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SELF-SIMILAR SOLUTIONS AND TRANSLATING SOLITONS FOR LAGRANGIAN MEAN CURVATURE FLOW, AND MEAN CURVATURE FLOW IN SUBMANIFOLDS

by

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Preface

This thesis is based on two chapters:

Chapter 1. "Some examples of self-similar solutions and translating solitons for Lagrangian mean curvature flow",

Chapter 2. "Mean curvature flow in submanifolds".

A revised version of the author's master thesis is written in Chapter 1 and a new result of his research is in Chapter 2. The former constructs new self-similar solutions and translating solitons for the Lagrangian mean curvature flow. One of these constructions has a relationship with the result of Lotay and Neves [11] which says that Lagrangian self-expanders with zero Maslov class in $\mathbb{C}^n$ which are asymptotic to a pair of planes transversely intersecting are locally unique or unique, where a Lagrangian self-expander is a Lagrangian submanifold L, to be defined below, of which the mean curvature vector in $\mathbb{C}^n$ is equal to $\alpha F^\perp$ for some $\alpha > 0$ and $F^\perp$ is the projection of the position vector F in $\mathbb{C}^n$ to the normal bundle of L. Our construction shows that without the smoothness assumption of the Lagrangian submanifolds the uniqueness does not hold, where a Lagrangian submanifold is a real n-dimensional submanifold in $\mathbb{C}^n$ on which the standard symplectic form $\sum_{j=1}^n dx_j \wedge dy_j$ vanishes. The latter gives some explicit mean curvature flows on the inside of some Lagrangian submanifolds which are explained in the next page of this preface.

The author was very lucky to find his study of the mean curvature flow, to be explained later. He began to investigate the mean curvature flow, the Lagrangian mean curvature flow, of which name is based on the fact that the mean curvature flow preserves the Lagrangian condition defined above, and their self-similar solutions about four years ago by an introduce of his supervisor. The author thought that this topic is natural and it is worth investigating it. He has read many articles and books and has learned many things of it and its neighborhood since then.

For example, he studied Joyce's constructions of special Lagrangian submanifolds [6] and found that Medoš and Wang discovered the following fantastic result [13]. If a symplectomorphism f of $\mathbb{CP}^n$ has a pinched condition then the Lagrangian mean curvature flow of the graph $\{\Sigma_t\}_t$ in $(\mathbb{CP}^n \times \mathbb{CP}^n, \pi_1^*\omega_{FS} - \pi_2^*\omega_{FS})$ converges smoothly to a graph of a biholomorphic isometry as $t \to \infty$, where π_1 and π_2 are the projections to the first and second factors and ω_{FS} is the Fubuni-Study metric. Wang studied the mean curvature flow of graphs of maps between riemaniann manifolds and he found many theorems [16]. Huisken also showed wonderful theorems that if a hypersurface in $\mathbb{R}^n$ has some convex condition then it's mean curvature flow converges to a single point and a rescaling limit at the point is a sphere [3], and any central blow-up of finite time singularity of the mean curvature flow is a self-similar solution [4]. Joyce, Lee and Tsui construct explicit self-similar solutions and traslating solitons in $\mathbb{C}^n$, which are Lagrangian submanifolds, in [8] by the method improving Joyce's construction of special Lagrangian submanifolds [6]. Lee and Wang gave constructions of noncompact eternal solutions for Brakke flow $\{V_t\}_{t\in\mathbb{R}}$ that is a generalization of the mean curvature flow in [10] by using that kind of self-similar solutions. In fact, self-similar solutions are classified as self-shrinkers and self-expanders. The former ones are the solutions of the mean curvature flow which are shrinking under preserving a condition of similar figures. The later ones are also the solutions of the flow which are expanding under the condition. Their eternal solutions for Brakke flow $\{V_t\}_{t\in\mathbb{R}}$ glue self-shrinkers $\{V_t\}_{t<0}$ and self-expanders $\{V_t\}_{t>0}$ together at $t = 0$. Since mean curvature flow is a volume decreasing flow, we can see that V_0 which is a Schoen-Wolfson cone is not area-minimizing [9]. This result has analogies to the Feldman-Ilmanen-Knopf gluing construction for the Kähler-Ricci flows [2]. (We often contrast mean curvature flow with Ricci flow.) Neves and Tian showed the important theorems that translating solutions to the Lagrangian mean curvature flow with an L^2 bound on the mean curvature are planes and almost calibrated translating solutions to the flow which are static are also planes in [14]. Moreover the work of Joyce, Lee and Tsui [8] shows that these conditions are optimal.

There are many examples of self-similar solutions and translating solitons in the Euclidean space. Many facts of the mean curvature flow and their proofs are given in [1].

In this thesis, we always consider submanifolds of the Euclidean space. Mean curvature flow is the smoothly moving submanifolds which goes to the direction of those mean curvature vectors. This is the most important flow in all flows of submanifolds. It is known that mean curvature flow appeared from the study of annealing metals in physics. So mean curvature flow has strong relationship with physics.

Now we start to consider the following submanifold L. Let Σ be a hypersurface on $\mathbb{R}^n$ and φ_s a one-parameter family of immersions form $\mathbb{R}^n$ to $\mathbb{C}^n$, where $s \in \mathbb{R}$. So we can write $\Sigma \subset \mathbb{R}^n$ and $\varphi_s : \mathbb{R}^n \to \mathbb{C}^n$. We define the submanifold L in $\mathbb{C}^n$ by

$$L = \bigcup_s \varphi_s(\Sigma).$$

This is a submanifold constructed by sweeping Σ out in $\mathbb{C}^n$ by φ_s. Since $(n-1)+1 = n$, we get a real n-dimensional submanifold in $\mathbb{C}^n$. It is difficult to compute the necessary and sufficient condition of the submanifold being Lagrangian. Joyce considered the family of linear or affine maps for φ_s and completed some special Lagrangians which is minimal Lagrangian submanifolds in [6]. For a very simple case, we put $\Sigma = \mathcal{S}^{n-1} \subset \mathbb{R}^n$ which is the round sphere of radius one with origin $0 \in \mathbb{R}^n$ and

$$\varphi_s = \begin{pmatrix} w(s) & 0 & \cdots & 0 \\ 0 & w(s) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & w(s) \end{pmatrix}, \tag{1}$$

where w is a function from an interval I to $\mathbb{C} \setminus \{0\}$. Then

$$\begin{aligned} L = \bigcup_{s \in I} \varphi_s(\Sigma) &= \{\varphi_s(x) \in \mathbb{C}^n; s \in I,\, x = (x_1, \cdots x_n) \in \mathbb{R}^n,\, \sum_{j=1}^n x_j^2 = 1\} \\ &= \{(x_1 w(s), \ldots, x_n w(s)) \in \mathbb{C}^n; s \in I,\, x_1, \cdots x_n \in \mathbb{R},\, \sum_{j=1}^n x_j^2 = 1\}. \end{aligned}$$

This submanifold L is Lagrangian for any smooth function $w : I \to \mathbb{C} \setminus \{0\}$ such that $\dot{w}(s) \neq 0$ for all $s \in I$. General settings of this example are considered in Chapter 1 and this example appears in Chapter 2. Next we consider one-parameter family of immersions ψ_s, $s \in \mathbb{R}$, from $\mathbb{R}^{n-1}$ to $\mathbb{C}^n$ rather than $\varphi_s : \mathbb{R}^n \to \mathbb{C}^n$ and a hypersurface Σ. Then we also obtain some real n-dimensional submanifolds

$$L' = \bigcup_s \psi_s(\mathbb{R}^{n-1}).$$

Let ψ_s be maps defined by

$$\psi_s \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} x_1 w(s) \\ \vdots \\ x_{n-1} w(s) \\ -1/2 \sum_{k=1}^{n-1} x_k^2 + \int \overline{w(s)} \dot{w}(s) ds \end{pmatrix},$$

where w is a function from an interval I to $\mathbb{C} \setminus \{0\}$. Then we can easily see that $L' = \bigcup_{s \in I} \psi(\mathbb{R}^{n-1})$ satisfies the Lagrangian condition and can find a function w that gives L' the property of translating soliton. The submanifold can be found in Chapter 1.

2014,
Hiroshi Nakahara

Chapter 1

SOME EXAMPLES OF SELF-SIMILAR SOLUTIONS AND TRANSLATING SOLITONS FOR LAGRANGIAN MEAN CURVATURE FLOW

HIROSHI NAKAHARA

Abstract. We construct examples of self-similar solutions and translating solitons for Lagrangian mean curvature flow by extending the method of Joyce, Lee and Tsui. Those examples include examples in which the Lagrangian angle is arbitrarily small as the examples of Joyce, Lee and Tsui. The examples are non-smooth Lagrangian self-expanders which are zero-Maslov class and asymptotic to a pair of planes transversely intersecting.

1. Introduction

In recent years the Lagrangian mean curvature flow has been extensively studied, as it is a key ingredient in the Strominger-Yau-Zaslow Conjecture [18] and Thomas-Yau Conjecture [19]. Strominger-Yau-Zaslow Conjecture explains Mirror Symmetry of Calabi-Yau 3-folds. In [8], Joyce, Lee and Tsui constructed many examples of self-similar solutions and translating solitons for Lagrangian mean curvature flow. Those Lagrangian submanifolds L are the total space of a 1-parameter family of quadrics Q_s, $s \in I$, where I is an open interval in $\mathbb{R}$. In this paper, we construct examples of those Lagrangian submanifolds that associate with the examples of Lagrangian submanifolds given in [7], [8], [9], [10] and so on. To do so we improve theorems in [8] by describing Lagrangian submanifolds of the forms of [8, Ansatz 3.1 and Ansatz 3.3].

Our ambient space is always the complex Euclidean space $\mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$ and the standard symplectic form $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$. A *Lagrangian submanifold* L is a real n-dimensional submanifold in $\mathbb{C}^n$ on which the symplectic form ω vanishes. On L, we can define *Lagrangian angle* $\theta : L \to \mathbb{R}$ or $\theta : L \to \mathbb{R}/2\pi\mathbb{Z}$ by the relation

$$dz_1 \wedge \cdots \wedge dz_n|_L \equiv e^{i\theta}\mathrm{vol}_L,$$

and the mean curvature vector H by

$$H = J\nabla\theta, \tag{2}$$

where ∇ is the gradient on L and J is the standard complex structure in $\mathbb{C}^n$. Equation (2) implies that a Lagrangian submanifold remains Lagrangian under the mean curvature flow, as in Smoczyk [17]. The Maslov class on L is defined by the cohomology class of $d\theta$. Hence L is zero-Maslov class when θ is a single-valued function. A Lagrangian submanifold L is called *Hamiltonian stationary* if the Lagrangian angle θ is harmonic, that is, if $\Delta\theta = 0$, and L is called a *special Lagrangian submanifold* if θ is a constant function. A Hamiltonian stationary Lagrangian submanifold is a critical point of the volume functional among all Hamiltonian deformations, and a special Lagrangian is a volume minimizer in its homology class.

DEFINITION 1.1. Let $L \subset \mathbb{R}^N$ be a submanifold in $\mathbb{R}^N$. L is called a *self-similar solution* if $H \equiv \alpha F^{\perp}$ on L for some constant $\alpha \in \mathbb{R}$, where $F^{\perp}$ is the orthogonal projection of the position vector F in $\mathbb{R}^N$ to the normal bundle of L, and H is the mean curvature vector of L in $\mathbb{R}^N$. It is called a *self-shrinker* if $\alpha < 0$ and a *self-expander* if $\alpha > 0$. On the other hand $L \subset \mathbb{R}^N$ is called a *translating soliton* if there exists a constant vector T in $\mathbb{R}^N$ such that $H \equiv T^{\perp}$, where $T^{\perp}$ is the orthogonal projection of the constant vector T in $\mathbb{R}^N$ to the normal bundle of L, and we call T a *translating vector*.

It is well known that if F is a self-similar solution then $F_t = \sqrt{2\alpha t}F$ is moved by the mean curvature flow, and if F is a translating soliton then $F_t = F + tT$ is moved by the mean curvature flow. By Huisken [4], any central blow-up of a finite-time singularity of the mean curvature flow is a self-similar solution.

First we consider self-similar solutions.

THEOREM 1.2. *Let $C, \lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus \{0\}$, $\alpha, \psi_1, \ldots, \psi_n \in \mathbb{R}$, $a_1, \ldots, a_n > 0$, and $E > 1$ be constants. Let $I \subset \mathbb{R}$ be a connected open neighborhood of $0 \in \mathbb{R}$ such that $\inf_{s\in I}(E\{\prod_{k=1}^{n}(1 + a_k\lambda_k s)\}e^{\alpha s} - 1)$ and $\inf_{s\in I}(1/a_j + \lambda_j s)$ are positive for any $1 \le j \le n$. Define $r_1, \ldots, r_n : I \to \mathbb{R}$ by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + \lambda_j s}, \quad j = 1, \ldots, n, \tag{3}$$

and $\phi_1, \ldots, \phi_n : I \to \mathbb{R}$ by

$$\phi_j(s) = \psi_j + \frac{\lambda_j}{2}\int_0^s \frac{dt}{(1/a_j + \lambda_j t)\sqrt{E\{\prod_{k=1}^{n}(1 + a_k\lambda_k t)\}e^{\alpha t} - 1}}, \tag{4}$$

$j = 1, \ldots, n$. Then the submanifold L in $\mathbb{C}^n$ given by

$$L = \{(x_1 r_1(s)e^{i\phi_1(s)}, \ldots, x_n r_n(s)e^{i\phi_n(s)}); \sum_{j=1}^{n} \lambda_j x_j^2 = C, x_j \in \mathbb{R}, s \in I\}$$

is an immersed Lagrangian submanifold diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m+1}$, where m is the number of positive λ_j/C, $1 \le j \le n$, and the mean curvature vector H satisfies $CH \equiv \alpha F^{\perp}$ for the position vector F. That is, L is a self-expander when $\alpha/C > 0$ and a self-shrinker when $\alpha/C < 0$. When $\alpha = 0$ the Lagrangian angle θ is constant, so that L is special Lagrangian.

The following Theorem 1.3 is slightly generalized from [8, Theorem C].

Theorem 1.3. *Let* $a_1, \ldots, a_n > 0$, $\psi_1, \ldots, \psi_n \in \mathbb{R}, E \geq 1$, *and* $\alpha \geq 0$ *be constants. Define* $r_1, \ldots, r_n : \mathbb{R} \to \mathbb{R}$ *by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + s^2}, \tag{5}$$

and $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$ *by*

$$\phi_j(s) = \psi_j + \int_0^s \frac{|t|dt}{(1/a_j + t^2)\sqrt{E\{\prod_{k=1}^n (1 + a_k t^2)\}e^{\alpha t^2} - 1}}. \tag{6}$$

Then the submanifold L *in* $\mathbb{C}^n$ *given by*

$$L = \{(x_1 r_1(s) e^{i\phi_1(s)}, \ldots, x_n r_n(s) e^{i\phi_n(s)}); \sum_{j=1}^n x_j^2 = 1, x_j \in \mathbb{R}, s \in \mathbb{R},\ s \neq 0\} \tag{7}$$

is an embedded Lagrangian diffeomorphic to $(\mathbb{R} \setminus \{0\}) \times \mathcal{S}^{n-1}$, *and the mean curvature vector* H *satisfies* $H \equiv \alpha F^{\perp}$, *where* F *is the position vector of* L. *If* $\alpha > 0$, *it is a self-expander, and if* $\alpha = 0$ *it is special Lagrangian. When* $E = 1$ *the construction reduces to that of Joyce, Lee and Tsui* [8, Theorem C]. *So the condition* $s \neq 0$ *on the definition of* L *is not necessary if* $E = 1$.

Remark 1.3.1. In the situation of Theorem 1.3, define $\bar{\phi}_1, \ldots, \bar{\phi}_n > 0$ by

$$\bar{\phi}_j = \int_0^\infty \frac{|t|dt}{(1/a_j + t^2)\sqrt{E\{\prod_{k=1}^n (1 + a_k t^2)\}e^{\alpha t^2} - 1}}.$$

We put $\alpha > 0$ and $E > 0$. From (15), the third equation of (14) and the proof of Theorem 1.3, the Lagrangian angle θ satisfies

$$\begin{aligned} \theta(s) &= \sum_j \phi_j(s) + \arg(s + i\frac{|s|}{\sqrt{E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1}}) \quad \text{and} \\ \dot{\theta}(s) &= \frac{-\alpha|s|}{\sqrt{E\{\prod_{k=1}^n (1 + a_k s^2)\}e^{\alpha s^2} - 1}}. \end{aligned} \tag{8}$$

It follows that θ is strictly decreasing. We define the submanifolds L_1 and L_2 of L so that $s > 0$ on L_1, and $s < 0$ on L_2, respectively. Therefore we have $L = L_1 \cup L_2$. We rewrite θ_1, θ_2 as the Lagrangian angle of L_1, L_2, respectively. Then $\lim_{s \to +\infty} \theta_1(s) < \theta_1(s) < \lim_{s \to +0} \theta_1(s)$ and $\lim_{s \to -0} \theta_2(s) < \theta_2(s) < \lim_{s \to -\infty} \theta_2(s)$ hold. So from the first equation of (8) we have

$$\begin{aligned} &\sum_j \psi_j + \sum_j \bar{\phi}_j < \theta_1(s) < \sum_j \psi_j + \tan^{-1}\frac{1}{\sqrt{E-1}} \quad \text{and} \\ &\sum_j \psi_j + \pi - \tan^{-1}\frac{1}{\sqrt{E-1}} < \theta_2(s) < \sum_j \psi_j + \pi - \sum_j \bar{\phi}_j. \end{aligned} \tag{9}$$

Therefore we can make the oscillations of the Lagrangian angles of L_1 and L_2 arbitrarily small by taking E close to ∞ and hence $\tan^{-1}(1/\sqrt{E-1})$ close to 0. Furthermore, we can prove that the map

$$\Phi : (0,\infty)^n \to \{(y_1,\dots,y_n) \in (0,\tan^{-1}\frac{1}{\sqrt{E-1}})^n;\ 0 < \sum_{j=1}^n y_j < \tan^{-1}\frac{1}{\sqrt{E-1}}\}$$

defined by $\Phi(a_1,\dots,a_n) = (\bar{\phi}_1,\dots,\bar{\phi}_n)$ gives a diffeomorphism similarly to the proof of in [8, Theorem D]. Therefore we also can make the oscillations of the Lagrangian angles of L_1 and L_2 arbitrarily small by taking $\sum_j \bar{\phi}_j$ close to $\tan^{-1}(1/\sqrt{E-1})$.

For understanding Theorem 1.3, we compute

$$\begin{aligned}\frac{dF}{ds} =&(x_1(\dot{r}_1 + ir_1\dot{\phi}_1)e^{i\phi_1},\dots,x_n(\dot{r}_n + ir_n\dot{\phi}_n)e^{i\phi_n})\\ =&\Bigg(x_1e^{i\phi_1}\left(\frac{s}{\sqrt{1/a_1+s^2}} + i\frac{|s|}{\sqrt{(1/a_1+s^2)[E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1]}}\right),\dots\\ &,x_ne^{i\phi_n}\left(\frac{s}{\sqrt{1/a_n+s^2}} + i\frac{|s|}{\sqrt{(1/a_n+s^2)[E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1]}}\right)\Bigg)\\ =&\left(s + i\frac{|s|}{\sqrt{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1\}}}\right)\cdot\left(\frac{x_1e^{i\phi_1}}{\sqrt{1/a_1+s^2}},\dots,\frac{x_ne^{i\phi_n}}{\sqrt{1/a_n+s^2}}\right).\end{aligned}$$

Then we have

$$\left|\frac{dF}{ds}\right| = |s|\sqrt{\left(1+\frac{1}{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}\right)\cdot\sum_j\frac{x_j^2}{1/a_j+s^2}}.$$

So we obtain

$$\begin{aligned}\lim_{s\to+0}\frac{1}{|dF/ds|}\cdot\frac{dF}{ds} =& \left(\frac{1}{\sqrt{1+1/(E-1)}} + i\frac{1/\sqrt{E-1}}{\sqrt{1+1/(E-1)}}\right)\frac{1}{\sqrt{\sum_j a_jx_j^2}}\\ &\cdot(x_1e^{i\psi_1}\sqrt{a_1},\dots,x_ne^{i\psi_n}\sqrt{a_n})\end{aligned}$$

and

$$\begin{aligned}\lim_{s\to-0}\frac{1}{|dF/ds|}\cdot\frac{dF}{ds} =& \left(\frac{-1}{\sqrt{1+1/(E-1)}} + i\frac{1/\sqrt{E-1}}{\sqrt{1+1/(E-1)}}\right)\frac{1}{\sqrt{\sum_j a_jx_j^2}}\\ &\cdot(x_1e^{i\psi_1}\sqrt{a_1},\dots,x_ne^{i\psi_n}\sqrt{a_n}).\end{aligned}$$

Thus we get

$$\lim_{s\to+0}\frac{1}{|dF/ds|}\cdot\frac{dF}{ds} \neq \lim_{s\to-0}\frac{1}{|dF/ds|}\cdot\frac{dF}{ds}.$$

Therefore, if we remove the condition $s \neq 0$ from the definition of L, it is not smooth at any point $s = 0$. In [11], Lotay and Neves proved that if Lagrangian self-expanders in $\mathbb{C}^n$ are smooth, zero-Maslov class and asymptotic to a pair of planes transversely intersecting, then those are locally unique when $n > 2$ and unique when $n = 2$. It is easy to check that L is zero-Maslov class and asymptotic to a pair of planes intersecting transversely. By [8, Theorem C], we can construct a smooth Lagrangian self-expander asymptotic to any pair of Lagrangian planes in $\mathbb{C}^n$ which transversely intersect at the origin and have sum of characteristic angles less than π, where the characteristic angle is defined in Lawlor [15]. So Theorem 1.3 shows that, without the smoothness assumption, the uniqueness statement does not hold.

Remark 1.3.2. In the situation of Theorem 1.3, if we put $E = 1$ and $\alpha = 0$, then changing $0 \mapsto -\infty$ in the integral of (6) gives Joyce's example [7, Example 6.11].

Remark 1.3.3. In the situation of Theorem 1.2, if we take $C = \lambda_1 = \cdots = \lambda_n = 1$ and $\alpha \geq 0$, then the construction of L reduces to that of Theorem 1.3 where $s > 0$.

Next we turn to translating solitons.

THEOREM 1.4. *Fix $n \geq 2$. Let $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R} \setminus \{0\}$, $E > 1$, $a_1, \ldots, a_{n-1} > 0$, and $\alpha, \psi_1, \ldots, \psi_{n-1} \in \mathbb{R}$ be constants. Let $I \subset \mathbb{R}$ be a connected open neighborhood of $0 \in \mathbb{R}$ such that $\inf_{s \in I}(E\{\prod_{k=1}^{n-1}(1 + a_k\lambda_k s)\}e^{\alpha s} - 1)$ and $\inf_{s\in I}(1/a_j + \lambda_j s)$ are positive for any $1 \leq j \leq n$. Define $r_1, \ldots, r_{n-1} : I \to \mathbb{R}$ by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + \lambda_j s}, \quad j = 1, \ldots, n-1, \tag{10}$$

and $\phi_1, \ldots, \phi_{n-1} : I \to \mathbb{R}$ by

$$\phi_j(s) = \psi_j + \frac{\lambda_j}{2}\int_0^s \frac{dt}{(1/a_j + \lambda_j t)\sqrt{E\{\prod_{k=1}^{n-1}(1 + a_k\lambda_k t)\}e^{\alpha t} - 1}}, \tag{11}$$

$j = 1, \ldots, n-1$. Then the submanifold L in $\mathbb{C}^n$ given by

$$L = \{(x_1 r_1(s)e^{i\phi_1(s)}, \ldots, x_{n-1}r_{n-1}(s)e^{i\phi_{n-1}(s)}, -\frac{1}{2}\sum_{j=1}^{n-1}\lambda_j x_j^2 + \frac{s}{2} + \frac{i}{2}\int_0^s \frac{dt}{\sqrt{E\{\prod_{k=1}^{n-1}(1 + a_k\lambda_k t)\}e^{\alpha t} - 1}}); x_1, \ldots, x_{n-1} \in \mathbb{R}, s \in I\}$$

is an immersed Lagrangian submanifold diffeomorphic to $\mathbb{R}^n$, and the mean curvature vector H satisfies $H \equiv T^\perp$, where $T = (0, \ldots, 0, \alpha) \in \mathbb{C}^n$. When $\alpha = 0$ it is special Lagrangian.

The following Theorem 1.5 is slightly generalized from [8, Corollary I].

THEOREM 1.5. *Fix $n \geq 2$. Let $a_1, \ldots, a_{n-1} > 0$, $\psi_1, \ldots, \psi_{n-1} \in \mathbb{R}$, $E \geq 1$, and $\alpha \geq 0$ be constants. Define $r_1, \ldots, r_{n-1} : \mathbb{R} \to \mathbb{R}$ by*

$$r_j(s) = \sqrt{\frac{1}{a_j} + s^2}, \quad j = 1, \ldots, n-1, \tag{12}$$

and $\phi_1, \ldots, \phi_{n-1} : \mathbb{R} \to \mathbb{R}$ *by*

$$\phi_j(s) = \psi_j + \int_0^s \frac{|t|dt}{(1/a_j + t^2)\sqrt{E\{\prod_{k=1}^{n-1}(1 + a_k t^2)\}e^{\alpha t^2} - 1}}, \tag{13}$$

$j = 1, \ldots, n-1$. *Then the submanifold* L *in* $\mathbb{C}^n$ *given by*

$$L = \{(x_1 r_1(s)e^{i\phi_1(s)}, \ldots, x_{n-1}r_{n-1}(s)e^{i\phi_{n-1}(s)}, -\frac{1}{2}\sum_{j=1}^{n-1} x_j^2 + \frac{s^2}{2} + i\int_0^s \frac{|t|dt}{\sqrt{E\{\prod_{k=1}^{n-1}(1 + a_k t^2)\}e^{\alpha t^2} - 1}}); x_1 \ldots, x_{n-1} \in \mathbb{R}, s \in \mathbb{R},\ s \neq 0\}$$

is an embedded Lagrangian submanifold diffeomorphic to $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1}$, *and the mean curvature vector* H *satisfies* $H \equiv T^{\perp}$, *where* $T = (0, \ldots, 0, \alpha) \in \mathbb{C}^n$. *When* $\alpha = 0$ *it is special Lagrangian. When* $E = 1$ *and* $\psi_1 = \cdots = \psi_{n-1} = 0$, *the construction reduces to that of* [8, Corollary I]. *So the condition* $s \neq 0$ *on the definition of* L *is not necessary if* $E = 1$.

Remark 1.5.1. In the situation of Theorem 1.5, we define the submanifolds L_1 and L_2 of L so that $s > 0$ on L_1, and $s < 0$ on L_2, respectively. Similarly to Remark 1.3.1 if we fix $\alpha > 0$, then we can make the oscillations of the Lagrangian angles of L_1 and L_2 arbitrarily small.

Remark 1.5.2. In the situation of Theorem 1.4, if we put $\lambda_1 = \cdots = \lambda_{n-1} = 1$ and $\alpha \geq 0$, then the construction of L reduces to that of Theorem 1.5 where $s > 0$.

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2. Proofs for self-similar solutions

In order to prove Theorems 1.2 and 1.3, we use the following Lemma 2.1 that is generalized from [8, Theorem B]. The submanifolds in the following Lemma 2.1 are immersed Lagrangian self-similar solutions diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m+1}$, where $1 \leq m \leq n$.

Lemma 2.1. *Let* I *be an open interval in* $\mathbb{R}$ *and* D *a domain in* $\mathbb{R}^{n+2}$. *Let* $\alpha \in \mathbb{R}, \lambda_1, \ldots, \lambda_n, C \in \mathbb{R} \setminus \{0\}$ *and* $a_1, \ldots, a_n > 0$ *be constants, and* $f : I \times D \to \mathbb{C} \setminus \{0\}$ *a smooth function. Let* $u, \phi_1, \ldots, \phi_n, \theta : I \to \mathbb{R}$ *be smooth functions such that*

$\{(s, u(s), \phi_1(s), \ldots, \phi_n(s), \theta(s)); s \in I\} \subset I \times D$. *Suppose that*

$$\begin{cases} \dfrac{du}{ds} = 2\operatorname{Re}(f(s, u, \phi_1, \ldots, \phi_n, \theta)), \\ \dfrac{d\phi_j}{ds} = \dfrac{\lambda_j \operatorname{Im}(f(s, u, \phi_1, \ldots, \phi_n, \theta))}{1/a_j + \lambda_j u(s)}, \quad j = 1, \ldots, n, \\ \dfrac{d\theta}{ds} = -\alpha \operatorname{Im}(f(s, u, \phi_1, \ldots, \phi_n, \theta)), \end{cases} \tag{14}$$

hold in I. *We also suppose that*

$$\inf_{s \in I}(1/a_j + \lambda_j u(s)) > 0, \quad j = 1, \ldots, n,$$

and

$$\theta(s) = \sum_{j=1}^{n} \phi_j(s) + \arg(f(s, u(s), \phi_1(s), \ldots, \phi_n(s), \theta(s))) \tag{15}$$

hold in I. *Then the submanifold* L *in* $\mathbb{C}^n$ *given by*

$$L = \{(x_1\sqrt{1/a_1 + \lambda_1 u(s)}e^{i\phi_1(s)}, \ldots, x_n\sqrt{1/a_n + \lambda_n u(s)}e^{i\phi_n(s)}); \\ \sum_{j=1}^{n} \lambda_j x_j^2 = C, x_j \in \mathbb{R}, s \in I\} \tag{16}$$

is an immersed Lagrangian submanifold diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m+1}$, *where* m *is the number of positive* λ_j/C, $1 \leq j \leq n$, *with Lagrangian angle* $\theta(s)$ *at*

$$(x_1\sqrt{1/a_1 + \lambda_1 u(s)}e^{i\phi_1(s)}, \ldots, x_n\sqrt{1/a_n + \lambda_n u(s)}e^{i\phi_n(s)}) \in L,$$

and the mean curvature vector H *satisfies* $CH \equiv \alpha F^{\perp}$, *where* F *is the position vector of* L. *Note that* $\theta(s)$ *is a function depending only on* s, *and* L *is a self-expander when* $\alpha/C > 0$ *and a self-shrinker when* $\alpha/C < 0$. *When* $\alpha = 0$ *the Lagrangian angle* θ *is constant, so that* L *is special Lagrangian.*

Remark 2.1.1. In the situation of Lemma 2.1, if we set $a_1 = \cdots = a_n = 1$, $\alpha = -\sum_{k=1}^{n} \lambda_k$ and

$$\begin{cases} f(s, y_1, \ldots, y_{n+2}) = i, \\ u(s) = 0, \\ \phi_j(s) = \lambda_j s, \quad 1 \leq j \leq n, \\ \theta(s) = -\alpha s + \dfrac{\pi}{2} = \left(\displaystyle\sum_{k=1}^{n} \lambda_k\right) s + \dfrac{\pi}{2}, \end{cases} \tag{17}$$

then it is easily seen that this setting satisfies the assumptions of Lemma 2.1, and the construction is Hamiltonian stationary in addition to being self-similar and it reduces to that of Lee and Wang [10, Theorem 1.1]. If f is a real valued function, then the submanifold L is an open subset of the special Lagrangian n-plane

$$\{(y_1 e^{i\xi_1}, \ldots, y_n e^{i\xi_n}); y_j \in \mathbb{R},\ 1 \le j \le n\},$$

where $\xi_j \in \mathbb{R}$.

Proof of Lemma 2.1. Write

$$w_j(s) = \sqrt{1/a_j + \lambda_j u(s)}\, e^{i\phi_j(s)}, \quad 1 \le j \le n.$$

We compute

$$\begin{aligned}
\frac{dw_j}{ds} &= \frac{d}{ds}\left(\sqrt{1/a_j + \lambda_j u(s)}\right) \cdot e^{i\phi_j(s)} + \sqrt{1/a_j + \lambda_j u(s)} \cdot i \frac{d\phi_j}{ds} e^{i\phi_j(s)} \\
&= \left(\frac{\lambda_j \mathrm{Re}(f(s,u,\phi_1,\ldots,\phi_n,\theta))}{\sqrt{1/a_j + \lambda_j u(s)}} + i \frac{\lambda_j \mathrm{Im}(f(s,u,\phi_1,\ldots,\phi_n,\theta))}{\sqrt{1/a_j + \lambda_j u(s)}} \right) e^{i\phi_j(s)} \\
&= \frac{\lambda_j f(s,u,\phi_1,\ldots,\phi_n,\theta)}{\overline{w_j}}.
\end{aligned}$$

Thus we obtain

$$\begin{cases} \dfrac{dw_j}{ds} = \dfrac{\lambda_j f(s,u,\phi_1,\ldots,\phi_n,\theta)}{\overline{w_j}}, \ j = 1,\ldots,n, \\ \dfrac{d\theta}{ds} = -\alpha\, \mathrm{Im}(f(s,u,\phi_1,\ldots,\phi_n,\theta)). \end{cases} \tag{18}$$

By (15) and (18), we can prove this theorem similarly to the proof of [8, Theorem A]. The details are left to the reader. This finishes the proof of Lemma 2.1. □

Now we can show Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Define $\tilde{f} : I \to \mathbb{C} \setminus \{0\}$ by

$$\tilde{f}(s) = \frac{1}{2} + \frac{i}{2\sqrt{E\{\prod_{k=1}^n (1 + a_k \lambda_k s)\} e^{\alpha s} - 1}}$$

and $f : I \times \mathbb{R}^{n+2} \to \mathbb{C} \setminus \{0\}$ by $f(s, y_1, \ldots, y_{n+2}) = \tilde{f}(s)$. Note that f is a function depending only on $s \in I$. We also define $u : I \to \mathbb{R}$ by

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s,$$

and $\theta : I \to \mathbb{R}$ by

$$\theta(s) = \sum_{j=1}^n \phi_j(s) + \arg(\tilde{f}(s)). \tag{19}$$

Then we get

$$r_j(s) = \sqrt{\frac{1}{a_j} + \lambda_j u(s)}$$

and

$$\frac{d\phi_j}{ds} = \frac{\lambda_j \operatorname{Im}(\tilde{f})}{1/a_j + \lambda_j u} \tag{20}$$

for any $j = 1, \ldots, n$. By our assumption we have

$$\inf_{s \in I} (1/a_j + \lambda_j u(s)) = \inf_{s \in I} (1/a_j + \lambda_j s) > 0, \quad j = 1, \ldots, n.$$

Since

$$\begin{aligned}
\frac{d}{ds}\arg(\tilde{f}) =& \frac{d}{ds}\tan^{-1}\left(\frac{\operatorname{Im}(\tilde{f})}{\operatorname{Re}(\tilde{f})}\right) \\
=& \left(1 + \frac{\operatorname{Im}(\tilde{f})^2}{\operatorname{Re}(\tilde{f})^2}\right)^{-1} \cdot \frac{d}{ds}\left(\frac{\operatorname{Im}(\tilde{f})}{\operatorname{Re}(\tilde{f})}\right) \\
=& \left(1 + \frac{1}{E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s} - 1}\right)^{-1} \\
& \cdot \frac{d}{ds}\left(\frac{1}{\sqrt{E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s} - 1}}\right) \\
=& \frac{E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s} - 1}{E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s}} \cdot \frac{-1}{2[E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s} - 1]^{3/2}} \\
& \cdot [E\{\sum_{l=1}^n \frac{\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}a_l\lambda_l}{1 + a_l\lambda_l s}\}e^{\alpha s} + E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}\alpha e^{\alpha s}] \\
=& \frac{1}{E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s}} \cdot \frac{-1}{2\sqrt{E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s} - 1}} \\
& \cdot E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s}\left(\sum_{l=1}^n \frac{a_l\lambda_l}{1 + a_l\lambda_l s} + \alpha\right) \\
=& \frac{-1}{2\sqrt{E\{\prod_{k=1}^n (1 + a_k\lambda_k s)\}e^{\alpha s} - 1}}\left(\sum_{l=1}^n \frac{\lambda_l}{1/a_l + \lambda_l u} + \alpha\right) \\
=& -\operatorname{Im}(\tilde{f})\left(\sum_{l=1}^n \frac{\lambda_l}{1/a_l + \lambda_l u} + \alpha\right),
\end{aligned}$$

we obtain

$$\sum_{j=1}^n \frac{\lambda_j \operatorname{Im}(\tilde{f})}{1/a_j + \lambda_j u} + \frac{d}{ds}\arg(\tilde{f}) = -\alpha \operatorname{Im}(\tilde{f}). \tag{21}$$

From (19), (20) and (21), we get

$$\frac{d\theta}{ds} = -\alpha \operatorname{Im}(\tilde{f}(s)).$$

Accordingly,

$$\begin{cases} \dfrac{du}{ds} = 2\,\mathrm{Re}(f(s,u,\phi_1,\dots,\phi_n,\theta)), \\ \dfrac{d\phi_j}{ds} = \dfrac{\lambda_j \,\mathrm{Im}\,(f(s,u,\phi_1,\dots,\phi_n,\theta))}{1/a_j+\lambda_j u(s)}, \quad j=1,\dots,n, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta)). \end{cases}$$

Therefore we can apply Lemma 2.1 to the data f, u, ϕ_j, θ above. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. We define $\tilde{f} : \mathbb{R} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ by

$$\tilde{f}(s) = s + i\frac{|s|}{\sqrt{E\{\prod_{k=1}^{n}(1+a_k s^2)\}e^{\alpha s^2} - 1}}$$

and $f : (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n+2} \to \mathbb{C} \setminus \{0\}$ by $f(s, y_1, \dots, y_{n+2}) = \tilde{f}(s)$. We also define $u : \mathbb{R} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ by

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s^2$$

and $\theta : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$\theta(s) = \sum_{j=1}^{n} \phi_j(s) + \arg(\tilde{f}(s)). \tag{22}$$

Then we get $r_j(s) = \sqrt{1/a_j + u(s)}$ and

$$\frac{d\phi_j}{ds} = \frac{\mathrm{Im}\,(\tilde{f})}{1/a_j + u} \tag{23}$$

for any $j = 1, \dots, n$. It is clear that

$$\inf_{s \in \mathbb{R}\setminus\{0\}} (1/a_j + u(s)) = \inf_{s \in \mathbb{R}\setminus\{0\}} (1/a_j + s^2) = 1/a_j > 0, \quad j = 1, \dots, n.$$

Since

$$\begin{aligned}
\frac{d}{ds}\arg(\tilde{f}) =& \frac{d}{ds}\tan^{-1}\left(\frac{\operatorname{Im}(\tilde{f})}{\operatorname{Re}(\tilde{f})}\right) \\
=& \left(1+\frac{\operatorname{Im}(\tilde{f})^2}{\operatorname{Re}(\tilde{f})^2}\right)^{-1}\cdot\frac{d}{ds}\left(\frac{\operatorname{Im}(\tilde{f})}{\operatorname{Re}(\tilde{f})}\right) \\
=& \left(1+\frac{1}{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}\right)^{-1} \\
& \cdot\frac{d}{ds}\left(\frac{|s|}{s\sqrt{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}}\right) \\
=& \frac{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}}\cdot\frac{|s|}{s}\frac{d}{ds}\left(\frac{1}{\sqrt{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}}\right) \\
=& \frac{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}}\cdot\frac{|s|}{s}\cdot\frac{-1}{2[E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1]^{3/2}} \\
& \cdot[E\{\sum_{l=1}^n\frac{\{\prod_{k=1}^n(1+a_ks^2)\}2a_ls}{1+a_ls^2}\}e^{\alpha s^2}+E\{\prod_{k=1}^n(1+a_ks^2)\}2\alpha se^{\alpha s^2}] \\
=& \frac{1}{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}}\cdot\frac{|s|}{s}\cdot\frac{-1}{2\sqrt{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}} \\
& \cdot 2sE\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}\left(\sum_{l=1}^n\frac{a_l}{1+a_ls^2}+\alpha\right) \\
=& \frac{-|s|}{\sqrt{E\{\prod_{k=1}^n(1+a_ks^2)\}e^{\alpha s^2}-1}}\left(\sum_{l=1}^n\frac{1}{1/a_l+u}+\alpha\right) \\
=& -\operatorname{Im}(\tilde{f})\left(\sum_{l=1}^n\frac{1}{1/a_l+u}+\alpha\right),
\end{aligned}$$

we obtain

$$\sum_{j=1}^n\frac{\operatorname{Im}(\tilde{f})}{1/a_j+u}+\frac{d}{ds}\arg(\tilde{f})=-\alpha\operatorname{Im}(\tilde{f}). \tag{24}$$

From (22), (23) and (24), we have $d\theta/ds=-\alpha\operatorname{Im}(\tilde{f}(s))$. Thus

$$\begin{cases}
\dfrac{du}{ds}=2\operatorname{Re}(f(s,u,\phi_1,\dots,\phi_n,\theta)), \\
\dfrac{d\phi_j}{ds}=\dfrac{\operatorname{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta))}{1/a_j+u(s)}, \quad j=1,\dots,n, \\
\dfrac{d\theta}{ds}=-\alpha\operatorname{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta)).
\end{cases}$$

So we can apply Lemma 2.1 to the data $\lambda_1 = \cdots = \lambda_n = 1$ and f, u, ϕ_j, θ above. That L is embedded follows from the same argument as the proof of [8, Theorem C]. This completes the proof of Theorem 1.3. $\square$

3. Proofs for translating solitons

This section is analogous to Section 2. In order to prove Theorems 1.4 and 1.5, we use the following Lemma 3.1 that is generalized from [8, Corollary H]. The following Lemma 3.1 sets up the ordinary differential equations for immersed Lagrangian translating soliton diffeomorphic to $\mathbb{R}^n$.

Lemma 3.1. *Fix $n \geq 2$. Let I be an open interval in $\mathbb{R}$ and D a domain in $\mathbb{R}^{n+1} \times \mathbb{C}$. Let $\alpha \in \mathbb{R}, \lambda_1, \ldots, \lambda_{n-1}, C \in \mathbb{R}\backslash\{0\}$ and $a_1, \ldots, a_{n-1} > 0$ be constants, and $f : I \times D \to \mathbb{C}\backslash\{0\}$ a smooth function. Let $u, \phi_1, \ldots, \phi_{n-1},\ \theta : I \to \mathbb{R}$ and $\beta : I \to \mathbb{C}$ be smooth functions such that*
$\{(s, u(s), \phi_1(s), \ldots, \phi_{n-1}(s), \theta(s), \beta(s)); s \in I\} \subset I \times D$. Suppose that

$$(25)\qquad \begin{cases} \dfrac{du}{ds} = 2\operatorname{Re}(f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta)), \\ \dfrac{d\phi_j}{ds} = \dfrac{\lambda_j \operatorname{Im}\left(f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta)\right)}{1/a_j + \lambda_j u(s)}, \quad j = 1, \ldots, n-1, \\ \dfrac{d\theta}{ds} = -\alpha \operatorname{Im}(f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta)), \\ \dfrac{d\beta}{ds} = f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta). \end{cases}$$

hold in I. We also suppose that

$$\inf_{s\in I}(1/a_j + \lambda_j u(s)) > 0, \quad j = 1, \ldots, n-1,$$

and

$$(26)\qquad \theta(s) = \sum_{j=1}^{n-1} \phi_j(s) + \arg(f(s, u(s), \phi_1(s), \ldots, \phi_{n-1}(s), \theta(s), \beta(s)))$$

hold in I. Then the submanifold L in $\mathbb{C}^n$ given by

$$L = \{(x_1 r_1(s) e^{i\phi_1(s)}, \ldots, x_{n-1} r_{n-1}(s) e^{i\phi_{n-1}(s)}, -\frac{1}{2}\sum_{j=1}^{n-1} \lambda_j x_j^2 + \beta(s)); \\ x_1, \ldots, x_{n-1} \in \mathbb{R}, s \in I\}$$

is an immersed Lagrangian submanifold diffeomorphic to $\mathbb{R}^n$ with Lagrangian angle $\theta(s)$ at

$$(x_1 r_1(s) e^{i\phi_1(s)}, \ldots, x_{n-1} r_{n-1}(s) e^{i\phi_{n-1}(s)}, -1/2\sum_{j=1}^{n-1} \lambda_j x_j^2 + \beta(s)) \in L,$$

and the mean curvature vector H satisfies $H \equiv T^{\perp}$, where $T = (0, \dots, 0, \alpha)$. When $\alpha = 0$ it is special Lagrangian.

Remark 3.1.1. In the situation of Lemma 3.1, if we set $\alpha = -\sum_{k=1}^{n} a_j\lambda_k$ and

$$\begin{cases} f(s, y_1, \dots, y_{n+1}, z) = i, \\ u(s) = 0, \\ \phi_j(s) = a_j\lambda_j s, \quad 1 \le j \le n-1, \\ \theta(s) = -\alpha s + \dfrac{\pi}{2} = \left(\displaystyle\sum_{k=1}^{n-1} a_k\lambda_k\right) s + \dfrac{\pi}{2}, \\ \beta(s) = is, \end{cases} \tag{27}$$

then it is easy to check that this setting satisfies the assumptions of Lemma 3.1, and the construction is Hamiltonian stationary in addition to being translating solition. If f is a real valued function, then the submanifold L is an open subset of the special Lagrangian n-plane

$$\{(y_1 e^{i\xi_1}, \dots, y_{n-1}e^{i\xi_{n-1}}, y_n); y_j \in \mathbb{R},\ 1 \le j \le n\},$$

where $\xi_l \in \mathbb{R}, 1 \le l \le n-1$.

Proof of Lemma 3.1. Write

$$w_j(s) = \sqrt{1/a_j + \lambda_j u(s)}\, e^{i\phi_j(s)}, \quad 1 \le j \le n-1.$$

We compute

$$\begin{aligned} \frac{dw_j}{ds} &= \frac{d}{ds}\left(\sqrt{1/a_j + \lambda_j u(s)}\right) \cdot e^{i\phi_j(s)} + \sqrt{1/a_j + \lambda_j u(s)} \cdot i\frac{d\phi_j}{ds} e^{i\phi_j(s)} \\ &= \left(\frac{\lambda_j \mathrm{Re}(f(s, u, \phi_1, \dots, \phi_{n-1}, \theta, \beta))}{\sqrt{1/a_j + \lambda_j u(s)}} + i\frac{\lambda_j \mathrm{Im}(f(s, u, \phi_1, \dots, \phi_{n-1}, \theta, \beta))}{\sqrt{1/a_j + \lambda_j u(s)}}\right) e^{i\phi_j(s)} \\ &= \frac{\lambda_j f(s, u, \phi_1, \dots, \phi_{n-1}, \theta, \beta)}{\overline{w_j}}. \end{aligned}$$

Accordingly,

$$\begin{cases} \dfrac{dw_j}{ds} = \dfrac{\lambda_j f(s, u, \phi_1, \dots, \phi_{n-1}, \theta, \beta)}{\overline{w_j}}, \ j = 1, \dots, n-1, \\ \dfrac{d\theta}{ds} = -\alpha\, \mathrm{Im}(f(s, u, \phi_1, \dots, \phi_{n-1}, \theta, \beta)), \\ \dfrac{d\beta}{ds} = f(s, u, \phi_1, \dots, \phi_{n-1}, \theta, \beta). \end{cases} \tag{28}$$

By (26) and (28), we can prove this theorem similarly to the proof of [8, Theorem G]. This finishes the proof, the detailed verification being left to the reader. □

Now we can show Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Define $\tilde{f}: I \to \mathbb{C} \setminus \{0\}$ by

$$\tilde{f}(s) = \frac{1}{2} + \frac{i}{2\sqrt{E\{\prod_{k=1}^{n-1}(1+a_k\lambda_k s)\}e^{\alpha s} - 1}}$$

and $f: I \times \mathbb{R}^{n+1} \times \mathbb{C} \to \mathbb{C} \setminus \{0\}$ by $f(s, y_1, \ldots, y_{n+1}, z) = \tilde{f}(s)$. We also define

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s,$$

$$\theta(s) = \sum_{j=1}^{n-1} \phi_j(s) + \arg(\tilde{f}(s)),$$

and

$$\beta(s) = \int_0^s \tilde{f}(t)dt = \frac{s}{2} + \frac{i}{2}\int_0^s \frac{dt}{\sqrt{E\{\prod_{k=1}^{n-1}(1+a_k\lambda_k t)\}e^{\alpha t} - 1}}.$$

Then we get $r_j(s) = \sqrt{1/a_j + \lambda_j u(s)}$ and

$$\frac{d\phi_j}{ds} = \frac{\lambda_j \,\mathrm{Im}\,(\tilde{f})}{1/a_j + \lambda_j u}$$

for any $j = 1, \ldots, n-1$. By our assumption we have

$$\inf_{s\in I}(1/a_j + \lambda_j u(s)) = \inf_{s\in I}(1/a_j + \lambda_j s) > 0, \quad j = 1, \ldots, n-1.$$

We can check $d\theta/ds = -\alpha \mathrm{Im}(\tilde{f})$ similarly to the proof of Theorem 1.2. Thus we obtain

$$\begin{cases} \dfrac{du}{ds} = 2\,\mathrm{Re}(f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta)), \\ \dfrac{d\phi_j}{ds} = \dfrac{\lambda_j \,\mathrm{Im}\,(f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta))}{1/a_j + \lambda_j u(s)}, \quad j = 1, \ldots, n-1, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta)), \\ \dfrac{d\beta}{ds} = f(s, u, \phi_1, \ldots, \phi_{n-1}, \theta, \beta). \end{cases}$$

Therefore we can apply Lemma 3.1 to the data $f, u, \phi_j, \theta, \beta$ above. This finishes the proof of Theorem 1.4. □

Proof of Theorem 1.5. We define $\tilde{f}: \mathbb{R} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ by

$$\tilde{f}(s) = s + i\frac{|s|}{\sqrt{E\{\prod_{k=1}^{n-1}(1+a_k s^2)\}e^{\alpha s^2} - 1}}$$

and $f: (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n+1} \times \mathbb{C} \to \mathbb{C} \setminus \{0\}$ by $f(s, y_1, \ldots, y_{n+1}, z) = \tilde{f}(s)$. We also define

$$u(s) = 2\int_0^s \mathrm{Re}(\tilde{f}(t))dt = s^2,$$

$$\theta(s) = \sum_{j=1}^{n-1} \phi_j(s) + \arg(\tilde{f}(s)),$$

and

$$\beta(s) = \int_0^s \tilde{f}(t)dt = \frac{s^2}{2} + i\int_0^s \frac{|t|dt}{\sqrt{E\{\prod_{k=1}^{n-1}(1+a_k t^2)\}e^{\alpha t^2} - 1}}.$$

Then we have $r_j(s) = \sqrt{1/a_j + u(s)}$ and

$$\frac{d\phi_j}{ds} = \frac{\mathrm{Im}\,(\tilde{f})}{1/a_j + u}$$

for any $j = 1, \dots, n-1$. It is clear that

$$\inf_{s\in\mathbb{R}\setminus\{0\}}(1/a_j + u(s)) = \inf_{s\in\mathbb{R}\setminus\{0\}}(1/a_j + s^2) = 1/a_j > 0, \quad j = 1, \dots, n-1.$$

We can check $d\theta/ds = -\alpha\mathrm{Im}(\tilde{f})$ similarly to the proof of Theorem 1.3. Thus we obtain

$$\begin{cases} \dfrac{du}{ds} = 2\,\mathrm{Re}(f(s,u,\phi_1,\dots,\phi_n,\theta)), \\ \dfrac{d\phi_j}{ds} = \dfrac{\mathrm{Im}\,(f(s,u,\phi_1,\dots,\phi_n,\theta))}{1/a_j + u(s)}, \quad j = 1,\dots,n-1, \\ \dfrac{d\theta}{ds} = -\alpha\,\mathrm{Im}(f(s,u,\phi_1,\dots,\phi_n,\theta)), \\ \dfrac{d\beta}{ds} = f(s,u,\phi_1,\dots,\phi_{n-1},\theta,\beta). \end{cases}$$

So we can apply Lemma 3.1 to the data $\lambda_1 = \cdots = \lambda_{n-1} = 1$ and $f, u, \phi_j, \theta, \beta$ above. That L is embedded follows from the same argument as the proof of [8, Theorem C]. This completes the proof of Theorem 1.5. □

Chapter 2

MEAN CURVATURE FLOW IN SUBMANIFOLDS

ABSTRACT. We obtain explicit solutions of the mean curvature flow in some submanifolds of the Euclidean space. We give particularly an explicit solution of the mean curvature flow of the hypersurface $\{y = constant\}$ in the Lagrangian self-expander L which is constructed in Joyce, Lee and Tsui [8] for a special case. In addition, we show that the hypersurface $\{y = 0\}$ is minimal.

4. Introduction

Mean curvature flow evolves submanifolds of a riemannian manifold in the direction of their mean curvature vector. It is the steepest descent flow for the area functional and is described by a parabolic system of partial differential equations for the immersed map of evolving submanifolds. Put M_0 to be a hypersurface in $\mathbb{R}^{n+1}$ and $\{M_t\}_{t\in[0,\epsilon)}$ to be the solution of mean curvature flow. By the weak maximum principle of it [1], we can see that if the initial manifold M_0 is in an open ball $B(0, r)$, where $r > 0$, then $M_t \subset B(0, \sqrt{r^2 - 2nt})$, for any $t \in [0, \epsilon)$. Furthermore, other properties of the mean curvature flow in $\mathbb{R}^N$ have been extensively studied. For example, Wang investigates the mean curvature flow of graphs in [12] and the author constructs explicit self-similar solutions and translating solitons for the mean curvature flow in $\mathbb{C}^n (= \mathbb{R}^{2n})$ in Chapter 1. In this paper, however, we consider the mean curvature flow in some submanifolds of $\mathbb{R}^N$. We give explicit solutions of the mean curvature flow in some Lagrangian submanifolds L of $\mathbb{C}^n$.

5. Results and Proofs

In order to discuss the mean curvature flow in submanifolds, firstly, we consider the following well known Proposition.

PROPOSITION 5.1. *Let l, L be submanifolds in $\mathbb{C}^n$. Suppose that l is a submanifold in L. Put H to be the mean curvature vector of l in L, and $\bar{H}$ to be the mean curvature vector of l in $\mathbb{C}^n$. Fix $p \in l$. Then*

$$H(p) = \bar{H}(p) - \sum_j A_{L,\mathbb{C}^n}(e_j, e_j),$$

where $A_{L,\mathbb{C}^n}$ is the second fundamental form of L in $\mathbb{C}^n$ and $\{e_j\}_j$ is an orthonormal basis of T_pl. So we can see that

$$H(p) = \pi_{T_pL}(\bar{H}(p)),$$

where $\pi_{T_pL}(\bar{H}(p))$ is the orthogonal projection of $\bar{H}(p)$ to T_pL.

In this paper, if a manifold M is a submanifold in a riemannian manifold N, then we denote $A_{M,N}$ the second fundamental form of M in N and ∇^N, ∇^M the Levi-Civita connections on N and M respectively. Hence $A_{M,N} \in C^\infty(M, (TN/TM) \otimes T^*M \otimes T^*M)$.

PROOF. From the definitions of the mean curvature vector and the second fundamental form we have

$$\begin{aligned} H(p) &= \sum_j A_{l,L}(e_j, e_j) = \sum_j (\nabla^L_{e_j} e_j - \nabla^l_{e_j} e_j) = \sum_j (\nabla^{\mathbb{C}^n}_{e_j} e_j - A_{L,\mathbb{C}^n}(e_j, e_j) - \nabla^l_{e_j} e_j) \\ &= \sum_j (A_{l,\mathbb{C}^n}(e_j, e_j) - A_{L,\mathbb{C}^n}(e_j, e_j)) = \bar{H}(p) - \sum_j A_{L,\mathbb{C}^n}(e_j, e_j). \end{aligned}$$

This finishes the proof. □

In the following Theorem 5.2, from a direct calculation, the submanifolds L are Lagrangian submanifolds.

THEOREM 5.2. *Let I be an interval of $\mathbb{R}$ and $w : I \to \mathbb{C} \setminus \{0\}$ be a smooth function. Suppose that $\dot{w}(s) \neq 0$, for any $s \in I$. Define submanifolds l_s, for $s \in I$, in $\mathbb{C}^n$ by*

$$l_s = \{(x_1 w(s), \ldots, x_n w(s)); \sum_{j=1}^n x_j^2 = 1,\ x_1, \ldots, x_n \in \mathbb{R}\},$$

and submanifold L in $\mathbb{C}^n$ by

$$L = \bigcup_{s \in I} l_s.$$

(Clearly, $l_s \subset L \subset \mathbb{C}^n$.) Let H_s be the mean curvature vector of l_s in L. Then

$$H_s(x_1 w(s), \ldots, x_n w(s)) = -\frac{(n-1)\mathrm{Re}\,(\bar{w}(s)\dot{w}(s))}{|w(s)|^2 |\dot{w}(s)|^2} \cdot \frac{\partial}{\partial s} \tag{29}$$

holds, where $\partial/\partial s = (x_1 \dot{w}(s), \ldots, x_n \dot{w}(s)) \in T_{(x_1 w(s), \ldots, x_n w(s))} L$. Thus, by the definition of the mean curvature flow, if we suppose that f is a solution of the following ordinal differential equation

$$\frac{df(t)}{dt} = -\frac{(n-1)\mathrm{Re}\,(\bar{w}(f(t))\dot{w}(f(t)))}{|w(f(t))|^2 |\dot{w}(f(t))|^2},$$

then $\{l_{f(t)}\}_t$ is a mean curvature flow in L.

The following Lemma 5.3 is a lemma of Theorem 5.2.

LEMMA 5.3. *Let $\alpha \in \mathbb{C} \setminus \{0\}$ be a constant. Define a submanifold S in $\mathbb{C}^n$ by*

$$S = \{\alpha(x_1, \ldots, x_n) \in \mathbb{C}^n; \sum_{j=1}^n x_j^2 = 1,\ x_1, \ldots, x_n \in \mathbb{R}\}.$$

Fix $p \in S$. Then

$$H(p) = -\frac{n-1}{|\alpha|^2} p,$$

where $H(p)$ is the mean curvature vector of S at p.

PROOF. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of T_pS. Let V_j be the plane which is generated by e_j and $\overrightarrow{Op}$, where $O = (0, \ldots, 0) \in \mathbb{C}^n$. Since the intersection of S and V_j is a circle of radius $|\alpha|$ with center O, we can get curves $c_1, \ldots, c_{n-1} : \mathbb{R} \to S$ such that

$$c_j(0) = p, \quad \dot{c}_j(0) = e_j, \quad \ddot{c}_j(0) = -\frac{1}{|\alpha|^2}p,$$

for any j. We compute

$$H(p) = \sum_{j=1}^{n-1} A_{S,\mathbb{C}^n}(e_j, e_j) = \sum_{j=1}^{n-1} \left(\nabla^{\mathbb{C}^n}_{e_j} e_j\right)^{\perp} = \sum_{j=1}^{n-1} (\ddot{c}_j(0))^{\perp} = \sum_{j=1}^{n-1} \left(-\frac{1}{|\alpha|^2}p\right)^{\perp} = -\frac{n-1}{|\alpha|^2}p,$$

where $\perp$ is the orthogonal projection to $T_p^{\perp}S$. This completes the proof. □

Now we prove Theorem 5.2.

proof of Theorem 5.2. We denote by $\bar{H}_s$ the mean curvature vector of l_s in $\mathbb{C}^2$. Fix $p = (x_1w(s), \ldots, x_nw(s)) \in l_s$. By Lemma 5.3,

$$\bar{H}_s(p) = -\frac{n-1}{|w(s)|^2}(p).$$

By Proposition 5.1, we have

$$H(p) = \pi_{T_pL}(\bar{H}(p)) = -\frac{n-1}{|w(s)|^2} \cdot \pi_{T_pL}(p)$$

From a direct calculation, we can see $\partial/\partial s \perp T_pl_s$. Hence we obtain

$$H(p) = -\frac{n-1}{|w(s)|^2} \cdot \frac{p \cdot \partial/\partial s}{\partial/\partial s \cdot \partial/\partial s} \cdot \frac{\partial}{\partial s} = -\frac{(n-1)\mathrm{Re}\,(\bar{w}(s)\dot{w}(s))}{|w(s)|^2|\dot{w}(s)|^2} \cdot \frac{\partial}{\partial s}.$$

This finishes the proof. □

Next we consider the following Remark 5.3.1 and Figure 1. If we put $w_1 = \cdots = w_1$ in the construction of the Lagrangian self-expander given in [8, Thorem C], then we can find a minimal hypersurface in the self-expander.

Remark 5.3.1. Let $a > 0$ and $\alpha \geq 0$ be constants. Define $r : \mathbb{R} \to \mathbb{R}$ by $r(s) = \sqrt{1/a + s^2}$ and $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(s) = \int_0^s \frac{|t|dt}{(1/a + t^2)\sqrt{(1 + at^2)^n e^{\alpha t^2} - 1}}.$$

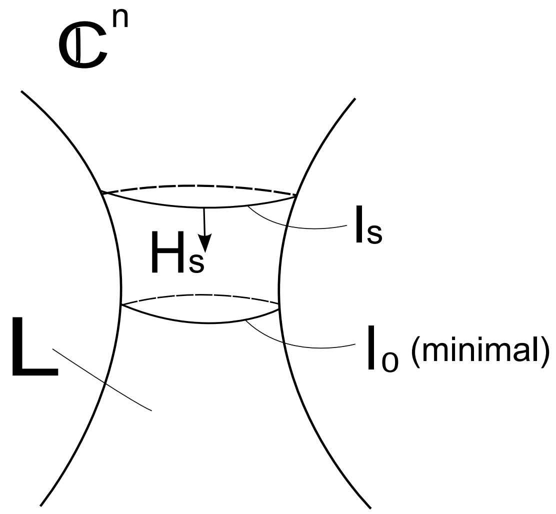

FIGURE 1. Remark 5.3.1

In the situation of Theorem 5.2, if we put $I = \mathbb{R}$ and $w(s) = r(s)e^{i\phi(s)}$, then L is the Lagrangian self-similar solution constructed in [8, Theorem C]. Then we compute

$$
\begin{aligned}
\frac{\operatorname{Re}(\bar{w}(s)\dot{w}(s))}{|w(s)|^2|\dot{w}(s)|^2} &= \frac{\operatorname{Re}\left(r(s)e^{-i\phi(s)}(\dot{r}(s)e^{i\phi(s)} + ir\dot{\phi}(s)e^{i\phi(s)})\right)}{r(s)^2 \cdot |\dot{r}(s)e^{i\phi(s)} + ir(s)\dot{\phi}(s)e^{i\phi(s)}|^2} \\
&= \frac{r(s)\dot{r}(s)}{r(s)^2 \cdot |\dot{r}(s) + ir(s)\dot{\phi}(s)|^2} \\
&= \frac{r(s)\dot{r}(s)}{r(s)^2\dot{r}(s)^2 + r(s)^4\dot{\phi}(s)^2} \\
&= \frac{s}{s^2 + s^2/((1+at^2)^n e^{\alpha s^2} - 1)} \\
&= \frac{1}{s + s/((1+at^2)^n e^{\alpha s^2} - 1)} \\
&= \frac{1}{s(1+as^2)^n e^{\alpha s^2}/((1+as^2)^n e^{\alpha s^2} - 1)} \\
&= \frac{(1+as^2)^n e^{\alpha s^2} - 1}{s(1+as^2)^n e^{\alpha s^2}} \\
&= \frac{(1+as^2)^n - e^{-\alpha s^2}}{s(1+as^2)^n}.
\end{aligned}
$$

By the equation (29) and L'Hôpital's rule, we obtain

$$\begin{aligned} H_0(x_1w(0),\ldots,x_nw(0)) =& -(n-1)\cdot\frac{\mathrm{Re}\,(\bar{w}(0)\dot{w}(0))}{|w(0)|^2|\dot{w}(0)|^2}\cdot\frac{\partial}{\partial s} \\ =& -(n-1)\cdot\lim_{s\to 0}\frac{(1+as^2)^n-e^{-\alpha s^2}}{s(1+as^2)^n}\cdot\frac{\partial}{\partial s} \\ =& -(n-1)\cdot\lim_{s\to 0}\frac{n(1+as^2)^{n-1}\cdot 2as+2\alpha se^{-\alpha s^2}}{(1+as^2)^n+s\cdot n(1+as^2)^{n-1}\cdot 2as}\cdot\frac{\partial}{\partial s} \\ =&0. \end{aligned}$$

Therefore, in this case, l_0 is minimal in L. Secondly, the author is going to prove a general version of the fact l_0 is volume-minimizing as well as minimal in his next paper. See also it. A bit of information is below. We can see $\pi^*(\mathrm{vol}_{l_0})$ is a calibration of L that is described in the section 4 of [6] and that is a little difficult to prove, and l_0 is the calibrated submanifold, where π is the projection from L to l_0 defined by

$$\pi(x_1w_1(s),\ldots,x_nw_n(s))=(x_1w_1(0),\ldots,x_nw_n(0))$$

and vol_{l_0} is the volume form of l_0 with respect to the induced metric of the Euclidean metric in $\mathbb{C}^n$. (Note that π is well-defined.) This implies that l_0 is volume-minimizing as well as minimal. In addition, without the restriction $w_1=\cdots=w_n$, the submanifold $\{y=0\}$ is a calibrated submanifold in the self-expander L [8, Thorem C].

Further we have to compute the following situation.

Remark 5.3.2. Let $a>0$, $E>1$ and $\alpha\geq 0$ be constants. Define $r:(0,\infty)\to\mathbb{R}$ by $r(s)=\sqrt{1/a+s^2}$ and $\phi:(0,\infty)\to\mathbb{R}$ by

$$\phi_E(s)=\int_0^s\frac{|t|dt}{(1/a+t^2)\sqrt{E(1+at^2)^ne^{\alpha t^2}-1}}.$$

In the situation of Theorem 5.2, if we put $I=(0,\infty)$ and $w(s)=r(s)e^{i\phi_E(s)}$, then L is the Lagrangian self-similar solution constructed in Theorem 1.3 in Chapter 1. Then we can see that

$$\lim_{s\to+0}H_s(x_1w(s),\ldots,x_nw(s))$$

is a non-zero vector. This proof is left to the reader.

6. Discussion

We can also consider the mean curvature flow in product manifolds, for example, cone manifolds, the paraboloid of revolution and so on, similarly to this paper and can obtain the solutions.

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