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An approximation of holomorphic 1-forms on Riemann surfaces by holomorphic 1-cochains

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CHAPTER 1

Preface

In this thesis, we study the relations between the conformal structure and a combinatorial structure, which is called the combinatorial Hodge theory, on a closed Riemann surface. The combinatorial Hodge theory on Riemannian manifolds with triangulations is constructed by Eckmann (1945), Dodziuk and Patodi (1976), Wilson (2007), etc. In 2008, using the combinatorial Hodge theory, Wilson defined holomorphic 1-cochains on closed Riemann surfaces with triangulations. Our goal is to show that holomorphic 1-cochains provide an approximation of holomorphic 1-forms on closed Riemann surfaces.

In [**6**], Eckmann observed that on a closed Riemannian manifold with a triangulation, an inner product in real cochain spaces of a finite simplicial complex gives rise to a combinatorial Hodge theory as follows. Using an inner product on cochains, we may obtain the adjoint operator of a coboundary operator on cochains. A harmonic cochain is defined as a cochain whose images of the coboundary operator and its adjoint operator both vanish. In a similar way to the smooth Hodge theory, an inner product on cochains gives rise to the Hodge decomposition of cochains.

For this combinatorial Hodge theory, Dodziuk [**3**] and Dodziuk and Patodi [**4**] studied the connection with the smooth Hodge theory and then they showed the following. Let K be a smooth triangulation of a compact oriented Riemannian manifold M. Then it is shown that the smooth Hodge theory on M is approximated by the combinatorial Hodge theory with a certain inner product on cochains. To show this, they employed the Whitney map W from cochains to differential forms, and the de Rham map R from differential forms to cochains. As a suitable inner product on cochains, Dodziuk and Patodi defined the Whitney inner product which is defined by using the inner product $\langle , \rangle_\Omega$ on smooth forms and the Whitney map. Note that we may define an inner product on smooth forms by using the Riemannian metric. Then they showed that for any smooth form ω, the corresponding cochain $R\omega$ is an approximation of ω, i.e., $\|\omega - WR\omega\|_\Omega$ converges to ω, as the mesh of a triangulation tends to zero, where $\| \cdot \|_\Omega$ is the $\mathcal{L}^2$-norm on differential forms. Also, each part of the Hodge decomposition of $R\omega$ converges to the corresponding part of the Hodge decomposition of ω. This implies that the Hodge decomposition of cochains, given by the

Whitney inner product, is an approximation of the Hodge decomposition of smooth differential forms.

In [**14**], Wilson developed the combinatorial analogues of several objects in differential geometry. Especially, he defined a combinatorial star operator on cochains which is analogue of the smooth Hodge star operator on differential forms. To define this star operator on cochains, he used a cup product on cochains defined by Whitney in [**13**], and an inner product on cochains. Using the approximation theorem proved by Dodziuk and Patodi, Wilson showed that his combinatorial star operator on cochains also converges to the smooth Hodge star operator. Also, Wilson showed that this approximation respects the Hodge decompositions of smooth forms and cochains.

In his another paper published in 2008, he applied this combinatorial Hodge theory to closed Riemann surfaces. To construct the combinatorial Hodge theory on closed Riemann surfaces, he extends all of the objects to complex settings. Since he also proved that Wilson's combinatorial star operator induces an isomorphism from harmonic cochains to harmonic cochains in [**14**], we define the combinatorial star operator on harmonic 1-cochains by this isomorphism. Then the space of holomorphic 1-cochains is defined as the span of the eigenvectors for non-positive imaginary eigenvalues of the isomorphism and the space of anti-holomorphic 1-cochains is defined as the eigenvectors for non-negative imaginary eigenvalues of the isomorphism. These are defined in [**15**] and also he proved that For a closed Riemann surface of genus g with a triangulation, these spaces have the following three properties: (i) the space of harmonic 1-cochains is decomposed into the spaces of holomorphic 1-cochains and anti-holomorphic 1-cochains, (ii) the dimensions of the spaces of holomorphic 1-cochains and anti-holomorphic 1-cochains are equal to g, (iii) complex conjugation maps holomorphic to anti-holomorphic and vice versa. However, in[**12**], Tanabe showed that complex conjugation dose not map holomorphic to anti-holomorphic in general and added further assumption as follows. A hermitian inner product on cochains is real-valued on real-cochains. This assumption is natural and the Whitney inner product satisfies it.

It is known that as an important property, holomorphic 1-forms satisfy Riemann's bi-linear relation which implies that for a canonical homology basis, holomorphic 1-forms are characterized by their periods. Indeed, for a canonical homology basis $\Sigma = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ of a closed Riemann surface of genus g, any holomorphic 1-form ω on the closed Riemann surface is characterized by their A-periods $\int_{a_1} \omega, \cdots, \int_{a_g} \omega$. Here we define the canonical basis $\{\theta_1, \cdots, \theta_g\}$ of holomorphic 1-forms by $\int_{a_k} \theta_j = \delta_{jk}$ and the period matrix $\Pi = (\pi_{jk})_{1 \leq j,k \leq g}$ by $\pi_{jk} = \int_{b_k} \theta_j$. For a holomorphic 1-cochain σ, we define the combinatorial periods of σ by $\sigma(a_j)$, $\sigma(b_j)$ for $1 \leq j \leq g$. Riemann's bi-linear relation also holds for $\sigma(a_j)$ and $\sigma(b_j)$. Therefore, we may show that any holomorphic 1-cochain σ is characterized by their combinatorial A-periods $\sigma(a_1), \cdots, \sigma(a_g)$ as well. Also, we define the canonical basis $\{\sigma_1, \cdots, \sigma_g\}$ of holomorphic 1-cochains by $\sigma_j(a_k) = \delta_{jk}$ and the combinatorial period matrix $\Pi_K = (\pi^K_{jk})_{1\le j,k\le g}$ by $\pi^K_{jk} = \sigma_j(b_k)$.

As a relation between (conformal) period matrices and combinatorial period matrices, Wilson proved that for a closed Riemann surface with a canonical homology basis and a triangulation, the combinatorial period matrix converges to the (conformal) period matrix as the mesh of the triangulation tends to zero. This implies that combinatorial period matrices give rise to an approximation of (conformal) period matrices.

As our main work of this thesis, using relations between (conformal) period matrices and combinatorial period matrices, we study the relations between holomorphic 1-forms and holomorphic 1-cochains. More precisely, we show that holomorphic 1-cochains are an approximation of holomorphic 1-forms. By the approximation result showed by Dodziuk and Patodi, for arbitrary holomorphic 1-form ω, we may obtain the cochain $R\omega$ which gives an approximation of ω. However, in general, it is unclear whether or not $R\omega$ is a holomorphic 1-cochain. To describe the relations between holomorphic 1-forms and holomorphic 1-cochains, we construct a new correspondence between them.

Now, we explain our main results which are given in Chapter 4 and 5. In Chapter 4, we show the further relation between (conformal) period matrices and combinatorial period matrices in Theorem 4.1. For a fixed triangulation of a closed Riemann surface M with a canonical homology basis Σ and a triangulation K, we prove the following matrix equation

$$\Pi = \overline{\Pi_K} - \overline{\Lambda_K},$$

where Λ_K is defined by $\Lambda_K = (\langle W\sigma_j, \star\theta_k\rangle_\Omega)_{1\le j,k\le g}$. This matrix equation indicates an explicit difference between Π and Π_K for a fixed triangulation. Since Π and Π_K lie in the Siegel upper half space, by using this matrix equation, we may see that Λ_K lies in the Siegel half space as well. Namely, a triple (M, Σ, K) determines three elements Π, Π_K and Λ_K in the Siegel upper half space.

In Chapter 5, we show that holomorphic 1-cochains provide an approximation of holomorphic 1-forms. First, we define a new correspondence between holomorphic 1-forms and holomorphic 1-cochains. For a holomorphic 1-form ω, we define the holomorphic 1-cochain ι_ω by

$$\iota_\omega(a_j) = \int_{a_j} \omega$$

for $1 \le j \le g$. Then the map $\omega \mapsto \iota_\omega$ is an isomorphism from holomorphic 1-forms to holomorphic 1-cochains. In Theorem 5.2, we prove that for any holomorphic 1-form ω, $\|W\iota_\omega - \omega\|_\Omega$ converges to 0, as the mesh of the triangulation tends to zero. To prove Theorem 5.2, we prove three theorems. In the first theorem (Theorem 5.4), using the matrix equation in Chapter 4, we evaluate $\|W\iota_\omega - \omega\|_\Omega$ for a fixed triangulation K. To evaluate this $\mathcal{L}^2$-norm, we use a vector $\Phi_K = (\varphi_1, \cdots, \varphi_g)$ which satisfies

$$\langle \bigstar\sigma_j, \sigma_j\rangle_C = \langle -i\varphi_j\sigma_j, \sigma_j\rangle_C$$

for $1 \leq j \leq g$, where $\langle , \rangle_C$ denotes the Whitney inner product on cochains and $\{\sigma_1, \cdots, \sigma_g\}$ is a basis of the space of holomorphic 1-cochains which satisfies $\sigma_j(a_k) = \delta_{jk}$. In the second theorem (Theorem 5.6), we prove that for any triangulation K of a closed Riemann surface of genus 1 (complex torus), Φ_K is always equal to 1. Then, combining Theorem 5.4 and 5.6, we see that for any triangulation of a complex torus, $\|W\iota_\omega - \omega\|_\Omega$ is always equal to 0. Finally, in the third theorem (Theorem 5.7), for $g > 1$, we prove that Φ_K converges to $(1, \cdots, 1)$, as the mesh of K tends to zero. Combine Theorem 5.4 and 5.7, we see that for $g > 1$, $\|W\iota_\omega - \omega\|_\Omega$ converges to 0.

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CHAPTER 2

Riemann surfaces

1. Introduction

We explain the fundamental theory of Riemann surfaces in this chapter. See [**7**] for details.

In Section 2, we recall some fundamental definitions and properties of Riemann surfaces. Riemann surfaces are real 2-dimensional oriented differentiable manifolds with complex structures. Using complex structures, we may define holomorphic functions and holomorphic 1-forms on surfaces. Also, we define the Hodge star operator on Riemann surfaces and then recall that the space of holomorphic 1-forms is characterized be the eigenvalues of the Hodge star operator.

In Section 3, we recall the fundamental properties of holomorphic 1-forms. Let g be the number of genus of a closed Riemann surfaces. Then the dimension of the space of holomorphic 1-forms is equal to g and holomorphic 1-forms satisfy an important relation which is called Riemann's bi-linear relation. For a canonical homology basis of a closed Riemann surface, Riemann's bi-linear relation gives rise to a unique matrix which lies in the Siegel upper half space and is called the period matrix.

2. Definitions

DEFINITION 2.1. *Let M be a two-dimensional manifold. A complex chart on M is a homeomorphism $z : U \to V$ of an open subset $U \subset M$ onto an open subset $V \subset \mathbb{C}$ such that two complex charts $z_1 : U_1 \to V_1$ and $z_2 : U_2 \to V_2$ satisfy the following, which is called holomorphically compatible:*

$$z_2 \circ z_1^{-1} : z_1(U_1 \cap U_2) \to z_2(U_1 \cap U_2)$$

is biholomorphic if $U_1 \cap U_2 \neq \emptyset$.

For an open cover $\{U_i\}_{i\in I}$ of M, i.e., $\bigcup_{i\in I} U_i = M$, we define a complex atlas on M by a system $\mathfrak{A} = \{z_i : U_i \to V_i,\ i \in I\}$ of charts which are holomorphically compatible.

For two complex atlases $\mathfrak{A}_1$ and $\mathfrak{A}_2$ on M, if every chart of $\mathfrak{A}_1$ is holomorphically compatible with every chart of $\mathfrak{A}_2$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are analytically equivalent. Then we define a complex structure of M is an equivalence class of analytically equivalent atlases on M.

A complex structure u on M contains a unique maximal atlas $\mathfrak{A}^*$. If $\mathfrak{A}$ is an arbitrary atlas in u, then $\mathfrak{A}^*$ consists of all complex charts on M which are holomorphically compatible with every chart of $\mathfrak{A}$.

DEFINITION 2.2. *A Riemann surface is a pair (M, u), where M is a connected two-dimensional manifold and u is a complex structure on M.*

DEFINITION 2.3. *Let M be a Riemann surface. A function $f : M \to \mathbb{C}$ is called holomorphic if for every chart $z : U \to V$ on M,*

$$f \circ z^{-1} : z(U \cap M) \to \mathbb{C}$$

is holomorphic in the usual sense on the open set $z(U \cap M) \subset \mathbb{C}$.

DEFINITION 2.4. *Let M be a Riemann surface. A 0-form on M is a function on M. A 1-form on M is an (ordered) assignment of two continuous functions f and g to each local coordinate $z = x + iy$ on M such that*

$$fdx + gdy.$$

A 2-form on M is an assignment of a continuous function f to each local coordinate $z = x + iy$ such that

$$fdx \wedge dy.$$

We write the set of j-forms on M, $j = 0, 1, 2$, by $\Omega^j(M)$ and $\Omega(M) = \oplus_{j=0,1,2}\Omega^j(M)$.

DEFINITION 2.5. *Let f be a C^2 function on M. We define the Laplacian of f, Δf in local coordinate $z = x + iy$ by*

$$\Delta f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy.$$

If the function f satisfies $\Delta f = 0$, then f is a harmonic function. Also, if a 1-form ω is locally given by df, ω is a harmonic 1-form. We write the set of the harmonic 1-forms of M by $\mathcal{H}\Omega^1(M)$.

DEFINITION 2.6. *A 1-form ω is a holomorphic 1-form if locally $\omega = df$ where f is holomorphic, and we write the set of holomorphic 1-forms on M by $\mathcal{H}\Omega^{1,0}(M)$.*

DEFINITION 2.7. *We define the Hodge star operator $\star$ on $\Omega(M)$ as follows. For a 1-form $\omega = fdx + gdy$, we define*

$$\star\omega = -gdx + fdy.$$

For a 0-form f and a 2-form A, we define

$$\star f = f(z)\big(\lambda(z)dx \wedge dy\big)$$

and

$$\star A = A/\lambda(z)dx \wedge dy,$$

where $\lambda(z)dx \wedge dy$ is a non-vanishing 2-form on M and the existence of such a canonical 2-form $\lambda(z)dx \wedge dy$ follows from IV.8. *in* [**7**].

It is clear that $\star^2 = (-1)^j$ on $\Omega^j(M)$.

Remark 2.8. *Using the Hodge star operator, we may write*

$$\mathcal{H}\Omega^1(M) = \big\{\omega \in \Omega^1(M)\big| d\omega = d\star\omega = 0\big\}$$

and

$$\mathcal{H}\Omega^{1,0}(M) = \big\{\omega \in \mathcal{H}\Omega^1(M)\big| \star\omega = -i\omega\big\}.$$

3. Closed Riemann surfaces

Let M be a closed Riemann surface of genus $g \geq 1$ and $H_1(M)$ be the first homology group of M. Then we obtain a basis $\Sigma = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ of $H_1(M)$ which satisfies the following intersection properties:

$$a_j \cdot b_k = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases}$$

and

$$a_j \cdot a_k = b_j \cdot b_k = 0,$$

where $a \cdot b$ is the intersection number of a and b. We call a basis which satisfies this intersection properties a canonical homology basis of $H_1(M)$.

In the case of closed Riemann surfaces, the dimensions of spaces $\mathcal{H}\Omega^1(M)$ and $\mathcal{H}\Omega^{1,0}(M)$ are determined by the genus g of each closed Riemann surface M.

Theorem 2.9. *On a closed Riemann surface M of genus g, the vector space $\mathcal{H}\Omega^1(M)$ of harmonic 1-forms has dimension $2g$.*

Theorem 2.10. *On a closed Riemann surface of genus g, the vector space $\mathcal{H}\Omega^{1,0}(M)$ of holomorphic 1-forms has dimension g.*

Note that the space $\mathcal{H}\Omega^1(M)$ of harmonic 1-forms has the following

$$\mathcal{H}\Omega^1(M) = \mathcal{H}\Omega^{1,0}(M) \oplus \mathcal{H}\Omega^{0,1}(M),$$

where $\mathcal{H}\Omega^{0,1}(M)$ is the space of anti-holomorphic 1-forms on M whose elements are complex conjugation of holomorphic 1-forms.

For a canonical homology basis $\Sigma = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$, we define the periods of a closed 1-form ω by

$$\int_{a_j} \omega, \quad \int_{b_j} \omega,$$

for $1 \leq j \leq g$. Then periods satisfy the following relations:

Proposition 2.11. *For two closed 1-forms ω_1 and ω_2,*

$$\iint_M \omega_1 \wedge \omega_2 = \sum_{j=1}^{g} \Big(\int_{a_j} \omega_1 \int_{b_j} \omega_2 - \int_{b_j} \omega_1 \int_{a_j} \omega_2 \Big).$$

Also, for holomorphic 1-forms, we have the following:

Corollary 2.12. *For two holomorphic* 1*-forms* ω_1 *and* ω_2*,*

$$\langle \star\omega_1, \omega_2\rangle_\Omega = \sum_{j=1}^{g}\left(\int_{a_j}\omega_1\int_{b_j}\overline{\omega_2} - \int_{b_j}\omega_1\int_{a_j}\overline{\omega_2}\right),$$

where the bar denotes the complex conjugation.

These relations are called Riemann's bi-linear relations. By Riemann's bi-linear relation of holomorphic 1-forms, for a holomorphic 1-form ω, we obtain

$$\|\omega\|_\Omega^2 = i\sum_{j=1}^{g}\left(\int_{a_j}\omega\int_{b_j}\overline{\omega} - \int_{b_j}\omega\int_{a_j}\overline{\omega}\right),$$

where $\|\omega\|_\Omega^2 = \langle\omega,\omega\rangle_\Omega$. This implies that if all A-periods $\int_{a_1}\omega, \cdots, \int_{a_g}\omega$ vanish. Then we obtain $\omega = 0$ and therefore holomorphic 1-forms are characterized by A-periods. Using this properties, we define a basis $\{\theta_1, \cdots, \theta_g\}$ of $\mathcal{H}\Omega^{1,0}(M)$ which satisfies $\int_{a_k}\theta_j = \delta_{jk}$ and is uniquely determined by M and Σ. This basis $\{\theta_1, \cdots, \theta_g\}$ is called the canonical basis of $\mathcal{H}\Omega^{1,0}(M)$ and gives rise to the period matrix Π:

$$\Pi = (\pi_{jk})_{1\le j,k\le g}, \quad \text{where } \pi_{jk} = \int_{b_k}\theta_j.$$

Also, Riemann's bi-linear relation of holomorphic 1-forms implies that Π is symmetric and its imaginary part is positive definite, and therefore period matrices lie in the Siegel upper half space.

It is known that period matrices is one of the characterizations of closed Riemann surfaces. It is clear that if two closed Riemann surfaces are conformally equivalent, then the two closed Riemann surfaces have the same period matrix. Conversely, in 1913, Torelli proved that two closed Riemann surfaces with the same period matrix are conformally equivalent. On the other hand, there is a problem, concerned with period matrix, which is called the Schottky problem. The Schottky problem is to determine which points in the Siegel upper half space represent the period matrix of a closed Riemann surface.

CHAPTER 3

Combinatorial Hodge theory

1. Introduction

In Section 2, we recall the combinatorial Hodge theory on Riemannian manifolds with triangulations. First, we recall that an inner product on cochains gives rise to the Hodge decomposition of each space of j-cochains. Then, to consider the relations between smooth differential forms and cochains, we induce the two maps, which are the Whitney map W on cochains and the de Rham map R on differential forms. By the result of Dodziuk and Patodi, for all smooth differential forms ω, the $\mathcal{L}^2$-norm $\|\omega - WR\omega\|_\Omega$ on differential forms converges to 0, as the mesh of a triangulation tends to zero. This implies that the cochain $R\omega$ provides an approximation of ω.

In Section 3, we recall the definition of a star operator on cochains which is analogue to the Hodge star operator on differential forms. This star operator is defined by using a cup product on cochains and provides an approximation of the Hodge star operator on differential forms.

In Section 4, we apply this combinatorial theory to closed Riemann surfaces and define holomorphic 1-cochains. Also, we recall some notions of holomorphic 1-cochains.

2. Combinatorial Hodge theory on Riemannian manifolds

In this section, we recall a combinatorial theory constructed by Eckmann, Dodziuk, Patodi, etc. This theory is constructed on Riemannian n-manifolds with triangulations, but we may apply this theory to Riemann surfaces with triangulations since we regard a Riemann surface as a Riemannain 2-manifold with a complex structure.

Let M be a closed smooth Riemannian n-manifold, and let $\Omega^j(M)$ be the space of smooth differential j-forms on M with the exterior derivative operator d. Then the Riemannian metric of M induces an inner product $\langle , \rangle_\Omega$ on $\Omega(M)$ and we may obtain the Hodge star operator $\star$. Now we define $d^* := (-1)^{j(j+1-n)} \star d \star$, which is the adjoint operator of d, and the space $\mathcal{H}\Omega^j(M)$ of harmonic j-forms on M by

$$\mathcal{H}\Omega^j(M) = \left\{ \omega \in \Omega^j(M) | d\omega = d^*\omega = 0 \right\}.$$

By the Hodge theory, we have the following decomposition

$$\Omega^j(M) = d\Omega^{j-1}(M) \oplus \mathcal{H}\Omega^j(M) \oplus d^*\Omega^{j+1}(M).$$

Next, for this classical Hodge theory of smooth differential forms, we recall an approximation to the Hodge decomposition of smooth differential forms by cochains. This approximation is constructed by Dodziuk and Patodi, see [**3, 4**] for details.

Let K be a C^∞ triangulation of M. Now we identify $|K|$ and M and fix an ordering of the vertices of K. Then we denote the i-th vertex of K by p_i and the barycentric coordinate corresponding to p_i by μ_i. Let $C^j(K)$ be the simplicial j-cochains of K with values in $\mathbb{R}$. Given the ordering of vertices, we have a coboudary operator $\delta : C^j(K) \to C^{j+1}(K)$. Since K is a finite complex, we can identify chains and cochains and then for $\sigma \in C^j(K)$, we may write

$$\sigma = \sum_\tau c_\tau \cdot \tau,$$

where $c_\tau \in \mathbb{R}$ and the sum is taken over all j-simplices of K. We write $\tau = [p_0, p_1, \cdots, p_j]$ of K with the vertices in an increasing sequence with respect to the ordering of vertices in K.

Definition 3.1. *For a triangulation K, we define the mesh $\eta(K)$ of K by*

$$\eta(K) = \sup r(p, q)$$

where r means the geodesic distance in M and the supremum is taken over all pairs of vertices p, q of a 1-simplex in K.

We define the fullness $\Theta(K)$ of K by

$$\Theta(K) = \inf \frac{vol(\sigma)}{\eta(K)^n},$$

where the inf *is taken over all n-simplices σ of K and $vol(\sigma)$ is the Riemannian volume of σ, as a Riemannian submanifold of M.*

Then we have the following lemma which is analogue of Whitney's result in [**13**].

Lemma 3.2. *Let M be a smooth Riemannian n-manifold.*

(1) *Let K be a smooth triangulation of M. Then, there is a positive constant $\Theta_0 > 0$ and a sequence of subdivisions $K_1, K_2, \cdots$ of K such that $\lim_{n\to\infty} \eta(K_n) = 0$ and $\Theta(K_n) \geq \Theta_0$ for all n.*
(2) *Let $\Theta_0 > 0$. There exist positive constants C_1, C_2 depending on M and Θ_0 such that for all smooth triangulations K of M satisfying $\Theta(K) \geq \Theta_0$, all n-simplices of $\sigma = [p_0, p_1, \cdots, p_n]$ and vertices p_k of σ, $vol(\sigma) \leq C_1 \cdot \eta(K)^n$ and $C_2 \cdot \eta(K) \leq r(p_k, \sigma_{p_k})$, where r is the Riemannian distance, $vol(\sigma)$ is the Riemannian volume, and $\sigma_{p_k} = [p_0, \cdots p_{k-1}, p_{k+1}, \cdots, p_n]$ is the face of σ opposite to p_k.*

Since any two metrics on M are commensurable, the lemma follows from Whitney's Euclidean result, see [**4**].

Here we assume that the fullness of any triangulation using this paper is bounded below by some constant $\Theta_0 > 0$. This implies that the shapes of all simplices of any triangulation do not become too thin.

Now suppose that the cochains $C(K)$ are equipped with a non-degenerate inner product $\langle , \rangle_C$ such that $C^j(K) \perp C^k(K)$ for $j \neq k$. Then we define the adjoint operator of δ:

DEFINITION 3.3. *The adjoint operator $\delta^* : C^j(K) \to C^{j-1}(K)$ of δ is defined by $\langle \delta^* \sigma_1, \sigma_2 \rangle_C = \langle \sigma_1, \delta \sigma_2 \rangle_C$.*

Then two operators δ and δ^* give rise to the harmonic cochains as follows.

DEFINITION 3.4. *We define the space $\mathcal{H}C^j(K)$ of harmonic j-cochains of K by*

$$\mathcal{H}C^j(K) = \left\{ \sigma \in C^j(K) | \delta \sigma = \delta^* \sigma = 0 \right\}.$$

Eckmann showed that an inner product $\langle , \rangle_C$ provides the Hodge decomposition of cochains.

THEOREM 3.5 ([**6**]). *There is an orthogonal direct sum decomposition*

$$C^j(K) = \delta C^{j-1}(K) \oplus \mathcal{H}C^j(K) \oplus \delta^* C^{j+1}(K)$$

and $\mathcal{H}C^j(K) \cong H^j(K)$, the cohomology of (K, δ) in degree j.

Note that the space $\mathcal{H}C^j(K)$ of harmonic j-cochains and the Hodge decompositions of cochains depend upon the choice of the inner product on cochains. Dodziuk and Patodi employed a particularly nice inner product on cochains, which is called the *Whitney inner product*. To define the Whitney inner product and the relations between the smooth Hodge theory and the combinatorial Hodge theory, we need to recall two maps between differential forms and cochains. First, we define a map W from $C^j(K)$ into $\mathcal{L}^2\Omega^j(M)$ which is the completion of $\Omega^j(M)$ with respect to $\langle , \rangle_\Omega$. The map W is called the *Whitney map.*

DEFINITION 3.6. *For a j-simplex $\tau = [p_0, \cdots, p_j]$ of K, we define $W\tau$ by*

$$W\tau = j! \sum_{i=0}^{j} (-1)^i \mu_i d\mu_0 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_j,$$

where $\widehat{}$ over a symbol means deletion. W is defined on $C(K) = \oplus_{j\in\{0,1,2\}} C^j(K)$ by extending linearly.

REMARK 3.7. *The barycentric coordinates μ_j are not even of class C^1, but they are of class C^∞ on the interior of any simplex of K. This implies that $d\mu_j$ is defined and $W\tau$ is well-defined. Therefore dW is also well-defined on $\mathcal{L}^2\Omega^j(M)$.*

The Whitney map W has several properties.

PROPOSITION 3.8 ([**13**]). *The following hold:*
(1) $W\tau = 0$ *on* $M\backslash\overline{St(\tau)}$,
(2) $dW = W\delta$,
where $\overline{St(\tau)}$ *is the closure of the open star* $St(\tau)$.

Next we define the *de Rham map* R from differential forms to cochains which is given by integration:

DEFINITION 3.9. *For any differential form* ω *and chain* c*, the de Rham map* R *is defined by*

$$R\omega(c) = \int_c \omega.$$

The de Rham map is a chain map:

LEMMA 3.10 ([**5**]). *The following holds:*

$$\delta R = Rd.$$

The Whitney map and the de Rham map satisfy the following relation, see [**3, 4, 13**]:

THEOREM 3.11. *The following holds:*

$$RW = Id.$$

In general, $WR \neq Id$. However, Dodziuk and Patodi [**4**] showed the following approximation theorem.

THEOREM 3.12 ([**3**]). *There exist a positive constant* C *and a positive integer* m*, independent of* K*, such that*

$$\|\omega - WR\omega\|_\Omega \leq C \cdot \|(Id + \Delta)^m \omega\|_\Omega \cdot \eta(K)$$

for all C^∞ *differential forms* ω *on* M.

In the right hand of the above theorem, the depending on the choice of triangulations is only the mesh. This implies that any smooth form ω can be approximated by the corresponding cochain $R\omega$ as the mesh tends to zero.

Next we define the Whitney inner product.

DEFINITION 3.13. *For two cochains* σ_1, σ_2*, we define*

$$\langle \sigma_1, \sigma_2 \rangle_C = \langle W\sigma_1, W\sigma_2 \rangle_\Omega.$$

In [**3**], Dodziuk showed that the Whitney inner product is non-degenerate. Also, Dodziuk and Patodi showed the following theorem which indicates that the Hodge decompositions of cochains are an approximation of the Hodge decompositions of smooth forms.

THEOREM 3.14 ([**4**]). *Let* $\omega \in \Omega^j(M)$ *and* $R\omega \in C^j(K)$ *have Hodge decompositions*

$$\begin{aligned} \omega &= d\omega_1 + \omega_2 + d^*\omega_3 \\ R\omega &= \delta a_1 + a_2 + \delta^* a_3. \end{aligned}$$

Then

$$\begin{aligned} \|d\omega_1 - W\delta a_1\|_\Omega &\leq \lambda \cdot \|(Id+\Delta)^m \omega\|_\Omega \cdot \eta(K) \\ \|\omega_2 - W a_2\|_\Omega &\leq \lambda \cdot \|(Id+\Delta)^m \omega\|_\Omega \cdot \eta(K) \\ \|d^*\omega_3 - W\delta^* a_3\|_\Omega &\leq \lambda \cdot \|(Id+\Delta)^m \omega\|_\Omega \cdot \eta(K) \end{aligned}$$

where λ and m are independent of ω and K.

3. Wilson's combinatorial Hodge star operator

In [**14**], Wilson developed this combinatorial theory as follows. Using a cup product on cochains, he defined a star operator on cochains which is an analogue of the Hodge star operator on smooth forms. Also, he showed that this star operator on cochains is an approximation of the Hodge star operator on smooth forms. First, we recall the definition of the cup product on cochains which is defined Whitney in [**13**].

Definition 3.15. *We define $\cup : C^j(K) \otimes C^k(K) \to C^{j+k}(K)$ by*

$$\sigma \cup \tau = R(W\sigma \wedge W\tau),$$

for $\sigma \in C^j(K)$ and $\tau \in C^k(K)$.

Since the de Rham map R and the Whitney map W are chain maps with respect to d and δ, we see that δ is a derivation of the cup product $\cup$ on cochains, i.e.,

$$\delta(\sigma \cup \tau) = \delta\sigma \cup \tau + (-1)^{deg(\sigma)} \sigma \cup \delta\tau.$$

By a theorem of Whitney in [**13**], this cup product induces the same map on cohomology as the usual (Alexander-Whitney) simplicial cochain product. Also, this cup product on cochains satisfies the following.

Theorem 3.16 ([**1**]). *Let $\sigma = [p_{\alpha_0}, \cdots, p_{\alpha_j}] \in C^j(K)$ and $\tau = [p_{\beta_0}, \cdots, p_{\beta_k}] \in C^k(K)$. Then $\sigma \cup \tau$ is zero unless σ and τ intersect in exactly one vertex and span a $(j+k)$-simplex v, in which case, for $\tau = [p_{\alpha_j}, \cdots, p_{\alpha_{j+k}}]$, we have*

$$\begin{aligned} \sigma \cup \tau &= [p_{\alpha_0}, \cdots, p_{\alpha_j}] \cup [p_{\alpha_j}, \cdots, p_{\alpha_{j+k}}] \\ &= \epsilon(\sigma, \tau) \frac{j!k!}{(j+k+1)!} [p_{\alpha_0}, \cdots, p_{\alpha_{j+k}}], \end{aligned}$$

where $\epsilon(\sigma, \tau)$ is determined by

orientation(σ)$\cdot$ orientation(τ)=$\epsilon(\sigma,\tau)\cdot$orientation($v$).

In [**14**], Wilson proved that the cup product $\cup$ on cochains is corresponding to the wedge product $\wedge$ on smooth differential forms.

Theorem 3.17 ([**14**]). *Let $\omega_1, \omega_2 \in \Omega(M)$. There exist a constant C and a positive integer m, independent of K, such that*

$$\|W(R\omega_1 \cup R\omega_2) - \omega_1 \wedge \omega_2\|_\Omega \leq C \cdot \lambda(\omega_1, \omega_2) \cdot \eta(K),$$

where

$$\lambda(\omega_1, \omega_2) = \|\omega\|_\infty \cdot \|(Id+\Delta)^m \omega_2\|_\Omega + \|\omega_2\|_\infty \cdot \|(Id+\Delta)^m \omega_1\|_\Omega$$

and $\|\cdot\|_\infty$ *is the uniform norm on* $\Omega(M)$.

Using the cup product on cochains, Wilson defined a star operator on cochains as follows:

DEFINITION 3.18. *Let* $\langle,\rangle_C$ *be a positive definite inner product on* $C(K)$ *such that* $C^j(K) \perp C^k(K)$ *for* $j \neq k$. *For* $\sigma \in C^j(K)$, *we define* $\bigstar\sigma \in C^{n-j}(K)$ *by*

$$\langle \bigstar\sigma, \tau\rangle_C = (\sigma \cup \tau)[M],$$

where $[M]$ *denotes the fundamental class of* M.

This star operator $\bigstar$ has several properties:

LEMMA 3.19 ([**14**]). *The following hold:*
(1) $\bigstar\delta = (-1)^{j+1}\delta^*\bigstar$, *i.e.* $\bigstar$ *is a chain map.*
(2) *For* $\sigma \in C^j(K)$ *and* $\tau \in C^{n-j}(K)$, $\langle\bigstar\sigma, \tau\rangle_C = (-1)^{j(n-j)}\langle\sigma, \bigstar\tau\rangle_C$, *i.e.* $\bigstar$ *is (graded) skew-adjoint.*
(3) $\bigstar$ *induces isomorphisms* $\mathcal{H}C^j(K) \to \mathcal{H}C^{n-j}(K)$ *on harmonic cochains.*

Using Theorem 3.12, Wilson showed that $\bigstar$ converges to the Hodge star operator $\star$ on $\Omega(M)$:

THEOREM 3.20 ([**14**]). *There exist a positive constant* C *and a positive integer* m, *independent of* K, *such that*

$$\|\star\omega - W\bigstar R\omega\|_\Omega \leq C \cdot \|(Id+\Delta)^m\omega\|_\Omega \cdot \eta(K),$$

for all C^∞ *differential forms* ω *on* M.

Under the assumption that the cochains $C(K)$ are equipped with the Whitney inner product, Wilson also showed that $\bigstar$ respects the Hodge decomposition of $C(K)$ and $\Omega(M)$:

THEOREM 3.21 ([**14**]). *Let* $\omega \in \Omega^j(M)$ *and* $R\omega \in C^j(K)$ *have the Hodge decompositions*

$$\begin{aligned}\omega &= d\omega_1 + \omega_2 + d^*\omega_3\\ R\omega &= \delta a_1 + a_2 + \delta^* a_3.\end{aligned}$$

Then there exist a positive constant C *and a positive integer* m, *independent of* K, *such that*

$$\begin{aligned}\|\star d\omega_1 - W\bigstar\delta a_1\|_\Omega &\leq C\cdot(\|(Id+\Delta)^m\omega\|_\Omega + \|(Id+\Delta)^m d\omega_1\|_\Omega)\cdot\eta(K),\\ \|\star\omega_2 - W\bigstar a_2\|_\Omega &\leq C\cdot(\|(Id+\Delta)^m\omega\|_\Omega + \|(Id+\Delta)^m \omega_2\|_\Omega)\cdot\eta(K),\\ \|\star d^*\omega_3 - W\bigstar\delta^* a_3\|_\Omega &\leq C\cdot(\|(Id+\Delta)^m\omega\|_\Omega + \|(Id+\Delta)^m d^*\omega_3\|_\Omega)\cdot\eta(K).\end{aligned}$$

For a Riemannian n-manifold M, it is known that the smooth Hodge star operator $\star$ on $\Omega^j(M)$ satisfies

$$\star^2 = (-i)^{j(n-j)} Id.$$

In [**14**], Wilson observed that $\bigstar^2$ on $C^j(K)$ and stated that $\bigstar^2$ is not equal to $\pm Id$ in general. In [**12**], Tanabe proved the following theorem.

THEOREM 3.22. *Let M be a Riemannian manifold with a triangulation K of mesh $\eta(K)$. There exist a positive constant C and a positive integer m, independent of K, such that*

$$\| \star^2 \omega - W \bigstar^2 R\omega \|_\Omega \leq C \cdot \|(Id + \Delta)^m \omega\|_\Omega \cdot \eta(K)$$

for all C^∞ differential forms ω on M.

Also, Tanabe showed that this approximation respects to the Hodge decompositions of $\Omega(M)$ and $C(K)$ in his paper [**12**].

In this combinatorial theory, there are some approximation questions. For instance, Dodziuk and Patodi asked whether or not the following holds:

$$\lim_{\eta(K)\to 0} \|W\delta^* R\omega - d^*\omega\|_\Omega = 0,$$

for all C^∞ differential forms ω. For this question, in Appendix II of [**4**], the authors suggest a counterexample to this question. However, Smits [**10**] pointed out the counterexample is not valid and showed that this approximation holds for all C^∞ differential 1-forms on surfaces under a certain restriction on the triangulations. In [**2**], we may have this approximation for more than two dimensions under a certain mesh condition. On the other hand, there are open questions. In [**14**], Wilson raised a question if either of $\delta\bigstar$ or $\bigstar\delta^*$ provide a good approximation to $d\star$ or $\star d^*$, respectively.

4. Combinatorial Hodge theory on closed Riemann surfaces

Using the combinatorial theory, Wilson constructed a combinatorial theory on closed Riemann surfaces in [**15**]. To construct the theory on closed Riemann surfaces, he extended to complex settings as follows. Let M be a closed Riemann surface of genus g with a triangulation K, $C(K) = \oplus_{j=0,1,2} C^j(K)$ the complex valued simplicial cochains with a non-degenerate positive definite hermitian inner product $\langle , \rangle_C$. Then Wilson defined the associated combinatorial star operator $\bigstar$ by

$$\langle \bigstar\sigma, \tau\rangle_C = (\sigma \cup \overline{\tau})[M],$$

where the bar denotes complex conjugation and $\cup$ is extended over $\mathbb{C}$ linearly. Also, for complex cochains, we have a Hodge decomposition

$$C^1(K) = \delta C^0(K) \oplus \mathcal{H}C^1(K) \oplus \delta^* C^2(K).$$

By Lemma 3.19, we regard $\bigstar$ as an isomorphism from $\mathcal{H}C^1(K)$ into $\mathcal{H}C^1(K)$ which is skew-adjoint. Then we define holomorphic 1-cochains as follows:

Definition 3.23. *Let $\langle , \rangle_C$ be a hermitian inner product on the complex valued simpicial* 1*-cochains which is* $\mathbb{R}$*-valued on* $\mathbb{R}$*-cochains. We define the space* $\mathcal{H}C^{1,0}(K)$ *of holomorphic* 1*-cochains to be the span of the eigenvectors for non-positive imaginary eigenvalues of* $\bigstar$ *and the space* $\mathcal{H}C^{0,1}(K)$ *of anti-holomorphic* 1*-cochains to be the span of the eigenvectors for non-negative imaginary eigenvalues of* $\bigstar$.

Now we assume that the cochains $C(K)$ are equipped with the Whitney inner product. Note that the Whitney inner product is $\mathbb{R}$-valued on $\mathbb{R}$-cochains. Then we have the following properties of $\mathcal{H}C^{1,0}(K)$ and $\mathcal{H}C^{0,1}(K)$, due to [**12, 15**].

Lemma 3.24. *Let M be a closed Riemann surface of genus g with a canonical homology basis Σ and a triangulation K. Then, the following hold:*

(1) $\mathcal{H}C^1(K) = \mathcal{H}C^{1,0}(K) \oplus \mathcal{H}C^{0,1}(K)$.

(2) $\dim_{\mathbb{C}} \mathcal{H}C^{1,0}(K) = \dim_{\mathbb{C}} \mathcal{H}C^{0,1}(K) = g$.

(3) *Complex conjugation maps* $\mathcal{H}C^{1,0}(K)$ *to* $\mathcal{H}C^{0,1}(K)$ *and vice versa.*

Remark 3.25. *The spaces of holomorphic* 1*-cochains and anti-holomorphic* 1*-cochains are defined by Wilson in* [**15**] *as follows. Since we redefined* $\bigstar$ *as the isomorphism on harmonic cochains induced by the combinatorial star operator,* $\bigstar$ *admits a unique polar decomposition* $\bigstar = HU$ *where H is symmetric positive definite and U is unitary. Also, since* $\bigstar$ *is skew-adjoint, so is U, and therefore the eigenvalues of U are $\pm i$. Then Wilson defined the spaces* $\mathcal{H}C^{1,0}(K)$ *and* $\mathcal{H}C^{0,1}(K)$ *by*

$$\mathcal{H}C^{1,0}(K) = \{\sigma \in \mathcal{H}C^1(K) \big| U\sigma = -i\sigma\},$$

and

$$\mathcal{H}C^{0,1}(K) = \{\sigma \in \mathcal{H}C^1(K) \big| U\sigma = i\sigma\}.$$

On the other hand, Wilson also gave an equivalent definition as above. However, he did not state the assumption of an inner product on cochains. In [**12**]*, Tanabe remarked that by this equivalent definition without the assumption, complex conjugation dose map* $\mathcal{H}C^{1,0}(K)$ *to* $\mathcal{H}C^{0,1}(K)$ *in general. To define* $\mathcal{H}C^{1,0}(K)$ *as the span of eigenvectors of* $\bigstar$*, we need to add the assumption that a hermitian inner product on cochains to be* $\mathbb{R}$*-valued on* $\mathbb{R}$*-cochains. This assumption is natural and the Whitney inner product satisfies it.*

Next we define *combinatorial periods*.

Definition 3.26. *Let M be a closed Riemann surface of genus g with a canonical homology basis Σ and a triangulation K. We define the combinatorial periods of $\sigma \in \mathcal{H}C^1(K)$ by the following complex numbers:*

$$\sigma(a_j),\ \sigma(b_j) \quad for\ 1 \le j \le g.$$

As an important property of holomorphic 1-cochains, Wilson showed that holomorphic 1-cochains also satisfy Riemann's bi-linear relations.

THEOREM 3.27 ([**15**]). *For σ, $\tau \in \mathcal{H}C^{1,0}(K)$, we have*

$$\sum_{j=1}^{g}(\sigma(a_j)\tau(b_j) - \sigma(b_j)\tau(a_j)) = 0.$$

To define combinatorial period matrices, Wilson [**15**] showed the following relation. For $\sigma, \tau \in \mathcal{H}C^{1,0}(K)$,

$$\langle \bigstar\sigma, \tau\rangle_C = \sum_{j=1}^{g}(\sigma(a_j)\overline{\tau(b_j)} - \sigma(b_j)\overline{\tau(a_j)}).$$

This yields the following.

COROLLARY 3.28 ([**15**]). *Let σ be a holomorphic 1-cochain.*
(1) *If all A-periods $\sigma(a_j)$, $1 \le j \le g$ or all B-periods $\sigma(b_j)$, $1 \le j \le g$ vanish, then $\sigma = 0$.*
(2) *If all A-periods $\sigma(a_j)$, $1 \le j \le g$ and all B-periods $\sigma(b_j)$, $1 \le j \le g$ are real, then $\sigma = 0$.*

For any basis $\{\tau_1, \cdots, \tau_g\}$ for $\mathcal{H}C^{1,0}(K)$, we consider the following equation of $(c_{ij})_{1\le i,j\le g}$:

$$\sum_{i=1}^{g} c_{ij}\tau_i(a_k) = \delta_{jk}.$$

By Corollary 3.28 (1), the matrix $(c_{ij})_{1\le i,j\le g}$ is uniquely determined by a triple (M, Σ, K). Then we put $\sigma_j = \sum_{j=1}^{g} c_{ij}\tau_i$. This basis $\{\sigma_1, \cdots, \sigma_g\}$ is called *the canonical basis of* $\mathcal{H}C^{1,0}(K)$.

Using the canonical basis, we define combinatorial period matrices as follows.

DEFINITION 3.29. *Let M be a closed Riemann surface of genus g with a canonical homology basis Σ and a triangulation K. Let $\{\sigma_1, \cdots, \sigma_g\}$ be the canonical basis of $\mathcal{H}C^{1,0}(K)$. Then the combinatorial period matrix $\Pi_K = (\pi^K_{jk})_{1\le j,k\le g}$ of M is defined by $\pi^K_{jk} = \sigma_j(b_k)$.*

THEOREM 3.30 ([**15**]). *Combinatorial period matrices are symmetric and their imaginary parts are positive definite.*

This theorem implies that combinatorial period matrices lie in the Siegel upper half space. Then Wilson showed that combinatorial period matrices are an approximation of conformal period matrices.

THEOREM 3.31 ([**15**]). *Let M be a closed Riemann surface with a canonical homology basis Σ, and let Π be the period matrix. Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of triangulations of M with mesh converging to zero. Then the combinatorial period matrices Π_{K_n} satisfy*

$$\lim_{n\to\infty} \Pi_{K_n} = \Pi.$$

Then Wilson mentioned that this theorem induces the following corollary which is related the Schottky problem.

COROLLARY 3.32. *Every conformal period matrix is the limit of a sequence of combinatorial period matrices.*

To prove Theorem 3.31, Wilson showed the following lemmas, which are also used for the proof of Theorem 5.7.

LEMMA 3.33 ([**15**]). *Let M be a closed Riemann surface with a triangulation K. For any $\omega \in \mathcal{H}\Omega^{1,0}(M)$ which has the following decomposition*

$$R\omega = \delta g + h_1 + h_2 + \delta^* k,$$

where $h_1 \in \mathcal{H}C^{1,0}(K)$ and $h_2 \in \mathcal{H}C^{0,1}(K)$, there exists positive constant C, dependent on ω but independent of K, such that

$$\|Wh_1 - \omega\|_\Omega \leq C \cdot \eta(K).$$

Also, the original proof of the above lemma in [**15**] provides the following lemma:

LEMMA 3.34. *Let M be a closed Riemann surface with a triangulation K. For any $\omega \in \mathcal{H}\Omega^{1,0}(M)$ which has the following decomposition*

$$R\omega = \delta g + h_1 + h_2 + \delta^* k,$$

where $h_1 \in \mathcal{H}C^{1,0}(K)$ and $h_2 \in \mathcal{H}C^{0,1}(K)$, there exists positive constant C, dependent on ω but independent of K, such that

$$\|W\bigstar h_1 - \star\omega\|_\Omega \leq C \cdot \eta(K).$$

PROOF. By Theorem 3.14 and 3.21, there is a positive constant C, independent of K, such that

$$\begin{aligned}
C \cdot \eta(K) &\geq \|W\bigstar(h_1 + h_2) - \star\omega\|_\Omega + \|\omega - W(h_1 + h_2)\|_\Omega \\
&= \|W\bigstar(h_1 + h_2) - \star\omega\|_\Omega + \|\star\,\omega + iW(h_1 + h_2)\|_\Omega \\
&\geq \|W\bigstar h_1 + W\bigstar h_2 + iW(h_1 + h_2)\|_\Omega \\
&= \|\bigstar h_1 + \bigstar h_2 + i(h_1 + h_2)\|_C.
\end{aligned}$$

Let $\phi_1, \cdots, \phi_g$ be an orthogonal eigenbasis of $\mathcal{H}C^{1,0}(K)$ for $\bigstar$, with eigenvalues $-i\lambda_1, \cdots, -i\lambda_g$ ($\lambda_j > 0$), and let $\widetilde{\phi}_1, \cdots, \widetilde{\phi}_g$ be an orthogonal eigenbasis of $\mathcal{H}C^{0,1}(K)$ for $\bigstar$, with eigenvalues $i\widetilde{\lambda}_1, \cdots, i\widetilde{\lambda}_g$ ($\widetilde{\lambda}_j > 0$). Then we may write $h_1 = \sum_{j=1}^g c_j\phi_j$ and $h_2 = \sum_{j=1}^g \widetilde{c}_j\widetilde{\phi}_j$.

Since $\mathcal{H}C^{1,0}(K)$ and $\mathcal{H}C^{0,1}(K)$ are orthogonal, we have

$$\begin{aligned} C^2 \cdot \eta(K)^2 &\geq \left\| \sum_{j=1}^{g} (1-\lambda_j) c_j \phi_j \right\|_C^2 + \left\| \sum_{j=1}^{g} (1+\widetilde{\lambda}_j) \widetilde{c}_j \widetilde{\phi}_j \right\|_C^2 \\ &= \sum_{j=1}^{g} (1-\lambda_j)^2 |c_j|^2 \|\phi_j\|_C^2 + \sum_{j=1}^{g} (1+\widetilde{\lambda}_j)^2 |\widetilde{c}_j|^2 \|\widetilde{\phi}_j\|_C^2 \\ &\geq \sum_{j=1}^{g} \widetilde{\lambda}_j^2 |\widetilde{c}_j|^2 \|\widetilde{\phi}_j\|_C^2 \\ &= \|\bigstar h_2\|_C^2. \end{aligned}$$

Hence we conclude

$$\|W\bigstar h_1 - \star\omega\|_\Omega \leq \|W\bigstar(h_1+h_2) - \star\omega\|_\Omega + \|\bigstar h_2\|_C \leq 2C \cdot \eta(K).$$

□

In the proof of Theorem 3.31 (Theorem 7.2 in [**15**]), for a sequence $\{K_n\}$ of triangulations with the mesh converging to zero and the holomorphic part h_j^n of $R^n\theta_j \in C^1(K_n)$, Wilson stated that Lemma 3.33 (Lemma 7.1 in [**15**]) provides

$$\lim_{n\to\infty} h_j^n(a_k) = \int_{a_k} \theta_j = \delta_{jk}. \tag{4.1}$$

However, in [**16**], Wilson remarked that (4.1) dose not follows from the lemma since the convergence in the lemma is with respect to $\mathcal{L}^2$-norm, and the integration is not a bounded operator on smooth forms with respect to the norm. Also, Wilson stated that using the following lemma, (4.1) holds since we are considering smooth differential forms that are closed.

Lemma 3.35 ([**16**]). *Let ω_n be a sequence of smooth closed differential forms on a closed Riemannian manifold which converge in L^2 to a smooth form ω. Then for any cycle, the sequence $\int_C \omega_n$ converges to $\int_C \omega$.*

CHAPTER 4

A matrix equation

1. Introduction

In this chapter, we refer to [**17**]. We describe the more relation between conformal period matrices and combinatorial period matrices. More precisely, for a fixed triangulation of a closed Riemann surface, we show the following matrix equation which includes the conformal period matrix and the combinatorial period matrix.

THEOREM 4.1. *Let M be a closed Riemann surface of genus g with a canonical homology basis Σ and a triangulation K, and let Π be the period matrix and Π_K the combinatorial period matrix. Let $\{\theta_1, \cdots, \theta_g\}$ be the canonical basis of $\mathcal{H}\Omega^{1,0}(M)$ and $\{\sigma_1, \cdots, \sigma_g\}$ the canonical basis of $\mathcal{H}C^{1,0}(K)$. Then the following equation holds:*

$$\Pi = \overline{\Pi_K} - \overline{\Lambda_K},$$

where $\Lambda_K = (\langle W\sigma_j, \star\theta_k\rangle_\Omega)_{1\le j,k\le g}$.

The matrix equation implies the difference between the conformal period matrix and the combinatorial period matrix for a fixed triangulation. In [**17**], we call the matrix Λ_K the *associate matrix* of K. Although two matrices Π and Π_K lie in the Siegel upper half space, by using this matrix equation, the associate matrix Λ_K lies in the Siegel upper half space as well. This implies that for a closed Riemann surface, when we fix a canonical homology basis and a triangulation, we may obtain three elements in the Siegel upper half space which satisfy the matrix equation in Theorem 4.1.

2. Proofs

PROOF OF THEOREM 4.1. Set

$$\widetilde{C}_K := \frac{1}{2i}(\Pi - \Pi_K)(\mathrm{Im}\Pi)^{-1},$$

and

$$C_K := E - \widetilde{C}_K$$

where E is the $(g \times g)$ identity matrix.
Note that since $\mathrm{Im}\Pi$ is positive definite, there exists $(\mathrm{Im}\Pi)^{-1}$.

We compute

$$\begin{aligned}\Pi_K &= \Pi - 2i\widetilde{C}_K \mathrm{Im}\Pi \\ &= (C_K + \widetilde{C}_K)\Pi - 2i\widetilde{C}_K \mathrm{Im}\Pi \\ &= C_K \Pi + \widetilde{C}_K \overline{\Pi}.\end{aligned}$$

Let c_{jk} be the (j,k)-entry of C_K and $\widetilde{c}_{jk}$ the (j,k)-entry of $\widetilde{C}_K$. Then, for each j, k, we have

$$\int_{b_k} W\sigma_j = \sum_{m=1}^{g} c_{jm} \int_{b_k} \theta_m + \sum_{m=1}^{g} \widetilde{c}_{jm} \int_{b_k} \overline{\theta_m},$$

and

$$\int_{b_k} \left(W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm}\overline{\theta_m} \right) = 0.$$

On the other hand, we compute

$$\begin{aligned}&\int_{a_k} \left(W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm}\overline{\theta_m} \right) \\ = \; &\int_{a_k} W\sigma_j - \sum_{m=1}^{g} c_{jm} \int_{a_k} \theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm} \int_{a_k} \overline{\theta_m} \\ = \; &\delta_{jk} - \sum_{m=1}^{g} (c_{jm} + \widetilde{c}_{jm})\delta_{km} \\ = \; &\delta_{jk} - \sum_{m=1}^{g} \delta_{jm}\delta_{km} \\ = \; &0.\end{aligned}$$

Namely, all A-periods and B-periods of $W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm}\overline{\theta_m}$ are zero. By Proposition 3.8 and $\delta\sigma_j = 0$, we have $dW\sigma_j = W\delta\sigma_j = 0$ on the interior τ^i of any n-simplex τ in K, where $n = 1, 2$. This implies that $W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm}\overline{\theta_m}$ is closed on τ^i. By de Rham's theorem, the closed form $W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm}\overline{\theta_m}$ is exact: there exists df_j such that

$$W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm}\overline{\theta_m} = df_j,$$

on M$\setminus\{p \in M | p : vertex\ in\ K\}$.
Since $\{p \in M | p : vertex\ in\ K\}$ is a null set, we have

$$\begin{aligned}
\langle df_j, \theta_k \rangle_\Omega &= \langle W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g} \widetilde{c}_{jm}\overline{\theta_m}, \theta_k \rangle_\Omega \\
&= \langle W\sigma_j, \theta_k \rangle_\Omega - \sum_{m=1}^{g} c_{jm} \langle \theta_m, \theta_k \rangle_\Omega \\
&\quad - \sum_{m=1}^{g} \widetilde{c}_{jm} \langle \overline{\theta_j}, \theta_k \rangle_\Omega .
\end{aligned}$$

Since $\overline{\theta_m} \in \mathcal{H}\Omega^{0,1}(M)$ and $d^*\theta_k = 0$, we obtain $\langle \overline{\theta_m}, \theta_k \rangle_\Omega = 0$ and $\langle \theta_m, df_j \rangle_\Omega = \langle d^*\theta_m, f_j \rangle_\Omega = 0$. So, we have

$$\langle W\sigma_j, \theta_k \rangle_\Omega = \sum_{m=1}^{g} c_{jm} \langle \theta_m, \theta_k \rangle_\Omega .$$

By Riemann's bi-linear relation of holomorphic 1-forms, we obtain

$$\begin{aligned}
\langle \theta_m, \theta_k \rangle_\Omega &= i \langle -i\theta_m, \theta_k \rangle_\Omega \\
&= i \langle \star\theta_m, \theta_k \rangle_\Omega \\
&= i \sum_{s=1}^{g} \left(\int_{a_s} \theta_m \int_{b_s} \overline{\theta_k} - \int_{b_s} \theta_m \int_{a_s} \overline{\theta_k} \right) \\
&= i(\overline{\pi_{km}} - \pi_{mk}) \\
&= i(\overline{\pi_{mk}} - \pi_{mk}) \\
&= 2\mathrm{Im}\pi_{mk} .
\end{aligned}$$

Note that since the period matrix Π lies in the Siegel upper half space, the period matrix Π is symmetric and we obtain $\pi_{km} = \pi_{mk}$. Thus, we have

$$\begin{aligned}
\Lambda_K &= (\langle W\sigma_j, \star\theta_k \rangle_\Omega)_{1 \le j,k \le g} \\
&= i(\langle W\sigma_j, \theta_k \rangle_\Omega)_{1 \le j,k \le g} \\
&= i \left(2 \sum_{m=1}^{g} c_{jm} \mathrm{Im}\pi_{mk} \right)_{1 \le j,k \le g} \\
&= 2i(c_{jm})_{1 \le j,m \le g} (\mathrm{Im}\pi_{mk})_{1 \le m,k \le g} \\
&= 2iC_K \mathrm{Im}\Pi .
\end{aligned}$$

So we conclude

$$\begin{aligned}
\Pi_K &= \Pi - 2i\widetilde{C}_K \mathrm{Im}\Pi \\
&= \Pi - 2i(E - C_K)\mathrm{Im}\Pi \\
&= \Pi - 2i\mathrm{Im}\Pi + 2iC_K \mathrm{Im}\Pi \\
&= \overline{\Pi} + \Lambda_K .
\end{aligned}$$

□

As a corollary of Theorem 4.1, we prove that for any triangulation K, Λ_K lies in the Siegel upper half space as well.

COROLLARY 4.2. *For a closed Riemann surface M of genus g with a canonical homology basis Σ and a triangulation K, Λ_K lies in the Siegel upper half space.*

PROOF. By Theorem 4.1, we have

$$\Lambda_K = \Pi_K - \overline{\Pi} = (\mathrm{Re}\Pi_K - \mathrm{Re}\Pi) + i(\mathrm{Im}\Pi_K + \mathrm{Im}\Pi).$$

For any $x \in \mathbb{R}^g$, we see that

$$\begin{aligned} {}^t x(\mathrm{Im}\Lambda_K)x &= {}^t x(\mathrm{Im}\Pi_K + \mathrm{Im}\Pi)x \\ &= {}^t x(\mathrm{Im}\Pi_K)x + {}^t x(\mathrm{Im}\Pi)x > 0. \end{aligned}$$

This implies that Λ_K is symmetric and $\mathrm{Im}\Lambda_K$ is positive definite, so Λ_K is an element of the Siegel upper half space.
Next, we assume that Λ_K is equal to Π. Then, by Theorem 4.1, we have

$$\Pi_K = \overline{\Pi} + \Lambda_K = \overline{\Pi} + \Pi = 2\mathrm{Re}\Pi.$$

This is a contradiction, because the imaginary part of a combinatorial period matrix is not equal to zero matrix. In a similar way, one can check that Λ_K is not equal to Π_K as well. $\square$

CHAPTER 5

The asymptotic behavior of holomorphic 1-cochains

1. Introduction

In this chapter, we refer to [**18**]. Using the matrix equation in Theorem 4.1, we show that holomorphic 1-cochains provide an approximation of holomorphic 1-forms. To describe the approximation, we now introduce a neq correspondence between holomorphic 1-forms and holomorphic 1-cochains as follows.

Definition 5.1. *For $\omega \in \mathcal{H}\Omega^{1,0}(M)$, we define $\iota_\omega \in \mathcal{H}C^{1,0}(K)$ which satisfies*

$$\iota_\omega(a_j) = \int_{a_j} \omega,$$

for $1 \leq j \leq g$.

Since both holomorphic 1-forms and holomorphic 1-cochains are characterized by the A-periods, this correspondence is a natural relation between holomorphic 1-forms and holomorphic 1-cochains. Then our main result is as follows.

Theorem 5.2. *Let M be a closed Riemann surface of genus g with a canonical homology basis Σ, ω an arbitrary holomorphic 1-form on M. In the case of $g = 1$, for any triangulation K of M, we have*

$$\|W\iota_\omega - \omega\|_\Omega = 0.$$

In the case of $g > 1$, for any sequence $\{K_n\}_{n\in\mathbb{N}}$ of triangulations of M with the mesh converging to zero, we have

$$\lim_{n\to\infty} \|W\iota_\omega^n - \omega\|_\Omega = 0,$$

where $\iota_\omega^n \in \mathcal{H}C^{1,0}(K_n)$.

2. Proofs

To prove Theorem 5.2, we need to study some relations between holomorphic 1-cochains and holomorphic 1-forms. First of all, we prove that for all holomorphic 1-forms ω, the map $\omega \mapsto \iota_\omega$ is an isomorphism.

Lemma 5.3. *The map $\iota : \mathcal{H}\Omega^{1,0}(M) \to \mathcal{H}C^{1,0}(K)$ defined by $\omega \mapsto \iota_\omega$ is an isomorphism.*

Proof. It is clear from the following diagram.

$$
\begin{tikzcd}
\mathbb{C}^g \arrow[r, "\mathrm{Id}"] & \mathbb{C}^g \\
\mathcal{H}\Omega^{1,0}(M) \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{H}C^{1,0}(K) \arrow[u] \\
\omega \arrow[u, phantom, "\Cup"] & \iota_\omega \arrow[u, phantom, "\Cup"]
\end{tikzcd}
$$

where the isomorphisms from $\mathcal{H}\Omega^{1,0}(M)$ to $\mathbb{C}^g$ and from $\mathcal{H}C^{1,0}(K)$ to $\mathbb{C}^g$ are as follows:

$$\mathcal{H}\Omega^{1,0}(M) \ni \omega \longmapsto \left(\int_{a_1} \omega, \cdots, \int_{a_g} \omega\right) \in \mathbb{C}^g,$$

and

$$\mathcal{H}C^{1,0}(K) \ni \sigma \longmapsto (\sigma(a_1), \cdots, \sigma(a_g)) \in \mathbb{C}^g.$$

$\square$

Now, to prove Theorem 5.2, we show three theorems. In the first theorem, we give an estimation of difference between holomorphic 1-forms and holomorphic 1-cochains with respect to the $\mathcal{L}^2$-norm. To evaluate the difference, we use the diagonal elements $\mathrm{Im}\pi^K_{jj}$ of the imaginary part $\mathrm{Im}\Pi_K$ of the combinatorial period matrix of K and the eigenvalues of the canonical basis $\{\sigma_1, \cdots, \sigma_g\}$ of $\mathcal{H}C^{1,0}(K)$ for $\bigstar$. Note that since the space $\mathcal{H}C^{1,0}(K)$ of holomorphic 1-cochains is the span of eigenvectors of $\bigstar$, it is unclear whether or not each σ_j has the eigenvalues of $\bigstar$. However, since $\{\sigma_1, \cdots, \sigma_g\}$ is a basis, we may obtain a vector $\Phi_K = (\varphi_1, \cdots, \varphi_g)$ such that

$$\langle \bigstar\sigma_j, \sigma_j\rangle_C = \langle -i\varphi_j\sigma_j, \sigma_j\rangle_C$$

for all j. Then, using $\mathrm{Im}\pi^K_{11}, \cdots, \mathrm{Im}\pi^K_{gg}$ and Φ_K, we have the following theorem.

THEOREM 5.4. *Let M be a closed Riemann surface of genus g with a canonical homology basis Σ and a triangulation K, and let $\{\theta_1, \cdots, \theta_g\}$ be the canonical basis of $\mathcal{H}\Omega^{1,0}(M)$, $\{\sigma_1, \cdots, \sigma_g\}$ the canonical basis of $\mathcal{H}C^{1,0}(K)$ and $\Pi_K = (\pi^K_{jk})_{1\le j,k\le g}$ the combinatorial period matrix. Then there exists a vector $\Phi_K = (\varphi_1, \cdots, \varphi_g) \in (0, 1]^g$ such that*

$$\langle \bigstar\sigma_j, \sigma_j\rangle_C = \langle -i\varphi_j\sigma_j, \sigma_j\rangle_C.$$

In addition, we have

$$\|W\sigma_j - \theta_j\|_\Omega = \sqrt{2\mathrm{Im}\pi^K_{jj}\left(\frac{1}{\varphi_j} - 1\right)},$$

and

$$\|W\iota_\omega - \omega\|_\Omega \leq \sum_{j=1}^{g} \left| \int_{a_j} \omega \right| \cdot \sqrt{2\mathrm{Im}\pi_{jj}^K \left(\frac{1}{\varphi_j} - 1 \right)},$$

for all $\omega \in \mathcal{H}\Omega^{1,0}(M)$.

PROOF. By Theorem 4.1, we have

$$\mathrm{Im}\Pi = -\mathrm{Im}\Pi_K + \mathrm{Im}\Lambda_K$$

and

$$\mathrm{Im}\pi_{jj} = -\mathrm{Im}\pi_{jj}^K + \mathrm{Im}\langle W\sigma_j, \star\theta_j \rangle_\Omega, \tag{2.1}$$

for $1 \leq j \leq g$. Using Riemann's bi-linear relation, we compute

$$\begin{aligned} \langle \theta_j, \theta_j \rangle_\Omega &= i\langle -i\theta_j, \theta_j \rangle_\Omega \\ &= i\langle \star\theta_j, \theta_j \rangle_\Omega \\ &= i\sum_{k=1}^{g} \left(\int_{a_k} \theta_j \int_{b_k} \overline{\theta_j} - \int_{b_k} \theta_j \int_{a_k} \overline{\theta_j} \right) \\ &= i(\overline{\pi_{jj}} - \pi_{jj}) \\ &= 2\mathrm{Im}\pi_{jj}. \end{aligned}$$

Similary, we obtain $i\langle \bigstar\sigma_j, \sigma_j \rangle_C = 2\mathrm{Im}\pi_{jj}^K$. Also we compute

$$\begin{aligned} 2\mathrm{Im}\langle W\sigma_j, \star\theta_j \rangle_\Omega &= i\big(\overline{\langle W\sigma_j, \star\theta_j \rangle_\Omega} - \langle W\sigma_j, \star\theta_j \rangle_\Omega\big) \\ &= i\big(\langle \star\theta_j, W\sigma_j \rangle_\Omega - \langle W\sigma_j, \star\theta_j \rangle_\Omega\big) \\ &= \langle \theta_j, W\sigma_j \rangle_\Omega + \langle W\sigma_j, \theta_j \rangle_\Omega. \end{aligned}$$

By (2.1),

$$\langle \theta_j, \theta_j \rangle_\Omega - \langle \theta_j, W\sigma_j \rangle_\Omega - \langle W\sigma_j, \theta_j \rangle_\Omega = -i\langle \bigstar\sigma_j, \sigma_j \rangle_C.$$

Then we have

$$\|W\sigma_j - \theta_j\|_\Omega^2 = \langle W\sigma_j - \theta_j, W\sigma_j - \theta_j \rangle_\Omega = \langle \sigma_j, \sigma_j \rangle_C - i\langle \bigstar\sigma_j, \sigma_j \rangle_C. \tag{2.2}$$

Here we define φ_j by

$$\varphi_j = 1 - \left(\frac{\|W\sigma_j - \theta_j\|_\Omega}{\|\sigma_j\|_C} \right)^2.$$

By this definition,

$$\langle \sigma_j, \sigma_j \rangle_C - i\langle \bigstar\sigma_j, \sigma_j \rangle_C = (1 - \varphi_j)\langle \sigma_j, \sigma_j \rangle_C,$$

and therefore

$$\langle \bigstar\sigma_j, \sigma_j \rangle_C = \langle -i\varphi_j\sigma_j, \sigma_j \rangle_C.$$

Since Π_K is an element of the Siegel upper half space, the diagonal elements $\mathrm{Im}\pi_{jj}^K$ $(1 \leq j \leq g)$ of $\mathrm{Im}\Pi_K$ are all positive. Thus, by (2.2),

$$\|\sigma_j\|_C^2 - \|W\sigma_j - \theta_j\|_\Omega^2 = i\langle \bigstar\sigma_j, \sigma_j \rangle_C = 2\mathrm{Im}\pi_{jj}^K > 0,$$

and so

$$0 \leq \frac{\|W\sigma_j - \theta_j\|_\Omega^2}{\|\sigma_j\|_C^2} < 1.$$

This implies that $0 < \varphi_j \leq 1$ $(1 \leq j \leq g)$ and therefore $\Phi_K \in (0,1]^g$. By the definition of φ_j,

$$\|W\sigma_j - \theta_j\|_\Omega = \sqrt{1-\varphi_j}\|\sigma_j\|_C.$$

Since

$$\|\sigma_j\|_C^2 = \frac{i\langle \bigstar\sigma_j, \sigma_j\rangle_C}{\varphi_j} = \frac{2\mathrm{Im}\pi_{jj}^K}{\varphi_j}$$

and $\|\sigma_j\|_C > 0$, we have

$$\|W\sigma_j - \theta_j\|_\Omega = \sqrt{1-\varphi_j}\|\sigma_j\|_C = \sqrt{1-\varphi_j}\cdot\sqrt{\frac{2\mathrm{Im}\pi_{jj}^K}{\varphi_j}} = \sqrt{2\mathrm{Im}\pi_{jj}^K\Big(\frac{1}{\varphi_j} - 1\Big)}.$$

For $\omega \in \mathcal{H}\Omega^{1,0}(M)$ and $\iota_\omega \in \mathcal{H}C^{1,0}(K)$, we may write

$$\omega = \sum_{j=1}^g \Big(\int_{a_j} \omega\Big) \cdot \theta_j$$

and

$$\iota_\omega = \sum_{j=1}^g \Big(\int_{a_j} \omega\Big) \cdot \sigma_j.$$

Hence we conclude

$$\begin{aligned}
\|W\iota_\omega - \omega\|_\Omega &= \Big\|\sum_{j=1}^g \Big(\int_{a_j} \omega\Big)(W\sigma_j - \theta_j)\Big\|_\Omega \\
&\leq \sum_{j=1}^g \Big|\int_{a_j} \omega\Big| \cdot \|W\sigma_j - \theta_j\|_\Omega \\
&= \sum_{j=1}^g \Big|\int_{a_j} \omega\Big| \cdot \sqrt{2\mathrm{Im}\pi_{jj}^K\Big(\frac{1}{\varphi_j} - 1\Big)}.
\end{aligned}$$

□

By Theorem 3.31, each $\mathrm{Im}\pi_{jj}^K$ converges to $\mathrm{Im}\pi_{\mathrm{jj}}$ as the mesh of K tends to zero and since period matrices lie in the Siegel upper half space, $\mathrm{Im}\pi_{\mathrm{jj}} > 0$. Therefore, to show Theorem 5.2, we need to study the behavior of Φ_K.

In the case of genus 1, we show that $\Phi_K = \varphi_1 = 1$. This implies that for any triangulation K of a complex torus, $\mathcal{H}C^{1,0}(K)$ is the eigenspace of $\bigstar$ for the eigenvalue $-i$, i.e.,

$$\mathcal{H}C^{1,0}(K) = \{\sigma \in \mathcal{H}C^1(K) | \bigstar\sigma = -i\sigma\}.$$

To prove this, we show the following lemma which is a characterization of $\Phi_K = (1, \cdots, 1)$.

LEMMA 5.5. *Let M be a closed Riemann surface with a canonical homology basis Σ and a triangulation K, and let $\{\theta_1, \cdots, \theta_g\}$ be the canonical basis of $\mathcal{H}\Omega^{1,0}(M)$. Let Φ_K be the vector as in Theorem 5.4. Then the following three conditions are equivalent:*

(a) $\Phi_K = (1, \cdots, 1)$.

(b) $\mathcal{H}C^{1,0}(K) = \{\sigma \in \mathcal{H}C^1(K) | \bigstar\sigma = -i\sigma\}$.

(c) $WR\theta_j = \theta_j$ *a.e. on M for all j.*

PROOF. (a)⇒(c): By Theorem 5.4, we obtain

$$\|W\sigma_j - \theta_j\|_\Omega = 0,$$

and so $W\sigma_j = \theta_j$ *a.e.* on M. By Theorem 3.11; $RW = Id$, we have

$$WR\theta_j = WRW\sigma_j = W\sigma_j = \theta_j$$

a.e. on M.

(c)⇒(b): For any $\sigma \in C^1(K)$, we compute

$$\begin{aligned}
\langle \bigstar R\theta_j, \sigma\rangle_C &= \iint_M WR\theta_j \wedge \overline{W\sigma} \\
&= \iint_M \theta_j \wedge \overline{W\sigma} \\
&= \langle \star\theta_j, W\sigma\rangle_\Omega \\
&= -i\langle \theta_j, W\sigma\rangle_\Omega \\
&= -i\langle WR\theta_j, W\sigma\rangle_\Omega \\
&= -i\langle R\theta_j, \sigma\rangle_\Omega \\
&= \langle -iR\theta_j, \sigma\rangle_\Omega.
\end{aligned}$$

This implies that $\bigstar R\theta_j = -iR\theta_j$. By Lemma 3.10, we have

$$\delta R\theta_j = Rd\theta_j = 0,$$

and by Lemma 3.19 (1),

$$\delta^* R\theta_j = i\delta^*\bigstar R\theta_j = i\bigstar\delta R\theta_j = 0.$$

Thus all $R\theta_j$ $(1 \le j \le g)$ are harmonic 1-cochains which have eigenvalues $-i$ of $\bigstar$, and therefore they are holomorphic 1-cochains. Since $\{\theta_1, \cdots, \theta_g\}$ is a basis of $\mathcal{H}\Omega^{1,0}(M)$ and $WR = Id$, $\{R\theta_1, \cdots, R\theta_g\}$ is a basis of $\mathcal{H}C^{1,0}(K)$. Since every eigenvalue of $R\theta_j$ is $-i$,

$$\mathcal{H}C^{1,0}(K) = \left\{\sigma \in \mathcal{H}C^1(K) | \bigstar\sigma = -i\sigma\right\}.$$

(b) ⇒ (a): Since all elements of $\mathcal{H}C^{1,0}(K)$ have the same eigenvalue $-i$ of $\bigstar$, we obtain $\bigstar\sigma_j = -i\sigma_j$ for all j. By the definition of $\Phi_K = (\varphi_1, \cdots, \varphi_g)$, we have

$$\begin{aligned}
\varphi_j\|\sigma_j\|_C^2 &= i\langle -i\varphi_j\sigma_j, \sigma_j\rangle_C \\
&= i\langle \bigstar\sigma_j, \sigma_j\rangle_C \\
&= i\langle -i\sigma_j, \sigma_j\rangle_C \\
&= \|\sigma_j\|_C^2.
\end{aligned}$$

Since $\{\sigma_1, \cdots, \sigma_g\}$ is a basis, we conclude $\varphi_j = 1$ for all j. $\square$

THEOREM 5.6. *Let M be a closed Riemann surface of genus 1 (complex torus) with a canonical homology basis Σ, K a triangulation of M, and $\varphi_1 \in (0,1]$ which satisfies $\langle \bigstar\sigma_1, \sigma_1\rangle_C = \langle -i\varphi_1\sigma_1, \sigma_1\rangle_C$, where $\{\sigma_1\}$ is the canonical basis of $\mathcal{H}C^{1,0}(K)$. Then $\varphi_1 = 1$.*

PROOF. Since the canonical basis of $\mathcal{H}\Omega^{1,0}(M)$ is $\{cdz\}$ where z is a local coordinate and c is some complex number, by Lemma 5.5, it is enough to show that $WRdz = dz$ *a.e.* on M.

First of all, we express $WRdz$ by

$$WRdz = W\biggl(\sum_\tau Rdz(\tau)\cdot\tau\biggr) = \sum_\tau Rdz(\tau)\cdot W\tau,$$

where the sum is taken over all 1-simplices τ of K. Let $[p_0, p_1, p_2]$ be any 2-simplex of K with the barycentric coordinates μ_0, μ_1 and μ_2. Since $W\tau = 0$ on $M\backslash\overline{St(\tau)}$, on the interior of $[p_0, p_1, p_2]$, we compute

$$\begin{aligned}
WRdz &= Rdz([p_0,p_1])\cdot W[p_0,p_1] + Rdz([p_1,p_2])\cdot W[p_1,p_2] + Rdz([p_2,p_0])\cdot W[p_2,p_0]\\
&= \biggl(\int_{[p_0,p_1]} dz\biggr)\cdot W[p_0,p_1] + \biggl(\int_{[p_1,p_2]} dz\biggr)\cdot W[p_1,p_2] + \biggl(\int_{[p_2,p_0]} dz\biggr)\cdot W[p_2,p_0]\\
&= (p_1-p_0)\cdot W[p_0,p_1] + (p_2-p_1)\cdot W[p_1,p_2] + (p_0-p_2)\cdot W[p_2,p_0]\\
&= (p_1-p_0)\cdot(\mu_0 d\mu_1 - \mu_1 d\mu_0) + (p_2-p_1)\cdot(\mu_1 d\mu_2 - \mu_2 d\mu_1)\\
&\quad +(p_0-p_2)\cdot(\mu_2 d\mu_0 - \mu_0 d\mu_2)\\
&= p_0(\mu_2 d\mu_0 - \mu_0 d\mu_2 - \mu_0 d\mu_1 + \mu_1 d\mu_0) + p_1(\mu_0 d\mu_1 - \mu_1 d\mu_0 - \mu_1 d\mu_2 + \mu_2 d\mu_1)\\
&\quad +p_2(\mu_1 d\mu_2 - \mu_2 d\mu_1 - \mu_2 d\mu_0 + \mu_0 d\mu_2)
\end{aligned}$$

On the interior of $[p_0, p_1, p_2](\ni z)$, the barycentric coordinates satisfy

$$\mu_0(z) + \mu_1(z) + \mu_2(z) = 1$$

and

$$d\mu_0 + d\mu_1 + d\mu_2 = 0.$$

Using these relations among μ_0, μ_1 and μ_2, we compute

$$\begin{aligned}
\mu_2 d\mu_0 - \mu_0 d\mu_2 - \mu_0 d\mu_1 + \mu_1 d\mu_0 &= (\mu_1+\mu_2)d\mu_0 - \mu_0(d\mu_1 + d\mu_2)\\
&= (1-\mu_0)d\mu_0 - \mu_0(-d\mu_0)\\
&= d\mu_0.
\end{aligned}$$

In similar ways, we obtain

$$\mu_0 d\mu_1 - \mu_1 d\mu_0 - \mu_1 d\mu_2 + \mu_2 d\mu_1 = d\mu_1$$

and

$$\mu_1 d\mu_2 - \mu_2 d\mu_1 - \mu_2 d\mu_0 + \mu_0 d\mu_2 = d\mu_2.$$

Since $z = \sum_{j=0}^2 p_j\mu_j(z)$, we have

$$WRdz = \sum_{j=0}^{2} p_j\cdot d\mu_j(z) = d\biggl(\sum_{j=0}^{2} p_j\mu_j(z)\biggr) = dz.$$

On the other hand, the union of the sets of all vertices (0-simplices) and 1-simplices of K is a finite set and therefore it is a null set. Hence $WRdz = dz$ *a.e.* on M. By Lemma 5.5, we conclude that $\varphi_1 = 1$. $\square$

In [**8**] and [**9**], Mercat constructed a different discrete structure on surfaces which is called a *discrete Riemann surface.* Especially, Mercat defined a *discrete period matrix* on a discrete Riemann surface and then he showed that discrete period matrices are also an approximation of (conformal) period matrices on Riemann surfaces. In the case of genus 1, all discrete periods coincide with (conformal) periods, but Wilson did not show that all combinatorial periods also coincide with (conformal) periods in [**15**]. However, combining Theorem 4.1 and 5.6, we see that all combinatorial periods of genus 1 coincide with (conformal) periods as well. This implies that although the constructions of Mercat's discrete period matrices and Wilson's combinatorial period matrices are different, they have the same properties.

Finally we consider the behavior of Φ_K for higher genus. The following theorem implies that eigenvalues of $\bigstar$ on $\mathcal{H}C^{1,0}(K)$ converge to $-i$, as the mesh tends to zero.

THEOREM 5.7. *Let M be a closed Riemann surface of genus $g > 1$ with a canonical homology basis Σ and $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of triangulations of M with the mesh converging to zero. Let $\Phi_{K_n} = (\varphi_1^n, \cdots, \varphi_g^n)$ be the vector in $(0,1]^g$ such that*

$$\langle \bigstar \sigma_j^n, \sigma_j^n \rangle_C = \langle -i\varphi_j^n \sigma_j^n, \sigma_j^n \rangle_C,$$

where each $\{\sigma_1^n, \cdots, \sigma_g^n\}$ be the canonical basis of $\mathcal{H}C^{1,0}(K_n)$. Then we have

$$\lim_{n\to\infty} \Phi_{K_n} = (1, \cdots, 1).$$

PROOF. Let $\{\omega_1, \cdots, \omega_g\}$ be an orthogonal basis of $\mathcal{H}\Omega^{1,0}(M)$ and R^n the de Rham map from $\Omega(M)$ to $C(K_n)$. By the Hodge decomposition and $\mathcal{H}C^1(K_n) = \mathcal{H}C^{1,0}(K_n) \oplus \mathcal{H}C^{0,1}(K_n)$, we obtain

$$R^n\omega_j = \delta^* k_j^n + h_j^n + \widetilde{h}_j^n + \delta g_j^n$$

for any $n \in \mathbb{N}$, where $h_j^n \in \mathcal{H}C^{1,0}(K_n)$ and $\widetilde{h}_j^n \in \mathcal{H}C^{0,1}(K_n)$.

First, we show that the number of K_n, such that $\{h_1^n, \cdots, h_g^n\}$ is not a basis of $\mathcal{H}C^{1,0}(K_n)$, is finite. Now we assume that the number is infinite. Then there exist $j \in \{1, \cdots, g\}$ and a subsequence $\{K_m\}$ of $\{K_n\}$ such that each h_j^m is generated by the other elements, i.e.,

$$h_j^m = \sum_{p\neq j} c_{jp}^m h_p^m,$$

for all $m \in \mathbb{N}$, where $c_{jp}^m \in \mathbb{C}$. Since $h_j^m - \sum_{p \neq j} c_{jp}^m h_p^m$ is the holomorphic part of $R^m(\omega_j - \sum_{p \neq j} c_{jp}^m \omega_p)$, by Lemma 3.33, we have

$$\lim_{m \to \infty} \left\| \omega_j - \sum_{p \neq j} c_{jp}^m \omega_p \right\|_\Omega = 0. \tag{2.3}$$

Also, since $\{\omega_1, \cdots, \omega_g\}$ is an orthogonal basis,

$$\left\| \omega_j - \sum_{p \neq j} c_{jp}^m \omega_p \right\|_\Omega^2 = \|\omega_j\|_\Omega^2 + \sum_{p \neq j} |c_{jp}^m|^2 \|\omega_p\|_\Omega^2, \tag{2.4}$$

and therefore

$$0 \leq |c_{jp}^m|^2 \|\omega_p\|_\Omega^2 \leq \left\| \omega_j - \sum_{p \neq j} c_{jp}^m \omega_p \right\|_\Omega^2.$$

By (2.3), we obtain $\lim_{m \to \infty} |c_{jp}^m| = 0$ and (2.4) implies that $\|\omega_j\|_\Omega = 0$. This is a contradiction since $\{\omega_1, \cdots, \omega_g\}$ is a basis.

Here we assume that $\{h_1^n, \cdots, h_g^n\}$ is a basis of $\mathcal{H}C^{1,0}(K_n)$ for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we may write

$$\sigma_j^n = \sum_{\ell=1}^g \widetilde{c}_{j\ell}^n h_\ell^n,$$

for $1 \leq j \leq g$, where $\widetilde{c}_{j\ell}^n \in \mathbb{C}$. Next we consider $\lim_{n \to \infty} \widetilde{c}_{j\ell}^n$. Let $(\widetilde{d}_{\ell j}^n)_{1 \leq \ell, j \leq g}$ be the inverse matrix of $(\widetilde{c}_{j\ell}^n)_{1 \leq j, \ell \leq g}$. This matrix provides

$$h_\ell^n = \sum_{j=1}^g \widetilde{d}_{\ell j}^n \sigma_j^n,$$

and

$$h_\ell^n(a_k) = \sum_{j=1}^g \widetilde{d}_{\ell j}^n \sigma_j^n(a_k) = \sum_{j=1}^g \widetilde{d}_{\ell j}^n \cdot \delta_{jk} = \widetilde{d}_{\ell k}^n,$$

for $1 \leq \ell, k \leq g$. By Lemma 3.33 and 3.35, we have

$$\lim_{n \to \infty} \widetilde{d}_{\ell k}^n = \lim_{n \to \infty} h_\ell^n(a_k) = \lim_{n \to \infty} RWh_\ell^n(a_k) = \int_{a_k} \omega_\ell, \tag{2.5}$$

for $1 \leq \ell, k \leq g$. Note that each Wh_ℓ^n is neither smooth nor closed, but it can be approximated by a sequence of closed smooth forms and therefore we may apply Lemma 3.35 to Wh_ℓ^n. Thus (2.5) implies that the matrix $(\widetilde{d}_{\ell j}^n)_{1 \leq \ell, j \leq g}$ converges to $(\int_{a_j} \omega_\ell)_{1 \leq \ell, j \leq g}$, and therefore $(\widetilde{c}_{j\ell}^n)_{1 \leq j, \ell \leq g} = (\widetilde{d}_{\ell j}^n)_{1 \leq \ell, j \leq g}^{-1}$ also converges to a matrix $(s_{j\ell})_{1 \leq j, p \leq g}$, where each $s_{j\ell}$ is determined by $\int_{a_1} \omega_1, \cdots, \int_{a_g} \omega_1, \cdots, \int_{a_1} \omega_g, \cdots, \int_{a_g} \omega_g$.

Using the Cauchy-Schwarz inequality, we compute

$$\begin{aligned} 0 \le (1-\varphi_j^n)\|\sigma_j^n\|_C^2 &= \langle \sigma_j^n, \sigma_j^n\rangle_C - i\langle -i\varphi_j^n\sigma_j^n, \sigma_j^n\rangle_C \\ &= \langle \sigma_j^n, \sigma_j^n\rangle_C - i\langle \bigstar\sigma_j^n, \sigma_j^n\rangle_C \\ &= \langle \sigma_j^n - i\bigstar\sigma_j^n, \sigma_j^n\rangle_C \\ &\le \|\sigma_j^n - i\bigstar\sigma_j^n\|_C \cdot \|\sigma_j^n\|_C, \end{aligned}$$

and then

$$0 \le (1-\varphi_j^n)\|\sigma_j^n\|_C \le \|\sigma_j^n - i\bigstar\sigma_j^n\|_C. \tag{2.6}$$

Since $\sigma_j^n = \sum_{\ell=1}^g \widetilde{c}_{\ell j}^n h_\ell^n$, we have

$$\begin{aligned} \|\sigma_j^n - i\bigstar\sigma_j^n\|_C &= \left\| \sum_{\ell=1}^g \widetilde{c}_{j\ell}^n \big(h_\ell^n - i\bigstar h_\ell^n\big)\right\|_C \\ &\le \sum_{\ell=1}^g |\widetilde{c}_{j\ell}^n| \cdot \|Wh_\ell^n - iW\bigstar h_\ell^n\|_\Omega \\ &= \sum_{\ell=1}^g |\widetilde{c}_{j\ell}^n| \cdot \|Wh_\ell^n - \omega_\ell + i\star\omega_\ell - iW\bigstar h_\ell^n\|_\Omega \\ &\le \sum_{\ell=1}^g |\widetilde{c}_{j\ell}^n| \cdot \Big(\|Wh_\ell^n - \omega_\ell\|_\Omega + \|\star\omega_\ell - W\bigstar h_\ell^n\|_\Omega \Big). \end{aligned}$$

Note that any holomorphic 1-form ω satisfies $\omega - i\star\omega = 0$.
By Lemma 3.33 and 3.34, there exist positive constants C_ℓ, independent of $\{K_n\}$, such that

$$\|Wh_\ell^n - \omega_\ell\|_\Omega + \|W\bigstar h_\ell^n - \star\omega_\ell\|_\Omega \le C_\ell \cdot \eta(K_n).$$

Thus we have

$$\|\sigma_j^n - i\bigstar\sigma_j^n\|_C \le \sum_{\ell=1}^g |\widetilde{c}_{j\ell}^n| \cdot C_\ell \cdot \eta(K_n),$$

and

$$\lim_{n\to\infty} \|\sigma_j^n - i\bigstar\sigma_j^n\|_C = 0.$$

Namely, by (2.6), we have

$$\lim_{n\to\infty} (1-\varphi_j^n)\|\sigma_j^n\|_C = 0.$$

By Riemann's bi-linear relations and $0 < \varphi_j^n \le 1$,

$$\|\sigma_j^n\|_C^2 = \frac{i}{\varphi_j^n}\langle \bigstar\sigma_j^n, \sigma_j^n\rangle_C = \frac{2\mathrm{Im}\pi_{jj}^{K_n}}{\varphi_j^n} \ge 2\mathrm{Im}\pi_{jj}^{K_n},$$

and therefore $\lim_{n\to\infty}\|\sigma_j^n\|_C^2 \ge 2\mathrm{Im}\pi_{jj} > 0$ by Theorem 3.31. Hence we conclude $\lim_{n\to\infty}\varphi_j^n = 1$ for all j. □

Combine Theorem 5.4, 5.6 and 5.7, we can easily show Theorem 5.2.

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