論文 / 著書情報
Article / Book Information

題目(和文)	交渉問題へのゲーム理論の応用
Title(English)	Application of Game Theory to Negotiation Problems
著者(和文)	傅靖
Author(English)	Jing Fu
出典(和文)	学位:博士(工学), 学位授与機関:東京工業大学, 報告番号:甲第9565号, 授与年月日:2014年3月26日, 学位の種別:課程博士, 審査員:武藤　滋夫,樋口　洋一郎,大和　毅彦,松井　知己,河崎　亮
Citation(English)	Degree:Doctor (Engineering), Conferring organization: Tokyo Institute of Technology, Report number:甲第9565号, Conferred date:2014/3/26, Degree Type:Course doctor, Examiner:,,,,
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Application of Game Theory to Negotiation Problems

Jing Fu

Department of Social Engineering

Tokyo Institute of Technology

A thesis submitted for the degree of

Doctor of Engineering

March, 2014

Acknowledgements

First and foremost I want to thank my advisor Shigeo Muto. It has been an honor to be his first foreign Ph.D. student. His enthusiastic and positive attitude toward research and life was always the inspiration for me during the past four and a half years. I appreciate all his contributions of time, ideas, and encouragement to make my Ph.D. experience productive and stimulating.

For this dissertation, I would like to thank the other members of the committee, Professor Takehiko Yamato, Professor Yoichiro Higuchi, Professor Tomomi Matsui and Associate Professor Ryo Kawasaki for their time, insightful comments and suggestions on revision. I would also like to thank Assistant Professor Shin Kishimoto for his continuous support and helpful comments in seminars.

In my work related to DEA game, I am particularly indebted to Satoru Sekine. He formulated the endogenous weight assignment mechanism as a strategic form game and showed the uniqueness condition for the coalition-proof Nash equilibrium. He also proposed a draft idea on the TU game.

The study of service quality's effects on an airline's decision making in partner selection process would not have been possible without the constructive suggestions from Professor Mikio Nakayama at Keio University. I very much appreciated his willingness to frequently discuss with me on this topic for nearly three months.

The members of Muto lab have contributed immensely to my personal and professional time at Tokyo Tech. They have been a source of friendship as well as good collaboration. I am especially grateful for their help at the beginning of my graduate life when I was the only foreign member and knew little Japanese. My time at Tokyo Tech was also made enjoyable in large part due to many friends and groups that became a part of my life. Special thanks to my friends from Sanshin club and TISA.

I gratefully acknowledge the funding source–Japanese government (Monbukagakusho: MEXT) scholarship, that made my Ph.D. work possible.

Lastly, I would like to thank my family for their love, support and encouragement all the time.

Abstract

Game theory is a study of strategic decision making–"the study of mathematical models of conflict and cooperation between intelligent rational decision makers" (Myerson [1991]). Since von Neumann introduced game theory as a specific field firstly in 1928, it has been widely applied in economics, political science, psychology, as well as logic and biology.

In an economic system that composed of multiple intelligent agents, it is often the case that several of them cooperate to maximize the profit or minimize the cost by negotiation. Since the agent is autonomous and intelligent, it is reasonable to assume that each of them chooses the behavior to bring itself the maximal benefit. Thus, the cooperation can be achieved successfully if the coordination mechanism–the allocation of profit or cost–is wisely designed.

In this dissertation, we pay a particular attention to the application of game theory in three multi-agent negotiation problems: minimum cost spanning tree, data envelopment analysis (DEA), and partner selection in airline alliances, where the first two problems assume for an abstract agent. In Chapter 2, we characterize the decentralized rule in the minimum cost spanning tree problem, which was introduced by Feltkamp et al. [1994b]. In Chapter 3, we first improve the DEA game in Nakabayashi and Tone [2006] by proposing an alternative scheme and then focus on analyzing the solutions by our new game theoretic approaches to weight assignments in DEA problems. Lastly, the main concern in Chapter 4 is how the service quality might affect an airline's decision making in the selection of its partner during the formation of airline alliances.

Contents

Chapter 1

Introduction

This dissertation advances game theory by analyzing its application in three different multi-agent negotiation problems: minimum cost spanning tree (MCST), data envelopment analysis (DEA), and partner selection in airline alliances. The approaches followed here differ in the three fields due to the maturity and completeness of the respective literature research, consisting of formal characterization of the cost allocation rule, improvement on the model in literature for a more natural approach, and proposal of a new three-stage analysis scheme. The main method employed to conduct this research has been the mathematical analysis.

Game theory has been a major method used in mathematical economics and business for modeling competing behaviors of interacting agents. The MCST problem is aimed to find a spanning tree with weight less than or equal to the weight of every other spanning tree. DEA is often used for benchmarking in operations management, where a set of criteria is selected to benchmark and evaluate the performance of multiple agents in voting, manufacturing, service operations, and etc. At least in certain contexts, it has received empirical support that an airline is evaluating the service quality of its optional partners before the formation of an alliance; that is, the service quality is affecting an airline's decision making in the partner selection. All three negotiation problems assume for an egoistic agent, and involve the interaction, i.e., the cooperation among intelligent agents, or players.

The contribution of this dissertation is three-fold. First, we have characterized the decentralized rule in the MCST problem by six properties, which was an open question in Feltkamp et al. [1994b]. Next, we have improved the DEA game proposed by Nakabayashi and Tone [2006] by re-assigning the total weight or power for the coalition members and studied the solutions and equilibria of the game following our new approach. Finally, we have showed the strategic effects of the service quality on our proposed complementary airline alliances by a three-stage analysis framework.

This chapter first introduces the field-oriented literature and motivation, then gives some background knowledge and concepts of game theory to be used in our analysis, and finally provides an overview of the dissertation.

1.1 Literature and motivation

1.1.1 MCST problem

1.1.1.1 Literature

Consider a group of agents demanding a particular service that is provided by a common supplier, called the source. Agents can be served through connections to the source either directly or via other agents. Connections are costly, and the cost might be reduced by cooperation. This situation gives rise to two targets: minimize the cost of connecting all agents to the source, and allocate the associated connection cost to the agents in a reasonable way. This kind of problem is called the MCST problem. It has direct applications in the design of networks, including computer networks, telecommunication networks, transportation networks, water supply networks, and etc. Bergantinos and Lorenzo [2004] studied a real case where villagers had to pay the cost of constructing pipes from their respective houses to a water supplier. Dutta and Kar [2004] gave an example of power plant. Many situations in other fields can also be modeled in this way. Xu et al. [2002] represented a set of multi-dimensional gene expression data as a MCST.

The first algorithm for obtaining a MCST was designed by Boruvka [1926b], where it was introduced as a method of constructing an efficient electricity network for Moravia. Later, Kruskal [1956], Prim [1957], and Dijkstra [1959] found similar algorithms. A historic overview for the MCST problem can be found in

Graham and Hell [1985].

Once a MCST is constructed, its associated cost has to be allocated among the agents if it is not sponsored by the government or any organization. This problem was first introduced by Claus and Kleitman [1973]. Bird [1976] suggested a game theoretic approach to the problem and proposed a cost allocation scheme that consists of assigning to each agent (vertex in the graph) the cost of the edge incident upon the vertex on the unique path from the source to the vertex in a MCST. Feltkamp et al. [1994a] introduced a rule for the MCST problem called the equal remaining obligations rule. Initially, each agent has an obligation of 1 and the network is empty. Applying Kruskal's algorithm, the obligation of each agent decreases when for each edge added to the network. At each step, each agent pays some proportion of the cost of the additional edge induced from the difference between its obligation before and after the edge was added. The concept of remaining obligation is also associated with both the proportional rule and the decentralized rule in Feltkamp et al. [1994b]. Other interesting cost sharing methods include the Kar solution (Kar [2002]), Bergantinos and Vidal-Puga [2007].

1.1.1.2 Motivation

In Feltkamp et al. [1994b], two refinements of the irreducible core (Feltkamp et al. [1994a]) for the minimum cost spanning extension (MCSE) problem were introduced, respectively, the proportional rule and the decentralized rule. The MCSE problem differs from MCST problem in the assumption of an initially existing network with free edges that can be used by the agents, which does not affect the characterization. Thus we restrict our context on the MCST problem and do not notify specifically about this difference. The proportional rule is based on Kruskal's algorithm, whereas the decentralized rule arises from Boruvka's algorithm. Feltkamp et al. [1994b] provided an axiomatic characterization on the proportional rule. Chapter 2 is to give a formal characterization on the decentralized rule.

1.1.2 DEA problem

1.1.2.1 Literature

DEA is a linear programming methodology to measure the efficiency of multiple decision-making units (DMUs) when the production process presents a structure of multiple inputs and outputs. Generally DEA is to minimize the "inputs" and maximize the "outputs"; in other words, lower inputs producing higher outputs indicates better performance or efficiency. Based upon Cook et al. [2014], although DEA has a strong link to production theory in economics, the tool is also used for benchmarking in operations management, where a set of criteria is selected to benchmark the performance of the DMUs, or players (the term we are to use in Chapter 3). In order to apply a proper DEA, we need to clarify the inputs and outputs. For example, when evaluating the performance of a set of students, if we consider the score of an subject as an outcome from their effort, it can rightly be viewed as an output. At the same time, the ideas and feedbacks from the students also help in improving the subject, and can therefore be viewed as a resource, or input to the process.

Recently game theoretic approaches to DEA problems have been often observed. These include, in part, Nakabayashi and Tone [2006], Wu et al. [2008], Wu et al. [2009], Liang et al. [2008], Zhu [2004], etc. Among them, Nakabayashi and Tone [2006] studied the problem of allocating a fixed amount of reward to players who are evaluated by multiple criteria. They proposed a new scheme for allocating the reward to the players based on cooperative game theory and DEA. Later, Wu et al. [2008] and Wu et al. [2009] applied the game model by Nakabayashi and Tone [2006] to evaluate the cross efficiency of players by using solutions in cooperative games, the nucleolus and the Shapley value. Liang et al. [2008] viewed the efficiency assessments in two-stage processes in terms of a game approach. In the problem of selection and negotiation of purchasing bids, Zhu [2004] proposed a buyer-seller game model with a more effective evaluation on the alternative bids compared to the existing methods, which is grounded in a revised DEA concept.

1.1.2.2 Motivation

The game proposed by Nakabayashi and Tone [2006] was, however, sub-additive. Namely, players lose their power when they cooperate. The reason is clear. Before forming a coalition, each player has a weight of one and puts it on his/her most preferable criterion. To increase their bargaining power, some of the players choose to form a coalition; but in Nakabayashi and Tone's assumption, the coalition is only given a weight of one in total though each of the members' pre-coalition weight was one, which causes the sub-additivity. To make the game super-additive, they took the dual of the game, called the DEA min game, and study solutions such as the core, the Shapley value and the nucleolus. In the DEA min game, each player and each coalition pick up the weight that minimizes their evaluation. No reasonable justification was given in their paper for picking up the minimizing weight under the assumption that players are egoistic and want to maximize their own evaluation. The purpose of Chapter 3 is to propose an alternative, and more natural, cooperative game scheme that fits for the problem. We will start with a strategic form game describing the problem posed by Nakabayashi and Tone [2006]; then construct a cooperative game from the strategic form game based on the procedures by von Neumann and Morgenstern [1944].

1.1.3 Partner selection problem in airline alliances

1.1.3.1 Literature

An airline alliance is an agreement between two or more airlines to cooperate on a substantial level (e.g., codeshare flights, ticketing systems, maintenance facilities, ground handling personnel, check-in and boarding staff, and etc.) to provide a network of convenient and seamless connectivity for passengers. At present, most major airlines belong to one of the three largest airline alliances: *Star Alliance*, *Oneworld*, and *SkyTeam*. One of the fundamental building blocks of an airline alliance is the codeshare flights. Codeshare is an aviation business agreement where two or more airlines share the same flight. A seat purchased from one airline's ticketing system is actually operated by its partner airline under a different flight number or code. Take three big Asian airlines of Star Alliance as an example, passengers' demand from Beijing (PEK) to Tokyo (NRT) can be satisfied either by a direct flight under ANA (NH), or an optional transit flight with the first leg Beijing (PEK) to Seoul (ICN) operated by Air China (CA), and the second leg Seoul (ICN) to Tokyo (NRT) operated by Asiana Airlines (OZ). Under codeshare agreement, this interline product is marketed by both Air China and Asiana Airlines, and generates profit for both carriers.

Airline alliances can be categorized from different aspects, i.e., commercial or strategic, passenger or cargo, and etc. From the competitiveness of the pre-alliance market, it can be classified as parallel or complementary (Park [1997]). A parallel alliance refers to collaboration between two or more airlines competing on the same route. The pre-alliance market is duopoly or oligopoly. Usually the domestic alliances follow this pattern and are dominated by codeshared routes operated by a single airline. The complementary alliance refers to the case where two airlines link up their existent networks providing an interline service to the passengers, where the pre-alliance market might be monopoly. The international alliances generally involve two or more vertically connecting operating airlines. In reality, two airlines might form both a parallel and a complementary alliance. The example of Air China (CA) and Asiana Airlines (OZ) mentioned above is complementary, while in fact the first leg is sometimes under a codeshare flight operated by Asiana Airlines (OZ), in this case, the two airlines can be viewed as parallel from Beijing (PEK) to Seoul (ICN). In Chapter 4, we focus on the complementary alliance, and leave parallel alliance as a future extension.

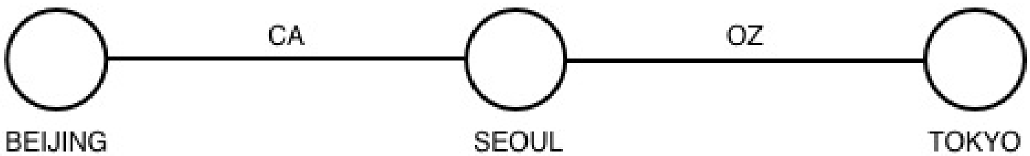

Figure 1.1: An example of complementary alliance

Other related management literature in the alliance formation area include Gulati [1995], Dyer and Singh [1998] and Chung et al. [2000], where they also focused on studying the dyadic alliance relations from the parallel vs. complementary perspective. Evidence from multiple industries imply that complementary alliances are more successful, by allowing partners to extend their network coverage. In the context of airline alliance, Gimeno [2004] examined the content and intensity of dyadic relations, and showed that partner selection is dependent on the extent of alliance co-specialization.

Many strategic factors encourage airlines to join an airline alliance, such as increasing profit opportunities, reducing the cost of airline operations, chance of gaining entry in international markets without obtaining the right through country wise negotiated bilateral agreements (Gudmundsson and Dawna [2001]), and etc. Hence, the main stream of work in the economics literature on airline alliances is grounded in revenue management. Vinod [2005] suggested a bid price scheme for partners in alliance revenue management, which was later analyzed by Wright et al. [2010]. Wright et al. [2010] attempted to analyze airline alliance agreements in conjunction with revenue management. They formulated a model for a two-partner alliance, and derived an equilibrium decision rule for the static proration scheme prevalent in practice, as well as for several dynamic schemes based on suggestions from practitioners. Vulcano et al. [2010] proposed an algorithm to resolve the unobserved no-purchases in estimating discrete choice models using transaction data, and demonstrated that this choice-based model improves revenues by $1-5\%$ on the city-pairs in their paper. Other related literature includes Hu et al. [2010], Talluri and Ryzin [2004].

1.1.3.2 Motivation

From the literature, we can see that little attention has been paid to the initial intent to form an airline alliance: providing a network of connectivity and convenience with better service quality for passengers. The delivery of high service quality is essential for an airline's survival and competitiveness. One of the distinguishing features of our dissertation is to discuss about the service quality's effects on the formation of airline alliances.

Service quality is a consumer's overall impression of the relative inferiority and superiority of the organization and its services (Bitner and Hubbert [1994]).

It can be regarded as a comparison of expectations with the actual performance. Airline service quality is different from services in other industries, comprising tangible and intangible attributes, i.e., reservations, ticketing, baggage handling, seat pitch and size, in-flight service, flight frequency, on time performance, and etc. Doubtlessly, safety cannot miss the first place within all these factors.

The airline service quality can be evaluated by the five-star quality rating system, i.e., the Skytrax Airline Rating System. The idea that the service quality has an important effect on the partner selection came from the member airlines' service quality rating data of the three largest passenger alliances.

Table 1.1: Service quality rating data of the three largest airline alliances

Rating	*Star Alliance* (28 members)	*Oneworld* (12 members)	*SkyTeam* (19 members)	Rest of Industry
5-Star	40%	20%	0%	40%
4-Star	29.03%	12.9%	12.9%	45.17%
3-Star	13.93%	4.91%	9.01%	72.15%
2-Star	0%	0%	0%	100%
1-Star	0%	0%	0%	100%

*Data source: IATA, as of June 2012

The rest of the industry belongs to none of the airline alliances above. From the data in Table 1.1, we can see that the three main airline alliances do not accept airlines rating lower than 3-star as its partner, and the average service quality rate of Star Alliance is obviously higher than that of the other two alliances, which indicates that the service quality of an airline affects, at least to some extent, on the alliance formation. Airlines with high service quality tend to cooperate with each other. So far as the literature we know, our dissertation is the first to formally analyze its effect on the selection of alliance partners by our proposed three-stage analysis framework, namely pre-alliance equilibria analysis, alliance equilibria analysis and criteria verification.

Colonques and Fillol [2005] analyzed the profitability of two alliances from the pricing aspect. Their model was less general because of the specific assumption of monopoly pre-alliance market. Another feature of our paper is a general network topology allowing for both monopoly and duopoly pre-alliance market. The analysis for the oligopoly case is similar but a little bit complicated compared to that of the duopoly one, which is an important extension to pursue in the future.

1.2 Basic concepts of game theory

Game theory deals with interactive decision making where two or more players make decisions that affect each other. A game is cooperative if the players are able to form binding commitments, and non-cooperative if they make decisions independently. Often it is assumed that communication among players is allowed in cooperative games, but not in non-cooperative ones.

1.2.1 Cooperative game theory

In cooperative game, groups of players, or coalitions may enforce cooperative behavior. The game is a competition between coalitions of players, rather than individual players. Most cooperative games are presented in the characteristic function form, which is often assumed to be superadditive (Owen [1995]).

Cooperative games are most generally defined as non-transferable utility (NTU) games. The term transferable utility refers to the fact that only the total payoff for the coalition is specified, and the players within the coalition can transfer the gained utility amongst themselves. It is often assumed in cooperative games, which we call the transferable utility (TU) games.

The main assumption in cooperative game theory is the formation of the grand coalition. The challenge is then to allocate the payoff among players. The core is the set of feasible allocations that cannot be improved upon by any coalition. A coalition is said to improve upon or block a feasible allocation if the members of that coalition are better off under another feasible allocation. The definition of the core in our dissertation follows Gillies [1959]. The Shapley value, named in honor of Lloyd Shapley (Shapley [1953]), assigns a unique distribution (among the players) of the total surplus generated by the coalition of all players. It emphasizes the importance of each player to the overall cooperation. Nucleolus, first introduced in Schmeidler [1969], is the lexicographically minimal imputation; in other words, it is trying to minimize the maximum dissatisfaction of the coalitions regarding an proposed imputation.

1.2.2 Non-cooperative game theory

Non-cooperative game theory deals with the situation where binding agreements are not possible, whether it is because communication is impossible, or agreements are illegal, or there is no authority that can enforce compliance. It is usually represented in the strategic form.

Nash equilibrium, named after John Forbes Nash (Nash [1951]), is a solution concept of the non-cooperative game in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. Strong Nash equilibrium is a Nash equilibrium in which no coalition, given the current strategies, can cooperatively deviates in a way that benefits all of its members (Aumann [1959]). It is criticized too "strong" in that the environment allows for unlimited private communication. A relatively weaker Nash stability concept is called coalitional-proof Nash equilibrium, in which the equilibria are immune to multilateral deviations that are self-enforcing (Bernheim et al. [1987]).

1.2.3 Relationship of cooperative and non-cooperative game theory

Let G be a strategic form (non-cooperative) game. Assuming that coalitions have the ability to enforce coordinated behavior, then it comes the cooperative game associated with G. The α-coalitional game associates with each coalition the sum of benefit its members can 'assure' the coalition to get. The β-coalitional game associates with each coalition the sum of benefit that the outsiders cannot 'prevent' the coalition to get.

1.3 Outline of the dissertation

The structure of this dissertation is as follows: Chapter 2 characterizes the decentralized rule in the MCST problem. In Sections 2.1 and 2.2 we start with the description for the MCST problem, and then revisit the proportional rule by Feltkamp et al. [1994b]. Section 2.3 formally introduces the decentralized rule, and then gives a definition of the decentralized solution. Section 2.4 illustrates the properties to be used in characterizing the decentralized solution. Section 2.5 presents the main characterization result. In Section 2.6 we discuss about the independence of the properties. Chapter 3 first improves the DEA game proposed by Nakabayashi and Tone [2006] by re-assigning the total weight for the coalition members, and then studies the solutions and equilibria of the DEA game under our new scheme. In Sections 3.1 and 3.2, we review the DEA problem and the DEA game proposed by Nakabayashi and Tone [2006]. After giving a definition of the strategic form game in Section 3.3, we present a cooperative TU game representation in Section 3.4, and then analyze its properties and solutions in depth in Sections 3.5 and 3.6. Next we define the NTU coalitional game in both α and β fashion in Section 3.7, and prove the existence of the α-core and give a sufficient condition under which the β-core is non-empty in Section 3.8. The Nash equilibrium, strong Nash equilibrium, and coalition-proof Nash equilibrium for the strategic form game are studied in Section 3.9. Chapter 4 explores service quality's effects on the selection of a partner airline in the formation of airline alliances. In Section 4.1, we describe our general network model, and then outline the main assumptions and decision criteria. Section 4.2 exposes the three-stage analysis framework for three types of pre-alliance market respectively: Monopoly–Monopoly, Monopoly–Duopoly, and Duopoly–Duopoly. The optimal strategy for each airline is discussed in Section 4.3. Chapter 5 summarizes our dissertation and identifies areas for future extension work.

Chapter 2

A characterization of the decentralized rule in the minimum cost spanning tree problem

This chapter characterizes the decentralized rule in the MCST problem by Feltkamp et al. [1994b]. We begin with the definition for the MCST problem and the MCST game. Then we summarize the proportional rule in Feltkamp et al. [1994b] for the comparison study in our axiomatization for the decentralized solution. Next we present the decentralized rule and define the decentralized solution, coming with an in-depth illustration on the six properties to be used in the axiomatization of the decentralized solution and our main characterization result. The last section of this chapter discusses about the axiom independence.

2.1 MCST problem

2.1.1 Notations and preliminaries

We denote by V the set of vertices, and E the set of edges. Some of the notations and preliminaries that we will use in our model are shown as below:

- $G =< V, E >$: a graph consisting of a set V of vertices and a set E of edges. A graph $G' =< V', E' >$ is said to a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$ are satisfied, and each edge of G' has the same ends in G' as in G.

- $e_{ij} = \{i, j\}$: an edge e incident to two vertices i and j.

- $E(V') = \{e \in E | e \subseteq V'\}$: for a graph $G =< V, E >$ and a set $V' \subseteq V$, $E(V')$ is the set of edges linking two vertices in V'.

- $V(E') = \{v \in V | \textit{ there exists an edge } e \in E' \textit{ with } v \in e\}$: the set of vertices incident with $E' \subseteq E$.

- $G_V =< V, E_V >$: the complete graph on a vertex set V, where $E_V = \{e_{ij} = \{i, j\} | i, j \in V \textit{ and } i \neq j\}$.

- Path and cycle: a path from i to j in a graph $G =< V, E >$ is a sequence $(i = i_0, i_1, \ldots, i_k = j)$ of vertices such that for all $1 \leq l \leq k$, the edge $e_{i_{l-1}i_l}$ lies in E. A cycle is a path where $i = j$.

- Spanning tree: a spanning tree of a graph $G =< V, E >$ is a spanning subgraph of G that is a tree, in other words, it is a cycle-free subgraph of G that spans all the vertices.

- V/E: the set of connected components of the graph $< V, E >$. Two vertices $i, j \in V$ are connected in a graph $< V, E >$ if there is a path from i to j in $< V, E >$. A subset V' of V is connected in $< V, E >$ if every two vertices $i, j \in V'$ are connected in the subgraph $< V', E(V') >$. A connected set V' is a *connected component* of the graph $< V, E >$ if no super set of V' is connected. A *connected graph* is a graph $< V, E >$ with V connected in $< V, E >$.

2.1.2 The MCST problem

A MCST problem $\mathcal{M} =< N, *, w, E >$ consists of a finite set N of agents, each of whom wants to be connected to a common source, denoted by $*$. The non-negative cost of constructing a link e_{ij} between the vertices i and j in $N^* = N \cup \{*\}$ is denoted by $w(e_{ij})$. There is a set E of initially existing edges, which can be used for free in connecting agents to the source.

If we shrink each component not containing the source into a single player, and shrink the source component into a new source, as is mentioned in the introduction, it does not affect the characterization in our dissertation, and thus we do not specify the difference between MCST and MCSE problems.

Let $T^{\mathcal{M}}$ be the set of spanning trees associated with a given MCST problem $\mathcal{M} =< N, *, w, E >$. The cost associated with $t \in T^{\mathcal{M}}$ can be represented as

$$c(N^*, w, t) = \sum_{e_{ij} \in t} w(e_{ij})$$

Without any ambiguity, we write $c(t)$ instead of $c(N^*, w, t)$ here. Then the MCST $mt \in T^{\mathcal{M}}$ associated with $\mathcal{M}$ is the spanning tree with the cost $c(mt) = min_{t \in T^{\mathcal{M}}} c(t)$.

The problem we address here is how to construct a network connecting all agents to the source, in the cheapest possible way, and allocate the costs of such a network among the agents. Hence a general solution of a MCST problem can be defined as

Definition 2.1 (general solution)
A general solution of a MCST problem $\mathcal{M} =< N, *, w, E >$ by some algorithm is a function ϕ assigning to $\mathcal{M}$ a sequence of edges (edge sets) ε and a vector x of cost allocation

$$\phi(\mathcal{M}) \subseteq \left\{ (\varepsilon, x) \;\middle|\; \begin{array}{l} < N^*, E \cup \varepsilon_1 \cup \cdots \cup \varepsilon_\tau > \\ is\ a\ connected\ graph\ and\ x \in R^N satisfies \\ \sum_{i \in N} x_i \geq \sum_{t=1}^{\tau} \sum_{C^{t-1} \in N^*/E^{t-1}, * \notin C^{t-1}} w(e^t_{C^{t-1}}) \end{array} \right\}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\tau) = (\{e^1_{C^0} \mid C^0 \in N^*/E^0, * \notin C^0\}, ..., \{e^\tau_{C^{\tau-1}} \mid C^{\tau-1} \in N^*/E^{\tau-1}, * \notin C^{\tau-1}\})$, τ is the number of the total steps in finding the MCST for $\mathcal{M}$ by the algorithm. Here C^{t-1} denotes a connected component not containing the source at the beginning of stage t; and $e^t_{C^{t-1}}$ denotes an edge constructed at stage t, which connects C^{t-1} to another component, where $1 \leq t \leq \tau$.

In our dissertation, the cost allocations are analyzed with game theoretic methods. Next we will introduce the MCST game.

2.1.3 The MCST game

The concept of MCST game was first introduced by Bird [1976], and it is defined as below:

Definition 2.2 (MCST game)
Let $< N, c^{\mathcal{M}} >$ *be a cooperative cost game associated with the MCST problem* $\mathcal{M} =< N, *, w, E >$*, where* $c^{\mathcal{M}}(S)$*, the value of the coalition* S*, is the minimum cost of building edges between components containing members of* $S^* = S \cup \{*\}$*, so as to make* S *connected to the source. The formal definition of* $c^{\mathcal{M}}(S)$ *is*

$$c^{\mathcal{M}}(S) = min\left\{\sum_{e\in E'} w(e)|S^* \in C_*^{E'}\right\}$$

where E' *contains only edges between components containing members of* S^**, and* $C_*^{E'}$ *is the component of the source* $*$ *in the graph* $< N^*, E \cup E' >$*.*

Next we define the core of the cooperative cost game $(N, c^{\mathcal{M}})$.

Definition 2.3 (core of $(N, c^{\mathcal{M}})$)

$$Core(N, c^{\mathcal{M}}) = \left\{x \in \mathcal{R}^N | \sum_{i\in N} x_i = c^{\mathcal{M}}(N) \ and \ \sum_{i\in S} x_i \leq c^{\mathcal{M}}(S) \ \forall S \subseteq N\right\}$$

Bird [1976] indicated a method to find the core element of the MCST game when a MCST is given. Further he also introduced the irreducible core of a MCST game, which is a subset of the core.

Definition 2.4 (irreducible core)
Given a MCST problem $\mathcal{M} =< N, *, w, E >$ *and a MCST* mt*, the irreducible core* $IC(\mathcal{M}, mt)$ *of* $\mathcal{M}$ *with respect to* mt *is defined as: let* $Var(\mathcal{M}, mt)$ *be the set of all MCST problems obtained from* $\mathcal{M}$ *by varying the weight* $w(e)$ *of edges* $e \notin mt$*, that still have* mt *as MCST. Then* $IC(\mathcal{M}, mt)$ *is the intersection of the cores of all MCST games associated with a MCST problem in* $Var(\mathcal{M}, mt)$*, that is*

$$IC(\mathcal{M}, mt) = \cap\{Core(N, c^{\mathcal{M}'}) \mid \mathcal{M}' \in Var(\mathcal{M}, mt)\}$$

If the set of E is empty, the definition above coincides with the one in Bird [1976]. Before introducing the decentralized rule, let us first make a review on the proportional rule.

2.2 Proportional rule revisited

If an agent is initially in the source component, it has no obligation to contribute for the network construction; whereas if it is not in the source component, then the component of that agent has to pay for building one edge, and this obligation is shared equally among the component agents. Based on this consideration, Feltkamp et al. [1994a] introduced the initial obligation, a concept to be used in both the proportional rule and the decentralized rule.

Definition 2.5 (initial obligation)
The initial obligation o_i for an agent $i \in N$ in a MCST problem $\mathcal{M} =< N, *, w, E >$ is defined as

$$o_i^0 = \begin{cases} \frac{1}{|C_i^0|} & \textit{if } * \notin C_i^0 \\ 0 & \textit{if } * \in C_i^0 \end{cases}$$

where C_i^0 denotes the component containing i in the initial graph $< N^*, E >$, and $|C_i^0|$ denotes the number of agents in that component.

The proportional solution is constructed by the following algorithm: construct the edges of a MCST as in Kruskal's algorithm. Each time an edge is constructed, its cost is divided proportionally to the remaining obligations, among the agents in the components being linked.

Algorithm 2.1 (proportional rule)

Input: *a MCST problem $\mathcal{M} =< N, *, w, E >$.*

Output: *a MCST formed by adding a sequence of edges and a cost allocation for all the related agents.*

(1) Given $\mathcal{M} =< N, *, w, E >$, define

$t = 0$*: the initial stage;*

$\tau = |N^*/E| - 1$: the number of stages;

$E^0 = E$: the initial edge set;

o_i^0: the initial obligation for all $i \in N$ (Definition 2.5).

(2) $t := t + 1$.

(3) At stage t, given E^{t-1}, choose a cheapest edge e^t such that the graph $< N^*, E^{t-1} \cup \{e^t\} >$ does not contain more cycles than $< N^*, E^{t-1} >$.

(4) If C^t is the connected component just formed by adding the edge e^t to the graph $< N^*, E^{t-1} >$, define the vector $f^t = (f_i^t)_{i \in N}$ of fractions the agents contribute by

$$f_i^t = \begin{cases} \frac{o_i^{t-1}}{\sum_{l \in C^t} o_l^{t-1}} & \textit{if } i \in C^t \\ 0 & \textit{if } i \notin C^t \end{cases}$$

(5) Define the remaining obligation after stage t by $o_i^t = o_i^{t-1} - f_i^t$ for all $i \in N$.

(6) Define $E^t := E^{t-1} \cup \{e^t\}$.

(7) If $t < \tau$, go back to stage 2.

(8) Define $\varepsilon = (e^1, \ldots, e^\tau)$.

(9) Define $PRO^\varepsilon(\mathcal{M}) = \sum_{t=1}^{\tau} f^t w(e^t)$.

Definition 2.6 (proportional solution)
The proportional solution is defined by

$$PRO(\mathcal{M}) = \cup\{(\varepsilon, PRO^\varepsilon(\mathcal{M}) \mid \varepsilon \textit{ is obtained by Algorithm } 2.1)\}$$

Feltkamp et al. [1994b] showed that the set of allocations generated by the proportional solution is a refinement of the irreducible core, and in particular, they are all core elements of the MCST game. The following theorem is their main result on the axiomatization of the proportional solution.

Theorem 2.1 (axiomatic characterization of the proportional rule)
The unique solution of MCST problems that satisfies Eff (efficiency), MC (minimal contribution), FSC (free for source component), ET (equal treatment), ES (equal share), Loc (locality) and CoCons (converse consistency) is the proportional solution.

where the definitions of Eff, MC, FSC, ET, Loc are the same as in Section 2.4. In order to define ES and CoCons, let us first introduce the edge-reduced MCST problem.

Definition 2.7 (edge-reduced MCST problem)
Given a MCST problem $\mathcal{M} =< N, *, w, E >$, *assume that an edge* $e_{ij} = \{i, j\}$ *connects two components* $C_i \ni i$ *and* $C_j \ni j$ *of* $< N^*, E >$, *the edge-reduced MCST problem is*

$$\mathcal{M}^{e_{ij}} =< N, *, w, E \cup \{e_{ij}\} >$$

The edge-reduced MCST problem is a smaller problem than the original problem such that less edges have to be constructed, while the number of agents remains the same. Then ES and CoCons can be defined as below.

Definition 2.8 (equal share)
ϕ *satisfies equal share* **(ES)** *if for any* $\mathcal{M} =< N, *, w, E >$, *for all* $(\varepsilon, x) \in \phi(\mathcal{M})$ *with* e^1_{ij} *connecting two components* $C^0_i \ni i$ *and* $C^0_j \ni j$, *there exists* $(\tilde{\varepsilon}, \tilde{x}) \in \phi(\mathcal{M}^{e^1_{ij}})$ *such that*

$$\sum_{i \in C^0_i} (x_i - \tilde{x}_i) = \sum_{j \in C^0_j} (x_j - \tilde{x}_j)$$

In effect, ES requires that the two components connected in the first step of a solution participate in equal amounts in the cost of the edge which connects them.

Definition 2.9 (converse consistency)
ϕ *satisfies converse consistency* **(CoCons)** *if for all* $\mathcal{M} =< N, *, w, E >$, *for all* $(\varepsilon, x) \in E^{\tau}_{N^*} \times \mathcal{R}^N$ *such that the solution* ϕ' *defined by*

$$\phi'(\mathcal{M}') = \begin{cases} \phi(\mathcal{M}) \cup \{(\varepsilon, x)\} & \text{if } \mathcal{M}' = \mathcal{M} \\ \phi(\mathcal{M}') & \text{if } \mathcal{M}' \neq \mathcal{M} \end{cases}$$

satisfies Eff, MC, FSC, ET, ES and Loc, it holds that

$$(\varepsilon, x) \in \phi(\mathcal{M})$$

The upshot of CoCons is that one should not be able to enlarge a solution without losing at least one of Eff, MC, FSC, ET, ES, and Loc. For more details regarding Theorem 2.1, please refer to Feltkamp et al. [1994b].

2.3 The decentralized rule and solution

2.3.1 The decentralized rule

The proportional rule is a centralized algorithm, in the sense that one edge is constructed per stage. However, one might imagine the situation that each component greedily builds the cheapest edge that links itself to another component and finally to the source during the construction of the MCST. If two components want to build the same edge, they meet in the middle, and each pays half of the construction cost. If one component wants to link itself to another component, which has other better options or is the source component, it has to pay for the cost of the whole edge. This is the basic idea behind the decentralized rule, which dates back to Boruvka [1926b]. The decentralized algorithm will build a network in fewer stages than the centralized algorithms. However, it has a shortcoming such that when applied to an arbitrary MCST problem, it might generate a network with cycles. In this paper, we assume for a generic MCST problem, where all weights are different, and then the algorithm works pretty well. The decentralized solution is derived from the following decentralized rule.

Algorithm 2.2 (decentralized rule)

Input: *a MCST problem* $\mathcal{M} =< N, *, w, E >$.

Output: *a MCST formed by adding a sequence of edge sets at each stage and a cost allocation for all the related agents.*

(1) Given $\mathcal{M} =< N, *, w, E >$, *define*

$t = 0$: *the initial stage;*

$E^0 = E$: *the initial edge set;*

o_i^0: *the initial obligation for all* $i \in N$ *(Definition 2.5).*

(2) $t := t + 1$.

(3) *At stage* t*, each component* C^{t-1} *of* $< N^*, E^{t-1} >$ *that does not contain the source chooses a cheapest edge* $e^t_{C^{t-1}}$ *linking* C^{t-1} *to another component of* $< N^*, E^{t-1} >$.

(4) *Define the vector* $f^t = (f_i^t)_{i \in N}$ *of fractions by*

$$f_i^t = \begin{cases} o_i^{t-1} & \text{if no other component chooses } e^t_{C_i^{t-1}} \\ o_i^{t-1}/2 & \text{if another component also chooses } e^t_{C_i^{t-1}} \\ 0 & \text{if } i \in C_*^{t-1} \end{cases}$$

for all $i \in N$. C_i^{t-1} *denotes the component containing* i *in the graph* $< N^*, E^{t-1} >$ *constructed at stage* $t - 1$.

(5) *Define the remaining obligation after stage* t *by* $o_i^t = o_i^{t-1} - f_i^t$ *for all* $i \in N$.

(6) *Define* $E^t := E^{t-1} \cup \{e^t_{C^{t-1}} | C^{t-1} \in N^*/E^{t-1} \text{ and } * \notin C^{t-1}\}$.

(7) *If the graph* $< N^*, E^t >$ *is not yet connected, go back to stage 2.*

(8) *Define* τ *to be the number of stages.*

(9) *Define the decentralized value* $DEC(\mathcal{M})$ *by*

$$DEC_i(\mathcal{M}) = \sum_{s=1}^{\tau} f_i^s w(e^s_{C_i^{s-1}})$$

for all $i \in N$.

2.3.2 The decentralized solution

Definition 2.10 (decentralized solution)

A decentralized solution $DEC(\mathcal{M})$ is a general solution $\phi(\mathcal{M})$ satisfying the conditions below

(1) For all t such that $1 \leq t \leq \tau$, for all $C^{t-1} \in N^*/E^{t-1}$ that does not contain the source, $w(e^t_{C^{t-1}}) = min\ w(e_{ab})$, where $a \in C^{t-1}$ and $b \notin C^{t-1}$.

(2) $\forall i \in N, x_i = DEC_i(\mathcal{M})$.

The first condition describes each component's greedy nature of selecting the cheapest edge connecting itself to another component. The second condition is the cost allocation generated by Algorithm 2.2. Before introducing the properties to be used in the characterization of the decentralized solution, let us first see an example.

2.3.3 An example

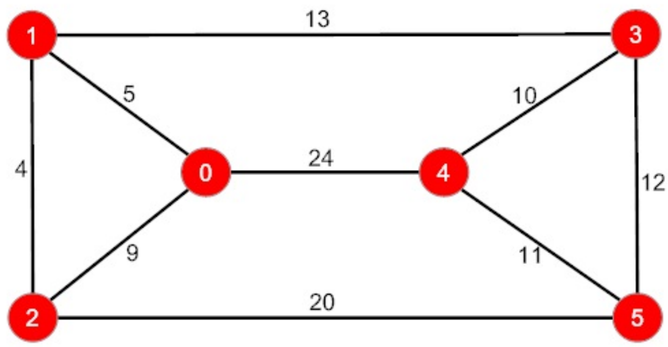

Figure 2.1: An example of the decentralized solution

(1) Given the MCST problem $\mathcal{M}$ shown in Figure 2.1, initialize the parameters as

$t = 0$: the initial stage;

$E^0 = \emptyset$: the initial edge set;

$o_i^0 = \begin{cases} 1 & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases}$: the initial obligation.

(2) $t := t + 1 = 1$.

(3) At stage 1, $w(e^1_{C^0_1}) = w(e^1_{C^0_2}) = 4, w(e^1_{C^0_3}) = w(e^1_{C^0_4}) = 10, w(e^1_{C^0_5}) = 11$.

(4) The vector of fractions is

$$f^1 = (f^1_i)_{i \in \{1,2,3,4,5\}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$$

(5) The remaining obligation after stage 1 is

$$o_i^1 = \begin{cases} \frac{1}{2} & \text{if } i \neq 0 \text{ or } 5 \\ 0 & \text{if } i = 0 \text{ or } 5 \end{cases}$$

(6) $E^1 = E^0 \cup \{e_{12}, e_{34}, e_{45}\} = \{e_{12}, e_{34}, e_{45}\}$.

(7) The graph is not yet connected, $t := t + 1 = 2$.

(8) At stage 2, $w(e^2_{C^1_1}) = w(e^2_{C^1_2}) = 5, w(e^2_{C^1_3}) = w(e^2_{C^1_4}) = w(e^2_{C^1_5}) = 13$.

(9) The vector of fractions is

$$f^2 = (f^2_i)_{i \in \{1,2,3,4,5\}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$$

(10) The remaining obligation after stage 2 is $o_i^2 = 0, \forall i \in \{1, 2, 3, 4, 5\}$.

(11) $E^2 = E^1 \cup \{e_{01}, e_{13}\} = \{e_{12}, e_{34}, e_{45}, e_{01}, e_{13}\}$.

(12) The graph is connected and $\tau = 2$.

(13) The implemented MCST is shown in Figure 2.2 and the decentralized value $DEC(\mathcal{M})$ is

$$DEC_1(\mathcal{M}) = DEC_2(\mathcal{M}) = \frac{9}{2}, DEC_3(\mathcal{M}) = DEC_4(\mathcal{M}) = \frac{23}{2}, DEC_5(\mathcal{M}) = 11$$

The decentralized solution is:

$$DEC(\mathcal{M}) = ((\{e_{12}, e_{34}, e_{45}\}, \{e_{01}, e_{13}\}), (\frac{9}{2}, \frac{9}{2}, \frac{23}{2}, \frac{23}{2}, 11))$$

It is easy to show that in a generic MCST problem, the proportional rule and the decentralized rule are inducing exactly the same MCST, while the cost allocation might be different. In this example, the MCST by the proportional rule is the same as shown in Figure 2.2, and the proportional solution is

$$PRO(\mathcal{M}) = ((e_{12}, e_{01}, e_{34}, e_{45}, e_{13}), (\frac{9}{2}, \frac{9}{2}, 11, 11, 12))$$

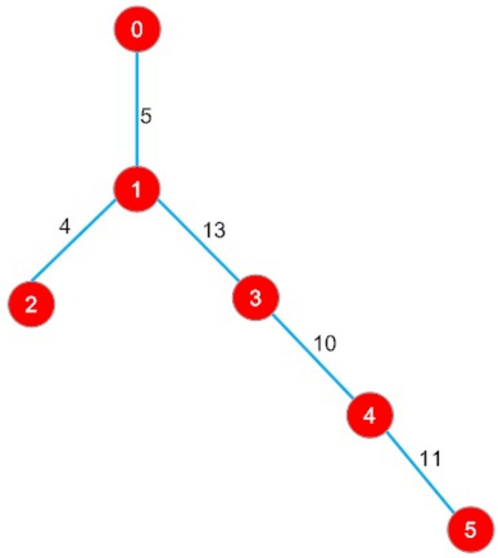

Figure 2.2: The MCST by the decentralized rule

2.4 Properties characterizing the decentralized solution

Feltkamp et al. [1994b] proved that on the class of generic MCST problems, the cost allocations in the decentralized solution are elements of the irreducible core.

In Theorem 2.1, they used seven properties characterizing the proportional solution, namely, Eff, MC, FSC, ET, ES, Loc and CoCons. However, two components connected in the first step by the decentralized rule do not necessarily share the same amount of the cost of the edge that connects them, i.e., e_{45} constructed at stage 1 is totally paid by agent 5 in the example above; thus the decentralized solution does not satisfy ES, and hence CoCons is not satisfied as well. We cannot expect to reproduce the characterization for the proportional rule.

2.4.1 Efficiency

ϕ is *efficient* (**Eff**) if for all $\mathcal{M} =< N, *, w, E >$, for all $(\varepsilon, x) \in \phi(\mathcal{M})$, $E \cup \varepsilon_1 \cup \cdots \cup \varepsilon_\tau$ is a MCST and

$$\sum_{i \in N} x_i = \sum_{t=1}^{\tau} \sum_{C^{t-1} \in N^*/E^{t-1}, * \notin C^{t-1}} w(e^t_{C^{t-1}})$$

2.4.2 Minimal contribution

ϕ has the *minimal contribution* (**MC**) property if for all $\mathcal{M} =< N, *, w, E >$, for all $(\varepsilon, x) \in \phi(\mathcal{M})$, for each component $C \in N^*/E$ that does not contain the source

$$\sum_{i \in C} x_i \geq min\{w(e)|e \ connects \ two \ components \ of \ < N^*, E >\}$$

For a general solution of a MCST problem, if every component that does not initially contain the source contributes at least the cost of a minimum cost edge that connects two components, this solution has the MC property.

2.4.3 Free for source component

ϕ has the *free for source component* (**FSC**) property if for all $\mathcal{M} =< N, *, w, E >$, for all $(\varepsilon, x) \in \phi(\mathcal{M})$, we have

$$x_i = 0$$

for all $i \in C_*$.

This property states that any agent in the component of the source does not contribute in any cost of edge construction. The subset of agents in the source component is indifferent to any new connection. However, agents that want to be connected to the source can potentially make use of the existing free links in the source component to reduce their cost. The bargaining game between the agents in the source component and the rest of agents is an interesting extension to pursue in our future research.

2.4.4 Equal treatment

ϕ has the *equal treatment* (**ET**) property if for all $\mathcal{M} =< N, *, w, E >$, for all $(\varepsilon, x) \in \phi(\mathcal{M})$, for all components $C \in N^*/E$, and for all agents i and $j \in C$

$$x_i = x_j$$

This property specifies that any pair of agents initially in the same component will participate in the same amount of the cost for the network construction.

2.4.5 Locality

ϕ is *local* (**Loc**) if for all $\mathcal{M} =< N, *, w, E >$, for all $(\varepsilon, x) \in \phi(\mathcal{M})$, assume that at stage 1, a set of edges $\varepsilon_1 \ni e^1_{ij}$ is constructed, where e^1_{ij} connects two components $C^0_i \ni i$ and $C^0_j \ni j$ of $< N^*, E >$ into a new component C^1_{ij}, then there exists an $\tilde{x} \in R^{C^1_{ij}}$ such that

$$((\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{\tau'}), (\tilde{x}, x^{N \setminus C^1_{ij}})) \in \phi(\mathcal{M}^{e^1_{ij}})$$

where $\tilde{\varepsilon}_1 \cup \ldots \cup \tilde{\varepsilon}_{\tau'} = \varepsilon_1 \cup \ldots \cup \varepsilon_\tau \setminus \{e^1_{ij}\}$, and $\tau' = \tau - 1$ or τ.

This property requires that adding an extra edge into the current network should not affect agents outside the component formed by adding this edge.

2.4.6 Unilateral self-interest

ϕ has the *unilateral self-interest* (**USI**) property if for all $\mathcal{M} =< N, *, w, E >$, for all $(\varepsilon, x) \in \phi(\mathcal{M})$, for all t such that $1 \leq t \leq \tau$ and for all components $C^{t-1} \in N^*/E^{t-1}$ that does not contain the source, the following condition is satisfied

$$|\varepsilon_t| \in [\lceil |C^{t-1}|/2 \rceil, |C^{t-1}|]$$

and at stage 1, a set of edges $\varepsilon_1 \ni e^1_{ij}$ is constructed, where e^1_{ij} connects two components $C^0_i \ni i$ and $C^0_j \ni j$ of $< N^*, E >$ into a new component C^1_{ij}, then the cost allocation satisfies

$$\begin{cases} \exists(\tilde{\varepsilon}, \tilde{x}) \in \phi(\mathcal{M}^{e^1_{ij}}) \ s.t. \ \sum_{i \in C^0_i}(x_i - \tilde{x}_i) = \sum_{j \in C^0_j}(x_j - \tilde{x_j}) & \text{if } |\varepsilon_1| = |C^0|/2 \\ \exists C^{0*}_i \ s.t. \ x_k = \frac{w(e^1_{ij})}{|C^{0*}_i|} \ \forall k \in C^{0*}_i & \text{if } |\varepsilon_1| > |C^0|/2 \end{cases}$$

where $|\varepsilon_t| \in \mathbb{Z}^+$ denotes the number of edges constructed at stage t, and $|C^{t-1}| \in \mathbb{Z}^+$ denotes the number of components not containing the source at the beginning of stage t.

This property depicts the self-interest nature of each component not connected to the source from two aspects: first, at any stage t, the number of edges constructed should be at least half of, and at most equal to the number of the components not connected to the source at the beginning of stage t. In Example 2.3.3 for the decentralized rule, at the beginning of the first stage, there are 5 components not yet connected to the source, then the number of edges to be constructed is 3, or 4, or 5. Second, from the cost sharing perspective, if the number of edges constructed in the first stage is exactly half of the number of the components not connected to the source in the initial network, then any pair of components connected at this stage participate in equal amounts in the cost of the edge which connects them. However, if the number is more than half, there exists at least one component in the original network paying for the whole edge which connects itself to another.

Unlike other centralized cost allocation rules, this property emphasizes the subject “component” choosing the “edge” at each stage based on its own interest, instead of a "central operator" adding one edge per stage for the network's overall interest. In other words, the decentralized process is one such that the decisions by the components are made without centralized control or processing. "Unilateral" is to describe the decision made by the component itself, without the agreement of others. Although it might result in bilateral cooperation, we still name this property as unilateral self-interest as the bilateral cooperation is grounded in each component's individual interest to share less in the construction cost.

2.5 Axiomatic characterization of the decentralized solution

2.5.1 Main result

In Section 2.4, we illustrated the properties to be used in the axiomatization. We formally characterize the decentralized solution in this section.

Proposition 2.1

The decentralized solution satisfies Eff, MC, FSC, ET, Loc, USI.

Proof. Feltkamp et al. [1994a] showed that the irreducible core satisfies Eff, MC and FSC. In Feltkamp et al. [1994b], they proved that the allocations generated by the decentralized rule are elements of the irreducible core. Hence the decentralized solution has the properties of **Eff**, **MC** and **FSC**.

ET: in the decentralized solution, for all $\mathcal{M} =< N, *, w, E >$, for all $(\varepsilon, x) \in DEC(\mathcal{M})$, for all components $C^0 \in N^*/E$, and for all agents i and $j \in C^0$

$$o_i^0 = o_j^0 = \begin{cases} \frac{1}{|C^0|} & \text{if } * \neq C^0 \\ 0 & \text{if } * \in C^0 \end{cases}$$

and for all t such that $1 \leq t \leq \tau$

$$f_i^t = f_j^t$$

is satisfied as i and j remain in the same component and always make the same decision on the edge selection. Following the definition of the decentralized solution, we have

$$DEC_i(\mathcal{M}) = DEC_j(\mathcal{M})$$

Hence the decentralized solution has the property of ET.

Loc: take $((\varepsilon_1, \ldots, \varepsilon_\tau), x) \in DEC(\mathcal{M})$. Assume that $e^1_{ij} \in \varepsilon_1$ connects two components $C^0_i \ni i$ and $C^0_j \ni j$ of $< N^*, E >$ into a new component C^1_{ij}. Based on our generic assumption on the MCST problem, $\tilde{\varepsilon} = (\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{\tau'})$ will lead to a MCST of $\mathcal{M}^{e^1_{ij}}$, where $\tilde{\varepsilon}_1 \cup \ldots \cup \tilde{\varepsilon}_{\tau'} = \varepsilon_1 \cup \ldots \cup \varepsilon_\tau \setminus \{e^1_{ij}\}$, and $\tau' = \tau$ or $\tau - 1$.

Let $\tilde{\mathcal{F}}$ be the unique sequence of fraction vectors corresponding to $\tilde{\varepsilon}$ following Algorithm 2.2, and define $\tilde{x} = x^{\tilde{\varepsilon}, \tilde{\mathcal{F}}}$. For any agent $k \notin C^1_{ij}$, its initial obligation and vector of contribution fractions are identical to that of ε, and thus $\tilde{x}_k = x_k$. The decentralized solution has the property of Loc.

USI: in Algorithm 2.2, at any stage t such that $1 \leq t \leq \tau$, each of the components $C^{t-1} \in N^*/E^{t-1}$ not connected to the source is greedily choosing a cheapest edge connecting itself to another component. If $|C^{t-1}| = 2n$, the number of components not containing the source is even, the minimum number of edges n is formed when all of them are in a pairwise cooperation, the maximum number of edges $2n$ is formed if no pairs of the components cooperate bilaterally. Similarly, if $|C^{t-1}| = 2n - 1$, the minimum number of edges formed at stage t should be n, and the maximum number should be $2n - 1$, where $n \in \mathbb{Z}^+$.

In the first stage, if the number of edges constructed $|\varepsilon_1|$ is exactly $|C^0|/2$, which means that there are even numbers of agents not connected to the source in the initial network, then the components are cooperating in a pairwise mode and none of them will be connected to the source at this stage. Suppose $e^1_{ij} \in \varepsilon_1$ connects two components $C^0_i \ni i$ and $C^0_j \ni j$ of $< N^*, E >$ into a new component C^1_{ij}, then C^0_i and C^0_j share $w(e^1_{ij})$ equally. Consider the edge-reduced MCST problem $\mathcal{M}^{e^1_{ij}} =< N, *, w, E \cup \{e^1_{ij}\} >$. $\forall i, j \in C^1_{ij}$, its initial obligation is $\frac{1}{|C^0_i| + |C^0_j|}$. On the generic assumption, for any sequence of edge sets $\tilde{\varepsilon}$ constructed by Algorithm

2.2, where $\tilde{\varepsilon}_1 \cup \ldots \cup \tilde{\varepsilon}_{\tau'} = \varepsilon_1 \cup \ldots \cup \varepsilon_\tau \setminus \{e^1_{ij}\}$, the remaining obligations at stage t are only dependent on the remaining obligations in the previous stage, by the inductive method, i and j have the same remaining obligations throughout all stages in $\mathcal{M}^{e^1_{ij}}$. Since in the unique sequence of fraction vectors $\tilde{\mathfrak{F}}$ corresponding to $\tilde{\varepsilon}$ following Algorithm 2.2, the fractions of edges that i and j pay are proportional to the remaining obligations at each stage. It follows that $\tilde{f}^t_i = \tilde{f}^t_j$ for all t such that $1 \leq t \leq \tau$, and $x_i^{\tilde{\varepsilon},\tilde{\mathfrak{F}}} = x_j^{\tilde{\varepsilon},\tilde{\mathfrak{F}}}$. Thus

$$\sum_{i \in C_i^0} (x_i - x_i^{\tilde{\varepsilon},\tilde{\mathfrak{F}}}) = \sum_{j \in C_j^0} (x_j - x_j^{\tilde{\varepsilon},\tilde{\mathfrak{F}}}) = \frac{w(e^1_{ij})}{2}$$

However, if the number of edges constructed in the first stage $|\varepsilon_1|$ is more than $|C^0|/2$, there is at least one component not connected to the source in the initial network which has a unilateral connection, and is responsible for the cost of that connection totally by itself.

■

Theorem 2.2
A solution ϕ satisfies Eff, MC, FSC, ET, Loc, USI if and only if it is a decentralized solution.

Proof. Existence. Proposition 2.1 proves that a decentralized solution satisfies Eff, MC, FSC, ET, Loc, USI.

Uniqueness. Consider a solution ϕ satisfying these properties. We first take $(\varepsilon, x) \in \phi(\mathcal{M})$, and show that (ε, x) is a decentralized solution.

Suppose $e^1_{ij} \in \varepsilon_1$ connects two components $C_i^0 \ni i$ and $C_j^0 \ni j$ of $< N^*, E >$ into C^1_{ij}. By the property of Loc, there exists an $\tilde{x} \in R^{C^1_{ij}}$ such that

$$((\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{\tau'}), (\tilde{x}, x^{N \setminus C^1_{ij}})) \in \phi(\mathcal{M}^{e^1_{ij}})$$

where $\tilde{\varepsilon}_1 \cup \ldots \cup \tilde{\varepsilon}_{\tau'} = \varepsilon_1 \cup \ldots \cup \varepsilon_\tau \setminus \{e^1_{ij}\}$, and $\tau' = \tau - 1$ or τ.

Hence there exists a sequence of fraction vectors $(\tilde{f}^1, \ldots, \tilde{f}^{\tau'})$ corresponding to $(\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{\tau'})$, such that

$$(\tilde{x}, x^{N\setminus C^1_{ij}}) = x^{(\tilde{\varepsilon}_1,\ldots,\tilde{\varepsilon}_{\tau'}),(\tilde{f}^1,\ldots,\tilde{f}^{\tau'})}$$

As ϕ satisfies Eff, the sum of the cost allocated to each agent is equal to the sum of the edge weights of ε in $\mathcal{M}$, and $\tilde{\varepsilon}$ in $\mathcal{M}^{e^1_{ij}}$, that is

$$\sum_{i\in N} x_i = \sum_{t=1}^{\tau}\sum_{e\in\varepsilon_t} w(e)$$

$$\sum_{i\in N} \tilde{x}_i = \sum_{t=1}^{\tau'}\sum_{e\in\tilde{\varepsilon}_t} w(e)$$

Both $\phi(\mathcal{M})$ and $\phi(\mathcal{M}^{e^1_{ij}})$ satisfy Eff and Loc, we have

$$\sum_{k\in C^1_{ij}} (x_k - \tilde{x}_k) = w(e^1_{ij})$$

We distinguish three cases:

Case 1: If $|\varepsilon_1| = |C^0|/2$, the number of edges constructed at the first stage is exactly half of the number of the components not containing the source in the initial network, then any of the connections $e^1_{ij} \in \varepsilon_1$ indicates a bilateral cooperation. By the properties of ET and USI

$$x_k(e^1_{ij}) - \tilde{x}_k(e^1_{ij}) = \begin{cases} \frac{w(e^1_{ij})}{2|C^0_i|} & \text{if } k \in C^0_i \\ \frac{w(e^1_{ij})}{2|C^0_j|} & \text{if } k \in C^0_j \end{cases}$$

In this case, define f^1 by

$$f^1_k(e^1_{ij}) = \begin{cases} \frac{1}{2|C^0_i|} & \text{if } k \in C^0_i \\ \frac{1}{2|C^0_j|} & \text{if } k \in C^0_j \\ 0 & \text{if } k \notin C^1_{ij} \end{cases}$$

Case 2: If $|\varepsilon_1| \in (|C^0|/2, |C^0|)$, by the property of USI, define $\mathcal{C}^{0*} = \{C^{0*}_i\}$ as the set of components such that $x_k = \frac{w(e^1_{ij})}{|C^{0*}_i|}$ $\forall k \in C^{0*}_i$. On the generic assumption, with the property of ET, for all $C_i^{0*} \in \mathcal{C}^{0*}$, for all $k \in C_i^{0*}$, the only condition such that $x_k = \frac{w(e_{ij}^1)}{|C_i^{0*}|}$ is when C_i^{0*} pays for the whole edge which connects itself to another component in the first stage. Thus for all $k \in C_i^0$

$$x_k(e_{ij}^1) - \tilde{x}_k(e_{ij}^1) = \begin{cases} \frac{w(e_{ij}^1)}{|C_i^0|} & \text{if } C_i^0 \in \mathcal{C}^{0*} \\ \frac{w(e_{ij}^1)}{2|C_i^0|} & \text{if } C_i^0 \notin \mathcal{C}^{0*} \text{ and } C_j^0 \notin \mathcal{C}^{0*} \\ 0 & \text{if } C_j^0 \in \mathcal{C}^{0*} \end{cases}$$

In this case, define f^1 by

$$f_k^1(e_{ij}^1) = \begin{cases} \frac{1}{|C_i^0|} & \text{if } C_i^0 \in \mathcal{C}^{0*} \\ \frac{1}{2|C_i^0|} & \text{if } C_i^0 \notin \mathcal{C}^{0*} \text{ and } C_j^0 \notin \mathcal{C}^{0*} \\ 0 & \text{if } C_j^0 \in \mathcal{C}^{0*} \end{cases}$$

For $k \in C_j^0$, the definition follows the same pattern above; and for $k \notin C_{ij}^1$, $f_k^1(e_{ij}^1) = 0$.

Case 3: If $|\varepsilon_1| = |C^0|$, the number of edges constructed at the first stage is equal to the number of components not connected to the source in the initial network, then any of the connections $e_{ij}^1 \in \varepsilon_1$ indicates a unilateral selection. Either C_i^0 or C_j^0, say C_j^0, unilaterally selects e_{ij}^1, by the properties of ET and USI, we obtain

$$x_k(e_{ij}^1) - \tilde{x}_k(e_{ij}^1) = \begin{cases} 0 & \text{if } k \in C_i^0 \\ \frac{w(e_{ij}^1)}{|C_j^0|} & \text{if } k \in C_j^0 \end{cases}$$

In this case, define f^1 by

$$f_k^1(e_{ij}^1) = \begin{cases} 0 & \text{if } k \notin C_j^0 \\ \frac{1}{|C_j^0|} & \text{if } k \in C_j^0 \end{cases}$$

Then $x = x^{(\varepsilon_1),(f^1)}$, and $((\varepsilon_1), x) \in DEC(\mathcal{M})$.

For Cases 1 and 2, f^2 can be defined by evaluating the first stage of $\phi(\mathcal{M}^{e_{ij}^1})$ $\forall e_{ij}^1 \in \varepsilon_1$. $f^3 \ldots$ can be defined inductively via the corresponding edge-reduced MCST problem. Until for some stage, the edge-reduced MCST problem falls in Case 3, and we find f^τ.

Let $\mathcal{F} = (f^1, \dots, f^\tau)$ be the sequence of share vectors corresponding to $(\varepsilon_1, \dots, \varepsilon_\tau)$ in $\phi(\mathcal{M})$, we have

$$((\varepsilon_1, \dots, \varepsilon_\tau), x^{(\varepsilon_1, \dots, \varepsilon_\tau), \mathcal{F}}) \in DEC(\mathcal{M})$$

Next we assume that $(\varepsilon, x), (\varepsilon', x') \in \phi(\mathcal{M})$ satisfying Eff, MC, FSC, ET, Loc and USI, then both of them are decentralized solutions. Let us show that (ε, x) and (ε', x') coincide with each other. We only need to prove that $\varepsilon = \varepsilon'$, then $x = x'$ is obviously satisfied based on the generic assumption. Assume that $\varepsilon \neq \varepsilon'$, if step t is the first step such that there exists $e^t_{C^{t-1}} \in \varepsilon_t$ and $\notin \varepsilon'_t$, that means, $\varepsilon_r = \varepsilon'_r$ for all $r \leq t-1$. We distinguish two cases:

Case 1: If $e^t_{C^{t-1}} \notin \varepsilon'_s$ $\forall s \in [1, \tau']$, then there exists no component C^{s-1} selecting edge $e^t_{C^{t-1}}$ in ε', which violates the generic assumption in our research. Either $E \cup \varepsilon_1 \cup \dots \cup \varepsilon_\tau$ or $E \cup \varepsilon'_1 \cup \dots \cup \varepsilon'_{\tau'}$ cannot form a MCST.

Case 2: If $e^t_{C^{t-1}} \in \varepsilon'_s$, where $s > t$, then ε' must violate the first condition in Definition 2.10 such that it does not select the cheapest edge at stage t.

Hence we have $(\varepsilon, x) = (\varepsilon', x')$.

■

2.5.2 Remarks

Remark 2.1

Besides the five properties we mentioned above, the decentralized solution also satisfies the continuity (Con) property. For all $i \in N, x_i(N, *, w, E)$ is continuous on w.

Proof. For any $\mathcal{M} = (N, *, w, E)$, we define $\mathcal{M}^{+\Delta} = (N, *, w^{+\Delta}, E)$, and $\mathcal{M}^{-\Delta} = (N, *, w^{-\Delta}, E)$, where for each $i, j \in N^*, w^{+\Delta}_{ij} = w_{ij} + \Delta$, and $w^{-\Delta}_{ij} = max\{0, w_{ij} - \Delta\}$. Here we assume $\Delta > 0$. It is easy to show that a MCST for $\mathcal{M}$ is also a MCST for both $\mathcal{M}^{+\Delta}$ and $\mathcal{M}^{-\Delta}$. The allocation generated by the decentralized solution for $\mathcal{M}^{+\Delta}$ is

$$
\begin{aligned}
x_i(N,*,w^{+\Delta},E) &= \sum_{s=1}^{\tau} f_i^s(w(e^s_{C_i^{s-1}})+\Delta) \\
&= \sum_{s=1}^{\tau} f_i^s w(e^s_{C_i^{s-1}})+o_i\Delta \\
&= x_i(N,*,w,E)+o_i\Delta
\end{aligned}
$$

The allocation generated by the decentralized solution for $\mathcal{M}^{-\Delta}$ is

$$
\begin{aligned}
x_i(N,*,w^{-\Delta},E) &= \sum_{s=1}^{\tau} f_i^s max\{0,(w(e^s_{C_i^{s-1}})-\Delta)\} \\
&\geq \sum_{s=1}^{\tau} f_i^s w(e^s_{C_i^{s-1}})-o_i\Delta \\
&= x_i(N,*,w,E)-o_i\Delta
\end{aligned}
$$

Consider a sequence of cost matrices $\{w^{\Delta}\}$ where $|w^{\Delta}_{ij}-w_{ij}|<\Delta$ for each $i,j\in N^*$

$$
\begin{aligned}
x_i(N,*,w,E)-o_i\Delta &\leq x_i(N,*,w^{-\Delta},E) \\
&\leq x_i(N,*,w^{\Delta},E) \\
&\leq x_i(N,*,w^{+\Delta},E) \\
&= x_i(N,*,w,E)+o_i\Delta
\end{aligned}
$$

We proved that $|x_i(N,*,w^{\Delta},E)-x_i(N,*,w,E)|\leq o_i\Delta\leq\Delta$. Hence the decentralized solution is continuous. ■

We know that Theorem 2.2 holds for generic cost matrices $\mathcal{G}^N$, where the cost for constructing the edges differs from each other. Notice that $\mathcal{G}^N$ is a dense subset of arbitrary cost matrices $\mathcal{A}^N$. For any $w\in\mathcal{A}^N\backslash\mathcal{G}^N$ and the corresponding MCST mt obtained by the decentralized rule, we can find a sequence of matrices $\{w^m\}_{m=1}^{+\infty}\subset\mathcal{G}^N$ such that

(1) $w^m\to w$;

(2) mt is also the MCST in w^m for all m.

Under these two conditions, $lim_{m\to+\infty} DEC(w^m) = DEC(w)$. If the properties above are true in $\mathcal{G}^{\mathcal{N}}$, the Con implies that the result might also be true for all $\mathcal{A}^N$ on refinement. It suggests a future extension for the characterization of the decentralized rule without the constraint of generic cost matrix. For example, a possible refinement on Algorithm 2.2 is that we may define the ranking rule for any $i, j \in N$ as $r_i < r_j$ if $w_{0i} \leq w_{0j}$. At each step, each component not connected to the source greedily chooses a cheapest edge connecting itself to another component. If two edges have the same cost, link from or to the agent with the lower rank.

Remark 2.2 (the proportional solution and the decentralized solution) *Compared with the characterization of the proportional solution, we can see that both solutions satisfy the properties of Eff, MC, FSC, ET, and Loc. However, two components connected in the first step by the decentralized rule do not necessarily share the same amount of the cost of the edge that connects them (Example 2.3.3); thus the decentralized solution does not satisfy ES, and hence CoCons is not satisfied as well. The property of USI is instead defining the cost sharing aspect of a decentralized solution. Both ES and USI only constrain the cost sharing at the first stage of an algorithm; but USI requests a judgement on the number of edges constructed at the first stage due to the decentralization.*

2.6 Independence of the axioms

Theorem 2.3

Eff, MC, FSC, ET, Loc, USI satisfy axiom independence.

Proof.

(1) **Eff is independent from MC, FSC, ET, Loc, USI.** If we add a fixed charge Δw to each agent newly connected to the source component, then this revised decentralized rule satisfies MC, FSC, ET, Loc, USI, but violating Eff.

(2) **MC is independent from Eff, FSC, ET, Loc, USI.** If we re-define the total initial obligation of 0 for the component $\underline{C}^0 \in N^*/E$ with the cheapest

direct connection to the source component, and 2 for the component $\overline{C}^0 \in N^*/E$ with the most expensive direct connection to the source component, then this revised decentralized rule satisfies Eff, FSC, ET, Loc, and USI, but violating MC.

(3) **FSC is independent from Eff, MC, ET, Loc, USI.** If the source component, i.e., the government, volunteers to fund $(1-\alpha)\sum_{i \in C^0} x_i, \alpha \in (0,1)$ for certain components $C^0 \in N^*/E$ not able to afford the allocated cost, i.e., poor states, assuming $\alpha \sum_{i \in C^0} x_i \geq min\{w(e) \mid e \ connects \ two \ components \ of < N^*, E >\}$, and $(1-\alpha)\sum_{i \in C^0} x_i$ is shared equally among all the related agents, then this revised decentralized rule satisfies Eff, MC, ET, Loc, USI, but violating FSC.

(4) **ET is independent from Eff, MC, FSC, Loc, USI.** If we define the initial obligation for agent $i \in C^0$ which has the cheapest adjacent potential edge compared to all other agents in $C^0 \in N^*/E$ to be 0; and assume that the total initial obligation 1 for C^0 is shared equally among all other agents $j \in C^0$ and $j \neq i$, then this revised decentralized rule satisfies Eff, MC, FSC, Loc, USI, but violating ET.

(5) **Loc is independent from Eff, MC, FSC, ET, USI.** Assume that the initial network is large enough and at stage 1, $e^1_{ij} \in \varepsilon_1$ connects two components $C^0_i \ni i$ and $C^0_j \ni j$ into a new component C^1_{ij}. If C^1_{ij} enforces $(1-\beta)w(e^1_{ij}), \beta \in (0,1)$ to components which have to make use of the network of C^1_{ij} in order to be connected to the source, assuming $\beta w(e^1_{ij}) \geq min\{w(e) \mid e \ connects \ two \ components \ of < N^*, E >\}$, and $(1-\beta)w(e^1_{ij})$ is shared equally among all the related agents, then this revised decentralized rule satisfies Eff, MC, FSC, ET, USI, but violating Loc.

(6) **USI is independent from Eff, MC, FSC, ET, Loc.** The proportional rule is a typical example satisfying Eff, MC, FSC, ET, Loc, but violating USI.

Hence Eff, MC, FSC, ET, Loc, USI satisfy axiom independence. ■

The example showing the independence of MC is not a representative one, and we will leave it as an open question and make further revision in the future work.

Chapter 3

Game Theoretic Approaches to Weight Assignments in DEA Problems

This chapter deals with the problem of fairly allocating a certain amount of divisible goods or burdens among individuals or organizations in the multi-criteria environment. It is analyzed within the framework of DEA. We improve the game proposed by Nakabayashi and Tone [2006] and develop an alternative scheme by re-assigning the total weight or power for the coalition members. Under our new proposition, we analyze the solutions for both TU and NTU game, as well as the equilibria of the strategic form game in the DEA problems.

3.1 The model

Let $E(> 0)$ denote the fixed amount of benefit to be allocated to players $1, \ldots, n$. Players' contributions are evaluated by multiple criteria and summarized as the score matrix $C = (c_{ij})_{i=1,\ldots,m,\ j=1,\ldots,n}$, where c_{ij} is player j's contribution measured by criterion i, called the evaluation index. The problem is to find a weight vector on the criteria determined endogenously by players themselves, and reasonable allocations of E based on the weight vector. Following the DEA analysis, each player k chooses a nonnegative weight vector $w^k = (w_1^k, \ldots, w_m^k)$ such that

$\sum_{i=1}^m w_i^k = 1,\ w_i^k \geq 0\ \forall i = 1, \ldots, m$, where w_i^k is the weight given to criterion i by player k. Then the contribution of player k relative to the total contribution of all players measured by the weight vector w^k is given by

$$\frac{\sum_{i=1}^m w_i^k c_{ik}}{\sum_{i=1}^m w_i^k (\sum_{j=1}^n c_{ij})}$$

Player k chooses the weight vector that maximizes this ratio. The weight vector is found by solving the following fractional program

$$max_{w^k} \frac{\sum_{i=1}^m w_i^k c_{ik}}{\sum_{i=1}^m w_i^k (\sum_{j=1}^n c_{ij})} \quad s.t. \quad \sum_{i=1}^m w_i^k = 1,\ w_i^k \geq 0\ \forall i = 1, \ldots, m$$

Each of the other players similarly maximizes the ratio produced by his/her own weight vector.

This maximization problem can be reformulated as the following much simpler form. First for each row $(c_{i1}, \ldots, c_{in})$, $i = 1, \ldots, m$, divide each element by the row-sum $\sum_{j=1}^n c_{ij}$. By Charnes-Cooper transformation(Charnes et al. [1978]), the maximization problem above is not affected by this operation. Let

$$c'_{ij} = c_{ij} / \sum_{j=1}^n c_{ij} \quad i = 1, \ldots, m.$$

The matrix $C' = (c'_{ij})_{i=1,\ldots,m,\ j=1,\ldots,n}$ is called the normalized score matrix and $\sum_{j=1}^n c'_{ij} = 1$ is satisfied. Then

$$\frac{\sum_{i=1}^m w_i^k c_{ik}}{\sum_{i=1}^m w_i^k (\sum_{j=1}^n c_{ij})} = \frac{\sum_{i=1}^m w_i^k c_{ik} / \sum_{j=1}^n c_{ij}}{\sum_{i=1}^m w_i^k (\sum_{j=1}^n c_{ij}) / \sum_{j=1}^n c_{ij}} = \frac{\sum_{i=1}^m w_i^k c'_{ik}}{\sum_{i=1}^m w_i^k}$$

Due to $\sum_i^m w_i^k = 1$, the fractional maximization program above can be expressed as the following linear maximization program.

$$max_{w^k} \sum_{i=1}^m w_i^k c'_{ik} \quad s.t. \sum_i^m w_i^k = 1,\ w_i^k \geq 0\ \forall i = 1, \ldots, m$$

Let $c(k)$ be the maximal value of the program. Apparently the maximal value is attained by letting $w^k_{i(k)} = 1$ for the criterion $i(k)$ such that $c'_{i(k)k} = max_{i=1,\dots,m}\ c'_{ik}$ and letting $w^k_i = 0$ for all other criteria $i \neq i(k)$. Thus $c(k)$ is the highest relative contribution of player k. Namely

$$c(k) = max_{i=1,\dots,m}\ c'_{ik}$$

Nakabayashi and Tone [2006] showed that if each player k claims the portion $c(k)$ of E, the sum of the claims generally exceeds the total benefit E. Then the problem arises: how to allocate E reasonably to players? To find a fair allocation of E, they proposed to apply cooperative game theory. Let us review their cooperative game model that they call a DEA game.

In the following sections, we assume that the score matrix is given in the normalized form. That is, $C = (c_{ij})_{i=1,\dots,m,\ j=1,\dots,n}$, where $\sum_{j=1}^{n} c_{ij} = 1\ \forall i = 1,\dots,m;\ c_{ij} \geq 0\ \forall i = 1,\dots,m,\ \forall j = 1,\dots,n$. Then the fractional maximization program can be re-represented by

$$max_{w^k} \sum_{i=1}^{m} w^k_i c_{ik} \quad s.t. \sum_{i}^{m} w^k_i = 1,\ w^k_i \geq 0\ \forall i = 1,\dots,m$$

3.2 The DEA game by Nakabayashi and Tone

Nakabayashi and Tone [2006] constructed a characteristic function form game (N, c) in the following manner. $N = \{1,\dots,n\}$ is the set of players and c is the characteristic function that gives each coalition a value it obtains. For each single player coalition $\{k\}$, $c(\{k\})$ is given by the $c(k)E$. Similarly for each coalition $S \subseteq N$, $c(S)$ is given by the maximum value of the linear maximization program

$$max \sum_{i=1}^{m}(w^S_i \sum_{j\in S} c_{ij})E \quad s.t. \sum_{i=1}^{m} w^S_i = 1,\ w^S_i \geq 0\ \forall i = 1,\dots,m$$

where $w^S = (w^S_1,\dots,w^S_m)$ is the weight vector chosen by S. Here they assume players' evaluation is transferable and take a total of players' evaluation in coalition S. Hereafter we call this characteristic function form game NT (Nakabayashi and Tone) DEA game.

Nakabayashi and Tone [2006] showed a counter-intuitive fact that the NT DEA game is sub-additive. Namely players lose their value by forming a coalition. The reason is quite simple. Consider the following score matrix.

Table 3.1: Score matrix for single player coalition case

	Player 1	Player 2	Player 3	row-sum
Criterion 1	6/10	2/10	2/10	1
Criterion 2	2/10	7/10	1/10	1
Criterion 3	1/10	1/10	8/10	1

It is easily seen that the optimal weight vector for each player is $(1, 0, 0)$ for 1, $(0, 1, 0)$ for 2 and $(0, 0, 1)$ for 3, and thus $c(\{1\}) = 0.6$, $c(\{2\}) = 0.7$ and $c(\{3\}) = 0.8$. Suppose players 1 and 2 form a coalition and aim at maximizing their joint evaluation for higher bargaining power. Then we have the following score matrix.

Table 3.2: Score matrix for the coalition $\{1, 2\}$ case

	Coalition $\{1, 2\}$	Player 3	row-sum
Criterion 1	8/10	2/10	1
Criterion 2	9/10	1/10	1
Criterion 3	2/10	8/10	1

Coalition $\{1, 2\}$'s optimal weight vector is $(0, 1, 0)$ and $c(\{1, 2\}) = 0.9$, which is less than $c(\{1\}) + c(\{2\}) = 1.3$; thus c is sub-additive.

The reason is intuitively clear. Either of players 1 and 2 has a weight of one and puts it on his/her most preferable criterion respectively before forming a coalition. To increase their bargaining power, they choose to form a coalition; but after forming a coalition, they are only given a total weight of one and put it on their most preferable criterion that maximizes their total evaluation. This is the primary reason for the sub-additivity. The coalition of players 1 and 2 should be given weight of two since their total weights are two before forming a coalition. For example, in voting decisions, it is considered to be fair if two voters each having one vote form a coalition, the coalition should be given two votes.

On the basis of this consideration, we introduce a new approach to endogenously determine the weight vector. Each player is equally given a weight of one and he/she chooses a weight vector, the component of which is a portion of the weight given to each criterion. The sum of the weights given to all criteria must be one. Then by taking a simple average of all players' weight vectors, we obtain the weight vector that will be used to calculate each player's average contribution. Here "simple average" implies that each player has equal influence, in other words, players are treated equally. To illustrate the procedure, suppose, for example, in Table 3.1, players 1, 2 and 3 choose weight vectors $(1/3, 1/3, 1/3), (1/2, 1/2, 0)$ and $(1/6, 1/2, 1/3)$, respectively. Then the weight vector for average contribution calculation is their simple average, i.e., $((1/3+1/2+1/6)/3, (1/3+1/2+1/2)/3, (1/3+0+1/3)/3) = (1/3, 4/9, 2/9)$; and player 1's weighted average contribution is $6/10 \times 1/3 + 2/10 \times 4/9 + 1/10 \times 2/9 = 28/90$. Similarly the average contributions of player 2 and 3 are 36/90 and 26/90, respectively.

By using this framework, we propose an alternative characteristic function form game. First we construct a strategic form game. Each player's strategy is a weight vector; and the payoff is his/her weighted average contribution calculated by using the simple average of all players' weight vectors. From this strategic form game, we derive a characteristic function form game following the procedure by von Neumann and Morgenstern [1944], which will be explained in detail in the following sections.

3.3 A strategic form DEA game

Let $N = \{1, \ldots, n\}$ be the set of players and $M = \{1, \ldots, m\}$ be the set of criteria. The basic DEA model stated in Section 3.1 is as follows. Each player $j \in N$ chooses a weight vector $w^j = (w_1^j, \ldots, w_m^j)$ with $w_1^j + \ldots + w_m^j = 1, w_i^j \geq 0 \; \forall i \in M$ on the criteria so as to maximize the weighted sum of his/her relative evaluation indices, $\sum_{i=1}^m w_i^j c_{ij}$. The fixed amount of reward E is shared by the players according to their weighted sums of the evaluation indices.

Therefore the strategic form game naturally reflecting the DEA model will be

$$G^{DEA} = (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$$

where $N = \{1, \ldots, n\}$ is the set of players, $W^j = \{w^j = (w_1^j, \ldots, w_m^j) | w_1^j + \ldots + w_m^j = 1, w_i^j \geq 0 \ \forall i \in M\}$ is the strategy set of player $j \in N$, and $f^j : W = W^1 \times \ldots \times W^n \to \Re$ is the payoff function of player $j \in N$, which is given by

$$f^j(w^1, \ldots, w^n) = (\sum_{i=1}^{m} (\frac{1}{n} \sum_{j=1}^{n} w_i^j) c_{ij}) E$$

The reward E is shared by players in proportion to the weighted sum of their evaluation indices where the weights are the average weights over all players. [1]

Hereafter we call this game the strategic form DEA game. The equilibria in the strategic form game will be studied in Section 3.9.

3.4 TU DEA game

In the characteristic function form game theory, it is commonly considered that the characteristic function gives each coalition the value that the coalition can surely win by itself. Here the term "by themselves" means that they can win the value even if outsiders' behavior is least favorable to them. In our game, every player wishes to maximize his/her evaluation so as to maximize the share of E. Hence the least favorable behavior of outsiders is to minimize the coalition's evaluation, i.e., the weighted sum of the players' evaluation indices inside the coalition. This behavior is also most favorable to the players outside the coalition since in our game players share a fixed amount of reward E. Namely, minimizing the coalition's evaluation leads to the maximization of the outsiders' evaluation.

[1] Another possible definition of f^j is

$$f^j(w^1, \ldots, w^n) = \frac{\sum_{i=1}^{m} w_i^j c_{ij}}{\sum_{j'=1}^{n} (\sum_{i=1}^{m} w_i^{j'} c_{ij'})} E$$

That is, the reward E is shared by players in proportion to their weighted evaluation indices with their own weights. Both payoff functions lead to the same conclusions with respect to the solution concepts.

Therefore following von Neumann and Morgenstern [1944], the characteristic function v gives the following value $v(S)$ to each coalition $S \subseteq N$.

$$\begin{aligned}
v(S) &= max_{(w^j)_{j\in S}} min_{(w^j)_{j\in N\setminus S}} \sum_{j\in S} f^j(w^1, \ldots, w^n) \\
&= max_{(w^j)_{j\in S}} min_{(w^j)_{j\in N\setminus S}} (\sum_{i=1}^{m} (\frac{1}{n} \sum_{j=1}^{n} w_i^j) c_{ij}) E \\
&= max_{(w^j)_{j\in S}} (\sum_{i=1}^{m} (\frac{1}{n} \sum_{j\in S} w_i^j) c_{ij}) E + min_{(w^j)_{j\in N\setminus S}} (\sum_{i=1}^{m} (\frac{1}{n} \sum_{j\in N\setminus S} w_i^j) c_{ij}) E \\
&= ((\frac{s}{n}) \times max_{i=1,\ldots,m} \sum_{j\in S} c_{ij} + (\frac{n-s}{n}) \times min_{i=1,\ldots,m} \sum_{j\in S} c_{ij}) E
\end{aligned}$$

where s is the number of players in S. From the perspective of the bargaining story we mentioned in the previous section, the worst situation for S is when outsiders put their whole weights on the criterion with the lowest evaluation for the coalition. For example, in Table 3.2, coalition $\{1, 2\}$'s maximum and minimum evaluation indices are 0.9 and 0.2. Therefore the value for coalition $\{1, 2\}$ is properly measured by the weighted average of 0.9 and 0.2 with weights of 2/3 and 1/3, respectively. Thus we would propose $(2/3)\times 0.9+(1/3)\times 0.2 = 2/3$ as the value of the characteristic function for coalition $\{1, 2\}$. Similarly for the single player coalition $\{3\}$, the value would be $(2/3) \times 0.1 + (1/3) \times 0.8 = 1/3$ since player 3's maximum and minimum evaluation indices are 0.8 and 0.2.

Note that $v(N) = E$ since $\sum_{j\in N} c_{ij} = 1 \ \forall i = 1, \ldots, m$. We call the characteristic function form game (N, v) the TU DEA game since we allow for side payments among players assuming transferable utility. Non-transferable utility case will be studied in the future extension work.

3.5 Properties of the TU DEA Game

TU DEA games satisfy interesting properties. First TU DEA games are superadditive.

Definition 3.1 (superadditivity)
A characteristic function form game (N, v) is **superadditive** *if*

$$v(S \cup T) \geq v(S) + v(T)$$

holds for all $S, T \subseteq N$ *with* $S \cap T = \emptyset$.

Theorem 3.1
Let (N, v) *be a TU DEA game; then* (N, v) *is superadditive.*

Proof. It is obvious that

$$max_{i=1,\dots,m} \sum_{j \in S \cup T} c_{ij} \geq max_{i=1,\dots,m} \sum_{j \in S} c_{ij} + min_{i=1,\dots,m} \sum_{j \in T} c_{ij}$$
$$max_{i=1,\dots,m} \sum_{j \in S \cup T} c_{ij} \geq max_{i=1,\dots,m} \sum_{j \in T} c_{ij} + min_{i=1,\dots,m} \sum_{j \in S} c_{ij}$$

Then by simple calculation, we can get

$$\begin{aligned}
v(S \cup T) - v(S) - v(T) &= ((\frac{s+t}{n}) \times max_{i=1,\dots,m} \sum_{j \in S \cup T} c_{ij} \\
&\quad +(\frac{n-s-t}{n}) \times min_{i=1,\dots,m} \sum_{j \in S \cup T} c_{ij} \\
&\quad -(\frac{s}{n}) \times max_{i=1,\dots,m} \sum_{j \in S} c_{ij} \\
&\quad -(\frac{n-s}{n}) \times min_{i=1,\dots,m} \sum_{j \in S} c_{ij} \\
&\quad -(\frac{t}{n}) \times max_{i=1,\dots,m} \sum_{j \in T} c_{ij} \\
&\quad -(\frac{n-t}{n}) \times min_{i=1,\dots,m} \sum_{j \in T} c_{ij})E \\
&\geq (\frac{n-s-t}{n}) \times (min_{i=1,\dots,m} \sum_{j \in S \cup T} c_{ij} \\
&\quad -min_{i=1,\dots,m} \sum_{j \in S} c_{ij} - min_{i=1,\dots,m} \sum_{j \in T} c_{ij})E \\
&\geq 0
\end{aligned}$$

$\blacksquare$

Furthermore TU DEA games are constant-sum.

Definition 3.2 (constant-sum)
A characteristic function form game (N, v) is **constant-sum** *if*

$$v(S) + v(N \setminus S) = v(N)$$

holds for all $S \subseteq N$.

Theorem 3.2
Let (N, v) be a TU DEA game; then (N, v) is constant-sum.

Proof. Since in our game $\sum_{j \in N} c_{ij} = 1 \ \forall i = 1, \ldots, m$, thus we have

$$max_{i=1,\ldots,m} \sum_{j \in S} c_{ij} + min_{i=1,\ldots,m} \sum_{j \in N \setminus S} c_{ij} = 1$$
$$min_{i=1,\ldots,m} \sum_{j \in S} c_{ij} + max_{i=1,\ldots,m} \sum_{j \in N \setminus S} c_{ij} = 1$$

Then it is quite simple to show that

$$v(S) + v(N \setminus S) = (\frac{s}{n} + \frac{n-s}{n})E = E = v(N)$$

$\blacksquare$

3.6 Solutions to the TU DEA Game

3.6.1 Core

In characteristic function form games, solutions are considered within the concept of imputations.

Definition 3.3 (imputation)
In a characteristic function form TU game (N, v), a payoff vector $x = (x_j)_{j \in N}$ is called an **imputation** *if it satisfies*

(1) (group rationality) $\sum_{j \in N} x_j = v(N)$*;*
(2) (individual rationality) $x_j \geq v(\{j\})\ \forall j \in N$*.*

The set of imputations of (N, v) is denoted by $A(v)$.

Definition 3.4 (core)
The set

$$C(v) = \{x \in A(v) | \sum_{j \in S} x_j \geq v(S)\ \forall S \subseteq N\}$$

is called the **core** *of* (N, v)*.*

Definition 3.5 (inessentiality)
A characteristic function form game (N, v) *is* **inessential** *if*

$$v(S) = \sum_{j \in S} v(\{j\})$$

holds for all $S \subseteq N$*. It is* **essential** *if* $v(N) > \sum_{j \in N} v(\{j\})$*.*

If a characteristic function form game (N, v) is inessential, then its imputation set is a singleton $(v(\{j\})_{j \in N}$. Each player can receive the "safety" amount, guaranteeing the individual value not worse off compared with the pre-coalition amount. Hence it is essential games that are interest to us. The following theorem is well known; for the proof see Owen [1995].

Theorem 3.3
Suppose a characteristic function form game (N, v) *is constant-sum. Then if it is essential, its core* $C(v) = \emptyset$*.*

As proved in the previous section, the TU DEA game (N, v) is superadditive and constant-sum; and thus the core is nonempty only when the game is inessential, which is equal to the unique imputation set. The following theorem characterizes the inessential TU DEA game.

Theorem 3.4
A TU DEA game is inessential if and only if for all $j \in N$*,* $c_{ij} = c_{i'j}$ *holds for all* $i, i' = 1, \ldots, m$*.*

Proof. First let us see the sufficient condition. If $c_{ij} = c_{i'j}$ $\forall j \in N$ $\forall i, i' = 1, \ldots, m$ holds, by simple calculation

$$\begin{aligned} v(S) &= ((\frac{s}{n}) \times max_{i=1,\ldots,m} \sum_{j\in S} c_{ij} + (\frac{n-s}{n}) \times min_{i=1,\ldots,m} \sum_{j\in S} c_{ij})E \\ &= (\sum_{j\in S} c_{ij})E = \sum_{j\in S} v(\{j\}) \end{aligned}$$

For the necessary condition, assume $\exists k \in N$ and $\exists \hat{i} \in M$ such that $c_{\hat{i}k} > c_{ik}$ (or $c_{\hat{i}k} < c_{ik}$, the proof is the same), and $c_{ik} = c_{i'k}$ $\forall i, i' \in M \backslash \{\hat{i}\}$. For any other player $j \in N \setminus \{k\}$, the condition in the theorem is satisfied; and thus for the coalition $S' \in \{S \subseteq N | \{k\} \subsetneq S\}$, we have $max_{i=1,\ldots,m} \sum_{j\in S'} c_{ij} > min_{i=1,\ldots,m} \sum_{j\in S'} c_{ij}$. Then

$$\begin{aligned} v(S') &= ((\frac{s'}{n}) \times max_{i=1,\ldots,m} \sum_{j\in S'} c_{ij} + (\frac{n-s'}{n}) \times min_{i=1,\ldots,m} \sum_{j\in S'} c_{ij})E \\ &= (\sum_{j\in S'\backslash\{k\}} c_{ij} + (\frac{s'}{n})c_{\hat{i}k} + (\frac{n-s'}{n})c_{ik})E \\ &> (\sum_{j\in S'\backslash\{k\}} c_{ij} + (\frac{1}{n})c_{\hat{i}k} + (\frac{n-1}{n})c_{ik})E = \sum_{j\in S'} v(\{j\}) \end{aligned}$$

which contradicts with the definition of inessentiality. ■

Therefore in the TU DEA game, the core is non-empty if and only if the evaluation indices for all the criteria are identical for each player. It is usually not the case in reality and thus the core is generally empty. We can see that the allocation by the core concept is not of significant meaning for the TU DEA game.

3.6.2 Shapley value

If the concept of the core is to give a set of stable imputations without distinguishing the most preferable payoff vector, although it might be empty, then the Shapley value is trying to assign the game a specific payoff vector. It is defined below.

Definition 3.6 (Shapley value)
In a characteristic function form game (N, v), for each $j \in N$

$$\phi_j(v) = \sum_{S \subseteq N, j \in S} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{j\}))$$

is called the Shapley value for player j. The vector $\phi(v) = (\phi_j(v))_{j \in N}$ is called the **Shapley value**.

By applying the constant-sum property of the TU DEA game, the Shapley value can be represented in a much simpler form, which is also an exercise problem in Owen [1995].

Theorem 3.5
In the TU DEA game (N, v), the Shapley value is given by $\phi(v) = (\phi_j(v))_{j \in N}$, where

$$\phi_j(v) = 2 \sum_{S \subseteq N, j \in S} (\frac{(s-1)!(n-s)!}{n!} v(S)) - v(N)$$

Proof. From the constant-sum property, we get $v(S \setminus \{j\}) = v(N) - v(N \setminus \{S \setminus \{j\}\})$, and we also know the combination equation can be represented as

$$\begin{pmatrix} n-1 \\ s-1 \end{pmatrix} = \frac{(n-1)!}{(s-1)!(n-s)!}$$

Then the calculation is shown as below

$$\begin{aligned}
\phi_j(v) &= \sum_{S \subseteq N, j \in S} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(N) + v(N \setminus \{S \setminus \{j\}\})) \\
&= \sum_{S \subseteq N, j \in S} \frac{(s-1)!(n-s)!}{n!}(v(S) + v(N \setminus \{S \setminus \{j\}\})) \\
&\quad - \sum_{s=1}^{n} \begin{pmatrix} n-1 \\ s-1 \end{pmatrix} \frac{(s-1)!(n-s)!}{n!} v(N) \\
&= 2 \sum_{S \subseteq N, j \in S} (\frac{(s-1)!(n-s)!}{n!} v(S)) - v(N)
\end{aligned}$$

for any $s = 1, ..., n$, $S \cap \{N \setminus \{S \setminus \{j\}\}\} = \{j\}$ is always satisfied. ■

Example 3.1
Applying the simplified formula above, and using the data set in Table 3.1, the imputation by Shapley value is $(9/30, 10/30, 11/30)$*, assuming* $E = 1$.

3.6.3 Nucleolus

Instead of applying a general axiomatization of fairness to a value function defined by the set of characteristic functions, the nucleolus looks at a fixed characteristic function, v, and try to find an imputation $x = (x_j)_{j \in N}$ that minimizes the worst inequity. That is, we ask each coalition $S \subseteq N$ how dissatisfied it is with the proposed imputation x and we try to minimize the maximum dissatisfaction.

Definition 3.7 (excess)
As a measure of the inequity of an imputation $x = (x_j)_{j \in N}$ *for a coalition* $S \subseteq N$*, the* **excess** *is defined as*

$$e(x, S) = v(S) - \sum_{j \in S} x_j$$

which measures the amount by which coalition S *falls short of* $v(S)$ *by the proposed allocation* x.

From the definition of the core, we immediately know that an imputation x is in the core if and only if all its excesses are negative or zero.

Define $O(x)$ as the vector of excesses arranged in non-increasing order. On the vectors $O(x)$ we use the lexographic order, i.e., $z >_L y$ if $\exists k \in \{1, \ldots, 2^n - 2\}$, such that $z_i = y_i \; \forall i \in \{1, \ldots, k-1\}$; and $z_k > y_k$. We may omit the empty set and the grand coalition from consideration since their excesses are always zero. The nucleolus is an efficient allocation that minimizes $O(x)$ in the lexicographic ordering.

Definition 3.8 (nucleolus)
Let $X = \{x = (x_j)_{j \in N} | \sum_{j=1}^{n} x_j = v(N)\}$ *be the set of efficient allocations. We say that a vector* $\nu \in X$ *is a* **nucleolus** *if for each* $x \in X$*, we have* $O(\nu) \leq_L O(x)$.

Table 3.3: The nucleolus calculation

S	v(S)	e(x, S)	$(\frac{10}{30}, \frac{10}{30}, \frac{10}{30})$	$(\frac{9}{30}, \frac{10}{30}, \frac{11}{30})$
{1, 2}	$\frac{20}{30}$	$\frac{20}{30} - x_1 - x_2$	0	$\frac{1}{30}$
{1, 3}	$\frac{21}{30}$	$\frac{21}{30} - x_1 - x_3$	$\frac{1}{30}$	$\frac{1}{30}$
{2, 3}	$\frac{22}{30}$	$\frac{22}{30} - x_2 - x_3$	$\frac{2}{30}$	$\frac{1}{30}$

The property such that $e(x, S) + e(x, N \setminus S) = 0$ can reduce the computational complexity of the nucleolus, which can be proved by the constant-sum property of the TU DEA game. Thus we just have to check the excesses for half of the coalitions.

Example 3.2

Based on the data set in Table 3.1, consider an arbitrary point, say $(\frac{10}{30}, \frac{10}{30}, \frac{10}{30})E$. *Assuming* $E = 1$, *the nucleolus* $(\frac{9}{30}, \frac{10}{30}, \frac{11}{30})$, *which coincides with the imputation by Shapley value, can be calculated as in Table 3.3.*

Theorem 3.6

In the 3-player TU DEA game (N, v), *the allocations by the Shapley value and the nucleolus coincide.*

Proof. Assuming $E = 1$, the characteristic function for each coalition $S \in \{1, 2, 3\}$ is

$$
\begin{aligned}
v(\emptyset) &= 0 \\
v(\{1\}) &= \tfrac{1}{3} max_{i \in M}\ c_{i1} + \tfrac{2}{3} min_{i \in M}\ c_{i1} \\
v(\{2\}) &= \tfrac{1}{3} max_{i \in M}\ c_{i2} + \tfrac{2}{3} min_{i \in M}\ c_{i2} \\
v(\{3\}) &= \tfrac{1}{3} max_{i \in M}\ c_{i3} + \tfrac{2}{3} min_{i \in M}\ c_{i3} \\
v(\{1, 2\}) &= \tfrac{2}{3} max_{i \in M}\ (c_{i1} + c_{i2}) + \tfrac{1}{3} min_{i \in M}\ (c_{i1} + c_{i2}) = 1 - v(\{3\}) \\
v(\{1, 3\}) &= \tfrac{2}{3} max_{i \in M}\ (c_{i1} + c_{i3}) + \tfrac{1}{3} min_{i \in M}\ (c_{i1} + c_{i3}) = 1 - v(\{2\}) \\
v(\{2, 3\}) &= \tfrac{2}{3} max_{i \in M}\ (c_{i2} + c_{i3}) + \tfrac{1}{3} min_{i \in M}\ (c_{i2} + c_{i3}) = 1 - v(\{1\}) \\
v(\{1, 2, 3\}) &= 1
\end{aligned}
$$

The Shapley value allocation $\phi(v)$ is decided by

$$\begin{aligned}
\phi_1(v) &= \tfrac{2}{3}v(\{1\}) + \tfrac{1}{3}(v(\{1,2\}) + v(\{1,3\})) - \tfrac{1}{3} \\
\phi_2(v) &= \tfrac{2}{3}v(\{2\}) + \tfrac{1}{3}(v(\{1,2\}) + v(\{2,3\})) - \tfrac{1}{3} \\
\phi_3(v) &= \tfrac{2}{3}v(\{3\}) + \tfrac{1}{3}(v(\{1,3\}) + v(\{2,3\})) - \tfrac{1}{3}
\end{aligned}$$

With the allocation above, the excess for each single-player coalition is

$$\begin{aligned}
e(\phi(v),\{1\}) &= \tfrac{1}{3}v(\{1\}) - \tfrac{1}{3}(v(\{1,2\}) + v(\{1,3\})) + \tfrac{1}{3} \\
e(\phi(v),\{2\}) &= \tfrac{1}{3}v(\{2\}) - \tfrac{1}{3}(v(\{1,2\}) + v(\{2,3\})) + \tfrac{1}{3} \\
e(\phi(v),\{3\}) &= \tfrac{1}{3}v(\{3\}) - \tfrac{1}{3}(v(\{1,3\}) + v(\{2,3\})) + \tfrac{1}{3}
\end{aligned}$$

It can be easily verified that $e(\phi(v),\{1\}) = e(\phi(v),\{2\}) = e(\phi(v),\{3\})$ is satisfied

$$\begin{aligned}
e(\phi(v),\{1\}) - e(\phi(v),\{2\}) &= \tfrac{1}{3}(v(\{1\}) - v(\{2\})) - \tfrac{1}{3}(v(\{1,3\}) - v(\{2,3\})) = 0 \\
e(\phi(v),\{1\}) - e(\phi(v),\{3\}) &= \tfrac{1}{3}(v(\{1\}) - v(\{3\})) - \tfrac{1}{3}(v(\{1,2\}) - v(\{2,3\})) = 0
\end{aligned}$$

By the constant-sum property, any different allocation from the allocation $\phi(v)$ will increase at least one of the excesses. Hence in the 3-player TU DEA game, the allocations by the Shapley value and the nucleolus always coincide. ■

For the cases that players are more than 3, here is a counter example showing that the theorem above does not necessarily hold.

Example 3.3

The Shapley value allocation is $(\frac{16}{60}, \frac{16}{60}, \frac{15}{60}, \frac{13}{60})$ *with the data set in Table 3.4. It can be easily verified that the maximum excess is not minimized with the allocation above; and hence the nucleolus allocation does not coincide with the Shapley value allocation in this example.*

Table 3.4: Incoincidence of the Shapley value and nucleolus

	Player 1	Player 2	Player 3	Player 4	row-sum
Criterion 1	6/10	2/10	1/10	1/10	1
Criterion 2	2/10	6/10	1/10	1/10	1
Criterion 3	1/10	1/10	5/10	3/10	1

3.7 NTU DEA game

By Theorem 3.3 and 3.4 in the previous section, we know that the core of the TU DEA game is empty in most games. This section turns the focus on the NTU DEA game, and mainly contributes to the analysis of the existence of α-core and β-core. First let us present the definition of the NTU coalitional game.

Definition 3.9 (NTU coalitional game)
The pair (N, V) is called NTU coalitional game if and only if V is a correspondence from any coalition $S \subseteq N$ into a set of real vectors $V(S) \subseteq \Re^N$ such that

(1) If $S \neq \emptyset, V(S)$ is a non-empty closed subset of $\Re^N$; and $V(\emptyset) = \emptyset$.

(2) $\forall x, y \in \Re^N$, if $x \in V(S)$, and $x^j \geq y^j$ $\forall j \in S$, then $y \in V(S)$.

(3) $\forall j \in N, \exists V^j \in \Re$ such that $\forall x \in \Re^N : x \in V(\{j\})$ if and only if $x^j \leq V^j$.

(4) $\{x \in V(N) : x^j \geq V^j\}$ is a compact set.

The interpretation of the NTU coalitional game (N, V) is that $V(S)$ is the set of feasible payoff vectors for the coalition S if that coalition forms. Only the coordinates for players $j \in S$ in elements of $V(S)$ matter. A consequence of property (2) is that, if x is feasible for S, then any $y \leq x$ is feasible as well for S; this property is often called *comprehensiveness*, and it can be interpreted as free disposability of utility. Property (4) ensures that the individually rational part of $V(N)$ is bounded.

Next, from the strategic form DEA game, the NTU DEA game can be defined in both α and β fashion depending on a coalition's reactions against the deviations by its counter-coalition.

Definition 3.10 (α-coalitional NTU DEA game)
Suppose a strategic form DEA game $G^{DEA} = (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ is given. The α-coalitional NTU game (N, V_α) associated with G^{DEA} is a correspondence from any coalition $S \subseteq N$ into a set of real vectors $V_\alpha(S) \subseteq \Re^N$, which satisfies

(1) For any non-empty $S \subsetneq N, x \in V_\alpha(S)$ if and only if there exists $w^S \in W^S$ such that, for all $j \in S$ and all $u^{N\setminus S} \in W^{N\setminus S}, x^j \leq f^j(w^S, u^{N\setminus S})$.

(2) $x \in V_\alpha(N)$ if and only if there exists $w^N \in W^N$ such that, for all $j \in N, x^j \leq f^j(w^N)$.

where $W^S = \prod_{j\in S} W^j$. Coalition $S \subseteq N$ is said to α-improve upon $x \in V_\alpha(S)$ if there exists $w^S \in W^S$ such that for any $u^{N\setminus S} \in W^{N\setminus S}$

$$x^j < f^j(w^S, u^{N\setminus S}) \quad \forall j \in S$$

*The α-**core** is the set of payoff vectors $x \in V_\alpha(N)$ upon which no coalition α-improves.*

Definition 3.11 (β-coalitional NTU DEA game)
Suppose a strategic form DEA game $G^{DEA} = (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ is given. The β-coalitional NTU game (N, V_β) associated with G^{DEA} is a correspondence from any coalition $S \subseteq N$ into a set of real vectors $V_\beta(S) \subseteq \Re^N$, which satisfies

(1) For any non-empty $S \subsetneq N, x \in V_\beta(S)$ if and only if for all $u^{N\setminus S} \in W^{N\setminus S}$, there exists $w^S \in W^S$ such that for all $j \in S, x^j \leq f^j(w^S, u^{N\setminus S})$.

(2) $x \in V_\beta(N)$ if and only if there exists $w^N \in W^N$ such that, for all $j \in N, x^j \leq f^j(w^N)$.

where $W^S = \prod_{j\in S} W^j$. Coalition $S \subseteq N$ is said to β-improve upon $x \in V_\beta(S)$ if for any $u^{N\setminus S} \in W^{N\setminus S}$ there exists $w^S \in W^S$ such that

$$x^j < f^j(w^S, u^{N\setminus S}) \; \forall j \in S$$

*The β-**core** is the set of payoff vectors $x \in V_\beta(N)$ upon which no coalition β-improves.*

Remark 3.1
From the definition of the α- and β-coalitional DEA game, we get

$$\beta - core \subseteq \alpha - core$$

3.8 Solutions to the NTU DEA game

Scarf [1971] introduced a beautiful theorem on the existence of an α-core.

Theorem 3.7 (Scarf)
Assume that for each $j \in N$, W^j is a compact convex set, and f^j is quasi-concave in $w \in W$. Then the α-core is non-empty.

Applying the theorem above, we can easily show the non-emptiness of the α-core in the NTU DEA game.

Theorem 3.8
The α-core of the NTU DEA game is non-empty.

Proof. For all $j \in N$, for all $w^j \in W^j$, we have $w_i^j \in [0,1]\ \forall i \in M$ and $\sum_{i=1}^m w_i^j = 1$, thus W^j is a compact convex set.

From the definition of the strategic form DEA game, we know that

$$f^j(w^1, \dots, w^n) = (\sum_{i=1}^m (\frac{1}{n} \sum_{j=1}^n w_i^j) c_{ij}) E$$

Let $w, v \in W$, and $\lambda \in (0,1)$, then

$$\begin{aligned} & f^j(\lambda w^1 + (1-\lambda)v^1, \dots, \lambda w^n + (1-\lambda)v^n) \\ = \ & (\sum_{i=1}^m (\frac{1}{n} \sum_{j=1}^n (\lambda w_i^j + (1-\lambda)v_i^j)) c_{ij}) E \\ \geq \ & min\{f^j(w^1, \dots, w^n), f^j(v^1, \dots, v^n)\} \end{aligned}$$

Thus f^j is quasi-concave in $w \in W$. The α-core of the NTU DEA game is non-empty. ■

Next, we will present a condition under which the β-core of the NTU DEA game is non-empty.

Theorem 3.9
For all $j \in N$, let $\bar{C}^j = \{i^*(j) \in M \mid c_{i^*(j)j} \geq c_{ij} \ \forall i \in M\}$. Then β-core of the NTU DEA game is non-empty if $\bar{C}^N = \bar{C}^1 \cap \bar{C}^2 \cap \ldots \cap \bar{C}^n \neq \emptyset$.

Proof. From the definition, we know that $\bar{C}^j$ is the set of criteria giving player j the highest evaluation. $\forall S \subseteq N$, the payoff function for each player $k \in S$ under the strategy profile $(w^S, u^{N\backslash S})$ can be re-written as

$$f^k(w^S, u^{N\backslash S}) = (\sum_{i=1}^{m}(\frac{1}{n}(\sum_{j\in S} w_i^j + \sum_{j\in N\backslash S} u_i^j))c_{ik})E$$

Assume $\bar{C}^N = \bar{C}^1 \cap \bar{C}^2 \cap \ldots \cap \bar{C}^n$ is non-empty and take a strategy profile w^{*N} such that each player $k \in N$ puts all of his/her weight on the criterion $i^*(k) \in \bar{C}^N$. Then its corresponding payoff vector $x^* = (x^{*k})_{k\in N}$ satisfies

$$x^{*k} = f^k(w^{*N}) = (\sum_{i=1}^{m}(\frac{1}{n}\sum_{j=1}^{n} w_i^{*j})c_{ik})E = c_{i^*(k)k}E$$

We can see that $x^* \in V_\beta(N)$. Take any $S \subseteq N$, for all $u^{N\backslash S} \in W^{N\backslash S}$, for all $w^S \in W^S$ and for all $k \in S$

$$\begin{aligned} x^{*k} - f^k(w^S, u^{N\backslash S}) &\geq f^k(w^{*S}, u^{N\backslash S}) - f^k(w^S, u^{N\backslash S}) \\ &= (\sum_{i=1}^{m}(\frac{1}{n}(\sum_{j\in S} w_i^{*j} - \sum_{j\in S} w_i^j))c_{ik})E \\ &\geq 0 \end{aligned}$$

None of the coalitions can β-improve upon x^*, thus x^* is in the β-core. Hence a sufficient condition for a non-empty β-core is that $\bar{C}^N = \cap\{\bar{C}^j\}_{j\in N} \neq \emptyset$. ■

3.9 Equilibria in the Strategic Form Game

In some situations, the players might be prohibited to form any form of coalition or binding agreement in the DEA process, then we need to look into the equilibria in the strategic form game, which is defined in Section 3.3. First let us start with the definition of Nash equilibrium in our strategic form DEA game.

3.9.1 Nash Equilibrium

Definition 3.12 (Nash equilibrium)
In a strategic form game $(N, \{W^j\}_{j\in N},$
$\{f^j\}_{j\in N})$*, a strategy combination* $(w^{*1}, \ldots, w^{*n})$ *is called a Nash equilibrium if for all* $j \in N$

$$f^j(w^{*j}, w^{*-j}) \geq f^j(w^j, w^{*-j}) \ \forall w^j \in W^j$$

holds. Here $w^{*-j} = (w^{*1}, \ldots, w^{*j-1}, w^{*j+1}, \ldots, w^{*n})$.

The Nash equilibrium is a strategy combination in which no player gains more by his/her unilateral deviation. Then we have the following theorem. That is, in Nash equilibrium, every player puts positive weights only on the criteria that give him/her the highest evaluation. Let $\bar{C}^j$ be the set of criteria that give player j the maximum evaluation index, that is, $\bar{C}^j = \{i \in M | c_{ij} \geq c_{i'j} \ \forall i' \in M\}$.

Theorem 3.10
In a strategic form DEA game $(N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$*, the Nash equilibrium is of the form*

$$w^* = (w^{*1}, \ldots, w^{*n}), \ w_i^{*j} = 0 \ \forall i \notin \bar{C}^j \ \forall j \in N$$

Proof. First let us prove the sufficient condition. By definition, we know that for a strategy combination $(w^{*1}, \ldots, w^{*n})$ to be a Nash equilibrium, $f^j(w^{*j}, w^{*-j}) \geq f^j(w^j, w^{*-j}) \ \forall w^j \in W^j$ should be satisfied. Assume $w^{-j'} = (w^1, \ldots, w^{j'-1}, w^{j'+1}, \ldots, w^n)$ is determined, then for player $j' \in N$

$$f^{j'}(w^1,\ldots,w^n) = (\sum_{i=1}^{m}(\frac{1}{n}(\sum_{j\in N,\ j\neq j'} w_i^j + w_i^{j'}))c_{ij'})E$$

$c_{ij'}(\forall i \in M)$ is given, in order to maximize $f^{j'}(w^1,\ldots,w^n)$, player j' should choose a weight vector maximizing $\sum_{i=1}^{m}(w_i^{j'}c_{ij'})$. It is obvious that only criteria with the highest evaluation indices should be assigned with positive weights. Applying this procedure for all other players, the Nash equilibrium should be $w^* = (w^{*1},\ldots,w^{*n}),\ w_i^{*j} = 0\ \forall j \in N\ \forall i \notin \bar{C^j}$.

For the necessary condition, assume that player $j' \in N$ deviates from $w^{*j'}$ to $w^{j'}$ such that $\exists i' \notin \bar{C^{j'}}$ with $w_{i'}^{j'} > 0$. Because $w_1^{j'} + \ldots + w_m^{j'} = 1$, there must exist at least one criterion $i'' \in \bar{C^{j'}}$ to which the weight assigned becomes less than $w_{i''}^{*j'}$; let us first assume such criteria set is a singleton, which means $w_{i''}^{j'} = w_{i''}^{*j'} - w_{i'}^{j'}$. Then

$$f^{j'}(w^{*1},\ldots,w^{*n}) - f^{j'}(w^{*1},\ldots,w^{j'},\ldots,w^{*n}) = \frac{1}{n}w_{i'}^{j'}(c_{i''j'} - c_{i'j'})E$$

By the definition of $\bar{C^{j'}}$, we know that $c_{i''j'} > c_{i'j'}$. Hence player j' gains less if he/she made this deviation. The proof for the condition when the weight assignments for multiple criteria become less than that before the deviation is similar. ■

Thus the Nash equilibrium exists, and the players choose putting positive weights only on the criteria with the highest evaluation indices.

3.9.2 Cooperative Behavior in the Strategic Form DEA Game

Next we examine players' cooperative behavior in the strategic form DEA game by using the concept of strong Nash equilibrium. First let us define the deviation.

Definition 3.13 (deviation)
Take a strategy combination $w = (w^1,\ldots,w^n)$. Coalition $T \subseteq N$ has a deviation

Table 3.5: A counter example for the existence of SNE

	Player 1	Player 2	Player 3	row-sum
Criterion 1	1/10	2.5/10	6.5/10	1
Criterion 2	3/10	1/10	6/10	1
Criterion 3	2.5/10	2/10	5.5/10	1

$u^T = (u^j)_{j \in T}$ *from* w *if*

$$f^j(u^T, w^{-T}) > f^j(w) \ \forall j \in T$$

where $w^{-T} = (w^j)_{j \in N \setminus T}$.

Definition 3.14 (strong Nash equilibrium)
In a strategic form game $(N, \{W^j\}_{j \in N}, \{f^j\}_{j \in N})$, *a strategy combination* $w^* = (w^{*1}, \dots, w^{*n})$ *is called a strong Nash equilibrium if no coalition* $T \subseteq N$ *has a deviation from* w^*.

Thus in the strong Nash equilibrium, no group of players has an incentive to unilaterally deviate from the strategy combination. The strong Nash equilibrium must be a Nash equilibrium.

The strong Nash equilibrium does not always exist as showed in the following example.

Example 3.4
Suppose a strong Nash equilibrium exists with the data set in Table 3.5, and $E = 1$. *Then it must be a Nash equilibrium; thus by the theorem above, it must be of the form* $w^* = ((0,1,0),(1,0,0),(1,0,0))$. *Criterion 1 is chosen with probability 2/3 and criterion 2 is chosen with probability 1/3. By simple calculation, we get that players 1, 2, and 3 gain 5/30, 6/30, and 19/30, respectively. Suppose players 1 and 2 jointly deviate and choose* $w^{\{1,2\}} = (0,0,2)$. *Then since* $w^{*3} = (1,0,0)$, *criteria 1 and 3 are chosen with probability 1/3 and 2/3, respectively. Hence player 1 gains 6/30 and player 2 gains 6.5/30; both players are better off. Therefore there is no strong Nash equilibrium in this example.*

We next analyze the coalition-proof Nash equilibrium. The strong Nash equilibrium assumes that the deviation is binding, i.e., deviated coalition never breaks up. The coalition-proof Nash equilibrium supposes the possibility of further deviation inside deviated coalitions; and considers only credible deviations, i.e., deviations from which no further deviation takes place.

Definition 3.15 (credible deviation)
We say $T \subseteq N$ has a credible deviation from a strategy profile $w = (w^1, \ldots, w^n)$ if (1) T has a deviation $u^T = (u^j)_{j \in T} \in W^T$ from w, and (2) there is no $R \subsetneq T$ which has a credible deviation from (u^T, w^{-T}). When $T = \{j\}$, j has a credible deviation u^j from w if and only if $f^j(u^j, w^{-j}) > f^j(w)$. For T with $|T| > 1$, the definition of a credible deviation follows inductively.

Definition 3.16 (coalition-proof Nash equilibrium)
In a strategic form game $(N, \{W^j\}_{j \in N}, \{f^j\}_{j \in N})$, a strategy combination $w^* = (w^{*1}, \ldots, w^{*n})$ is called a coalition-proof Nash equilibrium if no coalition $T \subseteq N$ has a credible deviation from w^*.

Same as the strong Nash equilibrium, the coalition-proof Nash equilibrium in the strategic form DEA game does not always exist as well. Here is a counter example.

Example 3.5
Suppose a coalition-proof Nash equilibrium exists with the data set in Table 3.6, and $E = 1$. Then it must be a Nash equilibrium; thus by the theorem above, it must be of the form $w^* = ((w_{11}, 1 - w_{11}, 0), (w_{12}, 0, 1 - w_{12}), (0, w_{23}, 1 - w_{23}))$. Criterion 1 is chosen with probability $(w_{11} + w_{12})/3$, criterion 2 is chosen with probability $(1 - w_{11} + w_{23})/3$, and criterion 3 is chosen with probability $(2 - w_{12} - w_{23})/3$. By simple calculation following our basic proposal, we get that players 1, 2, and 3 gain $(2w_{12} + 2w_{23} + 8)/30$, $(2w_{11} - 2w_{23} + 10)/30$, and $(-2w_{11} - 2w_{12} + 12)/30$, respectively.

Assume initially $w_{11} \in (0, 1)$, and $w_{12} \in (0, 1)$ as well. If players 1 and 2 form a coalition, they can jointly maximize their total as well as individual payoff by putting their whole weight on the first criterion, namely, letting $w_{11} = w_{12} = 1$. Then players 1 and 2 can respectively gain $(2w_{23} + 10)/30$ and $(-2w_{23} + 12)/30$. Both players are better off. However, player 3 can only receive $8/30$.

We can also see that each player's payoff is irrelevant to its own choice on the weight vector, thus neither $\{1\}$ nor $\{2\}$ has a credible deviation from $w_{11} = w_{12} = 1$. Hence the deviation above is credible.

Table 3.6: A counter example for the existence of CPNE

	Player 1	Player 2	Player 3	row-sum
Criterion 1	4/10	4/10	2/10	1
Criterion 2	4/10	2/10	4/10	1
Criterion 3	2/10	4/10	4/10	1

Starting from this deviated strategy profile $((1,0,0),(1,0,0),(0,w_{23},1-w_{23}))$. *If players 1 and 3 form a coalition and choose to put their whole weight on the second criterion, then they can gain* $12/30$ *and* $10/30$, *respectively. Both players 1 and 3 are better off, and this deviation is credible as well.*

In this manner, we can show that there is always a credible deviation from any Nash equilibrium. Hence the coalition-proof Nash equilibrium does not exist in this example.

Next we give a condition under which the coalition-proof Nash equilibrium exists and is uniquely determined.

Theorem 3.11

In a strategic form DEA game $(N,\{W^j\}_{j\in N},\{f^j\}_{j\in N})$, *if* $\bar{C}^j$ *is a singleton set for each* $j\in N$, *and* $w^*=(w^{*1},\ldots,w^{*n})$ *is the Nash equilibrium. Then* w^* *is the unique coalition-proof Nash equilibrium, where* $\bar{C}^j=\{i\in M|c_{ij}\geq c_{i'j}\ \forall i'\in M\}$.

Proof. Let $c_{\overline{i(j)}j}$ denote the maximum evaluation index for player $j\in N$. If $\bar{C}^j$ is a singleton set, then $c_{\overline{i(j)}j}>c_{i'j}\ \forall i'\neq\overline{i(j)}\in M \quad \forall j\in N$ must be satisfied. By Theorem 3.10, the unique Nash equilibrium is

$$w^*=(w^{*1},\ldots,w^{*n}),\ w^{*j}_{\overline{i(j)}}=1,\ w^{*j}_{i'}=0\ \ \forall i'\neq\overline{i(j)}\in M\ \forall j\in N$$

Assume $\exists T\subseteq N$ such that T has a credible deviation from w^*, and let $u^T=(u^j)_{j\in T}\in W^T$ denote the deviation. Then some $\{j\}\subsetneq T$ must has a deviation at (u^T,w^{*-T}) by Theorem 3.10. This deviation is a credible one by Definition 3.15. Therefore u^T is not a credible deviation at w^*. Hence w^* is the unique coalition-proof Nash equilibrium. ■

Theorem 3.11 shows that if for each player the criteria set giving him/her the maximum evaluation index is a singleton, then the coalition-proof Nash equilibrium exists and is uniquely determined.

Chapter 4

Service Quality's Effects on the Selection of a Partner Airline in the Formation of Airline Alliances

This chapter mainly concerns how an airline's service quality might affect the selection of its partner during the formation of airline alliances. Within our proposed three-stage analysis framework, we have shown the strategic effects of the service quality on the complementary airline alliances, where the pre-alliance market of the potential alliance members is either monopoly or duopoly. The main finding in this chapter is that an airline will cooperate with the one which has the same service quality level if the pre-alliance service quality distribution of the airlines in the whole market differs greatly, while it tends to choose the one with similar (either higher or lower) service quality level as its partner if the distribution is approximately uniform.

4.1 General network model

4.1.1 Network

We consider a simple network with 3 airports A, B, and C. There is direct flight(s) between airport A and B, also B and C, but no direct flight between A and C. Passengers wishing to fly from A to C (or C to A) have to transit once in airport B. The airline industry of A–B, and B–C can either be monopoly or duopoly, then three types of basic pre-alliance markets are formed as below:

Figure 4.1: Monopoly–Monopoly airline market

There are two airlines in the Monopoly–Monopoly case, where each airline owns monopoly power in their respective market.

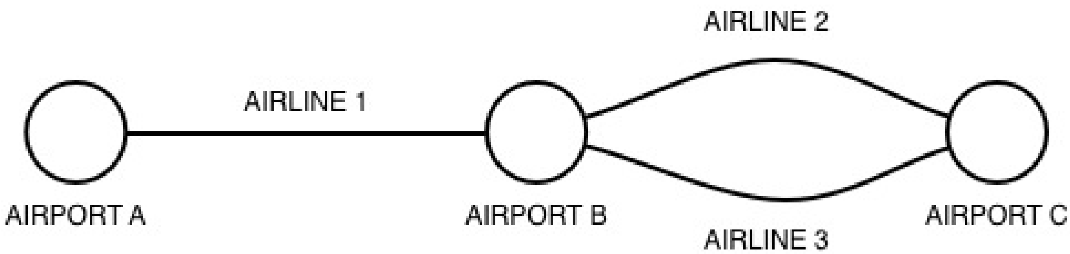

Figure 4.2: Monopoly–Duopoly airline market

For the Monopoly–Duopoly case, the market of airline 1 is monopoly, while airlines 2 and 3 are competing on the same leg B–C.

Finally for the Duopoly–Duopoly case, airlines 1 and 4, and airlines 2 and 3 are competing on the same leg, respectively.

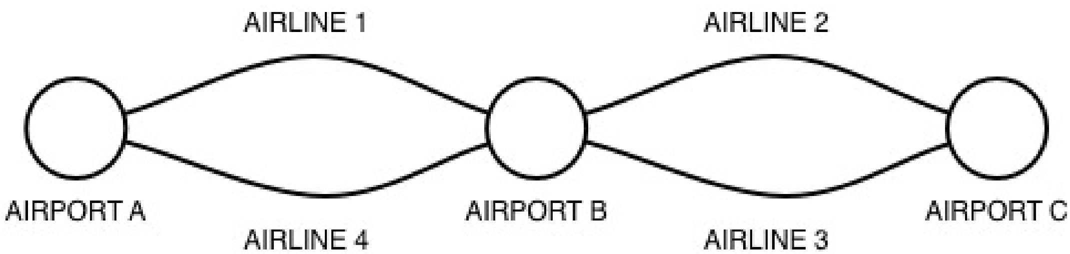

Figure 4.3: Duopoly–Duopoly airline market

4.1.2 Notations and preliminaries

We denote by $\mathcal{A}$ the set of airlines, indexed by $i = 1, 2, 3, 4$ in the analysis. Some notations that we will use to model the structure of the alliance are shown as below:

- d_i: passengers' demand for airline i.

- d_{ij}: the pre-alliance passengers' demand for market A–C, where i denotes the airline of market A–B, and j of market B–C.

- $d^{a_{ij}}$: passengers' demand for alliance $i-j$ if airlines i and j form an alliance. It does not include the demand for each airline's self-operated market. We assume that each airline's strategy and demand in their respective individual market are not affected by the decision of the alliance. The superscript a is used to denote quantities associated with *Alliance*.

- C_i: the overall operational cost for airline i.

- Π_i: the pre-alliance profit of airline i. Π_i^* denotes the equilibrium profit.

- $\Pi^{a_{ij}}$: the joint profit of alliance $i-j$ if airlines i and j form an alliance, including the profit generated in each airline's self-operated market. $\Pi^{a_{ij}*}$ denotes the equilibrium joint profit.

- $\Pi_i^{a_{ij}*}$,$\Pi_j^{a_{ij}*}$: the profit allocated to airline i, j respectively, if airlines i and j form an alliance.

- p_i: the fare charged for passengers by airline i.
- $p^{a_{ij}}$: the fare decided and charged jointly by alliance $i-j$ if airlines i and j form an alliance.
- $q_i = q_g$ or q_b: the service quality of airline i, which is assumed to be either q_g or q_b in this chapter. The subscript g and b are used to denote "good" and "bad" service quality, respectively. Here we consider a discrete service quality factor to simplify the mathematical calculation.
- m: a positive parameter that measures the market size.
- γ^d: a positive parameter in the demand function which measures the effect of service quality on the demand. The superscript d is used to denote the quantity associated to *Demand*. Assuming identical passengers, this effect does not differ among airlines.
- $\gamma_i^c(q_i)$: a positive parameter in the cost function which measures the effect of service quality on the cost of airline i. The superscript c is used to denote quantities associated to *Cost*.
- θ: a positive parameter measuring the improvement of the alliance service quality over two individually operated airlines, assuming $\theta \in (1/2, 1)$.
- $\beta_i^{a_{ij}}$, $\beta_j^{a_{ij}}$: the fraction of the joint profit $\Pi^{a_{ij}*}$ collected by airline i, j respectively, if airlines i and j form an alliance, where $\beta_i^{a_{ij}} + \beta_j^{a_{ij}} = 1$. We denote by $\mathcal{R}$ the profit allocation rule, and $\mathcal{R}_p$ the proportional rule.

The rest of the notations will be introduced in the corresponding sections.

4.1.3 The partner selection game

A partner selection problem $\mathcal{P} =< \mathcal{A}, Q >$ consists of a finite set $\mathcal{A} = \mathcal{A}^{AB} \cup \mathcal{A}^{BC}$ of airlines, where $\mathcal{A}^{AB}$ denotes the set of airlines in market A–B, and $\mathcal{A}^{BC}$ denotes the set of airlines in market B–C. As we only focus on the complementary alliances, $\mathcal{A}^{AB} \cap \mathcal{A}^{BC} = \emptyset$. Each of the airlines $i \in \mathcal{A}$ wants to find a partner from its complementary market to extend its current network coverage. $Q \in R^{|\mathcal{A}|}$ is the present service quality vector for all of the airlines.

Then the alliance profit associated with the partner selection problem $\mathcal{P} =< \mathcal{A}, Q >$ can be represented as

$$\Pi^{\mathcal{P}} = \{\Pi^{a_{ij}}\}_{\forall i \in \mathcal{A}^{AB}, \forall j \in \mathcal{A}^{BC}}$$

Definition 4.1 (partner selection game)
Let $(\mathcal{A}, \Pi^{\mathcal{P}^*})$ be a matching game associated with the partner selection problem $\mathcal{P} =< \mathcal{A}, Q >$, where $\Pi_{ij}^{\mathcal{P}^*}$ is the value of alliance i–j if airlines $i \in \mathcal{A}^{AB}$ and $j \in \mathcal{A}^{BC}$ cooperate. $\Pi_{ij}^{\mathcal{P}^*}$ can be formally defined as

$$\Pi_{ij}^{\mathcal{P}^*} = \{\Pi^{a_{ij}} \in \Pi^{\mathcal{P}} \mid \Pi^{a_{ij}} \geq \Pi_i^* + \Pi_j^* \quad \forall i \in \mathcal{A}^{AB} \ \forall j \in \mathcal{A}^{BC}\}$$

Definition 4.2 (stability)
An alliance i–j $\forall i \in \mathcal{A}^{AB} \ \forall j \in \mathcal{A}^{BC}$ is **stable** *if for all $k \in \mathcal{A}^{BC} \setminus \{j\}$ and for all $l \in \mathcal{A}^{AB} \setminus \{i\}$, the following condition is satisfied*

$$\Pi_i^{a_{ij}*} > \Pi_i^{a_{ik}*} \ and \ \ \Pi_j^{a_{ij}*} > \Pi_j^{a_{lj}*}$$

Next we define the core of the partner selection game $(\mathcal{A}, \Pi^{\mathcal{P}^*})$.

Definition 4.3 (core of $(\mathcal{A}, \Pi^{\mathcal{P}^*})$)

$$Core(\mathcal{A}, \Pi^{\mathcal{P}^*}) = \left\{ x \in R^{|\mathcal{A}|} \;\middle|\; \begin{array}{l} x_i = \Pi_i^{a_{ij}*} \ \forall i \in \mathcal{A}^{AB} \ if \ \exists j \in \mathcal{A}^{BC} \ s.t. \ x_i + x_j \in \Pi_{ij}^{\mathcal{P}^*} \\ x_i \geq \Pi_i^*, x_j \geq \Pi_j^*, \ and \ (x_i, x_j) \ is \ stable. \\ \\ x_i = \Pi_i^* \ \forall i \in \mathcal{A}^{AB} \ otherwise. \\ \\ x_j \ \forall j \in \mathcal{A}^{BC} \ follows \ the \ same \ pattern \ as \ x_i. \end{array} \right\}$$

This definition is a bit different from a typical matching game as the number of airlines in the two markets does not have to be equal and a matching is not a result for must. An airline can choose to operate by itself if no matching can bring more profit. This is the general definition for the partner selection game, in this chapter, we have not yet included the analysis for oligopoly pre-alliance market.

4.1.4 Demand and cost function

In this section, we first define the demand functions in the individual market without and with competition, the pre-alliance complementary market, as well as the alliance market. Then we introduce a simple cost function to be applied in our model.

Definition 4.4 (individual demand function)
The demand function for airline i is linear as follows:

(1) Monopoly market:

$$d_i = m - p_i + \gamma^d q_i$$

(2) Duopoly market of airlines i and k:

$$d_i = m - p_i + p_k + \gamma^d q_i - \gamma^d q_k$$

In the monopoly market, the demand of airline i is decreasing with the fare it charges for passengers, and increasing with its service quality. In the duopoly market, the demand of airline i is increasing with the fare its rival k charged, and decreasing with the rival's service quality. For simplicity, we assume for linear demand functions and the parameter measuring the effect of price is 1. The demand function for the the duopoly market reflects that composite products are substitutes for one another. Economides and Salop [1992] illustrated similar results on complementary goods with the linear demand system above, where the parameter for service quality was not included.

Definition 4.5 (pre-alliance demand function)
The pre-alliance demand function of passengers between airport A and C is:

$$d_{ij} = m - (p_i + p_j) + \gamma^d (\frac{q_i + q_j}{2})$$

where airline i operates leg A–B, and airline j operates leg B–C.

Before forming any alliance, airlines choose simultaneously and non-cooperatively their respective profit-maximizing fares and service quality. The perceived service quality for passengers between A and C is assumed to be the average service quality of the two airlines.

Definition 4.6 (alliance demand function)
The demand function for alliance $i-j$ is:

$$d^{a_{ij}} = m - p^{a_{ij}} + \gamma^d \theta(q_i + q_j)$$

where γ^d is a parameter measuring the effect of service quality on the demand, and $\theta \in (1/2, 1)$.

If airlines i and j form an alliance, airlines i and j set the fare for flight from A to C cooperatively, while the competition remains if another option, either individually operated or codeshare flight, still exists. The perceived service quality for passengers of market A–C is higher than that before forming an alliance, for reasons like no necessity of luggage claim during transit, faster mileage accumulation, and etc. Thus we assume $\theta \in (1/2, 1)$.

Definition 4.7 (cost function)
The cost function for airline i is:

$$C_i = \gamma_i^c(q_i)q_i$$

where $\gamma_i^c(q_i)$ is a parameter measuring the effect of service quality on the cost of airline i.

$\gamma_i^c(q_i)$ is increasing with the level of the service quality (Jamasb et al. [2012]). In our discrete setting for service quality, it is assumed that $\gamma_i^c(q_i) = \gamma_g^c$ if $q_i = q_g$, and $\gamma_i^c(q_i) = \gamma_b^c$ if $q_i = q_b$, where $\gamma_g^c > \gamma_b^c$. For simplicity, we write γ_i^c instead of $\gamma_i^c(q_i)$ hereafter. The alliance formation cost is neglectable compared to the operational cost, i.e., the integration of the ticketing system, share of check-in and boarding staff, and etc. It is assumed to be 0 in this chapter.

4.1.5 Assumptions about profit allocation

In general, the proration scheme $\mathcal{R}$ used by the alliance will influence both the overall profit of the alliance and the allocated profit to each airline. It is reasonable to assume that airlines are seeking a strategy that increases the joint profit by forming an alliance and the ultimate aim of each egoistic airline is to maximize its own profit. The decision making on the profit allocation mechanism is actually a bargaining process. On deciding whether to form an alliance or not, each of the potential partner airlines needs to know the portion of the profit it can receive. Assume that each of them can claim a proration scheme, it can be modeled as a sequential move bargaining game between two airlines i and j:

(1) Airline i begins in the first round by proposing a proration scheme $\mathcal{R}_i^1$.

(2) If Airline j accepts, the deal is struck. If Airline j rejects, another bargaining round may be played. In round 2, Airline j proposes $\mathcal{R}_j^2$.

(3) If Airline i accepts, the deal is struck. Otherwise, it is round 3 and Airline i gets to make another proposal.

(4) Bargaining continues in this manner until a deal is struck, or no agreement is reached and each receives their disagreement value–the pre-alliance profit.

However, this bargaining process is inducing great complexity in our model if we consider the value diminishing factor and is thus left as an extension work. In this chapter, we assume that the proportional rule $\mathcal{R}_p$ is the default proration scheme, and primarily focus on examining how the service quality affects the selection of a partner airline.

Definition 4.8 (proportional rule)
Given two airlines i and j as potential partners of an alliance, the proportional rule $\mathcal{R}_p$ is defined as:

$$\beta_i^{a_{ij}} = \frac{\Pi_i^*}{\Pi_i^* + \Pi_j^*}$$
$$\beta_j^{a_{ij}} = \frac{\Pi_j^*}{\Pi_i^* + \Pi_j^*}$$

where Π_i^* and Π_j^* are the pre-alliance optimal profit for airlines i and j, respectively.

It is reasonable to assume that the amount each airline may receive on alliance formation is proportional to its respective pre-alliance profit as the airline with higher pre-alliance profit usually has higher bargaining power.

4.1.6 Decision criteria

The fundamental questions faced by airline i with service quality q_i are:

(1) Whether to form an alliance with another airline.

(2) If yes, which airline should be chosen as the partner.

For the first question, airline i will form an alliance with airline j only if the cooperation is to bring more profit for i than that of the pre-alliance equilibria. Both collective and individual rationality should be satisfied.

Definition 4.9 (collective rationality)
For two airlines i and j, they are to form an alliance only if the joint profit of the alliance is more than the sum of their pre-alliance profit.

$$\Pi^{a_{ij}*} > \Pi_i^* + \Pi_j^*$$

Definition 4.10 (individual rationality)
For two airlines i and j, they are to form an alliance only if the alliance profit allocated to each of them is more than that of their respective pre-alliance profit.

$$\Pi_i^{a_{ij}*} > \Pi_i^*$$
$$\Pi_j^{a_{ij}*} > \Pi_j^*$$

However, as the proportional rule $\mathcal{R}_p$ is assumed to be adopted as the proration scheme in this chapter, these two criteria coincide with each other. We are only to verify the collective rationality in the following analysis.

For the second question, if airline i has two options, namely airlines j and k, it will select the one which brings more profit to itself as the partner. The stability of each proposed formation should be checked, and the more stable alliance will be formed.

Definition 4.11 (stability)
For airline i with two potential partner airlines j and k, the stability of alliance $i-j$ is higher than that of alliance $i-k$ if and only if

$$\Pi_i^{a_{ij}*} > \Pi_i^{a_{ik}*}$$

4.2 Analysis: a three-stage framework

As mentioned above, we proceed to the analysis for the equilibria of three types of pre-alliance markets: Monopoly–Monopoly, Monopoly–Duopoly, and Duopoly–Duopoly by our proposed three-stage framework, namely, pre-alliance equilibria, alliance equilibria, and criteria verification.

4.2.1 Monopoly–Monopoly

For the pre-alliance Monopoly–Monopoly situation, airlines 1 and 2 both own monopoly power for the leg A–B and B–C, respectively. From the service quality's perspective, each airline's rate could either be q_g or q_b, thus three cases need to be analyzed:

- **Case 1:** $q_1 = q_g, q_2 = q_g$
- **Case 2:** $q_1 = q_b, q_2 = q_b$
- **Case 3:** $q_1 = q_g, q_2 = q_b$

It is easy to estimate that the equilibria of the first two cases are the same. Let us first give the analysis for the alliance of two airlines with high service quality.

Case 1: $q_1 = q_g, q_2 = q_g$

Stage 1: pre-alliance equilibria. We start by finding the pre-alliance total profit for airline i:

$$\Pi_i = p_i(d_i + d_{12}) - C_i$$

where d_i and d_{12} are defined in Definition 4.4 and 4.5, respectively.

By differentiation, we get:

$$\Pi_1^* = \Pi_2^* = \frac{8}{25}(m + \gamma^d q_g)^2 - \gamma_g^c q_g$$

Alliance equilibria. If airlines 1 and 2 form an alliance, the total profit that the alliance might receive is:

$$\Pi^{a_{12}} = p_1 d_1 + p_2 d_2 + p^{a_{12}} d^{a_{12}} - C_1 - C_2$$

where $d^{a_{12}}$ is defined in Definition 4.6.

We get the following result:

$$\Pi^{a_{12}*} = \frac{12}{25}(m + \gamma^d q_g)^2 + (\frac{m}{2} + \theta\gamma^d q_g)^2 - 2\gamma_g^c q_g$$

The proportional rule $\mathcal{R}_p$ is applied to make the profit allocation, where $\beta_1^{a_{12}} = \beta_2^{a_{12}} = 1/2$. It yields,

$$\Pi_1^{a_{12}*} = \Pi_2^{a_{12}*} = \frac{\Pi^{a_{12}*}}{2}$$

Criteria verification. The Monopoly–Monopoly case is the simplest one in which neither of the airlines has an optional potential partner. Hence only the collective rationality needs to be verified. Straightforward calclulation shows that

$$\Pi^{a_{12}*} - \Pi_1^* - \Pi_2^* > 0$$

is satisfied. This cooperation is to bring more profit for both airlines.

For **case 2** and **case 3**, following the same three-stage analysis framework, the collective rationality can be verified and we get the same conclusion.

4.2.2 Monopoly–Duopoly

Next, we consider the pre-alliance Monopoly–Duopoly network, in which airlines 2 and 3 are competing in the B–C market, while airline 1 still enjoys the monopoly power as in the previous section. For a passenger of market A–C, there are two options:

(1) A–B by airline 1, B–C by airline 2.

(2) A–B by airline 1, B–C by airline 3.

These two options are assumed to be competitive with each other no matter for the pre-alliance market, or the re-formed market if airline 1 cooperates with another airline. In the Monopoly–Duopoly setting, where airlines 2 and 3 differ in service quality, which one is to be selected as airline 1's partner becomes our main concern. Note that there are 8 possible combinations here, only two representative cases will be analyzed:

- **Case 1**: $q_1 = q_g, q_2 = q_g, q_3 = q_b$

- **Case 2**: $q_1 = q_b, q_2 = q_g, q_3 = q_b$

Let us first discuss the case when 1 and 2 are airlines with high service quality, while 3 with low service quality.

Case 1: $q_1 = q_g, q_2 = q_g, q_3 = q_b$

Pre-alliance equilibria. Passengers' demand for market A–C is:

$$d_{12} = m - (p_1 + p_2) + (p_1 + p_3) + \gamma^d(\frac{q_1 + q_2}{2}) - \gamma^d(\frac{q_1 + q_3}{2})$$
$$d_{13} = m - (p_1 + p_3) + (p_1 + p_2) + \gamma^d(\frac{q_1 + q_3}{2}) - \gamma^d(\frac{q_1 + q_2}{2})$$

The definition above suggests that before any alliance is formed, the fare and service quality of airlines 2 and 3 interactively affect A–C passengers' choice.

The pre-alliance total profit for each airline is defined as:

$$\Pi_1 = p_1(d_1 + d_{12} + d_{13}) - C_1$$
$$\Pi_2 = p_2(d_2 + d_{12}) - C_2$$
$$\Pi_3 = p_3(d_3 + d_{13}) - C_3$$

The equilibria solutions by differentiation are:

$$\Pi_1^* = \frac{1}{4}(3m + \gamma^d q_g)^2 - \gamma_g^c q_g$$
$$\Pi_2^* = 2(m + \frac{1}{4}\gamma^d (q_g - q_b))^2 - \gamma_g^c q_g$$
$$\Pi_3^* = 2(m - \frac{1}{4}\gamma^d (q_g - q_b))^2 - \gamma_b^c q_b$$

Alliance equilibria. If airlines 1 and 2 form an alliance, passengers' demand for market A–C will be:

$$d^{a_{12}} = m - p^{a_{12}} + (p_1 + p_3) + \gamma^d \theta (q_1 + q_2) - \gamma^d (\frac{q_1 + q_3}{2})$$

The journey of two tickets issued by airlines 1 and 3 separately is still a competitive option for alliance 1-2. The total profit of alliance 1-2 is defined the same as in Section 4.2.1 and we can get $\Pi^{a_{12}*}$, the maximum alliance profit. Applying the proportional rule $\mathcal{R}_p$, the profit allocated to each airline under the cooperation scheme of 1-2 is:

$$\Pi_1^{a_{12}*} = \beta_1^{a_{12}} \Pi^{a_{12}*}$$
$$\Pi_2^{a_{12}*} = \Pi^{a_{12}*} - \Pi_1^{a_{12}*}$$

The calculation under the cooperation scheme of 1-3 can be done similarly.

Criteria verification. Let us verify the collective rationality first, assume $q_b = \alpha q_g$, where $\alpha \in (0, 1)$:

$$\Pi^{a_{12}*} - \Pi_1^* - \Pi_2^* > 0$$

is satisfied if and only if

$$q_g \in (\omega_{m-d}^{a_{12}}(m, \gamma^d, \theta, \alpha), +\infty)$$

where $\omega_{m-d}^{a_{12}}(m, \gamma^d, \theta, \alpha) \in R^+$. It indicates that an airline will consider forming an alliance with another if and only if its service quality reaches a certain level, i.e., low accident rate. Otherwise, it is difficult for any other airline to accept it as a partner. Also the airline itself is focusing on improving its service quality and rarely has spare capital to invest in alliance formation. For the stability of formation,

$$\Pi_1^{a_{12}*} - \Pi_1^{a_{13}*} > 0$$

is satisfied if and only if

$$\alpha \in (0, \upsilon_{m-d}(m, \gamma^d, q_g))$$

where $\upsilon_{m-d}(m, \gamma^d, q_g) \in (0,1)$ and is close to 1. The alliance structure of 1-2 is more stable than that of 1-3, and vice versa if $q_g \in (\omega_{m-d}^{a_{13}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (\upsilon_{m-d}(m, \gamma^d, q_g), 1)$, 1-3 is more stable. The conclusion of **case 2** is opposite to that of **case 1**.

4.2.3 Duopoly–Duopoly

In this section, we consider the network with four airlines shown in Figure 4.3, where the service quality of the two airlines competing on the same leg differs as in the previous section. This topology represents a typical situation of the airlines in or to-be-in the three big airline alliances. Before examining the specific strategy to be adopted by the three-stage analysis framework, we describe a simple example of two main airlines in Taiwan: EVA Air (BR) and China Airlines (CI). The network coverage of the two airlines is nearly the same. In other words, they are competing nearly on each route. China Airlines joined *SkyTeam* in 2011, and EVA Air joined *Star Alliance* later in 2013. As is known that China Airlines has records of many incidents and accidents since its formation, and was announced as the one with worst safety record among 60 international airlines by *Jet Airliner Crash Data Evaluation Centre (JACDEC)* in January, 2013. On the contrary, Eva Air has not had any aircraft losses or passenger fatalities in its operational history. From the perspective of the most important factor of service quality, safety, China Airlines' rate definitely cannot exceed that of EVA Air. Referring the three big airline alliances' service quality rating data, *Star Alliance* is doing better than *SkyTeam* as well. The analysis in this section can also be viewed as providing a theoretical support for the partner selection criteria by the three big airline alliances, if we take *Star Alliance* as an airline with high service quality, *SkyTeam* as one with low service quality.

In this section, we will study one **representative case**: $q_1 = q_g, q_2 = q_g, q_3 = q_b, q_4 = q_b$. For a passenger of market A–C, there are four pre-alliance options:

(1) A–B by airline 1, B–C by airline 2.

(2) A–B by airline 1, B–C by airline 3.

(3) A–B by airline 4, B–C by airline 2.

(4) A–B by airline 4, B–C by airline 3.

Pre-alliance equilibria. A–C Passengers' demand for the option is defined as:

$$\begin{aligned} d_{12} = m &- (p_1 + p_2) + (p_1 + p_3) + (p_4 + p_2) + (p_4 + p_3) \\ &+ \gamma^d(\frac{q_1 + q_2}{2}) - \gamma^d(\frac{q_1 + q_3}{2}) - \gamma^d(\frac{q_4 + q_2}{2}) - \gamma^d(\frac{q_4 + q_3}{2}) \end{aligned}$$

d_{13}, d_{42}, and d_{43} can be defined similarly as d_{12}.

The pre-alliance profit for airline 1 is:

$$\Pi_1 = p_1(d_1 + d_{12} + d_{13}) - C_1$$

Π_2, Π_3, and Π_4 can be defined respectively as well. We use $\Pi_1^*, \Pi_2^*, \Pi_3^*$ and Π_4^* to denote the equilibria solutions. The calculation is simple, and we are not to present the long equations here.

Alliance equilibria. If the alliance structure is 1-2 (high-high) and 4-3 (low-low), passengers' demand for market A–C will become:

$$\begin{aligned} d^{a_{12}} &= m - p^{a_{12}} + p^{a_{43}} + \gamma^d\theta(q_1 + q_2) - \gamma^d\theta(q_4 + q_3) \\ d^{a_{43}} &= m - p^{a_{43}} + p^{a_{12}} + \gamma^d\theta(q_4 + q_3) - \gamma^d\theta(q_1 + q_2) \end{aligned}$$

It is reasonable assuming passengers will not choose the option constituted by two airlines from different alliances. The alliance profit is defined the same as in Section 4.2.1. For alliance structure of 1-3 and 4-2, follow the same pattern above to make the definitions. By assuming $q_b = \alpha q_g, \gamma_b^c = \alpha \gamma_g^c$, where $\alpha \in (0, 1)$, we can get the equilibria solutions of $\Pi^{a_{12}*}, \Pi^{a_{43}*}, \Pi^{a_{13}*}$ and $\Pi^{a_{42}*}$. Applying the proportional rule $\mathcal{R}_p$, the profit allocated to each airline under different cooperation schemes can be denoted as $\Pi_1^{a_{12}*}, \Pi_2^{a_{12}*}, \Pi_4^{a_{43}*}, \Pi_3^{a_{43}*}, \Pi_1^{a_{13}*}, \Pi_3^{a_{13}*}, \Pi_4^{a_{42}*}$ and $\Pi_2^{a_{42}*}$.

Criteria verification. Let us verify the collective rationality first:

$$\Pi^{a_{12}*} - \Pi_1^* - \Pi_2^* > 0$$
$$\Pi^{a_{43}*} - \Pi_4^* - \Pi_3^* > 0$$

are satisfied if and only if

$$q_g \in (\omega_{d-d}^{a_{12}-a_{43}}(m, \gamma^d, \theta, \alpha), +\infty)$$

where $\omega_{d-d}^{a_{12}-a_{43}}(m, \gamma^d, \theta, \alpha) \in R^+$. For the stability of formation,

$$\Pi_1^{a_{12}*} - \Pi_1^{a_{13}*} > 0$$
$$\Pi_2^{a_{12}*} - \Pi_2^{a_{42}*} > 0$$
$$\Pi_3^{a_{43}*} - \Pi_3^{a_{13}*} > 0$$
$$\Pi_4^{a_{43}*} - \Pi_4^{a_{42}*} > 0$$

are satisfied if and only if

$$\alpha \in (0, \upsilon_{d-d}(m, \gamma^d, q_g))$$

where $\upsilon_{d-d}(m, \gamma^d, q_g) \in (0, 1)$ and is close to 1. The alliance structure of 1-2 and 4-3 is more stable than that of 1-3 and 4-2. Vice versa, the structure of 1-3 and 4-2 is more stable if

$$q_g \in (\omega_{d-d}^{a_{13}-a_{42}}(m, \gamma^d, \theta, \alpha), +\infty)$$
$$\alpha \in (\upsilon_{d-d}(m, \gamma^d, q_g), 1)$$

where $\omega_{d-d}^{a_{13}-a_{42}}(m, \gamma^d, \theta, \alpha) \in R^+$.

4.3 The optimal strategy

Proposition 4.1

For a pre-alliance Monopoly–Monopoly network consisted of airlines i and j, for any $q_i, q_j \in R^+$, assuming the proration scheme $\mathcal{R}$ is proportional, then

$$\Pi_i^{a_{ij}*} > \Pi_i^*$$
$$\Pi_j^{a_{ij}*} > \Pi_j^*$$

The optimal strategy of the two airlines is cooperation with each other.

This proposition indicates that for a Monopoly–Monopoly market, the cooperation will always bring more profit for each of its member, mainly due to the extension of network coverage for each airline, and demand increment because of the more convenient service during transit.

Proposition 4.2

For a pre-alliance Monopoly–Duopoly network consisted of airlines i, j and k, in which airline i's market is monopoly, for any $q_k = \alpha q_i = \alpha q_j$, where $\alpha \in (0, 1)$, assuming the profit allocation rule $\mathcal{R}$ is proportional, then if $q_i = q_j \in (\omega_{m-d}^{a_{ij}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (0, \upsilon_{m-d}(m, \gamma^d, q_g))$

$$\Pi_i^{a_{ij}*} > \Pi_i^*$$
$$\Pi_i^{a_{ij}*} > \Pi_i^{a_{ik}*}$$

Airline i's optimal strategy is to select airline j as its partner in the alliance formation. Vice versa, if $q_i = q_j \in (\omega_{m-d}^{a_{ik}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (\upsilon_{m-d}(m, \gamma^d, q_g), 1)$, the equilibrium alliance structure should be $i - k$.

If the pre-alliance service quality distribution differs greatly, the airline in the monopoly market will choose the one with the same service quality level as its partner, while if the distribution is approximately uniform, a combination of service quality and price competitiveness tends to be formed.

Proposition 4.3

For a pre-alliance Duopoly–Duopoly network consisted of airlines i, j, k and l, in which airlines i and l, airlines j and k each form a duopoly market, for any

$q_k = q_l = \alpha q_i = \alpha q_j$, where $\alpha \in (0,1)$, assuming the proration scheme $\mathcal{R}$ is proportional, then if $q_i = q_j \in (\omega_{d-d}^{a_{ij}-a_{lk}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (0, \upsilon(m, \gamma^d, q_g))$

$$
\begin{aligned}
\Pi_i^{a_{ij}*} &> \Pi_i^{a_{ik}*} \\
\Pi_j^{a_{ij}*} &> \Pi_j^{a_{lj}*} \\
\Pi_k^{a_{lk}*} &> \Pi_k^{a_{ik}*} \\
\Pi_l^{a_{lk}*} &> \Pi_l^{a_{lj}*}
\end{aligned}
$$

The equilibrium alliance structure should be $i - j$, and $k - l$. Vice versa, if $q_i = q_j \in (\omega_{d-d}^{a_{ik}-a_{lj}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (\upsilon(m, \gamma^d, q_g), 1)$, the equilibrium alliance structure should be $i - k$, and $l - j$.

This conclusion is intuitive. If the difference between airlines with high service quality and low service quality is large, airlines tend to form an alliance with another with the same service quality level. An airline with high service quality will not accept one with poor service quality to degrade itself too much. Whereas if the difference is relatively small, an airline with high service quality tends to select the one with price competitiveness as its partner, even if this kind of cooperation might reduce the overall rate of service quality a little bit.

Chapter 5

Concluding remarks

5.1 Summary

Traditionally, game theory has developed almost entirely from introspection and theoretical concerns, whilst this dissertation was initiated with the overall aim of advancing game theory by formally studying its applications in three different multi-agent negotiation problems, namely, the MCST problem, the DEA problem, and partner selection problem in airline alliances. There is clearly a lot to gain from the interaction of game theory and its applications in economical problems. With highly differentiated areas, these applications are allocated to three chapters, respectively.

In Chapter 2, we have presented the decentralized solution of the MCST problem and finalized its characterization. We treat the MCSE problem as a particular extension for the MCST problem and do not make a distinction specifically between them. The decentralized solution is a generalization of the Boruvka's rule and particularly appealing when considering the MCST problem under the game-theoretic framework: each connected component constructs links in a greedy pattern yielding the MCST.

The characterization constitutes of six main properties: Eff, MC, FSC, ET, Loc and USI, respectively. USI describes the primary difference between the decentralized solution and any centralized solution, which allows us to distinguish a decentralized cost allocation rule from a centralized one. It is also the property defining the cost sharing aspect in the decentralized solution based on the degree of decentralization at each stage.

In Chapter 3, we have improved the DEA min game proposed by Nakabayashi and Tone [2006] by re-assigning the total weight for the coalition members and developed a more natural, super-additive cooperative game scheme for this kind of problem. We first introduce a strategic form DEA game with the average weights over all players regarding each criterion; then define the TU DEA game based on the definition of the strategic form game, and study its properties and solutions. By applying the constant-sum property, we give the sufficient and necessary condition for the inessentiality of the TU DEA game, which is also the condition for a non-empty core. We next define the Shapley value and nucleolus in our new scheme, and also prove that the allocations by these two concepts coincide with each other in the 3-player setting.

On showing that the core is empty in most TU DEA games, we proceed to NTU DEA game, and mainly focus on proving the existence of α-core following Scarf's theorem and giving a condition under which the β-core is non-empty.

The final section analyzes the equilibria for the strategic form DEA game, in case players are not allowed to form any coalition during the DEA process. One main result in this section is the condition under which the coalition-proof Nash equilibrium exists and is uniquely determined.

The service quality rating data for the three big airline alliances suggests the need to understand the impact of service quality during the alliance formation. Chapter 4 proposes a framework studying service quality's effects on the selection of a partner airline. In particular, we model the optimal strategy decision process by a three-stage analysis framework. In the first stage, analyze the pre-alliance equilibria that each airline manages its own market in a non-cooperative fashion so as to maximize its expected profit. In the second stage, analyze the alliance equilibria under different cooperation schemes assuming a particular profit allocation rule. In the third stage, verify the collective rationality and stability to finalize the decision process.

We have discussed about airlines' optimal strategy in three types of pre-alliance markets. In a Monopoly–Monopoly market, the cooperation is always bringing more profit for both airlines, and thus forming an alliance is the optimal strategy for both airlines. In a Monopoly–Duopoly market, with the premise such that each airline's service quality has reached a certain level, if the pre-alliance service quality differentiation is high, the airline in the monopoly market will cooperate with the one with the same service quality; otherwise, it will choose a cooperation pattern of service quality and price competitiveness combination. The optimal strategy for airlines in a Duopoly–Duopoly market is similar to that of the Monopoly–Duopoly case.

These three main insights can be corroborated by airlines of the three big alliances, i.e., China Airlines and Eva Air. Basically airlines prefer to play with the one with the same service quality level. When the service quality of the airlines in the whole market does not differ too much with each other, the trend becomes a combination of service quality and price competitiveness.

5.2 Future extensions

In Chapter 2, we have assumed for a generic cost matrix, however, the property of Con and the limit condition indicate the room for future research without this constraint. An interesting avenue of research should aim to define the decentralized solution under an arbitrary cost matrix, and implement its axiomatization under the concept of irreducible core.

FSC consolidates the definite advantageous position of agents in the source component. In the real world, these components not only contribute nothing for the network construction, on the contrary, they may request extra payments from other agents who want to make use of their existing network in order to be connected to the source. The bargaining game between the agents in the source component and the outsiders is also an interesting extension to pursue in this context.

One more issue that deserves some attention is the application of this model to real economical problems, i.e., the effectiveness and efficiency study between the decentralized solution of MCST and mesh network for the smart community project.

In Chapter 3, we have gone into details for the solution concepts of core, Shapley value and nucleolus in the TU DEA game, and α-core and β-core in

NTU DEA game. One of our future research subjects is the role of other imputations, i.e., the proportional nucleolus (Young et al. [1981]), the kernel (Davis and Mascheler [1965]), the γ-core (Chander and Tulkens [1997]), the δ-core (Currarini and Marini [2004]) and etc.

The results in this chapter are theoretically validated based on its applications in the benefit allocation problem. We plan to explore its potential application in the voting game as well as its stability.

An important feature of the study in Chapter 4 is the more general network topology. It suggests an extension to the oligopoly pre-alliance market, which reflects the real situation better. This is more complicated compared to the analysis of the duopoly market, in this respect, our results can be viewed as the first step to understand how airlines with different service quality will act assuming a particular allocation scheme.

Another aspect of the model deserves some attention is the profit allocation rule. In the first stage, the proportional rule is assumed to be applied in our analysis. For the future research, with a refined model, the application of the equilibrium of bargaining game is an important extension to pursue.

Finally, in our model only complementary alliance is considered, the real network is the coexistence of complementary and parallel alliances among partner airlines. Such a scheme, however, requires more factors, i.e., the fleet size, the capacity, service frequency and etc., to be included in the model for analysis. Describing service quality's strategic effects under the coexistence scheme will be an interesting area of further study as well.

List of Figures

List of Tables

References

Aumann, R. (1959). *Acceptable points in general cooperative n-person games.* Contributions to the Theory of Games IV. Princeton University Press, Princeton, NJ.

Bergantinos, G. and L. Lorenzo (2004). *A non-cooperative approach to the cost spanning tree problem.* Math. Methods Operations Res., **59**, 393–403.

Bergantinos, G. and J. Vidal-Puga (2007). *A fair rule in minimum cost spanning tree problems.* Journal of Economic Theory, **137**, 326–352.

Bernheim, B. D., B. Peleg and M. D. Whinston (1987). *Coalition-Proof Equilibria I. Concepts.* Journal of Economic Theory, **42**, 1–12.

Bird, C. G. (1976). *On cost allocation for a spanning tree: a game theoretic approach.* Networks, **6**, 335–350.

Bitner, M. J. and A. R. Hubbert (1994). *Service quality: New directions in theory and practice.* Encounter satisfaction versus overall satisfaction versus quality (Rust, R. T. and R. Oliver, eds), 77. Sage Publications, Thousand Oaks, California.

Boruvka, O. (1926a). *O jistem problemu minimalnim (About a certain minimal problem).* Prace mor. Prirodoved. Spol. v Brne (in Czech, German summary), **3**, 37–58.

Boruvka, O. (1926b). *Prispevek k reseni otazky ekonomicke stavby elektrovodnich siti (Contribution to the solution of a problem of economical construction of electrical networks)*, Elektronicky Obzor, **15**, 153–154.

Chander, P. and H. Tulkens (1997). *The core of an economy with multiple externalities*. International Journal of Game Theory, **26**, 379–401.

Charnes, A., W. W. Cooper and E. Rhodes (1978) *Measuring the efficiency of decision making units*. European Journal of Operational Research, **2**, 429–444.

Chung, S., H. Singh, and K. Lee (2000). *Complementarity, status similarity and social capital as drivers of alliance formation*. Strategic Management Journal, **21 (1)**, 1–22.

Claus, A. and D. J. Kleitman (1973). *Cost allocation for a spanning tree*. Networks, **3**, 289-304.

Colonques, R. M. and R. F. Fillol (2005). *Strategic Effects Of International Airline Alliances*. Working Papers, serie AD 2005–15, Instituto Valenciano de Investigaciones Economicas, S.A. (Ivie).

Cook, W. D., K. Tone and J. Zhu (2014). *Data envelopment analysis: Prior to choosing a model*. OMEGA, **44**, 1–4.

Currarini, S. and M. Marini (2004). *A conjectural cooperative equilibrium for strategic games*. Game Practice and the Environment (Carraro, C. and V. Fragnelli, eds), Edward Elgar.

Davis, M. and M. Maschler (1965). *The kernel of the cooperative game*. Naval Research Logistics Quarterly, **12**, 223–259.

Dijkstra, E. W. (1959). *A note on two problems in connection with graphs* Numerische Mathematik, **1**, 269–271.

Dutta, B. and A. Kar (2004). *Cost monotonicity, consistency and minimum cost spanning tree games*. Games Econ. Behav., **48**, 223–248.

Dyer, J. H. and H. Singh (1998). *The relational view: Cooperative strategy and sources of interorganizational competitive advantage*. Academy of Management Review, **23 (4)**, 660–679.

Economides, N. and S. Salop (1992). *Competition and Integration among Complements, and Network Market Structure*. Journal of Industrial Economics, **40**, 105–123.

Feltkamp, V., S. Tijs and S. Muto (1994a). *On the irreducible core of a minimum cost spanning extension game*. Mimeo, Tilburg University, the Netherlands.

Feltkamp, V., S. Tijs and S. Muto (1994b). *Minimum cost spanning extension problems: the proportional rule and the decentralized rule*. CentER DP 9496, Tilburg University, the Netherlands.

Gillies, D. B. (1959). *Solutions to general non-zero-sum games*. Contributions to the Theory of Games IV, 47–85. Princeton University Press, Princeton, NJ.

Gimeno, J. (2004). *Competition within and between networks: The contingent effect of competitive embeddedness on alliance formation.* The Academy of Management Journal, **47 (6)**, 820–842

Graham, R. L. and P. Hell (1985). *On the history of the minimum spanning tree problem*. Annals of the History of Computing, **7**, 43–57.

Gudmundsson, S. V. and R. L. Dawna (2001). *Airline alliance survival analysis: typology, strategy and duration*. Journal of Air Transport Management, **8 (3)**, 209–218.

Gulati, R. (1995). *Social structure and alliance formation patterns: a longitudinal analysis*. Administrative Science Quarterly, **40 (4)**, 619–652.

Jamasb, T., L. Orea and M. Pollitt (2012). *Estimating the marginal cost of quality improvements: The case of the UK electricity distribution companies* Energy Economics, **34 (5)**, 1498–1506.

Hu, X., R. Caldentey and G. Vulcano (2010). *Revenue sharing in airline alliances.* Working Paper, New York University, New York City, NY.

Kar, A. (2002). *Axiomatization of the Shapley value on minimum cost spanning tree games*. Games and Economic Behavior, **38**, 265–277.

Kruskal, J. B. (1956). *On the shortest spanning subtree of a graph and the traveling salesman problem*. Proc. American Math. Soc., **7**, 48–50.

Liang, L., W. D. Cook and J. Zhu (2008). *DEA Models for Two-Stage Processes: Game Approach and Efficiency Decomposition*. Naval Research Logistics, **55**, 643–653.

Masuzawa, T. (2003). *Punishment strategies make the α-coalitional game ordinally convex and balanced*. International Journal of Game Theory, **32**, 479–483.

Myerson, R. B. (1991). *Game Theory: Analysis of Conflict*. Harvard University Press, 1.

Nakabayashi, K. and K. Tone (2006) *Egoist's dilemma: a DEA game*. The International Journal of Management Science (OMEGA), **34**, 135–148.

Nakayama, M. (1998). *Self-Binding Coalitions*. Keio Economic Studies, **35**, 1–8.

Nash, J. F. (1951). *Non-Cooperative Games*. The Annals of Mathematics, **54 (2)**, 286–295.

Owen, G. (1995). *Game Theory*. Emerald Group Publishing Limited, 3rd edition, 211–233.

Park, J. (1997). *The effect of airline alliances on markets and economic welfare*. Transportation Research, Part E, **33**, 181–195.

Prim, R. C. (1957). *Shortest connection networks and some generalizations*. Bell Systems Techn. J., **36**, 1389–1401.

Scarf, H. (1967). *The Core of an N-Person Game*. Econometirca, **35**, 50–67.

Scarf, H. (1971). *On the Existence of a Cooperative Solution for a General Class of N-Person Games*. Journal of Economic Theory, **3**, 169–181.

Schmeidler, D. (1969). *The nucleolus of a characteristic function game*. SIAM Journal of Applied Mathematics, **17 (6)**, 1163–1170.

Shapley, L. S. (1953). *A value for n-person games*. Contributions to the Theory of Games II (Kuhn, H. W. and A. W. Tucker, eds), 307–317. Princeton University Press, Princeton, NJ.

Talluri, K. T. and G. J. van Ryzin (2004). *The Theory and Practice of Revenue Management*. Kluwer Academic Publishers, Boston, MA.

Vinod, B. (2005). *Alliance revenue management*. Journal of Revenue and Pricing Management, **4**, 66–82.

von Neumann, J. (1928). *Zur Theorie der Gesellschaftsspiele (On the Theory of Games of Strategy)*. Mathematische Annalen, **100 (1)**, 295–320.

von Neumann, J. and O. Morgenstern (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Vulcano, G., G. van Ryzin and W. Chaar (2010). *Choice-based revenue management: An empirical study of estimation and optimization*. Manufacturing and Service Operations Management, **12 (3)**, 371–392.

Wright, C. P., H. Groenevelt and R. Shumsky (2010). *Dynamic revenue management in airline alliances*. Transportation Science, **44**, 15–37.

Wu, J., L. Liang and Y. C. Zha (2008). *Determination of the weights of ultimate cross efficiency based on the solution of nucleolus in cooperative game*. Systems Engineering - Theory and Practice, **28**, 92–97.

Wu, J., L. Liang and F. Yang (2009). *Determination of the weights of ultimate cross efficiency based on Shapley value in cooperative game*. Expert Systems with Applications, **36**, 872–876.

Xu, Y., V. Olman and D. Xu (2002). *Clustering gene expression data using a graph-theoretic approach: an application of minimum spanning trees*. Bioinformatics, **18 (4)**, 536–545.

Young, H., N. Okada and T. Hashimoto (1981). *Cost allocation in water resources development*. Water Resources Research, **18**, 463–475.

Zhu, J. (2004). *A buyer-seller game model for selection and negotiation of purchasing bids: extensions and new models*. Operations Research, **154**, 150–156.