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A Synthesis of Complex Allpass Circuits Using the Factorization of Scattering Matrices —— Explicit Formulae for Even-Order Real Complementary Filters Having Butterworth or Chebyshev Responses ——

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SUMMARY Low-sensitivity digital filters are required for accurate signal processing. Among many low-sensitivity digital filters, a method using complex allpass circuits is well-known. In this paper, a new synthesis of complex allpass circuits is proposed. The proposed synthesis can be realized more easily either only in the z-domain or in the s-domain than conventional methods. The key concept for the synthesis is based on the factorization of lossless scattering matrices. Complex allpass circuits are interpreted as lossless digital two-port circuits, whose scattering matrices are factored. Furthermore, in the cases of Butterworth, Chebyshev and inverse Chebyshev responses, the explicit formulae for multiplier coefficients are derived, which enable us to synthesize the objective circuits directly from the specifications in the s-domain. Finally design examples verify the effectiveness of the proposed method.

key words: *complex allpass circuit, scattering matrix, factorization, analog transfer function, characteristic function*

1. Introduction

Digital filters are usually designed assuming that the wordlength of the multiplier coefficients is infinite. Although the wordlength is desired to be as short as possible from the cost and speed points of view, the characteristics of the short wordlength digital filters get worse than those of the initially designed digital filter. Low-sensitivity digital filters are used to reduce the effect of finite wordlength. So far a number of low-sensitivity digital filters are proposed, which include wave digital lattice filters. Their structure is very simple because it is the parallel connection of two real allpass filters. Gazsi has proposed the explicit design formulae for designing wave digital lattice filters which realize odd-order Butterworth, Chebyshev, inverse Chebyshev and elliptic responses.(1) On the other hand, in the case of realizing even-order responses the real allpass filters are replaced with complex ones.(2) The resultant structure is the parallel connection of a conjugate pair of two complex allpass filters. Since the output of one of them is a complex-conjugate of the output of the other, the practical structure is reduced to single complex allpass filter. A given real transfer function is realized as its real or imaginary part. All the transfer functions treated by Ref. (2) are even-order Butterworth, Chebyshev, inverse Chebyshev and elliptic ones. They are obtained from analog transfer functions by the bilinear transformation. The design method proposed by Ref. (2) cannot synthesize such structures directly from the original analog transfer functions. On the other hand, the synthesis of the same structures based on the analog filter theory has been proposed,(3) which starts from an analog characteristic function. However the method for obtaining the complex constant of the gain term in a complex allpass function is the same as the one in Ref. (2), and is applied in the z-domain. Since Ref. (3) only classifies the poles of an analog transfer function into two groups in the s-domain, it is difficult to decide which of the two determination equations for such a constant is selected. Hence the design method proposed by Ref. (3) cannot explicitly synthesize the objective structures only in the s-domain. If they are synthesized only in the s-domain, the process of the bilinear transformation of analog transfer functions is deleted. This contributes to the simplification of the synthesis procedures. Moreover the process is more simplified if explicit design formulae like Ref. (1) can be derived.

The main purpose of this paper is to develop a simple design of the above-mentioned complex allpass filters in the s-domain, which includes explicit design formulae like Ref. (1). To this end, this paper first proposes a new synthesis method by the factorization of z-domain lossless scattering matrices.(4) The resultant procedures are similar to those of Ref. (2) except for the evaluation of the values of characteristic functions at the poles of a given transfer function. This method is accomplished by substitution. The corresponding operation in Ref. (2) is done by solving complex-coefficient algebraic equations. The substitution of poles is also possible instead of solving the equations, which is not mentioned in Ref. (2). Hence it can be said that the proposed procedures are easier than Ref. (2). Next, considering the correspondence between the z-domain and the s-domain, the proposed synthesis is completely transformed into the s-domain

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by the bilinear transformation. The obtained s-domain procedure makes it possible to synthesize the complex allpass filter from a given analog transfer function including the determination of the complex constants. Moreover, since the characteristic functions and the pole locations of classical analog transfer functions such as Butterworth, Chebyshev and inverse Chebyshev transfer functions are given in analytical forms, explicit formulae for the multiplier coefficients realizing the z-domain counterparts of these three types of transfer functions are derived. Therefore the proposed design is easier than Ref.(3), because the circuits can be synthesized directly from given specifications by using the explicit formulae.

2. Lossless Scattering Matrices

Figure 1 shows a digital two-port circuit. The relation between the inputs (X_1, X_2) and the outputs (Y_1, Y_2) of the digtal two-port circuit can be expressed by a scattering matrix $\boldsymbol{S}(z)$ as

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \boldsymbol{S}(z) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \qquad (1)$$

$\boldsymbol{S}(z)$ is paraunitary, if it satisfies

$$\boldsymbol{S}(z)\boldsymbol{S}^t(z^{-1}) = \boldsymbol{I}, \qquad (2)$$

where $\boldsymbol{S}^t(z)$ denotes the transposed matrix of $\boldsymbol{S}(z)$ and $\boldsymbol{I}$ denotes the 2×2 unit matrix. Then it is also called a lossless scattering matrix. The circuits realized from a lossless scattering matrix is well-known to have low-sensitivity in the passband.[(5),(6)]

A rational Nth-order IIR digital transfer function $H(z)$ is described by

$$H(z) = \frac{F(z)}{G(z)}, \qquad (3)$$

where $F(z)$ and $G(z)$ are Nth-order polynomials with respect to z^{-1}, and $G(z)$ is monic, i.e. the leading coefficient is equal to unity. Actual orders of $F(z)$ and $G(z)$ may be different to each other. In this paper, given transfer functions are assumed to have the same orders for simplicity.

To derive a lossless scattering matrix from a given transfer function, $K(z)$ is determined by

$$G(z)G(z^{-1}) = F(z)F(z^{-1}) + K(z)K(z^{-1}). \quad (4)$$

$K(z)$ is an Nth-order polynomial. The scattering matrix $\boldsymbol{S}(z)$ for $H(z)$ is constructed by $K(z)$, $F(z)$ and $G(z)$, as

$$\boldsymbol{S}(z) = \frac{1}{G(z)} \begin{bmatrix} F(z) & \sigma K(z^{-1})z^{-N} \\ K(z) & -\sigma F(z^{-1})z^{-N} \end{bmatrix}, \qquad (5)$$

where σ is plus or minus unity. The losslessness of $\boldsymbol{S}(z)$ is easily verified by Eq. (2).

When the scattering matrix is realized as the digital two-port circuits as shown in Fig. 1, the transfer function appears as

$$H(z) = \left.\frac{Y_1}{X_1}\right|_{X_2=0}. \qquad (6)$$

In addition, when $Q(z)$ is defined as

$$Q(z) \equiv \frac{K(z)}{G(z)}, \qquad (7)$$

$Q(z)$ appears as

$$Q(z) = \left.\frac{Y_2}{X_1}\right|_{X_2=0}. \qquad (8)$$

If $F(z)$ and $K(z)$ in Eq. (5) are symmetric polynomial, i.e. $F(z)$ and $K(z)$ are written as

$$F(z) = z^{-N}F(z^{-1}) \qquad (9)$$

$$K(z) = z^{-N}K(z^{-1}), \qquad (10)$$

Equation (5) becomes

$$\boldsymbol{S}(z) = \frac{1}{G(z)} \begin{bmatrix} F(z) & \sigma K(z) \\ K(z) & -\sigma F(z) \end{bmatrix}. \qquad (11)$$

This paper uses the even-order transfer functions which satisfy Eqs. (9) and (10).

By the way, it is very difficult to synthesize digital two-port circuits as shown in Fig. 1 directly from the high-order matrix of Eq. (11). To solve the problem, a cascade connection of first- or second-order lossless digital two-ports is often used.[(6)–(8)] Assuming that the scattering matrix $\boldsymbol{S}(z)$ derived from a given transfer function is factored into a product of $\boldsymbol{S}_1(z)$ and $\boldsymbol{S}_2(z)$ as

$$\boldsymbol{S}(z) = \boldsymbol{S}_1(z)\boldsymbol{S}_2(z), \qquad (12)$$

the cascade connection of the two-ports realizing $\boldsymbol{S}_1(z)$ and $\boldsymbol{S}_2(z)$ is shown in Fig. 2. Such a cascade connection is called a T-cascade connection.[(6)] If any two matrices in Eq. (12) are paraunitary, the other is also paraunitary.[(8)] Therefore the problem of synthesizing the digital two-port circuit as shown in Fig. 2 reduces to the problem of factoring the matrix of Eq. (11).

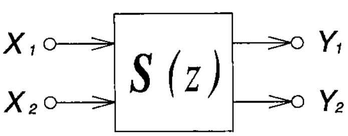

Fig. 1 Digital two-port circuit.

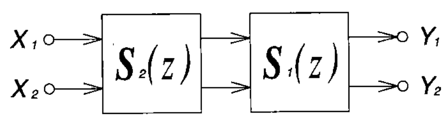

Fig. 2 Interconnecction of two-port circuits.

3. Factorization of Scattering Matrices

3.1 Conditions for Factorization

To factor the scattering matrix $S(z)$ into $S_1(z)$ and $S_2(z)$ as Eq. (12), it is the necessary and sufficient condition that the elements of $S_1(z)$ are irreducible, all the poles of $S_1(z)$ are included in those of $S(z)$ and the values of characteristic function of $S(z)$ at the poles of $S_1(z)$ are equal to the ones of $S_1(z)$ at the same values of z.[8] The characteristic function of the scattering matrix $S(z)$ is defined by

$$P(z)=\frac{K(z)}{F(z)}. \tag{13}$$

When $F(z)$ and $K(z)$ are symmetric polynomials, the substitution of Eqs. (9) and (10) for $F(z^{-1})$ and $K(z^{-1})$ in Eq. (4) gives

$$G(z)\,G(z^{-1})\,z^{-N}=F^2(z)+K^2(z). \tag{14}$$

Division of both sides of Eq. (14) by $F(z)$ leads to

$$\frac{G(z)\,G(z^{-1})\,z^{-N}}{F^2(z)}=1+P^2(z). \tag{15}$$

Let us assume that one of the poles of the transfer function $H(z)$ is denoted by $\sigma+j\omega$, which is a zero of $G(z)$. By substitution of $\sigma+j\omega$ for z in Eq. (15),

$$P(\sigma+j\omega)=\pm j \tag{16}$$

is obtained. Since $P(z)$ is a real polynomial, the relation of

$$P(\sigma+j\omega)=-P(\sigma-j\omega) \tag{17}$$

is satisfied. Because $H(z)$ is a real function, $\sigma-j\omega$ is necessarily a pole of $H(z)$. By Eqs. (16) and (17),

$$P(\sigma+j\omega)=\begin{cases} j, & \text{if } P(\sigma-j\omega)=-j \\ -j, & \text{if } P(\sigma-j\omega)=j \end{cases} \tag{18}$$

is derived.

Now it is assumed that $S_1(z)$ is a second-order lossless matrix, which has the same form as $S(z)$, with two poles at $\sigma\pm j\omega$. Its characteristic function is denoted as $P_1(z)$. For $P_1(z)$

$$P_1(\sigma+j\omega)=\begin{cases} j, & \text{if } P_1(\sigma-j\omega)=-j \\ -j, & \text{if } P_1(\sigma-j\omega)=j \end{cases} \tag{19}$$

holds. Hence, the condition to make the factorization possible is represented by

$$P_1(\sigma+j\omega)=P(\sigma+j\omega)=\pm j. \tag{20}$$

3.2 Scattering Matrices Derived from First-Order Complex Allpass Functions

A first-order complex allpass function whose pole is located at $z=a+jb$ is written as

$$A_{1P}(z)=\frac{z^{-1}-(a-jb)}{1-(a+jb)\,z^{-1}} \tag{21}$$

$$=\frac{1}{A_d(z)}(A_r(z)+jA_i(z)), \tag{22}$$

where

$$A_d(z)=1-2az^{-1}+(a^2+b^2)\,z^{-2} \tag{23}$$

$$A_r(z)=-a+(a^2-b^2+1)\,z^{-1}-az^{-2} \tag{24}$$

$$A_i(z)=b-2abz^{-1}+bz^{-2}. \tag{25}$$

The input-output relation of the complex allpass function $A_{1p}(z)$ is given by

$$A_{1P}(z)\,(X_r+jX_i)=Y_r+jY_i. \tag{26}$$

This relation is alternatively expressed in the form of a matrix multiplication as

$$\begin{bmatrix} Y_r \\ Y_i \end{bmatrix}=\boldsymbol{A}_{1P}(z)\begin{bmatrix} X_r \\ X_i \end{bmatrix}, \tag{27}$$

where $\boldsymbol{A}_{1p}(z)$ is given by

$$\boldsymbol{A}_{1P}(z)=\frac{1}{A_d(z)}\begin{bmatrix} A_r(z) & -A_i(z) \\ A_i(z) & A_r(z) \end{bmatrix}. \tag{28}$$

As $\boldsymbol{A}_{1P}(z)$ has the same form as Eq. (11) and satisfies Eq. (2), $\boldsymbol{A}_{1P}(z)$ can be regarded as a lossless scattering matrix.

Similarly a first-order complex allpass function with a pole located at $z=a-jb$ is written as

$$A_{1M}(z)=\frac{z^{-1}-(a+jb)}{1-(a-jb)\,z^{-1}} \tag{29}$$

$$=\frac{1}{A_d(z)}(A_r(z)-jA_i(z)), \tag{30}$$

and the scattering matrix derived from the function is given by

$$\boldsymbol{A}_{1M}(z)=\frac{1}{A_d(z)}\begin{bmatrix} A_r(z) & A_i(z) \\ -A_i(z) & A_r(z) \end{bmatrix}, \tag{31}$$

which is also lossless. $\boldsymbol{A}_{1P}(z)$ and $\boldsymbol{A}_{1M}(z)$ can be realized by the first-order complex allpass circuits in Ref. (9).

This paper adopts $\boldsymbol{A}_{1P}(z)$ and $\boldsymbol{A}_{1M}(z)$ as $S_1(z)$ in Eq. (12). Namely $\boldsymbol{A}_{1P}(z)$ and $\boldsymbol{A}_{1M}(z)$ are used to factor $S(z)$. Both $\boldsymbol{A}_{1P}(z)$ and $\boldsymbol{A}_{1M}(z)$ are second-order lossless matrices with the poles located at $z=a\pm jb$, and $A_d(z)$, $A_r(z)$ and $A_i(z)$ in Eqs. (23)-(25) are irreducible. For the factorization

$$P_{AP}(a+jb)=P(a+jb) \tag{32}$$

$$P_{AM}(a+jb)=P(a+jb) \tag{33}$$

must hold from Eq. (20), where $P_{AP}(z)$ and $P_{AM}(z)$ are the characteristic functions in regard to $\boldsymbol{A}_{1P}(z)$ and $\boldsymbol{A}_{1M}(z)$ and defined by

$$P_{AP}(z)=A_i(z)/A_r(z) \tag{34}$$

$$P_{AM}(z)=-A_i(z)/A_r(z) \tag{35}$$

respectively. By the way, $P_{AP}(a+jb)$ and $P_{AM}(a+jb)$ are calculated by Eqs. (23)-(25), (34) and (35) to be

$$P_{AP}(a+jb)=-j \tag{36}$$

$$P_{AM}(a+jb)=j. \tag{37}$$

By the substitution of Eqs. (36) and (37) in Eqs. (32) and (33) respectively, the condition of selecting $\boldsymbol{S}_1(z)$ is given by

$$\boldsymbol{S}_1(z)=\begin{cases}\boldsymbol{A}_{1P}(z), & \text{if } P(a+jb)=-j\\ \boldsymbol{A}_{1M}(z), & \text{if } P(a+jb)=j.\end{cases} \tag{38}$$

3.3 Factored Scattering Matrices

Suppose $\boldsymbol{A}_{1P}(z)$ is selected as $\boldsymbol{S}_1(z)$, then $\boldsymbol{S}_2(z)$ is described by Eq. (12) as

$$\boldsymbol{S}_2(z)=\boldsymbol{A}_{1P}^{-1}(z)\boldsymbol{S}(z). \tag{39}$$

By using Eqs. (11) and (28), $\boldsymbol{S}_2(z)$ is represented by

$$\boldsymbol{S}_2(z)=\frac{A_d(z)}{G(z)(A_r^2(z)+A_d^2(z))}\begin{bmatrix}S_{11}(z) & \sigma S_{21}(z)\\ S_{21}(z) & -\sigma S_{11}(z)\end{bmatrix}, \tag{40}$$

where $S_{11}(z)$ and $S_{21}(z)$ are $N+2$ th-order polynomials, and are given as

$$S_{11}(z)=A_r(z)F(z)+A_i(z)K(z) \tag{41}$$

$$S_{21}(z)=A_r(z)K(z)+A_i(z)F(z). \tag{42}$$

As the poles of $G(z)$ are located at $z=a\pm jb$, $G(z)$ is given by

$$\begin{aligned}G(z)&=\{1-2az^{-1}+(a^2+b^2)z^{-2}\}G_2(z)\\ &=A_d(z)G_2(z),\end{aligned} \tag{43}$$

where the order of $G_2(z)$ is $N-2$. In addition, since $\boldsymbol{A}_{1P}(z)$ has the same form as $\boldsymbol{S}(z)$, the application of Eq. (4) to $\boldsymbol{A}_{1P}(z)$ derives

$$A_d(z)A_d(z^{-1})=A_r(z)A_r(z^{-1})+A_i(z)A_i(z^{-1}). \tag{44}$$

Since $A_r(z)$ and $A_i(z)$ are symmetric polynomials, Eq. (44) is rewritten as

$$A_d(z)A_d(z^{-1})z^{-2}=A_r^2(z)+A_i^2(z). \tag{45}$$

$A_d(z)$ has only two zeros located at $z=a\pm jb$, and $A_d(z^{-1})z^{-2}$ has only two zeros located at $z=(a\pm jb)^{-1}$. Substituting Eqs. (43) and (45) into Eq. (40),

$$\boldsymbol{S}_2(z)=\frac{1}{G_2(z)A_d(z)A_d(z^{-1})z^{-2}}\cdot\begin{bmatrix}S_{11}(z) & \sigma S_{21}(z)\\ S_{21}(z) & -\sigma S_{11}(z)\end{bmatrix} \tag{46}$$

is obtained. Since $\boldsymbol{A}_{1P}(z)$ is selected for $P(a+jb)=P_{1P}(a+jb)=-j$, $S_{11}(z)$ and $S_{21}(z)$ in the matrix of the ritht-hand side of Eq. (46) have their zeros located at $z=a\pm jb$ and $z=(a\pm jb)^{-1}$.[8] Hence the right-hand side of Eq. (46) is reduced. As a result, the order of each element in $\boldsymbol{S}_2(z)$ is found to be $N-2$.

The characteristic function $P_2(z)$ of $\boldsymbol{S}_2(z)$ is derived from Eq. (46) as

$$P_2(z)=\frac{A_r(z)K(z)-A_i(z)F(z)}{A_r(z)F(z)+A_i(z)K(z)} \tag{47}$$

$$=\frac{P(z)-P_1(z)}{1+P(z)P_1(z)}. \tag{48}$$

Here assuming that two of the poles of the given transfer function $H(z)$ are located at $z=c\pm jd$ which are not equal to $z=a\pm jb$ and $P(c+jd)$ equals to j or $-j$, the following relations are derived from Eq. (48) as

$$P_2(c+jd)=\begin{cases}j, & \text{if } P(c+jb)=j\\ -j, & \text{if } P(c+jb)=-j.\end{cases} \tag{49}$$

If the value of $P(z)$ equals to j or $-j$ at the pole of $H(z)$, this relation is maintained after the separation of $\boldsymbol{A}_{1P}(z)$. When $\boldsymbol{A}_{1M}(z)$ is selected, the same result is obtained.

3.4 Simplified Synthesys

It is found from Sect. 3.3 that the factorization by $\boldsymbol{A}_{1P}(z)$ or $\boldsymbol{A}_{1M}(z)$ can be repeated until a constant matrix $\boldsymbol{A}_\beta$ appears, if the characteristic function $P(z)$ is equal to j or $-j$ at all the poles of the given transfer function $H(z)$. This procedure can be expressed in the form of the matrix product by

$$\boldsymbol{S}(z)=\boldsymbol{A}_1(z)\boldsymbol{A}_2(z)\cdots\boldsymbol{A}_{N/2-1}(z)\boldsymbol{A}_{N/2}(z)\boldsymbol{A}_\beta, \tag{50}$$

where $\boldsymbol{A}_k(z)$ is either $\boldsymbol{A}_{kP}(z)$ or $\boldsymbol{A}_{kM}(z)$ $(k=1, 2, \cdots, N/2)$. Cascading the circuitry realization of each scattering matrix, an overall structure is obtained as shown in Fig. 3. Equation (50) can be rewritten by the product of complex allpass transfer functions corresponding to the lossless scattering matrices as

$$A(z)=A_1(z)A_2(z)\cdots A_{N/2-1}(z)A_{N/2}(z)\beta, \tag{51}$$

where β is an complex constant whose modulus is equal to unity, and corresponds to $\boldsymbol{A}_\beta$. The given

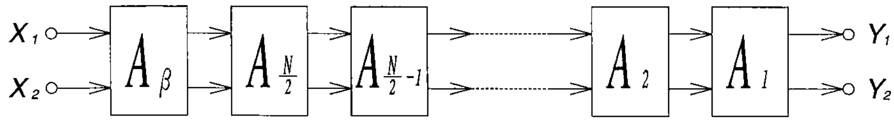

Fig. 3 Overall cascade structure.

transfer function is realized as the real or imaginary part of $A(z)$.

By the way, for the purpose of synthesizing only a complex allpass filter, the iteration process is not necessary. The selection of the required pole included in each $A_k(z)$ and the evaluation of β is formulated as follows.

(1) Among the poles of the given transfer function $H(z)$ all the poles $z_i(i=1, 2, \cdots, N/2)$ satisfying $P(z_i)=-j$ are selected.

(2) If they are described by $z_k(k=1, 2, \cdots, N/2)$, $A(z)=H(z)+jQ(z)$ can be written as

$$H(z)+jQ(z)=\beta\prod_{k=1}^{N/2}\frac{z^{-1}-z_k^*}{1-z_kz^{-1}}, \tag{52}$$

where β is unknown.

(3) Solving Eq. (52) at $z=1$ for β gives the value of β as

$$\beta=\{H(1)+jQ(1)\}\prod_{k=1}^{N/2}\frac{1-z_k}{1-z_k^*}. \tag{53}$$

This is similar to Ref.(2). A complex allpass function of Eq. (52) can be realized by the cascade connection of real digital two-port circuits as shown in Fig. 3.

When β is rewritten as

$$\beta=\gamma+j\delta, \tag{54}$$

where γ and δ are real numbers and $\gamma^2+\delta^2=1$ is satisfied, A_β is denoted by

$$A_\beta=\begin{bmatrix}\gamma & \sigma\delta\\ \delta & -\sigma\gamma\end{bmatrix}. \tag{55}$$

The value of σ in Eq.(55) corresponds to σ of $S(z)$ in Eq.(11). Since it is assumed that X_1 is the input and X_2 equals to zero in Fig. 3, the second column is not used. Therefore, the value of σ in Eq. (55) is not required. It is clear from the above result that either value of σ leads to the identical circuit. Hence it is unnecessary to determine the value of σ of $S(z)$.

The realized allpass filter is the same as the ones of Ref.(2). Reference (2) synthesizes the filter by computing the direct factorization of $H(z)+jQ(z)$. The proposed synthesis process only needs to select the poles satisfying $P(z_i)=-j$. Hence it is considered that the proposed method has the advantage over Ref.(2). The substitution of poles like this paper is also possible for the method in Ref.(2) instead of the factorization, which is not mentioned in that paper. If it is done, the synthesis procedure of this paper becomes identical to that of Ref.(2).

4. Relations between Analog and Digital Characteristic Functions

The above section has described the method to synthesize complex allpass circuits from given digital transfer functions. The realizable digital transfer functions are limited to the ones obtained by the bilinear transformation from analog transfer functions.[(2),(3)] This section describes the method to synthesize the circuits directly from original analog transfer functions.

The transfer function $S(s)$ of an analog filter is represented by

$$S(s)S(-s)=\frac{1}{1+\varepsilon^2U(s)U(-s)}, \tag{56}$$

where ε is a parameter to determine the passband ripple and $U(s)$ is the characteristic function in regard to $S(s)$. Using the bilinear transformation

$$s=\frac{1-z^{-1}}{1+z^{-1}}, \tag{57}$$

Equation (56) is rewritten as

$$S\left(\frac{1-z^{-1}}{1+z^{-1}}\right)S\left(\frac{1-z}{1+z}\right)=\frac{1}{1+\varepsilon^2U\left(\frac{1-z^{-1}}{1+z^{-1}}\right)U\left(\frac{1-z}{1+z}\right)} \tag{58}$$

and the digital transfer function $H(z)$ is described by

$$H(z)=S\left(\frac{1-z^{-1}}{1+z^{-1}}\right). \tag{59}$$

The substitution of $H(z)$ in Eq.(3) leads to

$$S\left(\frac{1-z^{-1}}{1+z^{-1}}\right)=\frac{F(z)}{G(z)}. \tag{60}$$

In addition, $C(z)$ is defined as

$$C(z)=\varepsilon U\left(\frac{1-z^{-1}}{1+z^{-1}}\right). \tag{61}$$

Equation (58) is rewritten by using Eqs.(60) and (61) as

$$\frac{F(z)F(z^{-1})}{G(z)G(z^{-1})}=\frac{1}{1+C(z)C(z^{-1})}. \tag{62}$$

By modifying Eq.(62), $C(z)C(z^{-1})$ is given as

$$C(z)C(z^{-1})=\frac{G(z)G(z^{-1})-F(z)F(z^{-1})}{F(z)F(z^{-1})}. \tag{63}$$

By using Eq. (4)

$$C(z)C(z^{-1})=\frac{K(z)K(z^{-1})}{F(z)F(z^{-1})} \tag{64}$$

is obtained. The comparison between Eqs. (13) and (64) leads to

$$P(z)=C(z). \tag{65}$$

Equation (65) shows that the bilinear transformed $\varepsilon U(s)$ equals to $P(z)$.

5. Selection of Poles from Analog Transfer Function

The poles of a digital transfer function are obtained by the bilinear transformation from the poles of the corresponding analog transfer function. In addition, $P(z)$ of the digital transfer function are obtained by the bilinear transformation from $\varepsilon U(s)$ of the analog transfer function. Hence the values of $\varepsilon U(s)$ at the analog poles coincide with the values of $P(z)$ at the digital poles.

This section provides the method of synthesizing a complex allpass circuit only from the given analog transfer function. The method is based on the selection of the analog poles. The analog poles at which the value of $\varepsilon U(s)$ equals to $-j$ are selected. The bilinear transformation of the selected analog poles gives the required digital poles.

The selection of the poles in some even-order filters is described as follows.

5.1 Butterworth Responses

$U(s)$ is written as s^N, and the poles of Butterworth responses are obtained by factoring the denominator of Eq. (56), i.e.

$$1+\varepsilon^2 s^{2N}=0 \tag{66}$$

is solved for s. The $2N$ roots are obtained as

$$s_{Ak}=\left(\frac{1}{\varepsilon}\right)^{1/N}\exp j\left(\frac{\pi+2\pi k}{2N}\right),\quad 0\leqq k\leqq 2N-1, \tag{67}$$

which lie on the circle with radius $(1/\varepsilon)^{1/N}$ in the s-plane. The required poles for the proposed method are obtained by selecting from the $2N$ roots on the condition of

$$\varepsilon s^N=-j, \tag{68}$$

and are written as

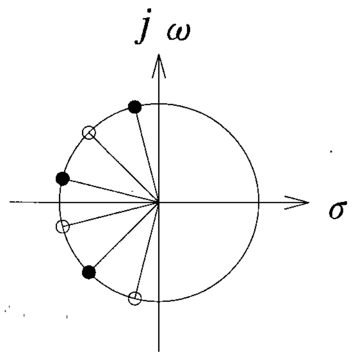

Fig. 4 Poles for sixth-order Butterworth response.

$$s_k=\left(\frac{1}{\varepsilon}\right)^{1/N}\exp j\left(\frac{3\pi/2+2\pi k}{N}\right),\quad 0\leqq k\leqq N-1. \tag{69}$$

Because of the stability condition, the poles of $S(s)$ are s_{Ak} in the left half s-plane among Eq. (67). Accordingly the required poles are s_k in the left half s-plane among Eq. (69). The following rule gives the required poles on the circle.

(1) The pole lying at the point with the angle $-\pi/(2N)$ from the negative real axis is selected.
(2) Each pole lying at the points with the angle $\pm 2\pi i/N\,(i=1, 2, \cdots, N/4)$ from the one of the previous step is selected.

The example in the sixth-order filter is shown in Fig. 4. The points marked with ● are the selected poles.

5.2 Chebyshev Responses

$U(s)$ is written as $T_N(s/j)$ by using the Chebyshev polynomial $T(x)$. As in the Butterworth case, the poles of Chevyshev responses are known to be obtained by solving for s

$$1+\varepsilon^2 T_N{}^2(s/j)=0. \tag{70}$$

The $2N$ roots are obtained as

$$s_{Ak}=\pm\sinh A\sin\left(\frac{\pi(1+2k)}{2N}\right)+j\cosh A\cos\left(\frac{\pi(1+2k)}{2N}\right),\quad 0\leqq k\leqq N-1 \tag{71}$$

where A is given as

$$A=\frac{1}{N}\sinh^{-1}\left(\frac{1}{N}\right). \tag{72}$$

These roots lie on the ellipse with the major axis $\cosh A$ and the minor axis $\sinh A$ in the s-plane.

The required poles for the proposed method are obtained by selecting from the $2N$ roots on the condition of

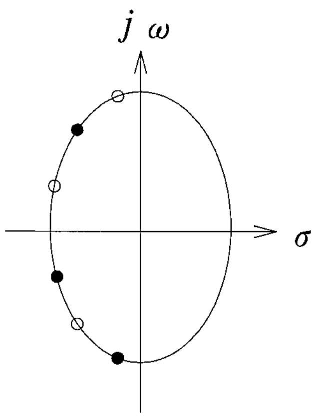

Fig. 5 Poles for sixth-order Chebyshev response.

$$\varepsilon T_N(s/j) = -j, \tag{73}$$

and are written as

$$s_k = (-1)^k \sinh A \sin\left(\frac{\pi(1+2k)}{2N}\right) + j\cosh A \cos\left(\frac{\pi(1+2k)}{2N}\right), \quad 0 \leqq k \leqq N-1. \tag{74}$$

Because of the stability condition, the required poles are s_k in the left half s-plane among Eq. (74). The following rule gives the required poles among the poles on the ellipse.

(1) The pole which lies nearest to the negative imaginary axis is selected.

(2) The every other poles lying clockwise from the one of the previous step are selected.

The example in the sixth-order filter is shown in Fig. 5. The points marked with ● are the required poles.

5.3 Inverse Chebyshev Responses

By subtracting the Chebyshev response from 1 and replacing s/j by $1/js$, the inverse Chebyshev response is given, i.e.

$$S(s)S(-s) = \frac{1}{1+\dfrac{1}{\varepsilon^2 T_N^2(1/js)}}. \tag{75}$$

Equations (56) and (75) prove that the required poles satisfy

$$\frac{1}{\varepsilon T_N(1/js)} = -j. \tag{76}$$

Equation (76) is rewritten as

$$\varepsilon T_N(1/js) = j. \tag{77}$$

By comparing Eq. (77) with Eq. (73), the required poles are written as

$$\frac{1}{s_k} = (-1)^{k-1} \sinh A \sin\left(\frac{\pi(1+2k)}{2N}\right) + j\cosh A \cos\left(\frac{\pi(1+2k)}{2N}\right), \quad 0 \leqq k \leqq N-1, \tag{78}$$

and are s_k in the left half s-plane. For the example in the sixth-order filter, the required poles are the reciprocal numbers of the points marked with ○ in Fig. 5.

6. Determination of Complex Constant and Synthesis of Circuit

$\varepsilon U(s)$ corresponds to $P(z)$ in a digital transfer function. Equations (3), (7) and (13) give the relation of

$$Q(z) = H(z)P(z). \tag{79}$$

If the selected poles are denoted as $s_k (k=1, 2, \cdots, N/2)$, the relation of

$$S(s) = j\varepsilon S(s)U(s) = \beta_S \prod_{k=1}^{N/2} \frac{s_k^* + s}{s_k - s} \tag{80}$$

holds, which is similar to Eq. (52). By solving Eq. (80) at $s=0$ for β_S, it is determined as

$$\beta_S = S(0)(1 + j\varepsilon U(0)) = \prod_{k=1}^{N/2} \frac{s_k}{s_k^*}. \tag{81}$$

The bilinear transformation modifies Eq. (80) into

$$H(z) + jU(z) = \beta_S \prod_{k=1}^{N/2} \frac{1-s_k^*}{1-s_k} \cdot \frac{z^{-1}-z_k^*}{1-z_k z^{-1}}. \tag{82}$$

The comparison between Eqs. (82) and (52) gives β as

$$\beta = \beta_S \prod_{k=1}^{N/2} \frac{1-s_k^*}{1-s_k}. \tag{83}$$

The multipliers in second-order digital two-port circuits can be computed explicitly by the bilinear transformation from the poles obtained in Sect. 5. In addition, β is also given explicitly by Eq. (83). Therefore the complex allpass circuit as shown Fig. 3 can be realized without referring to the digital transfer function.

By the way, the analog filter theory assumes the lowpass frequency responses with the cut-off frequency 1 (rad/s). So the digital frequency responses obtained by the bilinear transformation are the lowpass responses with the cut-off frequency $\pi/2$. If the other frequency responses are needed, the bilinear transformed poles are varied by using the well-known frequency transformation.[10] As this also transforms Eq. (82), β must be renewed. The detailed process is not described in this paper, but the results can be written generally as

$$\beta=\begin{cases}\beta_s\prod_{k=1}^{N/2}\frac{1-\eta s_k^*}{1-\eta s_k^*} & \text{for lowpass or bandstop}\\ \beta_s\prod_{k=1}^{N/2}-\frac{1-\eta s_k^*}{1-\eta s_k} & \text{for highpass or bandpass'}\end{cases} \tag{84}$$

where η is derived from the coefficients of each transformation formula.

It can be said that the method of Ref. (3) classifies the poles into two groups in accordance with the conditions of $\varepsilon U(s)=j$ and $\varepsilon U(s)=-j$. If the group with $\varepsilon U(s)=j$ is selected, $H(z)-jQ(z)$ must be realized, and if the group with $\varepsilon U(s)=-j$ is selected, $H(z)+jQ(z)$ must be realized. Reference (3) does not discuss which transfer function must be realized. This might make an error in computing β by using the same method in Sect. 3. Contrary to the Ref. (3), the proposed method gives the correct value. The proposed method has the advantage from this point of view.

By the way, the analog transfer functions to which the proposed method can be applied, have the condition of $\varepsilon U(s)=-j$ at the poles. If $U(s)=U(-s)$ is satisfied in Eq. (56), then such transfer functions satisfy this condition. Therefore the proposed method is applicable to even-order Butterworth, Chebyshev, inverse Chebyshev and elliptic responses.

7. Illustrative Examples

For two sixth-order Butterworth responses ($\varepsilon=1$), the complex allpass filters are synthesized by the proposed method from the specifications in the s-domain. Both real part outputs of these filters are lowpass frequency responses. One has the normalized cut-off frequency 0.25 Hz, and the other has the cut-off frequency 0.1 Hz. The computed poles and β in each case are shown as follows

The case of 0.25 Hz

$$\begin{cases}z_1=j0.414213562456\\ z_2=-j0.13165249735\\ z_3=-j0.767326988311\\ \beta=0.707106781083+j0.70710678129\end{cases}$$

The case of 0.1 Hz

$$\begin{cases}z_1=0.57149025128+j0.293599201014\\ z_2=0.51603470263-j0.097036735796\\ z_3=0.70219244536-j0.492788962142\\ \beta=0.31650043573446+j0.948592364597.\end{cases}$$

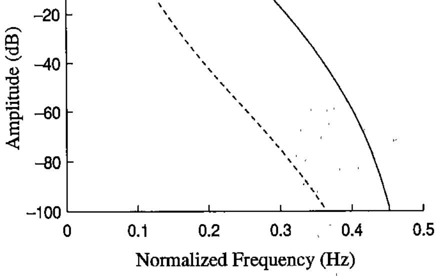

Fig. 6 Sixth-order Butterworth responses.

The obtained frequency responses are shown in Fig. 6. These examples verify the efficiency of the proposed method.

8. Concluding Remarks

A new synthesis of the known complex allpass circuits has been proposed, which can realize low-sensitivity real digital filters. This method is based on the factorization of lossless scattering matrices. It can synthesize such allpass circuits more easily in the z-domain than the conventional z-domain method. This paper also shows that the poles and the complex constant required by synthesized structures can be determined only in the s-domain. In the cases of Butterworth, Chebyshev and inverse Chebyshev responses, the explicit formulae for multiplier coefficients are derived, which enable us to synthesize the objective circuits directly from the specifications in the s-domain. These formulae make the proposed method more excellent than the conventional s-domain method. The examples in Sect. 7 verify the effectiveness of the proposed method.

The selection of the required poles in elliptic filters is the subject for a future study.

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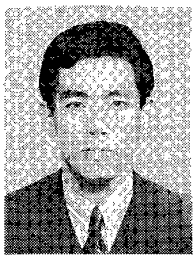

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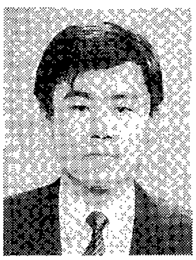

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