity region for the compound MAC problem.

Definition 9. A pair (R_1,R_2) is called *achievable* if there exists a sequence of $(n, M_n^{(1)}, M_n^{(2)}, (\varepsilon_n^{(i)})_{i=1}^m)$-codes such that

$$\lim_{n\to\infty} \varepsilon_n^{(i)} = 0 \ (^\forall i = 1,2,\cdots,m)$$
$$\liminf_{n\to\infty} R_n^{(1)} \geq R_1 \text{ and } \liminf_{n\to\infty} R_n^{(2)} \geq R_2.$$

The set of all achievable rates is called the *capacity region* of a pair of deterministic functions Ψ and a compound MAC with deterministic feedback, and denoted by $\mathscr{C}^\Psi(\{\mathbf{W}^{(i)}\}_{i=1}^m)$.

Our purpose in this section is to characterize the capacity region for the compound MAC with deterministic feedback. To this end, we show a relation between the capacity region for the compound MAC with deterministic feedback and the capacity region for the mixed MAC with deterministic feedback.

Theorem 9. We are given a pair of deterministic functions Ψ. Then, the capacity region $\mathscr{C}^\Psi(\mathbf{W})$ for a mixed MAC $\mathbf{W}$ of a MAC family $\{\alpha_i, \mathbf{W}^{(i)}\}_{i=1}^m$ is equal to the capacity region $\mathscr{C}^\Psi(\{\mathbf{W}^{(i)}\}_{i=1}^m)$ for the compound MAC $\{\mathbf{W}^{(i)}\}_{i=1}^m$.

Since the proof is very similar to the proof of [13, Theorem 3.3.5], we omit the proof.

By combining Theorem 8 and Theorem 9, we have the next corollary which clarifies the capacity region for compound MACs with deterministic feedback.

Corollary 3. We are given a pair of deterministic functions Ψ. Then, the capacity region $\mathscr{C}^\Psi(\{\mathbf{W}^{(i)}\}_{i=1}^m)$ for the compound MAC $\{\mathbf{W}^{(i)}\}_{i=1}^m$ is given by

$$\mathscr{C}^\Psi(\{\mathbf{W}^{(i)}\}_{i=1}^m) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^\Psi} \mathscr{R}_{\{\mathbf{W}^{(i)}\}_{i=1}^m}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}).$$

7. Some Coding Problems Related to MACs with Deterministic Feedback

In this chapter, we investigate the ε-coding problem, the strong converse property, and the cost constraint problem for general MACs with deterministic feedback.

7.1 ε-Coding for MACs with Deterministic Feedback

In this section, we show the capacity region for the ε-coding problem of general MACs with deterministic feedback.

We first define the capacity region for the ε-coding problem.

Definition 10. For a MAC $\mathbf{W}$ and a pair of deterministic functions Ψ, a rate pair (R_1,R_2) is called ε*-achievable* if there exists a sequence of $(n, M_n^{(1)}, M_n^{(1)}, \varepsilon_n)$-codes satisfying

$$\limsup_{n\to\infty} \varepsilon_n \leq \varepsilon, \ \ \liminf_{n\to\infty} R_n^{(1)} \geq R_1 \ \text{ and } \ \liminf_{n\to\infty} R_n^{(2)} \geq R_2.$$

The set of all ε-achievable rates is called the ε*-capacity region* of a pair of deterministic functions Ψ and a MAC $\mathbf{W}$ with deterministic feedback, and denoted by $\mathscr{C}^\Psi(\varepsilon|\mathbf{W})$.

We also define $J_{\mathbf{W}}(R_1,R_2|\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})$ by

$$J_{\mathbf{W}}(R_1,R_2|\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \triangleq \limsup_{n\to\infty} P_{X_1^nX_2^nY^n}\left(\frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{P_{Y^n\|X_2^n}(Y^n\|X_2^n)} \leq R_1 \right.$$
$$\text{or } \frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{P_{Y^n\|X_1^n}(Y^n\|X_1^n)} \leq R_2$$
$$\left. \text{or } \frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n,X_2^n)}{P_{Y^n}(Y^n)} \leq R_1+R_2 \right),$$

where (X_1^n, X_2^n, Y^n) is the random variable subject to the joint probability distribution $P_{X_1^nX_2^nY^n}$ such that $P_{X_1^nX_2^nY^n} = P_{X_1^n\|Y^{n-1}} \cdot P_{X_2^n\|Y^{n-1}} \cdot W_{Y^n\|X_1^nX_2^n}$. Then, we have the following theorem.

Theorem 10. The ε-capacity region $\mathscr{C}^\Psi(\varepsilon|\mathbf{W})$ of a pair of deterministic functions Ψ and a MAC $\mathbf{W}$ with deterministic feedback is given by

$$\mathscr{C}^\Psi(\varepsilon|\mathbf{W}) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})\in\mathscr{S}^\Psi} \mathrm{cl}\big(\{(R_1,R_2): R_1\geq 0,\ R_2\geq 0,\ J_{\mathbf{W}}(R_1,R_2|\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \leq \varepsilon\}\big).$$

Proof. By using Lemma 2, 7 and 8, we can prove the theorem by following the similar argument in [13, Theorem 7.11.1]. □

7.2 Strong Converse Theorem for MACs with Deterministic Feedback

In this section, we discuss the strong converse property of general MACs with deterministic feedback.

First, we define the strong converse property of MACs with deterministic feedback.

Definition 11. Let $\mathscr{C}^{\Psi}(\mathbf{W})$ be the capacity region of Ψ and a MAC $\mathbf{W}$ with deterministic feedback. If for any (R_1, R_2) satisfying $(R_1, R_2) \notin \mathscr{C}^{\Psi}(\mathbf{W})$, all the $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$-codes with

$$\liminf_{n\to\infty} R_n^{(1)} \geq R_1 \text{ and } \liminf_{n\to\infty} R_n^{(2)} \geq R_2$$

satisfy $\lim_{n\to\infty} \varepsilon_n = 1$, the MAC $\mathbf{W}$ is said to satisfy the *strong converse property*.

Next, we define $J_{\mathbf{W}}^*(R_1, R_2 | \mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})$ by

$$\begin{aligned} &J_{\mathbf{W}}^*(R_1, R_2 | \mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \\ &\triangleq \liminf_{n\to\infty} P_{X_1^n X_2^n Y^n}\left(\frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n, X_2^n)}{P_{Y^n\|X_2^n}(Y^n\|X_2^n)} \leq R_1 \right. \\ &\quad \text{or } \frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n, X_2^n)}{P_{Y^n\|X_1^n}(Y^n\|X_1^n)} \leq R_2 \\ &\quad \left. \text{or } \frac{1}{n}\log\frac{P_{Y^n\|X_1^nX_2^n}(Y^n\|X_1^n, X_2^n)}{P_{Y^n}(Y^n)} \leq R_1 + R_2 \right). \end{aligned} \tag{45}$$

We also define the region $\mathscr{R}_{\mathbf{W}}^*(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})$ by

$$\begin{aligned} \mathscr{R}_{\mathbf{W}}^*(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \triangleq &\mathrm{cl}(\{(R_1, R_2) : R_1 \geq 0,\ R_2 \geq 0, \\ &J_{\mathbf{W}}^*(R_1, R_2 | \mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) < 1\}). \end{aligned} \tag{46}$$

Then, we have the following theorem and corollary on the strong converse property by using Theorem 1, Lemma 2, 7 and 8, and following the proof of [13, Theorem 7.12.1 and Corollary 7.12.1].

Theorem 11. We are given a pair of deterministic functions Ψ. Then, a MAC $\mathbf{W}$ satisfies the strong converse property if and only if

$$\bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^{\Psi}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^{\Psi}} \mathscr{R}_{\mathbf{W}}^*(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}). \tag{47}$$

The next corollary shows the relation between the strong converse property and limit superior in probability of directed information.

Corollary 4. If

$$\bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^{\Psi}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^{\Psi}} \overline{\mathscr{R}}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \tag{48}$$

for a MAC $\mathbf{W}$ and Ψ, then $\mathbf{W}$ satisfies the strong converse property, where $\mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})$ is the region defined in Theorem 1 and

$$\begin{aligned} \overline{\mathscr{R}}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \triangleq \{&(R_1, R_2) : 0 \leq R_1 \leq \overline{I}(\mathbf{X}_1 \to \mathbf{Y}\|\mathbf{X}_2), \\ &0 \leq R_2 \leq \overline{I}(\mathbf{X}_2 \to \mathbf{Y}\|\mathbf{X}_1),\ R_1 + R_2 \leq \overline{I}(\mathbf{X}_1, \mathbf{X}_2 \to \mathbf{Y})\}. \end{aligned}$$

7.3 MACs with Cost Constraint and Deterministic Feedback

In this section, we consider the coding of MACs with input cost constraint when there exists deterministic feedback. First, we define cost functions $c_1^{(n)} : \mathscr{X}_1^n \to \mathbb{R}$ and $c_2^{(n)} : \mathscr{X}_2^n \to \mathbb{R}$ for encoders 1 and 2, where $\mathbb{R}$ denotes the set of all real numbers. If encoders $\{f_1^n[m_1] \in \mathscr{F}_1^n\}_{m_1 \in \mathscr{M}_n^{(1)}}$ and $\{f_2^n[m_2] \in \mathscr{F}_2^n\}_{m_2 \in \mathscr{M}_n^{(2)}}$ satisfy the input cost constraints

$$\frac{1}{n}c_1^{(n)}(f_1^n[m_1](\psi_1^{n-1}(y^{n-1}))) \leq \Gamma_1, \tag{49}$$

$$\frac{1}{n}c_2^{(n)}(f_2^n[m_2](\psi_2^{n-1}(y^{n-1}))) \leq \Gamma_2 \tag{50}$$

$$({}^{\forall}m_1 \in \mathscr{M}_n^{(1)},\ {}^{\forall}m_2 \in \mathscr{M}_n^{(2)},\ {}^{\forall}y^{n-1} \in \mathscr{Y}^{n-1}),$$

then the pair of encoders is called the (Γ_1, Γ_2)-encoder for the pair of deterministic functions Ψ. The set of all rate pairs (R_1, R_2) that are achievable by a sequence of $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$-codes restricted to the class of (Γ_1, Γ_2)-encoders for all $n = 1, 2 \cdots$ is called the (Γ_1, Γ_2)-capacity region of a pair of deterministic functions Ψ and a MAC $\mathbf{W}$ with deterministic feedback, and denoted by $\mathscr{C}_{\Gamma_1,\Gamma_2}^{\Psi}(\mathbf{W})$.

Let $\mathscr{S}_{\Gamma_1,\Gamma_2}^{\Psi}$ be the set of all pairs $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^{\Psi}$ satisfying

$$P_{X_1^n\|Y^{n-1}}(\mathscr{X}_1^n(\Gamma_1)\|y^{n-1}) = P_{X_2^n\|Y^{n-1}}(\mathscr{X}_2^n(\Gamma_2)\|y^{n-1}) = 1 \tag{51}$$

for all $n = 1, 2, \cdots$, where

$$\mathscr{X}_1^n(\Gamma_1) \triangleq \left\{ x_1^n \in \mathscr{X}_1^n : \frac{1}{n}c_1^{(n)}(x_1^n) \leq \Gamma_1 \right\},$$

$$\mathscr{X}_2^n(\Gamma_2) \triangleq \left\{ x_2^n \in \mathscr{X}_2^n : \frac{1}{n}c_2^{(n)}(x_2^n) \leq \Gamma_2 \right\}.$$

Then, we have the following theorem.

Theorem 12. The (Γ_1, Γ_2)-capacity region $\mathscr{C}_{\Gamma_1,\Gamma_2}^{\Psi}(\mathbf{W})$ for a pair of deterministic functions Ψ and a MAC $\mathbf{W}$ with deterministic feedback is given by

$$\mathscr{C}_{\Gamma_1,\Gamma_2}^{\Psi}(\mathbf{W}) = \bigcup_{(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}_{\Gamma_1,\Gamma_2}^{\Psi}} \mathscr{R}_{\mathbf{W}}(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}).$$

Proof. When $P_{F_k^n}$ is good with respect to $P_{X_k\|Y^{n-1}}$ such that

$$P_{X_k^n\|Y^{n-1}}(\mathscr{X}_k^n(\Gamma_k)\|y^{n-1}) = 1 \quad (k = 1, 2),$$

we have

$$\sum_{\substack{f_k^n \in \mathscr{F}_k^n: \\ \frac{1}{n}c_k^{(n)}(f_k^n(\psi_k^{n-1}(y^{n-1}))) \leq \Gamma_k}} P_{F_k^n}(f_k^n)$$

$$\overset{\text{(a)}}{=} \sum_{x_k^n \in \mathscr{X}_k^n(\Gamma_k)} P_{F_k^n}(\Upsilon_k^n(\psi_k^{n-1}(y^{n-1}), x_k^n))$$
$$= \sum_{x_k^n \in \mathscr{X}_k^n(\Gamma_k)} P_{X_k^n \| Y^{n-1}}(x_k^n \| y^{n-1}) = 1 \quad (k=1,2),$$

where (a) follows because

$$\mathscr{F}_k^n = \bigcup_{x_k^n \in \mathscr{X}_k^n} \Upsilon_k^n(\psi_k^{n-1}(y^{n-1}), x_k^n) \text{ and}$$
$$\Upsilon_k^n(\psi_k^{n-1}(y^{n-1}), x_k^n) \cap \Upsilon_k^n(\psi_k^{n-1}(y^{n-1}), \bar{x}_k^n) = \emptyset \quad (x_k^n \neq \bar{x}_k^n).$$

Thus, any f_k^n such that $P_{F_k^n}(f_k^n) > 0$ satisfies

$$\frac{1}{n} c_k^{(n)}(f_k^n(\psi_k^{n-1}(y^{n-1}))) \le \Gamma_k \quad (k=1,2). \tag{52}$$

By noting (52), the direct part is easily proved by following the proof of the direct part of Theorem 1 with restricting channel input distributions $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}})$ to $(\mathbf{P}_{\mathbf{X}_1\|\mathbf{Y}}, \mathbf{P}_{\mathbf{X}_2\|\mathbf{Y}}) \in \mathscr{S}^{\Psi}_{\Gamma_1,\Gamma_2}$.

To prove the converse part, we use Lemma 8. In Lemma 8, let $P_{F_1^n}$ and $P_{F_2^n}$ be uniform distributions over (Γ_1,Γ_2)-encoders $\{f_1^n[m_1] \in \mathscr{F}_1^n\}_{m_1 \in \mathscr{M}_n^{(1)}}$ and $\{f_2^n[m_2] \in \mathscr{F}_2^n\}_{m_2 \in \mathscr{M}_n^{(2)}}$, respectively. Since (Γ_1,Γ_2)-encoders satisfy (49) and (50), induced channel input distributions $Q_{X_1^n\|Y^{n-1}}$ and $Q_{X_2^n\|Y^{n-1}}$ satisfy

$$Q_{X_1^n\|Y^{n-1}}(\mathscr{X}_1^n(\Gamma_1) \| y^{n-1})$$
$$\overset{\text{(a)}}{=} \sum_{x_1^n \in \mathscr{X}_1^n(\Gamma_1)} P_{F_1^n}(\Upsilon_1^n(\psi_1^{n-1}(y^{n-1}), x_1^n))$$
$$= \sum_{\substack{f_1^n \in \mathscr{F}_1^n: \\ \frac{1}{n} c_1^{(n)}(f_1^n(\psi_1^{n-1}(y^{n-1}))) \le \Gamma_1}} P_{F_1^n}(f_1^n) = 1$$

and $Q_{X_2^n\|Y^{n-1}}(\mathscr{X}_2^n(\Gamma_2) \| y^{n-1}) = 1$ for all $n = 1, 2 \cdots$, where (a) comes from Lemma 1. Thus, by using Lemma 8 and following the proof of the converse part of Theorem 1, we can prove the converse part. □

8. Conclusion

We have shown that the capacity region for general MACs with deterministic feedback can be represented by the information-spectrum formula of directed information. We also have investigated the compound MAC problem, the ε-coding problem, and the cost constraint problem for general MACs with deterministic feedback, and showed the capacity region for these problems. Furthermore, we have showed the relation between the strong converse property and limit superior in probability of directed information.

As directions of future works, it is important to investigate relations between a general MAC with deterministic feedback and without deterministic feedback. For example, it is important to find a general MAC of which capacity region is enlarged by using deterministic feedback, and to investigate a relation between the class of general MACs with deterministic feedback and without deterministic feedback which satisfy the strong converse property. We are going to investigate some of these relations.

References

[1] N. Gaarder and J. Wolf, “The capacity region of a multiple-access discrete memoryless channel can increase with feedback,” IEEE Trans. Inf. Theory, vol.21, no.1, pp.100–102, Jan. 1975.
[2] T. Cover and C. Leung, “An achievable rate region for the multiple-access channel with feedback,” IEEE Trans. Inf. Theory, vol.27, no.3, pp.292–298, May 1981.
[3] G. Kramer, “Capacity results for the discrete memoryless network,” IEEE Trans. Inf. Theory, vol.49, no.1, pp.4–21, Jan. 2003.
[4] J. Massey, “Causality, feedback and directed information,” Proc. 1990 Int. Symp. on Inform. Theory and its Applications, pp.27–30, 1990.
[5] G. Kramer, Directed information for channels with feedback, Ph.D. thesis, ETH Zurich, Switzerland, Canada, 1998.
[6] H. Permuter, T. Weissman, and J. Chen, “Capacity region of the finite-state multiple-access channel with and without feedback,” IEEE Trans. Inf. Theory, vol.55, no.6, pp.2455–2477, June 2009.
[7] T.S. Han, “An information-spectrum approach to capacity theorems for the general multiple-access channel,” IEEE Trans. Inf. Theory, vol.44, no.7, pp.2773–2795, Nov. 1998.
[8] S. Tatikonda and S. Mitter, “Channel coding with feedback,” Proc. on ISIT 2001, p.126, June 2001.
[9] S. Verdú and T.S. Han, “A general formula for channel capacity,” IEEE Trans. Inf. Theory, vol.40, no.4, pp.1147–1157, July 1994.
[10] S. Tatikonda, Control under communication constraints, Ph.D. thesis, Ph.D. dissertation, Electr. Eng. Comput. Sci., Massachusetts Inst. Technol., Cambridge, MA, Aug. 2000.
[11] S. Tatikonda and S. Mitter, “The capacity of channels with feedback,” IEEE Trans. Inf. Theory, vol.55, no.1, pp.323–349, Jan. 2009.
[12] H. Permuter, T. Weissman, and A. Goldsmith, “Finite state channels with time-invariant deterministic feedback,” IEEE Trans. Inf. Theory, vol.55, no.2, pp.644–662, Feb. 2009.
[13] T.S. Han, Information-Spectrum Methods in Information Theory, Springer, 2003.

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