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On the boundary behavior of Teichmüller geodesic rays

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Chapter 1

Introduction

A Riemann surface is a connected 1-dimensional complex manifold, that is, it is a real 2-dimensional oriented differentiable manifold with a complex structure on it. A type of a Riemann surface is determined by its genus and the number of punctures. There exists an orientation preserving diffeomorphism between two Riemann surfaces of the same type, however it is not always biholomorphic. If there exists a biholomorphic mapping between them, we say that they are biholomorphically equivalent. Therefore, for any fixed type, to consider how many biholomorphically equivalence classes of Riemann surfaces exist is important. The set of all such classes is called the moduli space of the Riemann surfaces. For closed surfaces of genus $g \geq 2$, in 1857, Riemann concluded that the moduli space has $3g-3$ complex parameters. To analyze moduli spaces, we treat Teichmüller spaces which are universal covering spaces of moduli spaces. Teichmüller spaces were induced by Teichmüller in 1930s.

We consider the Teichmüller space $T(X)$ of an analytically finite Riemann surface X which has genus g with n punctures such that $3g-3+n>0$. In fact, if $g=0$ and $0 \leq n \leq 3$, or if $g=1$ and $n=0$, $T(X)$ is trivial or well-known. There is a representation of $T(X)$ which is used quasiconformal mappings on X. The concept of quasiconformal mappings appeared for the first time in Grötzsch's paper in 1928. He considered mappings of a square onto a rectangle such that vertexes of the square are mapped to vertexes of the rectangle. He found the extremal mapping which is the closest to conformal in such mappings, it is an affine mapping. In general, a quasiconformal mapping f of a planer domain D to another D' is an oriented preserving homeomorphism which is absolutely continuous on lines on D, and there exists $0 \leq k < 1$ such that the inequality $|f_{\bar{z}}/f_z| \leq k$ holds almost everywhere on D. In particular, if f is a diffeomorphism and satisfies the inequality of partial derivatives, then f is a quasiconformal mapping. The number k means an estimation of a gap between biholomorphic (conformal) mappings. The mapping f is biholomorphic if and only if k is attained as 0. The definition of quasiconformal mappings is extended naturally on Riemann surfaces. The Teichmüller space $T(X)$ is the set of equivalence classes of pairs (Y, f) where Y is a Riemann surface and f is a quasiconformal mapping of X onto Y. Here, the pairs (Y, f) and (Y', f') are equivalent if there exists a biholomorphic mapping h of Y onto Y' such that h is homotopic to $f' \circ f^{-1}$. A topology of $T(X)$ is induced by the complete distance on $T(X)$ which is called the Teichmüller distance. Its definition also uses quasiconformal mappings. By the Teichmüller distance, $T(X)$ is homeomorphic to $\mathbb{C}^{3g-3+n}$. In

1940s, Teichmüller generalized the Grötzsch's statement, that is, for a given quasiconformal mapping f of X onto Y, he found the unique extremal quasiconformal mapping which is homotopic to f. Furthermore, the extremal mapping is represented by an affine mapping by using natural coordinates on X, Y which are determined by holomorphic quadratic differentials on X, Y respectively. We call this mapping the Teichmüller mapping. In each homotopy class, the Teichmüller mapping has the smallest quasiconformal dilatation, it is an essence of the Teichmüller distance. In 1960s, Ahlfors and Bers showed that $T(X)$ is embedded in the space of holomorphic quadratic differentials on X as a bounded domain. This space is a complex $(3g-3+n)$-dimensional vector space by the Riemann-Roch theorem, then $T(X)$ has the same dimensional complex structure. The mapping class group of X is the covering transformation group of $T(X)$ and acts properly discontinuity on $T(X)$. It is also a discrete group of biholomorphic automorphisms of $T(X)$. Moreover, in 1971, Royden, and in 1974, Earle and Kra showed that the mapping class group of X and the group of biholomorphic automorphisms of $T(X)$ are group homomorphic in almost all cases. Then the corresponding moduli space is a $(3g-3+n)$-dimensional normal complex analytic space.

We also consider some boundaries of $T(X)$. Since $T(X)$ is not compact, to consider compactifications of $T(X)$ has a meaning. Teichmüller introduced a compactification of $T(X)$ if X is compact. He considered all images of Teichmüller mappings on X with any dilatation $0 \leq k < 1$ having any holomorphic quadratic differential with unit norm on X. Since the space of holomorphic quadratic differentials on X is $(3g-3)$-dimensional, the above correspondence induces a mapping of a complex $(3g-3)$-dimensional open unit ball onto $T(X)$. He showed that the mapping is a homeomorphism, and the compactification of $T(X)$ comes from the extended homeomorphism on the closed unit ball. The closure of $T(X)$ is compact, however the action of the mapping class group cannot be extended continuously to the boundary. Abikoff considered the augmented Teichmüller space $\hat{T}(X)$ which consists of the Teichmüller space together with the set of equivalence classes of pairs of a Riemann surface with nodes of the same type as X and a deformation of X onto it. This original concept came from the theory of Kleinian groups, and Bers gave the above representation. The boundary of $\hat{T}(X)$ consists of end points of all Teichmüller geodesic rays determined by Jenkins-Strebel differentials on X. In this time, the mapping class group can be extended continuously to the boundary of $\hat{T}(X)$. The augmented Teichmüller space $\hat{T}(X)$ is not compact, however the quotient space of $\hat{T}(X)$ is compact, i.e., it is a compactification of the moduli space. It coincides to the Deligne-Mumford compactification of the moduli space. Since X is a hyperbolic Riemann surface, there exist hyperbolic metrics on Riemann surfaces corresponding to points on $T(X)$. Thurston gave the embedding of $T(X)$ into the projective space of hyperbolic lengths of simple closed curves on Riemann surfaces corresponding to points on $T(X)$ ([FLP79]). He proved that the image of the embedding is relatively compact. The closure of the image is called the Thurston compactification of $T(X)$. The boundary is homeomorphic to the space of the projective measured foliations $\mathcal{PMF}(X)$ of X. The action of the mapping class group can be extended continuously to the boundary. However, Lenzhen found the Teichmüller geodesic ray which does not converges to the boundary ([Len06]). In 1991, Gardiner and Masur considered the compactification of $T(X)$ which is constructed by the similar way as in the case of Thurston, but they used extremal lengths instead of hyperbolic lengths ([GM91]). This is called the Gardiner-Masur compactification of $T(X)$. The action of the mapping class group can be extended continuously, and all Teichmüller geodesic rays converge to this boundary.

Let $T(X)$ be the Teichmüller space of an analytically finite Riemann surface X which has genus g with n punctures such that $3g - 3 + n > 0$. In this thesis, we consider Teichmüller geodesic rays and their geometric properties. A Teichmüller geodesic ray on $T(X)$ is determined by a holomorphic quadratic differential on an initial point. More precisely, each point of the Teichmüller geodesic ray is constructed by the image of the Teichmüller mapping from the initial Riemann surface with natural coordinates determined by the holomorphic quadratic differential. Then, we can say that holomorphic quadratic differentials determine directions of Teichmüller geodesic rays. There is a one-to-one correspondence between the space of holomorphic quadratic differentials on a Riemann surface and the space of measured foliations with singularities on a same surface. We use both representation depending on the situation. We are interested in the behavior of two geodesic rays near the boundary of the Teichmüller space.

An interesting question is when the Teichmüller distance between the given two Teichmüller geodesic rays are bounded, or diverge. This question is answered completely by Ivanov [Iva01], Lenzhen and Masur [LM10] and Masur [Mas75], [Mas80]. The details are the following. First, Masur showed that if the measured foliations of the given rays are determined by Jenkins-Strebel differentials and topologically equivalent, then the rays are bounded ([Mas75]). Masur also showed that if the measured foliations are uniquely ergodic and topologically equivalent with the condition that they have no simple closed curves consist of saddle connections, then the rays are asymptotic (and so bounded) ([Mas80]). Ivanov showed that if the measured foliations are absolutely continuous, then the rays are bounded, and if the measured foliations have non-zero intersection number, then the rays are divergent ([Iva01]). Finally, Lenzhen and Masur showed that if the measured foliations are not absolutely continuous, or the measured foliations are not topologically equivalent and have zero intersection number, then the rays are divergent ([LM10]).

On the other hand, Farb and Masur [FM10] considered two Teichmüller geodesic rays in the moduli space given by modularly equivalent Jenkins-Strebel differentials. They showed that under some conditions of initial points of two rays, the limit value of the distance between the rays in the moduli space is the distance between the end points of the rays in the boundary of the moduli space. If in addition, the end points coincide, then they are asymptotic.

In Chapter 3, we obtain the explicit limit value of the Teichmüller distance between two Teichmüller geodesic rays which are determined by Jenkins-Strebel differentials having a common end point in the augmented Teichmüller space (Theorem 3.1.1). Furthermore, we also obtain a condition under which these two rays are asymptotic (Corollary 3.1.2). This is similar to a result of Farb and Masur.

In Chapter 4, we obtain the limit value of the distance between any two Teichmüller geodesic rays determined by Jenkins-Strebel differentials (Theorem 4.1.1). This is the generalization of Theorem 3.1.1 in Chapter 3. We also consider the infimum of the limit value in shifting base points of the rays along the geodesics (Corollary 4.1.2). We show that the infimum is represented by two quantities. One is the detour metric between the end points of the rays on the Gardiner-Masur boundary of the Teichmüller space, and the other is the Teichmüller distance between the end points of the rays on the augmented Teichmüller space.

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Chapter 2

Preliminaries

2.1 Teichmüller spaces

Let X be a Riemann surface of type (g, n) with $3g - 3 + n > 0$. The *Teichmüller space* $T(X)$ of X is the set of all equivalence classes $[Y, f]$ of pairs of a Riemann surface Y and a quasiconformal mapping $f : X \to Y$. Two pairs (Y_1, f_1) and (Y_2, f_2) are equivalent if there is a conformal mapping $h : Y_1 \to Y_2$ such that $h \circ f_1$ is homotopic to f_2. The Teichmüller space $T(X)$ has a complete distance, called the *Teichmüller distance* $d_{T(X)}$. For any $p_1 = [Y_1, f_1]$, $p_2 = [Y_2, f_2] \in T(X)$, the distance is defined by

$$d_{T(X)}(p_1, p_2) = \frac{1}{2} \inf_h \log K(h),$$

where h ranges over all quasiconformal mappings $h : Y_1 \to Y_2$ such that $h \circ f_1$ is homotopic to f_2, and $K(h)$ means the maximal quasiconformal dilatation of h.

2.2 Holomorphic quadratic differentials

In this section, we refer to [Str84] for details.

A *holomorphic quadratic differential* q on X is a tensor which is represented locally by $q = q(z)dz^2$ where $q(z)$ is a holomorphic function of the local coordinate $z = x + iy$ on X. We allow holomorphic quadratic differentials to have simple poles at the punctures of X. For each holomorphic quadratic differential $q = q(z)dz^2$, $|q|$ is defined locally by the differential 2-form $|q(z)|dxdy$ on each coordinate z. The norm of q is defined by $\|q\| = \iint_X |q(z)|dxdy$. We treat holomorphic quadratic differentials whose norm are finite. A holomorphic quadratic differential q is called of *unit norm* if it satisfies $\|q\| = 1$. A *critical point* of q is a zero of q or a puncture of X. Each non-zero holomorphic quadratic differential has finitely many critical points. In a neighborhood of any non-critical point of q, there exists a local coordinate ζ on X such that $q = d\zeta^2$. Indeed, let p_0 be a non-critical point, then $q^{\frac{1}{2}} = q(z)^{\frac{1}{2}}dz$ has a single valued holomorphic branch in some neighborhood U around p_0. The new coordinate

$$\zeta(p) = \int_{p_0}^{p} q^{\frac{1}{2}}$$

is defined for any $p \in U$, and this coordinate satisfies $q = d\zeta^2$. We call the coordinate ζ a *q-coordinate* on X. For any two q-coordinates ζ_1, ζ_2 in a common neighborhood U, the equation $\zeta_2 = \pm\zeta_1 + c$ holds where $c \in \mathbb{C}$, because $q = d\zeta_1^2 = d\zeta_2^2$. A smooth path $z = \gamma(t)$ on X is called a *horizontal trajectory* of q if it is a maximal path which satisfies $q(\gamma(t))\dot{\gamma}(t)^2 > 0$. We notice that any horizontal trajectory of q consists of non-critical points of q, and it is represented by a Euclidean horizontal segment in q-coordinates. All horizontal trajectories of q are classified by the following three types. A horizontal trajectory of q is *critical* if it joins critical points of q, *closed* if it is a closed path, *recurrent* otherwise. The recurrent trajectory is dense on a subsurface of X whose boundary consists of critical trajectories of q. Let Γ_q be the set of all critical points and critical trajectories of q. Any component of $X - \Gamma_q$ is an *annulus* which is swept out by closed trajectories of q such that they are homotopic to each other, or a *minimal domain* which consists of infinitely many recurrent trajectories of q. If all components of $X - \Gamma_q$ are annuli, we call q a *Jenkins-Strebel differential.*

2.3 Measured foliations

In this section, we refer to [FLP79] for details.

A *foliation* $F = \{(U_j, z_j)\}_j$ *with singularities* on X is an atlas of X, and is given by the pair of an open covering $\{U_j\}_j$ of $X - \{\text{finite distinguished points}\}$ and each local coordinate $z_j = x_j + iy_j$ on U_j such that the equations $x_k = f_{jk}(z_j)$, $y_k = \pm y_j + c$ holds where $c \in \mathbb{C}$ if $U_j \cap U_k \neq \emptyset$. These distinguished points and punctures of X are called *singularities* of F. They are p-pronged singularities for $p \geq 1$, but the case of $p = 1, 2$ is attained only at the punctures of X. A maximal horizontal segment with respect to $\{z_j\}_j$ is called a *leaf* of F. A *transverse measure* μ of F is given by a measure on the set of all transversal arcs of leaves of F such that if transversal arcs α, β are moved to each other by a homotopy along leaves of the foliation F, then $\mu(\alpha) = \mu(\beta)$. The pair (F, μ) is called a *measured foliation* on X. Let $\mathcal{S}$ be the set of all homotopy classes of non-trivial and non-peripheral simple closed curves on X. For any measured foliation (F, μ) and any $\alpha \in \mathcal{S}$, we can define the *intersection number*

$$i((F, \mu), \alpha) = \inf_{\alpha' \in \alpha} \int_{\alpha'} d\mu,$$

where α' ranges over all simple closed curves in α. Two measured foliations (F_1, μ_1) and (F_2, μ_2) are equivalent if the equation

$$i((F_1, \mu_1), \alpha) = i((F_2, \mu_2), \alpha)$$

holds for any $\alpha \in \mathcal{S}$. We denote by $\mathcal{MF}(X)$ the set of all equivalence classes of measured foliations on X. The set $\mathcal{MF}(X)$ has the week-topology which is induced by intersection number functions in $\mathbb{R}_{\geq 0}^{\mathcal{S}}$. We denote by $[F, \mu]$ the equivalence class of (F, μ). We consider the space of measured foliations $\mathcal{MF}(Y)$ on any other Riemann surface Y of the same type as X. For any homeomorphism $f : X \to Y$, there exists a homeomorphism $f_* : \mathcal{MF}(X) \to \mathcal{MF}(Y)$ which is defined by $f_*([F, \mu]) = [f(F), \mu \circ f^{-1}] \in \mathcal{MF}(Y)$ for any $[F, \mu] \in \mathcal{MF}(X)$. After this, we denote by μ as the equivalence class of a measured foliation $[F, \mu] \in \mathcal{MF}(X)$. We can see that $\mathbb{R}_{\geq 0} \times \mathcal{S} \subset \mathcal{MF}(X)$. For any $b \geq 0$ and any $\gamma \in \mathcal{S}$, the measured foliation $b\gamma$ consists of closed leaves which are homotopic to γ and leaves tend to singular points. These closed leaves produce an annulus of height b, and other leaves are the boundary of it. This measured foliation satisfies $i(b\gamma, \alpha) = bi(\gamma, \alpha)$ for any $\alpha \in \mathcal{S}$. The right side of this equation is a geometric intersection number of simple closed curves. Thurston showed that $\overline{\mathbb{R}_{\geq 0} \times \mathcal{S}} = \mathcal{MF}(X)$. (For instance, we refer to [FLP79].) Then, we can see that intersection number functions are defined on $\mathcal{MF}(X) \times \mathcal{MF}(X)$. (We refer to [Ree81] for details.) We notice that for any homeomorphism $f : X \to Y$ and any $\mu, \nu \in \mathcal{MF}(X)$, the equation $i(f_*(\mu), f_*(\nu)) = i(\mu, \nu)$ holds.

Remark. The set $\mathcal{MF}(X)$ contains the zero-measured foliation, denoted by 0, that is, it has the zero intersection number with all measured foliations. Of course, for any $\mu \in \mathcal{MF}(X)$, we regard the measured foliation 0μ as 0.

2.4 Measured foliations and holomorphic quadratic differentials

For any non-zero holomorphic quadratic differential q on X, we can define the measured foliation $H(q) \in \mathcal{MF}(X)$ consists of all horizontal trajectories of q as leaves and $|dy|$ as a transverse measure where $z = x + iy$ is the q-coordinate. The singularities of $H(q)$ are the critical points of q.

Remark. There is a one-to-one correspondence between the set of all holomorphic quadratic differentials of finite norm on X and $\mathcal{MF}(X)$, we refer to [HM79].

If q has an annulus A in $X - \Gamma_q$, then the restriction to A of the measured foliation $H(q)$ can be written as $H(q)|_A = b\gamma$ where $\gamma \in \mathcal{S}$ corresponds to closed trajectories which sweep out in A, and $b > 0$ is the height of A with respect to the metric $|dy|$. If q has a minimal domain M in $X - \Gamma_q$, then the restriction to M of the measured foliation $H(q)$ can be written as $H(q)|_M = \sum_{i=1}^{p} b_i \mu_i$ where $p > 0$ is bounded by the number which is determined by the topology of M, $b_i \geq 0$ for any $i = 1, \cdots, p$ and $\{\mu_i\}_{i=1}^{p}$ is the set of ergodic transverse measures which are projectively-distinct and pairwise having zero intersection number (that is, for any $i \neq i'$ and any $k \geq 0$, $\mu_{i'} \neq k\mu_i$ and $i(\mu_i, \mu_{i'}) = 0$). A transverse measure μ is called an *ergodic transverse measure* if it is non-zero and cannot be written as a sum of projectively-distinct and non-zero measured foliations. (For ergodicity, we refer to [KH95] for more informations.) Any non-zero holomorphic quadratic differential q has finitely many critical points, so the measured foliation $H(q)$ has finitely many annuli or minimal domains. Therefore, the measured foliation $H(q)$ can be written as

$$H(q) = \sum_{j=1}^{k} b_j G_j,$$

where G_j is (a homotopy class of) a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number, and $b_j > 0$ if $G_j \in \mathcal{S}$, $b_j \geq 0$ if G_j is an ergodic measure. (About this notation, we refer the reader to [Iva92] for more details.) In particular, if q is a Jenkins-Strebel differential, then we can write

$$H(q) = \sum_{j=1}^{k} b_j \gamma_j,$$

where $b_1, \cdots, b_k$ are positive real numbers and $\gamma_1, \cdots, \gamma_k$ are distinct simple closed curves such that $i(\gamma_j, \gamma_{j'}) = 0$ for any $j \neq j'$, and in this situation, we also call that $H(q)$ is *Jenkins-Strebel*. If $X - \Gamma_q$ has only one minimal domain and the measured foliation $H(q)$ is represented by $b\mu$ where $b > 0$ and μ is an ergodic measure, and any other topologically equivalent measured foliation ν on X is represented by $\nu = b'\mu$ where $b' > 0$, then q and $H(q)$ are called *uniquely ergodic*. We come back to the general case and see that

$$i\left(\sum_{j=1}^{k} b_j G_j, \alpha\right) = \sum_{j=1}^{k} b_j i(G_j, \alpha)$$

for any $\alpha \in \mathcal{S}$.

Remark. If $G_j = \mu_j$ is ergodic, we regard G_j as $[G, \mu_j] \in \mathcal{MF}(X)$ where G is an unmeasured foliation which is $H(q)$ without the transverse measure. (μ_j is supported on a minimal domain.) Then, G_j is a measured foliation on the whole surface X.

For any $j = 1, \cdots, k$, we set

$$m_j = \frac{b_j}{i(G_j, V(q))},$$

where the measured foliation $V(q)$ is defined by $H(-q)$. If G_j is a simple closed curve $\gamma_j \in \mathcal{S}$, then $a_j := i(\gamma_j, V(q))$ means the infimum of the horizontal lengths of the simple closed curves in γ_j with respect to the metric $|dx|$, and $m_j = \frac{b_j}{a_j}$ means a modulus of the annulus which is generated by γ_j, that is, the ratio of the height and the circumference of the annulus.

Let q, q' be unit norm holomorphic quadratic differentials on X and $H(q)$, $H(q')$ be measured foliations in $\mathcal{MF}(X)$ constructed by q, q' respectively.

Definition 2.4.1. The pair of holomorphic quadratic differentials q, q' on X or measured foliations $H(q)$, $H(q') \in \mathcal{MF}(X)$ is called *topologically equivalent* if there is a homeomorphism $\alpha : X - \Gamma_q \to X - \Gamma_{q'}$ which is homotopic to the identity such that the leaves of $H(q)$ are mapped to the leaves of $H(q')$. In this situation, we can write the measured foliations as $H(q) = \sum_{j=1}^{k} b_j G_j$, $H(q') = \sum_{j=1}^{k} b'_j G_j$ where G_j is a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number, and b_j, $b'_j > 0$ if $G_j \in \mathcal{S}$, b_j, $b'_j \geq 0$ if G_j is an ergodic measure. The pair of holomorphic quadratic differentials q, q' or measured foliations $H(q)$, $H(q')$ is called *absolutely continuous* if q, q' are topologically equivalent and for the representations $H(q) = \sum_{j=1}^{k} b_j G_j$, $H(q') = \sum_{j=1}^{k} b'_j G_j$, the set of subscripts of non-zero coefficients b_j of $H(q)$ and the one of $H(q')$ coincide. In this situation, we can set b_j, $b'_j > 0$ for any $j = 1, \cdots, k$.

Remark. If two holomorphic quadratic differentials q, q' are Jenkins-Strebel and topologically equivalent, then they are absolutely continuous.

2.5 Teichmüller geodesic rays

Let q be a unit norm holomorphic quadratic differential on X. A quasiconformal mapping f on X is called a *Teichmüller mapping* for q if f has the Beltrami coefficient $-\frac{K(f)-1}{K(f)+1}\frac{\bar{q}}{|q|}$. The existence and uniqueness for Teichmüller mappings are the followings.

Theorem 2.5.1. (Teichmüller's existence theorem) *For any quasiconformal mapping $g : X \to Y$, there is a Teichmüller mapping f which is homotopic to g.*

Theorem 2.5.2. (Teichmüller's uniqueness theorem) *For any quasiconformal mapping $g : X \to Y$ which is homotopic to the Teichmüller mapping f, the inequality $K(f) \leq K(g)$ holds and the equality holds if and only if $f = g$.*

These facts are called Teichmüller's theorem. (We refer to [IT92] for details.) Therefore, a Teichmüller mapping is attained the infimum of the definition of the Teichmüller distance, i.e., for any $p_1 = [Y_1, f_1], p_2 = [Y_2, f_2] \in T(X)$, there is the Teichmüller mapping $h : Y_1 \to Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$ such that $d_{T(X)}(p_1, p_2) = \frac{1}{2}\log K(h)$.

For any $p = [Y, f] \in T(X)$, let q be a unit norm holomorphic quadratic differential on Y and $z = x + iy$ be any q-coordinate. For any $t \geq 0$, let Y_t be a Riemann surface determined by each local coordinate $z_t = e^{-t}x + ie^t y$ and $g_t : Y \to Y_t$ be a Teichmüller mapping which is determined by $z \mapsto z_t$. We assume that $Y_0 = Y$, $g_0 = id_Y$. We set $r(t) = [Y_t, g_t \circ f]$, then by Teichmüller's theorem, this mapping $r : \mathbb{R}_{\geq 0} \to T(X)$ is an isometry and is called the *Teichmüller geodesic ray* on $T(X)$ starting at p and having the initial holomorphic quadratic differential q. If the holomorphic quadratic differential q is of Jenkins-Strebel, the ray r is called a *Jenkins-Strebel ray*.

Definition 2.5.3. Let r, r' be Teichmüller geodesic rays on $T(X)$ starting at $r(0) = [Y, f]$, $r'(0) = [Y', f']$ and having initial unit norm holomorphic quadratic differentials q, q' on Y, Y' respectively. We denote by $H(q) \in \mathcal{MF}(Y)$, $H(q') \in \mathcal{MF}(Y')$ the measured foliations corresponding to q, q' respectively. We suppose that $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q')) \in \mathcal{MF}(X)$ are absolutely continuous, then the measured foliations are written as $f_*^{-1}(H(q)) = \sum_{j=1}^k b_j G_j$, $f_*'^{-1}(H(q')) = \sum_{j=1}^k b'_j G_j$, $H(q) = \sum_{j=1}^k b_j f_*(G_j)$ and $H(q') = \sum_{j=1}^k b'_j f'_*(G_j)$ where $b_1, \cdots, b_k$, $b'_1, \cdots, b'_k$ are positive real numbers and G_j is a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number. We set $m_j = \frac{b_j}{i(f_*(G_j),V(q))}$, $m'_j = \frac{b'_j}{i(f'_*(G_j),V(q'))}$ for any $j = 1, \cdots, k$. In this situation, the given rays r, r' are called *modularly equivalent* if there is $\lambda > 0$ such that $m'_j = \lambda m_j$ for any $j = 1, \cdots, k$.

Definition 2.5.4. The pair of Teichmüller geodesic rays r, r' on $T(X)$ are called *bounded* if there is $M > 0$ such that $d_{T(X)}(r(t), r'(t)) < M$ for any $t \geq 0$, *divergent* if $d_{T(X)}(r(t), r'(t)) \to +\infty$ as $t \to \infty$, and *asymptotic* if there is a choice of initial points $r(0)$, $r'(0)$ such that $d_{T(X)}(r(t), r'(t)) \to 0$ as $t \to \infty$, in other words, for the given rays $r(t)$, $r'(t)$, there is $\alpha \in \mathbb{R}$ such that $d_{T(X)}(r(t), r'(t + \alpha)) \to 0$ as $t \to \infty$.

2.6 The end point of a Jenkins-Strebel ray

In this section, we refer to [Abi77], [HS07] and [IT92].

Definition 2.6.1. (Riemann surfaces with nodes) A connected Hausdorff space R is called a *Riemann surface of type* (g, n) *with nodes* if R satisfies the following conditions:

1. Any $p \in R$ has a neighborhood which is homeomorphic to the unit disk $\mathbb{D}$ or the set $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1, z_1 \cdot z_2 = 0\}$. (In the latter case, p is called a *node* of R. We allow R to have finitely many nodes.)

2. Let $p_1, \cdots, p_k$ be nodes of R. We denote by $R_1, \cdots, R_r$ the connected components of $R - \{p_1, \cdots, p_k\}$. For any $i = 1, \cdots, r$, each R_i is of type (g_i, n_i) which satisfies $2g_i - 2 + n_i > 0$, $n = \sum_{i=1}^r n_i - 2k$ and $g = \sum_{i=1}^r g_i - r + k + 1$.

Remark. The condition (2) means that we get a Riemann surface of type (g, n) without nodes by opening each node of R. All Riemann surfaces of type (g, n) without nodes are included to this definition.

Definition 2.6.2. (Augmented Teichmüller spaces) Let X be a Riemann surface of type (g, n) without nodes which satisfies $3g - 3 + n > 0$. The *augmented Teichmüller space* $\hat{T}(X)$ of X is the set of all equivalence classes $[R, f]$ of pairs of a Riemann surface R of type (g, n) with nodes and a deformation $f : X \to R$ which is a mapping such that it contracts some disjoint loops on X to points (the nodes of R) and is a homeomorphism except on the loops. Two pairs (R_1, f_1) and (R_2, f_2) are equivalent if there is a biholomorphic mapping $h : R_1 \to R_2$ such that f_2 is homotopic to $h \circ f_1$. Here, for Riemann surfaces with nodes R and S, a homeomorphism $f : R \to S$ is called biholomorphic if each restricted mapping of f which maps a component of $R - \{\text{nodes of } R\}$ onto a component of $S - \{\text{nodes of } S\}$ is biholomorphic. A topology on $\hat{T}(X)$ is defined by the following neighborhoods. For any compact neighborhood V of the set of nodes in R and any $\varepsilon > 0$, a neighborhood $U_{V,\varepsilon}$ of a point $[R, f]$ is defined by

$$U_{V,\varepsilon} = \{[S, g] \in \hat{T}(X) \mid \text{there is a deformation } h : S \to R \text{ which is}$$
$$(1 + \varepsilon)\text{-quasiconformal on } h^{-1}(R - V) \text{ such that } f \text{ is homotopic to } h \circ g\}.$$

Let r be the Jenkins-Strebel ray on $T(X)$ starting at $r(0) = [Y, f]$ and having an initial unit norm Jenkins-Strebel differential q on Y. We set $r(t) = [Y_t, g_t \circ f]$ for any $t \geq 0$. We consider the end point of r in $\hat{T}(X)$. We denote by $H(q) = \sum_{j=1}^{k} b_j \gamma_j$ the measured foliation constructed by q. The surface $Y - \Gamma_q$ consists of annuli corresponding to $\gamma_1, \cdots, \gamma_k$, and let $m_1, \cdots, m_k$ be moduli of these annuli respectively. By q-coordinates, the annuli are represented by Euclidean cylinders. For each cylinder, we use polar coordinates and describe it as two round annuli:

$$A_j^1(0) = A_j^2(0) = \{w \in \mathbb{C} \mid \exp(-m_j \pi) \leq |w| < 1\}.$$

We glue $A_j^1(0)$ and $A_j^2(0)$ at the line of these inner boundary $|w| = \exp(-m_j\pi)$ by the mapping $w \mapsto \frac{\exp(-2m_j\pi)}{w}$ and denote by $A_j(0)$ the resulting surface. We obtain the base surface Y after gluing $\overline{A_1(0)}, \cdots, \overline{A_k(0)}$ by the gluing mappings which are determined by q. The Teichmüller mapping $g_t : Y \to Y_t$ is represented by $w = re^{i\theta} \mapsto r^{\exp(2t)} e^{i\theta}$ for any $t \geq 0$. We fix any $t \geq 0$ and define

$$A_j^1(t) = A_j^2(t) = \{w \in \mathbb{C} \mid \exp(-e^{2t} m_j \pi) \leq |w| < 1\}.$$

The gluing mapping of $A_j^1(t)$ onto $A_j^2(t)$ is $w \mapsto \frac{\exp(-2e^{2t} m_j \pi)}{w}$ in their inner boundary $|w| = \exp(-e^{2t} m_j \pi)$, and we denote by $A_j(t)$ the resulting surface. The surface Y_t is obtained by $\overline{A_1(t)}, \cdots, \overline{A_k(t)}$ with the same gluing mappings as in the case of $t = 0$.

Remark. We notice that the modulus of $A_j(t)$ is equal to $e^{2t} m_j$ for any $t \geq 0$ and any $j = 1, \cdots, k$.

We consider the limit of $A_j(t)$ as $t \to \infty$ for any $j = 1, \cdots, k$. We set $A_j^1(\infty) = A_j^2(\infty) = \{w \in \mathbb{C} \mid 0 < |w| < 1\}$, and denote by $\{pt\}$ the set which consists of an arbitrary point. The disjoint union $A_j(\infty) = A_j^1(\infty) \cup A_j^2(\infty) \cup \{pt\}$ becomes a complex cone by the following chart:

$$\Phi : A_j(\infty) \to \{(w^1, w^2) \in \mathbb{C}^2 \mid |w^1| < 1, |w^2| < 1, w^1 \cdot w^2 = 0\},$$

$$\Phi|_{A_j^1(\infty)}(w) = (w, 0),\ \Phi|_{A_j^2(\infty)}(w) = (0, w),\ \Phi(pt) = (0, 0)$$

for any $j = 1, \cdots, k$. We denote by Y_∞ the surface constructed by $\overline{A_1(\infty)}, \cdots, \overline{A_k(\infty)}$ with the same gluing mappings as in the case of $t = 0$. The mapping $g_\infty : Y \to Y_\infty$ is constructed by the mapping of $A_j^l(0)$ onto $A_j^l(\infty) \cup \{pt\}$ by $w = re^{i\theta} \mapsto h_j(r)e^{i\theta}$ where $h_j : [\exp(-m_j\pi), 1) \to [0, 1)$ is an arbitrary monotonously increasing diffeomorphism for any $j = 1, \cdots, k$ and $l = 1, 2$. The homotopy class of g_∞ is independent of the choices of h_j for any $j = 1, \cdots, k$. Then, we obtain $[Y_\infty, g_\infty \circ f]$ in $\hat{T}(X)$ and denote it by $r(\infty)$. From the definition of a neighborhood of $[Y_\infty, g_\infty \circ f]$, the following proposition holds.

Proposition 2.6.3. ([HS07]) *The Jenkins-Strebel ray $r(t) = [Y_t, g_t \circ f]$ on $T(X)$ starting at $r(0) = [Y, f]$ and having the initial unit norm Jenkins-Strebel differential q on Y converges to a point $r(\infty) = [Y_\infty, g_\infty \circ f]$ in $\hat{T}(X)$.*

Remark. We can reconstruct the surface Y_t from Y_∞ for any $t \geq 0$. First, for any $j = 1, \cdots, k$ and $l = 1, 2$, we remove the punctured disk $\{w \in \mathbb{C} \mid 0 < |w| < \exp(-e^{2t}m_j\pi)\}$ from $A_j^l(\infty)$ of Y_∞. Then, the interior of each resulting annulus is biholomorphic to the interior of $A_j^l(t)$. We obtain Y_t by gluing $\{\overline{A_j^l(t)}\}_{j=1,\cdots,k}^{l=1,2}$ with the suitable gluing mappings.

2.7 Extremal lengths

Let ρ be a locally L^1-measurable conformal metric on a Riemann surface Y, i.e., it is represented by the form $\rho = \rho(z)|dz|$ on any local coordinate z of Y where $\rho(z)$ is a non-negative measurable function of z such that the equation $\rho(z)|dz| = \rho(w)|dw|$ holds for any local coordinates z, w on a common neighborhood of Y. For any $\gamma \in \mathcal{S}$, we define the ρ-length of γ on Y and the ρ-area of Y by

$$l_\rho(\gamma) = \inf_{\gamma' \in \gamma} \int_{\gamma'} \rho(z)|dz|,$$
$$A_\rho = \iint_Y \rho(z)^2 dxdy$$

respectively. The *extremal length* $\mathrm{Ext}_Y(\gamma)$ of γ on Y is defined by the following:

$$\mathrm{Ext}_Y(\gamma) = \sup_\rho \frac{l_\rho(\gamma)^2}{A_\rho},$$

where ρ ranges over all locally L^1-measurable conformal metrics on Y such that $0 < A_\rho < \infty$.

For any $p = [Y, f] \in T(X)$ and any $\alpha \in \mathcal{S}$ on X, we define

$$\mathrm{Ext}_p(\alpha) = \mathrm{Ext}_Y(f_*(\alpha)).$$

Theorem 2.7.1. ([Ker80]) *For any Riemann surface Y, each extremal length on Y extends continuously on $\mathcal{MF}(Y)$ such that the equation $\mathrm{Ext}_Y(t\nu) = t^2 \mathrm{Ext}_Y(\nu)$ holds for any $t \geq 0$ and any $\nu \in \mathcal{MF}(Y)$.*

Hence, for any $p = [Y, f] \in T(X)$ and any $\mu \in \mathcal{MF}(X)$, we define $\mathrm{Ext}_p(\mu) = \mathrm{Ext}_Y(f_*(\mu))$ and the equation $\mathrm{Ext}_p(t\mu) = t^2 \mathrm{Ext}_p(\mu)$ holds for any $t \geq 0$. We notice that for any $p \in T(X)$ and any $\mu \in \mathcal{MF}(X) - \{0\}$, the extremal length $\mathrm{Ext}_p(\mu)$ is positive. We set $\mathcal{MF}(X)^* = \mathcal{MF}(X) - \{0\}$.

Theorem 2.7.2. (Kerckhoff's formula for the Teichmüller distance [Ker80]) *For any $p_1, p_2 \in T(X)$, the Teichmüller distance between p_1 and p_2 is represented by*

$$d_{T(X)}(p_1, p_2) = \frac{1}{2} \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\mathrm{Ext}_{p_2}(\mu)}{\mathrm{Ext}_{p_1}(\mu)}.$$

Kerckhoff's formula is useful for finding lower bounds of the Teichmüller distance.

Chapter 3

On behavior of pairs of Teichmüller geodesic rays

3.1 Introduction

In this chapter, we consider two Teichmüller geodesic rays $r, r' : \mathbb{R}_{\geq 0} \to T(X)$ on the Teichmüller space $T(X)$ starting at $r(0) = [Y, f]$, $r'(0) = [Y', f']$ and having initial unit norm Jenkins-Strebel differentials q, q' on Y, Y' respectively. Let $r(\infty)$, $r'(\infty)$ be the end points of r, r' in the augmented Teichmüller space respectively. Let $H(q)$, $H(q')$ be measured foliations corresponding to q, q' respectively, and we suppose that the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are topologically equivalent. We denote by $m_1, \cdots, m_k$, $m'_1, \cdots, m'_k$ the moduli of annuli on Y, Y' determined by $H(q)$, $H(q')$ respectively. Under the above assumption, we determine the limit value of the Teichmüller distance between two points $r(t)$, $r'(t)$.

Theorem 3.1.1. *If $r(\infty) = r'(\infty)$, then*

$$\lim_{t \to \infty} d_{T(X)}(r(t), r'(t)) = \frac{1}{2} \log \max_{j=1,\cdots,k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}.$$

Furthermore, we obtain the following condition which is equivalent to the asymptoticity of two Jenkins-Strebel rays. This is similar to a theorem of Farb and Masur [FM10].

Corollary 3.1.2. *For any two Jenkins-Strebel rays r, r', they are asymptotic if and only if r, r' are modularly equivalent and $r(\infty) = r'(\infty)$.*

In Theorem 3.1.1, the limit value depends on the modulus of each annulus which is determined by the initial holomorphic quadratic differentials on the initial points $r(0)$, $r'(0)$ of the given rays. Therefore, this value depends on the choice of the initial points $r(0)$, $r'(0)$. We consider the minimum of the limit value of the Teichmüller distance between two rays when we shift the initial points of the rays. We show that the minimum is represented by the detour metric δ between the end points of the rays in the Gardiner-Masur boundary of $T(X)$.

Proposition 3.1.3. *Under the assumption of Theorem 3.1.1, the minimum of the limit value of the distance between the given rays $r(t)$, $r'(t)$ when we shift the initial points $r(0)$, $r'(0)$ is given by $\frac{1}{2}\delta(\xi, \xi')$ where ξ, ξ' are the end points of the rays r, r' in the Gardiner-Masur boundary of $T(X)$ respectively.*

3.2 Proof of Theorem

In this section, we prove Theorem 3.1.1 which is our main theorem. Let r, r' be Jenkins-Strebel rays on $T(X)$ starting at $r(0) = [Y, f]$, $r(0)' = [Y', f']$ and having initial unit norm Jenkins-Strebel differentials q, q' on Y, Y' respectively. Let $r(\infty)$, $r'(\infty)$ be the end points of r, r' in the augmented Teichmüller space $\hat{T}(X)$ respectively. We denote by $H(q) \in \mathcal{MF}(Y)$, $H(q') \in \mathcal{MF}(Y')$ the corresponding measured foliations and suppose that the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q')) \in \mathcal{MF}(X)$ are topologically equivalent. In this situation, we can write $f_*^{-1}(H(q)) = \sum_{j=1}^k b_j\gamma_j$, $f_*'^{-1}(H(q')) = \sum_{j=1}^k b_j'\gamma_j$ where $b_1, \cdots, b_k$, $b_1', \cdots, b_k'$ are positive real numbers and $\gamma_1, \cdots, \gamma_k$ are distinct and pairwise having zero intersection number. We denote by $m_j = \frac{b_j}{i(f_*(\gamma_j), V(q))}$, $m_j' = \frac{b_j'}{i(f_*'(\gamma_j), V(q'))}$ the moduli of the annuli of $H(q)$, $H(q')$ corresponding to $f_*(\gamma_j)$, $f_*'(\gamma_j)$ for any $j = 1, \cdots, k$.

Theorem 3.1.1. *If $r(\infty) = r'(\infty)$, then*

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) = \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m_j'}{m_j}, \frac{m_j}{m_j'}\right\}.$$

We represent the Jenkins-Strebel rays by $r(t) = [Y_t, g_t \circ f]$, $r'(t) = [Y_t', g_t' \circ f']$ for any $t \geq 0$, and their end points by $r(\infty) = [Y_\infty, g_\infty \circ f]$, $r'(\infty) = [Y_\infty', g_\infty' \circ f']$ respectively. Since the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are topologically equivalent, there is a homeomorphism $\alpha : X - \Gamma_{f_*^{-1}(H(q))} \to X - \Gamma_{f_*'^{-1}(H(q'))}$ which is homotopic to the identity such that the leaves of $f_*^{-1}(H(q))$ are mapped to the leaves of $f_*'^{-1}(H(q'))$. Then, the mapping $f' \circ \alpha \circ f^{-1}$ which is homotopic to $f' \circ f^{-1}$ maps the leaves of $H(q)$ to the leaves of $H(q')$. We denote by $A_j(0)$, $A_j'(0)$ the annuli corresponding to $f_*(\gamma_j)$, $f_*'(\gamma_j)$ respectively, for any $j = 1, \cdots, k$. They are the same as in §2.6. Then, the equations $f_*' \circ \alpha_* \circ f_*^{-1}(f_*(\gamma_j)) = f_*'(\gamma_j)$, $f' \circ \alpha \circ f^{-1}(A_j(0)) = A_j'(0)$ hold for any $j = 1, \cdots, k$. For any $t \geq 0$ and any $j = 1, \cdots, k$, the annuli $A_j(t)$, $A_j'(t)$ can be determined the same as in §2.6, and we can see that $(g_t' \circ f') \circ \alpha \circ (g_t \circ f)^{-1}(A_j(t)) = A_j'(t)$.

In order to prove the equation of Theorem 3.1.1, we consider the upper and lower estimates of the limit supremum and limit infimum of the distance between the rays respectively.

3.2.1 Upper estimate

First, we show that

$$\limsup_{t\to\infty} d_{T(X)}(r(t), r'(t)) \le \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}.$$

The idea of the following lemma comes from [Gup11].

Lemma 3.2.1. *Let us choose $0<\varepsilon<1$ arbitrary. Then, for any sufficiently large t, there is a quasiconformal mapping $F_t : Y_t \to Y'_t$ which is homotopic to $(g'_t\circ f')\circ(g_t\circ f)^{-1}$ such that the inequality $\lim_{t\to\infty} K(F_t) < \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}+\varepsilon$ holds.*

Proof. We set $M_j = \frac{m'_j}{m_j}$ for any $j=1,\cdots,k$. By the equation $r(\infty)=r'(\infty)$, there exists a biholomorphic mapping $h : Y_\infty \to Y'_\infty$ such that $h\circ g_\infty \circ f$ is homotopic to $g'_\infty \circ f'$. From §2.6, we can write

$$Y_\infty = \bigcup_{j=1}^{k} \overline{A_j^1(\infty)} \cup \overline{A_j^2(\infty)},$$
$$Y'_\infty = \bigcup_{j=1}^{k} \overline{A_j'^1(\infty)} \cup \overline{A_j'^2(\infty)},$$

where $A_j^l(\infty)$, $A_j'^l(\infty)$ are the punctured disks $\mathbb{D}^* = \{z\in\mathbb{C} \mid 0<|z|<1\}$ for any $j=1,\cdots,k$ and $l=1,2$. Now, we fix any $j=1,\cdots,k$ and $l=1,2$. We set $h_j^l = h|_{A_j^l(\infty)} : A_j^l(\infty) \to h(A_j^l(\infty)) \subset Y'_\infty$. Since h is a biholomorphic mapping, we can set $h_j^l(0)=0$ and $\frac{dh_j^l(z)}{dz}\big|_{z=0}\neq 0$. We describe $h_j^l(z) = c_j^l z + c_{j,2}^l z^2 + \cdots = c_j^l z + \psi_j^l(z)$ where $c_j^l\neq 0$, $-\pi < \arg c_j^1 \le \pi$ and $-\pi \le \arg c_j^2 < \pi$. For any $t\ge 0$, we set $\delta_j(t) = \exp(-e^{2t}m_j\pi)$, $\delta'_j(t) = \exp(-e^{2t}m'_j\pi)$, then $\delta'_j(t) = \delta_j(t)^{M_j}$. After this, we assume that $A_j^l(t) = \mathbb{D}^* - \mathbb{D}_{\delta_j(t)} = \{z\in\mathbb{C} \mid \delta_j(t)\le |z|<1\}$ and $A_j'^l(t) = \mathbb{D}^* - \mathbb{D}_{\delta'_j(t)} = \{z\in\mathbb{C} \mid \delta'_j(t)\le|z|<1\}$ for any $t\ge 0$. For sufficiently large t, we construct a quasiconformal mapping $F_{j,t}^l : \mathbb{D}^* - \mathbb{D}_{\delta_j(t)} \to h_j^l(\mathbb{D}^*) - \mathbb{D}_{\delta'_j(t)}$. We consider the following three cases (1), (2) and (3).

(1) In the case of $M_j>1$, we take X_j as

$$X_j < \frac{\log \frac{\varepsilon}{M_j+\varepsilon-1}}{\log M_j} < 0.$$

This is equivalent to

$$M_j^{X_j} < \frac{\varepsilon}{M_j + \varepsilon - 1} < 1$$

and

$$\frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon.$$

We take sufficiently large t such that the inequality $\delta_j(t)^{M_j} < |c_j^l|\delta_j(t)^{M_j^{X_j}}$ holds. We set $\Delta_j(t) = \delta_j(t)^{M_j^{X_j}}$. We construct $F_{j,t}^l$ by the following:

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) & (\delta_j(t) \leq |z| \leq \Delta_j(t)) & \text{(i)} \\ Q_{j,t}^l(z) & (\Delta_j(t) \leq |z| \leq 2\Delta_j(t)) & \text{(ii)} \\ h_j^l(z) & (2\Delta_j(t) \leq |z| < 1) & \text{(iii)} \end{cases}$$

(i) In $\delta_j(t) \leq |z| \leq \Delta_j(t)$, we set

$$P_{j,t}^l(z) = \Delta_j(t)^{\frac{1-M_j}{1-M_j^{X_j}}} \cdot c_j^{l\ \frac{1}{1-M_j^{X_j}} + \frac{\log|z|}{\log\Delta_j(t) - \log\delta_j(t)}} \cdot |z|^{-\frac{1-M_j}{1-M_j^{X_j}}} \cdot z$$

which satisfies $P_{j,t}^l(z) = \delta_j(t)^{M_j - 1} \cdot z$ on $|z| = \delta_j(t)$, $P_{j,t}^l(z) = c_j^l z$ on $|z| = \Delta_j(t)$. The mapping $P_{j,t}^l$ is conjugate to a one-to-one affine mapping by $\log z$. Then, $P_{j,t}^l$ is a quasiconformal mapping, and its dilatation is the following:

$$K(P_{j,t}^l) = \frac{\left|\frac{\log c_j^l}{2(\log\Delta_j(t) - \log\delta_j(t))} + \frac{\alpha_j}{2} + 1\right| + \left|\frac{\log c_j^l}{2(\log\Delta_j(t) - \log\delta_j(t))} + \frac{\alpha_j}{2}\right|}{\left|\frac{\log c_j^l}{2(\log\Delta_j(t) - \log\delta_j(t))} + \frac{\alpha_j}{2} + 1\right| - \left|\frac{\log c_j^l}{2(\log\Delta_j(t) - \log\delta_j(t))} + \frac{\alpha_j}{2}\right|},$$

where

$$\alpha_j = -\frac{1 - M_j}{1 - M_j^{X_j}}.$$

We see that $\log\Delta_j(t) - \log\delta_j(t) = (M_j^{X_j} - 1)\log\delta_j(t) \to +\infty$ and

$$K(P_{j,t}^l) \to \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon$$

as $t \to \infty$.

(ii) In $\Delta_j(t) \leq |z| \leq 2\Delta_j(t)$, we set

$$Q_{j,t}^l(z) = c_j^l z + \phi_{\Delta_j(t)}(|z|)\psi_j^l(z),$$

where $\phi_{\Delta_j(t)} : [\Delta_j(t), 2\Delta_j(t)] \to [0,1]$ is defined by

$$\phi_{\Delta_j(t)}(|z|) = \frac{|z|}{\Delta_j(t)} - 1.$$

This function satisfies $Q^l_{j,t}(z) = c^l_j z$ on $|z| = \Delta_j(t)$, $Q^l_{j,t}(z) = h^l_j(z)$ on $|z| = 2\Delta_j(t)$. We consider the partial derivatives of $Q^l_{j,t}$,

$$\partial_{\bar{z}} Q^l_{j,t} = \frac{1}{2\Delta_j(t)} z^{\frac{1}{2}} \bar{z}^{-\frac{1}{2}} \psi^l_j(z),$$
$$\partial_z Q^l_{j,t} = c^l_j + \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar{z}^{\frac{1}{2}} \psi^l_j(z) + \phi_{\Delta(t)}(|z|) \frac{d\psi^l_j(z)}{dz}.$$

These partial derivatives are continuous in $\Delta_j(t) \leq |z| \leq 2\Delta_j(t)$. There is $C > 0$ such that $|\psi^l_j(z)| \leq C\Delta_j(t)^2$ for sufficiently large t. We see that

$$\left| \frac{1}{2\Delta_j(t)} z^{\frac{1}{2}} \bar{z}^{-\frac{1}{2}} \psi^l_j(z) \right| = \left| \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} \bar{z}^{\frac{1}{2}} \psi^l_j(z) \right| = \frac{|\psi^l_j(z)|}{2\Delta_j(t)} \leq \frac{C\Delta_j(t)}{2} \to 0$$

as $t \to \infty$. Then, $|\partial_{\bar{z}} Q^l_{j,t}| \to 0$, $|\partial_z Q^l_{j,t}| \to |c^l_j| \neq 0$ as $t \to \infty$. For sufficiently large t, $\operatorname{Jac} Q^l_{j,t} = |\partial_z Q^l_{j,t}|^2 - |\partial_{\bar{z}} Q^l_{j,t}|^2 > 0$. Hence, $Q^l_{j,t}$ is a local C^1-diffeomorphism. We denote by D the closed set whose boundary consists of two components $Q^l_{j,t}(\{|z| = \Delta_j(t)\}) = \{|w| = |c^l_j|\Delta_j(t)\}$ and $Q^l_{j,t}(\{|z| = 2\Delta_j(t)\}) = h^l_j(\{|z| = 2\Delta_j(t)\})$, and its fundamental group is $\pi_1(D) = \mathbb{Z}$. The equation $Q^l_{j,t}(\{\Delta_j(t) \leq |z| \leq 2\Delta_j(t)\}) = D$ holds because $Q^l_{j,t}$ is a local C^1-diffeomorphism with the above boundary conditions. We regard the mapping $Q^l_{j,t} : \{\Delta_j(t) \leq |z| \leq 2\Delta_j(t)\} \to D$ as a covering. Let $Q^l_{j,t*} : \pi_1(\{\Delta_j(t) \leq |z| \leq 2\Delta_j(t)\}) \to \pi_1(D)$ be the group homomorphism induced by $Q^l_{j,t}$. We see that $Q^l_{j,t*}(\pi_1(\{\Delta_j(t) \leq |z| \leq 2\Delta_j(t)\})) = \mathbb{Z} \triangleleft \pi_1(D)$ because $Q^l_{j,t}(z) = c^l_j z$ on $|z| = \Delta_j(t)$. Then, the covering $Q^l_{j,t}$ is a regular covering, and its covering transformation group is $\mathbb{Z}/\mathbb{Z} = 1$. Therefore, $Q^l_{j,t}$ is a C^1-diffeomorphism. By the partial derivatives of $Q^l_{j,t}$, for sufficiently large t, it is a quasiconformal mapping such that its dilatation holds $K(Q^l_{j,t}) \to 1$ as $t \to \infty$.

(iii) In $2\Delta_j(t) \leq |z| < 1$, $F^l_{j,t}(z) = h^l_j(z)$ and $K(h^l_j) = 1$.

Therefore, for sufficiently large t, we obtain the quasiconformal mapping $F^l_{j,t}$ such that

$$K(F^l_{j,t}) = \max\{K(P^l_{j,t}), K(Q^l_{j,t})\} \to \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon$$

as $t \to \infty$.

(2) In the case of $M_j < 1$, we take X_j as

$$X_j > \frac{\log \frac{M_j \varepsilon}{\frac{1}{M_j} - 1 + \varepsilon}}{\log M_j} > 2.$$

This is equivalent to

$$M_j^{X_j} < \frac{M_j \varepsilon}{\frac{1}{M_j} - 1 + \varepsilon} < M_j^2$$

and

$$\frac{1 - M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j} + \varepsilon.$$

We take sufficiently large t such that the inequality $\delta_j(t)^{M_j} < |c_j^l| \delta_j(t)^{M_j^{X_j}}$ holds. We also set $\Delta_j(t) = \delta_j(t)^{M_j^{X_j}}$, and also construct $F_{j,t}^l$ by the following:

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) & (\delta_j(t) \le |z| \le \Delta_j(t)) \\ Q_{j,t}^l(z) & (\Delta_j(t) \le |z| \le 2\Delta_j(t)) \\ h_j^l(z) & (2\Delta_j(t) \le |z| < 1) \end{cases}$$

The functions $P_{j,t}^l$, $Q_{j,t}^l$ have the same notations as in the case of (1). The difference is only the dilatation of $P_{j,t}^l$. In this case, we see that

$$K(P_{j,t}^l) \to \frac{1 - M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j} + \varepsilon$$

as $t \to \infty$. Similarly as in the case of (1), for sufficiently large t, we obtain the quasiconformal mapping $F_{j,t}^l$ such that

$$K(F_{j,t}^l) = \max\{K(P_{j,t}^l), K(Q_{j,t}^l)\} \to \frac{1 - M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j} + \varepsilon$$

as $t \to \infty$.

(3) In the case of $M_j = 1$, we take sufficiently large t such that the inequality $\delta_j(t) < |c_j^l|\delta_j(t)^{\frac{1}{2}}$ holds and set $\Delta_j(t) = \delta_j(t)^{\frac{1}{2}}$. We set

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) = {c_j^l}^{2\left(1-\frac{\log|z|}{\log\delta_j(t)}\right)} z & (\delta_j(t) \le |z| \le \Delta_j(t)) \\ Q_{j,t}^l(z) & (\Delta_j(t) \le |z| \le 2\Delta_j(t)) \\ h_j^l(z) & (2\Delta_j(t) \le |z| < 1) \end{cases}$$

The functions $P_{j,t}^l$, $Q_{j,t}^l$ are constructed similarly as in the case of (1). On $P_{j,t}^l$, the above notation is obtained by changing M_j, $M_j^{X_j}$ to 1, $\frac{1}{2}$ respectively, in (i) of the case of (1). In this time,

$$K(P_{j,t}^l) = \frac{\left|1 - \frac{\log c_j^l}{\log\delta_j(t)}\right| + \left|\frac{\log c_j^l}{\log\delta_j(t)}\right|}{\left|1 - \frac{\log c_j^l}{\log\delta_j(t)}\right| - \left|\frac{\log c_j^l}{\log\delta_j(t)}\right|}$$

and then, $K(P_{j,t}^l) \to 1$ as $t \to \infty$. The function $Q_{j,t}^l$ also satisfying $K(Q_{j,t}^l) \to 1$ as $t \to \infty$. Therefore, for sufficiently large t, the quasiconformal mapping $F_{j,t}^l$ satisfies

$$K(F_{j,t}^l) = \max\{K(P_{j,t}^l), K(Q_{j,t}^l)\} \to 1$$

as $t \to \infty$.

Thus, by (1), (2) and (3), for sufficiently large t, we construct the quasiconformal mapping $F_t : Y_t \to Y_t'$ by gluing $\{F_{j,t}^l\}_{j=1,\cdots,k}^{l=1,2}$. For any case of (1), (2) and (3), each h_j^l is homotopic to $(g_t' \circ f') \circ (g_t \circ f)^{-1}$. Each $Q_{j,t}^l$ satisfies $K(Q_{j,t}^l) \to 1$ as $t \to \infty$ and the domain $\{\Delta_j(t) < |z| < 2\Delta_j(t)\}$ has the constant modulus for any t. Each $P_{j,t}^l$ produces the twist of angle $\arg c_j^l$ in the domain $\{\delta_j(t) < |z| < \Delta_j(t)\}$ and satisfies $|\arg c_j^1 + \arg c_j^2| < 2\pi$. Therefore, for sufficiently large t, the mapping F_t is homotopic to $(g_t' \circ f') \circ (g_t \circ f)^{-1}$. We conclude that

$$\lim_{t\to\infty} K(F_t) = \lim_{t\to\infty} \max_{j=1,\cdots,k, l=1,2} K(F_{j,t}^l) < \max_{j=1,\cdots,k} \left\{M_j, \frac{1}{M_j}\right\} + \varepsilon.$$

□

Therefore, by this lemma, for any sufficiently large t, the inequality

$$\limsup_{t\to\infty} d_{T(X)}(r(t), r'(t)) \le \lim_{t\to\infty} \frac{1}{2}\log K(F_t) < \frac{1}{2}\log\left(\max_{j=1,\cdots,k}\left\{\frac{m_j'}{m_j}, \frac{m_j}{m_j'}\right\} + \varepsilon\right)$$

holds. Since ε is arbitrary, we are done.

3.2.2 Lower estimate

Next, we show that

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \geq \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m'_j}{m_j}, \frac{m_j}{m'_j}\right\}. \tag{3.1}$$

We use the following theorems. Let r be a Teichmüller geodesic ray on $T(X)$ starting at $r(0) = [Y, f]$ and having an initial unit norm holomorphic quadratic differential q on Y with the corresponding measured foliation $H(q) = \sum_{j=1}^{k} b_j f_*(G_j) \in \mathcal{MF}(Y)$ where $b_1, \cdots, b_k$ are positive real numbers and G_j is a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number. We set $m_j = \frac{b_j}{i(f_*(G_j), V(q))}$ for any $j = 1, \cdots, k$.

Theorem 3.2.2. ([Wal12]) *If we set*

$$\mathcal{E}_r(\mu) = \left\{\sum_{j=1}^{k} m_j i(G_j, \mu)^2\right\}^{\frac{1}{2}}$$

for any $\mu \in \mathcal{MF}(X)$, then, the equation

$$\lim_{t\to\infty} e^{-2t}\,\mathrm{Ext}_{r(t)}(\mu) = \sum_{j=1}^{k} m_j i(f_*(G_j), f_*(\mu))^2 = \mathcal{E}_r(\mu)^2$$

holds.

Furthermore, let r' be a Teichmüller geodesic ray on $T(X)$ starting at $r'(0) = [Y', f']$ and having an initial unit norm holomorphic quadratic differential q' on Y', and $H(q') \in \mathcal{MF}(Y')$ be the corresponding measured foliation. We set $Z = \{\mu \in \mathcal{MF}(X)^* |\ \mathcal{E}_r(\mu) = \mathcal{E}_{r'}(\mu) = 0\}$.

Theorem 3.2.3. ([Wal12]) *If we can write $H(q') = \sum_{j=1}^{k} b'_j f'_*(G_j)$ where $b'_j \geq 0$, and set $m'_j = \frac{b'_j}{i(f'_*(G_j), V(q'))}$ for any $j = 1, \cdots, k$, then, the equation*

$$\sup_{\mu\in\mathcal{MF}(X)^* - Z} \frac{\mathcal{E}_{r'}(\mu)^2}{\mathcal{E}_r(\mu)^2} = \max_{j=1,\cdots,k} \frac{m'_j}{m_j}$$

holds. Otherwise, the supremum is $+\infty$.

Remark. If $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are absolutely continuous, then $\sup \frac{\mathcal{E}_{r'}(\mu)^2}{\mathcal{E}_r(\mu)^2}$ and $\sup \frac{\mathcal{E}_r(\mu)^2}{\mathcal{E}_{r'}(\mu)^2}$ are both finite.

We notice that the desired inequality (3.1) holds for the rays r, r' which satisfy that $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are absolutely continuous. So, we do not require the conditions that the rays are Jenkins-Strebel and $r(\infty) = r'(\infty)$. The general case is used on Proposition 3.3.15. We write $H(q) = \sum_{j=1}^k b_j f_*(G_j)$, $H(q') = \sum_{j=1}^k b_j' f_*'(G_j)$ where b_j, $b_j' > 0$ and set $m_j = \frac{b_j}{i(f_*(G_j), V(q))}$, $m_j' = \frac{b_j'}{i(f_*'(G_j), V(q'))}$ for any $j = 1, \cdots, k$. Therefore, in our case, by Kerckhoff's formula and the above two theorems,

$$\begin{aligned}
\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) &= \liminf_{t\to\infty} \frac{1}{2}\log \sup_{\mu\in\mathcal{MF}(X)^*} \frac{\operatorname{Ext}_{r'(t)}(\mu)}{\operatorname{Ext}_{r(t)}(\mu)} \\
&\geq \frac{1}{2}\log \sup_{\mu\in\mathcal{MF}(X)^*-Z} \liminf_{t\to\infty} \frac{e^{-2t}\operatorname{Ext}_{r'(t)}(\mu)}{e^{-2t}\operatorname{Ext}_{r(t)}(\mu)} \\
&= \frac{1}{2}\log \max_{j=1,\cdots,k} \frac{m_j'}{m_j}.
\end{aligned}$$

This inequality comes from the fact that the limit of the supremum is greater than or equal to the supremum of the limit. Because of the symmetry of the distance, the inequality (3.1) holds.

Remark. If $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are not absolutely continuous, then

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) = +\infty.$$

Proof of Theorem 3.1.1. Because of the upper and lower estimates, we obtain the desired equation. □

Corollary 3.1.2. *For any two Jenkins-Strebel rays r, r', they are asymptotic if and only if r, r' are modularly equivalent and $r(\infty) = r'(\infty)$.*

Proof. Under the assumption of Theorem 3.1.1, if in addition the given rays r, r' are modularly equivalent, there is $\lambda > 0$ such that $m_j' = \lambda m_j$ for any $j = 1, \cdots, k$. Then, for $\alpha = -\frac{1}{2}\log\lambda$,

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t+\alpha)) = \frac{1}{2}\log \max_{j=1\cdots,k} \left\{ \frac{e^{2\alpha}m_j'}{m_j}, \frac{m_j}{e^{2\alpha}m_j'} \right\} = \frac{1}{2}\log 1 = 0.$$

This means that the rays are asymptotic. Conversely, if two Jenkins-Strebel rays r, r' are asymptotic, we can set

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) = 0$$

without loss of generality. From the above remark, $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are absolutely continuous. By the inequality (3.1), $m_j = m'_j$ for any $j = 1, \cdots, k$. Therefore, the rays are modularly equivalent. Next, we show that the correspondence between the end points $r(\infty)$, $r'(\infty)$. The proof is similar to Proposition 2.6.3 ([HS07]). For any $\varepsilon > 0$ and any compact neighborhood V of the set of nodes in Y_∞, we recall the neighborhood $U_{V,\varepsilon}(r(\infty))$ of $r(\infty)$:

$U_{V,\varepsilon}(r(\infty)) = \{[S, g] \in \hat{T}(X) \mid$ there is a deformation $h : S \to Y_\infty$ which is $(1+\varepsilon)$-quasiconformal on $h^{-1}(Y_\infty - V)$ such that $g_\infty \circ f$ is homotopic to $h \circ g\}$.

We set $V = V_1 \cup \cdots \cup V_k$ where $V_j = V_j^1 \cup V_j^2 \cup \{pt\}$, $V_j^l = \{0 < |z| \leq \iota_j\} \subset A_j^l(\infty)$, $0 < \iota_j < 1$ for any $j = 1, \cdots, k$ and $l = 1, 2$. There exists $T > 0$ such that for any $t > T$ and any $j = 1, \cdots, k$, $\delta_j(t) < \iota_j$ and $d_{T(X)}(r(t), r'(t)) < \frac{1}{2}\log(1+\varepsilon)$, i.e., there exists a $(1+\varepsilon)$-quasiconformal mapping $\alpha_t : Y'_t \to Y_t$ such that $g_t \circ f$ is homotopic to $\alpha_t \circ g'_t \circ f'$. For any $t > T$, we want to show that $r'(t) \in U_{V,\varepsilon}(r(\infty))$. We construct the deformation $h : Y'_t \to Y_\infty$ as follows. For any $j = 1, \cdots, k$ and $l = 1, 2$,

$$h|_{\alpha_t^{-1}(A_j^l(t))}(w) = \begin{cases} \alpha_t(w) & (w \in \alpha_t^{-1}(\{\iota_j \leq |z| < 1\})) \\ h_{j,t}(|\alpha_t(w)|)e^{i\arg\alpha_t(w)} & (w \in \alpha_t^{-1}(\{\delta_j(t) < |z| < \iota_j\})) \\ pt & (w \in \alpha_t^{-1}(\{|z| = \delta_j(t)\})) \end{cases}$$

where $h_{j,t} : (\delta_j(t), \iota_j) \to (0, \iota_j)$ is an arbitrary monotonously increasing diffeomorphism. The mapping $h|_{\alpha_t^{-1}(A_j^l(t))}$ is equal to the $(1+\varepsilon)$-quasiconformal mapping α_t on $h|_{\alpha_t^{-1}(A_j^l(t))}{}^{-1}(A_j^l(\infty) - V_j^l)$. Since h is homotopic to $g_\infty \circ g_t^{-1} \circ \alpha_t$, we obtain that $h \circ g'_t \circ f'$ is homotopic to $g_\infty \circ f$. This means that $r'(t) \in U_{V,\varepsilon}(r(\infty))$ for any $t > T$. Since ε and V are arbitrary, we conclude that $r(\infty) = r'(\infty)$. $\square$

3.3 The detour metric

In this section, we obtain the minimum value of the equation of Theorem 3.1.1 when we shift the initial points of the given two rays. This value is represented by the detour metric between the end points of the rays in the Gardiner-Masur boundary of $T(X)$.

3.3.1 The horofunction boundary of metric spaces and the detour metric

We recall the definition of the detour metric in the case of a general metric space, and refer the reader to [Wal11] for more details. First, we consider the horofunction compactification of a metric space. This was given by Gromov [Gro81]. Let (X, d) be a metric space, b be a base point of X. The distance d on X is called *proper* if any closed ball with respect to d is compact and *geodesic* if for any two points in X, there exists a geodesic which joins them. Suppose that the distance d is proper and geodesic. It is well known that any Teichmüller distance has these properties. We denote by $C(X)$ the set of all continuous functions of X into $\mathbb{R}$ which is equipped with the topology of the uniform convergence on any compact set of X. We define a mapping $\psi : X \to C(X)$ by $z \mapsto \{\psi_z(x) := d(x, z) - d(b, z)\}_{x \in X}$.

Theorem 3.3.1. ([Gro81]) *The mapping ψ is an embedding and the set $\psi(X)$ is relatively compact on $C(X)$.*

By this theorem, the space X is identified with $\psi(X)$. The closure $\overline{\psi(X)}$ is called the *horofunction compactification* of X. The boundary $X(\infty) = \overline{\psi(X)} - \psi(X)$ is called the *horofunction boundary* of X. We call $\xi \in X(\infty)$ a *horofunction*. We can denote by $X \cup X(\infty)$ the horofunction compactification of X.

Remark. The topological space $X \cup X(\infty)$ satisfies the first countability axiom.

Definition 3.3.2. For any $\xi, \xi' \in X(\infty)$, we define the *detour cost*

$$H(\xi, \xi') = \sup_{W \ni \xi} \inf_{x \in W \cap X} (d(b, x) + \xi'(x)),$$

where W ranges over all neighborhoods of ξ in $X \cup X(\infty)$.

There is another definition of the detour cost.

Definition 3.3.3. For any $\xi, \xi' \in X(\infty)$,

$$H(\xi, \xi') = \inf_{\gamma} \liminf_{t \to \infty} (d(b, \gamma(t)) + \xi'(\gamma(t))),$$

where γ ranges over all paths $\gamma : \mathbb{R}_{\geq 0} \to X$ which converge to ξ.

Definition 3.3.4. Let $A \subset \mathbb{R}_{\geq 0}$ be an unbounded set which contains 0. A mapping $r : A \to X$ is called a *almost geodesic* on X if for any $\varepsilon > 0$, there exists $T \geq 0$ such that for any $s, t \in A$ with $T \leq s \leq t$,

$$|d(r(0), r(s)) + d(r(s), r(t)) - t| < \varepsilon.$$

Any geodesic is an almost geodesic. Rieffel proved that any almost geodesic on X converges to a point in $X(\infty)$ ([Rie02]). Let $X_B(\infty) \subset X(\infty)$ be the set of end points of all almost geodesics. Any $\xi \in X_B(\infty)$ is called a *Busemann point*.

Proposition 3.3.5. ([Wal11]) *For any $\xi, \xi', \xi'' \in X(\infty)$ and $x \in X$, the detour cost H satisfies the following properties:*

1. $\xi'(x) \leq H(\xi, \xi') + \xi(x)$
2. $H(\xi, \xi) = 0$ *if and only if* $\xi \in X_B(\infty)$,
3. $H(\xi, \xi') \geq 0$,
4. $H(\xi, \xi'') \leq H(\xi, \xi') + H(\xi', \xi'')$.

By this proposition, for any $\xi, \xi' \in X_B(\infty)$, the symmetrization $H(\xi, \xi') + H(\xi', \xi)$ satisfies the axiom of the distance.

Definition 3.3.6. For any $\xi, \xi' \in X_B(\infty)$,

$$\delta(\xi, \xi') = H(\xi, \xi') + H(\xi', \xi)$$

is a (possibly $+\infty$-valued) distance on $X_B(\infty)$. This δ is called the *detour metric* for (X, d) and the base point $b \in X$.

3.3.2 The Gardiner-Masur boundary of Teichmüller spaces

The Gardiner-Masur compactification and the Gardiner-Masur boundary of a Teichmüller space were induced by Gardiner and Masur [GM91]. Liu and Su [LS12] showed that the horofunction compactification of the Teichmüller space with the Teichmüller distance is the same as the Gardiner-Masur compactification. Let $T(X)$ be a Teichmüller space of X. We define a mapping $\tilde{\phi} : T(X) \to \mathbb{R}^{\mathcal{S}}_{\geq 0}$ by $p \mapsto \{\mathrm{Ext}_p^{\frac{1}{2}}(\gamma)\}_{\gamma \in \mathcal{S}}$, and denote by $\pi : \mathbb{R}^{\mathcal{S}}_{\geq 0} - \{0\} \to \mathrm{P}\mathbb{R}^{\mathcal{S}}_{\geq 0}$ a natural projection.

Theorem 3.3.7. ([GM91]) *The composition* $\phi = \pi \circ \tilde{\phi} : T(X) \to \mathrm{P}\mathbb{R}^{\mathcal{S}}_{\geq 0}$ *is an embedding and the closure* $\overline{\phi(T(X))}$ *is a compact set.*

This closure is called the *Gardiner-Masur compactification* of $T(X)$ and we denote by $\overline{T(X)}^{GM} = \overline{\phi(T(X))}$. The boundary $\partial_{GM}T(X) = \overline{\phi(T(X))} - \phi(T(X))$ is called the *Gardiner-Masur boundary* of $T(X)$.

We set the base point $b = [X, id] \in T(X)$. For any $p \in T(X)$ and any $\mu \in \mathcal{MF}(X)$, we define

$$\mathcal{E}_p(\mu) = \left\{ \frac{\mathrm{Ext}_p(\mu)}{K_p} \right\}^{\frac{1}{2}},$$

where $K_p = e^{2d_{T(X)}(b,p)}$. By the definition of the Gardiner-Masur compactification, the function $\pi(\{\mathcal{E}_p(\gamma)\}_{\gamma\in\mathcal{S}})$ corresponds to $p \in T(X)$.

Proposition 3.3.8. ([Miy08]) *For any* $\xi \in \partial_{GM}T(X)$*, there exists a continuous function* $\mathcal{E}_\xi : \mathcal{MF}(X) \to \mathbb{R}_{\geq 0}$ *which satisfies the following properties:*

1. $\mathcal{E}_\xi(t\mu) = t\mathcal{E}_\xi(\mu)$ *for any* $t \geq 0$ *and any* $\mu \in \mathcal{MF}(X)$*,*
2. $\pi(\{\mathcal{E}_p(\gamma)\}_{\gamma\in\mathcal{S}})$ *corresponds to* $\xi \in \partial_{GM}T(X)$*,*
3. *If a sequence* $\{x_n\} \subset T(X)$ *converges to* $\xi \in \partial_{GM}T(X)$*, then there exists a subsequence* $\{x_{n_j}\} \subset \{x_n\}$ *and* $t_0 > 0$ *which does not depend on* $\mathcal{MF}(X)$ *such that* $\mathcal{E}_{x_{n_j}}$ *converges to* $t_0\mathcal{E}_\xi$ *uniformly on any compact set of* $\mathcal{MF}(X)$*.*

For any $p \in \overline{T(X)}^{GM}$, we define

$$Q(p) = \sup_{\nu\in\mathcal{MF}(X)^*} \frac{\mathcal{E}_p(\nu)}{\mathrm{Ext}_b^{\frac{1}{2}}(\nu)}$$

and for any $\mu \in \mathcal{MF}(X)$,

$$\mathcal{L}_p(\mu) = \frac{\mathcal{E}_p(\mu)}{Q(p)}.$$

Proposition 3.3.9. ([LS12]) *For any* $\{p_n\} \subset \overline{T(X)}^{GM}$ *and any* $p \in \overline{T(X)}^{GM}$*,* p_n *converges to* p *as* $n \to \infty$ *if and only if* $\mathcal{L}_{p_n}$ *converges to* $\mathcal{L}_p$ *uniformly on any compact set of* $\mathcal{MF}(X)$ *as* $n \to \infty$*.*

For any $p \in \overline{T(X)}^{GM}$, we define a function $\psi_p : T(X) \to \mathbb{R}$ by

$$\psi_p(x) = \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\mathcal{L}_p(\mu)}{\mathrm{Ext}_x^{\frac{1}{2}}(\mu)}$$

for any $x \in T(X)$.

Remark. By the definition and Kerckhoff's formula, if $p \in T(X)$, then

$$\begin{aligned}
\psi_p(x) &= \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\mathcal{E}_p(\mu)}{\mathrm{Ext}_x^{\frac{1}{2}}(\mu)} - \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\mathcal{E}_p(\mu)}{\mathrm{Ext}_b^{\frac{1}{2}}(\mu)} \\
&= \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\mathrm{Ext}_p^{\frac{1}{2}}(\mu)}{\mathrm{Ext}_x^{\frac{1}{2}}(\mu)} - \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\mathrm{Ext}_p^{\frac{1}{2}}(\mu)}{\mathrm{Ext}_b^{\frac{1}{2}}(\mu)} \\
&= d_{T(X)}(x,p) - d_{T(X)}(b,p)
\end{aligned}$$

for any $x \in X$. Thus, we can consider the horofunction compactification of $T(X)$ by the function ψ_p for any $p \in T(X)$.

The following is deduced from Proposition 3.3.9.

Theorem 3.3.10. ([LS12]) *We define a mapping $\psi : \overline{T(X)}^{GM} \to C(T(X))$ by $p \mapsto \psi_p$. Then ψ is injective and continuous. In particular, $\overline{T(X)}^{GM}$ and $\psi(\overline{T(X)}^{GM})$ are homeomorphic. Furthermore, $\psi(\overline{T(X)}^{GM}) = \overline{\psi(T(X))}$.*

Therefore, the horofunction compactification $\overline{\psi(T(X))}$ of $T(X)$ can be identified with the Gardiner-Masur compactification $\overline{T(X)}^{GM}$. We denote by $T(X) \cup T(X)(\infty)$ the horofunction compactification of $T(X)$. Then, we can assume that $T(X)(\infty) = \partial_{GM} T(X)$.

3.3.3 The detour metric for the Teichmüller distance

We consider the detour metric in the case of the Teichmüller space with the Teichmüller distance. Its representation is given by Walsh [Wal12]. We notice that there exist non-Busemann points on $\partial_{GM} T(X)$ if $3g - 3 + n > 1$. This result is proved by Miyachi [Miy11].

Theorem 3.3.11. ([Wal12]) *For any point p in $T(X)$ and any Busemann point ξ, there exists a unique Teichmüller geodesic ray on $T(X)$ starting at p and converging to ξ.*

By this theorem, we can assume that the set of all Busemann points consists of end points of all Teichmüller geodesic rays.

Theorem 3.3.12. ([Wal12]) *Let r, r' be Teichmüller geodesic rays on $T(X)$ converging to Busemann points ξ, ξ' respectively. Then, the rays r, r' are modularly equivalent if and only if $\xi = \xi'$.*

Let ξ, ξ' be Busemann points and r, r' be Teichmüller geodesic rays on $T(X)$ starting at $b = [X, id]$ and converging to ξ, ξ' respectively. We denote by q, q' the unit norm holomorphic quadratic differentials on X corresponding to the given rays r, r' respectively. If the measured foliations $H(q)$, $H(q') \in \mathcal{MF}(X)$ are absolutely continuous, then we can write $H(q) = \sum_{j=1}^{k} b_j G_j$, $H(q') = \sum_{j=1}^{k} b'_j G_j$ where b_j, $b'_j > 0$ and G_j is a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number. We set $m_j = \frac{b_j}{i(G_j, V(q))}$, $m'_j = \frac{b'_j}{i(G_j, V(q'))}$ for any $j = 1, \cdots, k$.

Theorem 3.3.13. ([Wal12]) *If $H(q)$, $H(q')$ are absolutely continuous, then the detour metric δ between ξ and ξ' is represented by*

$$\delta(\xi, \xi') = \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{m'_j}{m_j} + \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{m_j}{m'_j}.$$

If $H(q)$, $H(q')$ are not absolutely continuous, then $\delta(\xi, \xi') = +\infty$.

In Theorem 3.3.13, for any given Busemann points, Walsh showed the representation of the detour metric between them. Now, for any given rays, we obtain the representation of the detour metric between the end points of the rays. Let r, r' be Teichmüller geodesic rays on $T(X)$ starting at $r(0) = [Y, f]$, $r'(0) = [Y', f']$ and having initial unit norm holomorphic quadratic differentials q, q' on Y, Y' and converging to Busemann points ξ, ξ' respectively. If the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q')) \in \mathcal{MF}(X)$ are absolutely continuous, then we can write $f_*^{-1}(H(q)) = \sum_{j=1}^{k} b_j G_j$, $f_*'^{-1}(H(q)) = \sum_{j=1}^{k} b'_j G_j$ where b_j, $b'_j > 0$, G_j is a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number. We set $m_j = \frac{b_j}{i(f_*(G_j), V(q))}$, $m'_j = \frac{b'_j}{i(f'_*(G_j), V(q'))}$ for any $j = 1, \cdots, k$. We combine Theorems 3.3.11, 3.3.12 and 3.3.13 to obtain the following.

Proposition 3.3.14. *If $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are absolutely continuous, then the detour metric between ξ and ξ' is also represented by*

$$\delta(\xi, \xi') = \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{m'_j}{m_j} + \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{m_j}{m'_j}.$$

If $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are not absolutely continuous, then $\delta(\xi, \xi') = +\infty$.

Proof. By Theorem 3.3.11, there exist two Teichmüller geodesic rays s, s' starting at $b = [X, id]$ and having initial unit norm holomorphic quadratic differentials φ, φ' on X and converging to Busemann points ξ, ξ' respectively. By Theorem 3.3.12, the two pairs r, s and r', s' are both modularly equivalent. If the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are absolutely continuous, then the measured foliations $H(\varphi)$, $H(\varphi') \in \mathcal{MF}(X)$ are also absolutely continuous, and can be written as $H(\varphi) = \sum_{j=1}^k c_j G_j$, $H(\varphi') = \sum_{j=1}^k c'_j G_j$ where c_j, $c'_j > 0$ for any $j = 1, \cdots, k$. Let $n_j = \frac{c_j}{i(G_j, V(\varphi))}$, $n'_j = \frac{c'_j}{i(G_j, V(\varphi'))}$ for any $j = 1, \cdots, k$, then there are $\lambda, \lambda' > 0$ such that $n_j = \lambda m_j$, $n'_j = \lambda' m'_j$ respectively. Therefore, by Theorem 3.3.13,

$$\begin{aligned} \delta(\xi, \xi') &= \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{n'_j}{n_j} + \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{n_j}{n'_j} \\ &= \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{m'_j}{m_j} + \frac{1}{2} \log \max_{j=1,\cdots,k} \frac{m_j}{m'_j}. \end{aligned}$$

If the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are not absolutely continuous, then $H(\varphi)$, $H(\varphi')$ are not also absolutely continuous, and we conclude that $\delta(\xi, \xi') = +\infty$. $\square$

We suppose that two rays r, r' show that $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are absolutely continuous. Let ξ, ξ' be Busemann points corresponding to the end points of r, r' respectively. By the lower estimate of Theorem 3.1.1, and

Proposition 3.3.14,

$$\begin{aligned}
\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) &\geq \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m_j'}{m_j}, \frac{m_j}{m_j'}\right\} \qquad (3.2)\\
&\geq \frac{1}{2}\log\left(\max_{j=1,\cdots,k}\sqrt{\frac{m_j'}{m_j}}\cdot \max_{j=1,\cdots,k}\sqrt{\frac{m_j}{m_j'}}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\log\max_{j=1,\cdots,k}\frac{m_j'}{m_j} + \frac{1}{2}\log\max_{j=1,\cdots,k}\frac{m_j}{m_j'}\right)\\
&= \frac{1}{2}\delta(\xi, \xi').
\end{aligned}$$

Now, we consider the minimum value of the equation of Theorem 3.1.1 when we shift the initial points of the given rays r, r'.

Proposition 3.1.3. *Under the assumption of Theorem 3.1.1, the minimum of the limit value of the distance between the given rays $r(t)$, $r'(t)$ when we shift the initial points $r(0)$, $r'(0)$ is given by $\frac{1}{2}\delta(\xi,\xi')$ where ξ, ξ' are the end points of the rays r, r' in the Gardiner-Masur boundary of $T(X)$ respectively.*

Proof. By Theorem 3.1.1 and the inequality (3.2), we see that

$$\begin{aligned}
\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) &= \frac{1}{2}\log \max_{j=1,\cdots,k}\left\{\frac{m_j'}{m_j}, \frac{m_j}{m_j'}\right\}\\
&\geq \frac{1}{2}\delta(\xi, \xi').
\end{aligned}$$

We notice that the detour metric is invariant when we shift the initial points of the rays r, r'. The equality holds if we set

$$\beta = \frac{1}{4}\log\frac{\max_{j=1,\cdots,k}\frac{m_j}{m_j'}}{\max_{j=1,\cdots,k}\frac{m_j'}{m_j}}$$

and consider the rays $r(t)$, $r'(t+\beta)$. In this situation, we compute that

$$\max_{j=1,\cdots,k}\frac{e^{2\beta}m_j'}{m_j} = \max_{j=1,\cdots,k}\left\{\frac{\sqrt{\max_{j=1,\cdots,k}\frac{m_j}{m_j'}}\cdot m_j'}{\sqrt{\max_{j=1,\cdots,k}\frac{m_j'}{m_j}}\cdot m_j}\right\} = \sqrt{\max_{j=1,\cdots,k}\frac{m_j'}{m_j}}\cdot\sqrt{\max_{j=1,\cdots,k}\frac{m_j}{m_j'}}$$

and similarly,

$$\max_{j=1,\cdots,k} \frac{m_j}{e^{2\beta}m'_j} = \sqrt{\max_{j=1,\cdots,k} \frac{m'_j}{m_j}} \cdot \sqrt{\max_{j=1,\cdots,k} \frac{m_j}{m'_j}}.$$

Therefore, we conclude that

$$\begin{aligned}
\lim_{t\to\infty} d_{T(X)}(r(t), r'(t+\beta)) &= \frac{1}{2}\log \max_{j=1,\cdots,k} \left\{ \frac{e^{2\beta}m'_j}{m_j}, \frac{m_j}{e^{2\beta}m'_j} \right\} \\
&= \frac{1}{2}\left(\frac{1}{2}\log \max_{j=1,\cdots,k} \frac{m'_j}{m_j} + \frac{1}{2}\log \max_{j=1,\cdots,k} \frac{m_j}{m'_j} \right) \\
&= \frac{1}{2}\delta(\xi, \xi').
\end{aligned}$$

□

We can also obtain the following. In Proposition 3.3.15, we do not require the conditions that the given rays are Jenkins-Strebel.

Proposition 3.3.15. *If two rays r, r' are asymptotic, then the end points satisfy that $\xi = \xi'$ and the rays are modularly equivalent.*

Proof. From the remark in the previous section, the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q'))$ are absolutely continuous. Then, the conclusion follows from Theorem 3.3.12 and the inequality (3.2). □

3.4 The tables of the classification about the behavior of two Teichmüller rays

In this section, we give the tables of the classification of the conditions under which given two Teichmüller geodesic rays are bounded, diverge, or asymptotic.

Let r, r' be two Teichmüller geodesic rays on $T(X)$ starting at $r(0) = [Y, f]$, $r'(0) = [Y', f']$ and having initial unit norm holomorphic quadratic differentials q, q' on Y, Y', and $H(q)$, $H(q')$ be measured foliations corresponding to q, q' respectively. We set $H = f_*^{-1}(H(q))$, $H' = f_*'^{-1}(H(q'))$. In the following tables, the notation "top.equi." means topologically equivalent, "abs.conti." means absolutely continuous, "J-S." means Jenkins-Strebel, "u.e." means uniquely ergodic, and "mod.equi." means modularly equivalent.

$$H, H' : \begin{cases} \text{top.equi. and} \begin{cases} \text{abs.conti.} \Rightarrow \text{bounded [Iva01]} \\ \\ \text{not abs.conti.} \Rightarrow \text{diverge [LM10]} \end{cases} \\ \\ \text{not top.equi. and } i(H, H') \begin{cases} \neq 0 \Rightarrow \text{diverge [Iva01]} \\ \\ = 0 \Rightarrow \text{diverge [LM10]} \end{cases} \end{cases}$$

By the above table,

$$r,\ r'\text{: bounded} \iff H,\ H'\text{: abs.conti.}$$

If H, H' are absolutely continuous and Jenkins-Strebel, we denote by $r(\infty)$, $r'(\infty)$ the end points of the rays r, r' in the augmented Teichmüller space $\hat{T}(X)$ respectively.

$$H, H' : \text{abs.conti. and} \begin{cases} \text{J-S. and} \begin{cases} r, r' : \text{mod.equi. and } r(\infty) = r'(\infty) \\ \Rightarrow \text{asymptotic (Cor.3.1.2)} \\ \\ \text{otherwise} \\ \Rightarrow \text{bounded but not asymptotic} \\ \qquad \text{(Cor.3.1.2)} \end{cases} \\ \\ \text{not J-S. and} \begin{cases} \text{u.e., and } \Gamma_q, \Gamma_{q'} \text{ have no simple} \\ \text{closed curves} \Rightarrow \text{asymptotic} \\ \qquad \text{[Mas80]} \\ \text{otherwise} \Rightarrow \text{unknown} \end{cases} \end{cases}$$

Let ξ, ξ' be the end points of the rays r, r' in the Gardiner-Masur boundary $\partial_{GM} T(X)$ respectively.

$$r, r' : \text{asymptotic} \underset{\text{(Prop.3.3.15)}}{\Longrightarrow} \xi = \xi' \underset{\text{[Wal12]}}{\Longleftrightarrow} r, r' : \text{mod.equi.}$$

Chapter 4

The limit value of the Teichmüller distance between Jenkins-Strebel rays

4.1 Introduction

In this chapter, we obtain the limit value of the distance between two Jenkins-Strebel rays. This is the generalization of Theorem 3.1.1 in the previous chapter. Let r, r' be Jenkins-Strebel rays on $T(X)$ starting at $r(0) = [Y, f]$, $r(0)' = [Y', f']$ and having initial unit norm Jenkins-Strebel differentials q, q' on Y, Y' respectively. Let $r(\infty)$, $r'(\infty)$ be the end points of r, r' in the augmented Teichmüller space $\hat{T}(X)$ respectively. We denote by $H(q) \in \mathcal{MF}(Y)$, $H(q') \in \mathcal{MF}(Y')$ the corresponding measured foliations. As we have seen the previous chapter, if the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q')) \in \mathcal{MF}(X)$ are topologically equivalent, we denote by m_j, m'_j the moduli of the annuli on Y, Y' determined by $H(q)$, $H'(q)$ respectively, for any $j = 1, \cdots, k$. In this situation, we can define the Teichmüller distance $d_{\hat{T}(X)}(r(\infty), r'(\infty))$ between the end points $r(\infty)$, $r'(\infty)$. We describe this definition later.

Our main result is the following:

Theorem 4.1.1. *For any two Jenkins-Strebel rays r, r',*

$$\lim_{t\to\infty} d_{T(X)}(r(t), r'(t)) = \begin{cases} \max\left\{\dfrac{1}{2}\log \max\limits_{j=1,\cdots,k}\left\{\dfrac{m'_j}{m_j}, \dfrac{m_j}{m'_j}\right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\} \\ \qquad \text{(if } f_*^{-1}(H(q)),\ f_*'^{-1}(H(q')) \text{ are topologically equivalent)} \\ +\infty \text{ (otherwise)} \end{cases} \tag{4.1}$$

Corollary 4.1.2. *The minimum value of the equation* (4.1) *when we shift the starting points of r, r' is given by*

$$\max\left\{\frac{1}{2}\delta(\xi, \xi'), d_{\hat{T}(X)}(r(\infty), r'(\infty))\right\}.$$

4.2 Definitions

4.2.1 The local coordinates at critical points of q

Let X be a Riemann surface of type (g, n) with $3g - 3 + n > 0$ and q be a holomorphic quadratic differential of finite norm on X. Suppose that p_0 is a critical point of q, and its order is $n \geq -1$. In a small neighborhood of p_0 which does not contain any other critical points of q, there exists a local coordinate z on X such that $z(p_0) = 0$ and $q = z^n dz^2$. For instance, we refer to [Str84]. For any non-critical point in the neighborhood of p_0, there exists a q-coordinate ζ. By $d\zeta^2 = z^n dz^2$, the transformation $\zeta = \frac{2}{n+2} z^{\frac{n+2}{2}}$ holds. For any $k = 0, \cdots, n+1$, the set $\{\frac{2\pi k}{n+2} \leq \arg z \leq \frac{2\pi(k+1)}{n+2}\}$ on the z-plane is mapped to the half plane $\{0 \leq \arg \zeta \leq \pi\}$ or $\{\pi \leq \arg \zeta \leq 2\pi\}$ on the ζ-plane. We can see the trajectory flow in the neighborhood of p_0 as the $n+2$ copies of the half plane with the gluing along each horizontal edge of the planes (Figure 4.1).

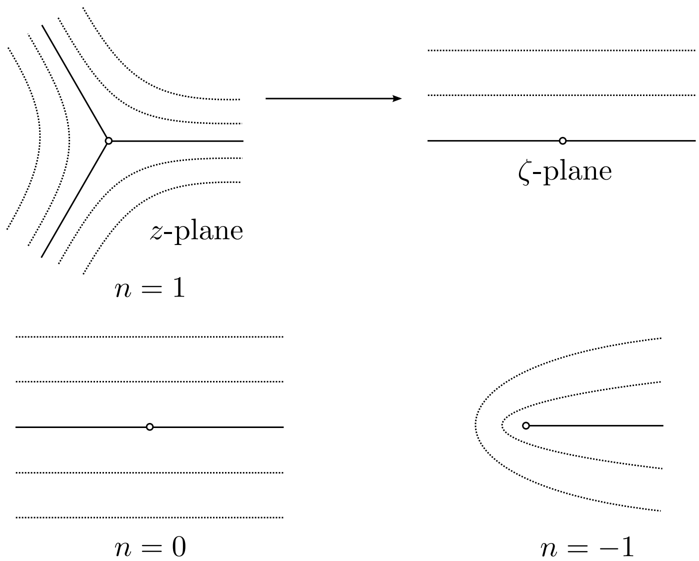

Figure 4.1: The trajectory flow in the neighborhood of p_0 in the case of $n = 1, 0, -1$

4.2.2 The distance between end points of Jenkins-Strebel rays

Suppose that r, r' are the Jenkins-Strebel rays on $T(X)$ in §4.1, and the measured foliations $f_*^{-1}(H(q))$, $f_*'^{-1}(H(q')) \in \mathcal{MF}(X)$ are topologically equivalent. We recall that there is a homeomorphism $\alpha : X - \Gamma_{f_*^{-1}(H(q))} \to X - \Gamma_{f_*'^{-1}(H(q'))}$ which is homotopic to the identity such that the leaves of $f_*^{-1}(H(q))$ are mapped to the leaves of $f_*'^{-1}(H(q'))$. We consider the distance between the end points of r, r'. We set $r(\infty) = [Y_\infty, g_\infty \circ f]$, $r'(\infty) = [Y'_\infty, g'_\infty \circ f']$ and let $\{Y_{\infty,\lambda}\}_{\lambda=1,\cdots,\Lambda}$, $\{Y'_{\infty,\lambda}\}_{\lambda=1,\cdots,\Lambda}$ be the components of $Y_\infty - \{\text{nodes of } Y_\infty\}$, $Y'_\infty - \{\text{nodes of } Y'_\infty\}$ respectively, such that $(g'_\infty \circ f') \circ \alpha \circ (g_\infty \circ f)^{-1}(Y_{\infty,\lambda}) = Y'_{\infty,\lambda}$ for any $\lambda = 1, \cdots, \Lambda$. We define the distance between $r(\infty)$, $r'(\infty)$ by

$$d_{\hat{T}(X)}(r(\infty), r'(\infty)) = \max_{\lambda=1,\cdots,\Lambda} \frac{1}{2} \inf_{h_\lambda} \log K(h_\lambda),$$

where h_λ ranges over all quasiconformal mappings $h_\lambda : Y_{\infty,\lambda} \to Y'_{\infty,\lambda}$ such that h_λ is homotopic to $(g'_\infty \circ f') \circ \alpha \circ (g_\infty \circ f)^{-1}$.

Remark. In fact, If X and Y have same genus and punctures, for any orientation preserving homeomorphism $g : X \to Y$, there exists a Teichmüller mapping $f : X \to Y$ which is homotopic to g (Theorem 1 in §1.5 of Chapter II of [Abi80]).

This definition means that the distance between end points of Jenkins-Strebel rays is the maximum of the distance between the corresponding points of the Teichmüller space of each component of the end points of the rays.

4.3 Proof of Theorem

Proof of Theorem 4.1.1. If $f_*^{-1}(H(q))$, $f'^{-1}_*(H(q'))$ are not topologically equivalent, the result is already known in §3. Then, we suppose that $f_*^{-1}(H(q))$, $f'^{-1}_*(H(q'))$ are topologically equivalent. First, we consider the upper estimate. Let $h_\lambda : Y_{\infty,\lambda} \to Y'_{\infty,\lambda}$ be the Teichmüller mapping which is homotopic to $(g'_\infty \circ f') \circ \alpha \circ (g_\infty \circ f)^{-1}$, and we set the mapping $h : Y_\infty \to Y'_\infty$ constructed by $\{h_\lambda\}_{\lambda=1,\cdots,\Lambda}$. We set $K = \exp(2d_{\hat{T}(X)}(r(\infty), r'(\infty))) = \max_{\lambda=1,\cdots,\Lambda} K(h_\lambda)$. We also use the representations of Y_∞, Y'_∞ as the unions of closed unit disks $\{\overline{A^l_j(\infty)}\}^{l=1,2}_{j=1,\cdots,k}$, $\{\overline{A'^l_j(\infty)}\}^{l=1,2}_{j=1,\cdots,k}$ respectively. For any $j = 1, \cdots, k$ and $l = 1, 2$, there is λ such that $h^l_j = h|_{A^l_j(\infty)}$ is equal to $h_\lambda|_{A^l_j(\infty)}$. We use the following lemma.

Lemma 4.3.1. *Let R, R' be Riemann surfaces with nodes and $f : R \to R'$ be a K-quasiconformal Teichmüller mapping. This means that f is a homeomorphism, each restricted mapping of f which maps a component of $R - \{$nodes of $R\}$ onto a component of $R' - \{$nodes of $R'\}$ is a Teichmüller mapping, and the maximum of maximal dilatations of such mappings is*

K. Then, for any sufficiently small $\varepsilon > 0$, there exists the $(K + O(\varepsilon))$-quasiconformal mapping $g : R \to R'$ such that g is conformal on a neighborhood of the set of nodes of R, and is homotopic to f.

The lemma is proved in the paper of [FM10], however we give a new proof of the latter part of their proof.

Proof of Lemma 4.3.1. Let μ be the Beltrami coefficient of f. For any $\varepsilon > 0$, we consider a new Beltrami coefficient

$$\mu_\varepsilon = \begin{cases} 0 \ (0 < |z| < \varepsilon) \\ \mu \ (\text{otherwise}) \end{cases}$$

on R, where z is each local coordinate near nodes of R and the domain $\{|z| < \varepsilon\}$ represents a neighborhood of nodes $z = 0$. Then, there exist a Riemann surface with nodes R_ε and a K-quasiconformal mapping $f_\varepsilon : R \to R_\varepsilon$ such that f_ε is conformal on the neighborhood of nodes of R. For sufficiently small ε, f_ε is close to f and R_ε is close to R'. The mapping $f \circ f_\varepsilon^{-1} : R_\varepsilon \to R'$ is K-quasiconformal in a small neighborhood of nodes of R_ε and is conformal on the outside of the neighborhood. We use local coordinates such that nodes of R_ε and R' correspond to 0, and $f \circ f_\varepsilon^{-1}(0) = 0$. Let $p_1, \cdots, p_k$ be all nodes of R_ε. For any $j = 1, \cdots, k$, we take small disks $N_j^1 = N_j^2 = \{|z| < \delta\}$ about p_j in R_ε where $f \circ f_\varepsilon^{-1}$ is K-quasiconformal in $\overline{N_j^1} \cup \overline{N_j^2}$. For any $j = 1, \cdots, k$ and $l = 1, 2$, the image $f \circ f_\varepsilon^{-1}(N_j^l)$ in R' is mapped to the disk $\{|z| < \delta\}$ by a conformal mapping $f_{j,\varepsilon}^l$ such that $f_{j,\varepsilon}^l(0) = 0$ and $f_{j,\varepsilon}^l(\delta) = \delta$. We denote simply by $F_\varepsilon := f_{j,\varepsilon}^l \circ f \circ f_\varepsilon^{-1}$. The family of K-quasiconformal mappings $\{F_\varepsilon\}$ is normal, then we can assume that F_ε converges to a K-quasiconformal mapping F_0 uniformly on any compact set of $\{|z| < \delta\}$ as $\varepsilon \to 0$. However, any point of $\{0 < |z| < \delta\}$ is a holomorphic point of F_ε for sufficiently small ε, then F_0 is holomorphic in $\{0 < |z| < \delta\}$. Since F_0 fixes 0 and δ, we can see that F_0 is an automorphism on $\{|z| < \delta\}$ and then it is the identity.

Now, we rescale $\{|z| < \delta\}$ to $\mathbb{D}$, and assume that $\{F_\varepsilon\}$ as mappings of $\mathbb{D}$ onto itself. We set a conformal mapping $\phi(z) := (z - i)/(z + i)$ of $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ onto $\mathbb{D}$. We consider mappings $f_\varepsilon := \phi^{-1} \circ F_\varepsilon \circ \phi$ of $\mathbb{H}$ onto itself.

Lemma 4.3.2. *We have*

$$\lim_{\varepsilon \to 0} \sup_{x,t} \frac{f_\varepsilon(x+t) - f_\varepsilon(x)}{f_\varepsilon(x) - f_\varepsilon(x-t)} = 1,$$

where the supremum ranges over all $x, t \in \mathbb{R}$ such that $t \neq 0$.

Proof of Lemma 4.3.2. We notice that $\frac{f_\varepsilon(x+t)-f_\varepsilon(x)}{f_\varepsilon(x)-f_\varepsilon(x-t)}$ is positive, and its reciprocal is the case of $-t$. Then, it suffice to show that for any $t > 0$. Let

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_1 - z_3} \cdot \frac{z_3 - z_4}{z_2 - z_4}$$

be a cross ratio for any $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$. We write $\phi(x) = e^{i\theta}$, $\phi(x+t) = e^{i(\theta+\varphi)}$ and $\phi(x-t) = e^{i(\theta-\psi)}$ where $0 \leq \theta < 2\pi$ and $\varphi, \psi > 0$. Since all Möbius transformation preserve cross ratios,

$$\begin{aligned} \frac{f_\varepsilon(x+t) - f_\varepsilon(x)}{f_\varepsilon(x) - f_\varepsilon(x-t)} &= -(f_\varepsilon(x), f_\varepsilon(x+t), f_\varepsilon(x-t), \infty) \\ &= -(F_\varepsilon \circ \phi(x), F_\varepsilon \circ \phi(x+t), F_\varepsilon \circ \phi(x-t), 1) \\ &= \left| \frac{F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})}{F_\varepsilon(e^{i\theta}) - F_\varepsilon(e^{i(\theta-\psi)})} \cdot \frac{F_\varepsilon(e^{i(\theta-\psi)}) - 1}{F_\varepsilon(e^{i(\theta+\varphi)}) - 1} \right|. \end{aligned} \tag{4.2}$$

We set $z = e^{iy}$. By $\log F_\varepsilon(e^{iy}) = i(\arg F_\varepsilon(e^{iy}) + 2n\pi)$, we have

$$\frac{d \arg F_\varepsilon(e^{iy})}{dy} = \frac{z \frac{dF_\varepsilon(z)}{dz}}{F_\varepsilon(z)}.$$

Since $F_\varepsilon(z)$ and $\frac{dF_\varepsilon(z)}{dz}$ converge to z and 1 uniformly on $\partial\mathbb{D}$ respectively, then

$$\begin{aligned} \sup_{0 \leq y < 2\pi} \left| \frac{d \arg F_\varepsilon(e^{iy})}{dy} - 1 \right| &= \sup \left| z \frac{dF_\varepsilon(z)}{dz} - F_\varepsilon(z) \right| \\ &\leq \sup \left(\left| \frac{dF_\varepsilon(z)}{dz} - 1 \right| + |z - F_\varepsilon(z)| \right) \to 0 \end{aligned}$$

as $\varepsilon \to 0$. For any $0 < E < 1$, we take sufficiently small ε such that

$$\left| \frac{d \arg F_\varepsilon(e^{iy})}{dy} - 1 \right| < E$$

holds for any $0 \le y < 2\pi$. Now, we calculate and estimate each term of (4.2).

$$\begin{aligned}
|F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})| &= 2\sin\frac{\arg F_\varepsilon(e^{i(\theta+\varphi)}) - \arg F_\varepsilon(e^{i\theta})}{2} \\
&= 2\sin\frac{\int_\theta^{\theta+\varphi} \frac{d\arg F_\varepsilon(e^{iy})}{dy} dy}{2} \\
&< 2\sin\frac{(1+E)\varphi}{2} \\
&= 2\sin\frac{(1+E)(\arg\phi(x+t) - \arg\phi(x))}{2} \\
&= 2\sin\frac{(1+E)\int_x^{x+t} \frac{d\arg\phi(y)}{dy} dy}{2} \\
&= 2\sin\frac{(1+E)\int_x^{x+t} \frac{\frac{d\phi(y)}{dy}}{i\phi(y)} dy}{2} \\
&= 2\sin\frac{(1+E)\int_x^{x+t} \frac{2}{1+y^2} dy}{2} \\
&= 2\sin\{(1+E)(\arctan(x+t) - \arctan x)\},
\end{aligned}$$

and similarly,

$$|F_\varepsilon(e^{i(\theta+\varphi)}) - F_\varepsilon(e^{i\theta})| > 2\sin\{(1-E)(\arctan(x+t) - \arctan x)\}.$$

For any $0 < \alpha \le \pi/2$,

$$\begin{aligned}
\left|\frac{\sin\{(1\pm E)\alpha\}}{\sin\alpha} - 1\right| &= \left|\frac{\sin\alpha\cos(E\alpha) \pm \cos\alpha\sin(E\alpha)}{\sin\alpha} - 1\right| \\
&= \left|\cos(E\alpha) \pm \cos\alpha\frac{\sin(E\alpha)}{\sin\alpha} - 1\right| \\
&\le |\cos(E\alpha) - 1| + \left|\frac{\sin(E\alpha)}{\sin\alpha}\right| \\
&= |\cos(E\alpha) - 1| + \left|\frac{\sin(E\alpha)}{E\alpha}\right|\left|\frac{\alpha}{\sin\alpha}\right| E \\
&\le E\alpha + \frac{\pi}{2}E \le \pi E \to 0
\end{aligned}$$

as $E \to 0$. This means that we can write $\sin\{(1\pm E)\alpha\} = (1 + O(E))\sin\alpha$.

Therefore,

$$\begin{aligned} & 2\sin\{(1\pm E)(\arctan(x+t)-\arctan x)\} \\ = \ & 2(1+O(E))\sin(\arctan(x+t)-\arctan x). \end{aligned}$$

We conclude that

$$|F_\varepsilon(e^{i(\theta+\varphi)})-F_\varepsilon(e^{i\theta})|<2(1+O(E))\sin(\arctan(x+t)-\arctan x)$$

and

$$|F_\varepsilon(e^{i(\theta+\varphi)})-F_\varepsilon(e^{i\theta})|>2(1+O(E))\sin(\arctan(x+t)-\arctan x).$$

Also we have similar estimates for

$$|F_\varepsilon(e^{i\theta})-F_\varepsilon(e^{i(\theta-\psi)})|,\ |F_\varepsilon(e^{i(\theta-\psi)})-1|, \text{ and } |F_\varepsilon(e^{i(\theta+\varphi)})-1|.$$

Finally, we can see that

$$\begin{aligned} & \frac{f_\varepsilon(x+t)-f_\varepsilon(x)}{f_\varepsilon(x)-f_\varepsilon(x-t)} \\ < \ & (1+O(E))\frac{\sin(\arctan(x+t)-\arctan x)\sin(\frac{\pi}{2}-\arctan(x-t))}{\sin(\arctan x-\arctan(x-t))\sin(\frac{\pi}{2}-\arctan(x+t))} \\ = \ & (1+O(E))\frac{\frac{t}{\sqrt{1+(x+t)^2}\sqrt{1+x^2}}\frac{1}{\sqrt{1+(x-t)^2}}}{\frac{t}{\sqrt{1+(x-t)^2}\sqrt{1+x^2}}\frac{1}{\sqrt{1+(x+t)^2}}} \\ = \ & 1+O(E). \end{aligned}$$

The lower estimate is similar. □

Lemma 4.3.2 implies that the mapping $f^l_{j,\varepsilon}\circ f\circ f_\varepsilon^{-1}$ is $(1+O(\varepsilon))$-quasisymmetric on the circle $\partial N^l_j=\{|z|=\delta\}$ for sufficiently small ε. We apply Lemma 5.1 in [Gup11]. Then, there exists a mapping $\eta^l_{j,\varepsilon}: N^l_j\to\{|z|<\delta\}$ which is $(1+O(\varepsilon))$-quasiconformal, $\eta^l_{j,\varepsilon}|_{\partial N^l_j}=f^l_{j,\varepsilon}\circ f\circ f_\varepsilon^{-1}|_{\partial N^l_j}$, and $\eta^l_{j,\varepsilon}$ is the identity in a sufficiently small neighborhood of 0. The mapping $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}: N^l_j\to f\circ f_\varepsilon^{-1}(N^l_j)$ is $(1+O(\varepsilon))$-quasiconformal and satisfies $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}|_{\partial N^l_j}=f\circ f_\varepsilon^{-1}|_{\partial N^l_j}$ and $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}(0)=0$. We consider the mapping of R_ε onto R' which is $(f^l_{j,\varepsilon})^{-1}\circ\eta^l_{j,\varepsilon}$ in N^l_j for any $j=1,\cdots,k$ and $l=1,2$, and is $f\circ f_\varepsilon^{-1}$ on $R_\varepsilon-\bigcup_{j=1,\cdots,k}^{l=1,2}N^l_j$. This mapping is $(1+O(\varepsilon))$-quasiconformal and is clearly homotopic to $f \circ f_\varepsilon^{-1}$. Therefore, we conclude that there exists a $(1+O(\varepsilon))$-quasiconformal Teichmüller mapping $h_\varepsilon : R_\varepsilon \to R'$ which is homotopic to $f \circ f_\varepsilon^{-1}$.

We deform h_ε to a $(1+O(\varepsilon))$-quasiconformal mapping which is conformal on a neighborhood of nodes of R_ε. We use $\varepsilon' < 1$ instead of $O(\varepsilon)$. Let q be the holomorphic quadratic differential on $R_\varepsilon - \{\text{nodes of } R_\varepsilon\}$ which is corresponding to the Teichmüller mapping h_ε. Let $z = x + iy$ be any q-coordinate, then h_ε is represented by $z \mapsto x + i(1+\varepsilon')y$. We consider the set $D_\varepsilon = \{-\varepsilon' \le x \le \varepsilon', 0 \le y \le \varepsilon'\} - \{0\}$ where 0 corresponds to a node of R_ε. Let H_ε be a mapping on the half set $\{0 \le x, y \le \varepsilon'\} - \{0\}$ which is defined the following:

$$H_\varepsilon(z) = \begin{cases} z & (0 \le x, y \le \varepsilon'^2, z \neq 0) \\ x + i\frac{y-\varepsilon'^3}{1-\varepsilon'} & (0 \le x \le \varepsilon'^2, \varepsilon'^2 \le y \le \varepsilon') \\ x + i\frac{x+1-\varepsilon'-\varepsilon'^2}{1-\varepsilon'}y & (\varepsilon'^2 \le x \le \varepsilon', 0 \le y \le \varepsilon'^2) \\ x + i\frac{(1-\varepsilon'^2-\varepsilon'(x+1-\varepsilon'-\varepsilon'^2))y+\varepsilon'^2(x-\varepsilon')}{(1-\varepsilon')^2} & (\varepsilon'^2 \le x, y \le \varepsilon') \end{cases}$$

An easy calculation shows that H_ε is $(1+O(\varepsilon))$-quasiconformal. We extend H_ε symmetrically to D_ε. We assume that $z = 0$ is a critical point of q of order $n \ge -1$. We consider the $n+2$ copies of D_ε around 0 whose horizontal segments of the boundary of D_ε lie on the horizontal trajectories of q which tend to 0, and vertical segments of the boundary of D_ε are joined adjacent to other D_ε (Figure 4.2).

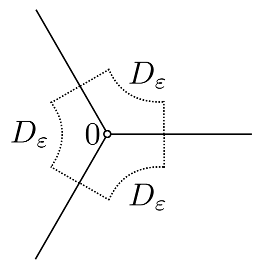

Figure 4.2: The gluing of D_ε in the case of $n = 1$

Finally, we extend the mapping H_ε as the original mapping h_ε on the outside of all D_ε, i.e., it is of the form $z \mapsto x + i(1+\varepsilon')y$. Then the mapping H_ε is defined on the whole surface R_ε. The quasiconformal mapping $H_\varepsilon \circ h_\varepsilon^{-1} : R' \to R'$ tends to the identity as $\varepsilon \to 0$. Hence, for sufficiently small ε, the quasiconformal mapping $H_\varepsilon \circ h_\varepsilon^{-1}$ is homotopic to the identity because each mapping class group on components of $R' - \{\text{nodes of } R'\}$ is discrete. We conclude that the composition $H_\varepsilon \circ f_\varepsilon$ is homotopic to f, and it is our desired mapping g. $\square$

We apply Lemma 4.3.1 to the mapping $h : Y_\infty \to Y'_\infty$. Hence, for any $j = 1, \cdots, k$ and $l = 1, 2$, we assume that h_j^l is $(K + O(\varepsilon))$-quasiconformal such that it is conformal in a neighborhood of 0 in $A_j^l(\infty)$ and is homotopic to $(g'_\infty \circ f') \circ \alpha \circ (g_\infty \circ f)^{-1}$. In this neighborhood, h_j^l is represented by a power series. Since for sufficiently large t and for any $j = 1, \cdots, k$ and $l = 1, 2$, the domain $\{\delta_j(t) \leq |z| \leq 2\Delta_j(t)\}$ of $A_j^l(t)$ on Y_t, defined in the proof of Theorem 3.1.1, is in the neighborhood, then we can use the same argument as in the case of Theorem 3.1.1. For sufficiently large t, there exists a mapping $F_t : Y_t \to Y'_t$ which is homotopic to $(g'_t \circ f') \circ (g_t \circ f)^{-1}$ such that

$$\lim_{t\to\infty} K(F_t) < \max\left\{ \max_{j=1,\cdots,k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}, K \right\} + O(\varepsilon)$$

and we have

$$\begin{aligned} &\limsup_{t\to\infty} d_{T(X)}(r(t), r'(t)) \\ &\leq \max\left\{ \frac{1}{2} \log \max_{j=1,\cdots,k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty)) \right\}. \end{aligned}$$

For the lower estimate, we can use the inequality (3.1),

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \geq \frac{1}{2} \log \max_{j=1,\cdots,k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}.$$

We recall that this estimation do not require the condition $r(\infty) = r'(\infty)$. Furthermore, we use the following fact.

Proposition 4.3.3. [Mas75] *We have*

$$\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \geq d_{\hat{T}(X)}(r(\infty), r'(\infty)).$$

Combine the above two inequalities, we obtain the inequality

$$\begin{aligned} &\liminf_{t\to\infty} d_{T(X)}(r(t), r'(t)) \\ &\geq \max\left\{ \frac{1}{2} \log \max_{j=1,\cdots,k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}, d_{\hat{T}(X)}(r(\infty), r'(\infty)) \right\}. \end{aligned}$$

□

Proof of Corollary 4.1.2. This is similar as in the case of Proposition 3.1.3. □

References

[Abi77] William Abikoff. Degenerating families of Riemann surfaces. *Ann. of Math. (2)*, 105(1):29–44, 1977.

[Abi80] William Abikoff. *The real analytic theory of Teichmüller space*, volume 820 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[FLP79] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Travaux de Thurston sur les surfaces*, volume 66 of *Astérisque*. Société Mathématique de France, Paris, 1979. Séminaire Orsay, With an English summary.

[FM10] Benson Farb and Howard Masur. Teichmüller geometry of moduli space, I: distance minimizing rays and the Deligne-Mumford compactification. *J. Differential Geom.*, 85(2):187–227, 2010.

[GM91] Frederick P. Gardiner and Howard Masur. Extremal length geometry of Teichmüller space. *Complex Variables Theory Appl.*, 16(2-3):209–237, 1991.

[Gro81] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 183–213. Princeton Univ. Press, Princeton, N.J., 1981.

[Gup11] Subhojoy Gupta. Asymptoticity of grafting and Teichmüller rays I. *arXiv:1109.5365v1*, 2011.

[HM79] John Hubbard and Howard Masur. Quadratic differentials and foliations. *Acta Math.*, 142(3-4):221–274, 1979.

[HS07] Frank Herrlich and Gabriela Schmithüsen. On the boundary of Teichmüller disks in Teichmüller and in Schottky space. In *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lect. Math. Theor. Phys.*, pages 293–349. Eur. Math. Soc., Zürich, 2007.

[IT92] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.

[Iva92] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[Iva01] Nikolai V. Ivanov. Isometries of Teichmüller spaces from the point of view of Mostow rigidity. In *Topology, ergodic theory, real algebraic geometry*, volume 202 of *Amer. Math. Soc. Transl. Ser. 2*, pages 131–149. Amer. Math. Soc., Providence, RI, 2001.

[Ker80] Steven P. Kerckhoff. The asymptotic geometry of Teichmüller space. *Topology*, 19(1):23–41, 1980.

[KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[Len06] Anna Lenzhen. *Teichmuller geodesics that do not have a limit in PMF*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–University of Illinois at Chicago.

[LM10] Anna Lenzhen and Howard Masur. Criteria for the divergence of pairs of Teichmüller geodesics. *Geom. Dedicata*, 144:191–210, 2010.

[LS12] Lixin Liu and Weixu Su. The horofunction compactification of Teichmüller metric. *arXiv:1012.0409v4*, 2012.

[Mas75] Howard Masur. On a class of geodesics in Teichmüller space. *Ann. of Math. (2)*, 102(2):205–221, 1975.

[Mas80] Howard Masur. Uniquely ergodic quadratic differentials. *Comment. Math. Helv.*, 55(2):255–266, 1980.

[Miy08] Hideki Miyachi. Teichmüller rays and the Gardiner-Masur boundary of Teichmüller space. *Geom. Dedicata*, 137:113–141, 2008.

[Miy11] Hideki Miyachi. Teichmüller space has non-Busemann points. *arXiv:1105.3070v1*, 2011.

[Ree81] Mary Rees. An alternative approach to the ergodic theory of measured foliations on surfaces. *Ergodic Theory Dynamical Systems*, 1(4):461–488 (1982), 1981.

[Rie02] Marc A. Rieffel. Group C^*-algebras as compact quantum metric spaces. *Doc. Math.*, 7:605–651 (electronic), 2002.

[Str84] Kurt Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1984.

[Wal11] Cormac Walsh. The horoboundary and isometry group of Thurston's Lipschitz metric. *arXiv:1006.2158v3*, 2011.

[Wal12] Cormac Walsh. The asymptotic geometry of the Teichmüller metric. *arXiv:1210.5565v1*, 2012.