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Convergence of loop erased random walks on a planar graph to a chordal SLE(2) curve

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Contents

Introduction

In this thesis, we show that the scaling limit of the loop-erased random walk on a planar graph in simply connected domain D connecting two distinct boundary points of D is a chordal SLE_2 curve. Our result is an extension of one due to Dapeng Zhan [17] where the problem is considered on the square lattice. A convergence to the radial SLE_2 has been obtained by Lawler, Schramm and Werner [4] for the square and triangular lattices and by Yadin and Yehudayoff [15] for a wide class of planar graphs.

The Scramm-Loewner evolutions (abbreviated as SLE), introduced by Oded Schramm, is a one parameter family of conformal maps and describes a random curve in a simply connected domain. The connection between SLE and discrete models in two dimensions is well known. There are a number of discrete models, e.g.random walks, loop-erased random walks, self avoiding walk, percolation, Ising models, that are expected to have conformally invariance. The SLE gives the scaling limit of a random simple curve obtained by such discrete models.

We will organize my thesis as follow: In Chapter 1, we present SLE theory. We first explain how Loewner's method can be used to describe a curve by a family of conformal maps. Then we consider a curve in a simply connected domain started from a boundary point as a change of a simply connected domains cutting by a curve. We define the SLE_κ curves driven by Brownian motion $\sqrt{\kappa}B_t$ of variance κ, and mention some of its properties. The behavior of the SLE_κ curves depends naturally on the value of the parameter κ. In particular, at $\kappa = 6$ and $\kappa = 8/3$, the SLE_κ curves have very specific properties. In Chapter 2, we consider the natural random walk on a planar graph and scale it by a small positive number δ. Given a simply connected domain D and its two boundary points a and b, we start the scaled walk at a vertex of the graph nearby a and condition it on its exiting D through a vertex nearby b, and prove that the loop erasure of the conditioned walk converges, as $\delta \to 0$, to the chordal SLE_2 that connects a and b in D, provided that an invariance principle is valid for both the random walk and the dual walk of it.

In Chapter 1, we refer to Lawler's book [2] mainly. Chapter 2 correspond to my paper [13].

Chapter 1

SLE theory

1.1 Loewner chain

1.1.1 Chordal Loewner equation

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane. A bounded subset $A \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $A = \overline{A} \cap \mathbb{H}$ (i.e.A is relatively closed in $\mathbb{H}$) and $\mathbb{H} \setminus A$ is a simply connected domain. Let $\mathcal{Q}$ denote the set of compact $\mathbb{H}$-hulls. For any $A \in \mathcal{Q}$, there exists a unique conformal map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ satisfying $|g_A(z) - z| \to 0$ as $z \to \infty$. The half-plane capacity $\mathrm{hcap}(A)$ is defined by

$$\mathrm{hcap}(A) := \lim_{z\to\infty} z(g_A(z) - z).$$

Then, g_A has the expansion

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

For example, if $A = \overline{\mathbb{D}} \cap \mathbb{H}$ or $A = (0, i]$, then

$$g_{\overline{\mathbb{D}}\cap\mathbb{H}}(z) = z + \frac{1}{z},\ g_{(0,i]} = \sqrt{z^2+1} = z + \frac{1}{2z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

Hence,

$$\mathrm{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = 1,\ \mathrm{hcap}((0, i]) = \frac{1}{2}.$$

The half-plane capacity has some properties and can be expressed by means of a Brownian motion.

Proposition 1.1.1. *If $r > 0, x \in \mathbb{R}, A \in \mathcal{Q}$, then*

$$\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A), \quad \mathrm{hcap}(A + x) = \mathrm{hcap}(A).$$

If $A, B \in \mathcal{Q}, A \subset B$, then

$$\mathrm{hcap}(B) = \mathrm{hcap}(A) + \mathrm{hcap}(g_A(B \setminus A)).$$

Let B_t be a 2-dimensional Brownian motion and τ be the smallest t with $B_t \in A\cup\mathbb{R}$. Then,

$$\mathrm{hcap}(A) = \lim_{y\to\infty} y\mathbf{E}^{iy}[\mathrm{Im}(B_\tau)].$$

For any $p \in \mathbb{R}, A \in \mathcal{Q}$, let $\mathrm{rad}_p(A) := \sup\{|z-p| : z \in A\}$. The next proposition is a uniform estimate for g_A and $\mathrm{hcap}(A)$.

Proposition 1.1.2. *For any $p \in \mathbb{R}, A \in \mathcal{Q}$,*

$$0 \leq \mathrm{hcap}(A) \leq \mathrm{rad}_p(A)^2.$$

There exists a constant $C > 0$ such that for any $p \in \mathbb{R}, A \in \mathcal{Q}$ and $|z-p| \geq 2\mathrm{rad}_p(A)$,

$$\left| g_A(z) - z - \frac{\mathrm{hcap}(A)}{z-p} \right| \leq C \frac{\mathrm{rad}_p(A)\mathrm{hcap}(A)}{|z-p|^2}.$$

Let γ be a simple curve with $\gamma(0) \in \mathbb{R}, \gamma(0,\infty) \subset \overline{\mathbb{H}}$. For any $t \geq 0$, let us denote by H_t a unbounded component of $\mathbb{H} \setminus \gamma(0,t]$. Let $K_t := \mathbb{H} \setminus H_t$ and $g_t := g_{K_t}$. We can define $U_t := g_t(\gamma(t))$ and U_t is a $\mathbb{R}$-valued continuous function. We call U_t the driving function of γ. Let $a(t) := \mathrm{hcap}(K_t)$.Then, $a(t)$ is a strictly increasing continuous function. We can reparametrize γ by the half-plane capacity (i.e. $a(t) = 2t$).

Proposition 1.1.3. *If γ is parametrized by the half-plane capacity, then for any $z \in \mathbb{H}$, $g_t(z)$ satisfies the following differential equation.*

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{1.1}$$

The equation (1.1) is called the chordal Loewner equation with the driving function U_t. A simple curve γ determines the driving function U_t, which actually drives the family $g_t(z), z \in \mathbb{H}$. Conversely, we can start from a driving function U_t as the following.

Proposition 1.1.4. *Suppose U_t is a $\mathbb{R}$-valued continuous function. For any $z \in \mathbb{H}$, let $g_t(z)$ denote the solution of the chordal Loewner equation with the driving function U_t. We set*

$$T_z := \sup\{t \geq 0 : |g_t(z) - U_t| > 0\},\ H_t := \{z \in \mathbb{H} : T_z > t\},\ K_t := \mathbb{H} \setminus H_t.$$

Then, g_t is the unique conformal map of H_t onto $\mathbb{H}$ which satisfies

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

A continuous function U_t generates the increasing hulls K_t through the chordal Loewner equation. We call these increasing hulls K_t a Loewner chain driven by a continuous function U_t. If there is a curve γ such that H_t is a unbounded component of $\mathbb{H} \setminus \gamma(0,t]$ for any $t \geq 0$, then we say that K_t is generated by a curve . Unfortunately, it is not true that every Loewner chain is generated by a curve. For example, logarithmic spiral is not generated by a curve (see Example 4.28. in [2]) . If a driving function U_t is sufficiently smooth, then a Loewner chain K_t is generated by a curve. Therefore, we may think that the driving function U_t identify a curve.

Proposition 1.1.5. *Let K_t be a Loewner chain driven by a continuous function U_t. Suppose for some $r < \sqrt{2}$ and all $s < t$,*

$$|U_t - U_s| \leq r\sqrt{t-s}.$$

Then K_t is generated by a simple curve.

We can characterize a Loewner chain K_t.

Proposition 1.1.6. *Let K_t be increasing hulls. Then the following are equivalent.*

1. *K_t is a Loewner chain driven by a continuous function.*
2. *For any $t \geq 0$, $\mathrm{hcap}(K_t) = 2t$, and for any $T > 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that for each $0 \leq t \leq T$ there is a bounded connected set $S \subset \mathbb{H} \setminus K_t$ such that $\mathrm{diam}(S) < \epsilon$ and S disconnects $K_{t+\delta} \setminus K_t$ from ∞ in $\mathbb{H} \setminus K_t$.*

Lemma 1.1.7. *There exists a constant $C > 0$ such that the following holds. Let K_t be a Loewner chain driven by a continuous function U_t. Set*

$$k(t) := \max\{\sqrt{t},\ \sup\{|U_s - U_0| : 0 \leq s \leq t\}\}.$$

Then, for any $t > 0$,

$$C^{-1}k(t) \leq \mathrm{diam}(K_t) \leq Ck(t).$$

1.1.2 Radial Loewner equation

There is a similar Loewner equation that describes the evolution of hulls growing from the boundary of the unit disk $\mathbb{D}$ towards the origin.

Let γ be a simple curve from $\partial\mathbb{D}$ to 0. Parametrize γ so that $g_t'(0) = e^t$, where g_t is the unique conformal map mapping $\mathbb{D} \setminus \gamma(0, t]$ onto $\mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) > 0$. We can define $\xi_t := g_t(\gamma(t))$ and $U_t := -i \log \xi_t$ is a $\mathbb{R}$-valued continuous function. We call ξ_t the driving function of γ.

Proposition 1.1.8. *For any $z \in \mathbb{D}$, $g_t(z)$ satisfies the following differential equation.*

$$\frac{\partial}{\partial t} g_t(z) = -g_t(z) \frac{g_t(z) + \xi_t}{g_t(z) - \xi_t}, \quad g_0(z) = z. \tag{1.2}$$

The equation (1.2) is called the radial Loewner equation with the driving function ξ_t. Conversely, we can start from a driving function ξ_t as the following.

Proposition 1.1.9. *Suppose U_t is a $\mathbb{R}$-valued continuous function. Let $\xi_t := e^{iU_t}$. For any $z \in \mathbb{D}$, let $g_t(z)$ denote the solution of the radial Loewner equation with the driving function ξ_t. We set*

$$T_z := \sup\{t \geq 0 : |g_t(z) - \xi_t| > 0\},\ D_t := \{z \in \mathbb{D} : T_z > t\},\ K_t := \mathbb{D} \setminus D_t.$$

Then, g_t is the unique conformal map of D_t onto $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) = e^t$.

Suppose $g_t(z)$ is the solution of the radial Loewner equation with the driving function $\xi_t := e^{iU_t}$. Then $h_t(z) := -i \log g_t(e^{iz})$ satisfies the equation

$$\frac{\partial}{\partial t} h_t(z) = \cot\left(\frac{h_t(z) - U_t}{2}\right),\ h_0(z) = z.$$

Note that when $h_t(z) - U_t$ is small, $\cot[(h_t(z) - U_t)/2]$ is approximately $2/(h_t(z) - U_t)$.

1.2 Scramm-Loewner evolutions

1.2.1 Chordal SLE in the upper half plane

The chordal Schramm-Loewner evolutions with parameter $\kappa > 0$ (abbreviated as chordal SLE$_\kappa$) is the random family of conformal maps g_t obtained from the chordal Loewner equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z \quad (z \in \mathbb{H})$$

where B_t is a one-dimensional standard Brownian motion with $B_0 = 0$. Let K_t be the Loewner chain corresponding to a chordal SLE$_\kappa$. It is not easy to see whether K_t is generated by a curve. However, the following results is well known.

Proposition 1.2.1. *With probability 1, the limit $\gamma(t) := \lim_{z \to 0} g_t^{-1}(z + \sqrt{\kappa} B_t)$ exits for any $t \geq 0$ and K_t is generated by the curve γ.*

This random curve γ is called a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞. We mention several properties of SLE$_\kappa$ curves.

Proposition 1.2.2. *Suppose that γ is a chordal SLE$_\kappa$ curve in $\mathbb{H}$ and $r > 0$. Let $\widehat{\gamma}(t) := r^{-1}\gamma(r^2 t)$. Then, $\widehat{\gamma}$ has the same distribution as γ.*

Proposition 1.2.3. *Suppose that γ is a chordal SLE$_\kappa$ curve in $\mathbb{H}$. Let τ be a stopping time. Let $\widehat{\gamma}(t) := g_\tau(\gamma(t+\tau)) - \sqrt{\kappa} B_\tau$. Then, $\widehat{\gamma}$ has the same distribution as γ.*

Suppose that g_t is a chordal SLE$_\kappa$. Let

$$h_t(z) := \frac{g_t(z) - \sqrt{\kappa} B_t}{\sqrt{\kappa}}.$$

By Ito' s formula,

$$\begin{aligned} dh_t(z) &= \frac{1}{\sqrt{\kappa}} \frac{2}{g_t(z) - \sqrt{\kappa} B_t} dt - dB_t \\ &= \frac{2}{\kappa} \frac{1}{h_t(z)} dt + dW_t, \end{aligned}$$

where $W_t := -B_t$ is a one-dimensional standard Brownian motion. Using Schwartz reflection principle, $h_t(z)$ can be extended to a real line and satisfies the foregoing SDE on it. For any $x \in \mathbb{R}$, $h_t(x)$ is a $(4/\kappa + 1)$-dim Bessel process. The hitting time of a d-dim Bessel process at 0 has three phases. The two phase transitions take place at the values $d = 2$ and $d = 3/2$. This fact leads to the following property of a chordal SLE$_\kappa$ curve.

Proposition 1.2.4. *Let γ be a chordal SLE$_\kappa$ curve in $\mathbb{H}$.*

- *If $0 < \kappa \leq 4$, then w.p.1, γ is a simple curve with $\gamma(0, \infty) \subset \mathbb{H}$.*
- *If $4 < \kappa < 8$, then w.p.1, $\gamma(0, \infty) \cap \mathbb{H} \neq \mathbb{H}$ and $\cup_{t>0} \overline{K_t} = \overline{\mathbb{H}}$.*
- *If $\kappa \geq 8$, then w.p.1, γ is a space-filling curve, i.e., $\gamma[0, \infty) = \overline{\mathbb{H}}$.*

If $\kappa \geq 8$, a chordal SLE_κ curve γ is space-filling, and therefore the Hausdorff dimension of the set $\gamma[0,\infty)$ is 2. But for $\kappa < 8$ the Hausdorff dimension of $\gamma[0,\infty)$ is not trivial. The following result has been proved by Beffara in [1].

Proposition 1.2.5. *Let γ be a chordal SLE_κ curve in $\mathbb{H}$. W.p.1,*

$$\dim_h(\gamma[0,\infty)) = (1+\frac{\kappa}{8}) \wedge 2,$$

where $\dim_h$ *denotes Hausdorff dimension.*

We can compute some crossing probabilities for a chordal SLE_κ curve for $\kappa > 4$. The following formula corresponds to Cardy's formula for percolation.

Proposition 1.2.6. *Let $\kappa > 4$ and $y > 0$. Suppose that γ is a chordal SLE_κ curve in $\mathbb{H}$. Put*

$$T_{-y} := \inf\{t \geq 0 : \gamma(t) \in (-\infty, -y)\},\ T_1 := \inf\{t \geq 0 : \gamma(t) \in (1,\infty)\}.$$

Then,

$$\mathbf{P}(T_{-y} > T_1) = \frac{\Gamma(2-\frac{8}{\kappa})}{\Gamma(1-\frac{4}{\kappa})^2}\int_0^{\frac{y}{y+1}} \frac{du}{u^{\frac{4}{\kappa}}(1-u)^{\frac{4}{\kappa}}}.$$

Let

$$t_* = \inf\{t \geq 0 : \gamma(t) \in [1,\infty)\}.$$

If $\kappa \geq 8$, then $\gamma(t_*) = 1$ w.p.1. If $4 < \kappa < 8$, then $\gamma(t_*)$ has a nontrivial distribution.

Proposition 1.2.7. *Let $4 < \kappa < 8$. Suppose that γ is a chordal SLE_κ curve in $\mathbb{H}$.*

$$\mathbf{P}(\gamma(t_*) < 1 + x) = \frac{\Gamma(\frac{4}{\kappa})}{\Gamma(\frac{8}{\kappa}-1)\Gamma(1-\frac{4}{\kappa})}\int_0^{\frac{x}{x+1}} \frac{du}{u^{2-\frac{8}{\kappa}}(1-u)^{\frac{4}{\kappa}}}.$$

The following martingale observable for SLE_κ is useful to consider the relationship between SLE_κ and discrete models.

Proposition 1.2.8. *Let g_t be a chordal SLE_κ and B_t be a one-dimensional standard Brownian motion. Let*

$$f(z,w) := \begin{cases} (z-w)^{1-\frac{4}{\kappa}} & (\kappa \neq 4) \\ \log|z-w| & (\kappa = 4) \end{cases}.$$

Then, for any $z \in \mathbb{H}$, $f(g_t(z), \sqrt{\kappa}B_t)$ is a local martingale.

1.2.2 Chordal SLE in simply connected domains

Let γ be a chordal SLE_κ curve in $\mathbb{H}$ from 0 to ∞. Let D be a simply connected domain and a, b be distinct points on ∂D. Let $\phi : D \to \mathbb{H}$ be a conformal map with $\phi(a) = 0, \phi(b) = \infty$. Although ϕ is not unique, any other such map ϕ' can be written as $r\phi$ for some $r > 0$. Therefore, $\phi^{-1}(\gamma)$ is independent of the choice of map up to a time change. We consider SLE_κ curve in D as unparametrized curves. A chordal SLE_κ curve in D from a to b is defined by $\phi^{-1}(\gamma)$.

The two properties stated in the next proposition, called the domain Markov property and conformal invariance, respectively, characterize the distribution of a chordal SLE_κ curve.

Proposition 1.2.9. *Let γ be a chordal SLE$_\kappa$ curve in D from a to b and $\mu_{a,b;D}$ be a law of γ. Let $f : D \to D'$ be a conformal map. Then,*

$$\mu_{a,b;D}(\cdot|\gamma(0,t]) = \mu_{\gamma(t),b;D\setminus\gamma(0,t]}(\cdot),$$

and

$$f \circ \mu_{a,b;D}(\cdot) = \mu_{f(a),f(b);D'}(\cdot).$$

The following reversibility of a chordal SLE$_\kappa$ curve is proved by Zhan in [16]

Proposition 1.2.10. *Let $\kappa \leq 4$. The time-reversal of a chordal SLE$_\kappa$ curve in D from a to b has the same distribution as a chordal SLE$_\kappa$ curve in D from b to a.*

If $\kappa > 8$, then a chordal SLE$_\kappa$ curve is not reversible. It is believed that a chordal SLE$_\kappa$ curve is reversible for $4 < \kappa < 8$. But this conjecture is open problem.

1.2.3 Locality and restriction

Two special value of κ are $\kappa = 6$ and $\kappa = 8/3$. At these values, a chordal SLE$_\kappa$ curve has some specific properties.

We first mention the locality property for $\kappa = 6$. Suppose γ is a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞ and g_t is the crresponding map. Let $\mathcal{N}$ be a neighborhood of 0 and Φ be a conformal map of $\mathcal{N}$ into $\mathbb{C}$ with $\Phi(\mathbb{R} \cap \mathcal{N}) \subset \mathbb{R}$ and $\Phi(\mathbb{H} \cap \mathcal{N}) \subset \mathbb{H}$. Let $t_0 := \inf\{t > 0 : \gamma(t) \notin \mathcal{N}\}$. For $t < t_0$, let $\gamma^*(t) := \Phi \circ \gamma(t)$. Let H_t^* be the unbounded component of $\mathbb{H} \setminus \gamma^*[0,t]$ and $g_t^* : H_t^* \to \mathbb{H}$ be the unique conformal map satisfying $|g_t^*(z) - z| \to 0$ as $z \to \infty$. Let $\Phi_t := g_t^* \circ \Phi \circ g_t^{-1}$. g_t^* satisfies the differential equation

$$\frac{\partial}{\partial t} g_t^*(z) = \frac{2\Phi_t'(U_t)^2}{g_t^*(z) - U_t^*}, \quad g_0(z) = z,$$

where $U_t = \sqrt{\kappa} B_t$ and $U_t^* = g_t^*(\gamma^*(t)) = \Phi_t(U_t)$. U_t^* satisfies the stochastic differential equation

$$dU_t^* = \left(\frac{\kappa}{2} - 3\right) \Phi_t''(U_t)dt + \sqrt{\kappa}\Phi_t'(U_t)dB_t.$$

We reparametrize the curve γ^* so that $\mathrm{hcap}(\gamma^*[0,t]) = 2t$. Define the change of time $r(t)$ by

$$t = \int_0^{r(t)} \Phi_s'(U_s)^2 ds.$$

Then $\tilde{\gamma} := \gamma^*(r(t))$ is parametrized by the half plane capacity and $\tilde{U}_t := U_{r(t)}^*$ satisfies the stochastic differential equation

$$d\tilde{U}_t = \left(\frac{\kappa}{2} - 3\right) \frac{\Phi_{r(t)}''(U_{r(t)})}{\Phi_{r(t)}'(U_{r(t)})^2} dt + \sqrt{\kappa} d\tilde{B}_t,$$

where $\tilde{B}_t := \int_0^{r(t)} \Phi_s'(U_s) dB_s$ is a standard Brownian motion. The map $\tilde{g}_t(z) := g_{r(t)}^*(z)$ corresponding to $\tilde{\gamma}$ satisfies the chordal Loewner equation

$$\frac{\partial}{\partial t} \tilde{g}_t(z) = \frac{2}{\tilde{g}_t(z) - \tilde{U}_t}, \; \tilde{g}_0(z) = z.$$

In particular, if $\kappa = 6$, then $\tilde{U}_t = \sqrt{\kappa} \tilde{B}_t$. Therefore, we can obtain the following property called the locality property for $\kappa = 6$.

Proposition 1.2.11. *If $\kappa = 6$, then γ^* has the same distribution as the time change of a chordal SLE_6 curve stopped at the first time it leave $\Phi(\mathcal{N})$.*

Let D be a simply connected domain and a, b, c be distinct boundary points of D. Suppose that γ is a chordal SLE_6 curve in D from a to b and γ^* is a chordal SLE_6 curve in D from a to c. Applying the locality property for suitable map, we find that two curves γ and γ^* have the same distribution up to the first time they reach the boundary between b and c not containg a.

Let $A \in \mathcal{Q}$ is bounded away from 0. Suppose that γ is a chordal SLE_6 curve in $\mathbb{H}$ from 0 to ∞ and γ^* is a chordal SLE_6 curve in $\mathbb{H} \setminus A$ from 0 to ∞. By the locality property, we find that γ has the same distribution as γ^* up to the first time the increasing hulls K_t generated by γ intersect A.

As we see above, the distribution of a chordal SLE_6 curve on the neighborhood of starting point is independent from conditions outside the neighborhood of starting point.

The crossing exponents exponents for chordal SLE_6 is defined by the following way. Let $L > 0$ and $\mathcal{R}_L := (0, L) \times (0, \pi)$ be the rectangle with vertical boundaries $\partial_1 = [0, i\pi], \partial_2 = [L, L + i\pi]$. Let γ be a chordal SLE_6 curve in $\mathcal{R}_L$ from $i\pi$ to $L + i\pi$. Let t_* be the first time that γ hits ∂_2. Let D be the connected component of $\mathcal{R}_L \setminus \gamma[0, t_*]$ whose boundary includes 0. We can conformally map D onto rectangle $(0, \mathcal{L}) \times (0, \pi)$ in such a way that $\partial_1 \cap \partial D$ maps onto $[0, i\pi]$ and $\partial_2 \cap \partial D$ maps onto $[\mathcal{L}, \mathcal{L} + i\pi]$. The length $\mathcal{L}$ of this rectangle is determined uniquely, is called π-extremal distance between $\partial_1 \cap \partial D$ and $\partial_2 \cap \partial D$ in D. Let E denote the event $\{\partial_2 \cap \partial D = \emptyset\}$.

Proposition 1.2.12. *For $b \geq 0$,*

$$\mathbf{E}[1_E \exp\{-b\mathcal{L}\}] \asymp \exp\{-\eta(b)L\}, \; L \to \infty,$$

where

$$\eta(b) = b + \frac{1}{6}(1 + \sqrt{1 + 24b}).$$

The exponents $\eta(b)$ is called the crossing exponent for chordal SLE_6. The major application of SLE is the determination of the intersection exponents for the planar Brownian motion by Lawler, Schramm and Werner in [5],[6],[7]. The close relationship of an SLE_6 curve and the Brownian motion can be used to derive the values of the intersection exponents for the planar Brownian motion from the crossing exponents for SLE_6. The determination of the intersection exponents leads to several fact. (e.g.,The Hausdorff dimension of the set of cut points of the planar Brownian motion is 3/4. The Hausdorff dimension of the outer boundary of the hull generated by the planar Brownian motion is 4/3.)

Last we mention the restriction property for $\kappa = 8/3$. We consider a random simple curve γ with $\gamma(0, \infty) \subset \mathbb{H}$ and $\gamma(0) = 0$. Suppose that $A \in \mathcal{Q}$ is bounded away from 0. Let $\Phi_A(z) := g_A(z) - g_A(0)$ and $V_A := \{\gamma[0, \infty) \cap A = \emptyset\}$. We say that γ satisfies the restriction property if the distribution of $\Phi_A \circ \gamma$ conditioned on the event V_A is the same as a time change of γ.

Proposition 1.2.13. *If $\kappa = 8/3$, then a chordal SLE_κ curve satisfies the restriction property.*

1.2.4 Radial SLE

The radial version of SLE$_\kappa$ is defined similarly to the chordal version.

The radial Schramm-Loewner evolutions with parameter $\kappa > 0$ (abbreviated as radial SLE$_\kappa$) is the random family of conformal maps g_t obtained from the radial Loewner equation

$$\frac{\partial}{\partial t} g_t(z) = -g_t(z) \frac{g_t(z) + e^{i\sqrt{\kappa}B_t}}{g_t(z) - e^{i\sqrt{\kappa}B_t}}, \quad g_0(z) = z, \quad (z \in \mathbb{D})$$

where B_t is a one-dimensional standard Brownian motion.

We can define the random curve γ corresponding to a radial SLE$_\kappa$ and γ grow from a boundary point of $\mathbb{D}$ to the origin 0 in the interior of $\mathbb{D}$. The random curve γ is called a radial SLE$_\kappa$ curve in $\mathbb{D}$. Locally the distribution of a radial SLE$_\kappa$ curve is absolutely continuous with respect to that obtained by taking the image of a chordal SLE$_\kappa$ curve under the map $z \mapsto e^{iz}$. Therefore, a radial SLE$_\kappa$ curve and a chordal SLE$_\kappa$ curve for the same value of κ are similar.

Chapter 2

Convergence of LERW

2.1 Introduction

The Schramm-Loewner evolutions driven by Brownian motion $\sqrt{\kappa}B(t)$ of variance κ, abbreviated as SLE_κ, introduced by Oded Schramm [10], have been studied from various points of view and are now recognized to well describe the scaling limits of certain lattice models of both physical and mathematical interest. Lawler, Schramm and Werner [4] have proved that the scaling limit of a loop erased random walk (or loop erasure of random walk, abbreviated as LERW) on either of the square and triangular lattices is the radial SLE_2. Dapeng Zhan [17] have studied LERW's on the square lattice but in a multiply connected domain and derived the convergence of them. In the case of a simply connected domain in particular, he has proved the convergence to the chordal SLE_2. Yadin and Yehudayoff [15] extend the result of [4], the convergence of LERW to a radial SLE to that for the natural random walks on planar graphs under a natural setting of the problem. In this paper we consider the LERW in a similar setting to [15] and show that LERW conditioned to connecting two boundary points in a simply connected domain converges to a chordal SLE_2 curve.

Here we state our result in an informal way by using the terminology familiar in the theory of SLE of which we shall give a brief exposition in the next section. Let V be the set of vertices of a planar graph on which a random walk (of discrete time) is defined and supposed to satisfy invariance principle in the sense that the linear interpolation of its space-scaled trajectory converges to that of Brownian motion (in a topology where two curves are identified if they agree by some change of time parameter). For each $\delta > 0$ we make the scale change of the space by δ : $V_\delta = \{\delta v : v \in V\}$, the set of scaled lattice points and accordingly we make the δ-scaling of our random walk so that it moves on V_δ. Given a simply connected bounded domain D and two distinct boundary points a and b of it, let γ_δ denote the loop erasure of the random walk scaled by δ that starts a vertex a_δ of V_δ nearby a and is conditioned to exits $D \cap V_\delta$ through a vertex b_δ nearby b so that γ_δ is a random self-avoiding path on $D \cap V_\delta$ connecting a_δ and b_δ, which may be regarded as a 'path' in the planar graph. We prove that the polygonal curve given by linearly interpolating γ_δ converges to the chordal SLE_2 path connecting a and b in D under a certain natural assumption on D, the pair a, b, the planar graph and the random walk (Theorem 2.5.6).

For obtaining the result as stated above we first prove the convergence of the driving function of the loop erasure (Theorem 2.5.1). The proof is made in a way similar to [4], [15] and [11]. In [11] the harmonic explorer, an evolution of a self avoiding random curve, is introduced and proved to converge to a chordal SLE_4 curve. For the proof a suitably chosen martingale associated with the evolving random curve, called martingale observable, plays a dominant role. Not as in [11] we take the martingale observable given by the ratio of harmonic measures of a (random) point relative to two points, the starting site of the walk and a suitably chosen site in a random domain defined by the loop erasure. This martingale is suggested in [4] as a suitable candidate of a martingale observable but we need to normalize it in an appropriate way; moreover we must change the normalization as the loop erasure grows. We apply the approximation result on the harmonic measure (Poisson kernel) proved in [15]. To this end we need a delicate probability estimate, since our random walk starts at a boundary point and we must deal with the conditional law given that it exits $D \cap V_\delta$ through another boundary point.

We deduce the convergence of the loop erasure in a uniform topology from that of the driving function under the hypothesis that not only the random walk but also the dual walk of it satisfy the invariance principle (Theorem 2.5.6). For the deduction we prove Proposition 2.4.1 asserting that the law of the time reversal of loop erasure of a walk agrees with the law of loop erasure of the time reversal of the same walk.

By the way, Proposition 2.4.1 provides an improvement of the convergence to a radial SLE_2. In [15] the loop erasure is unti-chronological (loops are discarded in the reverse order). The reason is that one wants to consider the loop erasure determined from the boundary. Because the radial SLE_2 starts at a boundary point and stops at an inner point, and one wants to use a domain Markov property of the loop erasure. In [4], they used the reversibility property of the loop erasure of a simple random walk proved by Lawler [3]. Proposition 2.4.1 implies that the convergence to SLE_2 in the result of Yadin and Yehudayoff is valid also for LERW with the loops discarded in the chronological order instead of unti-chronological order.

The rest of the paper is organized as follows. In Sections 2.2 and 2.3 we give brief expositions of the Loewner evolution and SLE, respectively, and the fundamental results relevant to the present issue or used in the proof of our results. In Section 2.4, consisting of three subsections, we first give the framework of our problem, the planar graph as well as the random walk on it, and bring in the LERW together with results associated with it (Subsection 2.4.1); we then present a martingale associated with the LERW (Subsection 2.4.2); we also present the result of [15] which asserts an approximation of the harmonic measure of our random walk by the classical Poisson kernel and a trivial lemma of the planar graph (Subsection 2.4.3). The statement and proof of the main result of the present paper are given in Section 2.5. The convergence of the loop erasure to SLE_2 curve with respect to the driving function is given in Subsection 2.5.1, where a certain probability estimate proved in Section 2.6 is taken for granted. The convergence of the loop erasure to SLE_2 curve in a uniform topology is given in Subsection 2.5.2, where we prove the invariance of law of LERW in (a double) time reversion. In Section 2.6 we verify the aforementioned probability estimate which plays an crucial role in the proof of our result, a probability estimate of the scaled random walk on $D \cap V_\delta$ starting at a boundary vertex under the conditional law given that it exists the domain through another boundary vertex that is specified in advance.

2.2 Loewner chain

In this section, consisting of four subsections, we give a brief exposition of the Loewner evolution and some results relevant to the present issue. The standard results in the theory as given in Lawler's book [2] are stated under the heading as **P 2.2.k** ($k = 1, 2, \dots$).

2.2.1 Conformal map and half-plane capacity

Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane. A bounded subset $A \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $A = \overline{A} \cap \mathbb{H}$ and $\mathbb{H} \setminus A$ is a simply connected domain. Let $\mathcal{Q}$ denote the set of compact $\mathbb{H}$-hulls. For any $A \in \mathcal{Q}$, there exists a unique conformal map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ satisfying $|g_A(z) - z| \to 0$ as $z \to \infty$. The half-plane capacity $\operatorname{hcap}(A)$ is defined by

$$\operatorname{hcap}(A) := \lim_{z \to \infty} z(g_A(z) - z).$$

Then, g_A has the expansion

$$g_A(z) = z + \frac{\operatorname{hcap}(A)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$

The half-plane capacity has some nice properties, of which we need the following.

P 2.2.1. *(p69-71) If $r > 0, x \in \mathbb{R}, A \in \mathcal{Q}$, then*

$$\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A), \quad \operatorname{hcap}(A + x) = \operatorname{hcap}(A).$$

If $A, B \in \mathcal{Q}, A \subset B$, then

$$\operatorname{hcap}(B) = \operatorname{hcap}(A) + \operatorname{hcap}(g_A(B \setminus A)).$$

2.2.2 Chordal Loewner Chain in the upper half plane

A chordal Loewner chain is the solution of a type of Loewner equation that describes the evolution of a curve growing from the boundary to the boundary of a domain in $\mathbb{C}$. In this section we consider the special case when the domain is $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, the upper half plane and the curve grows from the origin to the infinity in $\mathbb{H}$. Suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a simple curve with $\gamma(0) = 0, \gamma(0, \infty) \subset \mathbb{H}$. Then, for each $t \geq 0$, there exists a unique conformal map $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$ satisfying $|g_t(z) - z| \to 0$ as $z \to \infty$. It is noted that g_t can be continuously extended to the (two sided) boundary of $\mathbb{H} \setminus \gamma(0, t]$ along $\gamma(0, t]$. If γ is parametrized by half plane capacity (i.e., if $\lim_{z \to \infty} z(g_t(z) - z) = 2t$), g_t satisfies the following differential equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z, \tag{2.1}$$

where $U(t) = g_t(\gamma(t))$ and $U(\cdot)$ is a $\mathbb{R}$-valued continuous function (see [2]). We call the equation (2.1) the chordal Loewner equation and $U(\cdot)$ the driving function.

Conversely, suppose that $U(\cdot) : [0, \infty) \to \mathbb{R}$, a continuous function, is given in advance, for $z \in \mathbb{H}$, solve the ordinary differential equation (2.1) to obtain the solution $g_t(z)$ up to the time $T_z := \sup\{t > 0 : |g_t(z) - U(t)| > 0\}$ and put $K_t := \{z \in \mathbb{H} : T_z \leq t\}$. Then for $t > 0$, $g_t(z)$ is a conformal map from $\mathbb{H} \setminus K_t$ to $\mathbb{H}$. The family $(g_t)_{t\geq 0}$ describes the evolution of hulls $(K_t)_{t\geq 0}$ corresponding to $U(\cdot)$ and growing from the boundary to ∞. Therefore, we have a one-to-one correspondence between $U(\cdot)$ and $(K_t)_{t\geq 0}$. If $U(\cdot)$ is the driving function of a simple curve γ, we can recover γ from $U(\cdot)$ by the formula $\gamma(t) = g_t^{-1}(U(t))$ and we can write $K_t = \gamma(0, t]$. If $U(\cdot)$ is sufficiently nice, then $(K_t)_{t\geq 0}$ is generated by a curve γ with $\gamma(0) \in \mathbb{R}, \lim_{t\to\infty} \gamma(t) = \infty$ (i.e., for any $t \geq 0$, $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$). However, there exists a continuous function $U(\cdot)$ such that $(K_t)_{t\geq 0}$ can not be generated by a curve. There is known a sufficient condition for $U(\cdot)$ to drive a curve as given by

P 2.2.2. *(p108) Suppose for some $r < \sqrt{2}$ and all $s < t$,*

$$|U(t) - U(s)| \leq r\sqrt{t - s}.$$

Then $(K_t)_{t\geq 0}$ is generated by a simple curve.

The family $g_t, t \geq 0$ is called the (chordal) Loewner chain generated by a curve γ or driven by a function $U(t)$. In summary, a simple curve γ brings out a Loewner chain, whereby it determines the driving function $U(t)$, and conversely a continuous function $U(t)$ with appropriate regularity generates a curve through the Loewner chain driven by $U(t)$.

Proposition 2.2.3. *(Lemma 2.1. in [4]) There exists a constant $C > 0$ such that the following holds. Let K_t be the corresponding hull for a Loewner chain driven by a continuous function $U(t)$. Set*

$$k(t) := \sqrt{t} + \sup\{|U(s) - U(0)| : 0 \leq s \leq t\}.$$

Then, for any $t > 0$,

$$C^{-1}k(t) \leq \mathrm{diam}(K_t) \leq Ck(t).$$

2.2.3 Chordal Loewner chains in simply connected domains

Let $D \subsetneq \mathbb{C}$ be a simply connected domain and ∂D a set of prime ends. If D is a Jordan domain, then ∂D may be identified with the topological boundary of D. Let a, b be distinct points on ∂D. For $p \in D$, we define the inner radius of D with respect to p,

$$\mathrm{rad}_p(D) := \inf\{|z - p| : z \notin D\}.$$

Let $\phi : D \to \mathbb{H}$ be a conformal map with $\phi(a) = 0, \phi(b) = \infty$. Although ϕ is not unique, any other such map can be written as $r\phi$ for some $r > 0$. For a simple curve $\gamma : (0, T) \to D$ connecting a and b so that $\gamma(0+) = a$ and $\gamma(T-) = b$, let g_t be the Loewner chain generated by the curve $\phi \circ \gamma : (0, T) \to \mathbb{H}$ and put

$$\phi_t = g_t \circ \phi, \quad t \in [0, \infty).$$

We reparametrize the curve γ so that the curve $\phi \circ \gamma$ in $\mathbb{H}$ is parametrized by half plane capacity. Denote by $(\gamma(t))$ the function representing the curve in this parametrization, so that $2t = \mathrm{hcap}(\phi \circ \gamma[0, t])$. The driving function $U(t)$ of the chain g_t is then given by

$$U(t) = \phi_t(\gamma(t)).$$

The family of conformal maps $\phi_t, t \geq 0$ may also be called a chordal Loewner chain (in D) with driving function $U(t)$. For each $s > 0$, ϕ_s conformally maps $D(s) := D \setminus \gamma(0, s]$ onto $\mathbb{H}$ with $\phi_s(a_s) = U(s)$, $\phi_s(b) = \infty$, where $a_s = \gamma(s)$ and the curve $\gamma^{(s)}(t) := \gamma(s+t)$ connects a_s and b in $D(s)$. On putting

$$g_t^{(s)} = g_{s+t} \circ g_s^{-1} \quad \text{and} \quad \phi_t^{(s)} = \phi_{s+t}, \tag{2.2}$$

substitution into $U(s+t) = \phi_{s+t}(\gamma(s+t))$ yields

$$U(t+s) = \phi_t^{(s)}(\gamma^{(s)}(t)). \tag{2.3}$$

It follows from (2.2) that $\phi_t^{(s)} = g_t^{(s)} \circ \phi_s$ and $g_t^{(s)}$ (and $\phi_t^{(s)}$) is the Loewner chain generated by the curve $\gamma^{(s)}$; and also, from (2.3) that $U^{(s)}(t) := U(s+t)$ is the driving function of the chain $\phi_t^{(s)}$ in $D(s)$.

Define $p(t) \in D$ by

$$\phi_t(p(t)) = U(t) + i.$$

$p(t)$ serves as a reference point for the study of the conformal map ϕ_t. (See Proposition 2.4.5 and the remark advanced before Lemma 2.5.3.)

Lemma 2.2.4. *Let $T > 1$ and $\epsilon > 0$, and, given a pair (D, γ), put $\tilde{T} := \sup\{t \in [0, T] : |U(t)| < 1/\epsilon\}$. Then there exists a constant $c(T, \epsilon) > 0$, which may also depend on $(D, \gamma(0))$ but does not on $(\gamma(t), t > 0)$, such that*

$$\mathrm{rad}_{p(t)}(D(t)) \geq c(T, \epsilon)\, \mathrm{rad}_{p(0)}(D) \quad \textit{for} \quad t < \tilde{T}.$$

Proof. We claim that

$$|\phi(p(t)) - \phi(\gamma(t'))| \geq 2^{-1} e^{-4\tilde{T}} \quad \text{if} \quad t' \leq t < \tilde{T}. \tag{2.4}$$

Let $t' \leq t < \tilde{T}$ and $z = \phi(\gamma(t'))$, and put

$$y(s) = g_s(\phi(p(t))) - g_s(z), \qquad 0 \leq s \leq t.$$

We prove $|y(0)| = |\phi(p(t)) - z| \geq 2^{-1} e^{-4\tilde{T}}$. Recalling that $\mathrm{Im}\, g_s(w)$ is decreasing in s for any $w \in \mathbb{H}$, we see that

$$\mathrm{Im}\, g_s \circ \phi(p(t)) \geq \mathrm{Im}\, g_t \circ \phi(p(t)) = 1 \quad \text{if} \quad s \leq t. \tag{2.5}$$

Applying this with $s = 0$ we have $|y(0)| \geq 1/2$ if $\mathrm{Im}\, z \leq 1/2$. Let $\mathrm{Im}\, z > 1/2$ and define $\tau := \inf\{t \geq 0 : \mathrm{Im}\, g_t(z) = 1/2\}$. Then $\tau < t' \leq t$ (since $\mathrm{Im}\, g_{t'}(z) = 0$) and the Loewner equation together with the inequality (2.5) shows

$$\left| \frac{d}{ds} y(s) \right| = \frac{2|y(s)|}{|g_s \circ \phi(p(t)) - U(s)| \cdot |g_s(z) - U(s)|} \leq 4|y(s)| \quad \text{for} \quad 0 \leq s \leq \tau.$$

Hence $|y(s)|$ is absolutely continuous and satisfies $\frac{d}{ds}|y(s)| \leq 4|y(s)|$, so that

$$|y(\tau)| \leq |y(0)| e^{4\tau}.$$

Using (2.5) again we have $1/2 \leq \operatorname{Im} y(\tau)$ so that $1/2 \leq |y(0)|e^{4\tilde{T}}$, which is the same as what we need to prove. Thus the claim (2.4) is verified.

It is proved in [12] (the proof of Corollary 4.3) that the set $\{\phi(p(t)) : t < \tilde{T}\}$ is included in a compact set of $\mathbb{H}$ depending only on T and ε, whence according to the Koebe distortion theorem $\mathrm{rad}_{p(t)}(D) \geq c_0(T,\epsilon)\,\mathrm{rad}_{p(0)}(D)$ for some constant $c_0(T,\epsilon) > 0$. For the proof of the lemma it therefore suffices to show that

$$|p(t) - \gamma(t')| \geq c_1(T,\epsilon)\,\mathrm{dist}(p(t), \partial D) \quad \text{for} \quad t' \leq t < \tilde{T}.$$

To this end we may suppose $|p(t) - \gamma(t')| < 2^{-1}\mathrm{dist}(p(t), \partial D)$. Applying (2.4) and the distortion theorem in turn yields

$$2^{-1}e^{-4\tilde{T}} \leq |\phi(p(t)) - \phi(\gamma(t'))| \leq 16|p(t) - \gamma(t')| \cdot \frac{\mathrm{dist}(\phi(p(t)), \mathbb{R})}{\mathrm{dist}(p(t), \partial D)}.$$

We know that $\mathrm{dist}(\phi(p(t)), \mathbb{R}) \leq M$ for some constant $M = M(T,\epsilon) > 0$ from the result of [12] mentioned above. Hence $|p(t) - \gamma(t')| \geq [e^{-4T}/32M]\,\mathrm{dist}(p(t), \partial D)$ as desired. □

2.2.4 Metrics on curves

Let $\gamma, \gamma^j (j = 1, 2, \dots)$ be curves which generate the Loewner chains. Let $U(t)$ and $U_j(t)$ be driving functions corresponding to γ and γ^j, respectively. If $U_j(t)$ converges uniformly to $U(t)$ on any bounded interval, then we will say that γ^j converges to γ with respect to the driving function.

Next, we consider the metric on the space of unparametrized curves in $\mathbb{C}$. Let $f_1, f_2 : [0,1] \to \mathbb{C}$ be a continuous, non-locally constant functions. If there exists a continuously increasing bijection $\alpha : [0,1] \to [0,1]$ such that $f_2 = f_1 \circ \alpha$, then we will say f_1 and f_2 are the same up to reparametrization, denoted by $f_1 \sim f_2$. A unparametrized curve γ is defined to be an equivalence class modulo $\sim$. Let d_* be the spherical metric on $\widehat{\mathbb{C}}$. We define the metric on the space of unparametrized curves by

$$d_{\mathcal{U}}(\gamma_1, \gamma_2) := \inf_{\alpha} \left[\sup_{0 \leq t \leq 1} d_*(f_1(t), f_2 \circ \alpha(t)) \right], \tag{2.6}$$

where f_i any function in the equivalence class γ_i, and the infimum is taken over all reparametrizations α which are continuously increasing bijections of $[0,1]$. We often denote by the same notation γ a parametrized curve as well as an unparametrized curve. Let us denote by γ^- the time reversal of γ.

The convergence with respect to the driving function is weaker than the convergence with respect to the metric $d_{\mathcal{U}}$. We will consider a sufficiency condition for the convergence with respect to the metric $d_{\mathcal{U}}$ when we have the convergence with respect to the driving function. Let $D \subsetneq \mathbb{C}$ be a simply connected domain and ∂D be the set of prime ends of D. Let $a, b \in \partial D$ be distinct points. Let $\phi : D \to \mathbb{H}$ be a conformal map with $\phi(a) = 0, \phi(b) = \infty$. Let $\phi^- : D \to \mathbb{H}$ be a conformal map with $\phi^-(b) = 0, \phi^-(a) = \infty$.

Theorem 2.2.5. *(Theorem 1.2 in [12]) Let $\{\gamma^j\}$ be a sequence of simple curves travelling from a to b in D. Suppose that there exists simple curves γ and η such that $\phi \circ \gamma^j$ converges to $\phi \circ \gamma$ with respect to the driving function and $\phi^- \circ \gamma^{j-}$ converges to $\phi^- \circ \eta$ with respect to the driving function. Then $\gamma^- = \eta$ and γ^j converges to γ with respect to the metric $d_{\mathcal{U}}$.*

2.3 Schramm-Loewner evolutions

2.3.1 SLE in the upper half plane

Let B_t be a one-dimensional standard Brownian motion with $B_0 = 0$. A chordal Schramm-Loewner evolution with parameter $\kappa > 0$ (abbreviated as chordal SLE$_\kappa$) is the random family of conformal map g_t obtained from the chordal Loewner equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z \quad (z \in \mathbb{H}). \tag{2.7}$$

Let K_t be an evolving (random) hull corresponding to SLE$_\kappa$. Because B_t is not (1/2)-Hölder continuous, we can not use **P2.2.2** and it is not easy to see whether K_t is generated by a curve. However, according to the following results K_t is actually generated by a curve with full probability.

P 2.3.1. *(p148) With probability 1, the limit $\gamma(t) := \lim_{z\to 0} g_t^{-1}(z + \sqrt{\kappa} B_t)$ exists for any $t \geq 0$ and K_t is generated by the curve γ.*

This curve γ is called a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞. The following properties of SLE$_\kappa$ curves are easily verified.

P 2.3.2. *(p148) Suppose that γ is a chordal SLE$_\kappa$ curve in $\mathbb{H}$ and $r > 0$. Let $\widehat{\gamma}(t) := r^{-1}\gamma(r^2 t)$. Then, $\widehat{\gamma}$ has the same distribution as γ.*

P 2.3.3. *(p147) Suppose that γ is a chordal SLE$_\kappa$ curve in $\mathbb{H}$. Let τ be a stopping time. Let $\widehat{\gamma}(t) := g_\tau(\gamma(t+\tau)) - \sqrt{\kappa} B_\tau$. Then, $\widehat{\gamma}$ has the same distribution as γ.*

The behavior of a chordal SLE$_\kappa$ curve depends on the value of the parameter κ. There is three phases in the behavior of a chordal SLE$_\kappa$ curve. The two phases transitions take place at the values $\kappa = 4$ and $\kappa = 8$.

P 2.3.4. *(p150-151) Suppose that γ be a chordal SLE$_\kappa$ curve in $\mathbb{H}$.*

- *If $0 < \kappa \leq 4$, then w.p.1, γ is a simple curve with $\gamma(0, \infty) \subset \mathbb{H}$.*
- *If $4 < \kappa < 8$, then w.p.1, $\gamma(0, \infty) \cap \mathbb{H} \neq \mathbb{H}$ and $\cup_{t>0} \overline{K_t} = \overline{\mathbb{H}}$.*
- *If $\kappa \geq 8$, then w.p.1, γ is a space-filling curve, i.e., $\gamma[0, \infty) = \overline{\mathbb{H}}$.*

2.3.2 SLE in simply connected domains

Let γ be a chordal SLE$_\kappa$ curve in $\mathbb{H}$ from 0 to ∞. As in the subsection 2.3 let $D \subsetneq \mathbb{C}$ be a simply connected domain, ∂D a set of prime ends, a, b two distinct points on ∂D and $\phi : D \to \mathbb{H}$ a conformal map with $\phi(a) = 0, \phi(b) = \infty$. Although ϕ is not unique, any other such map $\tilde{\phi}$ can be written as $r\phi$ for some $r > 0$. By **P 2.3.2**, $\phi^{-1}(\gamma)$ is independent of the choice of the map up to a time change and we consider SLE$_\kappa$ curves in D as unparametrized curves. A chordal SLE$_\kappa$ curve in D from a to b is defined by $\phi^{-1}(\gamma)$.

The two properties stated in the next proposition, called the domain Markov property and conformal invariance, respectively, immediately follow from the definition of SLE.

P 2.3.5. *Let γ be a chordal SLE$_\kappa$ curve in D from a to b and $\mu_{a,b;D}$ be a law of γ. Let $f : D \to D'$ be a conformal map. Then,*

$$\mu_{a,b;D}(\cdot|\gamma(0,t]) = \mu_{\gamma(t),b;D\setminus\gamma(0,t]}(\cdot),$$

and

$$f \circ \mu_{a,b;D}(\cdot) = \mu_{f(a),f(b);D'}(\cdot).$$

In the theory of SLE, it is easier to prove the convergence with respect to the driving function than in the metric $d_{\mathcal{U}}$. Theorem 2.2.5 implies the following result, which we shall apply the following result to derive the convergence with respect to $d_{\mathcal{U}}$ of LERW from that of the driving function. Let $\phi^- : D \to \mathbb{H}$ be a conformal map with $\phi^-(b) = 0, \phi^-(a) = \infty$.

Theorem 2.3.6. *([12]) Let $\{\gamma^j\}$ be a sequence of simple random curves traveling from a to b in D. Let $\kappa \leq 4$, and $\gamma(a,b)$ be the chordal SLE$_\kappa$ curve in D from a to b. $\phi \circ \gamma^j$ and $\phi^- \circ \gamma^{j-}$ converge weakly to a chordal SLE$_\kappa$ curve in $\mathbb{H}$ with respect to the driving function. Then γ_j converges weakly to $\gamma(a,b)$ with respect to $d_{\mathcal{U}}$.*

The reversibility of SLE holds at least for $\kappa \leq 4$.

Theorem 2.3.7. *(Theorem 2.1 in [16]) Let $\kappa \leq 4$. The time-reversal of a chordal SLE$_\kappa$ curve in D from a to b has the same distribution as chordal SLE$_\kappa$ curve in D from b to a.*

If $\kappa > 8$, then SLE curve is not reversible.

2.4 Loop erased random walks

2.4.1 Some property of LERW

For any $u, v \in \mathbb{C}$, we write $[u,v] = \{(1-t)u + tv : 0 \leq t \leq 1\}$ for the line segment whose end points are u and v. Let $V \subset \mathbb{C}$ be a countable subset with $0 \in V$. Let $E : V \times V \to [0,\infty)$ and $E = \{(u,v) : E(u,v) > 0\}$. We call $G = (V,E)$ a directed weighted graph. We assume that $\sum_{v\in V} E(u,v) < \infty$ for every $u \in V$, and put

$$p(u,v) := \frac{E(u,v)}{\sum_{w\in V} E(u,w)}.$$

We call G that satisfies the following conditions a planar irreducible graph.

1. G is a planar graph.
   (i.e. for every distinct edges $(u,v),(u',v') \in E$, $[u,v] \cap [u',v'] \in \{\emptyset, \{u\}, \{v\}\}$.)

2. For any compact set $K \subset \mathbb{C}$, the number of vertices $v \in K$ is finite.

3. The Markov chain $S(\cdot)$ on V with transition probability $p(u,v)$ is irreducible.
   (i.e. for every $u, v \in V$, there exists $n \in \mathbb{N}$ such that $\mathbf{P}(S(n) = v \mid S(0) = u) > 0$.)

We call $S(\cdot)$ the natural random walk on G. For the reminder of this paper we think that G is a planar irreducible graph.

For any simply connected domain $D \subsetneq \mathbb{C}$, let $V(D) := V \cap D$. Define

$$\partial_{out} V(D) := \{(u,v) \in E : [u,v] \cap \partial D \neq \emptyset, u \in V(D)\}$$

and

$$\partial_{in} V(D) := \{(u,v) \in E : [u,v] \cap \partial D \neq \emptyset, v \in V(D)\}.$$

The first exit time from D is defined by

$$\tau_D := \begin{cases} \inf\{n \geq 1 : (S(n-1), S(n)) \in \partial V_{out}(D)\} & \text{if} \quad S(0) \in V(D) \\ \inf\{n \geq 2 : (S(n-1), S(n)) \in \partial V_{out}(D)\} & \text{if} \quad (S(0), S(1)) \in \partial_{in} V(D) \\ 0 \quad \text{otherwise} & \end{cases}.$$

We sometimes consider the edge $(u,v) \in \partial_{out} V(D)$ as the vertex v, and the edge $(u,v) \in \partial_{in} V(D)$ as the vertex u; e.g., we write $S(\tau_D) \in \partial_{out} V(D)$ and $S(0) \in \partial_{in} V(D)$ and for a set $J \subset \partial D$, we write $S(\tau_D) \in J$ instead of writing $[S(\tau_D - 1), S(\tau_D)] \cap J \neq \emptyset$.

Loop erasure. Let $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ be a finite sequence of points. Let $s_0 = \max\{k \geq 0 : \omega_0 = \omega_k\}$. Inductively, we define $s_m = \max\{k \geq 0 : \omega_{s_{m-1}+1} = \omega_k\}$. If $l = \min\{m \geq 0 : \omega_{s_m} = \omega_n\}$, then the loop erasure of ω is defined by

$$L[\omega] = (\omega_{s_0}, \omega_{s_1}, \ldots, \omega_{s_l}).$$

The time-reversal of ω is defined by

$$\omega^- = (\omega_n, \omega_{n-1}, \ldots, \omega_0).$$

It is readily recognized that the operations L and $^-$ are not commutable, namely, $L[\omega^-] \neq L[\omega]^-$ in general. If the transition probability $p(u,v)$ is symmetric, then the following result has been proved by Lawler in [3]. For our purpose, we prove the following result without assuming that $p(u,v)$ is symmetric.

Proposition 2.4.1. *Let $S(\cdot)$ be a natural random walk on G.*

$$\mathbf{P}(L[(S(0), S(1), \ldots, S(\tau_D))^-] = \omega) = \mathbf{P}(L[(S(0), S(1), \ldots, S(\tau_D))]^- = \omega).$$

Remark. Theorem 2.4.1 implies that the convergence to the radial SLE_2 in the result of Yadin and Yehudayoff (Theorem1.1 in [15]) is valid also for LERW with the loops discarded in the chronological order instead of unti-chronological order.

Proof. Let $\omega = (\omega_0, \ldots, \omega_n)$ and $\omega_1, \ldots, \omega_{n-1} \in V(D)$ be distinct and $(\omega_{n-1}, \omega_n) \in \partial_{out} V(D)$. Our task is to show the identity

$$\mathbf{P}(L[(S(0), \ldots, S(\tau_D))] = \omega) = \mathbf{P}(L[(S(0), \ldots, S(\tau_D))^-] = \omega^-). \tag{2.8}$$

Let $q : V \times V \to [0,1]$. Set

$$G_q(x; D) = 1 + \sum_{k=0}^{\infty} \sum_{\omega' \subset D : \omega'_0 = x, \omega'_k = x} q(\omega'_0, \omega'_1) \cdots q(\omega'_{k-1}, \omega'_k),$$

where the inner summation is taken over all paths $\omega' = (\omega'_0, \dots, \omega'_k)$ in D such that $\omega'_0 = x, \omega'_k = x$.

The probability of LERW is described by the following (See [3]).

$$\mathbf{P}(L[(S(0), \dots, S(\tau_D)] = \omega) = \prod_{j=0}^{n-1} p(\omega_j, \omega_{j+1}) G_p(\omega_j; D \setminus \{\omega_0, \dots, \omega_{j-1}\})$$

By the exchange lemma (the equation (12.2.3) in [3]), we get

$$\mathbf{P}(L[(S(0), \dots, S(\tau_D)] = \omega) = \prod_{j=0}^{n-1} p(\omega_j, \omega_{j+1}) G_p(\omega_j; D \setminus \{\omega_{j+1}, \dots, \omega_{n-1}\}) \tag{2.9}$$

On the other hand,

$$\begin{aligned}\mathbf{P}(L[(S(0), \dots, S(\tau_D))^-] = \omega^-) &= \sum_{\omega' \subset D : L[(\omega')^-] = \omega^-} \prod_{i=0}^{|\omega'|-1} p(\omega'_i, \omega'_{i+1}) \\ &= \sum_{\omega' \subset D : L[\omega'] = \omega^-} \prod_{i=0}^{|\omega'|-1} p^*(\omega'_i, \omega'_{i+1}),\end{aligned}$$

where $|\omega'|$ is the length of ω' and $p^*(x, y) := p(y, x)$. This equation and decomposing ω' between its last visit to $\omega_{n-1}, \dots, \omega_0$ imply that

$$\begin{aligned}\mathbf{P}(L[(S(0), \dots, S(\tau_D))^-] = \omega^-) &= \prod_{j=0}^{n-1} p^*(\omega_{j+1}, \omega_j) G_{p^*}(\omega_j; D \setminus \{\omega_{n-1}, \dots, \omega_{j+1}\}) \\ &= \prod_{j=0}^{n-1} p(\omega_j, \omega_{j+1}) G_{p^*}(\omega_j; D \setminus \{\omega_{j+1}, \dots, \omega_{n-1}\}). \quad (2.10)\end{aligned}$$

Finally observe that $G_p(x; D') = G_{p^*}(x; D')$. Thus, (2.9) and (2.10) imply (2.8). □

Let $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ be the loop erasure of the time-reversal of the natural random walk stopped on exiting D. By Proposition 2.4.1, we may think that γ is the time-reversal of the loop erasure. (In Section 5, we treat γ as the time-reversal of the loop erasure. But in this section, we treat γ as the loop erasure of the time-reversal because it is more suitable to consider the following properties of γ.)

Let $D_j := D \setminus \cup_{i=0}^{j-1} [\gamma_i, \gamma_{i+1}]$. For any $j \in \mathbb{N}$,

$$n_j := \min\{n \geq 0 : S(n) = \gamma_j\}.$$

Because the loop erasure γ is determined from the boundary, γ has the following Markov property.

Proposition 2.4.2. *(Lemma 3.2. in [4]) Conditioned on $\gamma[0, j]$, the following holds.*

1. *$S[0, n_j]$ and $S[n_j, \tau_D]$ are independent.*

2. *$\gamma[j, l]$ has the same distribution as the loop erasure of time-reversal of the natural random walk $S[0, \tau_{D_j}]$ conditioned to exit at γ_j.*

2.4.2 Martingale observable for LERW

Let $D \subsetneq \mathbb{C}$ be a simply connected domain. Let $S^x(\cdot)$ be a natural random walk on G started at $x \in V$. Let $v_0 \in V(D) \cup \partial_{in} V(D)$ and γ be the loop erasure of time-reversal of the natural random walk $S^{v_0}[0, \tau_D]$. Let $D_j := D \setminus \cup_{i=0}^{j-1} [\gamma_i, \gamma_{i+1}]$. The hitting probability $H_j(u, v)$ is defined by

$$H_j(u, v) := \mathbf{P}(S^u(\tau_{D_j}) = v).$$

Let $\mathcal{F}_j$ be a filtration generated by $\gamma[0, j]$.

Proposition 2.4.3. *For any $w \in V(D)$, let*

$$M_j := \frac{H_j(w, \gamma_j)}{H_j(v_0, \gamma_j)}.$$

Then, M_j is a martingale with respect to $\mathcal{F}_j$.

Lawler, Schramm and Werner [4] point out that the martingale M_j given above should be a possible martingale observable, although they don't adopt it but a martingale formed by the Green functions of evolving domains. They provide a curtailed proof that M_j is a martingale. Since M_j plays the central role in this paper we give a detailed proof of this fact.

Proof. First, we consider another representation of M_j. Let $\widehat{S}^x(\cdot)$ be a independent copy of $S^x(\cdot)$ and L_x be the loop erasure of the time-reversal of $\widehat{S}^x[0, \tau_D]$. We will denote by $\mathbf{Q}$ the law of $\widehat{S}$. Fix $\gamma[0, j]$. By proposition 2.4.2,

$$\begin{aligned}\frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j])}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])} &= \frac{\mathbf{Q}(\widehat{S}^w(\tau_{D_j}) = \gamma_j)\mathbf{Q}(L_{\gamma_j}[0, j] = \gamma[0, j])}{\mathbf{Q}(\widehat{S}^{v_0}(\tau_{D_j}) = \gamma_j)\mathbf{Q}(L_{\gamma_j}[0, j] = \gamma[0, j])} \\ &= \frac{H_j(w, \gamma_j)}{H_j(v_0, \gamma_j)}.\end{aligned}$$

Therefore, we can write

$$M_j = \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j])}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])}.$$

Hence,

$$\mathbf{E}[M_{j+1} | \gamma[0, j]] = \sum_{v \in V(D_j)} \mathbf{P}(\gamma_{j+1} = v | \gamma[0, j]) \cdot \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j], L_w(j+1) = v)}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j], L_{v_0}(j+1) = v)},$$

and, since $\mathbf{P}(\gamma_{j+1} = v | \gamma[0, j]) = \mathbf{Q}(L_{v_0}(j+1) = v | L_{v_0}[0, j] = \gamma[0, j])$, the right-hand side reduces to

$$\sum_{v \in V(D_j)} \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j], L_w(j+1) = v)}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])} = \frac{\mathbf{Q}(L_w[0, j] = \gamma[0, j])}{\mathbf{Q}(L_{v_0}[0, j] = \gamma[0, j])} = M_j.$$

Thus, M_j is a martingale. $\square$

2.4.3 Estimates of discrete harmonic measures

For $\delta > 0$, the graph $G_\delta = (V_\delta, E_\delta)$ defined by

$$V_\delta = \{\delta u : u \in V\}, \quad E_\delta = \{(\delta u, \delta v) : E(u,v) > 0\}.$$

Let the Markov chain $S_\delta(\cdot)$ on V_δ be the scaling of $S(\cdot)$ by a factor of δ. We call $S_\delta(\cdot)$ the natural random walk on G_δ. Let $S_\delta^x(\cdot)$ be a natural random walk on G_δ started at $x \in V_\delta$. Similarly, we can define $H_j^{(\delta)}(u,v)$, $V_\delta(D)$, $\partial_{out} V_\delta(D)$, $\partial_{in} V_\delta(D)$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc.

Definition 2.4.4. *If the family of the random walks S_δ^x satisfies the following condition, then we say that S_δ^x satisfies invariance principle:*
For any compact set $K \subset \mathbb{D}$ and $\epsilon > 0$, there is some $\delta_0 > 0$ such that the following holds. Let Z^x be a two-dimensional Brownian motion started at x stopped on exiting $\mathbb{D}$. For any $0 < \delta < \delta_0$ and $x \in K \cap V_\delta$, there exists a coupling of S_δ^x and Z^x satisfying

$$\mathbf{P}(d_{\mathcal{U}}(S_\delta^x[0, \tau_{\mathbb{D}}], Z^x) > \epsilon) < \epsilon.$$

In view of the Skorokhod representation theorem the above condition is equivalent to holding that S_δ^x weakly converges to Z^x uniformly for all $x \in K$.

In [15] (Lemma 1.2) the following result is proved.

Proposition 2.4.5. *Suppose that S_δ^x satisfies invariance principle. For any positive constants r, ε and $\eta < 1$, there exists some $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds. Let $D \subset \mathbb{D}$, let $p \in V_\delta(D)$ be such that $\mathrm{rad}_p(D) \geq r$, and let $\psi : D \to \mathbb{D}$ be a conformal map with $\psi(p) = 0$. Let $y \in V_\delta(D)$ be such that $|\psi(y)| < 1 - \eta$ and let $a \in \partial_{out} V_\delta(D)$. Then,*

$$\left| \frac{H_0^{(\delta)}(y,a)}{H_0^{(\delta)}(p,a)} - \frac{K_{\mathbb{D}}(\psi(y), \psi(a))}{K_{\mathbb{D}}(\psi(p), \psi(a))} \right| < \epsilon,$$

where $K_{\mathbb{D}}$ stands for the Poisson kernel of $\mathbb{D}$.

The Poisson kernel of $\mathbb{H}$ is given by

$$K_{\mathbb{H}}(u,v) := -\frac{1}{\pi} \mathrm{Im} \left(\frac{1}{u-v} \right) = \frac{1}{\pi} \frac{\mathrm{Im}\, u}{|u-v|^2}.$$

The result above may be translated in terms of $K_{\mathbb{H}}$. For our purpose we apply it in a rather trivial fashion. Let

Corollary 2.4.6. *Suppose that S_δ^x satisfies invariance principle. For any constants $r > 0$, $\epsilon > 0$, $\eta > 0$ and $\lambda > 1$, there exists some $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds. Let $D \subset \mathbb{D}$, let $p \in D$ be such that $\mathrm{rad}_p(D) \geq r$, and let $\phi : D \to \mathbb{H}$ be a conformal map with $\phi(p) = i$. Let $y, w \in V_\delta(D)$ be such that $\mathrm{Im}\,\phi(y) > \eta, \mathrm{Im}\,\phi(w) > \eta$ and $|\phi(y)| < \lambda, |\phi(w)| < \lambda$. Then, for all $a \in \partial_{out} V_\delta(D)$*

$$\left| \frac{H_0^{(\delta)}(w,a)}{H_0^{(\delta)}(y,a)} - \frac{K_{\mathbb{H}}(\phi(w), \phi(a))}{K_{\mathbb{H}}(\phi(y), \phi(a))} \right| < \epsilon.$$

Proof. Let $p_\delta \in V_\delta(D)$ be a nearest point of p. Applying Proposition 2.4.5 with $p = p_\delta$,

$$\frac{H_0^{(\delta)}(w,a)}{H_0^{(\delta)}(y,a)} = \frac{H_0^{(\delta)}(w,a)/H_0^{(\delta)}(p_\delta,a)}{H_0^{(\delta)}(y,a)/H_0^{(\delta)}(p_\delta,a)} = \frac{K_{\mathbb{D}}(\psi(w),\psi(a))}{K_{\mathbb{D}}(\psi(y),\psi(a))} + O(\epsilon).$$

Because the ratio of the Poisson kernel is conformal invariance, we find

$$\frac{K_{\mathbb{D}}(\psi(w),\psi(a))}{K_{\mathbb{D}}(\psi(y),\psi(a))} = \frac{K_{\mathbb{H}}(\phi(w),\phi(a))}{K_{\mathbb{H}}(\phi(y),\phi(a))}.$$

This completes the proof. □

Here we present the following trivial lemma for convenience of a later citation.

Lemma 2.4.7. *Suppose that S_δ^x satisfy invariance principle. For any $\epsilon > 0$, there exists δ_0 such that for all $0 < \delta < \delta_0$, the length of edges of G_δ in $\mathbb{D}$ is bounded above by ϵ.*

Proof. Suppose that this Lemma is not true. Then, there exists $\epsilon > 0$ such that for some sufficiently small δ, there exists an edge e of G_δ such that the length of e is bounded below by ϵ. Since G_δ is planar graph, S_δ^x can not cross the edge e, so that it cannot behave as a Brownian path and the invariance principle fails to hold. □

2.5 Scaling limit

2.5.1 Convergence with respect to the driving function

Let $D \subsetneq \mathbb{C}$ be a simply connected domain and a, b two distinct points on ∂D. We say that ∂D is locally analytic at $z \in \partial D$ if there exists a one-to-one analytic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = z$ and $f(\mathbb{D}) \cap D = f(\{w \in \mathbb{D} : \operatorname{Im} w > 0\})$. Let $G = (V, E)$ be a planar irreducible graph and S_δ^x a natural random walk on G_δ started at x (see Section 4 for detailed description). Let $\Gamma_\delta^{a,b}$ be a natural random walk on G_δ started at a_δ and stopped on exiting D and conditioned to hit ∂D at b_δ, where a_δ is a point of $\partial_{in} V_\delta(D)$ close to a and b_δ is a point of $\partial_{out} V_\delta(D)$ close to b such that there exists a path on G_δ connecting a_δ and b_δ in D. If ∂D is locally analytic at a and b, we can choose such a_δ and b_δ. Let $\gamma_\delta^{a,b}$ be the loop erasure of $\Gamma_\delta^{a,b}$.

Theorem 2.5.1. *Suppose that S_δ^x satisfy invariance principle. Let D be a bounded simply connected domain and a, b be distinct points on ∂D. Suppose that ∂D is locally analytic at a and b. Let $\phi : D \to \mathbb{H}$ be a conformal map with $\phi(a) = 0, \phi(b) = \infty$. Then, $\phi \circ (\gamma_\delta^{b,a})^-$ converges weakly to the chordal SLE_2 curve in $\mathbb{H}$ as $\delta \to 0$ with respect to the driving function.*

Remark. In order to assure the uniformity of invariance principle so imposed in Definition 2.4.4 it suffices to suppose it only for the walk starting at a point, e.g., the origin as is shown in [14].

We abbreviate $(\gamma_\delta^{b,a})^- = \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$. By Proposition 2.4.1, γ has the same distribution as the loop erasure of the time-reversal of $\Gamma_\delta^{b,a}$. Hence, it is possible for γ to use results in Section 4. Let $\mathcal{F}_j$ be a filtration generated by $\gamma[0,j]$. We may also think that $\gamma[0,j]$ is the simple curve that is a linear interpolation.

Let $U(t)$ be a driving function of $\phi(\gamma)$ and g_t be a Loewner chain driven by $U(t)$. Let $t_j := \frac{1}{2}\mathrm{hcap}\phi(\gamma[0,j])$ and

$$U_j := U(t_j), \quad \phi_j := g_{t_j} \circ \phi \ \text{ and } \ D_j := D \setminus \gamma[0,j].$$

Let $p_j := \phi_j^{-1}(i + U_j)$. p_j plays the role of a reference point, an 'origin', of D_j. In radial case, such a point is fixed at the origin. But in chordal case, p_j must be moved with j, so that there remains sufficient space around p_j in D_j, a sequence of reducing domains formed by encroachment of γ into D. (Cf. [11]).

We use the martingale introduced in Proposition 2.4.3, as in [4] and [15]. But we need to normalize it appropriately. We denote by S_δ^b a natural random walk on G_δ started at b_δ. Let $A := \phi^{-1}([-1,1])$ and the normalization is made by multiplying $\mathbf{P}(S_\delta^b(\tau_D) \in A)$, which we name M_j:

$$M_j := \frac{H_j^{(\delta)}(w, \gamma_j)}{H_j^{(\delta)}(b, \gamma_j)} H_0^{(\delta)}(b; A), \tag{2.11}$$

(for any $\delta > 0$ and $w \in V_\delta(D)$), where we write $H_0^{(\delta)}(b; A) := \mathbf{P}(S_\delta^b(\tau_D) \in A)$.

Let $D \subsetneq \mathbb{C}$ be a simply connected domain, a, b two distinct points on ∂D and $\phi : D \to \mathbb{H}$ a conformal map with $\phi(a) = 0, \phi(b) = \infty$ as before. Let $p = \phi^{-1}(i)$. Put $\Psi(z) = (z - i)/(z + i)$. Define $\psi := \Psi \circ \phi : D \to \mathbb{D}$, which is a conformal map with $\psi(b) = 1, \psi(p) = 0, \psi(a) = -1$. Let $\mathcal{D} = \mathcal{D}(r, R, \eta)$ be the collection of all quadruplets (D, a, b, p) such that $\mathrm{rad}_p(D) \geq r$ and $D \subset R\mathbb{D}$ and ψ^{-1} has analytic extension in $\{z \in \mathbb{C} : |z - 1| < \eta\}$.

In the rest of this section let r, R and η be arbitrarily fixed positive constants and suppose the same hypothesis of Theorem 2.5.1 to be valid. We write $\mathcal{D}$ for $\mathcal{D}(r, R, \eta)$ and consider $(D, a, b, p) \in \mathcal{D}$. For dealing with the martingale observable M_j defined above the following lemma plays a significant role and $\mathcal{D}(r, R, \eta)$ is introduced as a class for which the estimates given there is valid uniformly.

Lemma 2.5.2. *There exists a number $\lambda_0 = \lambda_0(\eta) > 1/2$ such that for any $\varepsilon > 0$ and $\lambda > \lambda_0$, there exists numbers $\delta_0 > 0$ and $\alpha \in (0, 1/2)$ such that if $(D, a, b, p) \in \mathcal{D}(r, R, \eta)$, $0 < \delta < \delta_0$ and $D' = D \setminus \phi^{-1}(\{z : |z| < 2\lambda\})$, then*

$$\mathbf{P}\left(\mathrm{Im}\, \phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_D) \in A\right) < \epsilon, \tag{2.12}$$

and, if $\mathrm{diam}(\phi(\gamma[0,j])) < 1$, *then*

$$\mathbf{P}\left(\mathrm{Im}\, \phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j\right) < \epsilon. \tag{2.13}$$

The proof of Lemma 2.5.2 is involved and postponed to the end of Section 6.

For any $\epsilon > 0$, let

$$m := \min\{j \geq 1 : t_j \geq \epsilon^2 \text{ or } |U_j - U_0| \geq \epsilon\}.$$

Lemma 2.5.3. *There exists a constant $C > 0$ and a number $\epsilon_0 > 0$ such that for each positive $\epsilon < \epsilon_0$, there exists $\delta_0 > 0$ such that if $(D, a, b, p) \in \mathcal{D}(r, R, \eta)$ and $0 < \delta < \delta_0$, then*

$$|\mathbf{E}[U_m - U_0]| \le C\epsilon^3,$$

and

$$|\mathbf{E}[(U_m - U_0)^2 - 2t_m]| \le C\epsilon^3.$$

(Although $U_0 = 0$, we write U_0 in the formulae above to indicate how they show be when the starting position $U_0 = \gamma_0$ is not mapped to the origin by ϕ.)

Proof. This proof is broken into four steps. It consists of certain estimations of the harmonic functions that constitutes the martingale observable defined by (2.11).

Step 1. In this step we derive an expression, given in (2.16) below, of the ratio

$$H_j^{(\delta)}(b, \gamma_j)/H_0^{(\delta)}(b; A).$$

We take sufficiently small $\epsilon_0 > 0$, which we need in this proof. Given $0 < \epsilon < \epsilon_0$ we take a number $\lambda = 1/\varepsilon^3$ that will be specified shortly. Let $D' := D \setminus \phi^{-1}(B(U_0, 2\lambda) \cap \mathbb{H})$ (note that $B(U_0, 2\lambda) = \{z : |z| < 2\lambda)\}$). In the following we consider for $j = 0, 1, 2, \ldots$, although we apply the resulting relation only for $j = 0, m$,

$$H_j^{(\delta)}(b, \gamma_j) = \sum_{y \in V_\delta(D)} \mathbf{P}(S_\delta^b(\tau_{D'}) = y, S_\delta^b(\tau_{D_j}) = \gamma_j)$$

We split the sum on the right-hand side into two parts according as y is close to the boundary of D or not. The part of those y which are close to the boundary must be negligible.

Proposition 2.2.3 and the definition of m imply that $\mathrm{diam}(\phi(\gamma[0, m-1])) = O(\epsilon)$. By Lemma 2.4.7, the harmonic measure from p of $\gamma[m-1, m]$ in D_m is $O(\epsilon)$ for sufficiently small $\delta > 0$. By conformal invariance of harmonic measure, the harmonic measure from $\phi_{m-1}(p)$ of $\phi_{m-1}(\gamma[m-1, m])$ in $\mathbb{H} \setminus \phi_{m-1}(\gamma[m-1, m])$ is $O(\epsilon)$. This implies that $\mathrm{diam}(\phi_{m-1}(\gamma[m-1, m])) = O(\epsilon)$, and we have

$$\mathrm{diam}(\phi(\gamma[0, m])) = O(\epsilon). \tag{2.14}$$

By (2.14) and Lemma 2.5.2, we can choose $\alpha = \alpha(\varepsilon) < 1/2$ so that for all sufficiently small $\delta > 0$, for $j = 0, m$,

$$\mathbf{P}\left(\mathrm{Im}\,\phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j\right) = O(\epsilon^3).$$

This implies

$$\begin{aligned}\frac{\mathbf{P}(\mathrm{Im}\,\phi(S_\delta^b(\tau_{D'})) < \alpha\lambda, S_\delta^b(\tau_{D_j}) = \gamma_j)}{\mathbf{P}(\mathrm{Im}\,\phi(S_\delta^b(\tau_{D'})) \ge \alpha\lambda, S_\delta^b(\tau_{D_j}) = \gamma_j)} &= \frac{\mathbf{P}(\mathrm{Im}\,\phi(S_\delta^b(\tau_{D'})) < \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j)}{\mathbf{P}(\mathrm{Im}\,\phi(S_\delta^b(\tau_{D'})) \ge \alpha\lambda \mid S_\delta^b(\tau_{D_j}) = \gamma_j)} \\ &= O(\epsilon^3).\end{aligned}$$

Therefore,

$$H_j^{(\delta)}(b, \gamma_j) = (1 + O(\epsilon^3)) \sum_{\substack{y \in V_\delta(D) \\ \mathrm{Im}\,\phi(y) \ge \alpha\lambda}} \mathbf{P}(S_\delta^b(\tau_{D'}) = y, S_\delta^b(\tau_{D_j}) = \gamma_j). \tag{2.15}$$

By strong Markov property,

$$\begin{aligned}\frac{\mathbf{P}(S_\delta^b(\tau_{D'})=y, S_\delta^b(\tau_{D_j})=\gamma_j)}{H_0^{(\delta)}(b;A)} &= \frac{\mathbf{P}(S_\delta^b(\tau_{D'})=y)\mathbf{P}(S_\delta^y(\tau_{D_j})=\gamma_j)}{\mathbf{P}(S_\delta^b(\tau_D)\in A)}\\ &= \frac{\mathbf{P}(S_\delta^b(\tau_{D'})=y)\mathbf{P}(S_\delta^y(\tau_D)\in A)}{\mathbf{P}(S_\delta^b(\tau_D)\in A)}\cdot\frac{\mathbf{P}(S_\delta^y(\tau_{D_j})=\gamma_j)}{\mathbf{P}(S_\delta^y(\tau_D)\in A)}.\end{aligned}$$

Therefore, (2.15) implies

$$\frac{H_j^{(\delta)}(b,\gamma_j)}{H_0^{(\delta)}(b;A)} = (1+O(\epsilon^3))\sum_{\substack{y\in V_\delta(D)\\ \operatorname{Im}\phi(y)\ge\alpha\lambda}}\mathbf{P}(S_\delta^b(\tau_{D'})=y \mid S_\delta^b(\tau_D)\in A)\cdot\frac{H_j^{(\delta)}(y,\gamma_j)}{H_0^{(\delta)}(y;A)}. \tag{2.16}$$

Step 2. Let $w\in V_\delta$ and $y\in V_\delta(D)$ satisfy

$$\operatorname{Im}\phi(w)\ge\frac{1}{2},\ |\phi(w)-U_0|\le 3;\quad \operatorname{Im}\phi(y)\ge\alpha\lambda,\ \lambda\le|\phi(y)-U_0|\le 2\lambda. \tag{2.17}$$

Applying Corollary 2.4.6 to the domain D with a reference point p,

$$\frac{H_0^{(\delta)}(w,\gamma_0)}{H_0^{(\delta)}(y,\gamma_0)} = \frac{\operatorname{Im}\phi(w)/|\phi(w)-U_0|^2}{\operatorname{Im}\phi(y)/|\phi(y)-U_0|^2}+O(\epsilon^3), \tag{2.18}$$

and the assumed invariance principle implies

$$H_0^{(\delta)}(y;A) = \frac{1}{\pi}\int_{-1}^{1}\frac{\operatorname{Im}\phi(y)}{|\phi(y)-x|^2}dx + O(\varepsilon^3\alpha/\lambda) \tag{2.19}$$

since $|\phi(y)-U_0|^2/\operatorname{Im}\phi(y)\le 2\lambda/\alpha$ (recall α/λ must get small together with ε). The relations (2.17), (2.18) and (2.19) together imply

$$\begin{aligned}\frac{H_0^{(\delta)}(w,\gamma_0)}{H_0^{(\delta)}(y,\gamma_0)}H_0^{(\delta)}(y;A) &= \frac{\operatorname{Im}\phi(w)}{\pi|\phi(w)-U_0|^2}\int_{-1}^{1}\frac{|\phi(y)-U_0|^2}{|\phi(y)-x|^2}dx+O(\epsilon^3)\\ &= \frac{2}{\pi}\frac{\operatorname{Im}\phi(w)}{|\phi(w)-U_0|^2}+O(\epsilon^3).\end{aligned} \tag{2.20}$$

From (2.16) and (2.20) we infer that

$$\frac{1}{M_0} = \frac{H_0^{(\delta)}(b,\gamma_0)}{H_0^{(\delta)}(b;A)H_0^{(\delta)}(w;\gamma_0)} = (1+O(\varepsilon^3))\sum_{\substack{y\in V_\delta(D)\\ \operatorname{Im}\phi(y)\ge\alpha\lambda}}p(y)\Big/\left[\frac{2}{\pi}\frac{\operatorname{Im}\phi(w)}{|\phi(w)-U_0|^2}+O(\epsilon^3)\right],$$

where $p(y)=\mathbf{P}(S_\delta^b(\tau_{D'})=y \mid S_\delta^b(\tau_D)\in A)$. In view of Lemma 2.5.2, we can suppose

$$\sum_{\substack{y\in V_\delta(D)\\ \operatorname{Im}\phi(y)\ge\alpha\lambda}}p(y) = 1+O(\epsilon^3), \tag{2.21}$$

by replacing α by smaller one if necessary. Since $\operatorname{Im}\phi(w)/|\phi(w)-U_0|^2$ is bounded by a universal constant, we now conclude

$$\begin{aligned} M_0 &= \frac{2}{\pi}\frac{\operatorname{Im}\phi(w)}{|\phi(w)-U_0|^2} + O(\epsilon^3) \\ &= \frac{2}{\pi}\operatorname{Im}\left(\frac{-1}{\phi(w)-U_0}\right) + O(\epsilon^3). \end{aligned} \tag{2.22}$$

Step 3. We derive an analogous formula for M_m. Lemma 2.2.3 and (2.14) imply

$$t_m = O(\epsilon^2), \quad |U(s)-U(0)| = O(\epsilon) \quad \text{for} \quad \forall s\in[0,t_m]. \tag{2.23}$$

The Loewner equation (2.1) shows that

$$|g_t(z)-z| \le t\cdot \sup_{0\le s\le t}\frac{2}{|g_s(z)-U(s)|}, \tag{2.24}$$

and, observing the imaginary part of the Loewner equation,

$$1 \ge \frac{\operatorname{Im} g_t(z)}{\operatorname{Im} z} \ge \exp\left(-t\cdot \sup_{0\le s\le t}\frac{2}{|g_s(z)-U(s)|^2}\right). \tag{2.25}$$

We also find $\frac{d}{dt}\operatorname{Im} g_t(z) \ge -2/\operatorname{Im} g_t(z)$, and this implies $\frac{d}{dt}(\operatorname{Im} g_t(z))^2 \ge -4$. By integrating this relation over $[0,t]$, we get $(\operatorname{Im} g_t(z))^2 \ge (\operatorname{Im} z)^2 - 4t$. Since $t_m = O(\epsilon^2)$, we have $\operatorname{Im} g_s\circ\phi(w) \ge 1/4$ for $0\le s\le t_m$. Therefore, (2.24) gives

$$|g_s\circ\phi(w)-\phi(w)| = O(\epsilon^2) \quad \text{for} \quad \forall s\in[0,t_m]. \tag{2.26}$$

Let $\sigma := \inf\{t\ge 0 : |g_t(z)-U(t)| \le \lambda/2\}$. Using (2.24), we get $|g_\sigma(z)-z| \le 4\sigma/\lambda$ and

$$|z-U(0)| \le \frac{4\sigma}{\lambda} + \frac{\lambda}{2} + |U(\sigma)-U(0)|.$$

Thus, if $|z-U(0)|>\lambda$, then $\sigma>t_m$. This implies $|g_s\circ\phi(y)-U(s)| \ge \lambda/2$ for $0\le s\le t_m$. Therefore, (2.24) and (2.25) lead to

$$|\phi_m(y)-\phi(y)| = O(\epsilon^3) \quad \text{and} \quad \frac{\operatorname{Im}\phi_m(y)}{\operatorname{Im}\phi(y)} = 1+O(\varepsilon^3). \tag{2.27}$$

(2.26) and (2.27) imply

$$\operatorname{Im}\phi_m(w) \ge \frac{1}{3},\ |\phi_m(w)-U_m| \le 4; \quad \operatorname{Im}\phi_m(y) \ge \frac{\alpha\lambda}{2},\ \frac{\lambda}{2} \le |\phi_m(y)-U_m| \le 3\lambda.$$

and it follows from Lemma 2.2.4 that $\operatorname{rad}_{p_m}(D_m) \ge r'$ for some $r'>0$. Therefore, we can apply Corollary 2.4.6 to the domain D_m with the reference point p_m, and hence the relation (2.19) implies

$$\begin{aligned} \frac{H_m^{(\delta)}(w,\gamma_m)}{H_m^{(\delta)}(y,\gamma_m)} H_0^{(\delta)}(y;A) =& \frac{\operatorname{Im}\phi_m(w)}{\pi|\phi_m(w)-U_m|^2}\int_{-1}^{1}\frac{\operatorname{Im}\phi(y)}{\operatorname{Im}\phi_m(y)}\cdot\frac{|\phi_m(y)-U_m|^2}{|\phi(y)-x|^2}dx \\ &+ O(\epsilon^3). \end{aligned}$$

Thus, from (2.16), (2.21) and (2.27) we get

$$M_m = \frac{2}{\pi}\mathrm{Im}\left(\frac{-1}{\phi_m(w) - U_m}\right) + O(\epsilon^3). \tag{2.28}$$

Step 4. Proposition 2.4.3 implies that M_j is a martingale. Because m is a bounded stopping time,

$$\mathbf{E}[M_m - M_0] = 0.$$

Thus, (2.22) and (2.28) lead to

$$\mathbf{E}\left[\mathrm{Im}\left(\frac{1}{\phi_m(w) - U_m}\right) - \mathrm{Im}\left(\frac{1}{\phi(w) - U_0}\right)\right] = O(\epsilon^3). \tag{2.29}$$

(2.23) and (2.26) imply

$$\frac{1}{g_s \circ \phi(w) - U(s)} = \frac{1}{\phi(w) - U_0} + O(\epsilon) \quad \text{for} \quad \forall s \in [0, t_m].$$

By integrating this relation over $[0, t_m]$, Loewner equation and (2.23) show that

$$\phi_m(w) = \phi(w) + \frac{2}{\phi(w) - U_0} \cdot t_m + O(\epsilon^3). \tag{2.30}$$

Let $f(u, v) = 1/(u - v)$. Using (2.23) and (2.30), we Taylor-expand $f(\phi_m(w), U_m) - f(\phi(w), U_0)$ with respect to $\phi_m(w) - \phi(w)$ and $U_m - U_0$, up to $O(\epsilon^3)$. Observing imaginary part of this Taylor expansion, from (2.29) and (2.30) we get

$$\mathrm{Im}\left(\frac{1}{(\phi(w) - U_0)^2}\right)\mathbf{E}[U_m - U_0] + \mathrm{Im}\left(\frac{1}{(\phi(w) - U_0)^3}\right)\mathbf{E}[(U_m - U_0)^2 - 2t_m] = O(\epsilon^3). \tag{2.31}$$

Now, we consider two different choices of w under the constraint $w \in V_\delta$ such that $\mathrm{Im}\,\phi(w) \geq \frac{1}{2}, |\phi(w)| \leq 3$. By the Koebe distortion theorem we can find w satisfying $\phi(w) - U_0 = i + O(\epsilon^3)$. Then, (2.31) implies

$$\mathbf{E}[(U_m - U_0)^2 - 2t_m] = O(\epsilon^3). \tag{2.32}$$

Similarly, we can find w satisfying $\phi(w) - U_0 = e^{i\frac{\pi}{3}} + O(\epsilon^3)$ and we get

$$\mathbf{E}[U_m - U_0] = O(\epsilon^3). \tag{2.33}$$

$\square$

As in Subsection 2.3, let $D(t) = D \setminus \gamma[0, t]$, $\phi_t = g_t \circ \phi$ and $p(t) = \phi_t^{-1}(i + U(t))$.

Lemma 2.5.4. *Let $T > 1$ and $\varepsilon > 0$, and, given a quadruplet $(D, a, b, p) \in \mathcal{D}$, put $\tilde{T} = \sup\{t \in [0, T] : |U(t)| < 1/\epsilon\}$. Then, there exists $\eta_1 = \eta_1(T, \epsilon) > 0$ and $r_1 = r_1(T, \varepsilon) > 0$ such that $(D(t), \gamma(t), b, p(t)) \in \mathcal{D}(r_1, R, \eta_1)$ for all $t < \tilde{T}$.*

Proof. Let $g_t^*(z) := g_t(z) - U(t)$. Put $\Psi(z) = (z-i)/(z+i)$. Define the conformal map $h_t : \mathbb{D} \setminus \psi(\gamma[0,t]) \to \mathbb{D}$ by

$$h_t(z) := \Psi \circ g_t^* \circ \Psi^{-1}(z).$$

Put $\psi_t(z) := h_t \circ \psi(z)$ so that $\psi_t : D(t) \to \mathbb{D}$ is a conformal map with $\psi_t(\gamma(t)) = -1, \psi_t(b) = 1, \psi_t(p(t)) = 0$. Clearly $\partial(\mathbb{D} \setminus \psi(\gamma[0,t]))$ is locally analytic at 1 and $h_t(1) = 1$. On using the Loewner equation we infer that $g_t'(z) = 1$ as $z \to \infty$, which implies $h_t'(1) = 1$. Now we can choose a positive $\eta_1 < \eta/4$ such that if $t < \tilde{T}$, then $\psi(\gamma[0,t])$ does not intersect with $B := \{z \in \mathbb{C} : |z-1| < 4\eta_1\}$. Thus, h_t is analytically extended to B for $t < \tilde{T}$, so that in view of Koebe's 1/4 theorem h_t^{-1} has an analytic extension in $\{z \in \mathbb{C} : |z-1| < \eta_1\}$ for $t < \tilde{T}$. Since $\psi_t^{-1} = \psi^{-1} \circ h_t^{-1}$ and ψ^{-1} is analytic on B, ψ_t^{-1} has an analytic extension in $\{z \in \mathbb{C} : |z-1| < \eta_1\}$ for $t < \tilde{T}$. The existence of r_1 is deduced from Lemma 2.2.4. Thus the assertion of the lemma has been proved. □

Proof of Theorem 2.5.1. Having proved Lemma 2.5.3 it is easy to adapt the arguments given in [11]. Let D be as in the theorem and take R so that $D \subset R\mathbb{D}$. Let $r := \mathrm{rad}_p(D)$. From our hypothesis of local analyticity of ∂D at b, the function ψ has an analytic extension in a neighborhood of b. Thus, we can choose $\eta > 0$ such that ψ^{-1} is analytic in $\{z \in \mathbb{C} : |z-1| < \eta\}$, hence $(D, a, b, p) \in \mathcal{D}(r, R, \eta)$.

Let $T > 1$ and $\epsilon_1 > 0$ and put $\tilde{T} = \sup\{t \in [0,T] : |U(t)| < 1/\epsilon_1\}$. Let $\epsilon > 0$ be small enough. Let $m_0 = 0$ and define m_n inductively by

$$m_n := \min\{j > m_{n-1} : t_j - t_{m_{n-1}} \geq \epsilon^2 \text{ or } |U_j - U_{m_{n-1}}| \geq \epsilon\}.$$

Let $N := \max\{n \in \mathbb{N} : t_{m_n} < \tilde{T}\}$. By Lemma 2.5.4, we can take some positive constants r_1 and η_1 such that $(D_{m_n}, \gamma_{m_n}, b, p_{m_n}) \in \mathcal{D}(r_1, R, \eta_1)$ for any $n \leq N$.

By the Markov property stated in Proposition 2.4.2, we find that $\gamma^{(t_{m_n})}(\cdot) = \gamma(t_{m_n} + \cdot)$ is the same distribution as the time-reversal of the loop erasure of a natural random walk on G_δ started at b_δ and stopped on exiting D_{m_n} and conditioned to hit ∂D_{m_n} at γ_{m_n}. We apply Lemma 2.5.3 with $(D_{m_n}, \gamma_{m_n}, b, p_{m_n})$ for any $n \leq N$. Then, we deduce from the fact stated at (2.3) that there exists $\delta_0 = \delta_0(\epsilon, \epsilon_1, T) > 0$ such that if $\delta < \delta_0$, then for any $n \leq N$

$$\mathbf{E}\Big[U_{m_{n+1}} - U_{m_n} \;\Big|\; \gamma[0, m_n]\Big] = O(\epsilon^3),$$

and

$$\mathbf{E}\Big[(U_{m_{n+1}} - U_{m_n})^2 \;\Big|\; \gamma[0, m_n]\Big] = \mathbf{E}\Big[2(t_{m_{n+1}} - t_{m_n}) \;\Big|\; \gamma[0, m_n]\Big] + O(\epsilon^3).$$

The rest of proof of Theorem 2.5.1 is the proof that $U(t)$ weakly converges to $\sqrt{2}B(t)$ uniformly on $[0,T]$ as $\delta \to 0$, where $B(t)$ is a one-dimensional standard Brownian motion with $B(0) = 0$. This proof follows from the above estimate and the Skorokhod embedding theorem as in [4] and [11]. (See Subsection 3.3 in [4] and Corollary 4.3 in [11].) □

2.5.2 Convergence with respect to the metric $d_{\mathcal{U}}$

Now, we assume that there exists an invariant measure π for a natural random walk $S(\cdot)$ on G such that $0 < \pi(v) < \infty$ for any $v \in V$. Let $p(u,v)$ be the transition probability for

$S(\cdot)$. We consider the dual walk $S^*(\cdot)$. The transition probability of $S^*(\cdot)$, denoted by $p^*(u,v)$, is given by

$$p^*(u,v) := \frac{\pi(v)}{\pi(u)} p(v,u).$$

Then, the dual walk $S^*(\cdot)$ is a natural random walk on some other planar irreducible graph. As in the case of $S(\cdot)$, we define $(S^*)^x_\delta, (\Gamma^*)^{a,b}_\delta, (\gamma^*)^{a,b}_\delta$ corresponding to $S^*(\cdot)$. The following lemma is a relation between the time-reversal and the dual walk.

Proposition 2.5.5. *Suppose that there exists an invariant measure π for a natural random walk $S(\cdot)$ on G such that $0 < \pi(v) < \infty$ for any $v \in V$. Then, the time-reversal of $\Gamma^{a,b}_\delta$ has the same distribution as $(\Gamma^*)^{b,a}_\delta$. Similarly, the time-reversal of $\gamma^{a,b}_\delta$ has the same distribution as $(\gamma^*)^{b,a}_\delta$.*

Proof. The first assertion immediately follows from the definition of the dual walk and the conditional probability. In addition to the first assertion, applying Proposition 2.4.1,

$$(\gamma^{a,b}_\delta)^- = L[\Gamma^{a,b}_\delta]^- \stackrel{d}{=} L[(\Gamma^{a,b}_\delta)^-] \stackrel{d}{=} L[(\Gamma^*)^{b,a}_\delta] = (\gamma^*)^{b,a}_\delta,$$

where $\stackrel{d}{=}$ means the same distribution. Hence, we get the second assertion. □

Let $\eta^{a,b}$ be a chordal SLE_2 curve in D from a to b. Recall the metric $d_{\mathcal{U}}$ defined by (2.6) in Subsection 2.4.

Theorem 2.5.6. *Suppose that there exists an invariant measure π for a natural random walk $S(\cdot)$ on G such that $0 < \pi(v) < \infty$ for any $v \in V$ and S^x_δ and $(S^*)^x_\delta$ satisfy invariance principle. Let D be a bounded simply connected domain and $a, b \in \partial D$ be distinct points. Suppose that ∂D is locally analytic at a and b. Then, $\gamma^{a,b}_\delta$ converges weakly to $\eta^{a,b}$ as $\delta \to 0$ with respect to the metric $d_{\mathcal{U}}$.*

Proof. Let $\phi : D \to \mathbb{H}$ be a conformal map with $\phi(a) = 0, \phi(b) = \infty$ and Let $\phi^- : D \to \mathbb{H}$ be a conformal map with $\phi(b) = 0, \phi(a) = \infty$. Theorem 2.5.1 implies that $\phi^- \circ (\gamma^{a,b}_\delta)^-$ converges weakly to a chordal SLE_2 with respect to the driving function. Because we also assume that $(S^*)^x_\delta$ satisfy invariance principle, Theorem 2.5.1 implies that $\phi \circ ((\gamma^*)^{b,a}_\delta)^-$ converges weakly to a chordal SLE_2 with respect to the driving function. By Proposition 2.5.5, $\gamma^{a,b}_\delta$ is the same distribution as $((\gamma^*)^{b,a}_\delta)^-$. Hence, $\phi \circ \gamma^{a,b}_\delta$ converges weakly to a chordal SLE_2 with respect to the driving function. Therefore, Theorem 2.3.6 completes the proof. □

2.6 Estimates of hitting probabilities of the random walk

In this section we prove Lemma 2.5.2. To this end it is convenient to work in the disc $\mathbb{D}$ instead of $\mathbb{H}$. Let $D \subsetneq \mathbb{C}$ be a simply connected domain and a, b be distinct points on ∂D. Let $\phi : D \to \mathbb{H}$ be a conformal map with $\phi(a) = 0, \phi(b) = \infty$. Let $p := \phi^{-1}(i)$. Put $\Psi(z) = (z-1)/(z+1)$ and $\psi = \Psi \circ \phi$ so that ψ is a conformal map of D onto $\mathbb{D}$ with $\psi(a) = -1, \psi(b) = 1, \psi(p) = 0$. Let S^b_δ be a natural random walk on G_δ started at b_δ, where b_δ is a point of $\partial_{in} V_\delta(D)$ close to b.

Recall the class $\mathcal{D}(r, R, \eta_0)$, which is the collection of all quadruplets (D, a, b, p) such that $\mathrm{rad}_p(D) \geq r$ and $D \subset R\mathbb{D}$ and ψ^{-1} has analytic extension in $\{z \in \mathbb{C} : |z-1| < \eta_0\}$. Throughout this section we consider the constants r, R and η_0 to be fixed and write $\mathcal{D}$ for $\mathcal{D}(r, R, \eta_0)$; also suppose that S^x_δ satisfies invariance principle.

For $(D, a, b, p) \in \mathcal{D}$ and $\eta < \eta_0 \wedge \frac{1}{2}$ put

$$U = U_\eta = \{z \in D : |\psi(z) - 1| < \eta\}$$

and for any number α from the open interval $(0, 1/2)$,

$$J_\alpha = \{z \in \partial U : \mathrm{dist}(\psi(z), \partial\mathbb{D}) < \alpha\eta, z \in D\}.$$

Proposition 2.6.1. *Let $U = U_\eta$ and J_α be as described above. Then for any $\varepsilon > 0$ there exists $\delta_0 = \delta(\varepsilon, \eta) > 0$ such that for all positive $\delta < \delta_0, \alpha < \delta_0$ and for all $(D, a, b, p) \in \mathcal{D}$,*

$$\mathbf{P}(S^b_\delta(\tau_U) \in J_\alpha \mid S^b_\delta(\tau_U) \in D) < \epsilon,$$

Here δ_0 may depend on the graph (V, E).

Remark. It is only for this proposition that we need the condition of the analyticity about b. Without that condition the estimate of the proposition is obtained by Uchiyama [14].

Proof. This proof is an adaptation of a part of the arguments given in [14]. Put

$$C = \{z \in \partial D : \mathrm{Im}\,\psi(z) > 0, |\psi(z) - 1| < \eta/3\},$$

and

$$B = \{z \in \mathbb{C} : |\psi(z) - 1| < \eta/3\} \setminus \overline{U}, \quad \Omega = B \cup C \cup U.$$

Let

$$C_\delta = \{v \in V_\delta(D) : [u, v] \cap C \neq \emptyset \text{ for some } u \in V_\delta(B)\},$$

and v^* be a vertex in C_δ such that $\mathrm{Im}\,\psi(v^*)$ is closest to $\eta/6$ among vertexes of C_δ.

Let L denote the last time when the walk $S^{v^*}_\delta$ in Ω killed when it crosses the boundary $\partial\Omega$ exits B:

$$L = \begin{cases} 1 + \max\{0 \leq n < \tau_\Omega : S^{v^*}_\delta(n) \in B\} & \text{if} \quad S^{v^*}_\delta(\tau_\Omega) \notin \partial B \\ \infty \quad \text{if} \quad S^{v^*}_\delta(\tau_\Omega) \in \partial B \end{cases}.$$

We write $T = \tau_U$. Putting $J^+_\alpha = J_\alpha \cap \mathbb{H}$ we compute $q = \mathbf{P}(S^{v^*}_\delta(\tau_\Omega) \in J^+_\alpha)$, the probability that the walk exits Ω through J^+_α, which we rewrite as

$$q = \mathbf{P}(S^{v^*}_\delta(T) \circ \theta_L \in J^+_\alpha, L < \tau_\Omega),$$

where the shift operator θ_L acts on T as well as on $S^{v^*}_\delta$. By employing the strong Markov property

$$\begin{aligned} q &= \sum_{n=0}^{\infty} \sum_{y \in C_\delta} \mathbf{P}(S^{v^*}_\delta(T) \circ \theta_n \in J^+_\alpha, L = n, S^{v^*}_\delta(n) = y) \\ &= \sum_{n=0}^{\infty} \sum_{y \in C_\delta} \mathbf{P}(S^{v^*}_\delta(T) \circ \theta_n \in J^+_\alpha, S^{v^*}_\delta(n) = y) \\ &= \sum_{n=0}^{\infty} \sum_{y \in C_\delta} \mathbf{P}(S^{v^*}_\delta(n) = y) \mathbf{P}(S^y_\delta(T) \in J^+_\alpha) \end{aligned}$$

The occurrence of the event $S_\delta^y(T) \in J_\alpha^+$ for $y \in C_\delta$ entails $S_\delta^y(T) \in D$, so that $\mathbf{P}(S_\delta^y(T) \in J_\alpha^+) = \mathbf{P}(S_\delta^y(T) \in J_\alpha^+, S_\delta^y(T) \in D)$. Hence, bringing in the conditional probability

$$p(y) = \mathbf{P}(S_\delta^y(T) \in J_\alpha^+ \mid S_\delta^y(T) \in D),$$

we infer that

$$q = \sum_{y \in C_\delta} G_\Omega(v^*, y)\mathbf{P}(S_\delta^y(T) \in D)p(y),$$

where G_Ω stands for the Green function of the walk killed on exiting Ω. We have

$$p(y) \geq p(b), \quad y \in C_\delta,$$

for, if γ^b denote a path joining b_δ with J_α^+ in $V_\delta(U)$, then the walk starting at $y \in C_\delta$ and conditioned on the event $S_\delta^y(T) \in D$ must hit $\gamma^b \cup J_\alpha^+$ before existing U. Observing the identity

$$\sum_{y \in C_\delta} G_\Omega(v^*, y)\mathbf{P}(S_\delta^y(T) \in D) = \mathbf{P}(S_\delta^{v^*}(\tau_\Omega) \in D),$$

we finally obtain

$$q \geq p(b)\mathbf{P}(S_\delta^{v^*}(\tau_\Omega) \in D).$$

This concludes $p(b) < \epsilon/2$ since $\mathbf{P}(S_\delta^{v^*}(\tau_\Omega) \in D) > 1/3$ and $q < \epsilon/6$ for all sufficiently small δ and α. Let $J_\alpha^- = J_\alpha \setminus J_\alpha^+$. On defining C with $\operatorname{Im}\psi(z) \leq 0$ in place of $\operatorname{Im}\psi(z) > 0$ we repeat the same argument to show that $\mathbf{P}(S_\delta^b(T) \in J_\alpha^- \mid S_\delta^b(T) \in D) < \epsilon/2$. □

Lemma 2.6.2. *Let $A := \phi^{-1}([-1, 1])$. For any $\epsilon > 0$, there exists $\delta_0 = \delta_0(\epsilon, \eta) > 0$ such that the following holds. Let $(D, a, b, p) \in \mathcal{D}$. Then, for all $0 < \delta < \delta_0$ and $0 < \alpha < \delta_0$,*

$$\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha \mid S_\delta^b(\tau_D) \in A) < \epsilon.$$

Proof. By the definition of the conditional probability and the strong Markov property,

$$\begin{aligned}
\frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha \mid S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_\alpha \mid S_\delta^b(\tau_D) \in A)} &= \frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha, S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_\alpha, S_\delta^b(\tau_D) \in A)} \\
&= \frac{\sum_{y \in J_\alpha} \mathbf{P}(S_\delta^b(\tau_U) = y)\mathbf{P}(S_\delta^y(\tau_D) \in A)}{\sum_{y \notin J_\alpha} \mathbf{P}(S_\delta^b(\tau_U) = y)\mathbf{P}(S_\delta^y(\tau_D) \in A)}.
\end{aligned}$$

Because we assume invariance principle, the hitting probability $\mathbf{P}(S_\delta^y(\tau_D) \in A)$ can be approximated by the same probability for a Brownian motion. Because the hitting probability for a Brownian motion is conformal invariant, we can calculate the hitting probability on the upper half plane instead of D. Therefore, we find that there exists a universal constant C such that for sufficiently small δ,

$$\frac{\sup_{y \in J_\alpha} \mathbf{P}(S_\delta^y(\tau_D) \in A)}{\inf_{y \notin J_\alpha} \mathbf{P}(S_\delta^y(\tau_D) \in A)} \leq C.$$

Thus, we obtain

$$\frac{\mathbf{P}(S_\delta^b(\tau_U) \in J_\alpha \mid S_\delta^b(\tau_D) \in A)}{\mathbf{P}(S_\delta^b(\tau_U) \notin J_\alpha \mid S_\delta^b(\tau_D) \in A)} \leq C \frac{\sum_{y \in J_\alpha} \mathbf{P}(S^b(\tau_U) = y)}{\sum_{y \notin J_\alpha} \mathbf{P}(S^b(\tau_U) = y)}.$$

Because

$$\frac{\sum_{y\in J_\alpha}\mathbf{P}(S^b(\tau_U)=y)}{\sum_{y\notin J_\alpha}\mathbf{P}(S^b(\tau_U)=y)}=\frac{\mathbf{P}(S^b_\delta(\tau_U)\in J_\alpha\mid S^b_\delta(\tau_U)\in D)}{\mathbf{P}(S^b_\delta(\tau_U)\notin J_\alpha\mid S^b_\delta(\tau_U)\in D)},$$

Proposition 2.6.1 completes the proof. □

Lemma 2.6.3. *For any $\epsilon>0$, there exists $\delta_0=\delta_0(\epsilon,\eta)>0$ such that the following holds. Let $(D,a,b,p)\in\mathcal{D}$. Then, for all $0<\delta<\delta_0$ and $0<\alpha<\delta_0$,*

$$\mathbf{P}(S^b_\delta(\tau_U)\in J_\alpha\mid S^b_\delta(\tau_D)=a_\delta)<\epsilon,$$

where a_δ is a point of $\partial_{out}V_\delta(D)$ close to a.

Proof. By the definition of the conditional probability,

$$\begin{aligned}\frac{\mathbf{P}(S^b_\delta(\tau_U)\in J_\alpha\mid S^b_\delta(\tau_D)=a_\delta)}{\mathbf{P}(S^b_\delta(\tau_U)\notin J_\alpha\mid S^b_\delta(\tau_D)=a_\delta)}&=\frac{\mathbf{P}(S^b_\delta(\tau_U)\in J_\alpha,S^b_\delta(\tau_D)=a_\delta)}{\mathbf{P}(S^b_\delta(\tau_U)\notin J_\alpha,S^b_\delta(\tau_D)=a_\delta)}\\&=\frac{\mathbf{P}(S^b_\delta(\tau_U)\in J_\alpha,S^b_\delta(\tau_D)=a_\delta\mid S^b_\delta(\tau_D)\in A)}{\mathbf{P}(S^b_\delta(\tau_U)\notin J_\alpha,S^b_\delta(\tau_D)=a_\delta\mid S^b_\delta(\tau_D)\in A)}.\end{aligned}$$

Since the random walk conditioned on exiting D through A is Markovian, the right-hand side above may be written as

$$\frac{\sum_{y\in J_\alpha}\mathbf{P}(S^b_\delta(\tau_U)=y|S^b_\delta(\tau_D)\in A)\mathbf{P}(S^y_\delta(\tau_D)=a_\delta|S^y_\delta(\tau_D)\in A)}{\sum_{y\notin J_\alpha}\mathbf{P}(S^b_\delta(\tau_U)=y|S^b_\delta(\tau_D)\in A)\mathbf{P}(S^y_\delta(\tau_D)=a_\delta|S^y_\delta(\tau_D)\in A)}.$$

By Lemma 5.8. in [15], there exists a universal constant C such that for sufficiently small δ,

$$\frac{\sup_{y\in J_\alpha}\mathbf{P}(S^y_\delta(\tau_D)=a_\delta\mid S^y_\delta(\tau_D)\in A)}{\inf_{y\notin J_\alpha}\mathbf{P}(S^y_\delta(\tau_D)=a_\delta\mid S^y_\delta(\tau_D)\in A)}\le C.$$

Hence, we obtain

$$\frac{\mathbf{P}(S^b_\delta(\tau_U)\in J_\alpha\mid S^b_\delta(\tau_D)=a_\delta)}{\mathbf{P}(S^b_\delta(\tau_U)\notin J_\alpha\mid S^b_\delta(\tau_D)=a_\delta)}\le C\frac{\mathbf{P}(S^b_\delta(\tau_U)\in J_\alpha\mid S^b_\delta(\tau_D)\in A)}{\mathbf{P}(S^b_\delta(\tau_U)\notin J_\alpha\mid S^b_\delta(\tau_D)\in A)}$$

Therefore, Lemma 2.6.2 completes the proof. □

Proof of Lemma 2.5.2. By the mapping $\Psi(z)=(z-i)/(z+i)$, the half disc $B_+(2\lambda):=B(U_0,2\lambda)\cap\mathbb{H}$ is mapped to a small disc of radius $\sim 1/2\lambda$ and centered at 1. For $1/2\lambda<\eta_0$, (2.12) follows from applying Lemma 2.6.2 with this small disc in place of U_η, the little discrepancy between them making no harm. If $\mathrm{diam}(\phi(\gamma[0,j]))<1$, the difference between $B_+(2\lambda)$ and $g_{t_j}(B_+(2\lambda))$ is insignificant for sufficiently large λ. Hence, we also have (2.13) by applying Lemma 2.6.3 with (D_j,γ_j,b,p_j), which is legitimate because of Lemma 2.5.4. □

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