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著者(和文)	岡野訓尚
Author(English)	Kunihisa Okano
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Networked Control of Uncertain Systems under Communication Constraints

by

Kunihisa Okano

A thesis submitted in conformity with the requirements
for the degree of Doctor of Engineering
Department of Computational Intelligence and Systems Science
Tokyo Institute of Technology

Contents

Abstract

We consider stabilization of uncertain control systems over communication channels. Due to the existence of the channels, communication between the plant and the controller is constrained. In particular, the data rate — the number of bits transmitted through a channel at a time — is limited and transmitted packets may be randomly lost. Hence, the controller cannot utilize complete information of the plant states. These constraints reflect bandwidth limitations and unreliability due to congestion, collision, or effects of noises, which are often observed in practical communications.

In addition to the incompleteness in state observation, we explicitly take account of the uncertainty of the plant model. Uncertainties have been an important subject in control theory. However, in the context of networked control, it has been commonly assumed that exact plant models are available.

The objective of the thesis is to clarify how large the amount of information communicated through the channel should be, especially in the presence of system uncertainties. We derive necessary conditions and sufficient conditions on the communication constraints for stability.

The necessity results provide limitations on the data rate, the packet loss probability, and the magnitude of the plant uncertainty. The limitations are characterized by the product of the eigenvalues of the plant similarly to the well-known data rate theorems for the known plants case. We also derive the optimal quantizer to achieve stability under the minimum data rate. It is shown that the quantizer takes a nonuniform structure due to the uncertainty.

The sufficient conditions provide stabilizing controllers, which can be less conservative compared with those given in existing results. The relationship between the convergence rate and the communication constraints is also given.

The thesis studies problems formulated in four setups. We first establish our results under the fundamental setup and then consider more general cases: One of them is for further data rate suppression. The other two setups are dedicated to deal with more practical classes of uncertain plants and packet loss behaviors, respectively.

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Notation

$\mathbb{R}$	set of real numbers
$\mathbb{Z}_+$	set of nonnegative integers
$\mathbb{N}$	set of natural numbers
$\log(\cdot)$	$\log_2(\cdot)$
$\mu(\cdot)$	Lebesgue measure on $\mathbb{R}$
$c(\cdot)$	midpoint of an interval on $\mathbb{R}$
$\bar{\mathcal{X}}$	supremum of an interval $\mathcal{X}$ on $\mathbb{R}$
$\underline{\mathcal{X}}$	infimum of an interval $\mathcal{X}$ on $\mathbb{R}$
$\lceil\cdot\rceil$	ceiling function (the smallest integer greater than or equal to a number)
$\lfloor\cdot\rfloor$	floor function (the largest integer less than or equal to a number)
$\lvert\cdot\rvert$	absolute value
$\lVert\cdot\rVert$	Euclidean norm
$[\cdot]_n$	residue modulo n
$\otimes$	Kronecker product
$\bullet^T$	transpose of a matrix
$\mathrm{diag}(\cdot)$	block diagonal matrix
$\lambda_i(\cdot)$	eigenvalue of a matrix
$\lambda_\bullet$	product of the eigenvalues of a matrix
$\rho(\cdot)$	spectral radius of a matrix
I_n	identity matrix of size n

Chapter 1

Introduction

1.1 Control systems containing communications

Due to the recent growth in communication technology and development of small and low-price computing devices, employing communication components in control systems is becoming more common. Fig. 1.1 illustrates an example of such control systems. Use of communication devices provides many benefits such as flexibility in spacial deployments and cost reduction in maintenance. Such control systems — systems whose components are connected through communication channels — are called as networked control systems and have attracted research interests during the last decade [1, 3].

The main difficulty in networked control systems is the constraints in the communication between the plant and the controller due to the existence of channels:

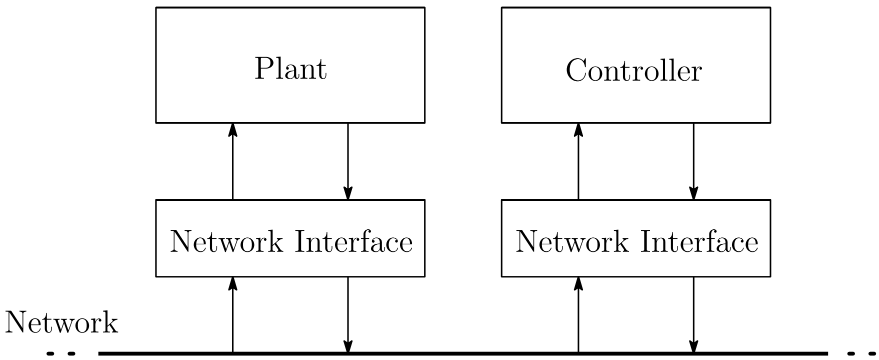


Figure 1.1: Control system containing communication

Recent communication links are often digital and hence transmitted data should be encoded to digital signals. These signals may contain errors to the original analog signals. It is possible to make the errors small if we were allowed to consume large data to represent the signals. However, the bit-length of the data is restricted by the bandwidth of the channel. In addition to the errors, transmitted signals may be delayed or even be lost because of congestion, collision, or effects of noises.

It is known that these constraints may be harmful for control. Though the capacity and reliability of recent communication channels are high, the number of components which communicate through the channel is also increasing. Thus, a component can only use a small portion of the channel resources. We also note that in some applications it is difficult to establish rich communication links due to environmental factors or battery capacities. For example, in [55], control of a group of autonomous underwater vehicles has been studied and it has been pointed out that bandwidth is severely limited underwater. In networked control systems, it is required to deal with these communication constraints explicitly and to establish communication rich enough to achieve given control objectives.

In this thesis, we study a stabilization problem of a system over a communication channel, where quantization is employed and transmitted packets may be lost. Our objective is to clarify how large the amount of information communicated through the channel should be to stabilize the system. This has been a central question in the field of networked control. The main feature of this research is that we explicitly consider uncertainty in plant models, which has commonly been ignored in the literature.

1.2 Related research

1.2.1 Limitations on communication constraints

As we have seen, in networked control systems, control-related information is incomplete to some degree due to the constraints induced by channels. The effects of such constraints on control performance have been actively studied.

We now provide an overview of the works in the area. Specifically, we review the references relevant to the topic of this thesis: control under the effects of quantization and packet losses.

Quantization

Control over quantized signals has been one of the central problems in networked control [27, 41]. The effects of quantization were traditionally modeled as additive white noise. An early work that considered quantization in a explicit form is [11]. In the work, it has been shown that quantization may cause chaotic behavior, which is quite different from what in the white noise model. Afterwards, the seminal work [62] has first provided the minimum quantization levels for stabilization. Stimulated by these pioneering works, control systems containing quantizers have been actively studied during the last decade. In the following, we introduce two approaches in this area.

Data rate: One approach to formulate control with quantized information is motivated by control based on finite data rate signals. The data rate is the number of bits transmitted through a channel at a time and is restricted by the bandwidth of the channel. This has been first attempted in [62] as a stabilization problem of a linear system. The work has succeeded to show the celebrated data rate theorem: There exists a critical value in the data rate to stabilize a feedback system, and the value depends only on the product of the unstable eigenvalues of the open-loop system. In [62], containability – existence of a control which remains any trajectory in a certain set – has been employed as the stability concept. On the other hand, [6] considers asymptotic stability and proposes quantizers with the variable input range, which is called a zoom-in/zoom-out function. In [57], the theorem has been extended to a general framework of stabilization of linear deterministic systems. Moreover, stabilization under stochastic disturbances and noises has been studied in [40]. This problem of limited data rate control has been solved in a wide variety of setups: control over unreliable channels [64], control of nonlinear systems [28, 32] and those with delays [10], and designing controllers for the case where available data rate may vary over time [37].

Coarseness: In [15], another approach based on the notion of coarseness has been introduced to capture the effects of quantization. In the work, the quantizer has infinite output symbols and coarseness is defined as the number of symbols per input interval. It has been shown that the structure of the coarsest quantizer is a logarithmic type, and moreover its coarseness is characterized by the product of the unstable eigenvalues of the plant. This result has been extended to the unreliable channel case [60]. Furthermore, the logarithmic quantizer has been employed in [18], where quantized control problem has been studied along the line of robust control [66].

In most of the works, the logarithmic quantizer is memoryless and static while the literature concerning the data rate theorem commonly considers dynamic quantizers with memories. We note that another memoryless quantizer can be found in [17]. The work provides a stabilization technique utilizing chaotic behavior induced by quantization.

Packet losses

The behavior of packet losses due to unreliability in communication is often treated as a random process. The limitations on packet loss probability have been studied both in stochastic stabilization problems [14, 25, 26] and in state estimation problems [24, 52, 53] (see also [23, 50] and the references therein). It is remarkable that similar to the minimum data rates, the maximum packet loss probability for achieving stabilization is characterized solely by the product of the unstable eigenvalues of the plant.

Though most of the literature has assumed that the packet losses occur independently for analytical convenience, several works have considered more advanced random processes. In particular, it is well known that Markov chains can be a useful model to express practical communication failures [16, 20]. In [53], the state estimation problem over Markovian losses is studied. The stabilization problem is also tackled both in infinite [21] and finite [65] data rate cases.

Other related works

In addition to the above two, there are several types of communication constraints. Recent works have dealt with multiple types of communication constraints simultaneously: In [42], quantization and scheduling of signal transmissions have been considered, and the work [7] studies control under packet losses and time-varying sampling intervals or delays.

It is interesting to note that several works dealing with limited data control problems have pointed out that notions and tools in information theory are useful in the analysis: In [58], rate distortion theory is employed to deal with the linear quadratic Gaussian problem over a channel. The work of [51] has studied performance limitations by entropy-based analysis. Further discussion can be found in [35].

1.2.2 Uncertain networked control systems

Uncertainties in the model of the system to be controlled is one of the important subjects studied in control theory, as we have seen the success of robust control theory [2, 4, 66]. However, in the field of networked control, the number of results involving uncertainty is fairly small considering the volume of this research area. Here we introduce some works concerning stabilization problems below.

In [48], linear time-invariant systems with norm bounded uncertainties are considered. An encoder-controller pair to robustly stabilize the system is proposed. In the work, the worst case in the state evolution is evaluated based on the norm of the uncertainty, and the coder-controller is designed in a somewhat conservative way to take care of all possible evolutions. In [34], scalar nonlinear systems with stochastic uncertainties and disturbances are dealt with. In the setup, the channel state is also stochastic; the available data rate may randomly vary. It has given sufficient data rates which are enough to achieve the mth moment stability. The work of [19] has employed the logarithmic quantizer and has studied using the techniques of robust control. On the other hand, [22] considers the problem from adaptive control viewpoints. Finally, in [61] control of uncertain systems by multiple controllers with supervisors has been studied.

In the existing results listed above, only sufficient conditions on data rates have been obtained, that is, these results have not been concerned with characterizing the minimum data rate. We also note that the above works consider the cases with lossless channels.

1.3 Contributions of the thesis

The focus of our study is to derive limitations on the data rates necessary for stabilization of uncertain systems over lossy channels. In particular, the limitation is characterized by the level of plant instability as in the data rate theorems mentioned above and is expressed in terms of data rates, packet loss probabilities, and the uncertainty bounds on plant parameters. Under the presence of uncertainties, in general, this is a difficult problem. The reason is that the combination of plant uncertainties and the nonlinearity in the system due to quantization complicates the analysis of state evolutions.

In the thesis, to overcome such difficulties, we formulate the problem based on two ideas as follows. First, we assume that the plant is a single-input and single-output system, and its uncertainties are parametric. In the analysis of such a system, the key parameter representing the product of eigenvalues in the plant can be expressed as a single parameter, which corresponds to the constant term in the denominator polynomial of the transfer function; for the case of known plants, this viewpoint has been proposed in [64]. The parameter can vary over time within a known bound, and in this sense the plant is parametrically uncertain. In the context of robust control, parametrically uncertain systems have been extensively studied (see, e.g., [2, 4]). The celebrated Kharitonov's theorem [30] provides an exact condition for robust stability for continuous-time systems, though its extensions to discrete-time systems are somewhat limited. Other developments include pole placement algorithms using linear programming in [54], stability tests based on μ-analysis [12, 13], linear matrix inequalities [5], and integral-quadratic-constraint approaches [36].

The second idea to address the uncertain systems case is the introduction of some structure into the controller by imposing restriction on the state estimation scheme. It is important to note that this controller class includes those that have appeared in minimum data rate results for known plants for derivation of sufficient conditions [56, 57, 64]. Therefore, when specialized to the case without uncertainties, our results coincide with those in such previous works and particularly [64]. An interesting aspect in the uncertain case is that the quantizer used in the encoder should not be restricted to the conventional uniform quantizers as in [34, 48]. We propose a new quantizer, which is in fact designed to compensate plant uncertainties and is capable of reducing the required data rate. Indeed, for scalar plants, this quantizer becomes optimal. This quantizer is a piecewise constant function whose step width shrinks as the input becomes larger in magnitude. In the special case of known plants, it becomes uniform, which supports the use of uniform quantizers in [64].

1.4 Overview of the thesis

This thesis is organized as follows. In Chapter 2, we show the first part of our results in the most fundamental setup. In the chapter, the data rate is assumed static, i.e., the bit-length of transmitted data is the same for every time step. Also, a part of the exact plant model is assumed to be known. In particular, we suppose that the input parameters are available. Furthermore, the packet losses are modeled as independent and identically distributed (i.i.d.) random processes.

In Chapter 3, we progress to a more general framework, where the data rate can be time varying. We show that the necessary bounds given in Chapter 2 are valid in this case also, but in the sufficiency result we can achieve stability with a smaller data rate by employing a communication protocol with a time-varying data rate.

In Chapter 4, we move on to deal with a more realistic class of uncertain plants. We consider the plant where not only state parameters but also input parameters are uncertain. That is, in the setup, the model of the actuator in the plant may include uncertainty. We generalize the results in Chapter 2, where the input parameters are assumed to be known. We show a necessary condition for stability, which clarify the effect of uncertain inputs on the data rate and the loss probability limitations. Also, a sufficient condition for this case is given.

Chapter 5 is devoted to another extension of the results in Chapter 2: The packet loss process model is extended from i.i.d. to Markov chain. Markovian packet losses can represent more practical situations including bursty communication failures. We provide a necessary condition and a sufficient condition for stability. Similar to the fundamental results, the necessary condition is given in the form of limitations on the data rate, the transition probability of the channel states, and the magnitude of the uncertainty.

Finally, we conclude in Chapter 6, where a summary of the thesis and open problems left for future research are stated.

Chapter 2

Control of uncertain systems over networks

In this chapter, we study limitations on communication for control of uncertain systems in the most fundamental setup. The stabilization problem of an uncertain system over a lossy and data rate limited channel is considered. The encoder and the decoder use a static data rate, i.e., the bit-length of transmitted data is static for every time step. The packet loss process is assumed to be independent and identically distributed (i.i.d.). Under the setup, a necessary condition and a sufficient condition for stabilization are presented. In particular, the necessity result provides limitations on the data rate, the loss probability, and the magnitude of uncertainties. The result can be considered as a generalization of the limitations which have been shown for the known plants case [57, 64].

This chapter is organized as follows. In Section 2.1, we formulate the stabilization problem. In Sections 2.2 and 2.3, we present the necessity result and the sufficiency result, respectively.

The results in this chapter have been partially presented in our works [43–45].

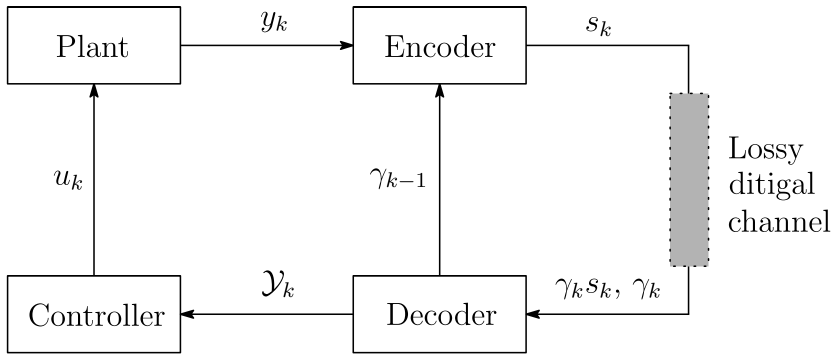


Figure 2.1: Networked control system

2.1 Problem formulation

We consider the networked system depicted in Fig. 2.1, where the plant connects with the controller by the communication channel. At time $k \in \mathbb{Z}_+$, the encoder observes the plant output $y_k \in \mathbb{R}$ and quantizes it to a discrete value. The quantized signal $s_k \in \Sigma_N$ is transmitted to the decoder through the channel. Here, the set Σ_N represents all possible outputs of the encoder and contains N symbols. Thus, the required data rate is expressed as $R := \log N$ [bits/sample]. The decoder receives the symbol s_k and decode it into the interval $\mathcal{Y}_k \subset \mathbb{R}$, which is an estimate of y_k. Finally, using the past and current estimates, the controller provides the control input $u_k \in \mathbb{R}$.

We now describe the details of each component in the system.

The plant is an n-dimensional autoregressive system which has uncertain parameters:

$$y_{k+1} = a_{1,k}y_k + a_{2,k}y_{k-1} + \cdots + a_{n,k}y_{k-n+1} + u_k. \tag{2.1}$$

Each parameter $a_{i,k}$ may be time varying and may vary within a known width. We denote the width of perturbation of the ith parameter by $2\epsilon_i$, where $\epsilon_i \geq 0$, and let a_i^* be its center, i.e.,

$$a_{i,k} \in \mathcal{A}_i := [a_i^* - \epsilon_i, a_i^* + \epsilon_i] \text{ for } i = 1, 2, \ldots, n. \tag{2.2}$$

These parameters are deterministic, and we do not make any assumption on the distribution within the above interval. The initial values y_k, $k = -n+1, \dots, -1, 0$, are bounded by known upper bounds as $|y_k| \le Y_k$, where $Y_k > 0$.

Remark 2.1. In (2.1), the parameter of the input u_k is scalar and is assumed to be known. However, the results in this chapter can easily be extended to the case where the plant has multiple b-parameters such as $y_{k+1} = \sum_{i=1}^{n_y} a_{i,k} y_{k-i+1} + \sum_{i=1}^{n_u} b_i u_{k-i+1}$, where $b_1, \dots, b_{n_u}$ are known parameters and $b_1 \neq 0$. In [39], a related class of plants is studied for limited data rate control. In Chapter 4, we consider the case that both parameters of the state and the input are uncertain.

The plant (2.1) can be written in the controllable canonical form:

$$x_{k+1} = A_k x_k + B u_k, \quad y_k = C x_k, \tag{2.3}$$

where the state $x_k := [y_{k-n+1} \; y_{k-n+2} \; \cdots \; y_k]^T$ and the system matrices are given by

$$\begin{aligned} A_k &:= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_{n,k} & a_{n-1,k} & \cdots & a_{1,k} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad B := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T \in \mathbb{R}^n, \\ C &:= B^T \in \mathbb{R}^{1 \times n}. \end{aligned} \tag{2.4}$$

Let A^* represent the nominal matrix of A_k, and let λ_{A^*} be the product of the eigenvalues of A^*:

$$A^* := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_n^* & a_{n-1}^* & \cdots & a_1^* \end{bmatrix}, \quad \lambda_{A^*} := \prod_{i=1}^{n} \lambda_i(A^*) = a_n^*,$$

where $\lambda_i(\cdot)$ represents an eigenvalue of a matrix.

The encoder quantizes the plant output y_k into the N-alphabet signal $s_k \in \Sigma_N$, where $\Sigma_N := \{1, 2, \dots, N\}$. The input range of the encoder is centered at the origin and the width is defined by a scaling parameter $\sigma_k > 0$. In particular, the output s_k of the encoder is given as

$$s_k = q_N\left(\frac{y_k}{\sigma_k}\right), \tag{2.5}$$

where $q_N(\cdot)$ is an N-level quantizer whose input range is $[-1/2, 1/2]$. In the quantizer q_N, it is assumed that boundaries of the quantization cells are symmetric about the origin. By its symmetry, the quantizer q_N can be expressed by the set of boundary points

$$h_l \in \mathbb{R},\ \ l \in \{0, 1, \ldots, \lceil N/2 \rceil\},$$

of nonnegative quantization cells. Here, $\lceil \cdot \rceil$ is the ceiling function. These points must satisfy

$$h_0 = 0, \quad h_{\lceil N/2 \rceil} = \frac{1}{2}, \quad h_l < h_{l+1}. \tag{2.6}$$

In the above, the first and the second equations follow since the nonnegative quantization cells must fill the interval $[0, 1/2]$. The last inequality means that the domain of each cell must not be empty. The origin h_0 is a boundary only when the number N of quantization cells is even. However, for simplicity, we use the same notation above even if N is odd. Denote the quantization cells determined by $\{h_l\}_l$, from the left to the right, by $\mathcal{C}_i$, $i = 1, 2, \cdots, N$:

(i) If N is odd, then

$$\mathcal{C}_i := \begin{cases} [-h_{\lceil N/2 \rceil - i + 1}, -h_{\lceil N/2 \rceil - i}) & \text{if } i \in \{1, 2, \cdots, \lceil \frac{N}{2} \rceil - 1\}, \\ [-h_1, h_1) & \text{if } i = \lceil N/2 \rceil, \\ [h_{i - \lceil N/2 \rceil}, h_{i + 1 - \lceil N/2 \rceil}) & \text{if } i \in \{\lceil \frac{N}{2} \rceil + 1, \lceil \frac{N}{2} \rceil + 2, \cdots, N-1\}, \\ [h_{\lceil N/2 \rceil - 1}, h_{\lceil N/2 \rceil}] & \text{if } i = N. \end{cases} \tag{2.7}$$

(ii) If N is even, then

$$\mathcal{C}_i := \begin{cases} [-h_{N/2-i+1}, -h_{N/2-i}) & \text{if } i \in \{1, 2, \cdots, N/2\}, \\ [h_{i-1-N/2}, h_{i-N/2}) & \text{if } i \in \{\frac{N}{2}+1, \frac{N}{2}+2, \cdots, N-1\}, \\ [h_{N/2-1}, h_{N/2}] & \text{if } i = N. \end{cases} \tag{2.8}$$

Then, for a given set of boundaries $\{h_l\}_l$ of the quantizer q_N and consequently the quantization cells (2.7) or (2.8), we define the outputs of the quantizer as follows:

$$q_N(y) := i \quad \text{if } y \in \mathcal{C}_i \text{ for } i = 1, 2, \cdots, N. \tag{2.9}$$

For simplicity of notation, both q_N and $\{h_l\}_l$ are used to refer to quantizers.

Due to unreliability in the channel, the transmitted signal s_k is randomly lost. We represent the state of the packet reception/loss at time k by the random variable $\gamma_k \in \{0, 1\}$. If $\gamma_k = 0$ then the packet is lost; otherwise, it arrives at the actuator side successfully. We assume that the process $\{\gamma_k\}_{k=0}^{\infty}$ is independent and identically distributed (i.i.d.) with the loss probability $p \in [0, 1)$, i.e., $\text{Prob}(\{\gamma_k = 0\}) = p$ for all $k \in \mathbb{Z}_+$. Furthermore, the encoder knows the previous loss state γ_{k-1} through the acknowledge (ACK) signal from the decoder.

The decoder converts the received signal $\gamma_k s_k$ to the interval $\mathcal{Y}_k \subset \mathbb{R}$. The interval $\mathcal{Y}_k$ provides an estimate of the set in which the plant output y_k should be contained. If the packet arrives ($\gamma_k = 1$), then $\mathcal{Y}_k$ corresponds to the quantization cell that y_k falls in. Otherwise $\mathcal{Y}_k$ is equal to the entire input range of the encoder $[-\sigma_k/2, \sigma_k/2]$, which is available at the decoder. Accordingly, the estimation set $\mathcal{Y}_k$ is given by

$$\mathcal{Y}_k := \begin{cases} [-\sigma_k/2, \sigma_k/2] & \text{if } \gamma_k = 0, \\ \sigma_k \mathcal{C}_{s_k} & \text{if } \gamma_k = 1, \end{cases} \tag{2.10}$$

where $\sigma_k \mathcal{C}_{s_k} := \{\sigma_k y : y \in \mathcal{C}_{s_k}\}$ and s_k is the transmitted signal defined in (2.5).

Here, we focus on the data rate expressing the state values. In practical situations, the encoder may include other data into transmitted signals such as headers and error detection/correction codes. However, we do not consider the bit-lengths of such data.

The controller provides the control input u_k based on the past and current estimates $\mathcal{Y}_{k-n+1}, \dots, \mathcal{Y}_k$ as

$$u_k = \sum_{i=1}^{n} f_{i,k}\left(\mathcal{Y}_{k-i+1}\right), \tag{2.11}$$

where $f_{i,k}(\cdot)$ is an arbitrary map from an interval on $\mathbb{R}$ to a real number.

We remark that the scaling parameter σ_k should be large enough to cover all possible inputs to the encoder. Otherwise, the quantizer may be saturated, in which case we lose track of the plant output y_k. On the other hand, if we take σ_k large, the quantization error also becomes large. Moreover, to achieve stabilization of the system, σ_k should decay to zero gradually.

We determine the scaling parameter σ_k as follows. At time k, the encoder and the decoder predict the next plant output y_{k+1} based on the estimates $\mathcal{Y}_{k-n+1}, \dots, \mathcal{Y}_k$. Let $\mathcal{Y}_{k+1}^- \subset \mathbb{R}$ be the set of all possible outputs y_{k+1} of the uncertain system (2.1). Then the scaling parameter σ_{k+1} is chosen such that

$$\sigma_{k+1} \geq \mu(\mathcal{Y}_{k+1}^-), \tag{2.12}$$

where $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$.

The prediction set of the plant output y_{k+1} constructed at time k is defined as follows:

$$\begin{aligned} \mathcal{Y}_{k+1}^- := \{ & a_1' y_k' + \cdots + a_n' y_{k-n+1}' : a_1' \in \mathcal{A}_1, \dots, a_n' \in \mathcal{A}_n, \\ & y_k' \in \mathcal{Y}_k, \dots, y_{k-n+1}' \in \mathcal{Y}_{k-n+1} \}. \end{aligned} \tag{2.13}$$

Under this definition, our prediction strategy is to use the information regarding $y_k, \dots, y_{k-n+1}$ independently such that $y_{k-i+1} \in \mathcal{Y}_{k-i+1}$ for each $i = 1, 2, \dots, n$, where $\mathcal{Y}_{k-i+1}$ is the interval received on the decoder side at time $k - i + 1$. Then, clearly, $\mu(\mathcal{Y}_{k+1}^-)$ is large enough to include y_{k+1}, and it is computable on both sides of the channel because of the ACK signal regarding γ_{k-1} from the decoder to the encoder.

We now introduce the definition of stability in the system.

Definition 2.1. The system depicted in Fig. 2.1 is mean square stable (MSS) if the plant output y_k asymptotically goes to zero in the mean square sense for all possible uncertainties within the bounds in (2.2). That is, for all (deterministic) parametric perturbations $a_{1,k} \in \mathcal{A}_1, a_{2,k} \in \mathcal{A}_2, \ldots, a_{n,k} \in \mathcal{A}_n$, it holds that $\lim_{k\to\infty} \mathrm{E}[|y_k|^2] = 0$, where the expectation is with respect to the packet losses.

The problem of the chapter is to give conditions on the data rate R and the packet loss probability p for the overall system in Fig. 2.1 to be MSS.

The problem setup given above is particularly affected by the consideration of uncertain plants. To overcome the difficulties, we have introduced some structures in the plant as well as the controller. In the plant (2.1), the product of the eigenvalues is represented as a single parameter $a_{n,k}$. It has been known that the product of the eigenvalues plays an important role to describe the bounds on the data rate and the loss probability [14, 40, 56, 57, 64]. Since our objective is to characterize the bounds by the product of the eigenvalues, the simple expression of the key parameter helps to reduce the complexity in the analysis.

Similarly, the classes of controllers (2.11) and prediction sets (2.13) are employed to pursue an analytical approach. Here, we use the information regarding $y_{k-n+1}, \ldots,$ y_k independently. This may make the state estimation somewhat conservative. If we use a more general controller or a prediction method that allows us to look at the correlations among them, then the estimation sets $\mathcal{Y}_{k-n+1}, \ldots, \mathcal{Y}_{k-1}$ from times before the current time k may be updated so that they shrink in size. As a result, the system can be stabilized under a smaller data rate compared with the case employing (2.11) and (2.13). We note that it may be possible to minimize the state estimation sets numerically [49]; however, in the case of uncertain plants, it is difficult to do this analytically.

When we know the exact plant model, i.e., $\epsilon_i = 0$ for $i = 1, 2, \ldots, n$, the following result given in [64] provides a necessary and sufficient condition for stability.

Proposition 2.1. Consider the system depicted in Fig. 2.1 where the exact plant model is known. Then the system is MSS if and only if the following inequalities are satisfied:

$$R > R^* := \log|\lambda^{\mathrm{u}}_{A^*}| + \log\sqrt{\frac{1-p}{1-p|\lambda^{\mathrm{u}}_{A^*}|^2}}, \tag{2.14}$$

$$p < p^* := \frac{1}{|\lambda^{\mathrm{u}}_{A^*}|^2}. \tag{2.15}$$

Here, $\lambda^{\mathrm{u}}_{A^*}$ denotes the product of the *unstable* eigenvalues of the nominal matrix A^*:

$$\lambda^{\mathrm{u}}_{A^*} := \prod_{i\,:\,|\lambda_i(A^*)|\geq 1} \lambda_i(A^*).$$

We will later see that our necessary conditions coincide with the above inequalities when the plant is known and all eigenvalues are unstable.

Here, regarding the set $\mathcal{Y}^-_{k+1}$, we introduce a useful lemma, which will be referred to in the following sections.

Lemma 2.1. The prediction set $\mathcal{Y}^-_{k+1}$ defined in (2.13) satisfies the following equality:

$$\mu(\mathcal{Y}^-_{k+1}) = \sum_{i=1}^{n} \mu\left(\mathcal{A}_i \mathcal{Y}_{k-i+1}\right), \tag{2.16}$$

where

$$\mathcal{A}_i\mathcal{Y}_{k-i+1} := \{a'y' : a' \in \mathcal{A}_i,\, y' \in \mathcal{Y}_{k-i+1}\} \tag{2.17}$$

for $i = 1, 2, \ldots, n$.

Proof. By applying the Brunn-Minkowski inequality [9] to (2.13), we have $\mu(\mathcal{Y}^-_{k+1}) \geq \sum_{i=1}^{n} \mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)$. Furthermore, the equality holds since $\mathcal{A}_i\mathcal{Y}_{k-i+1}$, $i = 1, \cdots, n$, are connected intervals in $\mathbb{R}$ by the definitions in (2.2) and (2.10). ∎

2.2 Limitations on data rate and loss probability

In this section, we present a necessary condition for the system to be MSS. The condition will be given in the form of a lower bound on the data rate R and an upper bound on the loss probability p expressed by the instability and uncertainty in the plant. The result provides an insight into the question that, to stabilize the uncertain system, at least how large the data rate should be. It is important to note that the existing works [19, 34, 48] dealing with uncertainties have studied only sufficient bounds, which have contained conservativeness. Moreover, we derive a quantizer which may reduce the necessary data rate. In particular, as we will see in Section 2.3, for scalar plants, the quantizer is optimal in the sense that we can stabilize the system under the lowest data rate.

Before providing the main result, we introduce an assumption regarding the instability of the plant (2.1) or (2.3).

Assumption 2.1. For every time $k \in \mathbb{Z}_+$, the matrix A_k has at least one unstable eigenvalue and the absolute value of the product of the eigenvalues is greater than 1. That is, we assume that the following inequality holds:

$$|a_n^*| - \epsilon_n > 1. \tag{2.18}$$

This assumption is required throughout the section since the objective here is to describe the limitations for stability by the product of eigenvalues λ_{A^*}. In the case of stable plants, the limitations are trivial since there is no need to control.

Remark 2.2. When the plant has both stable and unstable eigenvalues, we have to take account of the stable parts. In most of the literature which assumes exact plant models, stable eigenvalues have been omitted by applying transformation of the state coordinate [40, 56, 57, 64]. However, if the plant parameters contain uncertainties, we can not make such a coordinate transformation and thus, can not follow the technique. Hence, the plant considered here may have stable eigenvalues. If the plant has any stable modes, the limitations on the data rate and the loss probability given below necessarily become loose. The limitations are expressed by the product of the eigenvalues $\lambda_{A^*} = a_n^*$. Under the assumption (2.18), the product of the eigenvalues of the plant is always unstable. Hence, the limitations are not trivial, i.e., the bounds on the data rate and the loss probability take positive values. For a plant that does not satisfy (2.18), if we know its stable eigenvalues exactly, it may be possible to introduce a stable system to cancel those eigenvalues so that the resulting system has a larger $|a_n^*|$ for which (2.18) might hold.

2.2.1 Necessary condition for stability

We now introduce the following notations to represent the necessary bound:

$$\nu := \sqrt{\frac{1-p}{1-p(|\lambda_{A^*}|+\epsilon_n)^2}}, \quad r := \frac{|\lambda_{A^*}|-\epsilon_n}{|\lambda_{A^*}|+\epsilon_n}. \tag{2.19}$$

The following theorem is the necessity result.

Theorem 2.1. For the system in Fig. 2.1 satisfying Assumption 2.1, if the system is MSS with the static quantizer (2.9), then it holds that

$$R > R_{\text{nec}} := \begin{cases} \log \frac{\log(1-\epsilon_n \nu)^2}{\log r} & \text{if } \epsilon_n > 0, \\ \log |\lambda_{A^*}| + \log \nu & \text{if } \epsilon_n = 0, \end{cases} \tag{2.20}$$

$$0 \leq p < p_{\text{nec}} := \frac{1-\epsilon_n^2}{(|\lambda_{A^*}|+\epsilon_n)^2 - \epsilon_n^2}, \tag{2.21}$$

$$0 \leq \epsilon_n < 1. \tag{2.22}$$

One can easily confirm that the lower bound R_{nec} on data rate is monotonically increasing with respect to $|\lambda_{A^*}|$, p, and ϵ_n, and similarly p_{nec} is monotonically decreasing. Thus, more unstable dynamics or more uncertainty in the plant will result in higher requirement in communication with a larger data rate and a smaller loss probability. We remark that there is no gap between the two expressions in (2.20) since R_{nec} is right continuous with respect to ϵ_n at $\epsilon_n = 0$.

A special case of this result is when there is no plant uncertainty, i.e., $\epsilon_i = 0$, $\forall i$.

In such a case, the bounds in the theorem coincide with those in Proposition 2.1:

$$R_{\rm nec} \to R^*, \quad p_{\rm nec} \to p^* \text{ as } \epsilon_n \to 0. \tag{2.23}$$

Moreover, for an uncertain plant with $\epsilon_n = 0$ and $R_{\rm nec} > 1$, comparing the conditions in Theorem 2.1 with those in Proposition 2.1, we have that

$$R_{\rm nec} > \max_{|\lambda_A| \in [|\lambda_{A^*}| - \epsilon_n, |\lambda_{A^*}| + \epsilon_n]} R^*, \quad p_{\rm nec} < \min_{|\lambda_A| \in [|\lambda_{A^*}| - \epsilon_n, |\lambda_{A^*}| + \epsilon_n]} p^*.$$

Therefore, for uncertain systems, even if we assume the most conservative plant dynamics within the uncertainty of (2.2), the limitation given by Proposition 2.1 may not satisfy the necessary conditions (2.20) and (2.21). We note that even if $\epsilon_n = 0$ holds, the plant may be uncertain since other parameters $a_{1,k}, \ldots, a_{n-1,k}$ could be uncertain.

Now, we compare the necessary limitations in Theorem 2.1 with the result for known plants by Proposition 2.1 in a numerical example.

Example 2.1. Consider a plant with $\epsilon_n = 0.2$, and a lossy channel with the loss probability $p = 0.05$. Fig. 2.2 shows the necessary bounds on the data rate versus the product of the eigenvalues $|\lambda_{A^*}|$ of the nominal plant. Note that the bounds do not depend on the plant parameters except the product of the eigenvalues and its uncertainty. The solid line represents the necessary data rate $R_{\rm nec}$ in Theorem 2.1, and the dotted lines are those for known plants R^* given by Proposition 2.1. Due to the uncertainty on λ_A, the bound R^* varies within the shaded area. However, there exists a gap between the solid line and the dotted line representing the upper bound of the shaded area. Hence, the limitations for known plants are insufficient in the presence of uncertainties. We remark that the lines in the figure are not illustrated when the product of the eigenvalues does not satisfy the assumption (2.18).

Remark 2.3. The necessary data rate $R_{\rm nec}$ is not always greater than the bound R^* given in Proposition 2.1 even if the plant has a uncertain parameter. In particular, when the system satisfies that $R_{\rm nec} < 1$, then $R_{\rm nec}$ is lower than R^*. Thus, in such a

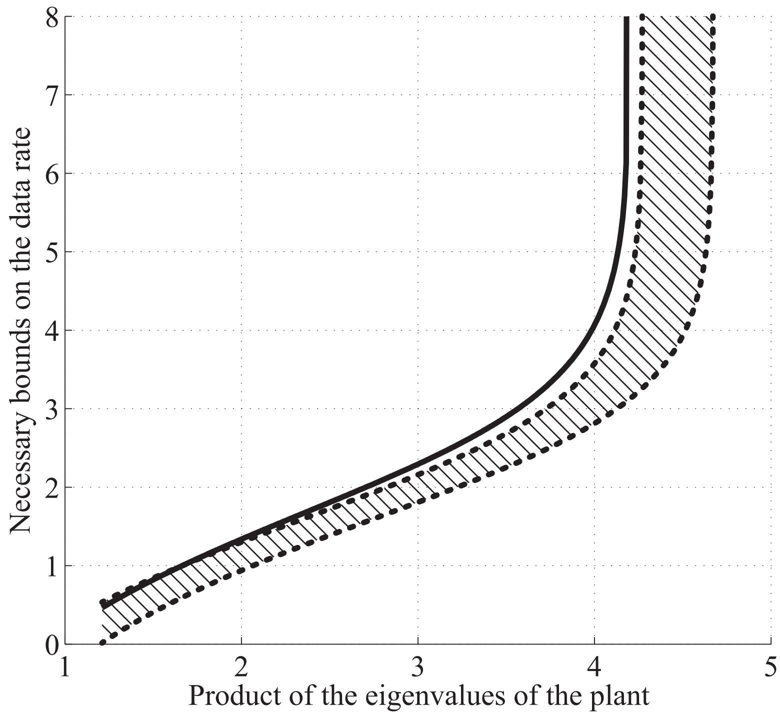


Figure 2.2: Comparison of necessary bounds on the data rate between those given by Theorem 2.1 (solid line) and by Proposition 2.1 (dotted lines and shaded area)

case, Proposition 2.1 provides a less conservative bound on the data rate compared with Theorem 2.1. We note that $R_{\mathrm{nec}} < 1$ is equivalent to $|\lambda_{A^*}| < 2/\sqrt{1+3p} - \epsilon_n$, and hence by the assumption (2.18), such systems are characterized as those satisfying $|\lambda_{A^*}| \in (1+\epsilon_n, 2/\sqrt{1+3p} - \epsilon_n)$; this interval may be empty for systems of a large magnitude of the uncertainty ϵ_n or loss probability p.

The work of [44] shows another special case of Theorem 2.1. A necessary condition for stabilization of uncertain systems is given for the case when the quantizer is uniform; the uniform quantizer is the simplest quantizer, which divides the input range into quantization cells of same lengths. If the plant is uncertain with $\epsilon_n > 0$, then the necessary data rate bound in [44] is higher than that in (2.20). Therefore, we may stabilize the system with a lower data rate by using a quantizer that is not uniform but more general.

2.2.2 Proof of Theorem 2.1

The proof of Theorem 2.1 consists of three steps. Throughout the proof, the central question is as follows: Under the situation where the estimation set of the plant state becomes large due to instability, at least how precise is the quantization required to be for making the estimation set gradually small? To answer this question, we focus on the effect of $a_{n,k}$ on the expansion of the estimation set, since $a_{n,k}$ is equal to the product of the eigenvalues of the plant; note that the product of the eigenvalues is the key factor to describe the bounds on the data rate [40, 57, 64]. Since we have considered a plant in the controllable canonical form, at time $k+1$, the effect of $a_{n,k}$ on the plant output appears as the coefficient of y_{k-n+1} (see (2.1)). Hence, the first step is to analyze this aspect.

Expansion rate of the estimation sets

Now we start the first step of the proof. We introduce the sequence w_l, $l = 0, 1, \dots, \lceil N/2 \rceil - 1$, and the random variable $\overline{\eta}$ for a given quantizer whose boundary points are $\{h_l\}_l$ as

$$w_l := \begin{cases} 2(|a_n^*| + \epsilon_n) h_{l+1} & \text{if } N \text{ is odd and } l = 0, \\ (|a_n^*| + \epsilon_n) h_{l+1} - (|a_n^*| - \epsilon_n) h_l & \text{else}, \end{cases} \tag{2.24}$$

$$\overline{\eta} := \begin{cases} |a_n^*| + \epsilon_n & \text{with prob. } p, \\ \max_{l \in \{0,1,\dots,\lceil N/2 \rceil - 1\}} w_l & \text{with prob. } 1 - p. \end{cases} \tag{2.25}$$

Then, the next lemma holds as a necessary condition for a given quantizer with $\{h_l\}_l$.

Lemma 2.2. If the system depicted in Fig. 2.1 is MSS, then it holds that

$$\mathrm{E}[\overline{\eta}^2] < 1. \tag{2.26}$$

Proof. First, we show that the mean square stability of the plant output y_k implies that the scaling parameter σ_k is also MSS. For the estimation set $\mathcal{Y}_k \subset \mathbb{R}$ at time k, it is obvious that $\max_{y'_k \in \mathcal{Y}_k} |y'_k| \geq \mu(\mathcal{Y}_k)/2$. Letting δ be the smallest width of the quantization cells, we have that $\mu(\mathcal{Y}_k) \geq \delta\sigma_k$. Hence, if $\lim_{k\to\infty} \mathrm{E}[|y_k|^2] = 0$, then $\lim_{k\to\infty} \mathrm{E}[\sigma_k^2] = 0$ holds.

In the rest of the proof, we show that (2.26) is a necessary condition for σ_k to be MSS. Notice from (2.12) that σ_{k+1} is bounded from below by $\mu(\mathcal{Y}_{k+1}^-)$. Substitution of (2.16) from Lemma 2.1 into (2.12) yields

$$\sigma_{k+1} \geq \sum_{i=1}^{n} \mu\left(\mathcal{A}_i \mathcal{Y}_{k-i+1}\right) \geq \mu\left(\mathcal{A}_n \mathcal{Y}_{k-n+1}\right). \tag{2.27}$$

We next evaluate the far right-hand side of the above inequality. The set $\mathcal{A}_n\mathcal{Y}_{k-n+1}$ depends on the boundaries of $\mathcal{A}_n$ and $\mathcal{Y}_{k-n+1}$. By (2.2), we have $\mathcal{A}_n = [a_n^* - \epsilon_n, a_n^* + \epsilon_n]$. The boundaries of $\mathcal{Y}_{k-n+1}$ may vary depending on whether the packet is lost or not at time $k-n+1$. If it is lost ($\gamma_{k-n+1} = 0$), then $\mathcal{Y}_{k-n+1} = [-\sigma_{k-n+1}/2, \sigma_{k-n+1}/2]$; otherwise ($\gamma_{k-n+1} = 1$), we can define the index l_{k-n+1} of $\mathcal{Y}_{k-n+1}$. The index represents which quantization cell is selected at the encoder at the time and is defined as follows: Consider the case when $\gamma_k = 1$. For given $\mathcal{Y}_k$ and σ_k, let l_k be the index of $\mathcal{Y}_k$ such that $\inf_{y'_k \in \mathcal{Y}_k} |y'_k/\sigma_k| = h_{l_k}$.

We claim that the width $\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1})$ can be written as

$$\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1}) = \eta_{k-n+1}\sigma_{k-n+1}, \tag{2.28}$$

where η_k is a random variable defined as

$$\eta_k := \begin{cases} |a_n^*| + \epsilon_n & \text{if } \gamma_k = 0, \\ w_{l_k} & \text{if } \gamma_k = 1, \end{cases}$$

with w_l from (2.24). Here, for simplicity, we assume that $a_n^* > 0$. Notice that from (2.18), $\mathcal{A}_n$ does not contain the origin. By replacing a_n^* with $|a_n^*|$ in the discussion, we can obtain the relations for the case $a_n^* < 0$.

Denote the infimum and the supremum of $\mathcal{Y}_{k-n+1}$ by $\underline{\mathcal{Y}}_{k-n+1}$ and $\overline{\mathcal{Y}}_{k-n+1}$, respectively. To derive (2.28), we consider the following three cases (i)–(iii).

(i) $0 \le \underline{\mathcal{Y}}_{k-n+1}$: In this case, it is necessary that $\gamma_{k-n+1} = 1$. Using basic results of interval arithmetics [38], from (2.17) and (2.18), we obtain

$$\begin{aligned}\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1}) &= (a_n^* + \epsilon_n)\overline{\mathcal{Y}}_{k-n+1} - (a_n^* - \epsilon_n)\underline{\mathcal{Y}}_{k-n+1} \\ &= \left\{(a_n^* + \epsilon_n)h_{l_{k-n+1}+1} - (a_n^* - \epsilon_n)h_{l_{k-n+1}}\right\}\sigma_{k-n+1}.\end{aligned}$$

Hence, by (2.24) we have (2.28).

(ii) $\underline{\mathcal{Y}}_{k-n+1} < 0 < \overline{\mathcal{Y}}_{k-n+1}$: In this case, we have

$$\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1}) = (a_n^* + \epsilon_n)(\overline{\mathcal{Y}}_{k-n+1} - \underline{\mathcal{Y}}_{k-n+1}). \tag{2.29}$$

Consider the following two cases:

(ii-1) If $\gamma_{k-n+1} = 0$, then $\mathcal{Y}_{k-n+1} = [-\sigma_{k-n+1}/2, \sigma_{k-n+1}/2]$. Hence, we have that $\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1}) = (a_n^* + \epsilon_n)\sigma_{k-n+1}$. Thus, (2.28) follows.

(ii-2) Otherwise, N must be odd and $l_{k-n+1} = 1$ from the condition (ii). In this case, (2.29) can be written as $\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1}) = 2(a_n^* + \epsilon_n)h_1\sigma_{k-n+1}$ since $\mathcal{Y}_{k-n+1} = [-h_1\sigma_{k-n+1}, h_1\sigma_{k-n+1})$. Hence, (2.28) holds for this case also.

(iii) $\overline{\mathcal{Y}}_{k-n+1} \le 0$: This case can be reduced to (i).

By (2.27) and (2.28), it holds that $\sigma_{k+1} \ge \eta_{k-n+1}\sigma_{k-n+1}$ and hence

$$\mathrm{E}[\sigma_{k+1}^2] \ge \mathrm{E}[\eta_{k-n+1}^2]\mathrm{E}[\sigma_{k-n+1}^2]. \tag{2.30}$$

Here, we used the fact that γ_{k-n+1} and σ_{k-n+1} are independent; this is because σ_{k-n+1} depends only on $\gamma_0, \ldots, \gamma_{k-n}$. The expectation $\mathrm{E}[\eta_{k-n+1}^2]$ may vary with $\mathcal{Y}_{k-n+1}$. By (2.25), it is clear that $\overline{\eta}$ is the maximum value of η_k.

Since σ_k is MSS for all possible parameters in (2.2) and initial values, by (2.30), we have that (2.26) is necessary. ■

Optimal quantizer

As the second step, we find the quantizer that minimizes $\overline{\eta}$ for a fixed N. To state such an optimal quantizer, we introduce the quantizer q_N^* represented by the boundary

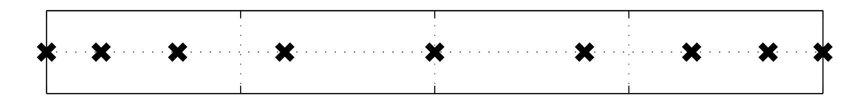


Figure 2.3: Boundaries of the quantizer q_N^* when $|\lambda_{A^*}| = 3.0$, $\epsilon_n = 0.5$, and $N = 8$

points $\{h_l^*\}_l$ as follows:

(i) If $\epsilon_n > 0$, then

$$h_l^* = \begin{cases} \frac{1}{2}\frac{1-tr^l}{1-tr^{\lceil N/2 \rceil}} & \text{if } N \text{ is odd,} \\ \frac{1}{2}\frac{1-r^l}{1-r^{\lceil N/2 \rceil}} & \text{if } N \text{ is even,} \end{cases} \tag{2.31}$$

where $t := |\lambda_{A^*}|/(|\lambda_{A^*}| - \epsilon_n)$.

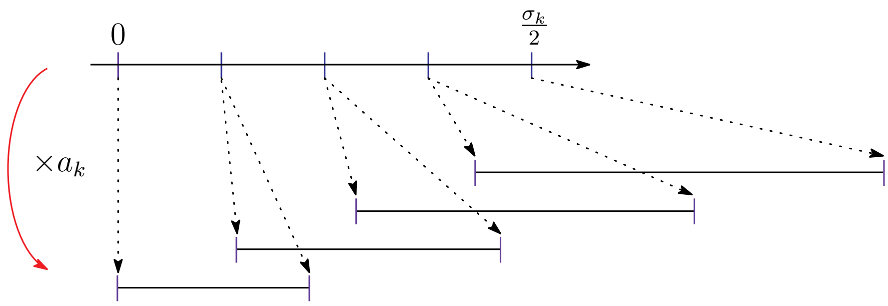
(ii) If $\epsilon_n = 0$, then

$$h_l^* = \begin{cases} \frac{1}{N}\left(l - \frac{1}{2}\right) & \text{if } N \text{ is odd,} \\ \frac{1}{N}l & \text{if } N \text{ is even.} \end{cases} \tag{2.32}$$

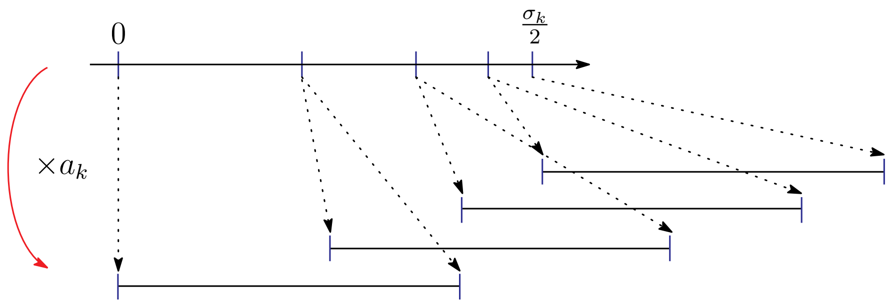
The following lemma holds.

Lemma 2.3. The quantizer q_N^* minimizes $\mathrm{E}[\overline{\eta}^2]$.

Fig. 2.3 illustrates the quantization boundaries $\{h_l^*\}_l$ of the optimal q_N^* when $|\lambda_{A^*}| = 3.0$, $\epsilon_n = 0.5$, and $N = 8$. We observe that the quantizer takes its quantization cells smaller towards the boundaries $\pm 1/2$ of the input range. We should note that this non-uniformity is an outcome of the minimization of $\max w_l$. More intuitively, this characteristic can be explained as follows. For simplicity, consider the case of a scalar plant where the parameter is given as $a_k \in \mathcal{A} = [a^* - \epsilon, a^* + \epsilon]$. Under the control scheme, the plant output is quantized and only the cell, or the interval $\mathcal{Y}_k$, to which it belongs is known to the controller. After one time step, because of the plant instability, the interval in which the output should be included, will expand in width. When the plant model is known, the expansion ratio is constant and is equal to $|a^*|$ for any cell. However, with plant uncertainties, the ratio depends on the location of



(a) Uniform quantizer case



(b) Optimal quantizer case

Figure 2.4: Expansion of the intervals in which the output should be included by the plant instability $a_k \in \mathcal{A} = [a^* - \epsilon, a^* + \epsilon]$ when $a^* > 0$

the cell. In particular, cells further away from the origin expands more. This fact is illustrated in Fig. 2.4(a) when the quantization is uniform. In contrast, when the proposed quantizer q_N^* is used, the intervals after one step have the same width (see Fig. 2.4(b)).

Furthermore, the quantizer q_N^* becomes more nonuniform in the presence of more uncertainties in the plant, expressed with a larger ϵ. This can be seen in the definition of the boundary points $\{h_l^*\}_l$, where the relative uncertainty r given in (2.19) determines the widths of the cells. Note that when $\epsilon = 0$, q_N^* becomes a uniform one as we have seen in (2.32).

For the general order plants case, the parameter a_k in the above explanation should be replaced with the nth parameter $a_{n,k}$, which is equal to the product of the eigenvalues of the plant and takes the nominal value as $|\lambda_{A^*}|$. Thus, the proposed quantizer q_N^* minimizes the maximum width of the intervals expanded by $a_{n,k}$ and other parameters do not affect the structure of q_N^*. This is because we have focused on the effect of $a_{n,k}$ in the stability analysis in the proof of Lemma 2.2. As a consequence, q_N^* is expressed in a simple form by $|\lambda_{A^*}|$ and its uncertainty ϵ_n.

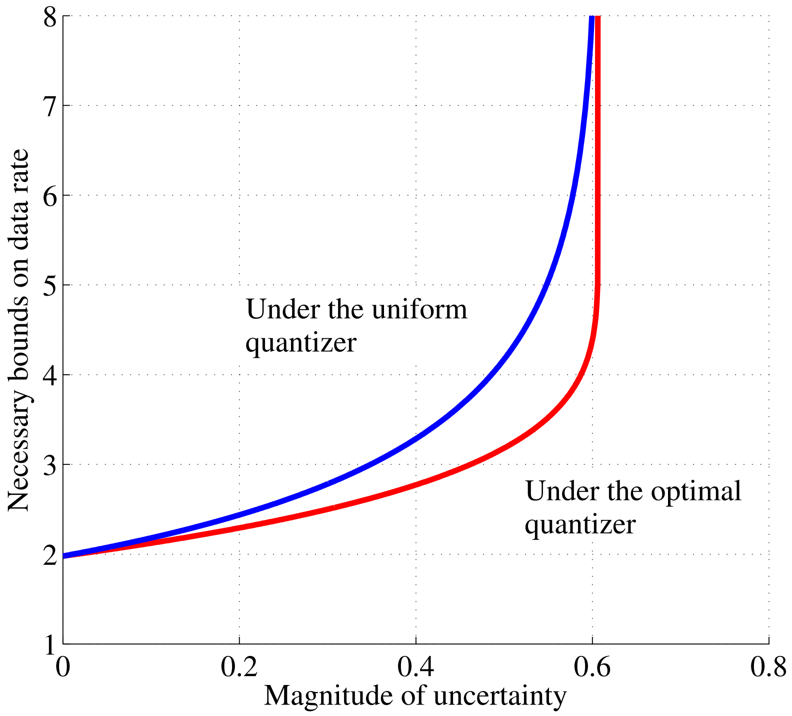


Figure 2.5: Comparison of necessary bounds on the data rate under the optimal quantizer q_N^* and the uniform one

It is interesting to note that as we see in Fig. 2.3, the quantizer q_N^* has a property in contrast with the logarithmic quantizer studied in [15, 18, 22, 29, 60]; in such quantizers, the quantization cells become small for inputs around the origin and grow exponentially as the input size increases. We note that the quantizer q_N^* and the logarithmic quantizer should not directly be compared since the problem setups are different: The logarithmic quantizer has infinite quantization cells and hence the data rate is not suitable to evaluate coarseness of the quantizer. There, the density of the cells has been proposed as a measure of coarseness.

Finally, we also note that the similar structure of the optimal quantizer q_N^* is found in [59], where a system identification problem based on quantized observations has been studied.

We confirm that the optimal quantizer q_N^* can suppress the required data rate compared with the case of the uniform one through a numerical example.

Example 2.2. Consider a plant with $a_n^* = 3.0$ and set the loss probability as $p = 0.05$.

In Fig. 2.5, we plot the necessary bounds on the data rate versus the magnitude of the uncertainty ϵ_n for both cases of the optimal quantizer and the uniform quantizer. These bounds are computed by substituting (2.31) and (2.32) into $\overline{\eta}$ in (2.26). In fact, as we will see in the last step of the proof of the theorem, the bound under the optimal quantizer coincides with R_{nec} defined in (2.20). Fig. 2.5 shows that if $\epsilon_n > 0$ then the optimal quantizer requires fewer data rates than the uniform one. It is interesting to note that there is no uncertainty, i.e., $\epsilon_n = 0$, the bounds are the same since q_N^* is equal to the uniform one.

Proof of Lemma 2.3. From (2.25), the realization of $\overline{\eta}$ corresponding to the case when a packet loss occurs (with probability p) does not depend on the quantizer. Hence, we focus on the minimization of $\max_l w_l$. After some calculation, we have that w_l is constant, i.e.,

$$w_l = w_{l'} \text{ for any } l, l' \in \{0, 1, \ldots, \lceil N/2 \rceil - 1\} \tag{2.33}$$

if and only if the quantizer is $\{h_l^*\}_l$. Therefore, it is enough to show that a quantizer which does not satisfy (2.33) yields a larger $\max_l w_l$ compared with the case $\{h_l^*\}_l$. We prove this by contradiction.

Let $w_l(h)$ denote the w_l when the quantizer is $\{h_l\}_l$. Assume that there exists a quantizer $\{g_l\}_l$ such that (2.33) is not satisfied and it holds that $\max_l w_l(g) < \max_l w_l(h^*)$. From the above, we have

$$w_l(g) \leq \max_{l'} w_{l'}(g) < \max_{l'} w_{l'}(h^*) = w_l(h^*) \tag{2.34}$$

for $l \in \{0, 1, \ldots, \lceil N/2 \rceil - 1\}$.

We now look at the relation between g_l and h_l^* for each l. From (2.6), we have

$g_0 = h_0^* = 0$. Substituting these equations into (2.24), we obtain

$$w_0(g) = \begin{cases} 2(|a_n^*| + \epsilon_n)g_1 & \text{if } N \text{ is odd,} \\ (|a_n^*| + \epsilon_n)g_1 & \text{else,} \end{cases}$$

$$w_0(h^*) = \begin{cases} 2(|a_n^*| + \epsilon_n)h_1^* & \text{if } N \text{ is odd,} \\ (|a_n^*| + \epsilon_n)h_1^* & \text{else.} \end{cases}$$

For the case $l = 0$ in (2.34), we have $w_0(g) < w_0(h^*)$. Thus, from the above equations, we have that

$$g_1 < h_1^*. \tag{2.35}$$

Furthermore, by (2.24), it follows that

$$w_l(g) = (|a_n^*| + \epsilon_n)g_{l+1} - (|a_n^*| - \epsilon_n)g_l,$$
$$w_l(h^*) = (|a_n^*| + \epsilon_n)h_{l+1}^* - (|a_n^*| - \epsilon_n)h_l^*$$

for $l \in \{1, 2, \ldots, \lceil N/2 \rceil - 1\}$. Substitution of these equations into (2.34) gives

$$g_{l+1} \leq r g_l + \frac{\max_{l'} w_{l'}(g)}{|a_n^*| + \epsilon_n}, \quad h_{l+1}^* = r h_l^* + \frac{\max_{l'} w_{l'}(h^*)}{|a_n^*| + \epsilon_n}.$$

By introducing the relation (2.35) to the above, we recursively obtain $g_l < h_l^*$ for all $l \in \{1, 2, \ldots, \lceil N/2 \rceil\}$. This contradicts $g_{\lceil N/2 \rceil} = h_{\lceil N/2 \rceil}^* = 1/2$ given in (2.6). Therefore, it follows that $\{h_l^*\}_l$ is the optimal quantizer. ■

Limitations under the use of the optimal quantizer

Since we found the optimal quantizer minimizing $\mathrm{E}[\overline{\eta}^2]$, the lower bound on N satisfying $\mathrm{E}[\overline{\eta}^2] < 1$ or (2.26) is the necessary condition on the data rate R ($= \log N$). This is to be proved as the third step.

Proof of Theorem 2.1. In this proof, we derive the bounds (2.20)–(2.22) from

(2.26) by using $\{h_l^*\}_l$ as the quantizer. First, suppose that $\epsilon_n > 0$. We consider the following two cases.

(i) N is odd: In this case, by the definition of $\{h_l^*\}_l$, we have that $h_l^* = (1 - tr^l)/(1 - tr^{\lceil N/2 \rceil})$. Thus, it holds that $\max_l w_l = \epsilon_n/(1 - tr^{\lceil N/2 \rceil})$. Consequently, the necessary condition (2.26) is equivalent to

$$\begin{aligned} \mathrm{E}[\overline{\eta}^2] &= p(|a_n^*| + \epsilon_n)^2 + (1-p)\left(\frac{\epsilon_n}{1 - tr^{\lceil N/2 \rceil}}\right)^2 < 1 \\ \Leftrightarrow N &> N_{\mathrm{nec}}^{(\mathrm{o})} := \frac{\log\{(1 - \epsilon_n \nu)/t\}^2}{\log r} - 1,\ p < p_{\mathrm{nec}}. \end{aligned}$$

(ii) N is even: Similarly, we have that the inequalities

$$N > N_{\mathrm{nec}}^{(\mathrm{e})} := \frac{\log(1 - \epsilon_n \nu)^2}{\log r}, \quad p < p_{\mathrm{nec}}$$

are necessary.

Comparing $N_{\mathrm{nec}}^{(\mathrm{o})}$ with $N_{\mathrm{nec}}^{(\mathrm{e})}$, by (2.18), we have $N_{\mathrm{nec}}^{(\mathrm{o})} > N_{\mathrm{nec}}^{(\mathrm{e})}$. Hence, $N > N_{\mathrm{nec}}^{(\mathrm{e})}$ and $p < p_{\mathrm{nec}}$ are necessary for both cases (i) and (ii). Using the relation $R = \log N$, we obtain the condition (2.20). Finally, since p_{nec} must be larger than zero, we arrive at $\epsilon_n < 1$ as in (2.22).

We next proceed to the case $\epsilon_n = 0$. It follows that $\max_l w_l = |a_n^*|/N$. Thus, (2.26) is equivalent to

$$\mathrm{E}[\overline{\eta}^2] = p|a_n^*|^2 + (1-p)\frac{|a_n^*|^2}{N^2} < 1.$$

Solving the above inequality with respect to N, we have the condition in (2.20) for the case $\epsilon_n = 0$. ■

2.3 Construction of stabilizing controllers

In this section, we present a sufficient condition for the existence of a stabilizing feedback control scheme. We will first present the proposed control law and then provide a result to analyze the stability based on the approach of Markov jump systems. In particular, the result shows that in the scalar system case, the data rate bound from Theorem 2.1 is sufficient for robust stabilization.

2.3.1 Sufficient condition for stability

Given a certain data rate R, or N, and a quantizer whose boundaries are $\{h_l\}_l$, we employ the control law as follows: In the encoder (2.5), the scaling parameter is determined by

$$\sigma_k = \mu(\mathcal{Y}_k^-), \tag{2.36}$$

and in the controller (2.11), the control is given as

$$u_k := -c(\mathcal{Y}_{k+1}^-). \tag{2.37}$$

Here, we denote the midpoint of the interval $\mathcal{Y}_{k+1}^-$ as $c(\mathcal{Y}_{k+1}^-)$.

Remark 2.4. The control low (2.37) brings the midpoint of the prediction set $\mathcal{Y}_{k+1}^-$ to the origin. We note that, for uncertain plants, this control low may be different from the dead-beat control based on the nominal parameters; for any interval $\mathcal{A}_i\mathcal{Y}_{k-i+1}$ consists of $\mathcal{Y}_{k+1}^-$, the midpoint of $\mathcal{A}_i\mathcal{Y}_{k-i+1}$ does not coincide with that of $a_i^*\mathcal{Y}_{k-i+1}$ in general (see the basic results about products of intervals [38]).

Next, we introduce some notations required for the analysis of the resulting system. For $i = 1, 2, \dots, n$, let the random variables $\theta_{i,k}$ be given by

$$\theta_{i,k} := \begin{cases} |a_i^*| + \epsilon_i & \text{if } \gamma_{k-i+1} = 0, \\ \overline{w}_i & \text{if } \gamma_{k-i+1} = 1. \end{cases} \tag{2.38}$$

Here, $\overline{w}_i$ is defined as follows for the given quantizer boundaries $\{h_l\}_l$:

$$\overline{w}_i := \begin{cases} \max\left\{\overline{w}_i^{(0)}, \overline{w}_i^{(1)}\right\} & \text{if } N \text{ is odd and } \mathcal{A}_i \not\ni 0, \\ \max\left\{\epsilon_i, \overline{w}_i^{(0)}\right\} & \text{if } N \text{ is odd and } \mathcal{A}_i \ni 0, \\ \overline{w}_i^{(1)} & \text{if } N \text{ is even and } \mathcal{A}_i \not\ni 0, \\ \epsilon_i & \text{if } N \text{ is even and } \mathcal{A}_i \ni 0, \end{cases}$$

where

$$\overline{w}_i^{(0)} := 2(|a_i^*| + \epsilon_i)h_1, \tag{2.39}$$

$$\overline{w}_i^{(1)} := \max_{l \in \{0,\ldots,\lceil N/2 \rceil - 1\}} \{(|a_i^*| + \epsilon_i)h_{l+1} - (|a_i^*| - \epsilon_i)h_l\}. \tag{2.40}$$

As we will see in the proof later, these are useful to bound the interval $\mathcal{A}_i\mathcal{Y}_{k-i+1}$ in (2.17) as $\mu(\mathcal{A}_i\mathcal{Y}_{k-i+1}) \leq \theta_{i,k}\sigma_{k-i+1}$. Moreover, define the random variable matrix H_{Γ_k} containing $\theta_{1,k}, \ldots, \theta_{n,k}$ by

$$H_{\Gamma_k} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \theta_{n,k} & \theta_{n-1,k} & \cdots & \theta_{1,k} \end{bmatrix}, \quad \text{where } \Gamma_k := [\gamma_{k-n+1}\, \gamma_{k-n+2} \cdots \gamma_k]. \tag{2.41}$$

Here, this process Γ_k is a Markov chain, which has 2^n states given by $\Gamma^{(1)} := [0 \cdots 0\, 0]$, $\Gamma^{(2)} := [0 \cdots 0\, 1], \ldots, \Gamma^{(2^n)} := [1 \cdots 1\, 1]$ and the transition probability matrix $P \in \mathbb{R}^{2^n \times 2^n}$ is given by

$$P := \begin{bmatrix} P' & & 0 \\ & \ddots & \\ 0 & & P' \\ P' & & 0 \\ & \ddots & \\ 0 & & P' \end{bmatrix} \in \mathbb{R}^{2^n \times 2^n}, \quad P' := \begin{bmatrix} p & 1-p \end{bmatrix} \in \mathbb{R}^{1 \times 2}, \tag{2.42}$$

where the (i, j) element of P is equal to the transition probability from $\Gamma^{(i)}$ to $\Gamma^{(j)}$. Note that H_{Γ_k} takes a controllable canonical form similar to A_k in (2.3). In fact, this matrix H_{Γ_k} corresponds to the A-matrix of a certain approximation of the overall uncertain system with the packet loss process. It is noted that this approximate system can be viewed as a Markov jump linear system. Though the only random process in the original system in Fig. 2.1 is that of the losses, which is i.i.d., the Markov property arises because the control input in (2.37) depends on the information received in the last n steps, as we can see in the prediction set $\mathcal{Y}_{k+1}^{-}$ of the output in (2.13). Finally, we define the matrix F using H_{Γ_k} and P by

$$F := F_1 F_2, \tag{2.43}$$

where

$$\begin{aligned} F_1 &:= P^T \otimes I_{n^2}, \\ F_2 &:= \operatorname{diag}\left(H_{\Gamma^{(1)}} \otimes H_{\Gamma^{(1)}}, \ldots, H_{\Gamma^{(2^n)}} \otimes H_{\Gamma^{(2^n)}}\right). \end{aligned} \tag{2.44}$$

Here, diag$(\cdot)$ denotes a block diagonal matrix and $\otimes$ is the Kronecker product. Also, let $\rho(\cdot)$ be the spectral radius of a matrix.

We are now ready to present the main theorem of this section. It employs results from the theory of Markov jump linear systems [8].

Theorem 2.2. Given the data rate $R = \log N$, the loss probability $p \in [0, 1)$, and the quantizer $\{h_l\}_l$, if the matrix F in (2.43) satisfies

$$\rho(F) < 1, \tag{2.45}$$

then under the control law using (2.36) and (2.37), the system depicted in Fig. 2.1 is MSS.

This theorem shows that the problem of seeking a sufficient data rate can be reduced to the stability test problem of the matrix F. The matrix F is of the size

$2^n n^2 \times 2^n n^2$ and appears in basic theorems to check stability of Markov jump linear systems.

2.3.2 Tightness in the scalar systems case

It is important to note that for the special case of scalar plants ($n = 1$), Theorem 2.1 provides a necessary and a sufficient condition. This fact is stated as a corollary below.

Corollary 2.1. In the system depicted in Fig. 2.1, if the plant is a scalar system and satisfying Assumption 2.1, then the following hold:

(i) If the data rate R satisfies $R > \lceil R_{\text{nec}} \rceil$, and (2.21) and (2.22) hold, then the system is MSS.

(ii) The quantizer q_N^* minimizes the required data rate for stability.

Proof. (i): From the proof of Theorem 2.1, if the inequalities $R > \lceil R_{\text{nec}} \rceil$, (2.21), and (2.22) hold, then under the quantizer q_N^* given in (2.31), we have that $\mathrm{E}[\overline{\eta}^2] < 1$, i.e., (2.26) in Lemma 2.2 holds. On the other hand, since $n = 1$, it follows that

$$H_{\Gamma^{(1)}} = |a_n^*| + \epsilon_n, \ H_{\Gamma^{(2)}} = \max_l w_l, \ P = \begin{bmatrix} p & 1-p \\ p & 1-p \end{bmatrix},$$

where w_l is defined in (2.24). Using the above matrices, we obtain that the inequality (2.26) is equivalent to the sufficient condition (2.45).

(ii): This is obvious from the fact that (2.45) is equivalent to (2.26), and that the quantizer q_N^* minimizes $\mathrm{E}[\overline{\eta}^2]$ by Lemma 2.3. ■

We now compare our result with those in the literature. In [48] and [34], sufficient conditions for stabilization of uncertain plants via lossless channels ($p = 0$) are given. For the case $n = 1$, the sufficient bound in [48] is

$$R_{\text{suf}} := \log_2 \frac{|\lambda_{A^*}| - \epsilon_1(|\lambda_{A^*}| + \epsilon_1)}{1 - \epsilon_1(2|\lambda_{A^*}| + 2\epsilon_1 + 1)}, \tag{2.46}$$

and the one from [34] becomes

$$R'_{\text{suf}} := \log_2 \frac{|\lambda_{A^*}|}{1 - \epsilon_1}. \tag{2.47}$$

On the other hand, from Corollary 2.1, we have that R_{nec} is a sufficient data rate bound for the case $n = 1$. It is easy to verify that $R_{\text{nec}} < R_{\text{suf}}, R'_{\text{suf}}$. Thus, our result shows that the known bounds (2.46) and (2.47) contain conservatism. However, for general order plants, it is difficult to compare Theorem 2.2 with the bounds in [48] and [34] since the types of uncertainties are different: In [48], unstructured uncertainties have been considered, and it is hard to describe the data rate limitation in an explicit form; while in [34], multi-dimensional plants have not been studied. The work of [44] gives another sufficient condition for stabilization via lossy channels, but the quantizer is constrained to be uniform; Theorem 2.2 is an extension to a more general quantizer case.

The following example is provided to illustrate the gap between the theoretical bounds on the data rate given in Theorems 2.1 and 2.2.

Example 2.3. Consider a second-order plant, where the plant parameter a_1^* is fixed as $a_1^* = 1.0$ and the uncertainty bounds are taken as $\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$, and a lossy channel with the loss probability $p = 0.05$. In Fig. 2.6, we plot the bounds on the data rate versus the product of the eigenvalues $|\lambda_{A^*}|$ of the nominal plant, which is equal to $|a_2^*|$. The vertical dash-dot line represents the supremum of $|\lambda_{A^*}|$ such that $p < p_{\text{nec}}$ holds. Hence, the necessary condition (2.21) is not satisfied on the right-hand side of this line. Here, we consider two different quantizers: the optimal one q_N^* and the uniform one. In the figure, the solid lines illustrate the bounds given by Theorems 2.1 and 2.2 when the quantizer is optimal, and the dotted lines are those for the case of the uniform quantizer studied in [44]. The figure shows that by using the optimal quantizer, we can stabilize the system under a lower data rate compared with the case using the uniform one. Here, the sufficient bounds take discrete values since the rates are rounded to integers. In Chapter 3, we will show that the increments due to rounding can be suppressed by employing time-varying data rate.

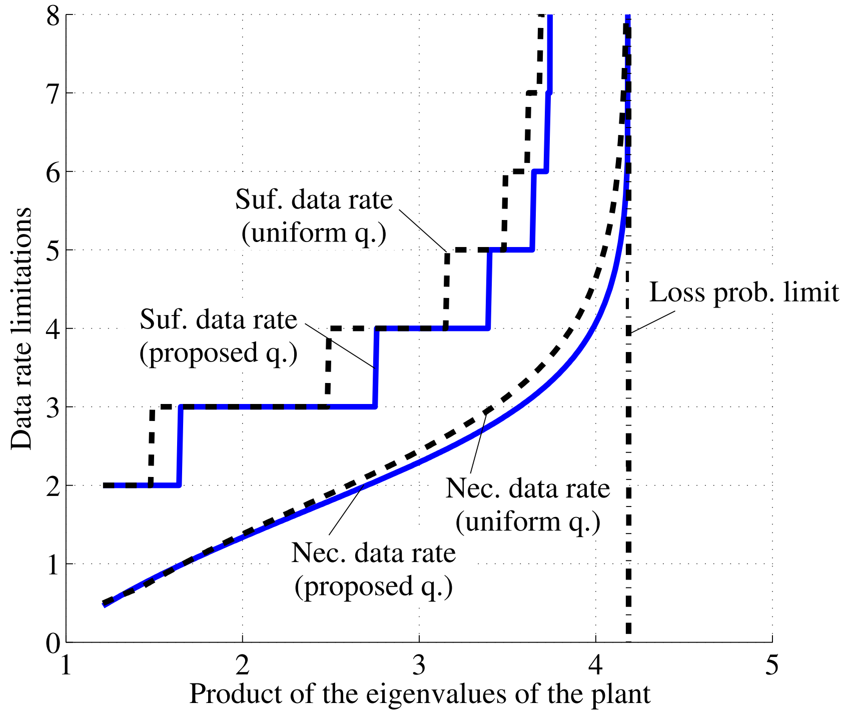


Figure 2.6: Bounds on the data rate ($n = 2$, $a_1^* = 1.0$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$, $p = 0.05$): The solid lines are the sufficient bound and the necessary bound when the quantizer is optimal q_N^*, while the dotted lines are those for the case of the uniform one

2.3.3 Proof of Theorem 2.2

Theorem 2.2 is directly obtained after the proof of the key lemma, which shall be soon given below. The lemma states that mean square stability of the feedback system can be reduced to stability of the following Markov jump linear system:

$$z_{k+1} = H_{\Gamma_k} z_k, \quad z_0 := [\sigma_{-n+1}\ \sigma_{-n}\ \cdots\ \sigma_0]^T. \tag{2.48}$$

Lemma 2.4. If the Markov jump linear system (2.48) is stable in the sense that $\mathrm{E}[z_k z_k^T]$ converges to the zero matrix, then the system in Fig. 2.1 can be MSS under the control law using (2.36) and (2.37).

Proof. First, we show that if $\mathrm{E}[\sigma_k^2] \to 0$ then $\mathrm{E}[|y_k|^2] \to 0$ as $k \to \infty$ under the control law. This is easy to establish because by substituting (2.37) to (2.1) and by referring to the definition (2.13) of $\mathcal{Y}_{k+1}^-$, we have that

$$\begin{aligned} |y_{k+1}| &= \left| a_{1,k} y_k + \cdots + a_{n,k} y_{k-n+1} - c(\mathcal{Y}_{k+1}^-) \right| \\ &\le \frac{\mu(\mathcal{Y}_{k+1}^-)}{2} = \frac{\sigma_{k+1}}{2}. \end{aligned} \tag{2.49}$$

Next, to establish that the stability of (2.48) implies that σ_k is MSS, we prove the following relation

$$\sigma_k \le (z_k)_n \text{ for } k = 0, 1, \ldots, \tag{2.50}$$

where $(\cdot)_n$ is the nth element of a vector.

By (2.36) and the equality in (2.16) from Lemma 2.1, we have

$$\sigma_{k+1} = \sum_{i=1}^{n} \mu\left(\mathcal{A}_i \mathcal{Y}_{k-i+1}\right). \tag{2.51}$$

For the ith term $\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)$, it holds that

$$\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)=\begin{cases}\left(|a_i^*|+\epsilon_i\right)\mu(\mathcal{Y}_{k-i+1}) & \text{if } \mathcal{Y}_{k-i+1}\ni 0,\\ |a_i^*|\mu(\mathcal{Y}_{k-i+1})+\epsilon|\overline{\mathcal{Y}}_{k-i+1}+\underline{\mathcal{Y}}_{k-i+1}| & \text{if } \mathcal{Y}_{k-i+1}\not\ni 0 \text{ and } \mathcal{A}_i\not\ni 0,\\ 2\epsilon_i\max\left\{|\overline{\mathcal{Y}}_{k-i+1}|,\ |\underline{\mathcal{Y}}_{k-i+1}|\right\} & \text{if } \mathcal{Y}_{k-i+1}\not\ni 0 \text{ and } \mathcal{A}_i\ni 0,\end{cases} \tag{2.52}$$

by using basic results in interval arithmetics [38] for $i=1,2,\ldots,n$. By taking the maximum of $\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)$ over all possible $\mathcal{Y}_{k-i+1}$, we can establish

$$\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)\le\theta_{i,k}\sigma_{k-i+1}, \tag{2.53}$$

where $\theta_{i,k}$ is given in (2.38). To show (2.53), consider the following three cases.

(i) $\gamma_{k-i+1}=0$: By the definition (2.10), we have that $\mathcal{Y}_{k-i+1}=[-\sigma_{k-i+1}/2,\sigma_{k-i+1}/2]\ni 0$. Thus, by (2.52), it follows that $\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)=(|a_i^*|+\epsilon_i)\sigma_{k-i+1}$. From the definition (2.38) of $\theta_{i,k}$, in this case we have $\theta_{i,k}=|a_i^*|+\epsilon_i$ and hence, (2.53) holds.

(ii) $\gamma_{k-i+1}=1$ and N is odd: To evaluate $\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)$, we consider the following two cases.

(ii-1) $\mathcal{Y}_{k-i+1}\ni 0$: By (2.7), it must hold $\mathcal{Y}_{k-i+1}=[-h_1\sigma_{k-i+1},h_1\sigma_{k-i+1})$. Thus, from (2.52), we have

$$\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)=(|a_i^*|+\epsilon_i)(h_1\sigma_{k-i+1}+h_1\sigma_{k-i+1})=\overline{w}_i^{(0)}\sigma_{k-i+1}, \tag{2.54}$$

where $\overline{w}_i^{(0)}$ is defined in (2.39).

(ii-2) $\mathcal{Y}_{k-i+1}\not\ni 0$: In this case, we have $\mathcal{Y}_{k-i+1}=[h_l\sigma_{k-i+1},h_{l+1}\sigma_{k-i+1})$ or $[-h_{l+1}\sigma_{k-i+1},-h_l\sigma_{k-i+1})$, where $l\in\{1,2,\ldots,\lceil N/2\rceil-1\}$. Thus, by (2.52), we obtain

$$\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right)=\begin{cases}\left\{(|a_i^*|+\epsilon_i)h_{l+1}-(|a_i^*|-\epsilon_i)h_l\right\}\sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1}\not\ni 0,\\ 2\epsilon_i h_{l+1}\sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1}\ni 0.\end{cases}$$

Taking the maximum of the right-hand side of the above equality over $l \in \{1, 2, \ldots, \lceil N/2 \rceil - 1\}$, we have

$$\max_{l} \mu\left(\mathcal{A}_i \mathcal{Y}_{k-i+1}\right) = \begin{cases} \overline{w}_i^{(1)} \sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1} \not\ni 0, \\ \epsilon_i \sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1} \ni 0, \end{cases} \tag{2.55}$$

where $\overline{w}_i^{(1)}$ is defined in (2.40).

Combing (2.54) and (2.55), we conclude that (2.53) holds in the case (ii) also.

(iii) $\gamma_{k-i+1} = 1$ and N is even: In this case, it holds that $\mathcal{Y}_{k-i+1} \not\ni 0$. Hence, this case can be reduced to (ii-2).

From (2.51) and (2.53), it follows that

$$\sigma_{k+1} \leq \sum_{i=1}^{n} \theta_{i,k} \sigma_{k-i+1}. \tag{2.56}$$

Notice that the right-hand side of (2.56) is equal to the nth low of $H_{\Gamma_k}[\sigma_k \ \sigma_{k-1} \ \cdots \ \sigma_{k-n+1}]^T$. Thus, from the definition of the Markov jump system (2.48), we obtain (2.50).

Finally, it is straight forward to show that if $\mathrm{E}[z_k z_k^T]$ goes to the zero matrix, then $\mathrm{E}[\sigma_k^2] \to 0$. ■

From [8, Theorem 3.9], it follows that the inequality (2.45) implies that (2.48) is stable. Hence, we conclude that Theorem 2.2 holds.

2.3.4 Convergence speed

The condition (2.45) in Theorem 2.2 provides us a criterion on the data rate or the loss probability for achieving the stability of the system. However, the condition does not say anything about performance of systems satisfying (2.45). It may be natural to expect that a system satisfies the condition by a certain margin, i.e., a system of smaller $\rho(F)$ will be more stable with rapid convergence. Here we show as a proposition that this is in fact true.

Proposition 2.2. Consider the system in Fig. 2.1 governed by the control law using (2.36) and (2.37), and suppose that (2.45) is satisfied. Given $\beta \in (\rho(F), 1)$, there exists a constant $\alpha > 0$ such that $\mathrm{E}[y_k^2] \leq \alpha\beta^k$.

In other words, the spectrum radius $\rho(F)$ is a lower bound on the convergence rate of the mean square of the output. Hence, we can evaluate the stability and the performance of a system by computing $\rho(F)$.

Proof. From the inequalities (2.49) and (2.50) in the proof of Lemma 2.4, we have

$$\mathrm{E}[y_k^2] \leq \mathrm{E}[\|z_k\|^2], \tag{2.57}$$

where z_k is the state of (2.48) and $\|\cdot\|$ represents the Euclidean norm.

In what follows, we evaluate the convergence speed of the Markov jump system (2.48). From [8, Proposition 3.25], if $\rho(F) < 1$ then the state z_k is bounded as

$$\mathrm{E}[\|z_k\|^2] \leq \overline{\alpha}\beta^k\|z_0\|^2$$

for some constants $\overline{\alpha} \geq 1$, $0 < \beta < 1$. By [31, Lemma 1], the constants are determined as follows: β is an arbitrary value in $(\rho(F), 1)$ and $\overline{\alpha}$ is a constant depending on β and $H_{\Gamma(1)}, \ldots, H_{\Gamma(2^n)}$. Therefore, by introducing $\alpha := \overline{\alpha}\|z_0\|^2$, we have $\mathrm{E}[\|z_k\|^2] \leq \alpha\beta^k$. Finally, from this inequality and (2.57), we conclude that the theorem holds. ■

Now we give a numerical example to see how the convergence rate varies with respect to the communication constraints on the channel.

Example 2.4. Consider a second-order plant, where $a_1^* = 1.0$, $\epsilon_1 = 0.1$, $a_2^* = 2.0$, $\epsilon_2 = 0.1$. We plot lower bounds on the convergence rate, i.e., the spectrum radius of F versus the data rate and the packet loss probability in Fig. 2.7. In the figure, the horizontal plane represents the level at $\rho(F) = 1$. Hence, the feedback system can be MSS when the data rate and the loss probability lie under the plane. We see that rich communication with a high data rate and a small packet loss probability results in a good performance of rapid convergence.

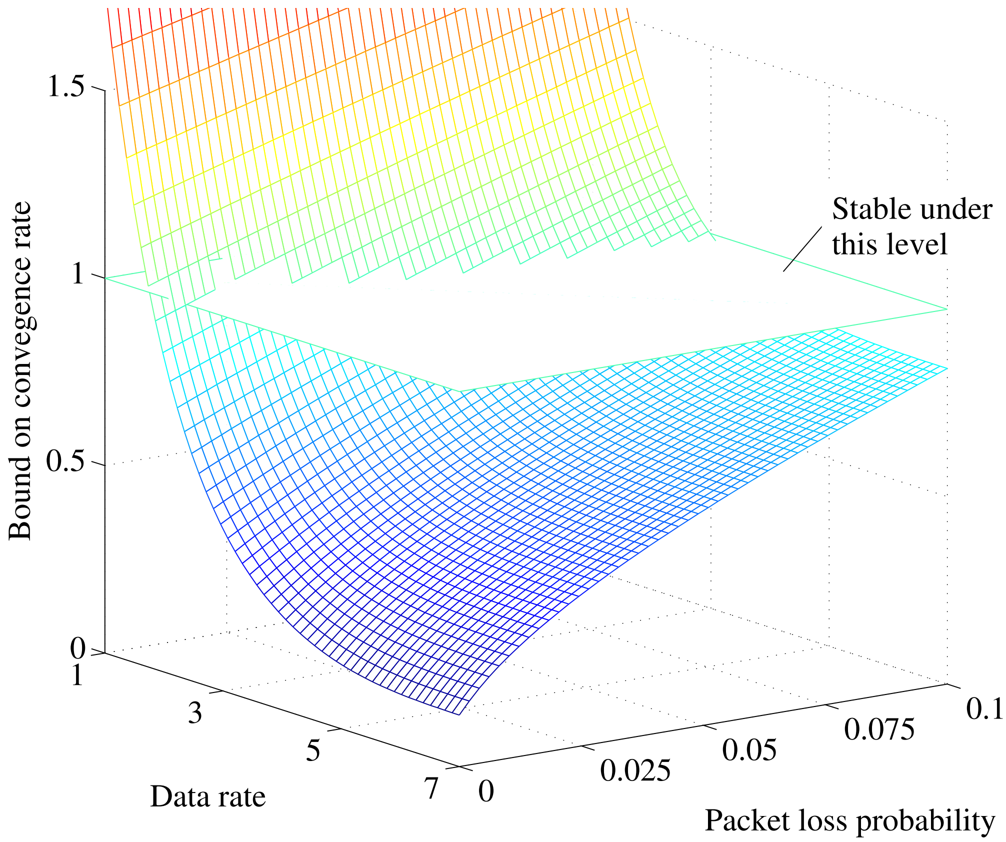


Figure 2.7: Lower bound on the convergence rate versus the data rate and the packet loss probability ($n = 2$, $a_1^* = 1.0$, $\epsilon_1 = 0.1$, $a_2^* = 2.0$, $\epsilon_2 = 0.1$)

In Figs. 2.8 and 2.9, we show the convergence rate bounds under the fixed packet loss probabilities ($p = 0.00, 0.05, 0.10$) and the fixed data rates ($R = 2, 4, 6$), respectively. Note that in Fig. 2.8 data rates must be taken as integers, where we plot the markers. In Fig. 2.8, the convergence rate bounds decay with respect to the data rate but the decreasing speeds are saturated when the data rate is large. By contrast, Fig. 2.9 shows that such saturation does not appear in the relation with the convergence rate and the loss probability. This difference between the effects of the data rate and that of the loss probability on the convergence rate can be explained by the limitations given in Theorem 2.1: The limitation (2.20) on the data rate decreases when the loss probability becomes smaller since ν depends on p. However, the limitation (2.21) on the loss probability does not depend on the data rate. Thus, even if we employ a large data rate, we cannot improve the margin on p from p_{nec}. On

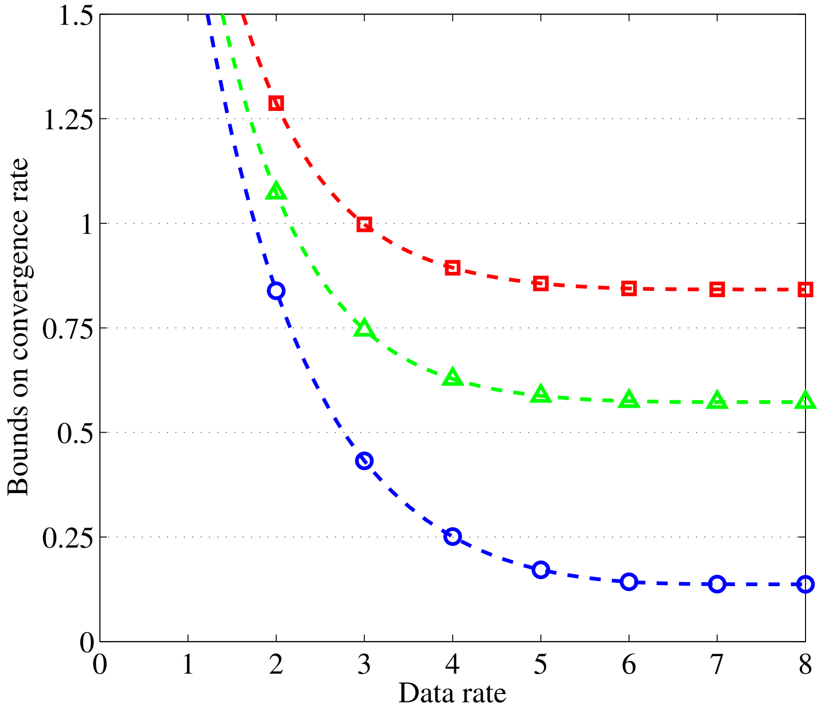


Figure 2.8: Lower bounds on the convergence rate versus data rate for cases of the loss probability $p = 0.00$ (circle marks), $p = 0.05$ (triangle marks), and $p = 0.10$ (square marks)

the contrary, if the loss probability becomes smaller, then both margins on R and p become larger from the limitations.

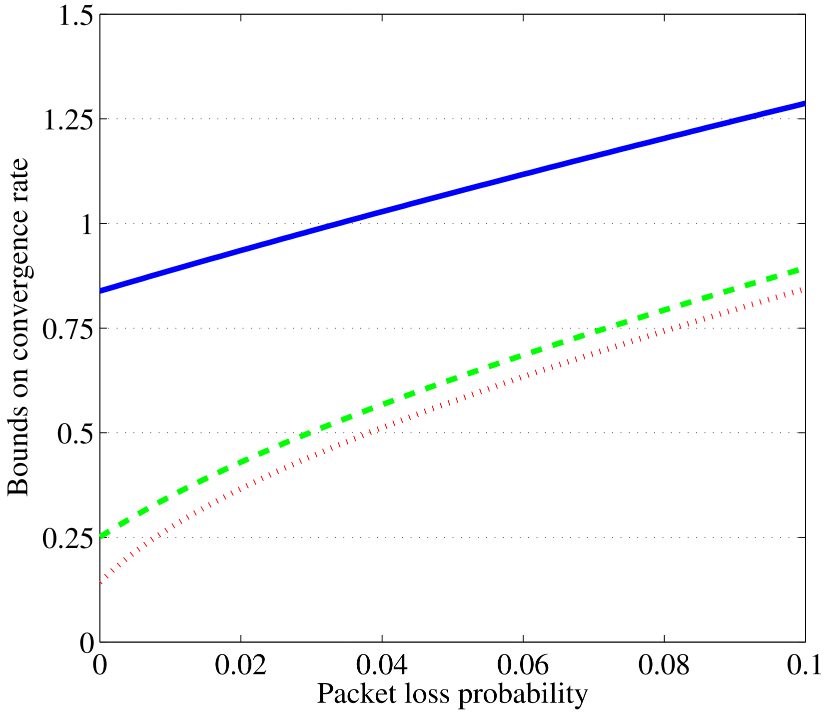


Figure 2.9: Lower bounds on the convergence rate versus packet loss probability for cases of the data rate $R = 2$ (solid line), $R = 4$ (dashed line), and $R = 6$ (dotted line)

Chapter 3

Control with time-varying data rate

In the previous chapter, we have studied limitations under the setup that the data rate is static, i.e., the bit-length of a packet is the same for every time step. In this section, we introduce a time-varying data rate and study limitations on the data rate for stability in the sense of average.

The motivation for introducing time-varying data rates is summarized as follows. When we consider a practical quantization and communication scheme, we cannot choose the data size of each transmission to be a noninteger. Hence, for a number R satisfying the sufficient condition in Theorem 2.2, the actual data rate required for stabilization is $\lceil R \rceil$, which may be larger than R. The sufficient bounds illustrated in Fig. 2.6 take discrete values due to this rounding and are greater than the minimum R satisfying (2.45) except the case that the minimum becomes an integer.

In various data rate results such as those in [40, 57, 64], it is known that when we know the exact plant model and employ time-varying data rates, then this gap on the data rate can be made arbitrarily small. That is, for any $R \in \mathbb{R}$ greater than the bound in Theorem 2.1 (with $\epsilon_n = 0$), there exists a feasible controller to stabilize the system. In this chapter, we follow such an approach for the case of uncertain plants and develop a control scheme with a time-varying data rate.

The results in this chapter have been partially presented in our works [43–45].

3.1 Problem formulation

We follow the setup in Chapter 2 where the main difference is that the encoder and the decoder can use time-varying data rates. At time k, the encoder quantizes the plant output y_k using a quantizer q_{k,N_k}, where q_{k,N_k} is a piece wise constant function which divides its input range $[-1/2, 1/2]$ into N_k cells. For each time step, the quantizer has the same structure of that introduced in Chapter 2: The quantization cells are symmetric about the origin, and the boundaries of the cells satisfy (2.6). We assume that the encoder and the decoder share the schedule of quantizers $\{q_{k,N_k}\}_k$ so that transmitted data can be successfully decoded.

Instead of the static data rate R, we study limitations on the average data rate defined as follows

$$\overline{R} := \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \log N_i.$$

In the following, we show a necessary condition and a sufficient condition for making the system in Fig. 2.1 MSS.

3.2 Limitations on average data rate

3.2.1 Necessary condition for stability

The following theorem provides necessary bounds on the average data rate, the loss probability, and the magnitude of the uncertainty.

Theorem 3.1. For the system in Fig. 2.1 satisfying Assumption 2.1, if the system is MSS with the time-varying data rate, then it holds that

$$\overline{R} > R_{\text{nec}}, \tag{3.1}$$

$$0 \leq p < p_{\text{nec}}, \tag{3.2}$$

$$0 \leq \epsilon_n < 1, \tag{3.3}$$

where the bounds R_{nec} and p_{nec} are defined in (2.20) and (2.21), respectively.

The above theorem implies that the necessary bounds given in Theorem 2.1 are valid even if we extend the class of communication schemes from static ones to time-varying ones. We note that in the literature there are two types of communication schemes employing time-varying data rates. The first type is the one considered here and also in [57]. Under the scheme, for every time step, the observation of the plant output is done and the plant is driven by the control input. On the other hand, another communication scheme has been developed in [40, 44, 64]. The idea is as follows: Divide the time into cycles of the same duration. At the initial step of each cycle, the encoder observes the plant output and then sends it slowly during the cycle. At the end of the cycle, the controller estimates the plant state and generates the input using the received information regarding the state data of the state from the initial time of the cycle. Except the last steps of the cycles, the control input is kept as zero.

An interesting point is that when the plant is known, these two schemes provide the same bound on the data rate, but for uncertain plants, the bounds are different. This is because in the latter scheme, the plant uncertainty causes accumulation of error in the state estimation, which affects the accuracy in the control input, since only the information at the beginning of the cycle is used.

3.2.2 Proof of Theorem 3.1

When the plant has no uncertainty, the proof of Theorem 3.1 follows in a straightforward manner from Theorem 2.1 or [64]. However, in the case of uncertain plants, we need a few additional steps. As in the the static data rate case, the nonuniform quantizer q_N^* in (2.31) will be shown to be optimal in the derivation, but its complex definition brings some analytical difficulties.

We will prove Theorem 3.1 in two steps in the following. We also have to consider the cases of $\epsilon_n > 0$ and $\epsilon_n = 0$ separately, since the definition of the optimal quantizer, which we employ in the following step, is different depending on ϵ_n. During the proof, we assume $\epsilon_n > 0$. For the case of $\epsilon_n = 0$, we can prove by following the same steps using the uniform quantizer, which is the optimal one in this case.

As the first step, we give another form of the necessary condition as a lemma. To state the lemma, we define the following random variable $\overline{\eta}_k$ for the number N_k of the quantization cells at time k:

$$\overline{\eta}_k := \begin{cases} |a_n^*| + \epsilon_n & \text{if } \gamma_k = 0, \\ v_k & \text{if } \gamma_k = 1, \end{cases} \tag{3.4}$$

where v_k corresponds to the maximum w_l in (2.24) when the quantizer is $q^*_{N_k}$, and is defined as follows:

$$v_k := \begin{cases} \frac{\epsilon_n}{1-tr^{(N_k+1)/2}} & \text{if } \epsilon_n > 0 \text{ and } N_k \text{ is odd}, \\ \frac{\epsilon_n}{1-r^{N_k/2}} & \text{if } \epsilon_n > 0 \text{ and } N_k \text{ is even}, \\ \frac{|a_n^*|}{N_k} & \text{if } \epsilon_n = 0. \end{cases} \tag{3.5}$$

Then the following lemma holds.

Lemma 3.1. If the system depicted in Fig. 2.1 is MSS with time-varying data rates, then for each $\alpha \in \{0, 1, \dots, n-1\}$, there exists a sequence of integers $\{m_i^{(\alpha)}\}_{i=0}^{\infty}$ such that

$$\prod_{j=0}^{m_i^{(\alpha)}-1} \mathrm{E}[\overline{\eta}^2_{n(\sum_{l=0}^{i-1} m_l^{(\alpha)}+j)+\alpha}] < 1, \ i = 0, 1, \dots, \tag{3.6}$$

where $\sum_{l=0}^{-1} m_l^{(\alpha)} := 0$ for all α.

This lemma is a modified version of Lemma 2.2 as the data rate is time varying and the quantizer is optimal, i.e., $q^*_{N_k}$. As we have seen in the proof of Theorem 2.1, we focus on the effect of the nth parameter $a_{n,k}$ on the expansion of the prediction set $\mathcal{Y}^-_{k+1}$. If the quantizer is static, then the mean square of the expansion rate $\overline{\eta}_{k-n+1}$ of $\mathcal{Y}^-_{k+1}$ must be smaller than 1 for every time step. On the other hand, when we employ a time-varying one, the expansion rate may be greater than 1 at a certain

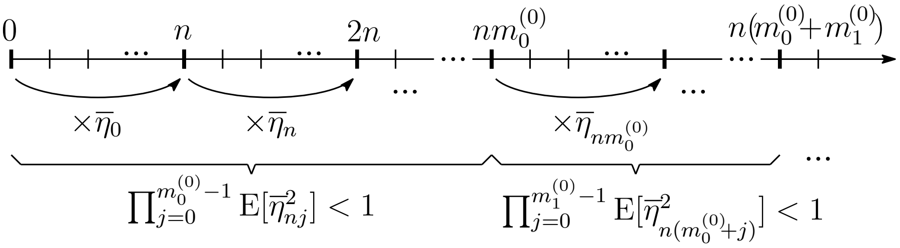


Figure 3.1: An interval such that the expansion rate of the prediction set from the initial step 0 to the last step $nm_0^{(0)}$ is smaller than 1

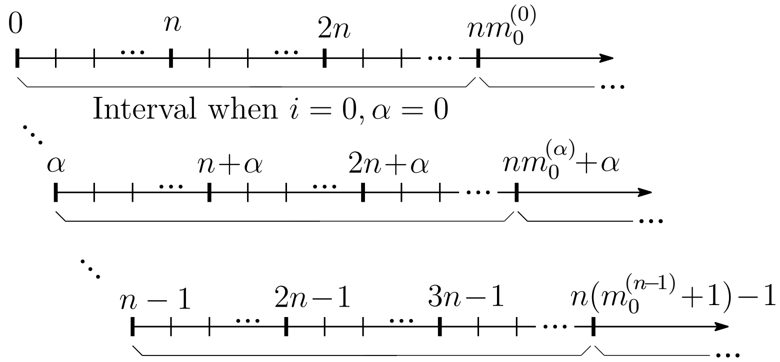


Figure 3.2: Intervals indexed by the initial time α

time. However, there must exist a time interval such that the expansion rate from a certain time k to the last step of the interval becomes smaller than 1 in expectation. In Fig. 3.1, we illustrate such an interval starting from $k = 0$. We denote the last step of the interval by $nm_0^{(0)}$, where $m_0^{(0)} \in \mathbb{N}$. If we take $nm_0^{(0)}$ as the initial time, we have another interval. Let the length of the interval be $nm_1^{(0)}$. Repeating the process we can divide the time into the intervals, where the ith interval starts at $n\sum_{l=0}^{i-1} m_l^{(\alpha)}$ and its length is $nm_i^{(0)}$. Furthermore, since the index of the expansion rate $\overline{\eta}_k$ is taken to be n periodic, there exist n series of such intervals depending on the initial time $\alpha \in \{0, 1, \dots, n-1\}$ of the first interval of $i = 0$ (see Fig. 3.2).

Proof of Lemma 3.1. By the proofs of Lemmas 2.2 and 2.3, we have that

$$\sigma_{k+1} \geq \overline{\eta}_{k-n}\sigma_{k-n} \geq \cdots \geq \prod_{j=0}^{\lfloor k/n \rfloor} \overline{\eta}_{nj+[k]_n}\sigma_{[k]_n},$$

where $[\cdot]_n$ is the residue modulo n and $\lfloor \cdot \rfloor$ is the floor function. Since the random variable $\overline{\eta}_k$ depends only on γ_k, by taking mean square of the above, we obtain

$$\mathrm{E}[\sigma_{k+1}^2] \geq \prod_{j=0}^{\lfloor k/n \rfloor} \mathrm{E}[\overline{\eta}_{nj+[k]_n}^2]\mathrm{E}[\sigma_{[k]_n}^2].$$

Note that $\mathrm{E}[\sigma_{k+1}^2] \to 0$ as $k \to \infty$ because of the stability of the system, but $\mathrm{E}[\sigma_{[k]_n}^2]$ remains positive since σ_k satisfies (2.12) and the lengths of the initial state estimation sets are positive (see Section 2.1). Taking the limits of both sides as $k \to \infty$, we have

$$\lim_{k \to \infty} \prod_{j=0}^{\lfloor k/n \rfloor} \mathrm{E}[\overline{\eta}_{nj+[k]_n}^2] = 0.$$

Hence, for each $[k]_n = \alpha \in \{0, 1, \ldots, n-1\}$, there exists an integer $m_0^{(\alpha)}$ such that

$$\prod_{j=0}^{m_0^{(\alpha)}-1} \mathrm{E}[\overline{\eta}_{nj+\alpha}^2] < 1.$$

Now, taking $m_0^{(\alpha)}$ as the initial time and applying the same procedure, we have that (3.6) holds. ∎

We can show the bounds (3.2) and (3.3) on p and ϵ_n by Lemma 3.1 and the following discussion: Taking the limit of $E[\overline{\eta}_j^2]$ as $N_j \to \infty$, we obtain

$$\begin{aligned}\lim_{N_j \to \infty} E[\overline{\eta}_j^2] &\geq p(|a_n^*| + \epsilon_n)^2 + \lim_{N_j \to \infty} (1-p) \left(\frac{\epsilon_n}{1 - r^{N_j/2}} \right)^2 \\ &= p\left((|a_n^*| + \epsilon_n)^2 - \epsilon_n^2 \right) + \epsilon_n^2. \end{aligned} \tag{3.7}$$

The first inequality is due to (3.4) and the fact that when $\epsilon_n > 0$, the variable v_k is greater than or equal to $\epsilon_n/(1-r^{N_j/2})$ for each N_j. From (3.7), we see that if $p \geq p_{\mathrm{nec}}$, then we must have $\mathrm{E}[\overline{\eta}_j^2] \geq 1$ for any large N_j. However this cannot hold under the assumption that the system is MSS. This is because (3.6) must hold for any $\{N_j\}$. Hence, we have $p < p_{\mathrm{nec}}$. Moreover, to make p_{nec} positive, (3.3) must be satisfied.

Now, we proceed to the second step to show the bound (3.1) on $\overline{R}$. In this step, based on (3.6), we evaluate the minimum of $\overline{R}$. In particular, we focus on each interval introduced in Fig. 3.2

$$\left\{ n \left(\sum_{l=0}^{i-1} m_l^{(\alpha)} + j \right) + \alpha : j = 0, 1, \ldots, m_i^{(\alpha)} - 1 \right\} \tag{3.8}$$

indexed by α and i and derive a lower bound on the average data rate used during the interval, that is, $1/m_i^{(\alpha)} \sum_{j=0}^{m_i^{(\alpha)}-1} \log N_{n(\sum_{l=0}^{i-1} m_l^{(\alpha)}+j)+\alpha}$. Here, the lower bound will be obtained by solving a certain minimization problem.

For simplicity of notation, let m be the number $m_i^{(\alpha)}$ of elements in the interval (3.8) and let $\check{N}_j$ be the number of quantization cells $N_{n(\sum_{l=0}^{i-1} m_l^{(\alpha)}+j)+\alpha}$. Then, we consider the following minimization problem:

$$\text{minimize} \quad \phi(\check{N}) := \prod_{j=0}^{m-1} \check{N}_j^{1/m}, \tag{3.9a}$$

$$\text{subject to} \quad \psi(\check{N}) := \prod_{j=0}^{m-1} \mathrm{E}[\check{\eta}_j^2] - 1 \leq 0. \tag{3.9b}$$

Here, we introduced the vector $\check{N} := [\check{N}_0 \ \check{N}_1 \ \cdots \ \check{N}_{m-1}]^T$ and

$$\check{\eta}_j := \begin{cases} |a_n^*| + \epsilon_n & \text{with prob. } p, \\ \frac{\epsilon_n}{1 - r^{\check{N}_j/2}} & \text{with prob. } 1-p. \end{cases} \tag{3.10}$$

As a constraint, we employ (3.9b) with $\hat{\eta}_j$ instead of (3.6) with $\overline{\eta}_j$. This is because $\overline{\eta}_j$ is difficult to treat since it depends on whether N_j is even or odd. One can easily verify that the constraint (3.9b) is looser than (3.6) because $\overline{\eta}_j \geq \check{\eta}_j$ holds in general. Thus, the solution $\check{N}^* = [\check{N}_0^* \ \check{N}_1^* \ \cdots \ \check{N}_{m-1}^*]^T$ of the minimization problem (3.9a) gives a lower bound on the average data rate.

We now show that the solution $\check{N}^*$ is represented by using $2^{R_{\text{nec}}}$.

Lemma 3.2. When p satisfies (3.2), the solution of the minimization problem (3.9a), (3.9b) is $\check{N}_j^* = 2^{R_{\text{nec}}}$ for all $j \in \{0, 1, \ldots, m-1\}$.

Proof. It can be verified that $\phi(\check{N})$ and $\psi(\check{N})$ are convex functions. Let $L(z, \lambda)$ be the Lagrangian of the minimization problem as

$$L(\check{N}, \lambda) := \phi(\check{N}) + \lambda\psi(\check{N})$$

and let

$$\check{N}' := \begin{bmatrix} 2^{R_{\rm nec}} \; 2^{R_{\rm nec}} \; \cdots \; 2^{R_{\rm nec}} \end{bmatrix}^T, \quad \lambda' := \frac{-\epsilon_n\nu}{m(1-\epsilon_n\nu)(1-p(|a_n^*|+\epsilon_n)^2)\log r}.$$

Then we have that

$$\frac{\partial}{\partial \check{N}_i} L(\check{N}, \lambda) = \frac{\phi(\check{N})}{m\check{N}_i} + \lambda \frac{\epsilon_n^2(1-p)(\log r) r^{\check{N}_i/2}}{(1-r^{\check{N}_i/2})^3} \prod_{\substack{j=0 \\ j\neq i}}^{m-1} \mathrm{E}[\check{\eta}_j^2]$$

and hence $\nabla_{\check{N}} L(\check{N}', \lambda') = 0$. Furthermore, because the loss probability p satisfies (3.2), it holds that $\lambda' > 0$. Thus, the pair $(\check{N}', \lambda')$ satisfies the KKT condition [33] and hence, $\check{N}'$ is the optimal solution. ■

By Lemma 3.2, we have that $\log(\prod_{j=0}^{m-1} \check{N}_j^*)^{1/m} = R_{\rm nec}$ is a lower bound on the average data rate of the interval of length m. Applying this result to all intervals of lengths $m_i^{(\alpha)}$, we obtain (3.1). This completes the proof of Theorem 3.1.

3.3 Sufficient condition

In this section, we present a sufficient condition for the case of time-varying quantization. Here, we follow the protocol proposed in [57], which is based on an m-periodic quantizer $\{q_{j,N_j}\}_{j=0}^{m-1}$; at time k, the quantizer is $q_{[k]_m, N_{[k]_m}}$, where $[\cdot]_m$ is the residue modulo m. Denote the set of boundary points of q_{j,N_j} as $\{h_{l,j}\}_l$. As the scaling parameter and the control input, we employ the ones given in (2.36) and (2.37) from the static quantization case. The sufficient condition can be proved in a way similar to that for the static data rate case studied in Section 2.3. Again the argument is based on representing the overall system by a Markov jump linear system. Thus, in what follows, we have to introduce slightly different notations.

For $i = 1, 2, \ldots, n$, we introduce the following random variables $\widetilde{\theta}_{i,k}$:

$$\widetilde{\theta}_{i,k} := \begin{cases} |a_i^*| + \epsilon_i & \text{if } \gamma_{k-i+1} = 0, \\ \overline{w}_{i,[k]_m} & \text{if } \gamma_{k-i+1} = 1, \end{cases}$$

where $\overline{w}_{i,j}$ is defined as

$$\overline{w}_{i,j} := \begin{cases} \max\left\{\overline{w}_{i,j}^{(0)}, \overline{w}_{i,j}^{(1)}\right\} & \text{if } N_j \text{ is odd and } \mathcal{A}_i \not\ni 0, \\ \max\left\{\epsilon_i, \overline{w}_{i,j}^{(0)}\right\} & \text{if } N_j \text{ is odd and } \mathcal{A}_i \ni 0, \\ \overline{w}_{i,j}^{(1)} & \text{if } N_j \text{ is even and } \mathcal{A}_i \not\ni 0, \\ \epsilon_i & \text{if } N_j \text{ is even and } \mathcal{A}_i \ni 0, \end{cases}$$

$$\overline{w}_{i,j}^{(0)} := 2(|a_i^*| + \epsilon_i)h_{1,j},$$

$$\overline{w}_{i,j}^{(1)} := \max_{l \in \{0,\ldots,\lceil N_j/2 \rceil - 1\}} \left\{(|a_i^*| + \epsilon_i)h_{l+1,j} - (|a_i^*| - \epsilon_i)h_{l,j}\right\}.$$

Moreover, define the random variable matrix $\widetilde{H}_{\widetilde{\Gamma}_k}$ containing $\widetilde{\theta}_{1,k}, \ldots, \widetilde{\theta}_{n,k}$ by

$$\widetilde{H}_{\widetilde{\Gamma}_k} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \widetilde{\theta}_{n,k} & \widetilde{\theta}_{n-1,k} & \cdots & \widetilde{\theta}_{1,k} \end{bmatrix},$$

where $\widetilde{\Gamma}_k := [[k]_m \ \gamma_{k-n+1} \ \gamma_{k-n+2} \ \cdots \ \gamma_k]$. Here, the process $\widetilde{\Gamma}_k$ is a Markov chain which has $m2^n$ states given by

$$\widetilde{\Gamma}^{(1)} := [0\ 0\ \cdots\ 0\ 0],\ \widetilde{\Gamma}^{(2)} := [0\ 0\ \cdots\ 0\ 1],\ \ldots,\ \widetilde{\Gamma}^{(2^n)} := [0\ 1\ \cdots\ 1\ 1],$$
$$\widetilde{\Gamma}^{(2^n+1)} := [1\ 0\ \cdots\ 0\ 0],\ \ldots,\ \widetilde{\Gamma}^{(m2^n)} := [m-1\ 1\ \cdots\ 1\ 1].$$

Furthermore, the transition probability matrix $\widetilde{P} \in \mathbb{R}^{m2^n \times m2^n}$ of the Markov chain is given by

$$\widetilde{P} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \otimes P,$$

where P is the matrix defined in (2.42). Using $\widetilde{H}_{\widetilde{\Gamma}_k}$ and $\widetilde{P}$, we now introduce the matrix $\widetilde{F}$ as follows:

$$\widetilde{F} := \widetilde{F}_1 \widetilde{F}_2, \tag{3.11}$$

where

$$\widetilde{F}_1 := \widetilde{P}^T \otimes I_{n^2},$$
$$\widetilde{F}_2 := \mathrm{diag}\left(\widetilde{H}_{\widetilde{\Gamma}^{(1)}} \otimes \widetilde{H}_{\widetilde{\Gamma}^{(1)}}, \ldots, \widetilde{H}_{\widetilde{\Gamma}^{(m2^n)}} \otimes \widetilde{H}_{\widetilde{\Gamma}^{(m2^n)}}\right).$$

We can now derive a sufficient condition for stabilization based on the time-varying quantization. It can be obtained by applying [8] as in Theorem 2.2.

Theorem 3.2. Given the duration $m \in \mathbb{N}$, the set of quantizers $\{q_{j,N_j}\}_{j=0}^{m-1}$ and loss probability $p \in [0, 1)$, if

$$\rho(\widetilde{F}) < 1, \tag{3.12}$$

then under the control law using (2.36) and (2.37), the system depicted in Fig. 2.1 is MSS.

We now show a numerical example and confirm that the time-varying protocol reduces the required data rate compared with the case of the static one. In particular, the sufficient bound given in Theorem 3.2 is strictly lower than that given in Theorem 2.2 except at the points where the bounds become integers.

Example 3.1. Consider a scalar uncertain plant, where $\epsilon_1 = 0.2$, and $p = 0.05$. In

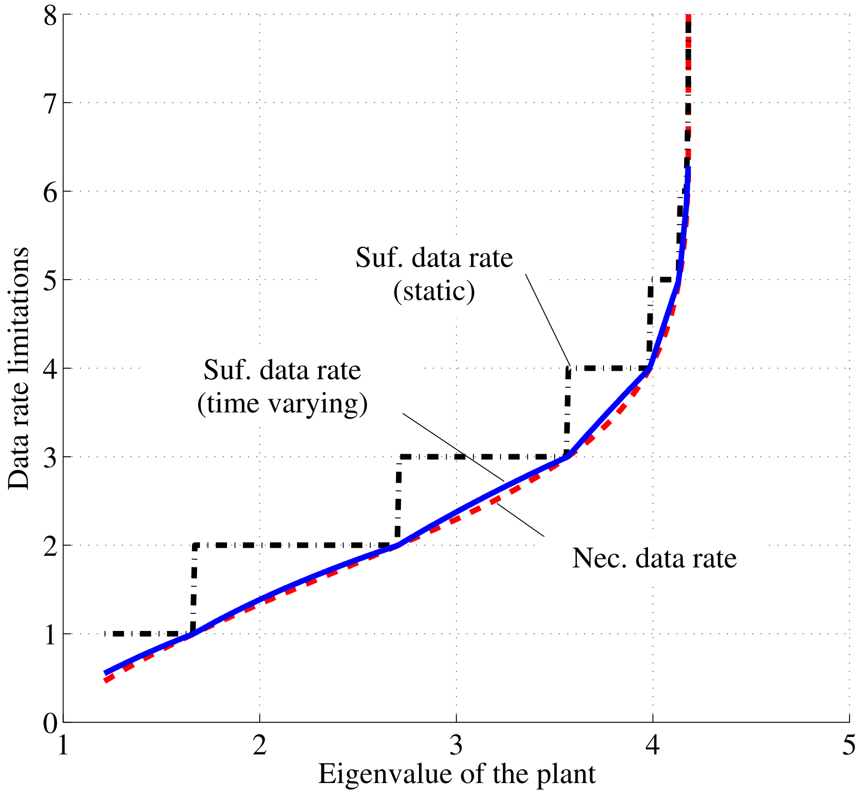


Figure 3.3: Limitations on the average data rate ($n = 1$, $\epsilon_1 = 0.2$, $p = 0.05$): The sufficient bound with the time-varying data rate (solid) is closer to the necessary bound (dashed) than that with static data rate (dash-dot)

Fig. 3.3, we plot the sufficient bound on the average data rate (by the solid line) versus the eigenvalue $|\lambda_{A^*}| = |a_1^*|$ of the nominal plant; we illustrate an achievable average data rate which is the minimum over the duration $m \leq 10^{16}$ and quantizers $\{q^*_{N_j}\}_j$. The dash-dot and the dashed lines represent the sufficient bound for the case of static protocol and the necessary bound R_{nec}, respectively. The figure shows that the sufficient average data rate (solid line) is smaller than that for static protocols. Moreover, it is close to the necessary bound (dashed line). Note that currently we consider the case of $n = 1$, and hence the sufficient bound is equal to $\lceil R_{\text{nec}} \rceil$ (see Corollary 2.1) but otherwise there exists a gap between them. When the plant has no uncertainty, the gap can be arbitrarily small. To reveal this gap in the uncertain case by an analytical approach is left for future research.

Chapter 4

Further results under uncertain control inputs

In this chapter, we study a more realistic class of uncertain plants in contrast with the fundamental class employed in Chapters 2 and 3. In the previous chapters, we have considered the uncertain plant defined in (2.1) and (2.2), where the uncertainty lies only on the coefficients $a_{i,k}$, $i = 1, \ldots, n$, of the past and the current plant states. We now investigate the case where the coefficient of the input may also be uncertain and may vary within a known interval. The setup change allows us to deal with uncertainties on the model of the actuator in the plant. Similar to the previous chapters, our objective is to study limitations on the data rate, the loss probability, and the uncertainties for stability. We derive a necessary condition and a sufficient condition for the mean square stability, which are generalized versions of those shown in Chapter 2.

We note that the existing work [48] has dealt with uncertainties on inputs: In [48], a norm bounded uncertain matrix has been given as the coefficient of the input. However, as we have seen in Chapter 2, [48] has focused only on deriving sufficient conditions and has not studied necessary ones.

The main difficulty in the setup can be described roughly as follows. As we have seen, the state estimation error, or the width of $\mathcal{Y}_k$, is the key in control under limited data rate. The error grows over time due to plant instability and is shrunk by state observations. So far we have assumed that the exact model of the actuator is available. Thus, the error has been invariant with respect to the control inputs. That is, we have been able to bring the state anywhere we want with no penalty. However, in the setup, the control inputs affect on the estimation error. In particular, we will see that large input will result in further growth in the estimation error. Evaluating such growth due to inputs is a new challenge in this chapter.

4.1 Problem formulation

Consider again the feedback system depicted in Fig. 2.1. We extend the classes of plants (2.1) and uncertainties (2.2) from those in Chapter 2. In this chapter, the plant may include an uncertain input parameter b_k:

$$y_{k+1} = a_{1,k}y_k + a_{2,k}y_{k-1} + \cdots + a_{n,k}y_{k-n+1} + b_k u_k, \tag{4.1}$$

$$a_{i,k} \in \mathcal{A}_i := [a_i^* - \epsilon_i, a_i^* + \epsilon_i] \text{ for } i = 1, 2, \ldots, n, \; b_k \in \mathcal{B} := [b^* - \delta, b^* + \delta], \tag{4.2}$$

where $\epsilon_i \geq 0$ and $\delta \geq 0$. We again assume that the plant is single input and single output.

To make the plant controllable for all time steps, we introduce the following assumption.

Assumption 4.1. For every time $k \in \mathbb{Z}_+$, the input parameter b_k is nonzero. That is, the following inequality hods:

$$|b^*| - \delta > 0. \tag{4.3}$$

Remark 4.1. In (4.1), the order of the input is restricted to one. For plants of two or more input parameters, it is difficult to characterize the limitations due to the analytical difficulties caused by the uncertainty. To derive a necessary condition, we have to study the optimal input for reducing state estimation errors and estimate the lower bound. When the order of the input is more than one, the analysis becomes a multi-parameter minimization problem with interval coefficients. It may be possible to solve the problem numerically but it is difficult to do so analytically. Since our main concern is to characterize limitations for stability in an explicit way, we introduced the restriction on the plant.

Regarding the quantizer, we employ the static one (2.9) and consider a static data rate R or a number N of the quantization cells.

In the following, we show a necessary condition and a sufficient condition for making the system in Fig. 2.1 MSS and see how the input uncertainty affects on the limitations for stability.

4.2 Limitations on data rate and loss probability under uncertain inputs

4.2.1 Necessary condition for stability

We now introduce the following notations to represent the necessary bound:

$$r_a := \frac{|\lambda_{A^*}| - \epsilon_n}{|\lambda_{A^*}| + \epsilon_n}, \quad r_b := \frac{|b^*| - \delta}{|b^*| + \delta}.$$

These notations represent relative uncertainties on λ_{A^*} (or $a_{n,k}$) and b_k. We note that r_a is equal to r defined in (2.19).

The following theorem is the necessity result for the uncertain input parameter case. Note that ν in theorem has been given in (2.19).

Theorem 4.1. For the system in Fig. 2.1 satisfying Assumptions 2.1 and 4.1, if the system is MSS with the static quantizer (2.9), then it holds that

$$R > \hat{R}_{\text{nec}} := \begin{cases} \log \frac{\log\{1-(\epsilon_n+\delta|\lambda_{A^*}|/|b^*|)\nu\}^2}{\log(r_a r_b)} & \text{if } \epsilon_n > 0 \text{ or } \delta > 0, \\ \log|\lambda_{A^*}| + \log\nu & \text{if } \epsilon_n = \delta = 0, \end{cases} \tag{4.4}$$

$$0 \le p < \hat{p}_{\text{nec}} := \frac{1 - (\epsilon_n + \delta|\lambda_{A^*}|/|b^*|)^2}{(|\lambda_{A^*}| + \epsilon_n)^2 - (\epsilon_n + \delta|\lambda_{A^*}|/|b^*|)^2}, \tag{4.5}$$

$$0 \le \epsilon_n + \delta\frac{|\lambda_{A^*}|}{|b^*|} < 1. \tag{4.6}$$

The theorem shows how the input uncertainty δ affects on the limitations on the communication constraints for stability. The minimum data rate $\hat{R}_{\text{nec}}$ is monotonically increasing with respect to δ, and similarly, the maximum loss probability $\hat{p}_{\text{nec}}$ is decreasing. Hence, more uncertainty on b_k results in higher requirements in the communication, higher data rate and less packet loss probability.

It is interesting to note that there is no explicit limitation on δ, but the sum $\epsilon_n + \delta|\lambda_{A^*}|/|b^*|$ of the uncertainties must be smaller than 1. Here, the product $\delta|\lambda_{A^*}|$ implies that the effect of δ on the limitations becomes greater when plant instability is large. Intuitively, this is because once a control input is applied to the plant state, then at the next time the state is amplified due to plant instability.

We confirm the limitations in the following example.

Example 4.1. Consider a plant with $a_n^* = 2.0$ and $b^* = 2.0$, and set the lossy probability as $p = 0.08$. Fig. 4.1 shows the necessary bound $\hat{R}_{\text{nec}}$ on the data rate given in Theorem 4.1 versus the uncertainties ϵ_n on $a_{n,k}$ and δ on b_k. For a large ϵ_n and δ such that $\epsilon_n + \delta|\lambda_{A^*}|/|b^*|$ takes a value close to 1, we need infinitely large data rate to stabilize the system.

4.2.2 Proof of Theorem 4.1

The main idea to prove Theorem 4.1 is the same as that for Theorem 2.1. The proof consists of the following three steps. First, we show an inequality regarding the expansion rate of the state estimation sets. Next, we derive the optimal structure in the quantization cells for data rate reduction. Finally, by substituting the structure of the optimal quantizer into the first inequality, the limitations are obtained.

The main difficulty in the case is that we have to take account of the expansion of the state estimation sets by control inputs. In Chapter 2, since we know the exact control input and can track the state, the width of the prediction set $\mathcal{Y}_{k+1}^{-}$ does not change by the control input applied to the plant. However, in the current setup, the prediction set will be expanded by the control input due to the existence of the uncertainty δ. Hence, the scaling parameter σ_{k+1} in the quantizer must be selected

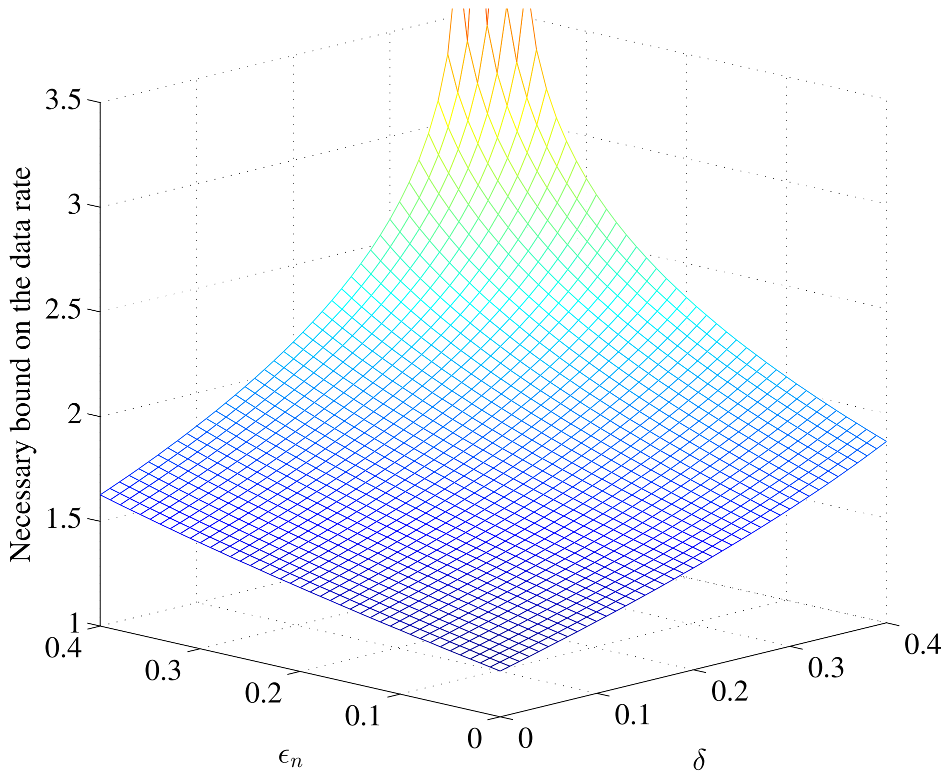


Figure 4.1: Necessary bound on the data rate versus the magnitudes of the uncertainties ϵ_n and δ ($a_n^* = 2.0$, $b^* = 2.0$, $p = 0.08$)

to cover the expansion due to the input in addition to the constraint (2.12). For a given control input u_k, the scaling parameter σ_{k+1} at the next time must satisfy the following inequality:

$$\sigma_{k+1} \geq 2 \max_{y^- \in \mathcal{Y}_{k+1}^-,\ b \in \mathcal{B}} |y^- + bu_k|. \tag{4.7}$$

The above inequality will be referred to in the first step of the proof.

Expansion rate of the estimation sets

The first step is to estimate the expansion rate of the prediction set. We first introduce the expansion rates $\hat{w}_l$, $l = 0, 1, \ldots, \lceil N/2 \rceil - 1$, for a given quantizer whose boundary points are $\{h_l\}_l$, and the maximum expansion rate $\bar{\hat{\eta}}$:

$$\hat{w}_l := \begin{cases} 2(|a_n^*| + \epsilon_n)h_{l+1} & \text{if } N \text{ is odd and } l = 0, \\ (|a_n^*| + \epsilon_n)(1 + \delta/|b^*|)h_{l+1} - (|a_n^*| - \epsilon_n)(1 - \delta/|b^*|)h_l & \text{else,} \end{cases} \tag{4.8}$$

$$\bar{\hat{\eta}} := \begin{cases} |a_n^*| + \epsilon_n & \text{with prob. } p, \\ \max_{l \in \{0,1,\ldots,\lceil N/2 \rceil - 1\}} \hat{w}_l & \text{with prob. } 1 - p. \end{cases} \tag{4.9}$$

Then, the following lemma holds.

Lemma 4.1. If the system depicted in Fig. 2.1 is MSS, then it holds that

$$\mathrm{E}[(\bar{\hat{\eta}})^2] < 1. \tag{4.10}$$

Proof. From the first part of the proof of Lemma 2.2, we have that if the feedback system is MSS, then the scaling parameter σ_k is also MSS. In the following, we will show that

$$\mathrm{E}[\sigma_{k+1}^2] \geq \mathrm{E}[(\bar{\hat{\eta}})^2]\mathrm{E}[\sigma_{k-n+1}^2] \tag{4.11}$$

and hence (4.10) must hold because of the stability of σ_k.

To show (4.11), we analyze the lower bound of σ_{k+1} given in (4.7). We claim that the right-hand side of (4.7) is bounded from below as follows:

$$\begin{aligned} 2 \max_{y^- \in \mathcal{Y}_{k+1}^-,\ b \in \mathcal{B}} |y^- + bu_k| &\geq 2 \max_{y^- \in \mathcal{Y}_{k+1}^-,\ b \in \mathcal{B}} |y^- + bu^*(\mathcal{Y}_{k+1}^-)| \\ &= \mu(\mathcal{Y}_{k+1}^-) + \frac{2\delta}{|b^*|}|c(\mathcal{Y}_{k+1}^-)|. \end{aligned} \tag{4.12}$$

Here, the control input u^* is defined as

$$u^*(\mathcal{Y}) := -\frac{c(\mathcal{Y})}{b^*} \tag{4.13}$$

for an interval $\mathcal{Y}$ on $\mathbb{R}$, and $c(\cdot)$ is the midpoint of an interval. The input $u^*(\mathcal{Y}_{k+1}^-)$

brings the midpoint of $\mathcal{Y}_{k+1}^-$ into the origin when the parameter b_k is equal to b^*.

To show (4.12), we first consider the case that $c(\mathcal{Y}_{k+1}^-) > 0$ and $b^* - \delta > 0$. For any input u_k, it can be written as $u_k = u^*(\mathcal{Y}_{k+1}^-) \pm \Delta_k$, where $\Delta_k \geq 0$. Substituting this into the left-hand side of (4.12), we have

$$\begin{aligned}
&2 \max_{y^- \in \mathcal{Y}_{k+1}^-,\ b \in \mathcal{B}} |y^- + b(u^*(\mathcal{Y}_{k+1}^-) \pm \Delta_k)| \\
&= 2 \max \Big[\overline{\mathcal{Y}}_{k+1}^- + (b^* - \delta) \left(u^*(\mathcal{Y}_{k+1}^-) \pm \Delta_k \right), \\
&\qquad - \left\{ \underline{\mathcal{Y}}_{k+1}^- + (b^* + \delta) \left(u^*(\mathcal{Y}_{k+1}^-) \pm \Delta_k \right) \right\} \Big] \\
&= 2 \max \Bigg[\frac{\overline{\mathcal{Y}}_{k+1}^- - \underline{\mathcal{Y}}_{k+1}^-}{2} + \frac{\delta}{b^*} \frac{\overline{\mathcal{Y}}_{k+1}^- + \underline{\mathcal{Y}}_{k+1}^-}{2} \pm (b^* - \delta) \Delta_k, \\
&\qquad \frac{\overline{\mathcal{Y}}_{k+1}^- - \underline{\mathcal{Y}}_{k+1}^-}{2} + \frac{\delta}{b^*} \frac{\overline{\mathcal{Y}}_{k+1}^- + \underline{\mathcal{Y}}_{k+1}^-}{2} \mp (b^* + \delta) \Delta_k \Bigg] \\
&= \max \left[\mu(\mathcal{Y}_{k+1}^-) + \frac{2\delta}{b^*} c(\mathcal{Y}_{k+1}^-) \pm 2(b^* - \delta)\Delta_k,\ \mu(\mathcal{Y}_{k+1}^-) + \frac{2\delta}{b^*} c(\mathcal{Y}_{k+1}^-) \mp 2(b^* + \delta)\Delta_k \right].
\end{aligned} \tag{4.14}$$

Here, the first equality follows from the conditions $c(\mathcal{Y}_{k+1}^-) > 0$ and $b^* - \delta > 0$ for this case. The far right-hand side of (4.14) takes its minimum when $\Delta_k = 0$, and hence we have (4.12).

For the case of $c(\mathcal{Y}_{k+1}^-) \leq 0$ or $b^* - \delta < 0$, it can be reduced to the above case by changing signs of $\overline{\mathcal{Y}}_{k+1}$, $\underline{\mathcal{Y}}_{k+1}$, and b^* appropriately.

We next evaluate the far right-hand side of (4.12) by focusing on the effect of $a_{n,k}$ on the prediction set $\mathcal{Y}_{k+1}^-$. In particular, we show that

$$\mu(\mathcal{Y}_{k+1}^-) + \frac{2\delta}{|b^*|} |c(\mathcal{Y}_{k+1}^-)| \geq \mu(\mathcal{A}_n \mathcal{Y}_{k-n+1}) + \frac{2\delta}{|b^*|} |c(\mathcal{A}_n \mathcal{Y}_{k-n+1})|. \tag{4.15}$$

The interval $\mathcal{A}_n \mathcal{Y}_{k-n+1}$ is a subset of the prediction set $\mathcal{Y}_{k+1}^-$ defined in (2.13). Thus, the above inequality implies that if we restrict the space for possible y^- from $\mathcal{Y}_{k+1}^-$ to $\mathcal{A}_n \mathcal{Y}_{k-n+1}$, then the furthest point $\max |y^- + bu^*|$ from the origin, which should be covered by σ_{k+1}, becomes closer to the origin. To show (4.15), consider the following claim: For any connected interval $\mathcal{A}_n \mathcal{Y}_{k-n+1}$ on $\mathbb{R}$, there exists a possible $\mathcal{Y}_{k-n+2}$ such that

$$\mu(\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2}+\mathcal{A}_n\mathcal{Y}_{k-n+1})+\frac{2\delta}{|b^*|}|c(\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2}+\mathcal{A}_n\mathcal{Y}_{k-n+1})|$$
$$\geq \mu(\mathcal{A}_n\mathcal{Y}_{k-n+1})+\frac{2\delta}{|b^*|}|c(\mathcal{A}_n\mathcal{Y}_{k-n+1})|. \tag{4.16}$$

Here, the interval $\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2}$ is defined in (2.17) and also the sum of intervals is defined as:

$$\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2}+\mathcal{A}_n\mathcal{Y}_{k-n+1} := \{x_1+x_2 : x_1\in\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2},\ x_2\in\mathcal{A}_n\mathcal{Y}_{k-n+1}\}.$$

It is clear that $\mu(\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2}+\mathcal{A}_n\mathcal{Y}_{k-n+1})\geq\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1})$ by the Brunn-Minkowski inequality (2.16). Hence, what we have to show is there exists a possible quantization cell such that the corresponding estimation set $\mathcal{Y}_{k-n+2}$ satisfies $|c(\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2}+\mathcal{A}_n\mathcal{Y}_{k-n+1})|\geq|c(\mathcal{A}_n\mathcal{Y}_{k-n+1})|$. Consider the following two cases (i) and (ii).

(i) $c(\mathcal{A}_n\mathcal{Y}_{k-n+1})\geq 0$: First, suppose that $c(\mathcal{A}_{n-1})\geq 0$. Then, there exists a quantization cell such that $c(\mathcal{Y}_{k-n+2})\geq 0$ since the cells are symmetric about the origin (see (2.7) and (2.8)). For such a cell, we have $c(\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2})\geq 0$.

For the case of $c(\mathcal{A}_{n-1})<0$, we also have $c(\mathcal{A}_{n-1}\mathcal{Y}_{k-n+2})\geq 0$ for a cell such that $c(\mathcal{Y}_{\mathcal{Y}_{k-n+2}})\leq 0$. Hence, there exists a possible $\mathcal{Y}_{k-n+2}$ and the inequality (4.16) holds.

(ii) $c(\mathcal{A}_n\mathcal{Y}_{k-n+1})<0$: This case can be reduced to (i).

Applying the same process to each product $\mathcal{A}_i\mathcal{Y}_{k-i+1}$ of intervals, which are by (2.13) part of $\mathcal{Y}^-_{k+1}$, for $i=n-1,n-2,\ldots,1$, we have (4.15).

The last step of the proof is calculating the right-hand side of (4.15) and obtain the following equality:

$$\mu(\mathcal{A}_n\mathcal{Y}_{k-n+1})+\frac{2\delta}{|b^*|}|c(\mathcal{A}_n\mathcal{Y}_{k-n+1})|=\hat{\eta}_{k-n+1}\sigma_{k-n+1}, \tag{4.17}$$

where $\hat{\eta}_k$ is a random variable defined as

$$\hat{\eta}_k := \begin{cases} |a_n^*| + \epsilon_n & \text{if } \gamma_k = 0, \\ \hat{w}_{l_k} & \text{if } \gamma_k = 1. \end{cases}$$

Here, $\hat{w}_l$ is defined in (4.8) and l_k represents the index of $\mathcal{Y}_k$, the quantization cell which y_k falls into at time k.

To derive (4.17), we consider the following three cases (i)–(iii). For simplicity, we assume that $a^* > 0$ and $b^* > 0$ in what follows.

(i) $0 \leq \mathcal{Y}_{k-n+1}$: In this case, it must hold that $\gamma_{k-n+1} = 1$. From the basic results for products of intervals [38], (2.17), and (2.18), the upper edge $\overline{\mathcal{A}_n \mathcal{Y}_{k-n+1}}$ and the lower edge $\underline{\mathcal{A}_n \mathcal{Y}_{k-n+1}}$ of the interval $\mathcal{A}_n \mathcal{Y}_{k-n+1}$ are as follows:

$$\overline{\mathcal{A}_n \mathcal{Y}_{k-n+1}} = (a_n^* + \epsilon_n)\overline{\mathcal{Y}}_{k-n+1}, \quad \underline{\mathcal{A}_n \mathcal{Y}_{k-n+1}} = (a_n^* - \epsilon_n)\underline{\mathcal{Y}}_{k-n+1}.$$

Substituting these equations into $\mu(\mathcal{A}_n \mathcal{Y}_{k-n+1})$ and $c(\mathcal{A}_n \mathcal{Y}_{k-n+1})$, we have

$$\begin{aligned} &\mu(\mathcal{A}_n \mathcal{Y}_{k-n+1}) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_n \mathcal{Y}_{k-n+1})| \\ &= (a^* + \epsilon_n)\overline{\mathcal{Y}}_{k-n+1} - (a^* - \epsilon)\underline{\mathcal{Y}}_{k-n+1} + \frac{2\delta}{b^*}\frac{(a^* + \epsilon_n)\overline{\mathcal{Y}}_{k-n+1} + (a^* - \epsilon)\underline{\mathcal{Y}}_{k-n+1}}{2} \\ &= \left\{(a^* + \epsilon_n)\left(1 + \frac{\delta}{b^*}\right)h_{l_{k-n+1}+1} - (a^* - \epsilon_n)\left(1 - \frac{\delta}{b^*}\right)h_{l_{k-n+1}}\right\}\sigma_{k-n+1}, \end{aligned}$$

where we used the relations $\overline{\mathcal{Y}}_{k-n+1} = h_{l_{l-n+1}+1}\sigma_{k-n+1}$ and $\underline{\mathcal{Y}}_{k-n+1} = h_{l_{l-n+1}}\sigma_{k-n+1}$ to obtain the second equality. Hence, (4.17) holds in this case.

(ii) $\underline{\mathcal{Y}}_{k-n+1} < 0 < \overline{\mathcal{Y}}_{k-n+1}$: In this case, we have

$$\overline{\mathcal{A}_n \mathcal{Y}_{k-n+1}} = (a_n^* + \epsilon_n)\overline{\mathcal{Y}}_{k-n+1}, \quad \underline{\mathcal{A}_n \mathcal{Y}_{k-n+1}} = (a_n^* + \epsilon_n)\underline{\mathcal{Y}}_{k-n+1}. \tag{4.18}$$

Because of the symmetry of the quantization cells, it holds that $c(\mathcal{A}_n \mathcal{Y}_{k-n+1}) = 0$ and hence $u^*(\mathcal{A}_n \mathcal{Y}_{k-n+1}) = 0$. Thus, the left-hand side of (4.17) is equal to $\mu(\mathcal{A}_n \mathcal{Y}_{k-n+1})$, which has been analyzed in the proof of Lemma 2.2.

Consider the following two cases:

(ii-1) If $\gamma_{k-n+1} = 0$, then $\mathcal{Y}_{k-n+1} = [-\sigma_{k-n+1}/2, \sigma_{k-n+1}/2]$. Hence, from (4.18), we have

$$\mu(\mathcal{A}_n \mathcal{Y}_{k-n+1}) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_n \mathcal{Y}_{k-n+1})| = (a^* + \epsilon_n)\sigma_{k-n+1}.$$

(ii-2) Otherwise, N must be odd and $l_{k-n+1} = 1$ from (2.7), (2.8), and the condition (ii). In this case,

$$\mu(\mathcal{A}_n \mathcal{Y}_{k-n+1}) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_n \mathcal{Y}_{k-n+1})| = 2(a^* + \epsilon_n)h_1\sigma_{k-n+1}. \tag{4.19}$$

since $\mathcal{Y}_{k-n+1} = [-h_1\sigma_{k-n+1}, h_1\sigma_{k-n+1})$. Hence, (4.17) holds for this case also.

(iii) $\overline{\mathcal{Y}}_{k-n+1} \leq 0$: This case can be reduced to (i).

From the constraint (4.7) on σ_{k+1} and the inequalities (4.12), (4.15), and (4.17), it follows that $\sigma_{k+1} \geq \hat{\eta}_{k-n+1}\sigma_{k-n+1}$. Noticing that $\{\gamma_k\}_k$ is i.i.d., then we have $\mathrm{E}[\sigma_{k+1}^2] \geq \mathrm{E}[\hat{\eta}_{k-n+1}^2]\mathrm{E}[\sigma_{k-n+1}^2]$. The expectation $\mathrm{E}[\hat{\eta}_{k-n+1}^2]$ may vary with $\mathcal{Y}_{k-n+1}$. By (4.9), it is clear that $\overline{\hat{\eta}}$ is the maximum of $\hat{\eta}_k$. Thus, we have (4.11).

Finally, since σ_k is MSS for all possible parameters and initial values, from (4.11), the inequality (4.10) must hold. ■

Optimal quantizer

Next, we find the quantizer that minimizes $\overline{\hat{\eta}}$ for a given number N of quantization cells. We first introduce the following quantizer $\hat{q}_N^*$ represented by the boundary points $\{h_l^*\}_l$ as follows:

(i) If $\epsilon_n > 0$ or $\delta = 0$, then

$$\hat{h}_l^* = \begin{cases} \frac{1}{2}\frac{1-\hat{t}(r_a r_b)^l}{1-\hat{t}(r_a r_b)^{\lceil N/2 \rceil}} & \text{if } N \text{ is odd,} \\ \frac{1}{2}\frac{1-(r_a r_b)^l}{1-(r_a r_b)^{\lceil N/2 \rceil}} & \text{if } N \text{ is even,} \end{cases} \tag{4.20}$$

where $\hat{t} := (1 + \delta/|b^*|) \,/\, (1 - \epsilon_n/|\lambda_{A^*}|)$.

(ii) If $\epsilon_n = \delta = 0$, then

$$\hat{h}_l^* = \begin{cases} \frac{1}{N}\left(l - \frac{1}{2}\right) & \text{if } N \text{ is odd,} \\ \frac{1}{N} l & \text{if } N \text{ is even.} \end{cases} \tag{4.21}$$

The following lemma holds.

Lemma 4.2. The quantizer $\hat{q}_N^*$ minimizes $\mathrm{E}[(\overline{\eta})^2]$.

The quantization cells of the optimal quantizer $\hat{q}_N^*$ are nonuniform when the plant is uncertain with $\epsilon_n > 0$ or $\delta > 0$. The structure of the cells are similar to q_N^*, which has been introduced in Chapter 2 and is illustrated in Fig. 2.3. The difference between q_N^* and $\hat{q}_N^*$ is the ratio of the cells' width; in q_N^*, it depends only on the relative uncertainty r on $a_{n,k}$ but in $\hat{q}_N^*$ the ratio is $r_a r_b$, which is the product of the relative uncertainties on $a_{n,k}$ and b_k. When there is no uncertainty in the plant, i.e., $\epsilon_n = \delta = 0$, then $\hat{q}_N^*$ becomes the well-known uniform quantizer. This is a common feature of q_N^* and $\hat{q}_N^*$.

Proof of Lemma 4.2. We follow the approach in the proof of Lemma 2.3 and consider the minimization of $\max_l \hat{w}_l$. One can easily confirm that $\hat{w}_l$ are the same for all l, i.e.,

$$\hat{w}_l = \hat{w}_{l'} \text{ for any } l, l' \in \{0, 1, \dots, \lceil N/2 \rceil - 1\} \tag{4.22}$$

if and only if the quantizer boundaries are $\{\hat{h}_l^*\}_l$. Therefore, we show that a quantizer which does not satisfy (4.22) will result in a larger $\max_l \hat{w}_l$ compared with the case that $\{\hat{h}_l^*\}_l$ is employed. We prove this by contradiction.

Let $\hat{w}_l(h)$ denote the $\hat{w}_l$ when the quantizer is $\{h_l\}_l$. Assume that there exists a quantizer $\{g_l\}_l$ such that (4.22) is not satisfied and it holds that $\max_l \hat{w}_l(g) < \max_l \hat{w}_l(h^*)$. Then, we have

$$\hat{w}_l(g) \leq \max_{l'} \hat{w}_{l'}(g) < \max_{l'} \hat{w}_{l'}(\hat{h}^*) = \hat{w}_l(\hat{h}^*) \tag{4.23}$$

for any $l \in \{0, 1, \ldots, \lceil N/2 \rceil - 1\}$.

We now look at the relation between g_l and $\hat{h}_l^*$ for each $l = 0, 1, \ldots, \lceil N/2 \rceil - 1$. From (2.6), we have $g_0 = \hat{h}_0^* = 0$. Substituting these equations into (4.8), we obtain

$$\hat{w}_0(g) = \begin{cases} 2(|a_n^*| + \epsilon_n) g_1 & \text{if } N \text{ is odd,} \\ (|a_n^*| + \epsilon_n)(1 + \delta/|b^*|) g_1 & \text{else,} \end{cases} \tag{4.24}$$

$$\hat{w}_0(\hat{h}^*) = \begin{cases} 2(|a_n^*| + \epsilon_n) \hat{h}_1^* & \text{if } N \text{ is odd,} \\ (|a_n^*| + \epsilon_n)(1 + \delta/|b^*|) \hat{h}_1^* & \text{else.} \end{cases} \tag{4.25}$$

On the other hand, we have $\hat{w}_0(g) < \hat{w}_0(\hat{h}^*)$ by considering the case $l = 0$ in (4.23). Thus, from (4.24), (4.25), and this inequality, the following relation holds:

$$g_1 < \hat{h}_1^*. \tag{4.26}$$

Furthermore, $\hat{w}_l(g)$ and $\hat{w}_l(\hat{h}^*)$ for $l \in \{1, 2, \ldots, \lceil N/2 \rceil - 1\}$ are defined in (4.8) as

$$\begin{aligned} \hat{w}_l(g) &= (|a_n^*| + \epsilon_n) \left(1 + \frac{\delta}{|b^*|}\right) g_{l+1} - (|a_n^*| - \epsilon_n) \left(1 - \frac{\delta}{|b^*|}\right) g_l, \\ \hat{w}_l(\hat{h}^*) &= (|a_n^*| + \epsilon_n) \left(1 + \frac{\delta}{|b^*|}\right) \hat{h}_{l+1}^* - (|a_n^*| - \epsilon_n) \left(1 - \frac{\delta}{|b^*|}\right) \hat{h}_l^*. \end{aligned}$$

By substituting these equations into (4.23), we have

$$g_{l+1} \leq r_a r_b g_l + \frac{\max_{l'} \hat{w}_{l'}(g)}{(|a_n^*| + \epsilon_n)(1 + \delta/|b^*|)}, \quad \hat{h}_{l+1}^* = r_a r_b \hat{h}_l^* + \frac{\max_{l'} \hat{w}_{l'}(\hat{h}^*)}{(|a_n^*| + \epsilon_n)(1 + \delta/|b^*|)}.$$

From (4.26) and the above, we recursively obtain $g_l < \hat{h}_l^*$ for all $l = 1, 2, \ldots, \lceil N/2 \rceil$. This contradicts $g_{\lceil N/2 \rceil} = \hat{h}_{\lceil N/2 \rceil}^* = 1/2$ given in (2.6). Therefore, it follows that $\{\hat{h}_l^*\}_l$ is the optimal quantizer. ■

Limitations under the use of the optimal quantizer

The last step of the proof of Theorem 4.1 is deriving (4.4)–(4.6) from Lemmas 4.1 and 4.2.

Proof of Theorem 4.1. It is sufficient to consider the case when $\delta > 0$. Other cases are from Theorem 2.1.

Consider the following two cases (i) and (ii).

(i) N is even: Substituting (4.20) to the definition (4.8) of $\hat{w}_l$, we have

$$\max_l \hat{w}_l = \frac{\epsilon_n + \delta |a_n^*|/|b^*|}{1 - (r_a r_b)^{\lceil N/2 \rceil}}.$$

Consequently, the inequality (4.10) is equivalent to

$$\mathrm{E}[(\bar{\hat{\eta}})^2] = p(|a_n^*| + \epsilon_n)^2 + (1-p)\left(\frac{\epsilon_n + \delta |a_n^*|/|b^*|}{1 - (r_a r_b)^{\lceil N/2 \rceil}}\right)^2 < 1.$$

Solving this inequality with respect to N, we have

$$N > \hat{N}_{\mathrm{nec}}^{(\mathrm{e})} := 2\frac{\log\{1 - (\epsilon_n + \delta |a_n^*|/|b^*|)\nu\}}{\log(r_a r_b)} \text{ and } 1 - \left(\epsilon_n + \delta\frac{|a_n^*|}{|b^*|}\right)\nu > 0.$$

(ii) N is odd: Similarly it follows that (4.10) is equivalent to

$$N > \hat{N}_{\mathrm{nec}}^{(\mathrm{o})} := 2\frac{\log\{1 - (\epsilon_n + \delta |a_n^*|/|b^*|)\nu\} - \log \hat{t}}{\log(r_a r_b)} - 1$$
$$\text{and } 1 - \left(\epsilon_n + \delta\frac{|a_n^*|}{|b^*|}\right)\nu > 0.$$

Comparing the lower bounds $\hat{N}_{\mathrm{nec}}^{(\mathrm{e})}$ and $\hat{N}_{\mathrm{nec}}^{(\mathrm{o})}$, by (2.18) and (4.3), we have $\hat{N}_{\mathrm{nec}}^{(\mathrm{o})} > \hat{N}_{\mathrm{nec}}^{(\mathrm{e})}$. Therefore, $N > \hat{N}_{\mathrm{nec}}^{(\mathrm{e})}$ must hold for both cases (i-1) and (i-2). This proves the inequality (4.4). Furthermore, from the inequality $1 - (\epsilon_n + \delta |a_n^*|/|b^*|)\,\nu > 0$, we have (4.5) and (4.6). ■

4.3 Construction of stabilizing controllers

In this section, we present a sufficient condition for the existence of a control scheme for the feedback system to be MSS. We follow the approach in Section 2.3 and extend the control scheme provided in the section to the case of uncertain input parameters.

4.3.1 Sufficient condition for stability

Given a data rate R and a quantizer with its boundaries $\{h_l\}_l$, we employ the control law as follows: In the encoder (2.5), the scaling parameter is determined by

$$\sigma_k = \mu(\mathcal{Y}_k^-) + \frac{2\delta}{|b^*|}|c(\mathcal{Y}_k^-)| \tag{4.27}$$

at each time k. Furthermore, in the controller (2.11), the control input is given as

$$u_k = -\frac{c(\mathcal{Y}_{k+1}^-)}{b^*}. \tag{4.28}$$

We note that, as we have seen in the proof of Lemma 4.1, the input (4.28) minimizes the lower bound shown in (4.7) on σ_k. Since the minimum is equal to the right-hand side of (4.27), equality holds in (4.7) under the control law.

Next, we introduce notations required to express the sufficient condition. For $i = 1, 2, \ldots, n$, let the random variables $\hat{\theta}_{i,k}$ be given by

$$\hat{\theta}_{i,k} := \begin{cases} |a_i^*| + \epsilon_i & \text{if } \gamma_{k-i+1} = 0, \\ \bar{\hat{w}}_i & \text{if } \gamma_{k-i+1} = 1. \end{cases} \tag{4.29}$$

Here, $\bar{\hat{w}}_i$ is defined as follows for the given quantizer boundaries $\{h_l\}_l$:

$$\bar{\hat{w}}_i := \begin{cases} \max\left\{\overline{w}_i^{(0)}, \bar{\hat{w}}_i^{(1)}\right\} & \text{if } N \text{ is odd and } \mathcal{A}_i \not\ni 0, \\ \max\left\{\epsilon_i + \delta|a_i^*|/|b^*|,\ \overline{w}_i^{(0)}\right\} & \text{if } N \text{ is odd and } \mathcal{A}_i \ni 0, \\ \bar{\hat{w}}_i^{(1)} & \text{if } N \text{ is even and } \mathcal{A}_i \not\ni 0, \\ \epsilon_i + \delta|a_i^*|/|b^*| & \text{if } N \text{ is even and } \mathcal{A}_i \ni 0, \end{cases}$$

where $\overline{w}_i^0$ is defined in (2.39) and $\bar{\hat{w}}_i^{(1)}$ is defined as

$$\bar{\hat{w}}_i^{(1)} := \max_{l \in \{0,\ldots,\lceil N/2 \rceil - 1\}} \left\{ (|a_i^*| + \epsilon_i)\left(1 + \delta\frac{|a_i^*|}{|b^*|}\right) h_{l+1} - (|a_i^*| - \epsilon_i)\left(1 - \delta\frac{|a_i^*|}{|b^*|}\right) h_l \right\}. \tag{4.30}$$

We will see in the proof later that these are used to express an upper bound on the scaling parameter (4.27) as $\sigma_{k+1} \leq \sum_{i=1}^{n} \hat{\theta}_{i,k}\sigma_{k-i+1}$ Moreover, define the random variable matrix $\hat{H}_{\Gamma_k}$ containing $\hat{\theta}_{1,k}, \ldots, \hat{\theta}_{n,k}$ by

$$\hat{H}_{\Gamma_k} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{\theta}_{n,k} & \hat{\theta}_{n-1,k} & \cdots & \hat{\theta}_{1,k} \end{bmatrix}. \tag{4.31}$$

Here, Γ_k is the vector of random variables defined in (2.41). The transition probability matrix P for the process Γ_k is given in (2.42).

Finally, we define the matrix $\hat{F}$ using $\hat{H}_{\Gamma_k}$ and P by

$$\hat{F} := F_1\hat{F}_2, \tag{4.32}$$

where F_1 is given in (2.44) and

$$\hat{F}_2 := \operatorname{diag}\left(\hat{H}_{\Gamma(1)} \otimes \hat{H}_{\Gamma(1)}, \ldots, \hat{H}_{\Gamma(2^n)} \otimes \hat{H}_{\Gamma(2^n)}\right).$$

Here, $\operatorname{diag}(\cdot)$ denotes a block diagonal matrix and $\otimes$ is the Kronecker product. Also, let $\rho(\cdot)$ be the spectral radius of a matrix.

We are now ready to present the main theorem of this section.

Theorem 4.2. Given the data rate $R = \log N$, the loss probability $p \in [0, 1)$, and the quantizer $\{h_l\}_l$, if the matrix $\hat{F}$ in (4.32) satisfies

$$\rho(\hat{F}) < 1, \tag{4.33}$$

then under the control law using (4.27) and (4.28), the system depicted in Fig. 2.1 is MSS.

4.3.2 Proof of Theorem 4.2

Following the approach for the proof of Theorem 2.2, we consider the Markov jump system

$$\hat{z}_{k+1} = \hat{H}_{\Gamma_k}\hat{z}_k, \quad \hat{z}_0 := [\sigma_{-n+1}\ \sigma_{-n}\ \cdots\ \sigma_0]^T. \tag{4.34}$$

We show that the stability of the feedback system can be reduced to stability of (4.34) in the lemma below.

Lemma 4.3. If the Markov jump linear system (4.34) is stable in the sense that $\mathrm{E}[\hat{z}_k\hat{z}_k^T]$ converges to the zero matrix, then the system in Fig. 2.1 can be MSS under the control law using (4.27) and (4.28).

Proof. We first show that if $\mathrm{E}[\sigma_k^2] \to 0$ then $\mathrm{E}[|y_k|^2] \to 0$ as $k \to \infty$ under the control law (4.27) and (4.28). Substituting (4.28) into (4.1), and by referring to the definition (2.13) of $\mathcal{Y}_{k+1}$, we have

$$\begin{aligned}|y_{k+1}| &= \left| a_{1,k}y_k + \cdots + a_{n,k}y_{k-n+1} - b_k \frac{c(\mathcal{Y}_{k+1}^-)}{b^*} \right| \\ &\le \frac{1}{2}\left(\mu(\mathcal{Y}_{k+1}^-) + \frac{2\delta}{|b^*|}|c(\mathcal{Y}_{k+1}^-)| \right) = \frac{\sigma_{k+1}}{2}.\end{aligned}$$

Next, to establish that the stability of (4.34) implies that σ_k is MSS, we prove the following relation

$$\sigma_k \le (\hat{z}_k)_n \text{ for } k = 0, 1, \ldots, \tag{4.35}$$

where $(\cdot)_n$ is the nth element of a vector.

The scaling parameter (4.27) can be decomposed as follows

$$\begin{aligned}\sigma_{k+1} &= \sum_{i=1}^{n} \mu(\mathcal{A}_i\mathcal{Y}_{k-i+1}) + \frac{2\delta}{|b^*|}|c(\mathcal{Y}_{k+1}^-)| \\ &\le \sum_{i=1}^{n} \left\{ \mu(\mathcal{A}_i\mathcal{Y}_{k-i+1}) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})| \right\}.\end{aligned} \tag{4.36}$$

Here, the equality follows from (2.16) and the inequality holds by applying the triangle inequality to the second term. For the term $\mu(\mathcal{A}_i\mathcal{Y}_{k-i+1})$, we have seen that

$$\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right) = \begin{cases} \left(|a_i^*| + \epsilon_i\right)\mu(\mathcal{Y}_{k-i+1}) & \text{if } \mathcal{Y}_{k-i+1} \ni 0, \\ |a_i^*|\mu(\mathcal{Y}_{k-i+1}) + \epsilon|\overline{\mathcal{Y}}_{k-i+1} + \underline{\mathcal{Y}}_{k-i+1}| & \text{if } \mathcal{Y}_{k-i+1} \not\ni 0 \text{ and } \mathcal{A}_i \not\ni 0, \\ 2\epsilon_i \max\left\{|\overline{\mathcal{Y}}_{k-i+1}|,\ |\underline{\mathcal{Y}}_{k-i+1}|\right\} & \text{if } \mathcal{Y}_{k-i+1} \not\ni 0 \text{ and } \mathcal{A}_i \ni 0, \end{cases} \tag{4.37}$$

for $i = 1, 2, \ldots, n$, in the proof of Lemma 2.4. Similarly, for the second term $2\delta|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})|/|b^*|$ in (4.36), we have

$$\begin{aligned} &|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})| \\ &= \begin{cases} 0 & \text{if } \mathcal{Y}_{k-i+1} \ni 0, \\ \Big[\left(|a_i^*| + \epsilon_i\right)\max\left\{|\overline{\mathcal{Y}}_{k-i+1}|,\ |\underline{\mathcal{Y}}_{k-i+1}|\right\} & \\ \quad +\left(|a_i^*| - \epsilon_i\right)\min\left\{|\overline{\mathcal{Y}}_{k-i+1}|,\ |\underline{\mathcal{Y}}_{k-i+1}|\right\}\Big]/2 & \text{if } \mathcal{Y}_{k-i+1} \not\ni 0 \text{ and } \mathcal{A}_i \not\ni 0, \\ |a_i^*|\max\left\{|\overline{\mathcal{Y}}_{k-i+1}|,\ |\underline{\mathcal{Y}}_{k-i+1}|\right\} & \text{if } \mathcal{Y}_{k-i+1} \not\ni 0 \text{ and } \mathcal{A}_i \ni 0, \end{cases} \end{aligned} \tag{4.38}$$

by using basic results in interval arithmetics [38] for $i = 1, 2, \ldots, n$.

The equations (4.37) and (4.38) are used to obtain the maximum of ith term in (4.36) over all possible $\mathcal{Y}_{k-i+1}$, i.e., the following inequality:

$$\mu(\mathcal{A}_i\mathcal{Y}_{k-i+1}) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})| \leq \hat{\theta}_{i,k}\sigma_{k-i+1}. \tag{4.39}$$

To show (4.39), consider the following three cases.

(i) $\gamma_{k-n+1} = 0$: In this case, we have $\mathcal{Y}_{k-i+1} = [-\sigma_{k-i+1}/2, \sigma_{k-i+1}/2] \ni 0$. Thus, by (4.37) and (4.38), it follows that

$$\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})| = (|a_i^*| + \epsilon_i)\sigma_{k-i+1.}$$

From the definition (4.29) of $\hat{\theta}_{i,k}$, in this case we have $\hat{\theta}_{i,k} = |a_i^*| + \epsilon_i$ and hence, (4.39) holds.

(ii) $\gamma_{k-i+1} = 1$ and N is odd: Consider the following two cases.

(ii-1) $\mathcal{Y}_{k-i+1} \ni 0$: By (2.7), it must hold $\mathcal{Y}_{k-i+1} = [-h_1\sigma_{k-i+1}, h_1\sigma_{k-i+1})$. Thus, from (4.37) and (4.38), we have

$$\mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})| = 2(|a_i^*| + \epsilon_i)h_1\sigma_{k-i+1} = \overline{w}_i^{(0)}\sigma_{k-i+1}, \tag{4.40}$$

where $\overline{w}_i^{(0)}$ is defined in (2.39).

(ii-2) $\mathcal{Y}_{k-i+1} \not\ni 0$: In this case, we have $\mathcal{Y}_{k-i+1} = [h_l\sigma_{k-i+1}, h_{l+1}\sigma_{k-i+1})$ or $[-h_{l+1}\sigma_{k-i+1}, -h_l\sigma_{k-i+1})$, where $l \in \{1, 2, \ldots, \lceil N/2\rceil - 1\}$. Thus, by (4.37) and (4.38), we obtain

$$\begin{aligned}
&\mu(\mathcal{A}_i\mathcal{Y}_{k-i+1}) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})| \\
&= \begin{cases} \left\{(|a_i^*| + \epsilon_i)\left(1 + \frac{\delta}{|b^*|}\right)h_{l+1} - (|a_i^*| - \epsilon_i)\left(1 + \frac{\delta}{|b^*|}\right)h_l\right\}\sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1} \not\ni 0, \\ 2\left(\epsilon_i + \delta|a_i^*|/|b^*|\right)h_{l+1}\sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1} \ni 0. \end{cases}
\end{aligned}$$

Taking the maximum of the right-hand side of the above equality over $l \in \{1, 2, \ldots, \lceil N/2\rceil - 1\}$, we have

$$\max_l \mu\left(\mathcal{A}_i\mathcal{Y}_{k-i+1}\right) + \frac{2\delta}{|b^*|}|c(\mathcal{A}_i\mathcal{Y}_{k-i+1})| = \begin{cases} \overline{\hat{w}}_i^{(1)}\sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1} \not\ni 0, \\ \left(\epsilon_i + \delta|a_i^*|/|b^*|\right)\sigma_{k-i+1} & \text{if } \mathcal{A}_{k-i+1} \ni 0, \end{cases} \tag{4.41}$$

where $\overline{\hat{w}}_i^{(1)}$ is defined in (4.30).

Combing (4.40) and (4.41), we conclude that (4.39) holds in the case (ii) also.

(iii) $\gamma_{k-i+1} = 1$ and N is even: In this case, we have $\mathcal{Y}_{k-i+1} \not\ni 0$. Hence, this case can be reduced to (ii-2).

From (4.36) and (4.39), it follows that

$$\sigma_{k+1} \le \sum_{i=1}^{n}\hat{\theta}_{i,k}\sigma_{k-i+1}. \tag{4.42}$$

Notice that the right-hand side of (4.42) is equal to the nth low of $\hat{H}_{\Gamma_k}[\sigma_k \ \sigma_{k-1} \ \cdots \ \sigma_{k-n+1}]^T$. Thus, from the definition of the Markov jump system (4.34), we obtain (4.35).

Finally, it is straight forward to show that if $\mathrm{E}[\hat{z}_k(\hat{z}_k)^T]$ goes to the zero matrix, then $\mathrm{E}[\sigma_k^2] \to 0$. ■

From [8, Theorem 3.9], it follows that the inequality (4.33) implies that (4.34) is stable. Hence, we conclude that Theorem 4.2 holds.

Chapter 5

Extensions for Markovian packet losses

So far, we have studied the limitations on the data rate and the packet loss probabilities under the assumption that packet loss process is independent and identically distributed (i.i.d.). This assumption is commonly employed to simplify the problem, but is restricting for modeling real communication channels.

In this chapter, we explore a more practical situation where the packet losses are governed by Markov chains, which are more general channel models. It has been known that Markov chains can express practical communication failures including bursty losses [16, 20]. The model has been employed in several researches in the field of networked control: In [53], the state estimation problem is studied, and the stabilization problem is tackled both in infinite [21] and finite [65] data rate cases.

Under the presence of the data rate constraint and the Markovian packet losses, we derive a necessary condition and a sufficient condition for the system to be mean square stable (MSS). The derived necessary limitations are expressed by the product of the eigenvalues and the uncertainty bounds of the plant. Moreover, for the case of scalar plants, the conditions are exact. These are considered as a generalization of the results in [65] to the uncertain plants case and also those in Chapter 2 to a more practical channel case.

This chapter is organized as follows. In Section 5.1, we state the problem setup

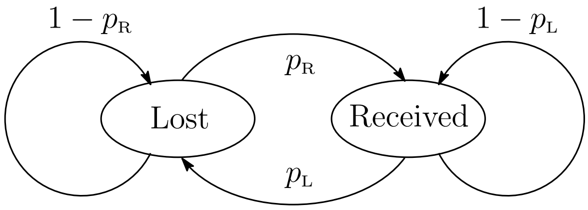


Figure 5.1: State transition probabilities of a Markov channel

especially regarding the channel model. In Section 5.2, the basic idea is proposed for the system with a scalar plant. In general, Markovian losses cause difficulties in the stability analysis since the channel states are no longer independent over time. To deal with this, we follow the approach of [65] and consider time intervals between successful transmissions, which become an i.i.d. process. The presented result will be extended to the general order plants case in Section 5.3.

The preliminary version of the results in this chapter can be found in [46].

5.1 Problem formulation

We consider the system in Fig. 2.1 described in Section 2.1; in this chapter, we consider the data rate R, or the number of the quantization cells N, is taken to be static.

Now, we introduce the Markovian packet loss model. It includes the i.i.d. losses given in Section 2.1 as a special case. We follow the representation used so far and denote the state of the packet reception/loss at time k by the random variable $\gamma_k \in \{0, 1\}$: If $\gamma_k = 0$ then the packet is lost; otherwise, it arrives successfully. The channel state process $\{\gamma_k\}_k$ is governed by a Markov chain which has two states: received and lost. Fig. 5.1 shows the states and the transition probabilities. In the figure, p_{L} is the loss probability when the previous packet has arrived, and p_{R}

represents the recovery probability from the loss state:

$$\begin{aligned}
&\mathrm{Prob}(\gamma_k = 0 \,|\, \gamma_{k-1} = 0) = 1 - p_{\mathrm{R}}, \\
&\mathrm{Prob}(\gamma_k = 1 \,|\, \gamma_{k-1} = 0) = p_{\mathrm{R}}, \\
&\mathrm{Prob}(\gamma_k = 0 \,|\, \gamma_{k-1} = 1) = p_{\mathrm{L}}, \\
&\mathrm{Prob}(\gamma_k = 1 \,|\, \gamma_{k-1} = 1) = 1 - p_{\mathrm{L}}.
\end{aligned}$$

To make the process $\{\gamma_k\}_k$ ergodic, we consider the case of $p_{\mathrm{L}}, p_{\mathrm{R}} \in (0, 1)$. Moreover, without loss of generality, we assume that the transmitted signal at the initial time is successfully received, i.e., $\gamma_0 = 1$. In the case of loss, redefine the initial time as the the first step of succeeded communication. There exists such a finite time with probability 1 since $\{\gamma_k\}_k$ is ergodic.

The problem of this chapter is to find limitations on the data rate R and the channel state transition probabilities $p_{\mathrm{L}}, p_{\mathrm{R}}$ for the overall system to be MSS.

5.2 Scalar plant case

In this section, we consider the simple case where the plant is a scalar system as follows:

$$y_{k+1} = a_k y_k, \quad a_k \in \mathcal{A} = [a^* - \epsilon, a^* + \epsilon], \quad \epsilon \geq 0. \tag{5.1}$$

We derive a necessary and sufficient condition for the system to be MSS. The condition is characterized by the data rate R, the transition probabilities $p_{\mathrm{L}}, p_{\mathrm{R}}$, the uncertain bound ϵ, and the plant instability $|a^*|$.

To describe the limitations, we introduce the following notation:

$$\widetilde{\nu} := \sqrt{\frac{1 - p_{\mathrm{L}} + (p_{\mathrm{R}} + p_{\mathrm{L}} - 1)(|a^*| + \epsilon)^2}{1 - (1 - p_{\mathrm{R}})(|a^*| + \epsilon)^2}}.$$

The parameter $\widetilde{\nu}$ corresponds to ν defined in (2.19) in Chapter 2. We also use r in (2.19) to represent the limitations; note that, in this section, $r = (|a^*| - \epsilon)/(|a^*| + \epsilon)$

since the plant is scalar system.

The following theorem holds for the scalar plants.

Theorem 5.1. Consider the system depicted in Fig. 2.1 where the plant is a scalar system (5.1) and satisfying Assumption 2.1. If the system is MSS, then the following inequalities hold:

$$R > \widetilde{R}_{\mathrm{nec}} := \begin{cases} \log \frac{\log(1-\epsilon\widetilde{\nu})^2}{\log r} & \text{if } \epsilon > 0, \\ \log |a^*| + \log \widetilde{\nu} & \text{if } \epsilon = 0, \end{cases} \tag{5.2}$$

$$p_{\mathrm{R}} > p_{\mathrm{R,nec}} := 1 - \frac{1}{(|a^*| + \epsilon)^2} \left(1 - \frac{\epsilon^2 p_{\mathrm{L}} \left((|a^*| + \epsilon)^2 - 1 \right)}{1 - \epsilon^2} \right), \tag{5.3}$$

$$0 \le \epsilon < 1. \tag{5.4}$$

Conversely, if these inequalities are satisfied for the data rate $R = \log N$ where N is an even number and the transition probabilities $p_{\mathrm{L}}, p_{\mathrm{R}} \in (0, 1)$, then there exists a control law such that the system is MSS.

For the case of $\epsilon = 0$, the above limitations $\widetilde{R}_{\mathrm{nec}}$ and $p_{\mathrm{R,nec}}$ are equal to those presented in [65], where the plant is assumed to be known. In addition, if $R \to \infty$ then the condition (5.3) on the recovery probability coincides with that in [24]. Thus, this theorem generalizes these existing results to the uncertain plants case. Since the bounds $\widetilde{R}_{\mathrm{nec}}$ and $p_{\mathrm{R,nec}}$ are increasing with respect to ϵ, as expected, plant uncertainty will result in higher requirements in communication with a large data rate and a high recovery probability. Furthermore, when the channel state process $\{\gamma_k\}_k$ is i.i.d., we have that $p_{\mathrm{R}} = 1 - p_{\mathrm{L}}$. In such a case, our problem can be reduced to that of Chapter 2, and the conditions (5.2)–(5.4) coincide with those given in Theorem 2.1.

To show this theorem, the approach developed in the analysis for the i.i.d. case is not enough to deal with the Markov channel. It is because in the present case, the channel states are not independent of those at the previous or next step, and consequently, it is difficult to evaluate the mean square of the state estimation error. To overcome this difficulty, we consider the packet receptions as random measurements. Then, it has been shown that the process of the sampling intervals becomes i.i.d. [63].

We formally state this fact as a lemma below. Let t_j, $j \in \mathbb{Z}_+$, be the sampling times, i.e., the times satisfying $\gamma_{t_j} = 1$. From the assumption in the previous section, we have that $\gamma_0 = 1$. Thus, it follows that $0 = t_0 < t_1 < \cdots < t_j < \cdots$. In addition, let τ_j denote the sampling interval defined as follows:

$$\tau_j := t_j - t_{j-1}, \ j \geq 1. \tag{5.5}$$

The following lemma holds [63].

Lemma 5.1. The process $\{\tau_j\}_j$ is i.i.d. and it holds that

$$\mathrm{Prob}(\tau_j = i) = \begin{cases} 1 - p_{\mathrm{L}} & \text{if } i = 1, \\ p_{\mathrm{L}} p_{\mathrm{R}} (1 - p_{\mathrm{R}})^{i-2} & \text{if } i > 1, \end{cases} \tag{5.6}$$

for all $j \geq 1$.

The proof of Theorem 5.1 consists of three steps. We first provide a condition for stability under a given quantizer whose boundaries are $\{h_l\}_l$. To derive the condition, we will analyze how the plant state estimation error grows by the instability of the plant and how precise the quantization should be to achieve stability. The rate of expansion of a quantization cell over time can be represented by w_l, $l = 0, 1, \ldots, \lceil N/2 \rceil - 1$, defined in (2.24) in Chapter 2. Then, the following lemma holds.

Lemma 5.2. The system depicted in Fig. 2.1 is MSS if and only if

$$\max_{0 \leq l \leq \lceil N/2 \rceil - 1} w_l \, \widetilde{\nu} < 1. \tag{5.7}$$

Proof. Necessity: First, we claim that the mean square stability of the plant output y_k implies that the scaling parameter σ_k is also MSS. For the estimation set $\mathcal{Y}_k \subset \mathbb{R}$ at time k, it is obvious that $\max_{y'_k \in \mathcal{Y}_k} |y'_k| \geq \mu(\mathcal{Y}_k)/2$. Letting δ be the smallest width of the quantization cells, we have that $\mu(\mathcal{Y}_k) \geq \delta \sigma_k$. Hence, if $\lim_{k\to\infty} \mathrm{E}[|y_k|^2] = 0$, then $\lim_{k\to\infty} \mathrm{E}[\sigma_k^2] = 0$ holds.

Next, we show that if σ_k is MSS then the inequality (5.7) holds. By (2.12) and (2.13), we have that

$$\sigma_{k+1} \geq \mu(\mathcal{A}\mathcal{Y}_k), \tag{5.8}$$

where $\mathcal{A}\mathcal{Y}_k := \{a'y' : a' \in \mathcal{A},\ y' \in \mathcal{Y}_k\}$. The interval $\mathcal{Y}_k$ depends on the channel state γ_k: If the packet is lost, i.e., $\gamma_k = 0$, then $\mathcal{Y}_k = [-\sigma_k/2, \sigma_k/2]$; otherwise, $\mathcal{Y}_k$ is the quantization cell which y_k fell into. Let $l_k \in \{0, 1, \ldots, \lceil N/2 \rceil - 1\}$ be the index of the cell corresponding to $\mathcal{Y}_k$ such that $\inf_{y' \in \mathcal{Y}_k} |y'/\sigma_k| = h_{l_k}$. Then, referring to basic results in interval arithmetic [38], for the case of $\gamma_k = 1$, we have that $\mu(\mathcal{A}\mathcal{Y}_k) = w_{l_k}\sigma_k$. Thus, the right-hand side of (5.8) is expressed as follows:

$$\mu(\mathcal{A}\mathcal{Y}_k) = \eta_k \sigma_k, \quad \text{where } \eta_k := \begin{cases} |a^*| + \epsilon & \text{if } \gamma_k = 0, \\ w_{l_k} & \text{if } \gamma_k = 1. \end{cases} \tag{5.9}$$

By (5.8) and (5.9), we have that $\sigma_{k+1} \geq \eta_k \sigma_k$. We will use this inequality and the stability of σ_k to show the condition (5.7). However, it is difficult to directly calculate the mean square of both sides of the inequality. Thus, we next derive the inequality described by $\sigma_{t_{j+1}}$ and σ_{t_j}; note that they are independent of each other by Lemma 5.1. Since $\gamma_{t_j} = 1$, referring to (5.9), we have

$$\mu(\mathcal{A}\mathcal{Y}_{t_j}) = w_{l_{t_j}} \sigma_{t_j}. \tag{5.10}$$

On the other hand, for all time $t' \in [t_j + 1, t_{j+1})$, it follows that $\gamma_{t'} = 0$, and hence

$$\mu(\mathcal{A}\mathcal{Y}_{t'}) = (|a^*| + \epsilon)\sigma_{t'}. \tag{5.11}$$

By (5.8), (5.10), and (5.11), we have

$$\sigma_{t_{j+1}} \geq (|a^*| + \epsilon)^{\tau_{j+1}-1} w_{l_{t_j}} \sigma_{t_j}, \tag{5.12}$$

where τ_{j+1} is the sampling interval defined in (5.5). Taking mean squares of both sides in (5.12), we obtain

$$\mathrm{E}[\sigma_{t_{j+1}}^2] \geq \mathrm{E}[(|a^*| + \epsilon)^{2(\tau_{j+1}-1)} w_{l_{t_j}}^2] \mathrm{E}[\sigma_{t_j}^2].$$

Note that σ_{t_j} depends only on $\tau_1, \dots, \tau_j$ and thus, is independent of τ_{j+1}. Since σ_k is MSS, the coefficient of the right-hand side should satisfy

$$\begin{aligned}\mathrm{E}[(|a^*| + \epsilon)^{2(\tau_{j+1}-1)} w_{l_{t_j}}^2] &= \mathrm{E}[(|a^*| + \epsilon)^{2(\tau_1-1)}] w_{l_{t_j}}^2 \\ &= \widetilde{\nu}^2 w_{l_{t_j}}^2 < 1,\end{aligned}$$

where the second equality holds by Lemma 5.1. In the above inequality, $w_{l_{t_j}}$ depends on the quantization cell where the output y_{t_j} fell into; however, the inequality must hold for all possible y_{t_j}. Thus, we arrive at the condition (5.7).

Sufficiency: By the necessity part, it is enough to prove that there exists a triple of an encoder, a decoder, and a controller such that the following holds: If σ_k is MSS then (i) y_k is also MSS, and (ii) the equality holds in (5.8).

The part (i) holds when we choose the control law as

$$u_k = -c(\mathcal{Y}_{k+1}^-), \tag{5.13}$$

where $c(\mathcal{Y}_{k+1}^-)$ represents the midpoint of $\mathcal{Y}_{k+1}^-$. This can be confirmed by substituting the above control law into (5.1), in which case we have

$$|y_{k+1}| = |a_k y_k + u_k| \leq \frac{\mu(\mathcal{Y}_{k+1}^-)}{2} \leq \frac{\sigma_{k+1}}{2},$$

where the last inequality holds from (2.12).

Furthermore, in the encoder and the decoder, we select the scaling parameter as $\sigma_{k+1} = \mu(\mathcal{Y}_{k+1}^-)$. Then, the (ii) holds. ■

Lemma 5.2 provides a condition for stability for a given quantizer.

The next step is to find the optimal quantizer, which minimizes the left-hand side of the inequality (5.7) under a fixed number N of quantization cells. Since $\widetilde{\nu}$ does not depend on the structure of the quantizer, the problem is to find the quantizer minimizing $\max_l w_l$. We introduce the quantizer q_N^* whose boundaries $\{h_l^*\}_l$ are given as follows:

(i) If $\epsilon > 0$, then

$$h_l^* = \begin{cases} \frac{1}{2}\frac{1-tr^l}{1-tr^{\lceil N/2\rceil}} & \text{if } N \text{ is odd,} \\ \frac{1}{2}\frac{1-r^l}{1-r^{\lceil N/2\rceil}} & \text{if } N \text{ is even,} \end{cases} \tag{5.14}$$

where $t := |a^*|/(|a^*| - \epsilon)$.

(ii) If $\epsilon = 0$, then

$$h_l^* = \begin{cases} \frac{1}{N}\left(l - \frac{1}{2}\right) & \text{if } N \text{ is odd,} \\ \frac{1}{N}l & \text{if } N \text{ is even.} \end{cases} \tag{5.15}$$

These are exactly the same as the optimal quantizer q_N^* given in (2.31) and (2.32).

The following lemma provides the optimal quantizer.

Lemma 5.3. The quantizer q_N^* minimizes $\max_l w_l$.

The optimal quantizer q_N^* has a nonuniform structure when the plant is uncertain, i.e., $\epsilon > 0$. In particular, it takes the width of the quantization cells smaller as its input becomes larger in magnitude. This structure helps to compensate the effect of the uncertainty in the expansion of estimation errors by plant instability; we note that in taking the product of intervals, the width of the resulting interval becomes large not only when the initial intervals are wide but also when they contain large values in magnitude. When there is no uncertainty in the plant, q_N^* is the uniform quantizer, which is commonly employed in the literature.

It is remarkable that the structure of q_N^* depends only on the level of instability $|a^*|$ and the uncertainty bound ϵ; the channel properties p_{L} and p_{R} do not affect it. Therefore, q_N^* is equal to the optimal quantizer for the case of i.i.d. channels, which has been studied in Chapter 2.

Proof of Lemma 5.3. See the proof of Lemma 2.3 in Chapter 2. ∎

The last step is deriving (5.2)–(5.4) from Lemmas 5.2 and 5.3.

Proof of Theorem 5.1. We consider the following two cases separately since the forms of q_N^* are different: $\epsilon > 0$ and $\epsilon = 0$.

First, assume that $\epsilon > 0$. Consider the two cases.

(i) N is even: Substituting (5.14) into the definition (2.24) of w_l, we have that

$$\max_{0 \le l \le \lceil N/2 \rceil - 1} w_l = \frac{\epsilon}{1 - r^{\lceil N/2 \rceil}}.$$

Thus, (5.7) is equivalent to

$$N > N_{\rm nec}^{\rm (e)} := \frac{\log(1 - \epsilon\widetilde{\nu})^2}{\log r} \text{ and } 1 - \epsilon\widetilde{\nu} > 0.$$

(ii) N is odd: By taking similar procedures, it follows that (5.7) is equivalent to

$$N > N_{\rm nec}^{\rm (o)} := \frac{\log\{(1 - \epsilon\widetilde{\nu})/t\}^2}{\log r} - 1 \text{ and } 1 - \epsilon\widetilde{\nu} > 0.$$

Noting that $N_{\rm nec}^{\rm (o)} > N_{\rm nec}^{\rm (e)}$ and $R = \log N$, we obtain the inequality (5.2). Furthermore, the inequality $1 - \epsilon\widetilde{\nu} > 0$ implies (5.3) and (5.4).

The above analysis can be directly applied to the case of $\epsilon = 0$, and consequently, we have (5.2)–(5.4) for this case also. ■

We now illustrate the limitations on the data rate $\widetilde{R}_{\rm nec}$ and the recovery probability $p_{\rm R,nec}$ by a numerical example.

Example 5.1. Consider a scalar plant of $a^* = 5$ and $\epsilon = 0.2$, and a channel where the loss probability is set as $p_{\rm L} = 0.1$. In Fig. 5.2, the solid line shows the limitation $\widetilde{R}_{\rm nec}$ versus the recovery probability $p_{\rm R}$, and the vertical dash-dot line represents $p_{\rm R,nec}$. The figure shows that larger data rate is required as the recovery probability becomes small toward $p_{\rm R,nec}$, and if $p_{\rm R} \le p_{\rm R,nec}$ then we can not stabilize the system for any data rate. The vertical dotted line corresponds to the probability when the channel state process is i.i.d., i.e., $p_{\rm R} = 1 - p_{\rm L} = 0.9$. From the figure, we observe that even if

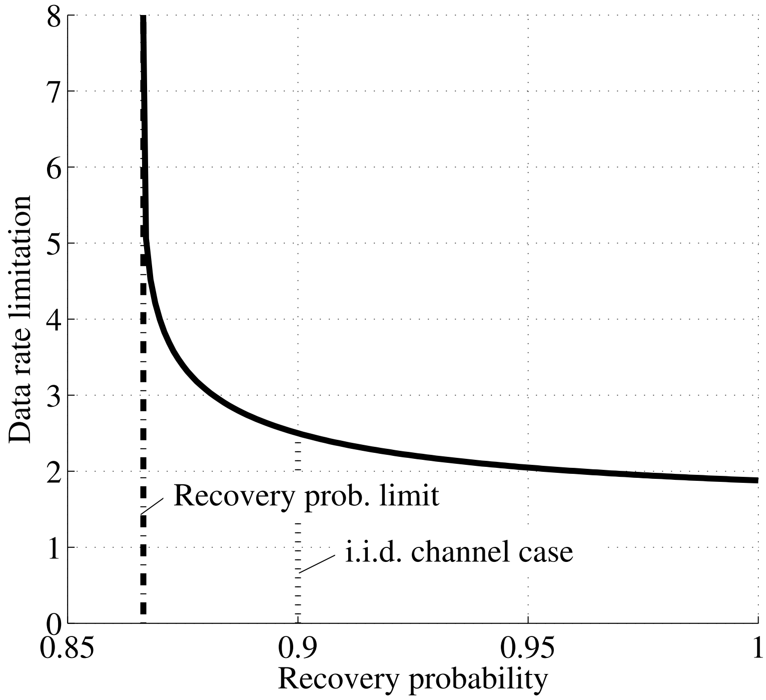


Figure 5.2: Data rate limitation versus the recovery probability p_{R}

the recovery probability is less than 0.9, in which case more bursty packet losses may occur, we can make the system stable by selecting the data rate large enough.

5.3 General order plant case

In this section, we develop a necessary condition and a sufficient condition for stabilization of multi-dimensional plants in (2.1). The necessary condition is derived by generalizing the result shown in the previous section. The sufficient condition is derived in the similar manner to that in Chapters 2 and 3, where the stability analysis of Markov jump linear systems is employed.

5.3.1 Necessary condition

To describe the necessary condition, we introduce the following notation corresponding to $\widetilde{\nu}$ in the previous section:

$$\check{\nu} := \sqrt{\frac{1 - \check{p}_{\mathrm{L}} + (\check{p}_{\mathrm{R}} + \check{p}_{\mathrm{L}} - 1)(|a^*| + \epsilon)^2}{1 - (1 - \check{p}_{\mathrm{R}})(|a^*| + \epsilon)^2}}, \tag{5.16}$$

where $\check{p}_{\mathrm{L}}$ and $\check{p}_{\mathrm{R}}$ are defined as

$$\check{p}_{\mathrm{L}} := \frac{1 - (1 - p_{\mathrm{L}} - p_{\mathrm{R}})^n}{p_{\mathrm{L}} + p_{\mathrm{R}}} p_{\mathrm{L}}, \quad \check{p}_{\mathrm{R}} := \frac{1 - (1 - p_{\mathrm{L}} - p_{\mathrm{R}})^n}{p_{\mathrm{L}} + p_{\mathrm{R}}} p_{\mathrm{R}}.$$

The following theorem provides the necessity result.

Theorem 5.2. For the system in Fig. 2.1 satisfying Assumption 2.1, if the system is MSS, then the following inequalities hold:

$$R > \check{R}_{\mathrm{nec}} := \begin{cases} \log \frac{\log(1 - \epsilon_n \check{\nu})^2}{\log r} & \text{if } \epsilon_n > 0, \\ \log |\lambda_{A^*}| + \log \check{\nu} & \text{if } \epsilon_n = 0, \end{cases} \tag{5.17}$$

$$\check{p}_{\mathrm{R}} > \check{p}_{\mathrm{R,nec}} := 1 - \frac{1}{(|\lambda_{A^*}| + \epsilon_n)^2} \left(1 - \frac{\epsilon_n^2 \check{p}_{\mathrm{L}} \left((|\lambda_{A^*}| + \epsilon_n)^2 - 1\right)}{1 - \epsilon_n^2}\right), \tag{5.18}$$

$$0 \leq \epsilon_n < 1. \tag{5.19}$$

This theorem provides the limitations characterized by the product λ_{A^*} of the eigenvalues of the nominal plant. For the class of systems considered, it can be viewed as an extension of the results in [65], where the known plants case has been studied. Let us compare the limitations $\check{R}_{\mathrm{nec}}$ and $\check{p}_{\mathrm{R,nec}}$ with those in [65] in the case of $\epsilon_n = 0$. For scalar plants case, i.e., $n = 1$, the inequalities (5.17)–(5.19) are equivalent to (5.2)–(5.4) in Theorem 5.1. Thus, $\check{R}_{\mathrm{nec}}$ and $\check{p}_{\mathrm{R,nec}}$ coincide with those in [65] as we mentioned in the previous section. However, when $n \geq 2$ and the channel state process is not i.i.d., i.e., $p_{\mathrm{L}} + p_{\mathrm{R}} \neq 1$, even if we assume that $\epsilon_n = 0$, the limitations $\check{R}_{\mathrm{nec}}, \check{p}_{\mathrm{R,nec}}$ may become larger than the bounds in [65]. In such a sense, Theorem 5.2 contains conservativeness. We would like to address this point in future research. On the other hand, when the channel state process is constrained to be i.i.d., the theorem is equivalent to Theorem 2.1.

Proof. In this proof, similarly to the scalar plants case, we analyze the expansion rate of the estimation set due to the plant instability. In particular, we focus on the effect of the nth plant parameter $a_{n,k}$ since it corresponds to the product of the eigenvalues of the plant, which is the key parameter to describe the limitations.

From the discussion in the beginning of the proof on Lemma 5.2, we have that if y_k is MSS then σ_k is also MSS. Hence, in the following, we show that if σ_k is MSS then the inequalities (5.17)–(5.19) hold.

We first establish that the scaling parameter σ_{k+1} is bounded from below by the width of the estimation set amplified by $a_{n,k}$. Applying the Brunn-Minkowski inequality (2.16) to the definition (2.13) of $\mathcal{Y}_{k+1}^{-}$, we have

$$\mu(\mathcal{Y}_{k+1}^{-}) \geq \sum_{i=1}^{n} \mu\left(\mathcal{A}_i \mathcal{Y}_{k-i+1}\right), \tag{5.20}$$

Substituting (5.20) into (2.12), we have

$$\sigma_{k+1} \geq \sum_{i=1}^{n} \mu\left(\mathcal{A}_i \mathcal{Y}_{k-i+1}\right) \geq \mu\left(\mathcal{A}_n \mathcal{Y}_{k-n+1}\right). \tag{5.21}$$

The far right-hand side corresponds to the effect of the expansion of the estimation set due to $a_{n,k}$. This term can be expressed in the form of the product of an random variable and σ_{k-n+1}. Let $w_{n,l}$, $l = 0, 1, \ldots, \lceil N/2 \rceil - 1$, be given as

$$w_{n,l} := \begin{cases} 2(|a_n^*| + \epsilon_n) h_{l+1} & \text{if } N \text{ is odd and } l = 0, \\ (|a_n^*| + \epsilon_n) h_{l+1} - (|a_n^*| - \epsilon_n) h_l & \text{else.} \end{cases}$$

This corresponds to w_l defined in (2.24) for the scalar plant case. Moreover, we introduce the random variable $\eta_{n,k}$ as

$$\eta_{n,k} := \begin{cases} |a_n^*| + \epsilon_n & \text{if } \gamma_k = 0, \\ w_{n,l_k} & \text{if } \gamma_k = 1, \end{cases} \tag{5.22}$$

where l_k is the index of the quantization cell which y_k fell into: The index is defined as the integer such that $\inf_{y' \in \mathcal{Y}_k} |y'/\sigma_k| = h_{l_k}$. Then, by using the result regarding products of intervals [38], we have that

$$\mu(\mathcal{A}_n \mathcal{Y}_{k-n+1}) = \eta_{n,k-n+1}\sigma_{k-n+1}. \tag{5.23}$$

Thus, by (5.21) and (5.23), it holds that

$$\sigma_{k+1} \geq \eta_{n,k-n+1}\sigma_{k-n+1}. \tag{5.24}$$

From the inequality (5.24) and the stability of σ_k, the n-down-sampled scalar system

$$\sigma'_{n(k+1)} = \eta_{n,nk}\sigma'_{nk}, \ \sigma'_0 = \sigma_0 \tag{5.25}$$

is MSS. Here, the process $\{\eta_{n,nk}\}_k$ is governed by the Markov chain $\{\gamma_{kn}\}_k$. Its transition probabilities are computed using the Chapman-Kolmogorov equation [47] as follows:

$$\begin{aligned} \text{Prob}(\gamma_{kn} = 0 \,|\, \gamma_{(k-1)n} = 0) &= 1 - \check{p}_{\text{R}}, \\ \text{Prob}(\gamma_{kn} = 1 \,|\, \gamma_{(k-1)n} = 0) &= \check{p}_{\text{R}}, \\ \text{Prob}(\gamma_{kn} = 0 \,|\, \gamma_{(k-1)n} = 1) &= \check{p}_{\text{L}}, \\ \text{Prob}(\gamma_{kn} = 1 \,|\, \gamma_{(k-1)n} = 1) &= 1 - \check{p}_{\text{L}}. \end{aligned}$$

Now, we can apply Theorem 5.1 in the previous section to the system (5.25) and consequently, we have the inequalities in the theorem. ■

5.3.2 Sufficient condition

We next present a sufficient condition for the existence of a stabilizing tuple of an encoder, a decoder, and a controller. First, the update law of the scaling parameter and the control law are proposed. Furthermore, the stability analysis is carried out based on the theory of Markov jump linear systems.

Given a certain data rate R, or N, and a quantizer expressed by the boundaries $\{h_l\}_l$, we determine the scaling parameter and the controller as those in (2.36) and (2.37), respectively. Moreover, we now introduce notations, which are partially extended from those in Section 2.3. Define the matrix G as

$$G := G_1 G_2, \tag{5.26}$$

where

$$\begin{aligned} G_1 &:= Q^T \otimes I_{n^2}, \\ G_2 &:= \operatorname{diag}\left(H_{\Gamma(1)} \otimes H_{\Gamma(1)}, \ldots, H_{\Gamma(2^n)} \otimes H_{\Gamma(2^n)}\right). \end{aligned}$$

The matrices $H_{\Gamma(1)}, \ldots, H_{\Gamma(2^n)}$ are realizations of the random matrix H_{Γ_k} defined in (2.41). Here, the transition matrix Q of the Markov chain Γ_k is given by

$$Q := \begin{bmatrix} Q' & & 0 \\ & \ddots & \\ 0 & & Q' \\ Q' & & 0 \\ & \ddots & \\ 0 & & Q' \end{bmatrix} \in \mathbb{R}^{2^n \times 2^n}, \quad Q' := \begin{bmatrix} 1-p_{\mathrm{R}} & p_{\mathrm{R}} & 0 & 0 \\ 0 & 0 & p_{\mathrm{L}} & 1-p_{\mathrm{L}} \end{bmatrix} \in \mathbb{R}^{2 \times 4}.$$

The following theorem provides the sufficient condition.

Theorem 5.3. Given the data rate $R = \log N$ and the quantizer whose boundaries

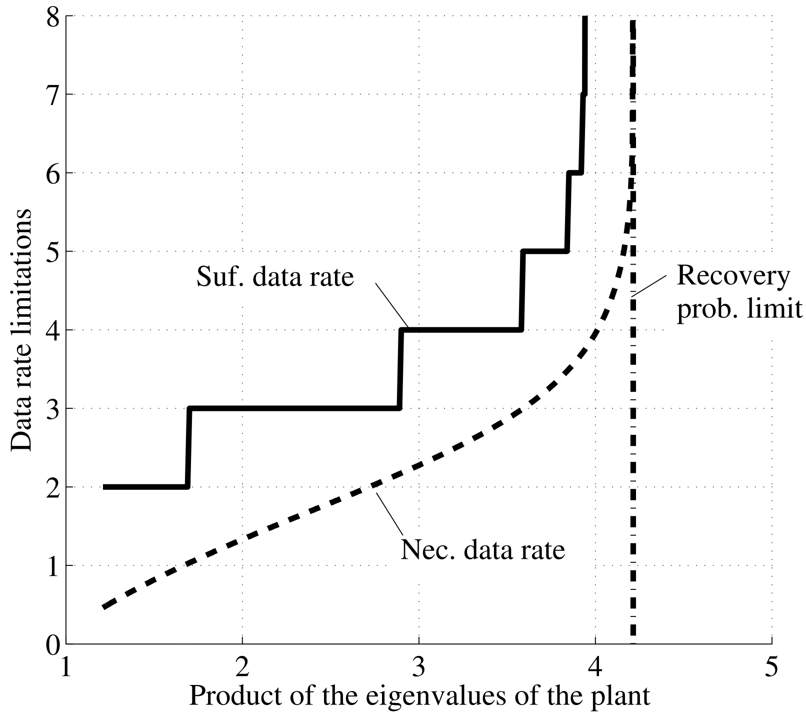


Figure 5.3: Data rate limitations versus the product of the eigenvalues of the nominal plant $|\lambda_{A^*}|$: $n = 2$, $a_1^* = 1.0$, $\epsilon_1 = 0.10$, $\epsilon_2 = 0.20$, $p_{\mathrm{L}} = 0.050$, $p_{\mathrm{R}} = 0.98$

are $\{h_l\}_l$, if the matrix G in (5.26) satisfies

$$\rho(G) < 1, \tag{5.27}$$

then under the control law using (2.36) and (2.37), the system depicted in Fig. 2.1 is MSS.

The proof of this theorem follows similarly to the sufficient result for the i.i.d. case in Section 2.3. In both cases, the Markov chain Γ_k represents packet reception/loss states for past n steps. Here, since the loss process is Markovian, we have taken different transition matrix Q from that in Section 2.3.

Now, we illustrate the bounds on the data rate shown in Theorems 5.2 and 5.3 by a numerical example.

Example 5.2. Consider a second-order plant with the parameters $a_1^* = 1.0$, $\epsilon_1 = 0.10$, and $\epsilon_2 = 0.20$. The transition probabilities are fixed as $p_{\mathrm{L}} = 0.050$, $p_{\mathrm{R}} = 0.98$. In

Fig. 5.3, we plot the bounds on the data rate versus the product of the eigenvalues $|\lambda_{A^*}|$ of the nominal plant, which is equal to $|a_2^*|$. The solid line is the sufficient bound where the quantizer is the optimal one q_N^* defined in (5.14). Furthermore, the dashed line represents the necessary bound. Note that by assumption, $|a_2^*| - \epsilon_2 > 1$. The figure shows that for sufficiently small $|\lambda_{A^*}|$, the gap between the sufficient bound and the necessary bound is one or two bits and is hence small. The vertical dash-dot line is the $|\lambda_{A^*}|$ such that the recovery probability p_{R} reaches $\check{p}_{\mathrm{R,nec}}$. Thus, for plants whose level of instability $|\lambda_{A^*}|$ are over the dash-dot line, we can not stabilize the system even if we use infinitely large data rate.

Chapter 6

Conclusion

6.1 Summary

This thesis explores the stabilization problem of uncertain systems over channels under communication constraints; in particular, finite data rate and lossy channels are considered. Throughout the thesis, our main interest lies in clarifying limitations on communication constraints for stability and how the plant uncertainty affects them.

The problem has been formulated in four different setups. First, we have started our analysis in the basic setup in Chapter 2, where the data rate is fixed as a constant value for all time steps, and the packet losses are drawn according to an i.i.d. process. Moreover, the plant uncertainty lies only on the state parameters and the input parameters are known. Under the setup, we have proposed a necessary condition (Theorem 2.1), which provides limitations on the data rate, the loss probability, and the magnitude of the plant uncertainty. These limitations are characterized by plant instability, in particular, the product of the eigenvalues of the plant. From the condition, we have found an interesting property of uncertain networked control systems: The limitations induced by the results for known plants cases are no longer valid and strictly larger amount of information should be conveyed through the channel. It is also interesting that for the uncertain plants case, we can reduce the required data rate by employing a nonuniform quantizer, which has been proposed in Lemma 2.3.

In addition to the necessary condition, we have given a sufficient condition in Theorem 2.2. In general, the sufficient data rate and the loss probability suggested by the condition are not equal to the necessary ones. However, for the special case of one-dimensional plants, they coincide with each other. The derivation of the sufficiency result is based on Markov jump linear systems theory and the convergence speed analysis of the system has been also shown by employing the results from theory.

In Chapter 3, we have addressed the problem induced by the constraint that the data rate should be taken as an integer. To achieve further data rate suppression, the control scheme has been generalized to employ time-varying data rate. We have presented a necessary condition (Theorem 3.1) and a sufficient condition (Theorem 3.2) concerning the average data rate. The conditions show that we can not overcome the limitations given in the static case even if we use the time-varying data rate though the sufficient data rate can be reduced. It is worth noting that there are two types of communication schemes and achievable data rates are different according to the schemes though the same limitation can be achieved when the plant is known.

Chapter 4 has been devoted to deal with plants with uncertain input parameters. In this setup, in contrast to Chapter 2, the exact control inputs are no longer available at the controller side. We have shown a necessary condition for stability in Theorem 4.1, which generalizes that for the case of known inputs. In the derivation of the condition, we have analyzed the effect of the input uncertainty on expansion of state estimation sets and have derived the optimal quantizer to minimize the expansion. Also, a sufficient condition (Theorem 4.2) has been shown for this case by following the approach based on Markov jump linear systems theory in Chapter 2.

In Chapter 5, we have relaxed the assumption on the packet loss process from i.i.d. to Markovian so that we can deal with practical situations like bursty losses. In the chapter, two Markov states determined by the previous channel state are considered, and a necessary condition and a sufficient condition for stability have been shown in Theorems 5.2 and 5.3, respectively. We have seen that if the transition probability from the lost state to the received state, or so-called the recovery probability, is sufficiently large, we can stabilize the system even if the data rate does not satisfy the limitation given in Chapter 2.

6.2 Future research

We list a few topics for future research inspired by this thesis.

Performance analysis: In the thesis, our main control objective has been set to stability of the closed-loop system. It would be interesting to study relationships between performance of control and communication constraints. We have discussed the effect of the data rate and the loss probability on the convergence speed in Chapter 2. Further analysis may be devoted to obtain the relation in a more explicit form.

Generalization of the plant class: We have considered single-input single-output autoregressive systems as the class of the plants. One way to extend the results could be to generalize the class to linear systems with multi inputs and multi output. On the other hand, the class of uncertainties may also be extended from the current parametric ones to, e.g., norm bounded or stochastic ones.

Generalization of the communication constraints types: As we have seen in Chapter 1, a wide variety of the types of communication constraints have been proposed. Though we have focused on the data rate and the packet losses, dealing with other constraints such as delays, perturbation of sampling intervals, and bit flipping might be a future research topic.

Control robust to cyber-attacks: There is an increasing demand for developing solutions to cyber-attacks against control systems employing networks. The attacks can be treated as communication constraints. Hence, constructing controllers robust to these attacks, and characterizing tolerance of the systems against attacks would be a possible future research.

Extension to multi-agent systems: In the thesis, the target system consists of a pair of the plant and the controller. Recently, multi-agent systems, where multiple autonomous agents are interconnected through channels, have been actively studied. In cooperative control of such systems, there exists a research interest common with that studied in the thesis: How does the communication constraints affect behavior of these networked systems? Applying the approach and the results presented in the thesis would be an interesting problem.

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