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LETTER

Graph Isomorphism Completeness for Trapezoid Graphs

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SUMMARY The complexity of the graph isomorphism problem for trapezoid graphs has been open over a decade. This paper shows that the problem is GI-complete. More precisely, we show that the graph isomorphism problem is GI-complete for comparability graphs of partially ordered sets with interval dimension 2 and height 3. In contrast, the problem is known to be solvable in polynomial time for comparability graphs of partially ordered sets with interval dimension at most 2 and height at most 2.
key words: *comparability graphs, graph isomorphism, interval dimension, trapezoid graphs*

1. Introduction

Let G be an undirected simple graph, and let $V(G)$ and $E(G)$ be the vertex set and the edge set of G, respectively. Two graphs G_1 and G_2 are said to be *isomorphic* if there is a bijection $\phi : V(G_1) \to V(G_2)$ such that for every pair of vertices $u, v \in V(G_1)$, $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. Such ϕ is called *isomorphism* from G_1 to G_2. We denote by $G_1 \simeq G_2$ if G_1 and G_2 are isomorphic. The *graph isomorphism problem* asks whether two given graphs are isomorphic. Although the problem is in NP, it is not known to be NP-complete or polynomial-time solvable.

The graph isomorphism problem for particular classes of graphs has been investigated. See [2], [14], [18] for survey. The problem for a graph class is said to be *GI-complete* if it is polynomial-time equivalent to the problem for general graphs. For some graph classes, the problem is not known to be GI-complete or polynomial-time solvable. One of such graph classes is trapezoid graphs [14], [18], [19]. Trapezoid graphs are natural generalization of two well-known graph classes, interval graphs and permutation graphs. Although the problem can be solved in linear time for interval graphs [8] and for permutation graphs [3], [13], the complexity of the problem for trapezoid graphs has been open over a decade. We show in this paper that the problem is GI-complete for trapezoid graphs.

2. GI-Completeness for Trapezoid Graphs

Let L_1 and L_2 be two lines parallel to x-axis in the xy-plane. A graph G is called a *trapezoid graph* [4], [6] if for each vertex $v \in V(G)$, there is a trapezoid T_v with parallel sides along L_1 and L_2 such that for any pair of vertices $u, v \in V(G)$, $uv \in E(G)$ if and only if T_u and T_v intersect. The set of trapezoids $\{T_v \mid v \in V(G)\}$ is called a *trapezoid representation* of G.

The *complement* of a graph G is the graph $\bar{G}$ such that $V(\bar{G}) = V(G)$ and for any pair of vertices $u, v \in V(G)$, $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. Notice that for any two graphs G_1 and G_2, $G_1 \simeq G_2$ if and only if $\bar{G_1} \simeq \bar{G_2}$. To prove the GI-completeness for trapezoid graphs, we consider the complements of trapezoid graphs, which are known as comparability graphs of partially ordered sets with interval dimension at most 2 [6].

A *partially ordered set* (*poset* for short) is a pair $P = (X, \preceq)$, where X is a finite set and $\preceq$ is a binary relation on X that is reflexive, antisymmetric, and transitive. We denote $x \prec y$ if $x \preceq y$ and $x \neq y$. Two elements $x, y \in X$ are said to be *comparable* in P if either $x \prec y$ or $x \succ y$, and are said to be *incomparable* otherwise. A subset $Y \subseteq X$ is called a *chain* of P if any pair of elements of Y are comparable in P. A chain of P is *maximum* if no other chain of P contains more elements than it. The *height* of P is the number of elements in a maximum chain of P.

A poset $P = (X, \preceq)$ is called an *interval order* if for each element $x \in X$, there is an interval $I_x = [l_x, r_x]$ on the real line such that for any pair of elements $x, y \in X$, $x \prec y$ if and only if $r_x < l_y$. Here, we use $<$ to denote the ordering of points on the real line, while $\prec$ indicates the relation of a poset. The set of intervals $\{I_x \mid x \in X\}$ is called an *interval representation* of P. A family of posets on the same set $\{P_i = (X, \preceq_i) \mid 1 \leq i \leq k\}$ is said to *realize* a poset $P = (X, \preceq)$ if for any $x, y \in X$, $x \preceq y$ if and only if $x \preceq_i y$ for every $i \in \{1, 2, \ldots, k\}$. The *interval dimension* of a poset P is the minimum number k of interval orders that realize P.

A graph G is called a *comparability graph* of a poset $P = (X, \preceq)$ if there is a bijection assigning each vertex $v \in V(G)$ to an element $x_v \in X$ such that for any $u, v \in V(G)$, $uv \in E(G)$ if and only if x_u and x_v are comparable in P. We have the following, which is proved in the next section.

Theorem 1. *The graph isomorphism problem is GI-complete for comparability graphs of posets with interval dimension 2 and height 3.* □

Since a graph is a comparability graph of a poset with interval dimension at most 2 if and only if it is the complement of a trapezoid graph [6], we have the following.

Corollary 2. *The graph isomorphism problem is GI-*

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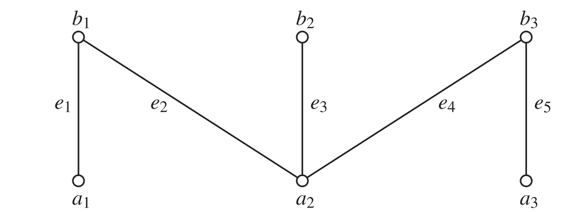
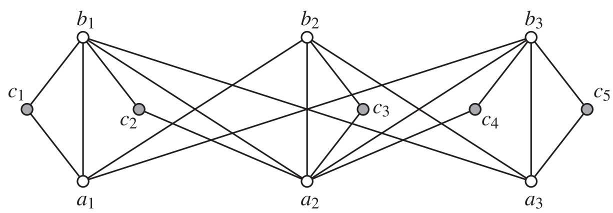
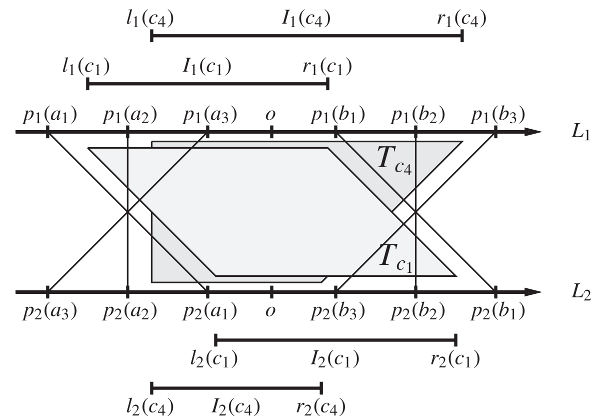


(a) A bipartite graph G_0 with bipartition $(\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$.



(b) The graph H_0 obtained from G_0. The vertex c_i, $1 \leq i \leq 5$, denoted by a gray point, corresponds to the edge e_i of G_0.



(c) The interval representations of P_1 and P_2 that realize P_{H_0} together with the trapezoid representation of the complement of H_0. The intervals and trapezoids corresponding to the vertices of $A \cup B$ are degenerated to the points and the line segments, respectively. The intervals and trapezoids corresponding to c_2, c_3, and c_5 are omitted for the simplicity.

Fig. 1 An example of the construction of the comparability graph (Figs. 1(a) and 1(b)) and the representation of the pair of interval orders that realizes the poset (Fig. 1(c)).

complete for trapezoid graphs. □

Theorem 1 also gives a dichotomy for the graph isomorphism problem for comparability graphs of posets with interval dimension at most 2, since the problem can be solved in polynomial time if the height of the poset is at most 2.

Proposition 3. *The graph isomorphism problem can be solved in $O(n^2)$ time for comparability graphs of posets with interval dimension at most 2 and height at most 2, where n is the number of vertices of a graph.*

Proof. The complements of comparability graphs of posets with interval dimension at most 2 and height at most 2 are circular-arc graphs with clique-cover number 2 [15], [16], for which the graph isomorphism problem is shown to be linear-time solvable by Eschen (as cited in [5]). Since it requires $O(n^2)$ time to take the complements of graphs, we have the proposition. □

Comparability graphs of posets with interval dimension at most 2 and height at most 2 are also known as *2-directional orthogonal ray graphs* [12], [15]. See [18] for more information on the isomorphism of these graphs.

3. Proof of Theorem 1

The graph isomorphism problem is GI-complete for connected bipartite graphs [2]. We show a polynomial-time reduction from the problem for connected bipartite graphs to the problem for comparability graphs of posets with interval dimension 2 and height 3. The reduction is similar to that of [17], [19].

Let G be a connected bipartite graph with bipartition (A, B) with $|A|, |B| \geq 3$. We construct a graph H from G

- by replacing each edge $e = ab$ of G with a vertex c_e together with two edges ac_e and bc_e, and
- by adding edges so that the subgraph induced by $A \cup B$ is a complete bipartite graph with bipartition (A, B).

See Figs. 1(a) and 1(b) for example of the construction. The graph H can be constructed in polynomial time. Let $C = \{c_e \mid e \in E(G)\}$, and we call (A, B, C) the *tripartition* of H.

We first show that two connected bipartite graphs G_1 and G_2 are isomorphic if and only if H_1 and H_2 are isomorphic. Since it is obvious that $H_1 \simeq H_2$ if $G_1 \simeq G_2$, we show the other direction. Let (A_i, B_i, C_i) be the tripartition of H_i for each $i \in \{1, 2\}$. The degree of all vertices of C_i are 2 and the degree of the other vertices are at least 3, since $|A_i|, |B_i| \geq 3$. Hence, an isomorphism from H_1 to H_2 maps the vertices of C_1 to the vertices of C_2 and maps the vertices of $A_1 \cup B_1$ to the vertices of $A_2 \cup B_2$. Since G_i can be obtained from H_i by deleting all edges between the vertices of $A_i \cup B_i$, and by deleting each $c \in C_i$ and adding an edge joining two vertices adjacent to c, we conclude that $G_1 \simeq G_2$ if $H_1 \simeq H_2$.

We next show that H is the comparability graph of a poset with interval dimension at most 2 and height 3. Let $P_H = (V(H), \preceq)$ be the poset obtained from H with tripartition (A, B, C) such that $a \prec b$, $a \prec c$, and $c \prec b$ for any $a \in A$, $b \in B$, and $c \in C$. It is easy to verify that the relation $\prec$ is transitive and the height of the poset P_H is 3.

Now, it suffices to show the interval representations of two interval orders P_1 and P_2 that realize P_H. Let $\{I_i(v) \mid v \in V(H)\}$ be the interval representation of P_i for each $i \in \{1, 2\}$. The interval $I_i(v)$ for any $v \in A \cup B$ is degenerated to the point $p_i(v)$ in the representations. For any $c \in C_i$, let $l_i(c)$ and $r_i(c)$ be the left and right end-point of $I_i(c)$, respectively. We denote the interval representations by a series of points on the real line. Let o be the origin of the real line, and let $A = \{a_1, a_2, \ldots, a_s\}$ and $B = \{b_1, b_2, \ldots, b_t\}$. We place the points corresponding to the elements of $A \cup B$ on the real line such that

$$p_1(a_1) < p_1(a_2) < \ldots < p_1(a_s) < o < p_1(b_1) < p_1(b_2) < \ldots < p_1(b_t)$$

and

$$p_2(a_s) < p_2(a_{s-1}) < \ldots < p_2(a_1) < o < p_2(b_t) < p_2(b_{t-1}) < \ldots < p_2(b_1).$$

We can verify that $a \prec b$ for any $a \in A$ and $b \in B$, any pair of elements of A are incomparable, and any pair of elements of B are incomparable.

Let $c \in C$ be a vertex of H adjacent to $a_i \in A$ and $b_j \in B$. We place the end-points $l_1(c)$ and $r_1(c)$ such that

$$p_1(a_i) < l_1(c) < p_1(a_{i+1}) \leq o \leq p_1(b_{j-1}) < r_1(c) < p_1(b_j)$$

and place the end-points $l_2(c)$ and $r_2(c)$ such that

$$p_2(a_i) < l_2(c) < p_2(a_{i-1}) \leq o \leq p_2(b_{j+1}) < r_2(c) < p_2(b_j),$$

where $p_1(a_{s+1}) = p_1(b_0) = o$ and $p_2(a_0) = p_2(b_{t+1}) = o$. When more than one vertex of C is adjacent to a vertex $a_i \in A$ (resp. $b_j \in B$), we place the left (resp. right) end-points any order on the intervals $[p_1(a_i), p_1(a_{i+1})]$ and $[p_2(a_i), p_2(a_{i-1})]$ (resp. on the intervals $[p_1(b_{j-1}), p_1(b_j)]$ and $[p_2(b_{j+1}), p_2(b_j)]$). It can be verified that $a_i \prec c \prec b_j$ and c is incomparable to any other element of $A \cup B$. Moreover, any pair of elements of C are incomparable, since any interval corresponding to an element of C contains the origin o. Hence, the interval orders P_1 and P_2 realize P_H, and we have Theorem 1.

Figure 1(c) shows the interval representations of the pair of posets that realizes the poset P_{H_0} in Fig. 1(b). The trapezoid representation of the complement of H_0 is also shown in Fig. 1(c).

4. Concluding Remarks

We show in this paper that the graph isomorphism problem is GI-complete for trapezoid graphs. Since the problem can be solved in linear time for interval graphs [8] and for permutation graphs [3], [13], it is an interesting open question to determine the complexity of the problem for graph classes between trapezoid graphs and interval graphs or between trapezoid graphs and permutation graphs. Examples of such graphs are *parallelogram graphs* [1], [7], [11], *triangle graphs* [4], [9] and *simple-triangle graphs* [4], [10]. Other open problems can be found in [14], [18].

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