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Phase synchronization between collective rhythms of globally coupled oscillator groups: Noisy identical case

Yoji Kawamura,[1,a)] Hiroya Nakao,[2] Kensuke Arai,[3] Hiroshi Kori,[4] and Yoshiki Kuramoto[5]

[1]*Institute for Research on Earth Evolution, Japan Agency for Marine-Earth Science and Technology, Yokohama 236-0001, Japan*
[2]*Department of Physics, Kyoto University, Kyoto 606-8502, Japan and CREST, JST, Kyoto 606-8502, Japan*
[3]*Brain Science Institute, RIKEN, Wako 351-0198, Japan*
[4]*Division of Advanced Sciences, Ochadai Academic Production, Ochanomizu University, Tokyo 112-8610, Japan and PRESTO, Japan Science and Technology Agency, Kawaguchi 332-0012, Japan*
[5]*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan and Institute for Integrated Cell-Material Sciences, Kyoto University, Kyoto 606-8501, Japan*

We theoretically investigate the collective phase synchronization between interacting groups of globally coupled noisy identical phase oscillators exhibiting macroscopic rhythms. Using the phase reduction method, we derive coupled collective phase equations describing the macroscopic rhythms of the groups from microscopic Langevin phase equations of the individual oscillators via nonlinear Fokker–Planck equations. For sinusoidal microscopic coupling, we determine the type of the collective phase coupling function, i.e., whether the groups exhibit in-phase or antiphase synchronization. We show that the macroscopic rhythms can exhibit effective antiphase synchronization even if the microscopic phase coupling between the groups is in-phase, and vice versa. Moreover, near the onset of collective oscillations, we analytically obtain the collective phase coupling function using center-manifold and phase reductions of the nonlinear Fokker–Planck equations. © *2010 American Institute of Physics*. [doi:10.1063/1.3491344]

Systems of limit-cycle oscillators are used to model various rhythmic phenomena in natural and artificial systems. When the interaction between the oscillators is weak, the system can generally be described as coupled phase oscillators. Qualitative understanding of synchronization phenomena, in particular the emergence of collective oscillations, has been achieved successfully through this type of reduced model. In the present paper, we develop collective phase reduction method for macroscopic oscillations arising from mutual synchronization of many microscopic oscillators within a group, which is typical in the real world. We show that weakly interacting groups of oscillators, each consisting of globally coupled noisy identical phase oscillators and exhibiting collective oscillations, can be reduced to coupled equations for collective phases. This makes it possible to analyze the nature of synchronization between the collective oscillations in a closed way at the macroscopic level.

I. INTRODUCTION

Populations of coupled rhythmic elements can exhibit macroscopic oscillations through mutual synchronization.[1–6] The phase oscillator models have played important roles in theoretically analyzing their behavior, and the special class of models given by globally coupled phase oscillators, in particular, was studied most intensively in the past.[7–12] Theoretical predictions based on such models have also

been experimentally validated, e.g., in electrochemical oscillator systems[13–17] and in discrete chemical oscillator populations.[18–20]

Recently, macroscopic synchronization between interacting groups of globally coupled phase oscillators exhibiting collective oscillations has attracted attention.[21–26] In most of the works so far, the macroscopic properties such as mutual entrainment between the groups have been analyzed through the microscopic individual phases. However, because we are interested in the macroscopic behavior of the collective rhythms exhibited by the oscillator groups, it should be much more convenient if each group of oscillators can be treated as a single macroscopic oscillator. Based on such consideration, we have developed collective phase reduction methods,[27–29] which provide us with the collective phase sensitivity of macroscopic rhythms of the oscillator group to weak perturbations.

In this paper, we employ the notion of collective phase description,[27–29] and formulate a theory for weakly interacting groups of globally coupled noisy identical phase oscillators in a closed form at the macroscopic level. Specifically, we derive coupled collective phase equations from microscopic Langevin phase equations describing weakly interacting groups of globally coupled phase oscillators. A general formula that gives collective phase coupling functions from the microscopic phase coupling functions between the individual oscillators is obtained, and for the case with sinusoidal coupling, the types of the collective phase coupling function are determined as a function of the coupling parameters. Near the onset of collective oscillations, we can even

a)Electronic mail: ykawamura@jamstec.go.jp.

analytically obtain the collective phase coupling function by the center-manifold and phase reductions. Based on the collective phase equations, we illustrate counterintuitive phenomena in which two oscillator groups become antiphase synchronized in spite of in-phase microscopic coupling between the groups, and vice versa (in-phase synchronization despite antiphase microscopic coupling).

In Ref. 30, we considered a similar problem, namely, collective phase synchronization between two groups of globally coupled oscillators. The crucial difference is that we treat noisy identical phase oscillators in the present work, whereas we analyzed noiseless nonidentical phase oscillators in Ref. 30. Although these two situations look similar, they are essentially different physical systems (i.e., stochastic versus deterministic) and mathematical treatments should be developed independently. Here we apply center-manifold reduction as well as phase reduction to nonlinear Fokker–Planck equations governing the oscillator groups, whereas we used the Ott–Antonsen ansatz in the analysis of the noiseless nonidentical system.[30] In both cases, despite the large difference in their mathematical structures, we obtain similar coupled collective phase equations describing macroscopic dynamics of the groups. Thus, the present paper and Ref. 30 are mutually complementary and together give deeper understanding of macroscopic collective phenomena.

The organization of the present paper is the following. In Sec. II, we introduce a model of weakly interacting groups of globally coupled noisy phase oscillators and illustrate both effective antiphase and in-phase collective phase synchronization between the groups by numerical simulations. In Sec. III, we develop a theory that derives coupled collective phase equations from the microscopic model and determine the effective type of phase coupling between collective oscillations. In Sec. IV, we analytically obtain the collective phase coupling function near the onset of collective oscillations and discuss several important cases. In Sec. V, we discuss a relation to noise-induced turbulence in a system of nonlocally coupled oscillators. Concluding remarks will be given in Sec. VI.

II. THE MODEL AND ITS DYNAMICS

A. Interacting groups of globally coupled phase oscillators

We consider two interacting groups of globally coupled noisy identical phase oscillators described by the following model:

$$
\dot{\phi}_j^{(\sigma)}(t) = \omega + \frac{1}{N}\sum_{k=1}^{N}\Gamma(\phi_j^{(\sigma)} - \phi_k^{(\sigma)}) + \sqrt{D}\xi_j^{(\sigma)}(t)
$$
$$
+ \frac{\epsilon}{N}\sum_{k=1}^{N}\Gamma_{\sigma\tau}(\phi_j^{(\sigma)} - \phi_k^{(\tau)}) \tag{1}
$$

for $j = 1, \ldots, N$ and $(\sigma, \tau) = (1,2)$ or $(2,1)$, where $\phi_j^{(\sigma)}(t)$ is the phase of the jth oscillator in the σth group consisting of N oscillators and ω is the natural frequency common to all oscillators. The second term on the right-hand side represents the internal coupling between the oscillators within the same

group, the third term represents the noise, and the last term gives the external coupling between the oscillators belonging to different groups. The internal phase coupling function $\Gamma(\phi)$ is assumed to be in-phase, $d\Gamma(\phi)/d\phi|_{\phi=0} < 0$,[2] namely, the oscillators within the same group tend to synchronize with each other. The external phase coupling function between the groups is described by $\Gamma_{\sigma\tau}(\phi)$. Characteristic intensity of the internal coupling within each group is scaled to unity, whereas that of the external coupling between the groups is given by $\epsilon \geq 0$. The noise $\xi_j^{(\sigma)}(t)$ is assumed to be white Gaussian,[31–33] whose statistics are given by

$$
\langle\xi_j^{(\sigma)}(t)\rangle = 0, \quad \langle\xi_j^{(\sigma)}(t)\xi_k^{(\tau)}(s)\rangle = 2\delta_{jk}\delta_{\sigma\tau}\delta(t-s). \tag{2}
$$

The noise intensity is characterized by $D \geq 0$. When the external coupling is absent, i.e., $\epsilon = 0$, Eq. (1) has a critical noise intensity D_c below which phase coherent states are realized, namely, collective oscillations arise when $0 \leq D < D_c$.[2] In the following, we assume that ϵ is sufficiently small and each group of oscillators exhibits stable collective oscillations.

B. Phase synchronization between collective oscillations

In the following numerical simulations, we assume that the phase coupling functions are sinusoidal (note, however, that our theory itself can be applied to general 2π-periodic phase coupling functions[28]). Without loss of generality, the natural frequency can be assumed to be zero, $\omega = 0$. The internal phase coupling function between the oscillators within the same group is given by

$$
\Gamma(\phi) = -\sin(\phi + \alpha), \quad |\alpha| < \frac{\pi}{2}, \tag{3}
$$

which is in-phase (attractive). In this case, the critical noise intensity is given by $D_c = (\cos\alpha)/2$, as explained in Sec. IV. The external phase coupling function is described by

$$
\Gamma_{\sigma\tau}(\phi) = -\sin(\phi + \beta), \quad |\beta| \leq \pi, \tag{4}
$$

which can be either in-phase (attractive) ($|\beta| < \pi/2$) or antiphase (repulsive) ($|\beta| > \pi/2$).[2] Introducing a complex order parameter $A^{(\sigma)}(t)$ with modulus $R^{(\sigma)}(t)$ and phase $\Theta^{(\sigma)}(t)$ through

$$
A^{(\sigma)}(t) = R^{(\sigma)}(t)e^{i\Theta^{(\sigma)}(t)} = \frac{1}{N}\sum_{k=1}^{N}e^{i\phi_k^{(\sigma)}(t)}, \tag{5}
$$

we can rewrite Eq. (1) with the sinusoidal coupling functions given in Eqs. (3) and (4) as follows:

$$
\dot{\phi}_j^{(\sigma)}(t) = \omega - R^{(\sigma)}\sin(\phi_j^{(\sigma)} - \Theta^{(\sigma)} + \alpha) + \sqrt{D}\xi_j^{(\sigma)}(t)
$$
$$
- \epsilon R^{(\tau)}\sin(\phi_j^{(\sigma)} - \Theta^{(\tau)} + \beta). \tag{6}
$$

Note that $R^{(\sigma)}$ quantifies the degree of synchronization and $\Theta^{(\sigma)}$ gives the *collective phase* of the σth group.

Focusing on weakly coupled collective oscillations, we carried out numerical simulations of Eq. (6) with Eq. (5) under the following conditions: the external coupling was assumed to be much weaker than the internal coupling, i.e., $\epsilon = 0.01 \ll 1$; we set $D = D_c/2 = (\cos\alpha)/4$ and $\alpha = 3\pi/8$; the
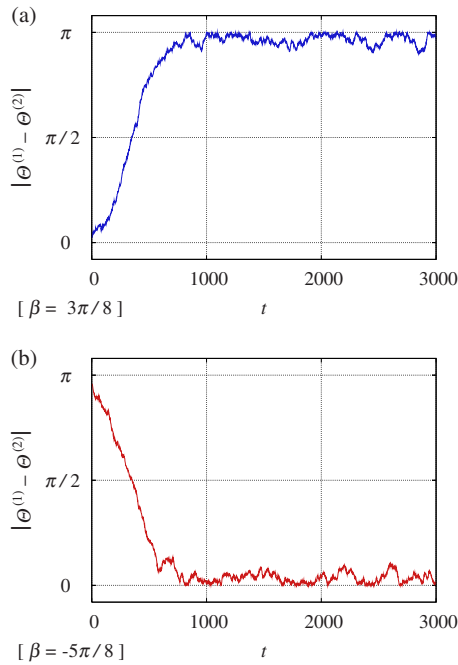
FIG. 1. (Color online) Time evolution of collective phase difference $|\Theta^{(1)}-\Theta^{(2)}|$. (a) Effective antiphase collective synchronization with microscopic in-phase external coupling, $\beta=3\pi/8$. (b) Effective in-phase collective synchronization with microscopic antiphase external coupling, $\beta=-5\pi/8$. The other parameters are $\epsilon=0.01$, $\omega=0$, $D=D_c/2=(\cos\alpha)/4$, and $\alpha=3\pi/8$. In numerical simulations, the number of oscillator in each group is $N=10\,000$.
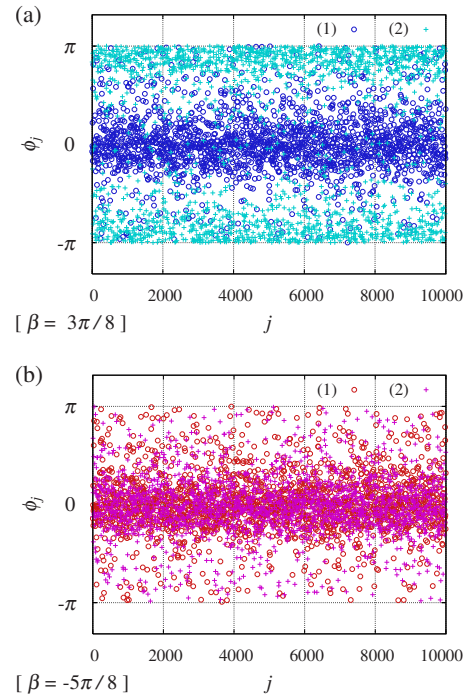

FIG. 2. (Color online) Snapshots of the asymptotic states of the individual oscillators in Fig. 1. Only one in every five oscillators is plotted. Open circle ($\bigcirc$) and plus ($+$) respectively indicate oscillator of group (1) and that of group (2). (a) Effective antiphase coupling with microscopic in-phase coupling, $\beta=3\pi/8$. (b) Effective in-phase coupling with microscopic antiphase coupling, $\beta=-5\pi/8$.

number of oscillators in each group was $N=10\,000$, which was sufficiently large to observe clear collective oscillations. We separately prepared two groups of phase oscillators exhibiting collective oscillations and used these states as the initial conditions of the simulations.

In Fig. 1(a), evolution of the collective phase difference $|\Theta^{(1)}-\Theta^{(2)}|$ from almost in-phase synchronized state of the groups is shown. In spite of the in-phase external phase coupling condition between individual oscillator pairs, $\beta=3\pi/8$, the collective phase difference $|\Theta^{(1)}-\Theta^{(2)}|$ eventually approached π, namely, the two groups became antiphase synchronized after some time. Thus, Fig. 1(a) indicates that the collective phase coupling function between the group is antiphase although microscopic external phase coupling functions are in-phase. In contrast, Fig. 1(b) shows evolution of $|\Theta^{(1)}-\Theta^{(2)}|$ from almost antiphase synchronized state of the groups with antiphase microscopic external phase coupling function, $\beta=-5\pi/8$, which eventually became in-phase synchronized.

Snapshots of the microscopic phase variables after the collective phase difference has reached the asymptotic value in Fig. 1 are displayed in Fig. 2. In Fig. 2(a), the two distributions of the oscillators are shifted by π, indicating antiphase synchronization between the groups. In contrast, the two distributions almost overlap in Fig. 2(b), i.e., they are in-phase synchronized. Note that oscillators from different groups do not synchronize with each other. In other words, the collective phase synchronization between the groups is not due to complete synchronization of individual oscillators at the microscopic level.

Thus, the type of the collective phase coupling functions can be effectively different from that of the microscopic external phase coupling functions, depending on the collective dynamics of the oscillators taking place in each group. We develop a theory that yields the collective phase coupling function from the microscopic model in Secs. III and IV.

III. COLLECTIVE PHASE REDUCTION

We derive coupled dynamical equations for the collective phase variables of the groups from the Langevin phase equations of individual oscillators through nonlinear Fokker–Planck equations and obtain a formula that relates the collective phase coupling function between the groups to the microscopic phase coupling function between individual oscillator pairs from different groups. Using them, we determine the type of the collective phase coupling function and explain the results of the numerical simulations in Sec. II.

A. Nonlinear Fokker–Planck equations

In the continuum limit, i.e., $N\to\infty$, the Langevin phase equations (1) can be transformed into the following coupled nonlinear Fokker–Planck equations:[2,21,27,28]

$$\frac{\partial}{\partial t}f^{(\sigma)}(\phi,t)$$

$$= -\frac{\partial}{\partial\phi}\left[\left\{\omega + \int_0^{2\pi}d\phi'\Gamma(\phi-\phi')f^{(\sigma)}(\phi',t)\right\}f^{(\sigma)}(\phi,t)\right]$$

$$+ D\frac{\partial^2}{\partial\phi^2}f^{(\sigma)}(\phi,t)$$

$$- \epsilon\frac{\partial}{\partial\phi}\left[\int_0^{2\pi}d\phi'\Gamma_{\sigma\tau}(\phi-\phi')f^{(\tau)}(\phi',t)f^{(\sigma)}(\phi,t)\right] \quad (7)$$

for $(\sigma,\tau)=(1,2)$ or $(2,1)$. Here, $f^{(\sigma)}(\phi,t)$ is the one-body probability density function of the individual oscillator phase ϕ in the σth group, which is normalized as $\int_0^{2\pi}d\phi f^{(\sigma)}(\phi,t) = 1$. The first two terms on the right-hand side represent internal dynamics of the σth group, and the third term represents weak interaction between σth group and τth group. The complex order parameter of Eq. (5) is now expressed as

$$A^{(\sigma)}(t) = R^{(\sigma)}(t)e^{i\Theta^{(\sigma)}(t)} = \int_0^{2\pi}d\phi e^{i\phi}f^{(\sigma)}(\phi,t). \quad (8)$$

When the external coupling between the groups is absent, each group of oscillators obeying Eq. (7) with $\epsilon=0$ exhibits collective rhythms under the condition $0 \le D < D_c$.[2,21,27,28] We assume that this situation persists even if ϵ becomes slightly positive and the two groups interact with each other weakly.

The collectively oscillating solution of the nonlinear Fokker–Planck equations (7) without external coupling ($\epsilon=0$) can be expressed as a steadily rotating wave packet on a periodic interval $[0,2\pi]$,

$$f^{(\sigma)}(\phi,t) = f_0(\varphi^{(\sigma)}), \quad \varphi^{(\sigma)} = \phi - \Theta^{(\sigma)}, \quad \dot{\Theta}^{(\sigma)} = \Omega \quad (9)$$

for $\sigma=1,2$, where the $f_0(\varphi)$ represents the steady functional shape of the wave packet, $\Theta^{(\sigma)}$ is the location of the wave packet at time t, namely, the collective phase of the σth group, and Ω is the collective frequency common to both groups.[27,28]

B. Collective phase equations

Let us assume $\epsilon=0$ and focus on a single group. The group index σ will be dropped for the moment. Inserting Eq. (9) into the nonlinear Fokker–Planck equation (7) with $\epsilon=0$, we find that $f_0(\varphi)(\varphi=\phi-\Theta)$ satisfies the following equation:

$$D\frac{d^2}{d\varphi^2}f_0(\varphi) + (\Omega-\omega)\frac{d}{d\varphi}f_0(\varphi) - \frac{d}{d\varphi}[g_0(\varphi)f_0(\varphi)] = 0, \quad (10)$$

where

$$g_0(\varphi) = \int_0^{2\pi}d\varphi'\Gamma(\varphi-\varphi')f_0(\varphi'). \quad (11)$$

Let $u(\varphi,t)$ represent small disturbance to the collectively oscillating solution and consider a slightly perturbed solution $f(\phi,t) = f_0(\varphi) + u(\varphi,t)$. Equation (7) with $\epsilon=0$ is linearized in

$u(\varphi,t)$, i.e., $\partial_t u(\varphi,t) = \hat{L}u(\varphi,t)$, where the linear operator $\hat{L}$ is given by

$$\hat{L}u(\varphi) = D\frac{d^2}{d\varphi^2}u(\varphi) + (\Omega-\omega)\frac{d}{d\varphi}u(\varphi) - \frac{d}{d\varphi}[g_0(\varphi)u(\varphi)]$$

$$- \frac{d}{d\varphi}\left[f_0(\varphi)\int_0^{2\pi}d\varphi'\Gamma(\varphi-\varphi')u(\varphi')\right]. \quad (12)$$

Defining the inner product as

$$[u^*(\varphi),u(\varphi)] = \int_0^{2\pi}d\varphi u^*(\varphi)u(\varphi), \quad (13)$$

we introduce an adjoint operator $\hat{L}^*$ of $\hat{L}$ by

$$[u^*(\varphi),\hat{L}u(\varphi)] = [\hat{L}^*u^*(\varphi),u(\varphi)]. \quad (14)$$

The adjoint operator $\hat{L}^*$ is explicitly given as

$$\hat{L}^*u^*(\varphi) = D\frac{d^2}{d\varphi^2}u^*(\varphi) - (\Omega-\omega)\frac{d}{d\varphi}u^*(\varphi) + g_0(\varphi)\frac{d}{d\varphi}u^*(\varphi)$$

$$+ \int_0^{2\pi}d\varphi'\Gamma(\varphi'-\varphi)f_0(\varphi')\frac{d}{d\varphi'}u^*(\varphi'). \quad (15)$$

In the calculation below, we need only zero eigenfunctions $u_0(\varphi)$ of $\hat{L}$ and $u_0^*(\varphi)$ of $\hat{L}^*$. Note that the right zero eigenfunction can be chosen as

$$\hat{L}u_0(\varphi) = 0, \quad u_0(\varphi) = \frac{d}{d\varphi}f_0(\varphi), \quad (16)$$

which follows from differentiation of Eq. (10) with respect to φ. The left zero eigenfunction is normalized as

$$\hat{L}^*u_0^*(\varphi) = 0, \quad [u_0^*(\varphi),u_0(\varphi)] = 1. \quad (17)$$

Now let us introduce weak external coupling, i.e., we assume $0 < \epsilon \ll 1$ and treat the last term in Eq. (7) as perturbations. Using the phase reduction method,[2,27,28] we can derive coupled collective phase equations from the nonlinear Fokker–Planck equations (7). Namely, we project the nonlinear Fokker–Planck equations (7) onto the unperturbed collectively oscillating solution as

$$\frac{d}{dt}(-\Theta^{(\sigma)})$$

$$= \left[u_0^*(\phi-\Theta^{(\sigma)}),\frac{\partial}{\partial t}f^{(\sigma)}(\phi,t)\right]$$

$$\simeq -\Omega - \epsilon\left[u_0^*(\varphi),\frac{d}{d\varphi}\int_0^{2\pi}d\varphi'\Gamma_{\sigma\tau}(\varphi-\varphi'-\Theta^{(\sigma)}-\Theta^{(\tau)})\right.$$

$$\times f_0(\varphi')f_0(\varphi)\bigg], \quad (18)$$

where we approximated $f^{(\sigma)}(\phi,t)$ by the unperturbed solution $f_0(\varphi^{(\sigma)})$ and used that $[u_0^*(\varphi),\dot{f}_0(\varphi)] = -\Omega$. Therefore, the collective phase equation takes the form

$$\dot{\Theta}^{(\sigma)} = \Omega + \epsilon\Gamma_{\sigma\tau}(\Theta^{(\sigma)}-\Theta^{(\tau)}), \quad (19)$$

where the *collective phase coupling function* is given by

$$F_{\sigma\tau}(\Theta^{(\sigma)} - \Theta^{(\tau)}) = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \Gamma_{\sigma\tau}(\varphi - \varphi' + \Theta^{(\sigma)} - \Theta^{(\tau)})$$

$$\times k_0(\varphi) f_0(\varphi') \quad (20)$$

for $(\sigma, \tau) = (1, 2)$ or $(2, 1)$. The function $k_0(\varphi)$ is defined by

$$k_0(\varphi) = -f_0(\varphi) \frac{d}{d\varphi} u_0^*(\varphi) \quad (21)$$

and normalized as

$$\int_0^{2\pi} d\varphi \, k_0(\varphi) = \int_0^{2\pi} d\varphi \, u_0^*(\varphi) u_0(\varphi) = 1, \quad (22)$$

which is the kernel function determining the *collective phase sensitivity* of the group as a convolution of the microscopic phase sensitivity.[28]

C. The case with sinusoidal coupling

When the microscopic external phase coupling function $\Gamma_{\sigma\tau}(\phi)$ is sinusoidal as given in Eq. (4), the collective phase coupling function also takes a sinusoidal form

$$F_{\sigma\tau}(\Theta) = -\rho \sin(\Theta + \delta), \quad (23)$$

because Eq. (20) is a double convolution of $\Gamma_{\sigma\tau}(\varphi - \varphi' + \Theta)$ with $k_0(\varphi) f_0(\varphi')$. Here, the parameter $\rho \cos \delta$ determines the effective type of the collective phase coupling function; it is in-phase when $\rho \cos \delta > 0$ and antiphase when $\rho \cos \delta < 0$. This quantity can be evaluated from the following equation:

$$\rho \cos \delta = -\left. \frac{dF_{\sigma\tau}(\Theta)}{d\Theta} \right|_{\Theta=0}$$

$$= -\int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \Gamma_{\sigma\tau}(\varphi - \varphi') k_0(\varphi) u_0(\varphi'). \quad (24)$$

Now we examine the case $D = D_c/2 = (\cos \alpha)/4$, which we considered in the numerical simulations shown in Fig. 1. Typical functional shapes of $f_0(\phi)$, $u_0(\phi)$, $u_0^*(\phi)$, and $k_0(\phi)$ in this case are illustrated in Fig. 3, which were numerically obtained from the nonlinear Fokker–Planck equation. Details of the numerical method are described in Ref. 27. From these functions, the dependence of $\rho \cos \delta$ on α and β was numerically evaluated by Eq. (24), as shown in Fig. 4(a). The type of the collective phase coupling function is represented in Fig. 4(b), where the solid curves satisfying $\rho \cos \delta = 0$ represent the borders between the in-phase and the antiphase parameter regions. The two sets of parameter values used in Fig. 1 are also plotted in Fig. 4(b). As can be seen, the set of parameters corresponding to Fig. 1(a) is in the antiphase region, $\rho \cos \delta < 0$, which yields effective antiphase collective phase coupling between the groups. Similarly, the parameter set corresponding to Fig. 1(b) is in the in-phase region, $\rho \cos \delta > 0$, yielding effective in-phase collective phase coupling. Thus, the collective phase reduction theory successfully explains the numerical results in Fig. 1.
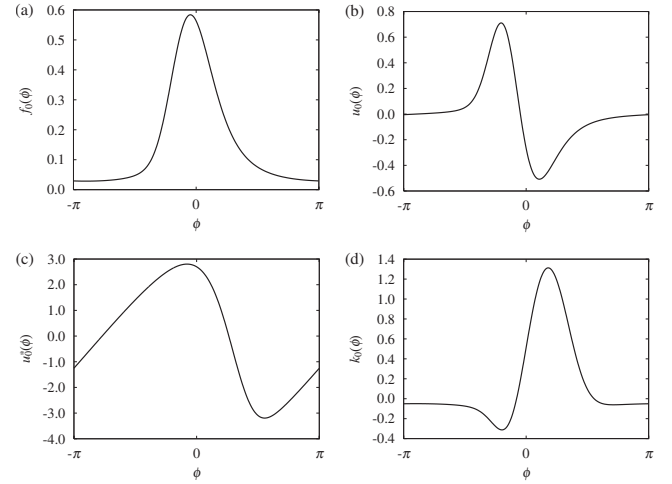

FIG. 3. (a) Distribution function $f_0(\phi)$. (b) Right zero eigenfunction $u_0(\phi)$. (c) Left zero eigenfunction $u_0^*(\phi)$. (d) Kernel function $k_0(\phi)$. Parameters are $D = D_c/2 = (\cos \alpha)/4$ and $\alpha = 3\pi/8$, where order parameter amplitude is $R = 0.653\,28$. $\beta = 3\pi/8$ gives $\rho \cos \delta = -0.322\,64$, whereas $\beta = -5\pi/8$ gives $\rho \cos \delta = 0.322\,64$.

IV. CENTER-MANIFOLD AND PHASE REDUCTIONS

In this section, we analytically determine the collective phase coupling function at the onset of collective oscillations by applying phase reduction to amplitude equations obtained by the center-manifold reduction of the nonlinear Fokker–Planck equations. This method gives analytical results without recourse to numerical determination of the kernel and other functions for general microscopic phase coupling functions, although restricted to the vicinity of the onset of collective oscillations.
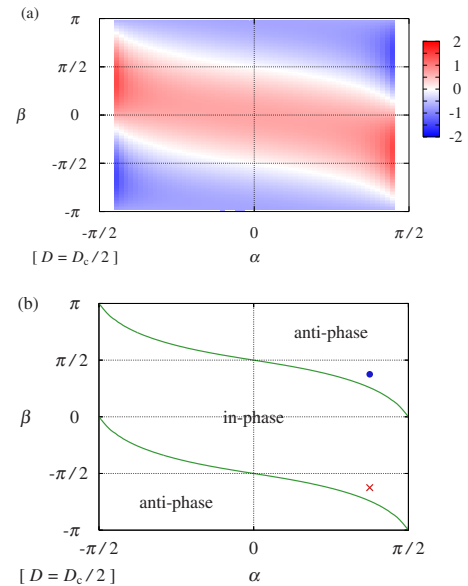

FIG. 4. (Color online) Effective type of phase coupling between collective oscillations with $\alpha \in (-\pi/2, \pi/2)$, $\beta \in [-\pi, \pi]$, and $D = D_c/2 = (\cos \alpha)/4$, which is numerically evaluated by Eq. (24). (a) Dependence of $\rho \cos \delta$ on α and β. (b) The solid curves are determined by $\rho \cos \delta = 0$. The filled circle (●) indicates $\alpha = \beta = 3\pi/8$ corresponding to Figs. 1(a) and 2(a). The cross (×) indicates $\alpha = 3\pi/8$ and $\beta = -5\pi/8$ corresponding to Figs. 1(b) and 2(b).

A. Amplitude equations near the onset of collective oscillations

We derive coupled amplitude equations that describe the macroscopic rhythms of the groups near the onset of collective oscillations. Expanding the 2π-periodic functions $f^{(\sigma)}(\phi,t)$, $\Gamma(\phi)$, and $\Gamma_{\sigma\tau}(\phi)$ into Fourier series as

$$f^{(\sigma)}(\phi,t) = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty} f_l^{(\sigma)}(t)e^{il\phi}, \quad f_l^{(\sigma)}(t) = \int_0^{2\pi} d\phi f^{(\sigma)}(\phi,t)e^{-il\phi},$$
(25)

$$\Gamma(\phi) = \sum_{l=-\infty}^{\infty} \Gamma_l e^{il\phi}, \quad \Gamma_l = \frac{1}{2\pi}\int_0^{2\pi} d\phi \Gamma(\phi)e^{-il\phi},$$
(26)

$$\Gamma_{\sigma\tau}(\phi) = \sum_{l=-\infty}^{\infty} \Gamma_{\sigma\tau,l} e^{il\phi}, \quad \Gamma_{\sigma\tau,l} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \Gamma_{\sigma\tau}(\phi)e^{-il\phi}, \quad (27)$$

the coupled nonlinear Fokker–Planck equations (7) can be expressed as

$$\dot{f}_l^{(\sigma)}(t) = -[Dl^2 + il(\omega + \Gamma_0 + \Gamma_l)]f_l^{(\sigma)} - il\sum_{m\neq 0,l}\Gamma_m f_m^{(\sigma)}f_{l-m}^{(\sigma)}$$
$$+ \epsilon\left[-il\Gamma_{\sigma\tau,0}f_l^{(\sigma)} - il\Gamma_{\sigma\tau,l}f_l^{(\tau)} - il\sum_{m\neq 0,l}\Gamma_{\sigma\tau,m}f_m^{(\tau)}f_{l-m}^{(\sigma)}\right].$$
(28)

When the noise intensity D is decreased below the critical value D_c in the absence of external coupling between the groups, $\epsilon=0$, the uniform solution $f^{(\sigma)}(\phi,t)=1/(2\pi)$ of Eq. (7), corresponding to the incoherent state, becomes unstable. Equivalently, the trivial solution $f_0^{(\sigma)}=1$, $f_l^{(\sigma)}=0 (l\neq 0)$ of Eq. (28) is destabilized and a pair of modes $f_{\pm l_c}^{(\sigma)}$ with critical nonzero wavenumbers $\pm l_c$ starts to grow. From the linear part of Eq. (28), instability of the mode l occurs when $Dl^2 - l\,\text{Im}\,\Gamma_l < 0$, namely, $D < \text{Im}\,\Gamma_l/l$. Thus, the most unstable wavenumbers $\pm l_c$ are those that maximize $\text{Im}\,\Gamma_l/l$. Generally, the fundamental harmonic components tend to be predominant in the phase coupling function, so that we obtain $l_c = \pm 1$ in most cases.

Assuming $l_c = \pm 1$, we introduce a complex amplitude $A^{(\sigma)}(t)$ of the fundamental harmonic modes of $f^{(\sigma)}(\phi,t)$ as

$$f^{(\sigma)}(\phi,t) = \frac{1}{2\pi} + \frac{1}{2\pi}(A^{(\sigma)}(t)e^{-i\phi} + \bar{A}^{(\sigma)}(t)e^{i\phi}),$$
(29)

where $\bar{A}^{(\sigma)}(t)$ is the complex conjugate of $A^{(\sigma)}(t)=f_{-1}^{(\sigma)}(t)$. We are concerned with weakly coupled collective oscillations, and thus consider the external interaction as perturbations. Using the center-manifold reduction method,[2] we can derive a pair of coupled complex amplitude equations from Eq. (28) in the following form:[21]

$$\dot{A}^{(\sigma)} = (\mu + i\Omega_c)A^{(\sigma)} - g|A^{(\sigma)}|^2 A^{(\sigma)} + \epsilon d_{\sigma\tau}A^{(\tau)}$$
(30)

for $(\sigma,\tau)=(1,2)$ or $(2,1)$, where the parameters are given by

$$\mu = D_c - D, \quad D_c = -\text{Im}\,\Gamma_{-1},$$
$$\Omega_c = \omega + \text{Re}\,\Gamma_{-1} + \Gamma_0 + \epsilon\Gamma_{\sigma\tau,0}$$
(31)

and

$$g = \frac{-\Gamma_{-1}(\Gamma_{-2}+\Gamma_1)}{2\,\text{Im}\,\Gamma_{-1} - i\,\text{Re}\,\Gamma_{-1} + i\Gamma_{-2}}, \quad d_{\sigma\tau} = i\Gamma_{\sigma\tau,-1}. \quad (32)$$

See Refs. 2, 21, and 27 for details of the derivation. We should note that Eq. (30) represents two coupled Stuart–Landau oscillators, each of which [i.e., $\dot{A}=(\mu+i\Omega_c)A - g|A|^2A$] describes collective oscillations of the respective oscillator group.

B. Phase reduction of the amplitude equations

Next, we derive coupled collective phase equations by reducing the coupled Stuart–Landau equations obtained above by assuming that the external interaction between the groups is sufficiently weak, i.e., ϵ is small. When the two groups are uncoupled, $\epsilon=0$, the limit-cycle solution $A_0(\Theta)$ of Eq. (30) is given by (σ is dropped again for the moment)

$$A_0(\Theta) = \sqrt{\frac{\mu}{\text{Re}\,g}}e^{i\Theta}, \quad \dot{\Theta} = \Omega = \Omega_c - \mu\frac{\text{Im}\,g}{\text{Re}\,g}. \quad (33)$$

The left and right Floquet eigenvectors of this limit-cycle solution associated with the zero eigenvalue can be written as

$$U_0(\Theta) = \frac{dA_0(\Theta)}{d\Theta} = i\sqrt{\frac{\mu}{\text{Re}\,g}}e^{i\Theta},$$

$$U_0^*(\Theta) = i\sqrt{\frac{\text{Re}\,g}{\mu}}\frac{g}{\text{Re}\,g}e^{i\Theta},$$
(34)

where the inner product of $U_0(\Theta)$ and $U_0^*(\Theta)$ satisfies the normalization condition

$$\text{Re}[\bar{U}_0^*(\Theta)U_0(\Theta)] = 1. \quad (35)$$

Although Eq. (34) is expressed in complex representation for the sake of convenience in analytical calculations performed below, they are equivalent to the known results.[2]

Now let us introduce weak external coupling as perturbations, i.e., we assume $0 < \epsilon \ll \mu \ll 1$. Using the phase reduction method,[2] we can derive the collective phase equation (19) from the amplitude equation (30). Namely, we project the amplitude equation (30) onto the unperturbed limit-cycle orbit as

$$\dot{\Theta}^{(\sigma)} = \text{Re}[\bar{U}_0^*(\Theta^{(\sigma)})\dot{A}^{(\sigma)}]$$
$$\simeq \Omega + \epsilon\,\text{Re}[\bar{U}_0^*(\Theta^{(\sigma)})d_{\sigma\tau}A_0(\Theta^{(\tau)})], \quad (36)$$

where we approximated $A^{(\sigma)}$ by the unperturbed solution $A_0(\Theta^{(\sigma)})$ and used that $\text{Re}[\bar{U}_0^*(\Theta)A_0(\Theta)]=\Omega$. Thus, the reduced equation is obtained in the form of Eq. (19), and the collective phase coupling function is given by

$$F_{\sigma\tau}(\Theta^{(\sigma)} - \Theta^{(\tau)}) = \mathrm{Re}[\bar{U}_0^*(\Theta^{(\sigma)})d_{\sigma\tau}A_0(\Theta^{(\tau)})]. \quad (37)$$

By inserting the expressions of Eqs. (31)–(34) into the formula equation (37), the collective phase coupling function $F_{\sigma\tau}(\Theta)$ can be analytically obtained, which takes a sinusoidal form

$$F_{\sigma\tau}(\Theta) = -\rho \sin(\Theta + \delta), \quad \rho e^{i\delta} = \frac{g\bar{d}_{\sigma\tau}}{\mathrm{Re}\,g}. \quad (38)$$

Note that we have not assumed that the external phase coupling function $\Gamma_{\sigma\tau}(\phi)$ is sinusoidal so far. The sinusoidal collective phase coupling function arises because we assume that collective oscillations exhibited by the groups of oscillators are near the supercritical Hopf bifurcation point.

When the phase coupling functions are given by the sinusoidal forms, Eqs. (3) and (4), the parameters of Eqs. (31)–(33) can be calculated as

$$D_c = \frac{\cos \alpha}{2}, \quad \Omega_c = \omega - \frac{\sin \alpha}{2},$$
$$\Omega = \omega - \frac{3 \sin \alpha}{4} + \frac{D \tan \alpha}{2} \quad (39)$$

and

$$g = \frac{1}{4 \cos \alpha - 2i \sin \alpha}, \quad d_{\sigma\tau} = \frac{1}{2}e^{-i\beta}. \quad (40)$$

Inserting Eq. (40) into Eq. (38), we obtain

$$\rho e^{i\delta} = \frac{1}{4}(2 \cos \beta - \tan \alpha \sin \beta)$$
$$+ \frac{i}{4}(2 \sin \beta + \tan \alpha \cos \beta). \quad (41)$$

Therefore, the type of the collective phase coupling function is analytically found from the following quantity:

$$\rho \cos \delta = \frac{1}{4}(2 \cos \beta - \tan \alpha \sin \beta). \quad (42)$$

Reflecting the symmetry of the original model of Eq. (6) with respect to $(\alpha, \beta) \to -(\alpha, \beta)$ and $\phi \to -\phi$, Eq. (42) is symmetric about the origin in the α-β plane. The type of the collective phase coupling function at the onset of collective oscillations, i.e., $D = D_c$, is represented in Fig. 5, which is very similar to Fig. 4.

C. Several important cases

Here, we consider three special and important cases of the collective phase coupling functions derived for the sinusoidal microscopic phase coupling functions, Eqs. (3) and (4), at the onset of collective oscillations.

(i)   The first case is $\alpha = 0$, which indicates that the internal phase coupling function within the same group is antisymmetric. Inserting $\alpha = 0$ into Eq. (41), we obtain the following result:
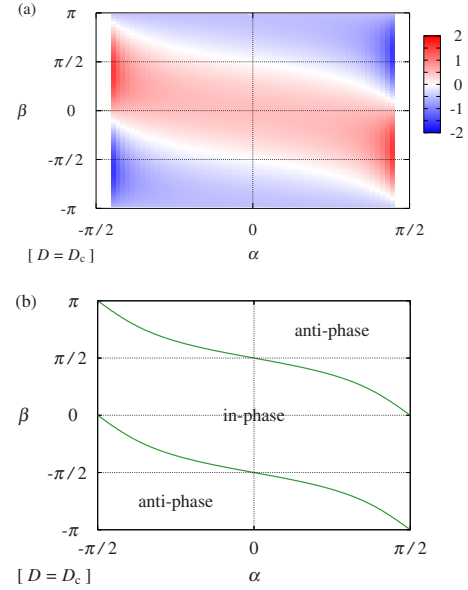

FIG. 5. (Color online) Effective type of phase coupling between collective oscillations in α and β with $D = D_c = (\cos \alpha)/2$, which is analytically given by Eq. (42), i.e., $\rho \cos \delta = (2 \cos \beta - \tan \alpha \sin \beta)/4$. (a) Dependence of $\rho \cos \delta$ on α and β. (b) The solid curves are determined by $\rho \cos \delta = 0$.

$$\rho e^{i\delta} = \frac{1}{2}e^{i\beta}, \quad (43)$$

so that $\rho \cos \delta = (\cos \beta)/2$. Thus, the collective phase coupling function has the same type as the microscopic external phase coupling function. The internal phase coupling function does not affect the type of the collective phase coupling. A similar scenario has been encountered in different models.[28–30]

(ii)   Several special values of the microscopic external coupling phase shift β comprise the second case. Inserting $\beta = 0, \pm \pi, \pm \pi/2$ into Eq. (41), we obtain the following results:

$$\beta = 0, \quad \rho e^{i\delta} = \frac{1}{2} + \frac{i}{4}\tan \alpha, \quad (44)$$

$$\beta = \pm \pi, \quad \rho e^{i\delta} = -\frac{1}{2} - \frac{i}{4}\tan \alpha, \quad (45)$$

$$\beta = \pm \frac{\pi}{2}, \quad \rho e^{i\delta} = \mp \frac{1}{4}\tan \alpha \pm \frac{i}{2}. \quad (46)$$

For antisymmetric external interactions, i.e., $\beta = 0, \pm \pi$, the type of the collective phase coupling function coincides with the microscopic external coupling and is not affected by the type of the microscopic internal coupling phase shift α. In contrast, for symmetric external interactions, i.e., $\beta = \pm \pi/2$, the type of the collective phase coupling function is solely determined by the internal coupling parameter α, which can be either in-phase or antiphase.

(iii)   The third case is $\beta = \alpha$, namely, when the external coupling has the same phase shift as the internal one. Inserting $\beta = \alpha$ into Eq. (41), we obtain the following result:

$$\rho e^{i\delta} = \frac{1}{4}(2\cos\alpha - \tan\alpha\sin\alpha) + i\frac{3\sin\alpha}{4}. \qquad (47)$$

Note that $|\beta| = |\alpha| < \pi/2$ in this case, namely, both internal and external coupling functions are in-phase. The type of the collective phase coupling function is antiphase when $\tan^2\alpha > 2$. As we discuss below, this condition is the same as the condition for noise-induced turbulence in nonlocally coupled phase oscillators.[27]

V. ON NOISE-INDUCED TURBULENCE

Finally, we briefly discuss the relation between "effective antiphase coupling" and "noise-induced turbulence." In this section, our arguments do not assume that collective oscillations are near the onset. In Ref. 27, we considered a system of nonlocally coupled noisy phase oscillators described by the following model:

$$\frac{\partial}{\partial t}\phi(\mathbf{r},t) = \omega + \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\Gamma(\phi(\mathbf{r},t) - \phi(\mathbf{r}',t))$$
$$+ \sqrt{D}\xi(\mathbf{r},t), \qquad (48)$$

where $\phi(\mathbf{r},t)$ represents the phase field of spatially extended oscillatory media, $G(\mathbf{r})$ is a nonlocal kernel function that decays with the distance $|\mathbf{r}|$, $\Gamma(\phi)$ is the phase coupling function, $\xi(\mathbf{r},t)$ represents spatiotemporally white Gaussian noise, and D is the noise intensity.

The Langevin phase equation (48) can be transformed into a nonlinear Fokker–Planck equation in the following form:

$$\frac{\partial}{\partial t}f(\phi,\mathbf{r},t) = -\frac{\partial}{\partial \phi}\left[\left\{\omega + \int_0^{2\pi} d\phi'\Gamma(\phi-\phi')\right.\right.$$
$$\times f(\phi',\mathbf{r},t)\Big\}f(\phi,\mathbf{r},t)\Big] + D\frac{\partial^2}{\partial\phi^2}f(\phi,\mathbf{r},t)$$
$$-\frac{\partial}{\partial\phi}\left[\int_0^{2\pi} d\phi'\Gamma(\phi-\phi')\right.$$
$$\times\{G_2\nabla^2 f(\phi',\mathbf{r},t)\}f(\phi,\mathbf{r},t)\Big] - \cdots. \qquad (49)$$

Here, we have expanded the nonlocal coupling term as

$$\int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')f(\phi',\mathbf{r}',t) = \sum_{n=0}^{\infty} G_{2n}\nabla^{2n}f(\phi',\mathbf{r},t), \qquad (50)$$

where G_{2n} is the $2n$th moment of $G(\mathbf{r})$.

The space-dependent order parameter $A(\mathbf{r},t)$ is defined by

$$A(\mathbf{r},t) = R(\mathbf{r},t)e^{i\Theta(\mathbf{r},t)}$$
$$= \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\int_0^{2\pi} d\phi' e^{i\phi'} f(\phi',\mathbf{r}',t), \qquad (51)$$

where $\Theta(\mathbf{r},t)$ can be considered as the space-dependent collective phase, $f(\phi,\mathbf{r},t) = f_0(\phi - \Theta(\mathbf{r},t))$. Applying the phase
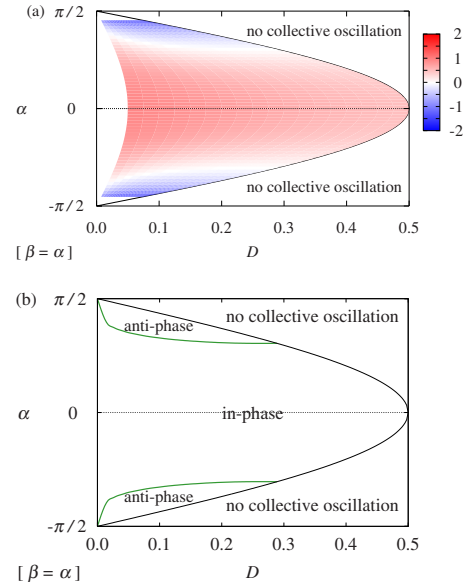

FIG. 6. (Color online) Effective type of phase coupling between collective oscillations in D and α with $\beta = \alpha$. (a) Dependence of $\rho\cos\delta$ on D and α. (b) The solid curves are determined by $\rho\cos\delta = 0$.

reduction method to Eq. (49), we obtained the following collective phase equation:

$$\frac{\partial}{\partial t}\Theta(\mathbf{r},t) = \Omega + \bar{\nu}\nabla^2\Theta(\mathbf{r},t) + \bar{\mu}(\nabla\Theta(\mathbf{r},t))^2 + \cdots, \qquad (52)$$

where $\bar{\nu}$ and $\bar{\mu}$ are coefficients. In particular, the collective phase diffusion coefficient $\bar{\nu}$ was given by

$$\bar{\nu} = -G_2\int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \Gamma(\varphi-\varphi')k_0(\varphi)u_0(\varphi'), \qquad (53)$$

which can be negative and then induce spatiotemporal chaos (turbulence). Details of the definitions and the derivations are given in Ref. 27.

Now, it is clear that Eqs. (7), (19), and (24) describing interacting groups of globally coupled noisy phase oscillators are similar to Eqs. (49), (52), and (53) describing a system of nonlocally coupled noisy phase oscillators. When the external phase coupling function is the same as the internal one, i.e., $\Gamma_{\sigma\tau}(\varphi) = \Gamma(\varphi)$, Eq. (24) is essentially equivalent to Eq. (53). Namely, the instability condition $-\Gamma'(0) < 0$ for in-phase collective synchronization between two groups, which gives the antiphase condition for the sinusoidal coupling, coincides with the instability condition $\bar{\nu} < 0$ for spatially uniform solutions of the collective phase equation. Therefore, the phase diagram plotted in Fig. 6 using D and α with $\beta = \alpha$ is the same as that we obtained for noise-induced turbulence in Ref. 27.

The above situation for collective oscillations at the macroscopic level is in parallel with the classical problem for phase oscillators at the microscopic level, in which the instability condition for in-phase synchronization of two coupled phase oscillators coincides with the instability condition for spatially uniform solutions of the phase diffusion equation.[2]
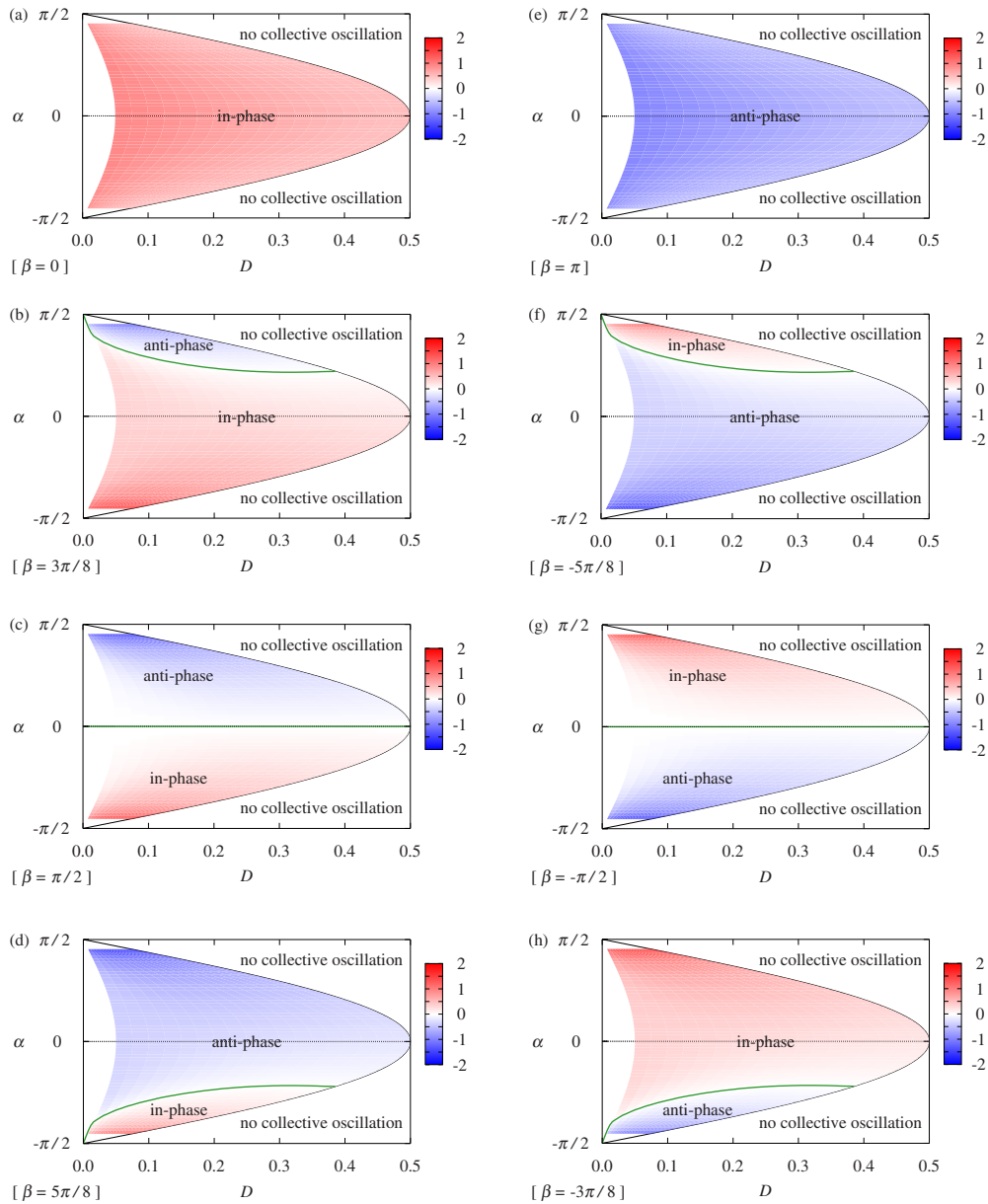
FIG. 7. (Color online) Phase diagram in D and α with (a) $\beta=0$, (b) $\beta=3\pi/8$, (c) $\beta=\pi/2$, (d) $\beta=5\pi/8$, (e) $\beta=\pi$, (f) $\beta=-5\pi/8$, (g) $\beta=-\pi/2$, and (h) $\beta=-3\pi/8$.

VI. CONCLUDING REMARKS

In the present paper, we considered two weakly interacting groups of globally coupled noisy identical phase oscillators undergoing collective oscillations. To analyze them, we adopted the idea of collective phase description, namely, we treated the collective oscillations of each group as a single macroscopic phase oscillator. We developed a theory that derives the collective phase coupling function between the groups from the microscopic external phase coupling function between individual oscillator pairs belonging to the different groups. Based on this theory, we illustrated counterintuitive situations in which the two groups become antiphase synchronized despite in-phase microscopic coupling, and vice versa. We also developed a theory that gives explicit analytical expressions of the collective phase coupling functions near the onset of collective oscillations. A complete

phase diagram in the case of the sinusoidal internal and external coupling functions is summarized in the Appendix.

In our companion work,[30] we considered two weakly interacting groups of globally coupled noiseless nonidentical phase oscillators and discussed their collective synchronization properties. The strong similarity in results between the two types of systems, one stochastic and the other deterministically random, is remarkable, while the theoretical methods employed are completely different between them. In particular, we found the same counterintuitive phenomena, namely, the disagreement of the types between the collective phase coupling function and the microscopic external phase coupling function.

The notion of collective phase description is convenient and powerful in analyzing complex macroscopic rhythms arising from systems of interacting microscopic dynamical

elements. Further development of the theories will provide useful viewpoints to understand various complex rhythms in real-world systems, in particular, their functional meaning.

APPENDIX: PHASE DIAGRAM FOR SINUSOIDAL COUPLING

We here present a complete phase diagram for the case of the sinusoidal internal and external coupling functions. The type of the collective phase coupling function is found from $\rho \cos \delta$ given in Eq. (24), which was numerically evaluated for $\beta \in [-\pi, \pi]$ in the parameter region of $D/D_c \in [0.1, 1.0]$ and $\alpha \in [-a\pi/2, a\pi/2]$ with $a=0.9$. In addition, we used the analytical formula equation (42) on the Hopf bifurcation line, i.e., $D=D_c=(\cos \alpha)/2$. Phase diagrams in D and α with several values of β are displayed in Fig. 7.

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