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著者(和文)	松野寛樹
Author(English)	Hiroki Matsuno
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Doctoral Thesis

Supersymmetric backgrounds from supergravities

Hiroki Matsuno

Department of Physics, Tokyo Institute of Technology

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Abstract

Supersymmetric field theories in various spacetime dimensions have been studied a great deal. Thanks to supersymmetry, which is an extension of the Poincaré symmetry consisting of the translational and Lorentz symmetry, it is easier to analyze them perturbatively and non-perturbatively than without the supersymmetry. In particular, the computation of the partition function for supersymmetric field theories on curved manifolds can be performed exactly due to the existence of the supersymmetry. The exact results can be used for checks of dualities, some of which are originated from superstring and M-theory.

In order to perform such calculation, we should construct a supersymmetric field theory on a curved manifold. If the manifold is characterized by some deformation parameters, the partition function is a function of these parameters, which gives us detailed information of the theory. However, a general curved manifold does not admit supersymmetry. It is important to derive conditions for the existence of supersymmetry and to clarify which deformation parameters the partition function depends on.

A systematic method for constructing supersymmetric field theories on curved manifolds by using supergravities was presented by Festuccia and Seiberg. In a supergravity, there are the metric and its superpartners. In this method, we treat them as background fields compatible with a supersymmetric transformation. By using this method, we can obtain general constraints for the background fields by imposing the condition that at least one supercharge is preserved. We can also discuss whether the partition function depends on each deformation of the background fields.

In this doctoral thesis, after reviewing the 4-dimensional and 3-dimensional manifolds, we study 5-dimensional curved manifolds which admit at least one supercharge by using a 5-dimensional supergravity. We also discuss the background (in)dependence of the partition function, and realize some simple backgrounds as examples.

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Contents

Chapter 1

Introduction

The goals of this thesis are to obtain general configurations for background fields in the Weyl and vector multiplets such that some of supercharges are preserved, and to clarify the background dependence of partition functions. These analyses are performed by using the framework of supergravity. In this chapter, we overview superstring theory, M-theory and supersymmetric field theories, and explain importance of supersymmetric field theories on curved manifolds.

1.1 Particle physics and superstring theory

There is no doubt that the Standard Model and General Relativity describe dynamics of certain sectors of our world. The Standard Model is a quantum field theory which describes the quarks, leptons and the Higgs particles with the electromagnetic, weak and strong interactions. General Relativity is a classical field theory which describes the gravitational interaction. These theories have been tested through a lot of experiments and confirmed to be highly reliable.

One might try to combine these theories and to obtain a theory which describes all four interactions. However, that challenge has not been succeeded. The gravitational interaction is non-renormalizable in the sense of the power-counting, at least.

A strong candidate of a unified theory is superstring theory. We introduce some basic facts about superstring theory below. For more details, see standard textbooks on superstring theory, for example [1, 2, 3]. Superstring theory is a theory of strings as fundamental objects with supersymmetry. There are two kinds of strings: open strings and closed strings. By the quantization of oscillations of strings, both massless and massive modes arise. From open strings, gauge fields, matter fields and their superpartners arise as the massless modes. Gravitons, various antisymmetric tensor fields and their superpartners arise as the massless modes of closed strings. Therefore, we can see that both the Standard Model fields and the gravitational field are included. This theory has only

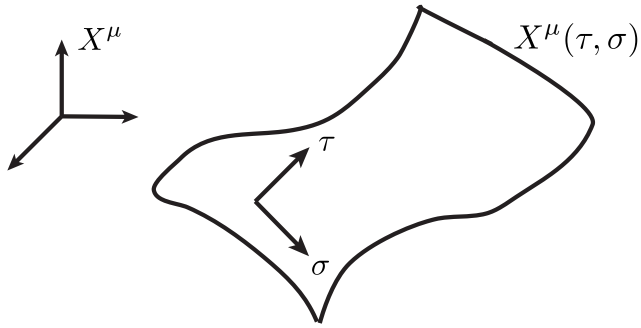

Figure 1.1: String worldsheet in a spacetime. The spacetime coordinates are written by X^μ ($\mu = 0, \ldots, 9$). The string worldsheet is represented by embedding a 2d space parameterized by (τ, σ) in the spacetime.

one parameter: the slope parameter α', which is related to the string scale ℓ_s by $\ell_s = \sqrt{\alpha'}$. α' is an inverse of the string tension. Furthermore, consistency of the theory determines its spacetime dimension to 10. The string coupling is also determined by the scalar field Φ originated from the massless modes of the closed string as $g_s = e^{-\Phi}$. These facts look nice as a unified theory.

The string worldsheet is a 2d subspace of the 10d spacetime. Thus it is represented as a function $X^\mu(\tau, \sigma)$ ($\mu = 0, \ldots, 9$) which maps the 2d space of to 10d, as shown in Figure 1.1. By using the string coordinates $X^\mu(\tau, \sigma)$ and their superpartners $\psi^\mu(\tau, \sigma)$ and $\widetilde{\psi}^\mu(\tau, \sigma)$, the string action can be written as

$$S = \frac{1}{4\pi} \int d^2 z \left(\frac{2}{\alpha'} \partial X^\mu \overline{\partial} X_\mu + \psi^\mu \overline{\partial} \psi_\mu + \widetilde{\psi}^\mu \partial \widetilde{\psi}_\mu \right), \tag{1.1}$$

where $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial \overline{z}$ with $z = \tau + i\sigma$ and $\overline{z} = \tau - i\sigma$.

In field theories, divergences of interactions come from their short-range (ultraviolet) effects. They arise due to interactions at points. These quantum effects appear in both the Standard Model and General Relativity. In the Standard Model, one can obtain finite quantities by renormalization, while one cannot in General Relativity. In superstring theory, as shown in Figure 1.2, a point-like interaction is resolved and thus one can obtain finite quantities.

In the low energy and weak coupling limit, massive excitations and quantum corrections are suppressed. The resulting theory is a 10d classical gravitational theory with supersymmetry, called 10d supergravity. The field content of the 10d supergravity is

- The metric $g_{\mu\nu}$,
- A scalar Φ called dilaton,
- An antisymmetric tensor $B_{\mu\nu}$ called NS-NS 2-form,

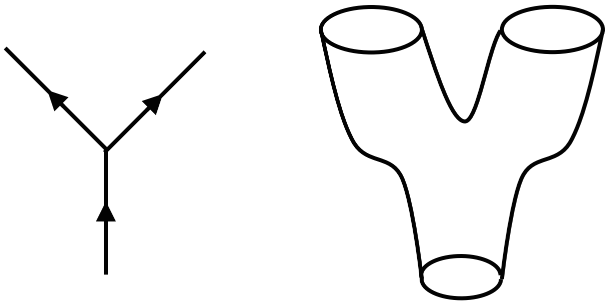

Figure 1.2: Interactions of point particles and strings. In the point particle interaction an infinity can appear from the interacting point , while it cannot in the string interaction.

- Antisymmetric tensors $A_{\mu_1 \cdots \mu_{p+1}}$, and
- A fermion Ψ_μ called gravitino.

Surprisingly, several kinds of extended objects in superstring theory had been discovered. We generally refer to objects extended along p spatial directions as p-branes. One of them found by Polchinski [4] is called D-brane or Dp-brane. A Dp-brane is defined as an object on which endpoints of open strings are bounded. Open strings bounded on a Dp-brane are represented by the Neumann and Dirichlet boundary condition as

$$\left.\frac{\partial X^\mu(\tau,\sigma)}{\partial\sigma}\right|_{\sigma=0} = \left.\frac{\partial X^\mu(\tau,\sigma)}{\partial\sigma}\right|_{\sigma=\pi} = 0, \quad \mu = 0,\ldots,p, \tag{1.2}$$

$$X^\mu(\tau,\sigma)|_{\sigma=0} = X^\mu(\tau,\sigma)|_{\sigma=\pi} = x^\mu, \quad \mu = p+1,\ldots,9, \tag{1.3}$$

where x^μ is a constant vector and the parameter region of σ is $0 \leq \sigma \leq \pi$. The endpoints of the open strings can move only along p spatial directions. By the quantization of open strings on Dp-branes, a gauge field and its superpartners arise as the massless modes.

On Dp-branes, a $(p+1)$-dimensional supersymmetric gauge theory arises as an effective theory. The world-volume action for a Dp-brane is the Dirac-Born-Infeld action

$$S_{\mathrm{D}p} = -\mu_p \int d^{p+1}\xi e^{-\Phi}\sqrt{-\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)}, \tag{1.4}$$

where μ_p is the Dp-brane tension, ξ^a $(a = 0,\ldots p)$ is the coordinates on the Dp-brane, G_{ab} is the induced metric on the Dp-brane. B_{ab} is a pull-back of the NS-NS 2-form, and F_{ab} is a field strength of the gauge field arising from the open string on the Dp-brane. When the energy scale we consider is sufficiently lower than the one of the string fluctuation, by expanding (1.4) we can obtain the Maxwell action for F_{ab}. The gauge coupling g is expressed in terms of α' and g_s as

$$g^2 \sim g_s \alpha'^{\frac{p-3}{2}}. \tag{1.5}$$

Massless modes of closed strings include antisymmetric tensor fields. Since they are generalizations of gauge fields, there can be charged objects. This is similar to the fact that a 1-form potential A_μ is coupled with particles. For instance, in 4d electrodynamics, the electric charge e and the magnetic charge m are defined by the Gauss law as

$$e = \int *F, \quad m = \int F, \tag{1.6}$$

where the integrations are performed over a closed surface surrounding the particle and

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad *F = \frac{1}{4} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma} dx^\mu \wedge dx^\nu. \tag{1.7}$$

We can generalize this argument for a $(p+1)$-form potential $A_{\mu_1 \cdots \mu_{p+1}}$ in spacetime dimension D. In general, p-branes are coupled with $(p+1)$-form potential electrically or $(D-p-3)$-form potential magnetically. In superstring theory,

- Dp-branes are coupled with the $(p+1)$-form potential eletrically or the $(7-p)$-form potential magnetically.
- A string is a 1-brane. It is coupled with the NS-NS 2-form electrically. The magnetically coupled 5-brane is called the NS5-brane.

Although we have tried to obtain a unified theory, there are five types of superstring theory due to its consistency. In four types of superstring theory we treat the left- and right-moving modes of the string oscillation independently, while we identify them in the other one. The former theories are called oriented string theories and the latter is called unoriented string theory. By taking the low energy limit, we can obtain the corresponding 10d supergravity. We list the five types below:

- Type IIA superstring: Oriented string theory with 10d $\mathcal{N}=2$ supersymmetry[1]. Two supercharges have the opposite chiralities. Dp-branes are stable if p is even.
- Type IIB superstring: Oriented string theory with 10d $\mathcal{N}=2$ supersymmetry. Two supercharges have the same chirality. Dp-branes are stable if p is odd.

[1] The number of supersymmetry $\mathcal{N}$ is explained in Section 1.3.

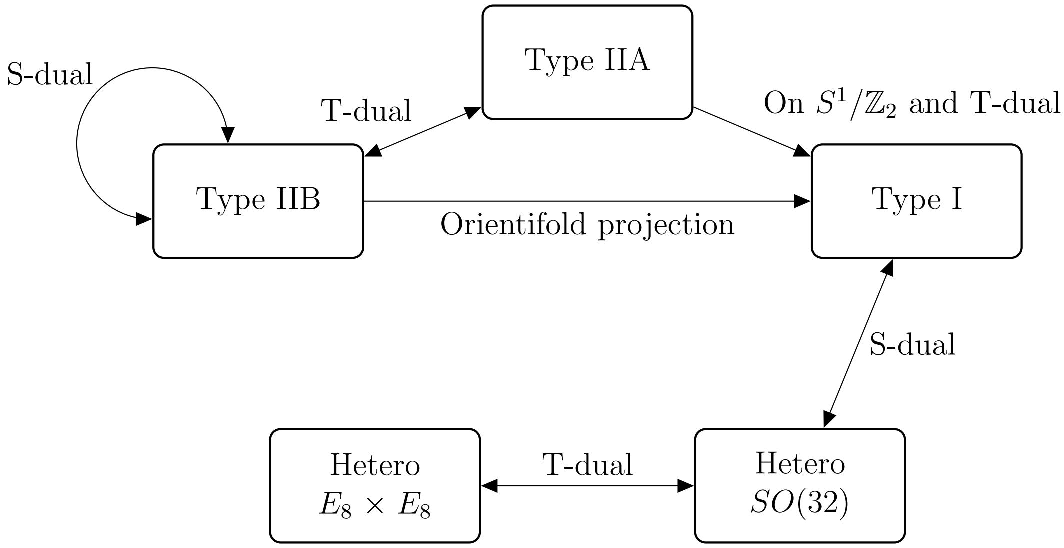

Figure 1.3: Five types of superstring theory and dualities.

- Type I superstring: Unoriented string theory with 10d $\mathcal{N} = 1$ supersymmetry and $SO(32)$ gauge symmetry. Dp-branes are stable if $p = 1, 5, 9$.
- Heterotic superstring with $SO(32)$ gauge symmetry: The left- and right-moving modes are bosonic and supersymmetric, respectively, and
- Heterotic superstring with $E_8 \times E_8$ gauge symmetry: The left- and right-moving modes are bosonic and supersymmetric, respectively.

They look apparently different. However, they are related by physical equivalence, called "duality." In Figure 1.3 we show how duality transforms one to another. Branes in each theory are also transformed by duality transformations. S-duality relates a weakly coupled theory to a strongly coupled theory. Below, we introduce T-duality as an example of duality.

Let us consider compactification of one spacetime direction to a circle with radius R. A closed string wrapping the circle can be characterized by the wrapping number $m \in \mathbb{Z}$. Moreover, by quantum mechanics, a momentum along the circle is quantized as

$$p = \frac{n}{R}, \quad n \in \mathbb{Z}. \tag{1.8}$$

By quantizing the oscillation of a closed string, the squared mass for the closed string characterized by (m, n) is

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'} mn. \tag{1.9}$$

Therefore, the the following two strings give the same squared mass:

- A closed string wrapping a circle with radius R with the wrapping number m and the quantized momentum $p = n/R$, and

- A closed string wrapping a circle with radius $R' = \alpha'/R$ with the wrapping number n and the quantized momentum $p = m/R'$.

T-duality is the physical equivalence of two compactifications with radius R and $R' = \alpha'/R$. The T-duality transformation changes the radius and swaps the momentum and the wrapping. Indeed, for a type of superstring theory, its T-dual leads another type of superstring theory.

1.2 M-theory

In 1995, Witten [5] suggested the existence of a theory which includes all the types of superstring theory. This theory is called M-theory. Conversely, each type of superstring theory can be obtained from a certain limit of M-theory. M-theory is believed to have the following properties:

- 11d theory.

- S^1 compactification of M-theory yields the type IIA superstring theory, and

- Its low-energy limit is 11d supergravity.

Since there is a three-form potential in the 11d supergravity, M-theory includes

- M2-brane, which is electrically coupled with the three-form potential, and

- M5-brane, which is magnetically coupled with the three-form potential.

In the case of superstring theory, an effective theory on D-branes is a supersymmetric gauge theory. By studying supersymmetric gauge theories, we can reveal properties of the D-branes and superstring theory. Now we would like to know field theories realized on a stack of M-branes. However, since we have not been succeeded to quantize M-branes, dynamics of M-branes is highly mysterious.

Nevertheless, effective field theories on a stack of M2-branes were proposed in [6, 7, 8, 9]. They are equipped with the properties which an effective theory on M2-branes must have. We only know the 6d field theory realized on a single M5-brane [10, 11]. An effective field theory on M5-branes is called the 6d $\mathcal{N} = (2, 0)$ theory due to its supersymmetry. There is no known Lagrangian description of "non-Abelian" 6d $\mathcal{N} = (2, 0)$ theory realized on a stack of M5-branes.

1.3 Supersymmetric field theories and exact computation

1.3.1 Supersymmetric field theories

Supersymmetric field theories have been studied in phenomenological and theoretical motivations. Theoretically, supersymmetric field theories are easier to analyze due to symmetries and related with properties of superstring and M-theory. In this thesis, we focus on the theoretical motivations.

Supersymmetry is an extension of a spacetime symmetry. The symmetry of a flat spacetime is the Poincaré symmetry, realized by the Poincaré algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\nu\sigma}M_{\mu\rho}, \tag{1.10}$$

$$[M_{\mu\nu}, P_\rho] = -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu. \tag{1.11}$$

In addition, the supersymmetry introduces fermionic generators Q:

$$\{Q_\alpha, Q_\beta\} = 2\left(\gamma^\mu\right)_{\alpha\beta} P_\mu, \tag{1.12}$$

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{\mu\nu}Q\right)_\alpha, \tag{1.13}$$

where we show the 4d minimal ($\mathcal{N} = 1$) supersymmetry as an example. In the language of field theories, the supersymmetry relates bosons and fermions. Then an action is invariant under the supersymmetric transformation, schematically written as

$$\delta_Q(\text{boson}) = \xi \times (\text{fermion}), \tag{1.14}$$

$$\delta_Q(\text{fermion}) = \xi \times (\text{boson}), \tag{1.15}$$

where ξ is a spinor parameter. A set of fields related by (1.14) and (1.15) is called a supermultiplet. The number of supersymmetry $\mathcal{N}$ is represented as the number of irreducible spinors in each dimension.

A benefit of considering supersymmetry is that theories are easier to analyze, while non-supersymmetric field theories are difficult to analyze, in particular for non-Abelian gauge theories. Supersymmetry sometimes enables us to analyze the theory exactly, including nonperturbative effects. One of the most important works were performed by Seiberg and Witten [12, 13]. In [12] the 4d $\mathcal{N} = 2$ supersymmetric $SU(2)$ pure Yang-Mills theory is considered. The theory includes a vector multiplet, which consists of a gauge field A_μ, two fermions ψ and λ, and a scalar field ϕ. All of them are in the adjoint representation of the $SU(2)$ gauge group. In the low energy, the $SU(2)$ gauge group is spontaneously broken to $U(1)$ by the vacuum expectation value of ϕ. Then the massive degrees of freedom can be integrated out. The low-energy effective action is written by a single function

$\mathcal{F}(A)$, called the prepotential, as

$$\mathcal{L}_{\rm eff} = \frac{1}{4\pi}{\rm Im}\left[\int d^4\theta \frac{\partial\mathcal{F}(A)}{\partial A}\overline{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2\mathcal{F}(A)}{\partial A^2}W^2\right], \tag{1.16}$$

where A is a massless $\mathcal{N}=1$ chiral multiplet, W is a field strength chiral superfield constructed from the $\mathcal{N}=1$ $U(1)$ vector multiplet and θ represents the fermionic components of the 4d $\mathcal{N}=1$ superspace. In [12] the prepotential $\mathcal{F}$ is determined exactly, including nonperturbative effects. [13] extends the result to the case with matter fields.

1.3.2 Supersymmetric localization

In this derivation, some physical assumptions are made. By a direct computation without any assumptions, Nekrasov [14] derived the partition function for 4d $\mathcal{N}=2$ supersymmetric gauge theories on a deformed $\mathbb{R}^4$, called the Ω-background. By taking the undeformed limit, it can be found that the result is consistent with the Seiberg-Witten's result. The method of the direct computation is called "supersymmetric localization." Let us explain what the supersymmetric localization is.

For example, we would like to compute the partition function

$$Z = \int \mathcal{D}\Phi \exp(-S[\Phi]), \tag{1.17}$$

where $S[\Phi]$ is the off-shell Euclidean action and Φ represents all dynamical fields in a theory. Let us introduce a one-parameter deformation of the partition function

$$Z(t) = \int \mathcal{D}\Phi \exp(-S[\Phi] - t\delta V[\Phi]), \tag{1.18}$$

where $V[\Phi]$ is a functional of Φ and δ is a transformation. δ and $V[\Phi]$ can be either bosonic or fermionic, while $\delta V[\Phi]$ must be bosonic. We assume that they satisfy

$$\delta S = \delta^2 V = 0, \quad \delta V|_{\rm bos} \geq 0, \tag{1.19}$$

where $\delta V|_{\rm bos}$ is a part of δV consisting of only bosonic fields. The t-derivative of $Z(t)$ vanishes:

$$\frac{dZ(t)}{dt} = -\int \mathcal{D}\Phi\delta V \exp(-S - t\delta V) \tag{1.20}$$

$$= -\int \mathcal{D}\Phi\delta(V \exp(-S - t\delta V)) = 0, \tag{1.21}$$

where we assumed that the measure $\mathcal{D}\Phi$ is invariant under δ. Therefore the one-parameter deformation $Z(t)$ of the partition function is equal to the original partition function. Of course we can take $t \to \infty$ limit:

$$Z = Z(t) = Z(t \to \infty). \tag{1.22}$$

Because $\delta V|_{\text{bos}} \geq 0$, field configuration Φ_0 satisfying $\delta V[\Phi_0] = 0$ becomes dominant in the path integral. Let Φ' be the fluctuation from Φ_0 and decompose Φ as

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\Phi'. \tag{1.23}$$

We formally expand $t\delta V[\Phi]$ around Φ_0:

$$t\delta V[\Phi] = \frac{1}{2}\left.\frac{d^2(\delta V)}{d\Phi^2}\right|_{\Phi=\Phi_0} \Phi'^2 + \mathcal{O}(t^{-1/2}), \tag{1.24}$$

because $\delta V[\Phi_0] = \left.\frac{d(\delta V)}{d\Phi}\right|_{\Phi=\Phi_0} = 0$. $\mathcal{O}(t^{-1/2})$ includes interaction terms, thus the $t \to \infty$ limit is a weak-coupling limit in some sense and the path integral over Φ' can be calculated explicitly.

Therefore, the partition function is written as

$$Z = \int \mathcal{D}\Phi_0 \exp(-S[\Phi_0]) Z_{\text{1-loop}}[\Phi_0], \tag{1.25}$$

$$Z_{\text{1-loop}}[\Phi_0] = \int \mathcal{D}\Phi' \exp(-S'[\Phi']), \tag{1.26}$$

$$S'[\Phi'] = \frac{1}{2}\left.\frac{d^2(\delta V)}{d\Phi^2}\right|_{\Phi=\Phi_0} \Phi'^2. \tag{1.27}$$

By choosing V and δ appropriately, the integral over Φ_0 becomes a finite-dimensional integral and/or some summations. Furthermore, this procedure does not depend on coupling constants and available even if the original theory is strongly-coupled.

1.3.3 Dualities

Once partition functions are computed, we can use them as tools for checking dualities since physical quantities for dual theories must be the same. Moreover, some kinds of dualities are thought to be originated in superstring or M-theory. As examples, we list some dualities related with 4d supersymmetric field theories below:

- AdS/CFT correspondence.

  The configuration in the original works [15, 16, 17] is interpreted in terms of a stack of N D3-branes in type IIB superstring theory. AdS/CFT says that the following two theories are related:

  – 4d $\mathcal{N}=4$ supersymmetric Yang-Mills theory on S^4 with $SU(N)$ gauge group and a coupling constant $g_{\rm YM}$, and
  – Type IIB superstring theory in $\mathrm{AdS}_5 \times S^5$ with the string coupling g_s and AdS radius $R = (4\pi N g_s)^{1/4}\,\ell_s$.

  On the gauge theory side, we can obtain the partition function $Z_{S^4}^{\mathcal{N}=4}$ exactly by the supersymmetric localization [18]. The string coupling is related with the Yang-Mills coupling as $g_{\rm YM}^2 = 2\pi g_s$.

  If we take a limit in which the latter becomes the classical supergravity, which is easy to analyze, the partition function becomes $Z_{\rm grav} = \exp(-I_{\rm cl})$ where $I_{\rm cl}$ is the on-shell action of the supergravity. This limit is realized by $R \gg \ell_s$ and $N \gg 1$. The first condition suppresses massive modes of strings and the second condition suppresses loop corrections of closed strings.

  We can compute $Z_{S^4}^{\mathcal{N}=4}$ in the corresponding parameter region $\lambda = N g_{\rm YM}^2 \gg 1$ and $N \gg 1$, then obtain [19]

$$-\log Z_{S^4}^{\mathcal{N}=4} = -\frac{N^2}{2}\log\lambda, \tag{1.28}$$

  which coincides with $I_{\rm cl}$ from the supergravity [20].[2]

- Seiberg duality [22], which relates the following in the low-energy region:

  – 4d $\mathcal{N}=1$ $SU(N)$ gauge theory with chiral multiplets Q^a and $\widetilde{Q}_a$ ($a = 1,\ldots,N_f$) in the fundamental and anti-fundamental representations, and
  – 4d $\mathcal{N}=1$ $SU(N_f - N)$ gauge theory with chiral multiplets q_a and $\widetilde{q}^a$ ($a = 1,\ldots,N_f$) in the fundamental and anti-fundamental representations and singlet chiral superfields M_b^a ($a, b = 1,\ldots,N_f$) with the superpotential $W = qM\widetilde{q}$.

  We can find that one of them is strongly coupled, hence we need non-perturbative analyses. As one of these analyses, correspondence of some of gauge invariant operators between these theories has been studied [22]. The supersymmetric localization is another tool for counting gauge invariant operators more generally.

  The partition function on $S^3 \times S^1$, or superconformal index, is computed in [23, 24]. In addition to the definition in terms of path integral, the

[2] Since AdS_5 has an infinite volume, the on-shell action diverges. In order to regularize the on-shell action, we have to introduce a counterterm action [20]. In the regularization, we should carefully relate the cutoff in the field theory to the one in the supergravity [19, 21].

superconformal index can be defined as a trace of a certain operator on $S^3 \times S^1$:

$$I(t, x, h_i) = \mathrm{tr}\left[(-1)^F q^{D-\frac{3}{2}R-2J_L} t^{R+2J_L} x^{2J_R} \prod_i h_i^{F_i}\right], \tag{1.29}$$

where F is the fermion number, D is the dilatation, R is the $U(1)_R$ symmetry, J_L and J_R are the Cartan generators[3] of the $SO(4) \sim SU(2)_L \times SU(2)_R$ subgroup of the superconformal algebra, and F_i is the i-th Cartan generator of flavor symmetries. Expanding the superconformal index, we obtain a polynomial of the variables. Each exponent of a variable means a symmetry charge of the corresponding state. Therefore we can count states with charges of global symmetries by calculating the partition function on $S^3 \times S^1$. Indeed, one can show the coincidence between the partition functions for dual theories.

4d $\mathcal{N} = 1$ gauge theories have realizations by brane configurations in the superstring theory [25]. Hence this duality relates two distinct brane configurations.

- Alday-Gaiotto-Tachikawa (AGT) correspondence [26].

  [26] shows that the instanton partition function for an $\mathcal{N} = 2$ gauge theory coincides with a conformal block of a certain 2d conformal field theory (CFT) on a Riemann surface Σ. Including the perturbative part, the S^4 partition function coincides with a four-point correlation function of the 2d CFT on Σ. This relation is interpreted in terms of M5-branes wrapping $S^4 \times \Sigma$ [27], as shown in Figure 1.4.

These dualities have been extended to various cases: other gauge groups, matter contents, manifolds, etc. We expect that studying various dualities may reveal more detailed properties of superstring and M-theory.

What we need for the localization computation is the off-shell action of a supersymmetric field theory on a compact manifold. In a flat space, the partition function diverges due to infrared (IR) and ultraviolet (UV) divergences. Computation on a compact (typically curved) space suppresses IR divergences and gives us finite results, while UV divergences are suppressed by supersymmetry. Since a square of supersymmetry transformations generates bosonic symmetries, we often use a supersymmetry transformation as δ satisfying $\delta^2 V = 0$ in the localization computation with an appropriate choice of a functional $V[\Phi]$.

[3]Cartan generators of a symmetry group are defined as Hermite generators H_i of the group satisfying $[H_i, H_j] = 0$. $SU(2)$ has one Cartan generator.

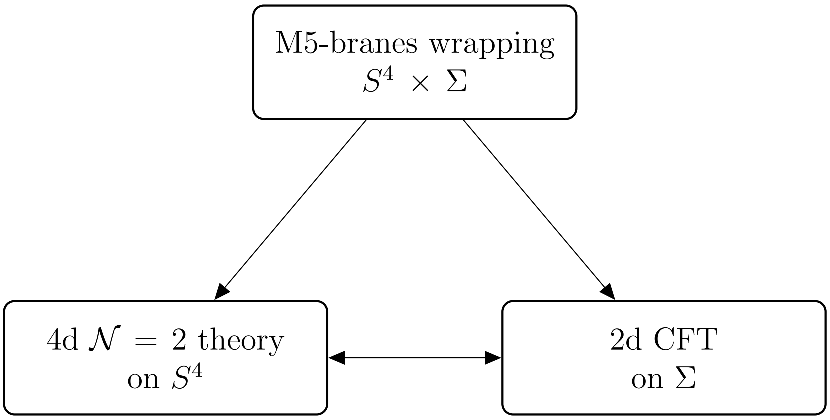

Figure 1.4: The 6d interpretation of the AGT correspondence.

1.3.4 Supersymmetric field theories on curved spaces

In order to compute the partition function, first of all we should construct a supersymmetric field theory on the curved manifold. We can adopt an intuitive method to construct a supersymmetric Lagrangian as follows. Let us consider construction of a supersymmetric field theory on a sphere with radius r from one on a flat space, for example. In the Lagrangian and supersymmetry transformation laws on the flat space, the Lorentz indices are contracted by the flat space metric $\eta_{\mu\nu}$. To construct the theory on the sphere, we firstly replace $\eta_{\mu\nu}$ into the sphere metric $g_{\mu\nu}$. Supersymmetry is not preserved if we only do that. To restore sypersymmetry, we add $\mathcal{O}(r^{-n})$ $(n = 1, 2, \ldots)$ corrections order by order and realize the supersymmetric field theory on the sphere.

Now we would like to introduce deformations for the manifold. Since supersymmetry is an extension of the spacetime symmetry, introducing deformations may break supersymmetry. However, if a part of supersymmetry is left we can perform the localization computation. Then we expect to obtain the partition function depending on the deformation parameters, which is a more detailed information of the theory. However, it is hard to construct a supersymmetric field theory on such manifold. Also, it is not clear whether a supersymmetric field theory on a complicated manifold can be constructed or not. Moreover, there are some cases in which a deformation of the manifold does not change the partition function [28, 29, 30]. In particular, for supersymmetric field theories on a manifold with the S^3 topology, it is shown in [31] that the partition function depends on background manifolds only through a single parameter.

Now we have two questions:

1. What kind of manifold we can construct supersymmetric field theories on? If possible, how is the action given?

2. Which parameters do partition functions depend on?

1.4 Importance of backgrounds

In this thesis, we often use the terminology "background" or "background fields." These mean "something non-dynamical which affects dynamics." Note that backgrounds are not affected by dynamics. In the following we mainly study non-dynamical fields such as the metric characterizing manifolds and gauge fields associated with global symmetries.

A typical example of backgrounds is the external electromagnetic field. In classical mechanics, motions of particles depend on their electric charges in the presence of the electromagnetic field. For a charged particle beam in the electromagnetic field with known velocity, for example, we can derive the ratio of its charge to mass by tracking its trajectory. Also, in quantum mechanics, the degenerating states of an atom become splitted by turning on a magnetic field (Zeeman effect). This reveals the spin structure of the atom. Thus we can know properties of each particle in detail by turning on background fields.

We mentioned above that backgrounds are not affected by dynamics. In the example above, the electromagnetic field may be affected by the motion of the charged particle in the framework of electromagnetism. In order to reduce the electromagnetic field into the external field, we should take an appropriate limit of parameters such as the dielectric constant.

Background fields play important roles in quantum field theory, too. As an example, let us consider computing an n-point correlation function $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$ of operators $\mathcal{O}_i(x_i)$ $(i = 1, \ldots, n)$. In order to compute this, we add the source terms to the original Lagrangian and define the generating functional

$$Z[J] = \int \mathcal{D}\Phi \exp\left[i \int dx \left(\mathcal{L} + J_i(x_i) \mathcal{O}_i(x_i) \right) \right], \tag{1.30}$$

where $J_i(x_i)$ is an i-th source as a background and Φ represents all dynamical fields in the theory. The n-point correlation function can be written as

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z[0]} \left. \frac{\delta^n Z[J]}{\delta J_1(x_1) \cdots \delta J_n(x_n)} \right|_{J=0}. \tag{1.31}$$

The background fields $J_i(x_i)$ behave as the sources generating $\mathcal{O}_i(x_i)$.

Next example is a conformal field theory. Conformal field theories arise in fixed points of the renormalization group flow. A conformal symmetry is generated by

- Translation $x^\mu \to x^\mu + a^\mu$,
- Lorentz transformation $x^\mu \to M^\mu{}_\nu x^\nu$,

- Dilatation $x^\mu \to \alpha x^\mu$, and
- Special conformal transformation $x^\mu \to \dfrac{x^\mu - (x \cdot x) b^\mu}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}$.

These transformations leave an angle between two arbitrary vectors invariant. Like supersymmetry, the conformal symmetry restricts theories and makes them easier to analyze. In conformal field theory, the trace of the energy-momentum tensor classically vanishes: ${T_\mu}^\mu = 0$. By quantum corrections, the conformal symmetry can be broken. The quantity $\langle {T_\mu}^\mu \rangle$, characterizing the breaking of the conformal symmetry is called the Weyl anomaly. The source corresponding to the operator $T^{\mu\nu}$ is $g_{\mu\nu}$, hence the Weyl anomaly is defined as the variation of the action under the infinitesimal Weyl transformation given by

$$\delta g_{\mu\nu}(x) = 2\Lambda(x) g_{\mu\nu}(x) \tag{1.32}$$

for small $\Lambda(x)$. In 4d, the contribution of the curved metric to the Weyl anomaly $\langle {T_\mu}^\mu \rangle_{\text{curved}}$ consists of two components with coefficients a and c as [32]

$$\langle {T_\mu}^\mu \rangle_{\text{curved}} = cF - aG, \tag{1.33}$$

$$F = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \tag{1.34}$$

$$G = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2, \tag{1.35}$$

where the theory is coupled with the external gravity with the Riemann tensor $R_{\mu\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R. Even if the conformal symmetry is quantum mechanically preserved, a and c are important quantities characterizing the theory. In particular, a monotonically decreases along the renormalization group flow and can be seen as an effective degrees of freedom of the theory [33, 34].

As shown above, introducing background fields can reveal properties of theories or particles in detail. In this reason, we would like to study supersymmetric field theories on curved spaces to obtain their properties and reveal dualities in detail.

1.5 Rigid supersymmetry from supergravity

1.5.1 Rigid supersymmetry from supergravity

Festuccia and Seiberg [35] proposed a systematic construction of supersymmetric field theories on curved manifolds by using off-shell supergravity. Since we use an off-shell formalism of field theory for the path integral, we begin with off-shell supergravity in this method.

Supergravity is a theory in which the gravity has the superpartner called gravitino. The supermultiplet including them is called the Weyl multiplet. Other supermultiplets including gauge fields and matter fields are also present.

In the Einstein-Hilbert action, a kinetic term of the gravity is

$$\frac{m_{\mathrm{Pl}}^2}{2}\int dx\sqrt{g}R, \tag{1.36}$$

where m_{Pl} is the Planck mass. In order to treat the gravity as a non-dynamical field without backreactions, we take the "rigid limit" $m_{\mathrm{Pl}} \to \infty$. Then the gravity decouples and we can fix the component fields of the Weyl multiplet as background fields. In particular, we choose the metric so that it realizes a manifold which we would like to construct. Now supersymmetry transformation acts only on dynamical fields, while the background Weyl multiplet is kept intact. Therefore, for the Lagrangian to be supersymmetric, original supersymmetry transformations of all components of the Weyl multiplet should vanish. Since we set the fermionic components of the Weyl multiplet to be zero, the supersymmetry transformations for the bosonic components are always zero. The condition that the supersymmetry transformation of the fermionic components in the Weyl multiplet should be zero is nontrivial because the bosonic components are generally nonzero. Therefore, supercharges corresponding to the spinor parameter ξ satisfying

$$\delta_Q(\text{fermions in the Weyl multiplet}) = \xi \times (\text{bosons in the Weyl multiplet}) = 0 \tag{1.37}$$

are only preserved. We use the terminology the "rigid" supersymmetry for the supersymmetry with the gravity fixed. For this formulation, see Figure 1.5.

Using this systematic method, we can study general backgrounds preserving supersymmetry. By requiring the existence of the solutions of (1.37), we can derive restriction for bosonic background fields in the Weyl multiplet. In [36, 37], the condition that a 4d Riemannian manifold $\mathcal{M}_4$ can realize rigid supersymmetry is studied by considering 4d $\mathcal{N} = 1$ supergravity with a $U(1)_R$ symmetry, called "the new minimal supergravity." One conserved supercharge restricts $\mathcal{M}_4$ to Hermitian. For more preserved supercharges, the condition is more restrictive.

Using the new minimal supergravity, rigid supersymmetry on S^4 cannot be realized because S^4 is not Hermitian. In [38], it was shown that a rigid supersymmetry can be realized on round and squashed S^4 by starting with a supergravity without the $U(1)_R$ symmetry, called "the old minimal supergravity." This fact shows that different results may be obtained by starting with different supergravities even if they are reduced into the same on-shell theory. This is because we do not impose equations of motion on auxiliary fields in the Weyl multiplet.

In [39], a similar condition for 3d Riemannian manifold $\mathcal{M}_3$ is solved by starting with a 3d $\mathcal{N} = 2$ supergravity with a $U(1)_R$ symmetry. By imposing

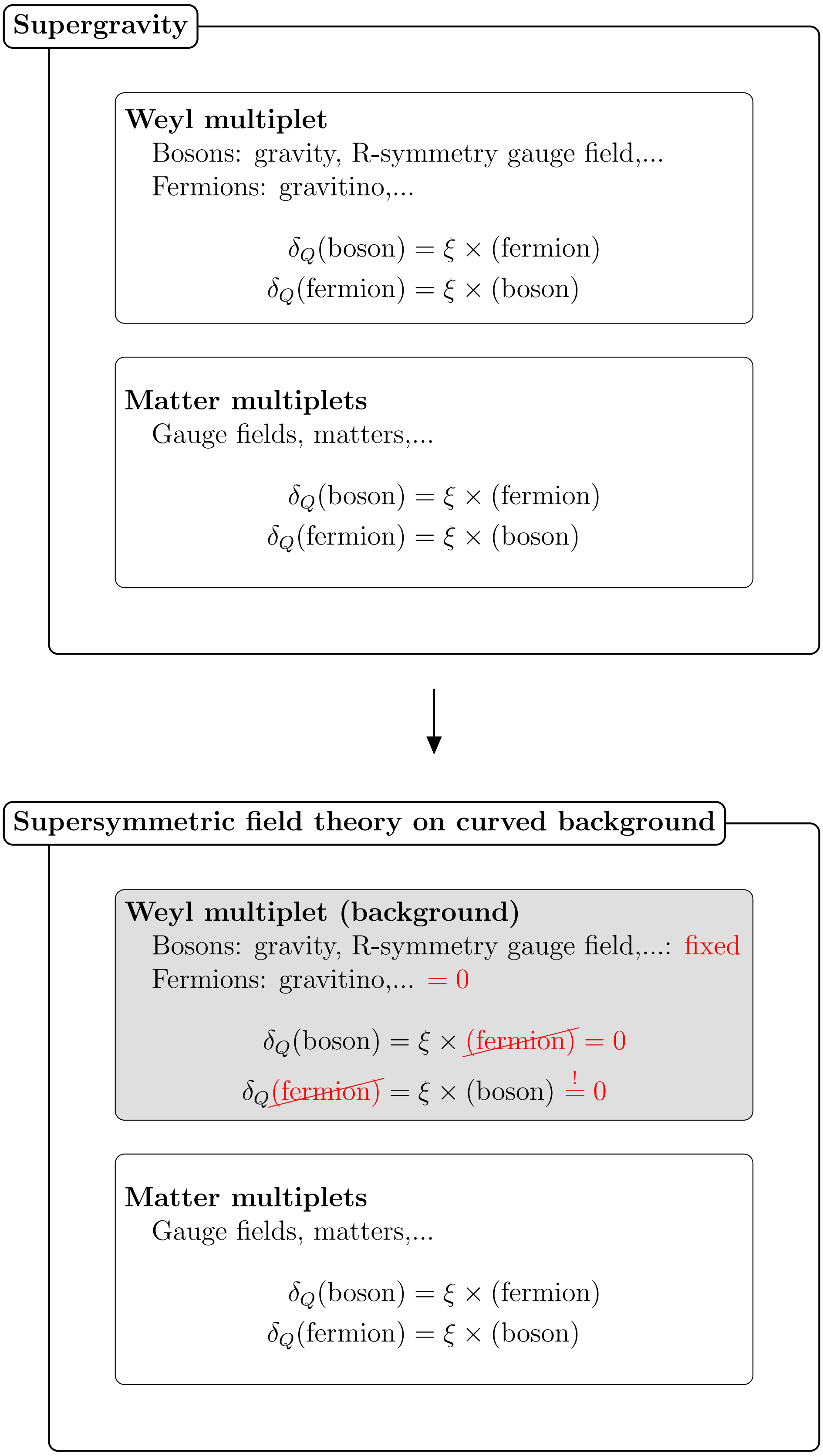

Figure 1.5: Supergravity to rigid supersymmetry. Imposing that the supersymmetry transformation of the fermions in the Weyl multiplet vanishes, we can obtain a rigid supersymmetry.

the condition that one supercharge is preserved, we can find that $\mathcal{M}_3$ admits a certain mathematical structure called transversely holomorphic foliation with a transversely Hermitian metric. Similarly to 4d, the condition is more restrictive for more preserved supercharges.

1.5.2 Parameter independence of partition function

In 4d and 3d, the resulting backgrounds are characterized by functional degrees of freedom. However, [40] show that only a small part of them contributes to the partition function. Let us consider a deformation of a background manifold. Then the Lagrangian also changes due to the deformation. If the variation of the Lagrangian can be written as $\Delta\mathcal{L} = \delta_Q(\cdots)$, such deformation does not affect the partition function. We call such deformation as a Q-exact deformation.

1.5.3 Analysis in 5d

Now we would like to analyze 5d supersymmetric field theories on curved manifolds by the same techniques. Some of 5d supersymmetric field theories are worth studying, in spite of their non-renormalizability. They have nontrivial dynamics in both UV and IR regime [41]. In particular, a global symmetry is enhanced to an exceptional group in a certain situation for some theories. Moreover, a certain 5d supersymmetric field theory is thought to be related with a mysterious 6d $\mathcal{N} = (2,0)$ theory, realized on a stack of M5-branes [42, 43]. The analysis of the condition for 5d supersymmetric backgrounds was studied partially in [44]. One of the goals of this thesis is to perform the complete analysis for 5d rigid supersymmetry [45]. The other one is to derive the background independence of the partition function, as in [40].

1.6 Organization of the thesis

In Chapter 2, we review the analyses for 4d supersymmetric backgrounds [37, 40]. We introduce the 4d $\mathcal{N} = 1$ new minimal supergravity, solve the conditions for the existence of a preserved supercharge and obtain the result that $\mathcal{M}_4$ should be Hermitian. Then we analyze the background dependence of the 4d partition function.

In Chapter 3, we perform a similar analysis for the 3d $\mathcal{N} = 2$ supergravity as a review of [39, 40]. We obtain the result that $\mathcal{M}_3$ should admit a certain mathematical structure. After that, we analyze the background dependence of the 3d partition function.

In preparation for 5d, we introduce 5d supersymmetric field theory in Chapter 4. We also show its several properties, some of which can be tested by partition functions.

Chapter 5 is the main part of this thesis and the original work of the author. This chapter is based on the reference [45]. We obtain the restriction for background fields and background (in)dependence of the partition function. After that, we construct several simple manifolds by using our formulation.

Useful formulas and basic facts are summarized in Appendices A, B and C. In Appendix D, we show the analysis of 4d $\mathcal{N} = 1$ supersymmetric backgrounds by using the 4d $\mathcal{N} = 1$ old minimal supergravity.

Chapter 2

4d $\mathcal{N} = 1$ supersymmetric backgrounds

4d $\mathcal{N} = 1$ supersymmetric field theories are studied vigorously based on various phenomenological and theoretical motivations. The Seiberg duality [22] relates different 4d $\mathcal{N} = 1$ gauge theories. The exact computation of $S^3 \times S^1$ partition function, called the superconformal index, is performed [23, 24] and the equivalence of dual theories is checked in [46]. The computation of the $S^3 \times S^1$ partition function by using another Q-exact term, called the Higgs branch localization, is performed in [47]. $S^1 \times M_3$ partition function is computed in [48], where M_3 is a circle bundle over a Riemann surface. $T^2 \times S^2$ partition function is computed in [49, 50].

A number of analyses for 4d $\mathcal{N} = 2$ gauge theories have been also performed. In [18], S^4 partition function is computed. For deformations of S^4, partition functions are computed in [51, 52]. These can be used for checks of the AGT correspondence [26]. This correspondence is thought to be related with some dynamics of 6d $\mathcal{N} = (2, 0)$ theories.

In this chapter, we review the analysis for 4d supersymmetric backgrounds [37, 40] by using the 4d $\mathcal{N} = 1$ new minimal supergravity. We introduce a supergravity with four supercharges, solve the condition that at least one supercharge is preserved, and consider the Q-exact deformations. In the following, we consider Euclidean spaces for the purpose of the computation of the partition function.

2.1 4d spinor

The notation of spinors in 4d is based on [53], except for difference between the Minkowski and Euclidean signature. 4d Euclidean space has the local Lorentz symmetry $SO(4) \sim SU(2)_+ \times SU(2)_-$. A spinor in 4d can be decomposed to a left-handed spinor ξ_α and a right-handed spinor $\overline{\xi}^{\dot{\alpha}}$. The former is an $SU(2)_+$ doublet and the latter is an $SU(2)_-$ doublet. ξ and $\overline{\xi}$ are related by the complex conjugation in the Minkowski signature. In the Euclidean signature, ξ and $\overline{\xi}$ are not related by the complex conjugation and we treat them independently.

The Hermite conjugate of spinors are complex cojugate of spinors as

$$\left(\xi^{\dagger}\right)^{\alpha} = (\xi_{\alpha})^{*}, \quad \left(\overline{\xi}^{\dagger}\right)_{\dot{\alpha}} = \left(\overline{\xi}^{\dot{\alpha}}\right)^{*}. \tag{2.1}$$

4d sigma matrices are defined by

$$\sigma^{\widehat{\mu}}_{\alpha\dot{\alpha}} = (\boldsymbol{\sigma}, -i), \quad \overline{\sigma}^{\widehat{\mu}\dot{\alpha}\alpha} = (-\boldsymbol{\sigma}, -i), \tag{2.2}$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. $\widehat{\mu} = \widehat{1}, \ldots, \widehat{4}$ is the local Lorentz index. $\sigma_{\widehat{\mu}}$ and $\overline{\sigma}_{\widehat{\mu}}$ satisfy

$$\sigma_{\{\widehat{\mu}}\overline{\sigma}_{\widehat{\nu}\}} = -\delta_{\widehat{\mu}\widehat{\nu}}, \quad \overline{\sigma}_{\{\widehat{\mu}}\sigma_{\widehat{\nu}\}} = -\delta_{\widehat{\mu}\widehat{\nu}}. \tag{2.3}$$

Some of products of the Pauli matrices are written as

$$\sigma_{\widehat{\mu}\widehat{\nu}} = \frac{1}{2}\sigma_{[\widehat{\mu}}\overline{\sigma}_{\widehat{\nu}]}, \quad \overline{\sigma}_{\widehat{\mu}\widehat{\nu}} = \frac{1}{2}\overline{\sigma}_{[\widehat{\mu}}\sigma_{\widehat{\nu}]}. \tag{2.4}$$

They are (anti-)self-dual:

$$\frac{1}{2}\epsilon_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\lambda}}\sigma^{\widehat{\rho}\widehat{\lambda}} = \sigma_{\widehat{\mu}\widehat{\nu}}, \quad \frac{1}{2}\epsilon_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\lambda}}\overline{\sigma}^{\widehat{\rho}\widehat{\lambda}} = -\overline{\sigma}_{\widehat{\mu}\widehat{\nu}}, \tag{2.5}$$

where the antisymmetric tensor is defined by $\epsilon^{\widehat{1}\widehat{2}\widehat{3}\widehat{4}} = 1$.

2.2 4d $\mathcal{N}=1$ new minimal supergravity

4d $\mathcal{N}=1$ supersymmetry algebra consists of the following symmetries:

- The translational symmetry,
- The Lorentz symmetry $SO(4) \sim SU(2)_+ \times SU(2)_-$,
- The supersymmetry, and
- $U(1)_R$ R-symmetry.

The 4d $\mathcal{N}=1$ supersymmetry algebra in the flat space is written as

$$\left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\} = 2\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}} P_{\mu}. \tag{2.6}$$

In 4d $\mathcal{N}=1$ supersymmetry, the supersymmetric transformation parameters ξ and $\overline{\xi}$ have R-charge $+1$ and -1, respectively. We can expect that 4d $\mathcal{N}=1$ supergravity is the theory in which these symmetries are gauged.

There are several versions of 4d $\mathcal{N}=1$ supergravity [54, 55, 56, 57, 58, 59, 60, 61, 62][1]. One of them is constructed in [59, 61] and called the new minimal supergravity. For constructing this, we consider the Noether current associated with the supersymmetry transformation. These supersymmetry currents $S_{\alpha\mu}$ and $\overline{S}^{\dot{\alpha}}_{\mu}$ belong to a supermultiplet. In the simplest case, other components of this supermultiplet are the energy-momentum tensor $T_{\mu\nu}$ and the $U(1)_R$ current $j^{(R)}_{\mu}$. However, these currents give a superconformally invariant theory. One way to break the conformal symmetry is introducing a closed 2-form field $\mathcal{F}_{\mu\nu}$ in the supermultiplet. The transformation law of this supermultiplet is

$$\delta_Q j^{(R)}_{\mu} = -i\xi S_{\mu} + i\overline{\xi}\overline{S}_{\mu}, \tag{2.7}$$

$$\delta_Q S_{\mu\alpha} = 2i\left(\sigma^{\nu}\overline{\xi}\right)_{\alpha}\overline{\mathcal{T}}_{\mu\nu}, \tag{2.8}$$

$$\delta_Q \overline{S}^{\dot{\alpha}}_{\mu} = 2i\left(\overline{\sigma}^{\nu}\xi\right)^{\dot{\alpha}}\mathcal{T}_{\mu\nu}, \tag{2.9}$$

$$\delta_Q T_{\mu\nu} = \frac{1}{2}\xi\sigma_{\mu\rho}\partial^{\rho}S_{\nu} + \frac{1}{2}\overline{\xi}\overline{\sigma}_{\mu\rho}\partial^{\rho}\overline{S}_{\nu} + (\mu\leftrightarrow\nu), \tag{2.10}$$

$$\delta_Q \mathcal{F}_{\mu\nu} = -\frac{i}{2}\xi\sigma_{\mu}\overline{\sigma}_{\rho}\partial_{\nu}S^{\rho} + \frac{i}{2}\overline{\xi}\overline{\sigma}_{\mu}\sigma_{\rho}\partial_{\nu}\overline{S}^{\rho} - (\mu\leftrightarrow\nu), \tag{2.11}$$

where $\mathcal{T}_{\mu\nu}$ and $\overline{\mathcal{T}}_{\mu\nu}$ are non-symmetric tensors defined by

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \frac{i}{4}\epsilon_{\mu\nu\rho\lambda}\mathcal{F}^{\rho\lambda} - \frac{i}{4}\epsilon_{\mu\nu\rho\lambda}\partial^{\rho}j^{(R)\lambda} - \frac{i}{2}\partial_{\nu}j^{(R)}_{\mu}, \tag{2.12}$$

$$\overline{\mathcal{T}}_{\mu\nu} = T_{\mu\nu} + \frac{i}{4}\epsilon_{\mu\nu\rho\lambda}\mathcal{F}^{\rho\lambda} - \frac{i}{4}\epsilon_{\mu\nu\rho\lambda}\partial^{\rho}j^{(R)\lambda} + \frac{i}{2}\partial_{\nu}j^{(R)}_{\mu}. \tag{2.13}$$

The supermultiplet which consists of the current fields

$$j^{(R)}_{\mu}, \quad S_{\mu\alpha}, \quad \overline{S}^{\dot{\alpha}}_{\mu}, \quad T_{\mu\nu}, \quad \mathcal{F}_{\mu\nu} \tag{2.14}$$

is called the $\mathcal{R}$-multiplet. The fields coupled with components in the $\mathcal{R}$-multiplet are given by

$$A_{\mu}, \quad \psi_{\mu\alpha}, \quad \overline{\psi}^{\dot{\alpha}}_{\mu}, \quad \Delta e^{\widehat{\mu}}{}_{\nu}, \quad B_{\mu\nu}, \tag{2.15}$$

where A_{μ} is the R-symmetry gauge field, $\psi_{\mu\alpha}$ and $\overline{\psi}^{\dot{\alpha}}_{\mu}$ are gravitinos, $\Delta e_{\mu}{}^{\widehat{\nu}}$ is the variation of the vielbein from the flat one and $B_{\mu\nu}$ is a two-form gauge field. The set of these fields is called the Weyl multiplet. Around the flat background, the linearized supergravity Lagrangian can be written as

$$\Delta\mathcal{L} = -\Delta e^{\widehat{\mu}}{}_{\nu}T^{\nu} + \psi^{\mu\alpha}S_{\mu\alpha} + \overline{\psi}^{\mu}_{\dot{\alpha}}\overline{S}^{\dot{\alpha}}_{\mu} + \left(A^{\mu} - \frac{3}{2}V^{\mu}\right)j^{(R)}_{\mu} + \frac{i}{4}\epsilon^{\mu\nu\rho\lambda}\mathcal{F}_{\mu\nu}B_{\rho\lambda}, \tag{2.16}$$

[1] In [63], it is shown that various versions of supergravity, including the new and old minimal supergravity, can be derived by so called superconformal tensor calculus and appropriate gauge fixing.

where V^μ is the dual of the three-form field strength of $B_{\mu\nu}$:

$$V^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}. \tag{2.17}$$

This is conserved as

$$\nabla_\mu V^\mu = 0, \tag{2.18}$$

where ∇_μ is the Lorentz covariant derivative, called the Levi-Civita connection. For spinors ξ and $\overline{\xi}$ the covariant derivative is defined by

$$\nabla_\mu \xi = \partial_\mu \xi + \frac{1}{2}\omega_{\mu\widehat{\nu}\widehat{\rho}}\sigma^{\widehat{\nu}\widehat{\rho}}\xi, \quad \nabla_\mu \overline{\xi} = \partial_\mu \overline{\xi} + \frac{1}{2}\omega_{\mu\widehat{\nu}\widehat{\rho}}\overline{\sigma}^{\widehat{\nu}\widehat{\rho}}\overline{\xi}. \tag{2.19}$$

$\omega_{\mu\widehat{\nu}\widehat{\rho}}$ is the spin connection defined by

$$\begin{aligned}\omega_{\mu\widehat{\nu}\widehat{\rho}} = \frac{1}{2}\Big(& e_{\widehat{\nu}}{}^{\lambda}\partial_\mu e_{\widehat{\rho}\lambda} - e_{\widehat{\nu}}{}^{\lambda}\partial_\lambda e_{\widehat{\rho}\mu} - e_{\widehat{\rho}}{}^{\lambda}\partial_\mu e_{\widehat{\nu}\lambda} + e_{\widehat{\rho}}{}^{\lambda}\partial_\lambda e_{\widehat{\nu}\mu} \\ & - e_{\widehat{\sigma}\mu} e_{\widehat{\nu}}{}^{\lambda} e_{\widehat{\rho}}{}^{\kappa}\partial_\lambda e_{\widehat{\sigma}\kappa} + e_{\widehat{\sigma}\mu} e_{\widehat{\nu}}{}^{\lambda} e_{\widehat{\rho}}{}^{\kappa}\partial_\kappa e_{\widehat{\sigma}\lambda}\Big).\end{aligned} \tag{2.20}$$

The linearized version of the supersymmetric transformation for the Weyl multiplet is given by requiring the supersymmetric invariance of $\Delta\mathcal{L}$. The nonlinear completion of this supergravity is constructed in [59, 61], called 4d $\mathcal{N} = 1$ new minimal supergravity.

In particular, the supersymmetric transformations for the gravitinos are written as follows:

$$\delta_Q \psi_\mu = -2\left(\nabla_\mu - iA_\mu\right)\xi - 2iV_\mu \xi - 2iV^\nu \sigma_{\mu\nu}\xi, \tag{2.21}$$

$$\delta_Q \overline{\psi}_\mu = -2\left(\nabla_\mu + iA_\mu\right)\overline{\xi} + 2iV_\mu \overline{\xi} + 2iV^\nu \overline{\sigma}_{\mu\nu}\overline{\xi}, \tag{2.22}$$

In these equations, we already set $\psi_\mu = \overline{\psi}_\mu = 0$. As explained in Section 1.5, in order to obtain the condition for existing supercharges in the rigid limit, we impose the existence of the solutions ξ and/or $\overline{\xi}$ for the equations $\delta_Q\psi_\mu = 0$ and/or $\delta_Q\overline{\psi}_\mu = 0$. In the Minkowski signature the vector fields A_μ and V_μ are real, but they may be generally complex as the background fields in the Euclidean signature. Only for the metric, we assume them to be real.

2.3 4d $\mathcal{N} = 1$ supersymmetric backgrounds

Let us consider the case in which at least one left-handed supercharge is preserved in the rigid limit for a 4d Riemannian manifold $\mathcal{M}_4$. It means that there is at least one solution ξ for the spinor equation $\delta_Q\psi_\mu = 0$. For the case of at least one right-handed supercharge, we can obtain the solution by swapping $SU(2)_+ \leftrightarrow SU(2)_-$ and the flip of the sign of the R-charge.

From $\delta_Q \psi_\mu = 0$, ξ satisfies

$$\nabla_\mu \xi = iA_\mu \xi - iV_\mu \xi - iV^\nu \sigma_{\mu\nu}\xi. \tag{2.23}$$

By taking the Hermite conjugate, $\xi^\dagger$ satisfies

$$\nabla_\mu \xi^\dagger = -iA^*_\mu \xi^\dagger + iV^*_\mu \xi^\dagger - iV^{*\nu}\xi^\dagger \sigma_{\mu\nu}. \tag{2.24}$$

Because $\delta_Q \psi_\mu = 0$ is the first-order differential equation for ξ, the nontrivial solution ξ is nowhere vanishing. By using ξ, we define the bilinears

$$|\xi|^2 = \xi^\dagger \xi, \tag{2.25}$$

$$J_{\mu\nu} = \frac{2i}{|\xi|^2}\xi^\dagger \sigma_{\mu\nu}\xi, \tag{2.26}$$

$$P_{\mu\nu} = \xi\sigma_{\mu\nu}\xi. \tag{2.27}$$

$|\xi|^2$ is a real scalar with R-charge zero. $J_{\mu\nu}$ is a real, self-dual two-form with R-charge zero. The following equation holds by the Fierz identity:

$$J^\mu{}_\nu J^\nu{}_\rho = -\delta^\mu{}_\rho. \tag{2.28}$$

The computation of this equation is shown in Appendix B.2. Therefore, $J^\mu{}_\nu$ is the almost complex structure. On every point on 4d space, we can decompose the complexified tangent space into the holomorphic and anti-holomorphic subspaces. For the almost complex structure and holomorphicity of vectors and one-forms, see Appendix C.2. A vector field X^μ is holomorphic if and only if

$$X_\mu \overline{\sigma}^\mu \xi = 0, \tag{2.29}$$

because

$$J^\mu{}_\nu X^\nu = iX^\mu \quad \Longleftrightarrow \quad \left(\xi^\dagger \sigma^\mu \overline{\sigma}_\nu \xi\right) X^\nu = 0. \tag{2.30}$$

$P_{\mu\nu}$ is a self-dual two-form with R-charge two. The following equation holds:

$$J_\mu{}^\rho P_{\rho\nu} = iP_{\mu\nu}. \tag{2.31}$$

This means that $P_{\mu\nu}$ is anti-holomorphic with respect to the almost complex structure $J^\mu{}_\nu$.

By using $\delta_Q \psi_\mu = 0$, we can show that $J^\mu{}_\nu$ is integrable. The definition of "integrable" is that for arbitrary holomorphic vector fields X^μ and Y^μ the Lie commutator

$$[X,Y]^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu \tag{2.32}$$

is also holomorphic. By using holomorphic vectors X^μ and Y^μ,

$$\begin{aligned} 0 &= X^\nu \nabla_\nu \left(Y^\mu \overline{\sigma}_\mu \xi \right) - Y^\nu \nabla_\nu \left(X^\mu \overline{\sigma}_\mu \xi \right) \\ &= [X,Y]^\mu \overline{\sigma}_\mu \xi - 2X^{[\mu} Y^{\nu]} \overline{\sigma}_\mu \nabla_\nu \xi. \end{aligned} \tag{2.33}$$

Therefore, $[X,Y]$ is holomorphic if and only if

$$X^{[\mu} Y^{\nu]} \overline{\sigma}_\mu \nabla_\nu \xi = 0. \tag{2.34}$$

By using $\delta_Q \psi_\mu = 0$, the left hand side of (2.34) is

$$X^{[\mu} Y^{\nu]} \overline{\sigma}_\mu \nabla_\nu \xi = X^{[\mu} Y^{\nu]} \overline{\sigma}_\mu \left(iA_\nu \xi - iV_\nu \xi - iV^\rho \sigma_{\nu\rho} \xi \right). \tag{2.35}$$

By using the assumption that X^μ and Y^μ are both holomorphic, the first two terms of the right hand side vanish. The last term also vanishes because

$$\overline{\sigma}_\mu \sigma_{\nu\rho} = \overline{\sigma}_{\nu\rho} \overline{\sigma}_\mu + g_{\mu\rho} \overline{\sigma}_\nu - g_{\mu\nu} \overline{\sigma}_\rho. \tag{2.36}$$

Therefore $[X,Y]$ is holomorphic and thus $J^\mu{}_\nu$ is integrable.

As another proof, we can show that the Nijenhuis tensor of $J^\mu{}_\nu$

$$N^\mu{}_{\nu\rho} = J^\lambda{}_\nu \nabla_\lambda J^\mu{}_\rho - J^\lambda{}_\rho \nabla_\lambda J^\mu{}_\nu - J^\mu{}_\lambda \nabla_\nu J^\lambda{}_\rho + J^\mu{}_\lambda \nabla_\rho J^\lambda{}_\nu \tag{2.37}$$

vanishes by using $\delta_Q \psi_\mu = 0$.

As a mathematical fact, for an almost complex manifold, the following statements are equivalent:

- The manifold is complex.
- There is an integrable almost complex structure.
- The Nijenhuis tensor vanishes.

Therefore, the 4d manifold $\mathcal{M}_4$ is complex. By using the almost complex structure $J^\mu{}_\nu$, we can introduce local holomorphic coordinates z^i ($i = 1, 2$). We use i, j as holomorphic indices and $\bar{i}, \bar{j}$ as anti-holomorphic indices. In these coordinates, the almost complex structure is represented as

$$J^i{}_j = i\delta^i{}_j, \quad J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}{}_{\bar{j}}. \tag{2.38}$$

A Hermitian metric can be defined on every complex manifold. Lowering the upper indices in (2.38) by the Hermitian metric, we obtain the relation

$$g_{i\bar{j}} = iJ_{i\bar{j}}, \quad g_{\bar{i}j} = -iJ_{\bar{i}j}. \tag{2.39}$$

Hence $J_{\mu\nu}$ is the Kähler form of the Hermitian metric $g_{\mu\nu}$.

In order to obtain the condition for V_μ, let us differentiate $J^\mu{}_\nu$. By using $\delta_Q\psi_\mu=0$ straightforwardly,

$$\nabla_\mu J^\mu{}_\nu = -\left(V_\nu+V_\nu^*\right)+i\left(V_\mu-V_\mu^*\right)J^\mu{}_\nu. \tag{2.40}$$

Multiplying $J^\nu{}_\rho$, we obtain

$$\left(\nabla_\mu J^\mu{}_\nu\right)J^\nu{}_\rho = -\left(V_\nu+V_\nu^*\right)J^\nu{}_\rho - i\left(V_\rho-V_\rho^*\right). \tag{2.41}$$

From these two equations, we can eliminate V_μ^* as

$$\nabla_\mu J^\mu{}_\rho - i\left(\nabla_\mu J^\mu{}_\nu\right)J^\nu{}_\rho = -2V_\rho+2iV_\mu J^\mu{}_\rho = -4V_\rho^+, \tag{2.42}$$

where V_ρ^+ represents the holomorphic part of V_ρ. Focusing on a holomorphic part of $\nabla_\mu J^\mu{}_\rho$, the left hand side becomes $2\nabla_\mu J^\mu{}_\rho$. As a result, (2.40) restricts the holomorphic part of V_μ as

$$V_\mu = -\frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu, \tag{2.43}$$

where U_μ is an undetermined anti-holomorphic vector; $J_\mu{}^\nu U_\nu = iU_\mu$. Due to (2.18), U_μ is conserved:

$$\nabla^\mu U_\mu = 0. \tag{2.44}$$

A Hermitian manifold is Kähler if and only if the almost complex structure satisfies $\nabla_\mu J_{\nu\rho}=0$. As seen in (2.40), it is not the case. Hence, it is desirable to introduce another connection ∇_μ^c satisfying $\nabla_\mu^c g_{\nu\rho}=0$ and $\nabla_\mu^c J_{\nu\rho}=0$. Such connection is called the Chern connection and defined by replacing the ordinary spin connection to

$$\omega^c_{\mu\nu\rho} = \omega_{\mu\nu\rho} - \frac{1}{2}J_\mu{}^\lambda\left(\nabla_\lambda J_{\nu\rho}+\nabla_\nu J_{\rho\lambda}+\nabla_\rho J_{\lambda\nu}\right). \tag{2.45}$$

By using this Chern connection, we can rewrite $\delta_Q\psi_\mu=0$ as

$$\left(\nabla_\mu^c - iA_\mu^c\right)\xi = 0, \tag{2.46}$$

where

$$A_\mu^c = A_\mu + \frac{1}{4}\left(\delta_\mu{}^\nu - iJ_\mu{}^\nu\right)\nabla_\rho J^\rho{}_\nu - \frac{3}{2}U_\mu. \tag{2.47}$$

In the case in which the 4d manifold is Kähler, V_μ vanishes and the refined spin connection (2.45) and the refined gauge field (2.47) reduce to the ordinary ones:

$$V_\mu=0,\quad \nabla_\mu^c=\nabla_\mu,\quad \omega^c_{\mu\nu\rho}=\omega_{\mu\nu\rho},\quad A_\mu^c=A_\mu, \tag{2.48}$$

if $U_\mu = 0$.

Let us derive a restriction due to $P_{\mu\nu}$. As mentioned above, $P_{\mu\nu}$ is an anti-holomorphic self-dual two-form. In the local holomorphic coordinates z^i, its nonzero component is only

$$p = P_{\overline{12}}. \tag{2.49}$$

If we would like to construct a scalar from p, we can see that $|p|^2/\sqrt{g}$ is a positive scalar on $\mathcal{M}_4$, where $g = \det g_{\mu\nu}$. Therefore, we can consider the nowhere vanishing scalar

$$s = pg^{-1/4} \tag{2.50}$$

with the R-charge two.

By using (2.46), p satisfies

$$\left(\nabla^c_\mu - 2iA^c_\mu\right) p = 0. \tag{2.51}$$

The action of the Chern connection to the anti-holomorphic two-form p is as follows:

$$\nabla^c_i p = \partial_i p, \quad \nabla^c_{\bar{i}} p = \partial_{\bar{i}} p - \frac{p}{2}\partial_{\bar{i}} \log p. \tag{2.52}$$

Therefore, by using (2.50), (2.51) and (2.52), we obtain A^c_μ and thus A_μ as

$$A_\mu = A^c_\mu - \frac{1}{4}\left(\delta_\mu{}^\nu - iJ_\mu{}^\nu\right)\nabla_\rho J^\rho{}_\nu + \frac{3}{2}U_\mu, \tag{2.53}$$

$$A^c_i = -\frac{i}{8}\partial_i \log g - \frac{i}{2}\partial_i \log s, \tag{2.54}$$

$$A^c_{\bar{i}} = \frac{i}{8}\partial_{\bar{i}} \log g - \frac{i}{2}\partial_{\bar{i}} \log s. \tag{2.55}$$

To summarize, the existence of the solution of $\delta_Q\psi_\mu = 0$ yields the existence of an integrable complex structure. Hence 4d manifold $\mathcal{M}_4$ is Hermitian. The holomorphic part of V_μ is defined by (2.43) and the R-symmetry gauge field is written by (2.53), (2.54) and (2.55). Note that S^4 is not complex manifold and does not admit the almost complex structure. Therefore we cannot construct supersymmetric field theories on S^4 by using this formulation.

So far we have shown that

$$\exists \text{ solution of } \delta_Q\psi_\mu = 0 \quad \Longrightarrow \quad \mathcal{M}_4 \text{ is Hermitian manifold.}$$

Conversely, we can show that there is at least one solution of $\delta_Q\psi_\mu = 0$ for a general Hermitian manifold with a metric $g_{\mu\nu}$ and complex structure $J^\mu{}_\nu$. We introduce a nowhere vanishing real scalar s and an anti-holomorphic conserved vector U_μ and set the background fields V_μ and A_μ as in (2.43), (2.53), (2.54) and (2.55). In a local frame, we take the vielbein as

$$ds^2 = e^{\hat{1}} e^{\hat{\bar{1}}} + e^{\hat{2}} e^{\hat{\bar{2}}}, \tag{2.56}$$

$$\frac{1}{\sqrt{2}} e^{\hat{1}} = \sqrt{g_{1\bar{1}}} dz^1 + \frac{g_{2\bar{1}}}{\sqrt{g_{1\bar{1}}}} dz^2, \quad \frac{1}{\sqrt{2}} e^{\hat{2}} = \frac{g^{1/4}}{\sqrt{g_{1\bar{1}}}} dz^2. \tag{2.57}$$

In such configuration, we can find that

$$\xi_\alpha = \frac{\sqrt{s}}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.58}$$

is always a solution of $\delta_Q \psi_\mu = 0$.

By imposing the condition that two and more number of supercharges are preserved, we can obtain more restrictive conditions. We list such result below:

- The presence of one supercharge ξ implies that $\mathcal{M}_4$ is Hermitian.
- The presence of two supercharges ξ and $\bar{\xi}$ with opposite $U(1)_R$ charge gives a complex Killing vector $K_\mu = \xi \sigma_\mu \bar{\xi}$. Hence $\mathcal{M}_4$ can be described as a torus fibration over an arbitrary Riemann surface.
- The presence of two supercharges with equal $U(1)_R$ charge implies that $\mathcal{M}_4$ is either of them:
  - A torus T^4 with flat metric,
  - A $K3$ surface with Ricci-flat Kähler metric, and
  - A discrete quotient of $S^3 \times S^1$ with the standard metric $ds^2 = d\tau^2 + r^2 d\Omega_3$,

  if $\mathcal{M}_4$ is compact.
- The presence of four supercharges implies that $\mathcal{M}_4$ is locally isometric to either of them:
  - $S^3 \times \mathbb{R}$,
  - Flat $\mathbb{R}^4$, and
  - $H^3 \times \mathbb{R}$, where H^3 is a 3d hyperbolic space.

There is another version of 4d $\mathcal{N} = 1$ off-shell supergravity, called the old minimal supergravity [56, 57]. We can work out a similar analysis by using this [38]. From that analysis, we can show that Hermitian manifolds and warped products $S^3 \times \mathbb{R}$ are allowed. This shows that S^4 is also allowed, while S^4 is not allowed by the new minimal supergravity. The analysis by using the old minimal supergravity is shown in Appendix D.

2.4 Background vector multiplet

If there are continuous flavor symmetries, we can obtain the flavor symmetry current j_μ by the Noether procedure. This current j_μ is included in a real linear multiplet $\mathcal{J}$. The real linear multiplet also includes fermions j_α and $\overline{j}^{\dot\alpha}$ and a scalar J. Their supersymmetry transformation laws are written as follows:

$$\delta_Q J = i\xi j - i\overline{\xi}\overline{j}, \tag{2.59}$$
$$\delta_Q j_\alpha = -i\left(\sigma^\mu\overline{\xi}\right)_\alpha \overline{\mathcal{J}}_\mu, \tag{2.60}$$
$$\delta_Q \overline{j}^{\dot\alpha} = -i\left(\overline{\sigma}^\mu\xi\right)^{\dot\alpha} \mathcal{J}_\mu, \tag{2.61}$$
$$\delta_Q j_\mu = -2\xi\sigma_{\mu\nu}\partial^\nu j - 2\overline{\xi}\overline{\sigma}_{\mu\nu}\partial^\nu\overline{j}, \tag{2.62}$$

where

$$\mathcal{J}_\mu = j_\mu - i\partial_\mu J, \quad \overline{\mathcal{J}}_\mu = j_\mu + i\partial_\mu J. \tag{2.63}$$

We can couple them to a vector multiplet, which consists of a gauge field A_μ, gauginos λ_α and $\overline{\lambda}^{\dot\alpha}$ and a scalar D. In order to introduce a background vector multiplet, both gauginos and their supersymmetry transformations should vanish. The supersymmetry transformation for the gaugino λ_α is written as

$$\delta_Q\lambda = i\xi D + \sigma^{\mu\nu}\xi F_{\mu\nu}(A), \tag{2.64}$$

where $F_{\mu\nu}(A)$ is the fields strength of A_μ.

Similar to the analysis above, we can obtain the restriction for the background vector multiplet. By multiplying ξ and $\xi^\dagger$ to (2.64), we obtain the conditions for D and $F_{\mu\nu}(A)$ as

$$F_{ij}(A) = 0, \quad D = \frac{1}{2}J^{\mu\nu}F_{\mu\nu}(A). \tag{2.65}$$

2.5 Q-exact deformations

As in Section 1.3 and 1.5, Q-exact deformations of an action $S \to S + \delta_Q V$ do not change the partition function. Therefore, when we consider a deformation of the theory, the partition function does not change if the variation of the action is Q-exact. In the above analysis, we have obtained the formulation in which parameters characterizing a theory can be treated as background fields in the supergravity. This makes the discussion whether a deformation of the theory gives a Q-exact deformation easier. By a small deformation for the background fields of the Weyl multiplet, the shift of the Lagrangian can be written by a linear coupling between the Weyl multiplet and the supercurrent multiplet. Therefore, we will discuss Q-exact deformations by using properties of the supercurrent multiplet. In the analysis [40] and here, we focus on the formulation by the new minimal supergravity, which is coupled with the $\mathcal{R}$-multiplet.

Moreover, in the following we only consider the case in which the 4d manifold is given by the small deformation around the flat space. However, the conclusion can be extended to general supersymmetric backgrounds. The supersymmetry Q can be a scalar by appropriate twisting by background gauge fields. Thus, similar to the topological field theory, the twisted energy-momentum tensor is Q-exact and the partition function is independent of the metric. Therefore it is sufficient to consider the small deformation around the flat space.

As shown in Section 2.3 and 2.4, the existence of at least one preserved supercharge restricts the background fields in the Weyl multiplet and vector multiplets. Keeping one supercharge preserved, we can freely choose the following background fields and parameters with a preserved supercharge:

- The integrable complex structure $J^{\mu}{}_{\nu}$,
- A compatible Hermitian metric $g_{i\bar{j}}$,
- The $(1,2)$-form W satisfying $\partial W = 0$,
- Abelian background gauge fields satisfying (2.65), and
- Coupling constants.

In the above the $(1,2)$-form W is defined by $W = *U$. By using (2.43), $H = dB$ can be expressed as

$$H = -\frac{1}{2}dJ + W. \tag{2.66}$$

The conservation condition $\nabla^{\mu}U_{\mu} = 0$ leads the condition for W as

$$\partial W = 0, \tag{2.67}$$

where ∂ is the Dolbeault operator.

There are functional degrees of freedom. However, as we will see, only the small number of the degrees of freedom affects the partition function.

Note that the notation in [40] is little bit different from the one in [37]. In this thesis we use the notation in [37].

2.5.1 Deformation theory

Before seeing deformation of a supersymmetric theory, let us review the properties of deformations of the complex structure, the Hermitian metric and gauge fields [64, 65].

Complex structures

Let us consider an infinitesimal deformation of the integrable complex structure $\Delta J^{\mu}{}_{\nu}$. Because $\Delta J^{\mu}{}_{\nu}$ is infinitesimal, we focus only on the linear order. We use the holomorphic coordinate z^i with respect to $J^{\mu}{}_{\nu}$.

$J^{\mu}{}_{\nu} + \Delta J^{\mu}{}_{\nu}$ should be an almost complex structure, satisfying

$$\left(J^{\mu}{}_{\nu} + \Delta J^{\mu}{}_{\nu}\right)\left(J^{\nu}{}_{\rho} + \Delta J^{\nu}{}_{\rho}\right) = -\delta^{\mu}_{\rho}. \tag{2.68}$$

This implies the condition

$$\Delta J^{i}{}_{j} = \Delta J^{\bar{i}}{}_{\bar{j}} = 0. \tag{2.69}$$

$J^{\mu}{}_{\nu} + \Delta J^{\mu}{}_{\nu}$ should be also integrable. This implies that the infinitesimal deformation of the Nijenhuis tensor leaves zero as

$$\partial_j \Delta J^{\bar{i}}{}_{k} - \partial_k \Delta J^{\bar{i}}{}_{j} = 0. \tag{2.70}$$

Its complex conjugate also holds. (2.70) is antisymmetric with respect to j and k. Hence we can introduce a $(1,0)$-form

$$\Theta^{\bar{i}} = \Delta J^{\bar{i}}{}_{j} dz^{j} \tag{2.71}$$

with coefficients in the anti-holomorphic tangent bundle $T^{0,1}\mathcal{M}_4$. By using $\Theta^{\bar{i}}$, (2.70) can be expressed as

$$\partial \Theta^{\bar{i}} = 0. \tag{2.72}$$

Not all deformations $\Delta J^{\mu}{}_{\nu}$ are meaningful deformations. There is a class of deformations which can be identified as infinitesimal translation of $J^{\mu}{}_{\nu}$. For an infinitesimal real vector ϵ^{μ}, the translation of $J^{\mu}{}_{\nu}$ along ϵ^{μ} is written as

$$\Delta J^{\bar{i}}{}_{j} = \left(\mathcal{L}_{\epsilon} J\right)^{\bar{i}}{}_{j} = 2i\partial_j \epsilon^{\bar{i}}. \tag{2.73}$$

Hence there is a trivial deformation of the complex structure written as

$$\Theta^{\bar{i}} = 2i\partial \epsilon^{\bar{i}}. \tag{2.74}$$

Therefore, quotienting a space parameterized by the infinitesimal $\Theta^{\bar{i}}$ by the identification $\Theta^{\bar{i}} \sim \Theta^{\bar{i}} + 2i\partial\epsilon^{\bar{i}}$, we can obtain a space parameterizing the deformations of the complex structure. This is described by the Dolbeault cohomology with coefficients in $T^{0,1}\mathcal{M}_4$ as

$$\left[\Theta^{\bar{i}}\right] \in H^{1,0}\left(\mathcal{M}_4, T^{0,1}\mathcal{M}_4\right). \tag{2.75}$$

$H^{1,0}\left(\mathcal{M}_4, T^{0,1}\mathcal{M}_4\right)$ is a complex vector space spanned by ∂-harmonic $(1,0)$-forms. Its dimension is finite if $\mathcal{M}_4$ is compact. Therefore, the deformation of the complex structure can be parametrized by the finite number of parameters.

Hermitian metrics

Next let us consider the deformation of the Hermitian metric. For a complex manifold, we can always take a Hermitian metric compatible with the complex structure as

$$g_{\mu\nu} J^{\mu}{}_{\rho} J^{\nu}{}_{\sigma} = g_{\rho\sigma}. \tag{2.76}$$

Consider the infinitesimal deformation of both the Hermitian metric and the complex structure. Then $g_{\mu\nu} + \Delta g_{\mu\nu}$ should be compatible with $J^{\mu}{}_{\nu} + \Delta J^{\mu}{}_{\nu}$. In the first order, the following equation holds:

$$g_{\mu\nu} \left(\Delta J^{\mu}{}_{\rho} J^{\nu}{}_{\sigma} + J^{\mu}{}_{\rho} \Delta J^{\nu}{}_{\sigma} \right) + \Delta g_{\mu\nu} J^{\mu}{}_{\rho} J^{\nu}{}_{\sigma} = \Delta g_{\rho\sigma}. \tag{2.77}$$

This equation implies the relation between Δg_{ij}, $\Delta g_{\bar{i}\bar{j}}$ and $\Delta J^{\mu}{}_{\nu}$ as

$$\Delta g_{ij} = -\frac{i}{2} \left(g_{i\bar{k}} \Delta J^{\bar{k}}{}_{j} + g_{j\bar{k}} \Delta J^{\bar{k}}{}_{i} \right), \quad \Delta g_{\bar{i}\bar{j}} = \frac{i}{2} \left(g_{k\bar{i}} \Delta J^{k}{}_{\bar{j}} + g_{k\bar{j}} \Delta J^{k}{}_{\bar{i}} \right), \tag{2.78}$$

while $\Delta g_{i\bar{j}} = \Delta g_{\bar{j}i}$ is not constrained.

Abelian gauge fields

Let us consider the deformation of an Abelian gauge field A_{μ} satisfying $F_{ij}(A) = 0$. Since $F_{ij}(A) = 0$, the holomorphic part of A_{μ} can be locally expressed by using a complex function $\lambda(z, \bar{z})$ as

$$A_i = \partial_i \lambda. \tag{2.79}$$

The complex function λ is defined only locally. If we allow a complexified $U(1)$ gauge transformation, we can take a gauge such that $A_i = 0$. A transition function $g(z, \bar{z})$ preserving such gauge choice satisfies

$$\partial_i g = 0, \tag{2.80}$$

which determines an anti-holomorphic line bundle over $\mathcal{M}_4$.

The structure of the anti-holomorphic line bundle depends only on A_i. Let us consider a deformation of A_i by ΔA_i which is a globally defined $(1,0)$-form. From $F_{ij}(A) = 0$, ΔA_i satisfies

$$\partial_i \Delta A_j - \partial_j \Delta A_i = 0. \tag{2.81}$$

Similar to the discussion about the deformation of the complex structure, a class of deformations can be absorbed by a gauge transformation. The gauge transformation yields the deformation of the gauge field as

$$\Delta A_i = \partial_i \epsilon, \tag{2.82}$$

where $\epsilon(z, \bar{z})$ is a globally defined complex function on $\mathcal{M}_4$. Therefore, meaningful deformations of A_i are parametrized by the Dolbeault cohomology:

$$[\Delta A_i] \in H^{1,0}(\mathcal{M}_4). \tag{2.83}$$

2.5.2 Parameter dependence

We would like to consider a deformation of the background Weyl multiplet, leaving one supercharge Q preserved. By the infinitesimal deformation, the Lagrangian is shifted by the linear combination between the shift of the Weyl multiplet and the $\mathcal{R}$-multiplet. Let us recall the properties of the $\mathcal{R}$-multiplet in the presence of the supercharge Q, which generates a supersymmetry transformation parameterized by ξ.

The $\mathcal{R}$-multiplet is introduced in Section 2.2. The supersymmetry transformation law parameterized by ξ is obtained by setting $\overline{\xi}=0$ in (2.7)-(2.11) as

$$\delta_Q j_\mu^{(R)} = -i\xi S_\mu, \tag{2.84}$$
$$\delta_Q S_{\mu\alpha} = 0, \tag{2.85}$$
$$\delta_Q \overline{S}_\mu^{\dot{\alpha}} = 2i\left(\overline{\sigma}^\nu \xi\right)^{\dot{\alpha}} \mathcal{T}_{\mu\nu}, \tag{2.86}$$
$$\delta_Q T_{\mu\nu} = \frac{1}{2}\xi\sigma_{\mu\rho}\partial^\rho S_\nu + \frac{1}{2}\xi\sigma_{\nu\rho}\partial^\rho S_\mu, \tag{2.87}$$
$$\delta_Q \mathcal{F}_{\mu\nu} = -\frac{i}{2}\xi\sigma_\mu\overline{\sigma}_\rho\partial_\nu S^\rho + \frac{i}{2}\xi\sigma_\nu\overline{\sigma}_\rho\partial_\mu S^\rho, \tag{2.88}$$

where $\mathcal{T}_{\mu\nu}$ is defined in (2.12).

We can find the eight bosonic Q-exact operators $\delta_Q \overline{S}_\mu^{\dot{\alpha}}$, which are supersymmetry transformations of the fermionic operators. Multiplying $|\xi|^{-2}\xi^\dagger\sigma_\rho$ by (2.86), we obtain

$$\delta_Q\left(\frac{1}{|\xi|^2}\xi^\dagger\sigma_\rho\overline{S}_\mu\right) = -2i\left(\delta^\nu{}_\rho - iJ^\nu{}_\rho\right)\mathcal{T}_{\mu\nu}. \tag{2.89}$$

Multiplying $\delta^\nu{}_\rho - iJ^\nu{}_\rho$ by $\mathcal{T}_{\mu\nu}$ leaves only the holomorphic part with respect to ν. Therefore, the eight bosonic Q-exact operators are $\mathcal{T}_{\mu i}$. In the holomorphic coordinates $z^1=w$ and $z^2=z$, each component of $\mathcal{T}_{\mu i}$ can be written as

$$\mathcal{T}_{\overline{w}w} = T_{\overline{w}w} - \frac{i}{2}\mathcal{F}_{z\overline{z}} - \frac{i}{2}\partial_w j_{\overline{w}}^{(R)} + \frac{i}{4}\partial_z j_{\overline{z}}^{(R)} - \frac{i}{4}\partial_{\overline{z}} j_z^{(R)}, \tag{2.90}$$
$$\mathcal{T}_{\overline{w}z} = T_{\overline{w}z} - \frac{i}{2}\mathcal{F}_{\overline{w}z} - \frac{3i}{4}\partial_z j_{\overline{w}}^{(R)} + \frac{i}{4}\partial_{\overline{w}} j_z^{(R)}, \tag{2.91}$$
$$\mathcal{T}_{ww} = T_{ww} - \frac{i}{2}\partial_w j_w^{(R)}, \tag{2.92}$$
$$\mathcal{T}_{wz} = T_{wz} + \frac{i}{2}\mathcal{F}_{wz} - \frac{i}{4}\partial_w j_z^{(R)} - \frac{i}{4}\partial_z j_w^{(R)}, \tag{2.93}$$

and the remaining four components can be obtained from the above by $w \leftrightarrow z$ and $\overline{w} \leftrightarrow \overline{z}$.

At the first order, the Lagrangian for the bosonic components of the Weyl multiplet is given by the linear combination between the Weyl multiplet and the

$\mathcal{R}$-multiplet as

$$\Delta\mathcal{L} = -\frac{1}{2}\Delta g^{\mu\nu}T_{\mu\nu} + A^{(R)\mu}j^{(R)}_{\mu} + \frac{i}{4}\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}\mathcal{F}_{\rho\lambda}, \tag{2.94}$$

where

$$A^{(R)}_{\mu} = A_{\mu} - \frac{3}{2}V_{\mu}. \tag{2.95}$$

We would like to consider the deformation of the Hermitian metric and the complex structure restricted by (2.78). Other bosonic components of the background Weyl multiplet V^{μ} and $A^{(R)\mu}$ are restricted as shown in Section 2.3. They can be written in terms of $\Delta g_{\mu\nu}$, $\Delta J^{\mu}{}_{\nu}$ and W as

$$V^{w} = \frac{1}{2}\partial_z\left(\Delta J^{z}{}_{\overline{w}} - \Delta J^{w}{}_{\overline{z}}\right) - 2i\left(\partial_{\overline{z}}\Delta g_{z\overline{w}} - \partial_{\overline{w}}\Delta g_{z\overline{z}}\right) + 4iW_{zw\overline{z}}, \tag{2.96}$$

$$V^{\overline{w}} = \frac{1}{2}\partial_{\overline{z}}\left(\Delta J^{\overline{z}}{}_{w} - \Delta J^{\overline{w}}{}_{z}\right) - 2i\left(\partial_{w}\Delta g_{z\overline{z}} - \partial_{z}\Delta g_{w\overline{z}}\right), \tag{2.97}$$

$$A^{(R)w} = \frac{1}{2}\partial_w\Delta J^{w}{}_{\overline{w}} - \frac{1}{4}\partial_z\Delta J^{z}{}_{\overline{w}} + \frac{3}{4}\partial_z\Delta J^{w}{}_{\overline{z}} - 3i\partial_{\overline{z}}\Delta g_{z\overline{w}} + 2i\partial_{\overline{w}}\Delta g_{z\overline{z}} - i\partial_{\overline{w}}\Delta g_{w\overline{w}}, \tag{2.98}$$

$$A^{(R)\overline{w}} = \frac{1}{2}\partial_{\overline{w}}\Delta J^{\overline{w}}{}_{w} + \frac{1}{4}\partial_{\overline{z}}\Delta J^{\overline{z}}{}_{w} + \frac{1}{4}\partial_{\overline{z}}\Delta J^{\overline{w}}{}_{z} + i\partial_w\Delta g_{w\overline{w}} + i\partial_z\Delta g_{w\overline{z}}, \tag{2.99}$$

and the remaining four components can be obtained from the above by $w \leftrightarrow z$ and $\overline{w} \leftrightarrow \overline{z}$. The $(1,2)$-form W satisfies $\partial W = 0$, thus can be locally written by $(0,2)$-form $\widetilde{B}_{\overline{i}\overline{j}}$ as $W = \partial\widetilde{B}$. By using (2.90)-(2.93), (2.96)-(2.99) and $W = \partial\widetilde{B}$ and dropping total derivatives, the Lagrangian (2.94) can be rewritten as

$$\begin{aligned}\Delta\mathcal{L} = &-\Delta g^{\overline{i}j}\mathcal{T}_{\overline{i}j} + i\widetilde{B}_{\overline{w}\overline{z}}\mathcal{F}_{wz} - i\sum_{j=\overline{j}}\Delta J^{i}{}_{\overline{j}}\mathcal{T}_{ji}\\ &+ i\Delta J^{\overline{w}}{}_{w}\left(T_{\overline{w}\overline{w}} + \frac{i}{2}\partial_{\overline{w}}j^{(R)}_{\overline{w}}\right) + i\Delta J^{\overline{z}}{}_{z}\left(T_{\overline{z}\overline{z}} + \frac{i}{2}\partial_{\overline{z}}j^{(R)}_{\overline{z}}\right)\\ &+ i\Delta J^{\overline{w}}{}_{z}\left(T_{\overline{w}\overline{z}} + \frac{i}{2}\mathcal{F}_{\overline{w}\overline{z}} - \frac{i}{4}\partial_{\overline{w}}j^{(R)}_{\overline{z}} + \frac{3i}{4}\partial_{\overline{z}}j^{(R)}_{\overline{w}}\right)\\ &+ i\Delta J^{\overline{z}}{}_{w}\left(T_{\overline{w}\overline{z}} - \frac{i}{2}\mathcal{F}_{\overline{w}\overline{z}} - \frac{i}{4}\partial_{\overline{z}}j^{(R)}_{\overline{w}} + \frac{3i}{4}\partial_{\overline{w}}j^{(R)}_{\overline{z}}\right).\end{aligned} \tag{2.100}$$

We can find that the first line of the right hand side of (2.100) is Q-exact, because $\mathcal{T}_{\mu i}$ and $\mathcal{F}_{wz} = i\left(\mathcal{T}_{zw} - \mathcal{T}_{wz}\right)$ are Q-exact. This fact implies that the first line does not affect the partition function. Deformations arising on the remaining part of (2.100) are only $\Delta J^{i}{}_{\overline{j}}$ and $\Delta J^{\overline{i}}{}_{j}$. Therefore, we obtain the (in)dependence of the partition function with respect to the deformation of the Hermitian metric and the complex structure as:

- For a fixed complex structure, a deformation of the Hermitian metric does not affect the partition function.
- The partition function depends only on the part of the deformation of the complex structure, $\Delta J^{i}_{\ \overline{j}}$ and $\Delta J^{\overline{i}}_{\ j}$. From the discussion about the deformation of the complex structure (2.75), the partition function is a locally anti-holomorphic function of the complex structure moduli.

Let us consider the (in)dependence of the partition function with respect to W. We find that a globally defined $(0,2)$-form $\widetilde{B}$ yields Q-exact deformation and does not change the partition function. Therefore,

- The partition function depends on W through its cohomology class $H^{1,2}\left(\mathcal{M}_4\right)$.

Let us discuss the deformation of the background Abelian vector multiplet. The current multiplet associated with the vector multiplet is the real linear multiplet $\mathcal{J}$, introduced in Section 2.4. In the presence of the preserved supercharge Q, which generates a supersymmetry transformation parameterized by ξ, the real linear multiplet is transformed as

$$\delta_Q J = i\xi j, \tag{2.101}$$

$$\delta_Q j_\alpha = 0, \tag{2.102}$$

$$\delta_Q \overline{j}^{\dot{\alpha}} = -i\left(\overline{\sigma}^\mu \xi\right)^{\dot{\alpha}} \mathcal{J}_\mu, \tag{2.103}$$

$$\delta_Q j_\mu = -2\xi\sigma_{\mu\nu}\partial^\nu j, \tag{2.104}$$

where $\mathcal{J}_\mu$ is defined in (2.63). We can find two bosonic Q-exact operators $\delta_Q \overline{j}^{\dot{\alpha}}$. Similar to the analysis above, by multiplying $|\xi|^{-2}\xi^\dagger\sigma_\nu$ to (2.103), we obtain

$$\delta_Q\left(\frac{1}{|\xi|^2}\xi^\dagger\sigma_\nu\overline{j}\right) = i\left(\delta^\mu_{\ \nu} - iJ^\mu_{\ \nu}\right)\mathcal{J}_\mu. \tag{2.105}$$

Thus we find that the holomorphic part $\mathcal{J}_i$ is Q-exact. The bosonic linearized coupling between a vector multiplet and corresponding real linear multiplet is written as

$$\Delta\mathcal{L} = A^\mu j_\mu - DJ. \tag{2.106}$$

In the presence of a supercharge corresponding to ξ, the background field D is written as (2.65), which is rewritten as

$$D = 2i\left(F_{w\overline{w}}(A) + F_{z\overline{z}}(A)\right), \tag{2.107}$$

in the holomorphic coordinates. Substituting it to (2.106) and dropping total derivatives, we obtain

$$\Delta\mathcal{L} = 2A_{\overline{w}}\mathcal{J}_w + 2A_{\overline{z}}\mathcal{J}_z + 2A_w\left(j_{\overline{w}} + i\partial_{\overline{w}}J\right) + 2A_z\left(j_{\overline{z}} + i\partial_{\overline{z}}J\right). \tag{2.108}$$

Because $\mathcal{J}_i$ is Q-exact, the first two terms of the right hand side do not affect the partition function. Therefore, we conclude that

- The partition function depends only on the holomorphic part of the background Abelian gauge field. From the discussion above (2.83), the partition function is a locally anti-holomorphic function of the corresponding anti-holomorphic line bundle moduli.

If we allow the complexified gauge transformation, the partition function depends on A_i through the cohomology class $H^{1,0}(\mathcal{M}_4)$.

In [40], the following is commented. The $\mathcal{R}$-multiplet can be embedded into the Ferrara-Zumino multiplet except for the following cases:

- An action includes the Feyet-Iliopoulos terms [66], and

- The Kähler form of the target space is not exact [67].

The Ferrara-Zumino multiplet is a supercurrent multiplet corresponding to the Weyl multiplet in the 4d $\mathcal{N} = 1$ old minimal supergravity. If we can embed the $\mathcal{R}$-multiplet into the Ferrara-Zumino multiplet, it can be shown that W does not affect the partition function.

Chapter 3

3d $\mathcal{N} = 2$ supersymmetric backgrounds

3d supersymmetric field theories are also studied vigorously in theoretical motivations. Dualities for them play important roles to reveal properties of superstring and M-theory. Similarly to 4d, there is the Seiberg-like duality [68] which relates different 3d $\mathcal{N} = 2$ gauge theories. A 3d effective theory on a stack of N M2-branes is proposed, in terms of 3d $\mathcal{N} = 2$ gauge theory [9]. This 3d theory is called the ABJM theory. Moreover, there are relations between 3d $\mathcal{N} = 2$ field theories and 3d $SL(2)$ Chern-Simons theories, called the 3d/3d correspondence [69, 70]. It may be related with a 3d compactification of the 6d $\mathcal{N} = (2, 0)$ theory.

Dualities can be checked by using exactly computed partition functions. S^3 partition function is computed in [71, 72, 73]. The relation between $S^3 \times S^1$ and S^3 partition functions is discussed in [74, 75, 76]. Partition functions for several kinds of deformations of S^3 are computed in [28, 31, 77, 78, 79, 80, 81]. $S^2 \times S^1$ partition function is computed in [82]. In particular the free energy of the ABJM theory $F = -\log Z_{S^3}$ behaves as $\mathcal{O}\left(N^{3/2}\right)$ [83], that is consistent with the behavior of a stack of N M2-branes through the AdS/CFT correspondence [84].

We would like to perform a similar analysis in 3d $\mathcal{N} = 2$ theory [39, 40] as in the previous chapter. Let us begin with introducing the properties of 3d spinors.

3.1 3d spinor

The Lorentz symmetry in 3d Euclidean space is $SO(3) \sim SU(2)$. Thus a spinor is doublet, written as ξ_α. Spinor indices are raised and lowered by the antisymmetric tensor $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$. In this thesis, we take $\epsilon^{12} = \epsilon_{12} = +1$.

In the Minkowski signature, 3d spinors are real. Due to $\mathcal{N} = 2$ supersymmetry, a supersymmetry transformation is parametrized by a complex spinor ξ, which consists of two real spinors. $\overline{\xi}$ is the complex conjugation of ξ. On the other hand, spinors are complex in the Euclidean signature. Thus we treat ξ and $\overline{\xi}$ as the independent spinors. The Hermitian conjugate of a spinor is related with the complex conjugate of it as

$$\left(\xi^{\dagger}\right)^{\alpha} = \left(\xi_{\alpha}\right)^{*}. \tag{3.1}$$

The gamma matrices are just the Pauli matrices:

$$\left(\gamma^{\widehat{\mu}}\right)_{\alpha}^{\ \beta} = \sigma^{\mu}. \tag{3.2}$$

They satisfy

$$\gamma^{\widehat{\mu}}\gamma^{\widehat{\nu}} = \delta^{\widehat{\mu}\widehat{\nu}} + i\epsilon^{\widehat{\mu}\widehat{\nu}\widehat{\rho}}\gamma^{\widehat{\rho}}, \tag{3.3}$$

where the antisymmetric tensor is normalized by $\epsilon^{\widehat{1}\widehat{2}\widehat{3}} = +1$.

3.2 3d $\mathcal{N} = 2$ new minimal supergravity

3d $\mathcal{N} = 2$ supersymmetry algebra consists of the following symmetries:

- The translational symmetry,
- The Lorentz symmetry $SO(3) \sim SU(2)$,
- The supersymmetry, and
- $U(1)_R$ R-symmetry.

The 3d $\mathcal{N} = 2$ supersymmetry algebra is written as

$$\left\{Q_{\alpha}, \overline{Q}_{\beta}\right\} = 2\left(\gamma^{\mu}\right)_{\alpha\beta} P_{\mu} + 2i\epsilon_{\alpha\beta} Z, \tag{3.4}$$

where Z is a central charge. The supercharges Q_{α} and $\overline{Q}_{\alpha}$ have the R-charge -1 and $+1$, respectively. The corresponding supersymmetric transformation parameters ξ and $\overline{\xi}$ have the R-charge $+1$ and -1, respectively.

Similar to the 4d case, by the Noether procedure, we construct the $\mathcal{R}$-multiplet which includes the energy-momentum tensor, the supersymmetry currents and the $U(1)_R$ current. Then we linearly couple the $\mathcal{R}$-multiplet and the Weyl multiplet and we can obtain the linearized supergravity. The general nonlinear description is formulated in [85][1].

The components of the $\mathcal{R}$-multiplet are

$$j_{\mu}^{(R)}, \quad S_{\mu\alpha}, \quad \overline{S}_{\mu\alpha}, \quad T_{\mu\nu}, \quad j_{\mu}^{(Z)}, \quad J^{(Z)}. \tag{3.5}$$

[1] At the time of [39, 40], there was no fully nonlinear component formulation of the 3d new minimal supergravity. In [39, 40] the discussions rely on the inputs from the linearized supergravity and the dimensional reduction of the 4d $\mathcal{N} = 1$ new minimal supergravity.

First four of them are the same notation as the 4d case. $j_\mu^{(Z)}$ is the current associated with the central charge. $J^{(Z)}$ is from the two-form conserved current $i\epsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}$. The supersymmetry transformation of the $\mathcal{R}$-multiplet is

$$\delta_Q j_\mu^{(R)} = -i\xi S_\mu + i\overline{\xi}\overline{S}_\mu, \tag{3.6}$$

$$\delta_Q S_{\mu\alpha} = \overline{\xi}_\alpha \left(2j_\mu^{(Z)} - i\epsilon_{\mu\nu\rho}\partial^\nu j^{(R)\rho}\right) + \left(\gamma^\nu\overline{\xi}\right)_\alpha \left(-2iT_{\mu\nu} + \partial_\nu j_\mu^{(R)} - \epsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}\right), \tag{3.7}$$

$$\delta_Q \overline{S}_{\mu\alpha} = \xi_\alpha \left(2j_\mu^{(Z)} + i\epsilon_{\mu\nu\rho}\partial^\nu j^{(R)\rho}\right) + \left(\gamma^\nu\xi\right)_\alpha \left(2iT_{\mu\nu} + \partial_\nu j_\mu^{(R)} - \epsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}\right), \tag{3.8}$$

$$\delta_Q T_{\mu\nu} = \frac{i}{4}\epsilon_{\mu\rho\lambda}\xi\gamma^\rho\partial^\lambda S_\nu - \frac{i}{4}\epsilon_{\mu\rho\lambda}\overline{\xi}\gamma^\rho\partial^\lambda\overline{S}_\nu + (\mu\leftrightarrow\nu), \tag{3.9}$$

$$\delta_Q j_\mu^{(Z)} = -\frac{i}{2}\xi\gamma^\nu\partial_\nu S_\mu + \frac{i}{2}\overline{\xi}\gamma^\nu\partial_\nu\overline{S}_\mu - \frac{1}{2}\epsilon_{\mu\nu\rho}\xi\partial^\nu S^\rho - \frac{1}{2}\epsilon_{\mu\nu\rho}\overline{\xi}\partial^\nu\overline{S}^\rho, \tag{3.10}$$

$$\delta_Q J^{(Z)} = -\frac{1}{2}\xi\gamma^\mu S_\mu - \frac{1}{2}\overline{\xi}\gamma^\mu\overline{S}_\mu. \tag{3.11}$$

The corresponding Weyl multiplet consists of the fields

$$A_\mu, \quad \psi_{\mu\alpha}, \quad \overline{\psi}_{\mu\alpha}, \quad \Delta e_\mu^{\ \widehat{\nu}}, \quad C_\mu, \quad B_{\mu\nu}. \tag{3.12}$$

The supersymmetry transformations for the gravitinos ψ_μ and $\overline{\psi}_\mu$ are

$$\delta_Q\psi_\mu = 2\left(\nabla_\mu - iA_\mu\right)\xi + H\gamma_\mu\xi + 2iV_\mu\xi + \epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\xi, \tag{3.13}$$

$$\delta_Q\overline{\psi}_\mu = 2\left(\nabla_\mu + iA_\mu\right)\overline{\xi} + H\gamma_\mu\overline{\xi} - 2iV_\mu\overline{\xi} - \epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\overline{\xi}, \tag{3.14}$$

where

$$V^\mu = -i\epsilon^{\mu\nu\rho}\partial_\nu C_\rho, \quad \partial_\mu V^\mu = 0, \tag{3.15}$$

$$H = \frac{i}{2}\epsilon^{\mu\nu\rho}\partial_\mu B_{\nu\rho}. \tag{3.16}$$

3.3 3d $\mathcal{N}=2$ supersymmetric backgrounds

We would like to obtain conditions for the bosonic background fields in the Weyl multiplet by imposing the existence of at least one solution ξ for $\delta_Q\psi_\mu = 0$. Because $\delta_Q\psi_\mu = 0$ is the homogeneous first order differential equation, the nontrivial solution ξ is nowhere vanishing.

Similar to 4d, by using ξ, we can define the bilinears

$$|\xi|^2 = \xi^\dagger\xi, \tag{3.17}$$

$$\eta_\mu = \frac{1}{|\xi|^2}\xi^\dagger\gamma_\mu\xi, \tag{3.18}$$

$$P_\mu = \xi\gamma_\mu\xi. \tag{3.19}$$

First two of them are real, nowhere vanishing and have R-charge zero. By the Fierz identity, η_μ satisfies

$$\eta^\mu \eta_\mu = 1. \tag{3.20}$$

Let us define a tensor $\Phi^\mu{}_\nu$ by using η_μ:

$$\Phi^\mu{}_\nu = \epsilon^\mu{}_{\nu\rho}\eta^\rho. \tag{3.21}$$

It satisfies

$$\begin{aligned}\Phi^\mu{}_\rho \Phi^\rho{}_\nu &= \epsilon^\mu{}_{\rho\sigma}\epsilon^\rho{}_{\nu\lambda}\eta^\sigma\eta^\lambda \\ &= -\delta^\mu{}_\nu + \eta^\mu\eta_\nu.\end{aligned} \tag{3.22}$$

This implies that the triple $\left(\eta_\mu, g_{\mu\nu}, \Phi^\mu{}_\nu\right)$ defines a mathematical structure on the 3d manifold $\mathcal{M}_3$, which is called an almost contact metric structure. For the definition of the almost contact metric structure, see Appendix C.3.1. The tensor $\Phi^\mu{}_\nu$ can be interpreted as an almost complex structure on the 2d space orthogonal to η^μ. When $\Phi^\mu{}_\nu X^\nu = iX^\mu$ or $\Omega_\mu \Phi^\mu{}_\nu = i\Omega_\nu$ for a vector X^μ or a one-form Ω_μ, we refer to it as holomorphic. Note that arbitrary (anti-)holomorphic vectors and one-forms are orthogonal to η_μ. P_μ is a complex anti-holomorphic one-form satisfying

$$P_\mu \Phi^\mu{}_\nu = -iP_\nu, \tag{3.23}$$

which is obtained by using the Fierz identity.

Now let us obtain conditions for the bosonic background fields in the Weyl multiplet, by imposing the existence of a solution ξ for $\delta_Q\psi_\mu = 0$. The equation $\delta_Q\psi_\mu = 0$ is written by

$$\left(\nabla_\mu - iA_\mu\right)\xi = -\frac{1}{2}H\gamma_\mu\xi - iV_\mu\xi - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\xi. \tag{3.24}$$

The background fields A_μ, V_μ and H are not completely fixed because the equation (3.24) is invariant under the following shift in terms of the complex scalar κ and the vector U^μ as

$$\begin{aligned}V^\mu &\to V^\mu + U^\mu + \kappa\eta^\mu, \\ H &\to H + i\kappa, \\ A_\mu &\to A_\mu + \frac{3}{2}\left(U_\mu + \kappa\eta_\mu\right),\end{aligned} \tag{3.25}$$

where κ and U^μ satisfy

$$\Phi^\mu{}_\nu U^\nu = iU^\mu, \quad \nabla_\mu\left(U^\mu + \kappa\eta^\mu\right) = 0. \tag{3.26}$$

First, we can compute the differential of η_μ by using (3.24) as

$$\nabla_\mu \eta_\nu = \frac{1}{2}(H + H^*)(\eta_\mu \eta_\nu - g_{\mu\nu}) + \frac{i}{2}(H - H^*)\Phi_{\mu\nu} + \frac{i}{2} g_{\mu\nu}\eta_\rho (V^\rho - V^{*\rho}) \\ - \frac{i}{2}\eta_\mu (V_\nu - V^*_\nu) + \frac{1}{2}\Phi_{\mu\nu}\eta_\rho (V^\rho + V^{*\rho}) + \frac{1}{2}\eta_\mu \Phi_{\nu\rho}(V^\rho + V^{*\rho}). \quad (3.27)$$

We can find that the solution of (3.27) can be written as

$$V^\mu = \epsilon^{\mu\nu\rho}\nabla_\nu \eta_\rho, \quad H = -\frac{1}{2}\nabla_\mu \eta^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho}\eta_\mu \nabla_\nu \eta_\rho, \quad (3.28)$$

up to the shift (3.25). By using (3.28), the symmetric part of (3.27) is

$$\begin{aligned} \nabla_\mu \eta_\nu + \nabla_\nu \eta_\mu &= (H + H^*)(\eta_\mu \eta_\nu - g_{\mu\nu}) + i g_{\mu\nu}\eta_\rho (V^\rho - V^{*\rho}) \\ &\quad - i\eta_{\{\mu}\left(V_{\nu\}} - V^*_{\nu\}}\right) + \eta_{\{\mu}\Phi_{\nu\}\rho}(V^\rho + V^{*\rho}) \\ &= (g_{\mu\nu} - \eta_\mu \eta_\nu)\nabla_\rho \eta^\rho + 2\eta_{\{\mu}\eta^\rho \nabla_\rho \eta_{\nu\}}. \end{aligned} \quad (3.29)$$

By using this, we can show an integrability condition

$$\Phi^\mu{}_\nu \mathcal{L}_\eta \Phi^\nu{}_\rho = 0, \quad (3.30)$$

where the Lie derivative along η^μ is given by

$$\mathcal{L}_\eta \Phi^\mu{}_\nu = \eta^\rho \nabla_\rho \Phi^\mu{}_\nu - \nabla_\rho \eta^\mu \Phi^\rho{}_\nu + \nabla_\nu \eta^\rho \Phi^\mu{}_\rho. \quad (3.31)$$

An almost contact metric structure satisfying (3.30) defines a mathematical structure called a transversely holomorphic foliation [40, 86]. The analysis for (3.30) is shown in Appendix C.3.2. From this, there exist the local coordinates $(\tau, z, \overline{z})$ with real τ and complex z on $\mathcal{M}_3$ satisfying the following properties:

- On local coordinates $(\tau, z, \overline{z})$,
  - The vector η^μ is written as
  $$\eta^\mu \partial_\mu = \partial_\tau. \quad (3.32)$$
  - A holomorphic one-form Ω_μ is written as
  $$\Omega_\mu dx^\mu = \omega(\tau, z, \overline{z})\, dz. \quad (3.33)$$
  - The metric is given with a complex function $h(\tau, z, \overline{z})$ and a real function $c(\tau, z, \overline{z})$ as
  $$ds^2 = \left(d\tau + h(\tau, z, \overline{z})\, dz + \overline{h}(\tau, z, \overline{z})\, d\overline{z}\right)^2 + c(\tau, z, \overline{z})^2\, dz d\overline{z}. \quad (3.34)$$

- For overlapping coordinates $(\tau, z, \overline{z})$ and $(\tau', z', \overline{z}')$, they are related by

$$\tau' = \tau + t(z, \overline{z}), \quad z' = f(z), \tag{3.35}$$

where $t(z, \overline{z})$ and $f(z)$ are real and holomorphic, respectively.

In the metric (3.34), η_μ and $\Phi^\mu{}_\nu$ are given by

$$\eta = d\tau + h dz + \overline{h} d\overline{z}, \quad \Phi^\mu{}_\nu = \begin{pmatrix} 0 & -ih & i\overline{h} \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{3.36}$$

Finally, let us obtain the expression for A_μ. For this, it is convenient to introduce a connection $\widehat{\nabla}_\mu$ satisfying $\widehat{\nabla}_\mu g_{\nu\rho} = 0$ and $\widehat{\nabla}_\mu \eta_\nu = 0$. This can be realized by replacing the usual spin connection $\omega_{\mu\nu\rho}$ with

$$\widehat{\omega}_{\mu\nu\rho} = \omega_{\mu\nu\rho} + \eta_\rho \nabla_\mu \eta_\nu - \eta_\nu \nabla_\mu \eta_\rho + 2W_\mu \Phi_{\nu\rho}, \quad W_\mu = -\frac{1}{4}\eta_\mu \epsilon^{\nu\rho\lambda}\eta_\nu \nabla_\rho \eta_\lambda. \tag{3.37}$$

By using the connection $\widehat{\nabla}_\mu$, the condition (3.24) is written as

$$\left(\widehat{\nabla}_\mu - i\widehat{A}_\mu\right)\xi = 0, \tag{3.38}$$

where

$$\widehat{A}_\mu = A_\mu - \frac{1}{2}\left(2\delta_\mu{}^\nu - i\Phi_\mu{}^\nu\right)V_\nu + \frac{i}{2}\eta_\mu H - W_\mu. \tag{3.39}$$

To determine $\widehat{A}_\mu$ or A_μ, let us consider the remaining bilinear P_μ. As shown above, this is anti-holomorphic and its non-zero component is only $p = P_{\overline{z}}$ in the coordinates $(\tau, z, \overline{z})$. From (3.38), p satisfies

$$\left(\widehat{\nabla}_\mu - 2i\widehat{A}_\mu\right)p = 0, \tag{3.40}$$

and thus we obtain

$$\widehat{A}_\mu = -\frac{i}{2}\widehat{\nabla}_\mu \log p. \tag{3.41}$$

Similar to the 4d case, let us define a scalar

$$s = \frac{1}{\sqrt{2}} p g^{-1/4}. \tag{3.42}$$

By using the metric (3.34), $\widehat{\nabla}_\mu s$ is given by

$$\widehat{\nabla}_\tau s = \partial_\tau s, \tag{3.43}$$

$$\widehat{\nabla}_z s = \partial_z s + \frac{s}{4}\left(\partial_z - \eta_z \partial_\tau\right)\log g, \tag{3.44}$$

$$\widehat{\nabla}_{\overline{z}} s = \partial_{\overline{z}} s - \frac{s}{4}\left(\partial_{\overline{z}} - \eta_{\overline{z}} \partial_\tau\right)\log g. \tag{3.45}$$

By (3.41)-(3.45), we therefore find that $\widehat{A}_\mu$ is given by

$$\widehat{A}_\mu = \frac{1}{8}\Phi_\mu{}^\nu \partial_\nu \log g - \frac{i}{2}\partial_\mu \log s. \tag{3.46}$$

Conversely, we can solve (3.24) in the situation such that the metric and background fields take the forms as obtained above. Let us take the vielbein as

$$e^{\hat{1}} = \eta, \quad e^{\hat{2}} - ie^{\hat{3}} = c\,(\tau, z, \bar{z})\, dz, \quad e^{\hat{2}} + ie^{\hat{3}} = c\,(\tau, z, \bar{z})\, d\bar{z}. \tag{3.47}$$

In this frame, we can find that

$$\xi_\alpha = \sqrt{s\,(\tau, z, \bar{z})}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.48}$$

is a solution.

By imposing the condition that two or more supercharges are preserved, we can obtain more restrictive conditions. We list the results below:

- The presence of one supercharge ξ implies that $\mathcal{M}_3$ admits a transversely holomorphic foliation with a transversely Hermitian metric.

- In the presence of two supercharges ξ and $\bar{\xi}$ with opposite $U(1)_R$ charge, we can define a vector field $K^\mu = \xi\gamma^\mu\bar{\xi}$. We can find that it is a Killing vector. If K^μ is real, $\mathcal{M}_3$ is an S^1 fibration over a Riemann surface, called a Seifert manifold. If K^μ is complex, it gives two independent isometries and yields a more restrictive result.

- The presence of four supercharges implies that $\mathcal{M}_3$ is locally isometric to either of them:

  - S^3, T^3, H^3,
  - $\mathbb{R} \times S^2$, $\mathbb{R} \times T^2$, $\mathbb{R} \times H^2$, and
  - A certain fibration over S^2, T^2, H^2.

3.4 Background vector multiplet

As in the 4d case, we can turn on background vector multiplets. Let us consider a condition in which at least one supercharge preserves in the presence of background $U(1)$ vector multiplets. A real linear multiplet $\mathcal{J}$ in 3d consists of the flavor symmetry current j_μ, the fermions j_α and $\bar{j}_\alpha$ and two scalars J and K.

Their supersymmetry transformations are written as

$$\delta_Q J = i\xi j - i\overline{\xi}\overline{j}, \tag{3.49}$$

$$\delta j_\alpha = i\left(\gamma^\mu\overline{\xi}\right)_\alpha (j_\mu + i\partial_\mu J) + \overline{\xi}_\alpha K, \tag{3.50}$$

$$\delta_Q \overline{j}_\alpha = -i\left(\gamma^\mu\xi\right)_\alpha (j_\mu - i\partial_\mu J) + \xi_\alpha K, \tag{3.51}$$

$$\delta_Q j_\mu = i\epsilon_{\mu\nu\rho}\xi\gamma^\rho\partial^\nu j - i\epsilon_{\mu\nu\rho}\overline{\xi}\gamma^\rho\partial^\nu\overline{j}, \tag{3.52}$$

$$\delta_Q K = -i\xi\gamma^\mu\partial_\mu j + i\overline{\xi}\gamma^\mu\partial_\mu\overline{j}. \tag{3.53}$$

The coupled vector multiplet consists of the gauge field A_μ, the gauginos λ_α and $\overline{\lambda}_\alpha$ and two scalars D and σ. The supersymmetric transformations of λ and $\overline{\lambda}$ are given by

$$\delta_Q\lambda = i\xi(D + \sigma H) - \frac{i}{2}\gamma_\mu\xi\epsilon^{\mu\nu\rho}F_{\nu\rho}(A) - i\gamma^\mu\xi\left(\partial_\mu\sigma + iV_\mu\sigma\right), \tag{3.54}$$

$$\delta_Q\overline{\lambda} = -i\overline{\xi}(D + \sigma H) + \frac{i}{2}\gamma_\mu\overline{\xi}\epsilon^{\mu\nu\rho}F_{\nu\rho}(A) + i\gamma^\mu\overline{\xi}\left(\partial_\mu\sigma - iV_\mu\sigma\right). \tag{3.55}$$

We would like to obtain constraints for the bosonic background fields by imposing the existence of a solution ξ for $\delta_Q\lambda = 0$. By multiplying ξ or $\xi^\dagger$ to $\delta_Q\lambda = 0$,

$$0 = \xi\delta_Q\lambda = -\frac{i}{2}\epsilon^{\mu\nu\rho}P_\mu F_{\nu\rho}(A) - iP^\mu\left(\partial_\mu\sigma + iV_\mu\sigma\right), \tag{3.56}$$

$$0 = |\xi|^{-2}\xi^\dagger\delta_Q\lambda = i(D + \sigma H) - \frac{i}{2}\Phi^{\mu\nu}F_{\mu\nu}(A) - i\eta^\mu\left(\partial_\mu\sigma + iV_\mu\sigma\right). \tag{3.57}$$

By using the coordinates $(\tau, z, \overline{z})$ and substituting the solution of V_μ and H, we obtain the conditions

$$F_{\tau z}(\mathcal{A}) = 0, \quad D = \frac{1}{2}\Phi^{\mu\nu}F_{\mu\nu}(\mathcal{A}) + \eta^\mu\partial_\mu\sigma + \sigma\left(\frac{1}{2}\nabla_\mu\eta^\mu - \frac{i}{2}\epsilon^{\mu\nu\rho}\eta_\mu\partial_\nu\eta_\rho\right), \tag{3.58}$$

where $\mathcal{A}_\mu$ is the shifted gauge field

$$\mathcal{A}_\mu = A_\mu + i\sigma\eta_\mu. \tag{3.59}$$

Note that the shift of the background fields in the Weyl multiplet (3.25) does not affect the solution of the background vector multiplets.

3.5 Q-exact deformations

3.5.1 Deformation theory

$\widetilde{\partial}$-cohomology

As in the 4d case, some cohomology plays an important role in the discussion of the parameter dependence of the partition function. For this, let us firstly define a complex projection operator

$$\Pi^{\mu}{}_{\nu} = \frac{1}{2}\left(\delta^{\mu}{}_{\nu} + i\Phi^{\mu}{}_{\nu} - \zeta^{\mu}\eta_{\nu}\right), \quad \Pi^{\mu}{}_{\nu}\Pi^{\nu}{}_{\rho} = \Pi^{\mu}{}_{\rho}. \tag{3.60}$$

This projection operator leaves an anti-holomorphic parts of vector fields and one-forms, with respect to $\Phi^{\mu}{}_{\nu}$. We refer to a vector field X^{μ} or a one-form $\omega^{0,1}_{\mu}$ as anti-holomorphic with respect to $\Pi^{\mu}{}_{\nu}$ if

$$\Pi^{\mu}{}_{\nu}X^{\nu} = X^{\mu}, \quad \omega^{0,1}_{\mu}\Pi^{\mu}{}_{\nu} = \omega^{0,1}_{\nu}. \tag{3.61}$$

In this section, we use "(anti-)holomorphic" in the meaning of "(anti-)holomorphic with respect to $\Pi^{\mu}{}_{\nu}$." In the coordinates $(\tau, z, \overline{z})$, they can be written as

$$X = X^{\overline{z}}\left(\partial_{\overline{z}} - \overline{h}\partial_{\tau}\right), \quad \omega^{0,1} = \omega^{0,1}_{\overline{z}}d\overline{z}. \tag{3.62}$$

The remaining part of one-forms is called holomorphic. A holomorphic one-form $\omega^{1,0}$ satisfies $\omega^{1,0}_{\mu}\Pi^{\mu}{}_{\nu} = 0$. In the coordinates $(\tau, z, \overline{z})$,

$$\omega^{1,0} = \omega^{1,0}_{\tau}\left(d\tau + \overline{h}d\overline{z}\right) + \omega^{1,0}_{z}dz. \tag{3.63}$$

The $(1,0)$-forms $\omega^{1,0}$ span a 2d subspace of the cotangent bundle on $\mathcal{M}_3$, while the $(0,1)$-forms $\omega^{0,1}$ span a 1d subspace of it. Note that $(1,0)$-forms and $(0,1)$-forms are not related with the simple complex conjugation. We can split differential forms as

$$\begin{aligned}
&\text{1-forms} &\rightarrow\quad &(1,0)\text{-forms and } (0,1)\text{-forms},\\
&\text{2-forms} &\rightarrow\quad &(1,1)\text{-forms and } (2,0)\text{-forms},\\
&\text{3-forms} &\rightarrow\quad &(2,1)\text{-forms}.
\end{aligned}$$

Each differential form is written as

$$\omega^{1,1} = \omega^{1,1}_{\tau\overline{z}}d\tau\wedge d\overline{z} + \omega^{1,1}_{z\overline{z}}dz\wedge d\overline{z}, \tag{3.64}$$

$$\omega^{2,0} = \omega^{2,0}_{\tau z}(d\tau + \overline{h}d\overline{z})\wedge dz, \tag{3.65}$$

$$\omega^{2,1} = \omega^{2,1}_{\tau z\overline{z}}d\tau\wedge dz\wedge d\overline{z}. \tag{3.66}$$

As differential forms on complex manifolds, we can define an operator $\widetilde{\partial}$ as

$$\widetilde{\partial}: \Lambda^{p,q} \rightarrow \Lambda^{p+1,q}, \quad \widetilde{\partial}\omega^{p,q} = d\omega^{p,q}|_{\Lambda^{p+1,q}}, \tag{3.67}$$

where $\Lambda^{p,q}$ is a set which has (p,q)-forms $\omega^{p,q}$ as elements. The operator $\widetilde{\partial}$ is the 3d analogue of the Doubeault operator ∂ in even dimensions. For (p,q)-forms $\omega^{p,q}$ (3.62)-(3.66), the operator $\widetilde{\partial}$ acts as

$$\widetilde{\partial}\omega^{0,0} = \partial_{\tau}\omega^{0,0}(d\tau + \overline{h}d\overline{z}) + \partial_{z}\omega^{0,0}dz, \tag{3.68}$$

$$\widetilde{\partial}\omega^{1,0} = \left(\partial_{\tau}\omega^{1,0}_{z} - \partial_{z}\omega^{0,1}_{\tau}\right)\left(d\tau + \overline{h}d\overline{z}\right)\wedge dz, \tag{3.69}$$

$$\widetilde{\partial}\omega^{0,1} = \partial_{\tau}\omega^{0,1}_{\overline{z}}d\tau\wedge d\overline{z} + \partial_{z}\omega^{0,1}_{\overline{z}}dz\wedge d\overline{z}, \tag{3.70}$$

$$\widetilde{\partial}\omega^{1,1} = \left(\partial_{\tau}\omega^{1,1}_{z\overline{z}} - \partial_{z}\omega^{1,1}_{\tau\overline{z}}\right)d\tau\wedge dz\wedge d\overline{z}, \tag{3.71}$$

$$\widetilde{\partial}\omega^{2,0} = \widetilde{\partial}\omega^{2,1} = 0. \tag{3.72}$$

Because $d^2=0$, $\widetilde{\partial}$ satisfies $\widetilde{\partial}^2=0$. Therefore we can define its cohomology as

$$H^{p,q}\left(\mathcal{M}_3\right)=\frac{\left\{\omega^{p,q}\in\Lambda^{p,q}\,\middle|\,\widetilde{\partial}\omega^{p,q}=0\right\}}{\widetilde{\partial}\Lambda^{p-1,q}}. \tag{3.73}$$

As in the 4d case, it is finite dimensional if $\mathcal{M}_3$ is compact.

Deformations of transversely holomorphic foliation

Let us consider infinitesimal deformations of the almost contact structure (η,ζ,Φ), satisfying

$$\Phi^{\mu}{}_{\nu}\Phi^{\nu}{}_{\rho}=-\delta^{\mu}{}_{\rho}+\zeta^{\mu}\eta_{\rho},\quad \eta_{\mu}\zeta^{\mu}=1, \tag{3.74}$$

by $\Delta\eta_\mu$, $\Delta\zeta^\mu$ and $\Delta\Phi^{\mu}{}_{\nu}$. By using the fact that $(\eta+\Delta\eta,\zeta+\Delta\zeta,\Phi+\Delta\Phi)$ is also an almost contact structure, at the first order, we obtain

$$\Delta\eta_\tau=-\eta_\mu\Delta\zeta^\mu, \tag{3.75}$$

$$\Delta\Phi^{z}{}_{\tau}=-i\Delta\zeta^{z}, \tag{3.76}$$

$$\Delta\Phi^{z}{}_{z}=-ih\Delta\zeta^{z}, \tag{3.77}$$

$$\Delta\Phi^{\tau}{}_{z}=-i\Delta\eta_z-ih\Delta\zeta^\tau-ih^2\Delta\zeta^z+h\Delta\Phi^{\tau}{}_{\tau}-\overline{h}\Delta\Phi^{\overline{z}}{}_{z}, \tag{3.78}$$

where we used the coordinates $(\tau,z,\overline{z})$, on which $\Phi^{\mu}{}_{\nu}$ is represented as in (3.36). The expressions for $\Delta\Phi^{\overline{z}}{}_{\tau}$, $\Delta\Phi^{\overline{z}}{}_{\overline{z}}$ and $\Delta\Phi^{\tau}{}_{\overline{z}}$ are obtained from (3.76)-(3.78) by the complex conjugation. Furthermore, by requiring that the deformed almost contact structure satisfies the integrability condition (3.30), we obtain

$$\left(\Delta\Phi^{\tau}{}_{\tau}-ih\Delta\zeta^z+i\overline{z}\Delta\zeta^{\overline{z}}\right)\partial_\tau h=0, \tag{3.79}$$

$$\partial_\tau\left(\Delta\Phi^{z}{}_{\overline{z}}-i\overline{h}\Delta\zeta^z\right)+2i\partial_{\overline{z}}\Delta\zeta^z=0. \tag{3.80}$$

(3.79) determines $\Delta\Phi^{\tau}{}_{\tau}$. To understand (3.80), let us introduce a $(1,0)$-form $\Theta^{\overline{z}}$ with coefficients in the anti-holomorphic tangent bundle $T^{0,1}\mathcal{M}_3$ as

$$\Theta^{\overline{z}}=2i\Delta\zeta^{\overline{z}}\left(d\tau+\overline{h}d\overline{z}\right)+\left(\Delta\Phi^{\overline{z}}{}_{z}+ih\Delta\zeta^{\overline{z}}\right)dz. \tag{3.81}$$

The condition (3.80) can be simply written in terms of $\Theta^{\overline{z}}$ as

$$\widetilde{\partial}\Theta^{\overline{z}}=0. \tag{3.82}$$

Hence $\Theta^{\overline{z}}$ is the 3d analogue of (2.71) satisfying (2.72). There is a trivial deformation

$$\Theta^{\overline{z}}=2i\widetilde{\partial}\epsilon^{\overline{z}}. \tag{3.83}$$

Therefore, nontrivial deformations of the transversely holomorphic foliation are parametrized by the $\widetilde{\partial}$-cohomology with coefficients in $T^{0,1}\mathcal{M}_3$ as

$$\left[\Theta^{\overline{z}}\right]\in H^{1,0}\left(\mathcal{M}_3,T^{0,1}\mathcal{M}_3\right). \tag{3.84}$$

Deformations of metric

There is a compatible metric $g_{\mu\nu}$ with the almost contact structure (η, ζ, Φ), satisfying

$$g_{\mu\nu}\Phi^{\mu}{}_{\rho}\Phi^{\nu}{}_{\sigma} = g_{\rho\sigma} - \eta_\rho\eta_\sigma. \tag{3.85}$$

η_μ and ζ^μ is related by $\eta_\mu = g_{\mu\nu}\zeta^\nu$. By imposing the condition that the deformation of the almost contact metric structure keeps (3.85), we obtain the following:

$$g_{\mu\nu}\Delta\Phi^{\mu}{}_{\rho}\Phi^{\nu}{}_{\sigma} + g_{\mu\nu}\Phi^{\mu}{}_{\rho}\Phi^{\nu}{}_{\sigma} + \Delta g_{\mu\nu}\Phi^{\mu}{}_{\rho}\Phi^{\nu}{}_{\sigma} = \Delta g_{\rho\sigma} - \Delta\eta_\rho\eta_\sigma - \eta_\rho\Delta\eta_\sigma. \tag{3.86}$$

In the undeformed coordinates $(\tau, z, \bar{z})$, this constrains $\Delta g_{\mu\nu}$, except for $\Delta g_{z\bar{z}}$, as

$$\Delta g_{\tau\tau} = -2\eta_\mu\Delta\zeta^\mu, \tag{3.87}$$

$$\Delta g_{\tau z} = \Delta\eta_z - h\eta_\mu\Delta\zeta^\mu - \frac{c^2}{2}\Delta\zeta^{\bar{z}}, \tag{3.88}$$

$$\Delta g_{zz} = \frac{ic^2}{2}\Delta\Phi^{\bar{z}}{}_{z} - \frac{hc^2}{2}\Delta\zeta^{\bar{z}} + 2h\Delta\eta_z, \tag{3.89}$$

while $\Delta g_{\tau\bar{z}}$ and $\Delta g_{\bar{z}\bar{z}}$ are obtained by the complex conjugation.

Abelian gauge fields

In Section 3.4, we have obtained the constraint for Abelian background vector multiplets. A background gauge field in the vector multiplet is restricted by $F_{\tau z}(\mathcal{A}) = 0$ with $\mathcal{A}_\mu = A_\mu + i\sigma\eta_\mu$. This means that the holomorphic part $\mathcal{A}^{1,0}_\mu$ satisfies $\widetilde{\partial}\mathcal{A}^{1,0} = 0$. Therefore, we can characterize a deformation of $\mathcal{A}^{1,0}$ by $\Delta\mathcal{A}^{1,0}$ in terms of the $\widetilde{\partial}$-cohomology as

$$\left[\Delta\mathcal{A}^{1,0}\right] \in H^{1,0}\left(\mathcal{M}_3\right). \tag{3.90}$$

3.5.2 Parameter dependence

We would like to consider a deformation of the background Weyl multiplet, leaving one supercharge Q preserved. By the infinitesimal deformation, the Lagrangian is shifted by a linear combination between the shift of the Weyl multiplet and the $\mathcal{R}$-multiplet. Let us recall properties of the $\mathcal{R}$-multiplet in the presence of the supercharge Q, which generates a supersymmetry transformation parameterized by ξ.

The $\mathcal{R}$-multiplet is introduced in Section 3.2. Its supersymmetry transformations parameterized by ξ are obtained by setting $\overline{\xi}=0$ in (3.6)-(3.11) as

$$\delta_Q j^{(R)}_\mu = -i\xi S_\mu, \tag{3.91}$$
$$\delta_Q S_{\mu\alpha} = 0, \tag{3.92}$$
$$\delta_Q \overline{S}_{\mu\alpha} = \xi_\alpha\left(2j^{(Z)}_\mu + i\epsilon_{\mu\nu\rho}\partial^\nu j^{(R)\rho}\right) + (\gamma^\nu\xi)_\alpha\left(2iT_{\mu\nu} + \partial_\nu j^{(R)}_\mu - \epsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}\right), \tag{3.93}$$
$$\delta_Q T_{\mu\nu} = \frac{i}{4}\epsilon_{\mu\rho\lambda}\xi\gamma^\rho\partial^\lambda S_\nu + \frac{i}{4}\epsilon_{\nu\rho\lambda}\xi\gamma^\rho\partial^\lambda S_\mu, \tag{3.94}$$
$$\delta_Q j^{(Z)}_\mu = -\frac{i}{2}\xi\gamma^\nu\partial_\nu S_\mu - \frac{1}{2}\epsilon_{\mu\nu\rho}\xi\partial^\nu S^\rho, \tag{3.95}$$
$$\delta_Q J^{(Z)} = -\frac{1}{2}\xi\gamma^\mu S_\mu. \tag{3.96}$$

We can find the six bosonic Q-exact operators $\delta_Q\overline{S}_{\mu\alpha}$, which are supersymmetry transformations of the fermionic operators. By calculating $\xi\delta_Q\overline{S}_\mu$ and $\xi^\dagger\delta_Q\overline{S}_\mu$, we can find that they are linear combinations of

$$\mathcal{T}_{\tau\tau} = T_{\tau\tau} - ij^{(Z)}_\tau + 2i\partial_{\overline{z}} j^{(R)}_z, \tag{3.97}$$
$$\mathcal{T}_{\tau z} = T_{\tau z} - \frac{i}{2}\partial_z j^{(R)}_\tau - \frac{1}{2}\partial_z J^{(Z)}, \tag{3.98}$$
$$\mathcal{T}_{\tau\overline{z}} = T_{\tau\overline{z}} - ij^{(Z)}_{\overline{z}} - \frac{i}{2}\partial_{\overline{z}} j^{(R)}_\tau - \frac{1}{2}\partial_{\overline{z}} J^{(Z)}, \tag{3.99}$$
$$\mathcal{T}_{zz} = T_{zz} - \frac{i}{2}\partial_z j^{(R)}_z, \tag{3.100}$$
$$\mathcal{T}_{z\overline{z}} = T_{z\overline{z}} - \frac{i}{2}\partial_z j^{(R)}_{\overline{z}} + \frac{1}{4}\partial_\tau J^{(Z)}, \tag{3.101}$$
$$\mathcal{J}^{(Z)}_z = j^{(Z)}_z + \partial_\tau j^{(R)}_z - \partial_z j^{(R)}_\tau + i\partial_z J^{(Z)}. \tag{3.102}$$

At the first order, the Lagrangian for the bosonic components of the Weyl multiplet is written by the linear combination between the Weyl multiplet and the $\mathcal{R}$-multiplet as

$$\Delta\mathcal{L} = -\frac{1}{2}\Delta g^{\mu\nu}T_{\mu\nu} + A^{(R)\mu}j^{(R)}_\mu + C^\mu j^{(Z)}_\mu + HJ^{(Z)}, \tag{3.103}$$

where

$$A^{(R)}_\mu = A_\mu - \frac{3}{2}V_\mu. \tag{3.104}$$

We would like to consider the deformation of the transversely holomorphic foliation and the compatible metric, around the flat space. Other bosonic components of the background Weyl multiplet $A^{(R)\mu}$, C^μ and H are deteremined in Section

3.3 as

$$A^{(R)\tau} = -\frac{i}{2}\partial_{\bar{z}}\Delta\zeta^{\bar{z}} + i\partial_z\Delta\eta_{\bar{z}} - i\partial_{\bar{z}}\Delta\eta_z + \frac{i}{2}\partial_\tau\Delta g_{z\bar{z}}, \tag{3.105}$$

$$A^{(R)z} = \frac{1}{2}\partial_z\Delta\Phi^z{}_{\bar{z}} - i\partial_{\bar{z}}\Delta g_{z\bar{z}} - 2i\partial_{\bar{z}}\Delta\zeta^\tau - 2i\partial_\tau\Delta\eta_{\bar{z}}, \tag{3.106}$$

$$A^{(R)\bar{z}} = i\partial_z\Delta g_{z\bar{z}} + \frac{1}{2}\partial_{\bar{z}}\Delta\Phi^{\bar{z}}{}_z, \tag{3.107}$$

$$C^\tau = -i\Delta\zeta^\tau + \widetilde{C}^\tau, \quad C^z = 2i\Delta\eta_{\bar{z}} + \widetilde{C}^z, \quad C^{\bar{z}} = 2i\Delta\eta_z + \widetilde{C}^{\bar{z}}, \tag{3.108}$$

$$H = -i\partial_\tau\Delta g_{z\bar{z}} - \frac{1}{2}\partial_z\Delta\zeta^z - \frac{1}{2}\partial_{\bar{z}}\Delta\zeta^{\bar{z}} + \partial_z\Delta\eta_{\bar{z}} - \partial_{\bar{z}}\Delta\eta_z + i\kappa, \tag{3.109}$$

where $\widetilde{C}_\mu$ is given by

$$-i\epsilon^{\mu\nu\rho}\partial_\nu\widetilde{C}_\rho = U^\mu + \kappa\eta^\mu. \tag{3.110}$$

U^μ and κ are the ambiguities of the solution of supersymmetric backgrounds. They satisfies

$$\Phi^\mu{}_\nu U^\nu = iU^\mu, \tag{3.111}$$

$$\nabla_\mu\left(U^\mu + \kappa\eta^\mu\right) = 0. \tag{3.112}$$

Instead of considering U^μ and κ separately, let us introduce a two-form

$$W_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho}\left(U^\rho + \kappa\eta^\rho\right). \tag{3.113}$$

(3.112) means that $W_{\mu\nu}$ is closed: $dW = 0$. From (3.111), U^μ has only a z-component, and hence $W_{\tau z} = 0$. Thus W is a $(1,1)$-form. The holomorphic components of dW also vanishes:

$$\widetilde{\partial}W = 0. \tag{3.114}$$

It seems that the partition function depends on W through the $\widetilde{\partial}$-cohomology. This will be shown later.

By substituting (3.105)-(3.109) into (3.103), we obtain the variation of the Lagrangian

$$\begin{aligned}\Delta\mathcal{L} = &-4\Delta g_{z\bar{z}}\mathcal{T}_{z\bar{z}} - 2\Delta\eta_z\left(\mathcal{T}_{\tau\bar{z}} - i\mathcal{J}_{\bar{z}}\right) - 2\Delta\eta_{\bar{z}}\mathcal{T}_{\tau z} \\ &+ \Delta\zeta^\tau\mathcal{T}_{\tau\tau} + \Delta\zeta^{\bar{z}}\mathcal{T}_{\tau\bar{z}} - i\Delta\Phi^{\bar{z}}{}_z\mathcal{T}_{\bar{z}\bar{z}} + \widetilde{C}^\mu j^{(Z)}_\mu + i\kappa J^{(Z)} \\ &+ \Delta\zeta^z\left(T_{\tau z} + \frac{1}{2}\partial_z J^{(Z)}\right) + i\Delta\Phi^z{}_{\bar{z}}\left(T_{zz} + \frac{i}{2}\partial_z j^{(R)}_z\right).\end{aligned} \tag{3.115}$$

This Lagrangian is supersymmetric if the integrability conditions (3.79) and (3.80) hold with $h = 0$, because we are considering the deformation around the flat space. Since the several terms in (3.115) is written by the Q-exact operators, we obtain the (in)dependence of the partition function with respect to the transversely holomorphic foliation and the compatible metric:

- For a fixed transversely holomorphic foliation, deformations of the compatible metric do not affect the partition function.

- The partition function does not depend on $\Delta\zeta^\tau$, $\Delta\zeta^{\bar{z}}$ and $\Delta\Phi^{\bar{z}}{}_z$.

$\Delta\zeta^z$ and $\Delta\Phi^z{}_{\bar{z}}$ are not independent due to (3.80). The deformations by them are characterized by the $(0,1)$-form Θ^z with coefficients in the holomorphic tangent bundle $T^{1,0}\mathcal{M}_3$. Therefore, we conclude the $\Delta\zeta^z$ and $\Delta\Phi^z{}_{\bar{z}}$ dependence of the partition function as

- The partition function depends on the transversely holomorphic foliation only through the cohomology class of Θ^z in $H^{0,1}\left(\mathcal{M}_3, T^{1,0}\mathcal{M}_3\right)$.

In order to show that the partition function depends on $\widetilde{C}^\mu$ and κ through the $\widetilde{\partial}$-cohomology class of the $(1,1)$-form W, let us assume that $W=\widetilde{\partial}\varphi^{1,0}$ for a globally defined $(1,0)$-form $\varphi^{1,0}$. Substituting this, we obtain

$$\widetilde{C}^\mu j_\mu^{(Z)} + i\kappa J^{(Z)} = 4i\varphi_z^{1,0}\mathcal{J}_{\bar{z}}^{(Z)} + (\text{total derivative}). \tag{3.116}$$

Therefore, we find that

- The partition function depends on W only through its cohomology class in $H^{1,1}\left(\mathcal{M}_3\right)$.

Let us discuss the deformation of a background Abelian vector multiplet. The current multiplet associated with the vector multiplet is the real linear multiplet $\mathcal{J}$, introduced in Section 3.4. In the presence of the preserved supercharge Q, which generates a supersymmetry transformation parameterized by ξ, the supersymmetry transformation of the real linear multiplet is represented by

$$\delta_Q J = i\xi j, \tag{3.117}$$
$$\delta_Q j_\alpha = 0, \tag{3.118}$$
$$\delta_Q \bar{j}_\alpha = -i\left(\gamma^\mu\xi\right)_\alpha\left(j_\mu - i\partial_\mu J\right) + \xi_\alpha K, \tag{3.119}$$
$$\delta_Q j_\mu = i\epsilon_{\mu\nu\rho}\xi\gamma^\rho\partial^\nu j, \tag{3.120}$$
$$\delta_Q K = -i\xi\gamma^\mu\partial_\mu j. \tag{3.121}$$

We can find two bosonic Q-exact operators $\delta_Q\bar{j}_\alpha$. By calculating $\xi\delta_Q\bar{j}$ and $\xi^\dagger\delta_Q\bar{j}$, they are given by

$$\mathcal{J}_z = \frac{i}{|P^\mu|}\xi\delta_Q\bar{j} = j_z - i\partial_z J, \tag{3.122}$$
$$\mathcal{K} = \frac{1}{|\xi|^2}\xi^\dagger\delta_Q\bar{j} = K - ij_\tau - \partial_\tau J. \tag{3.123}$$

The bosonic linearized couplings between a vector multiplet and corresponding real linear multiplet is written as

$$\Delta\mathcal{L} = A^{\mu} j_{\mu} + \sigma K + DJ. \tag{3.124}$$

In the presence of a supercharge corresponding to ξ, the background field D is written as (3.58), which is rewritten as

$$D = \partial_{\tau}\sigma - 2i\left(\partial_{z} A_{\overline{z}} + \partial_{\overline{z}} A_{z}\right), \tag{3.125}$$

in the coordinates $(\tau, z, \overline{z})$. Substituting it to (3.124) and dropping total derivatives, we obtain

$$\Delta\mathcal{L} = \mathcal{A}_{\tau} j_{\tau} + 2\mathcal{A}_{\overline{z}}\mathcal{J}_{z} + 2\mathcal{A}_{z}\left(j_{\overline{z}} + i\partial_{\overline{z}} J\right) + \sigma\mathcal{K}. \tag{3.126}$$

Since $\mathcal{J}_z$ and $\mathcal{K}$ are Q-exact, $\mathcal{A}_{\overline{z}}$ and σ do not affect the partition function. Background vector multiplets affect the partition function only through $\mathcal{A}_{\tau}$ and $\mathcal{A}_z$, or holomorphic part of $\mathcal{A}$. Therefore, similar to the above,

- The partition function depends on background Abelian vector multiplets only through their $\widetilde{\partial}$-cohomology classes in $H^{1,0}\left(\mathcal{M}_3\right)$.

Chapter 4

5d supersymmetric field theories

Before the analysis for 5d supersymmetric backgrounds, we introduce interesting properties of 5d supersymmetric field theories in this chapter. These properties indicate that 5d theories are worth studying. Some of them are obtained from superstring/M-theory and can be checked by using exact computations.

4.1 5d $\mathcal{N} = 1$ supersymmetry

In order to consider 5d supersymmetry, let us introduce spinors in 5d.

5d Dirac matrices are 4×4 matrices, similarly to the 4d case. It is sufficient to identify the chirality matrix γ^5 in 4d as one of the 5d Dirac matrices. We can impose neither the Weyl nor the Majorana condition for spinors. Instead, we introduce $Sp(1)_R \sim SU(2)_R$ symmetry and impose the symplectic Majorana condition

$$(\xi_{I\alpha})^* = \xi^{I\alpha} = \epsilon^{IJ} C^{\alpha\beta} \xi_{J\beta} \tag{4.1}$$

in the Euclidean signature, where C is the charge conjugation matrix in 5d. The indices take $\alpha, \beta = 1, 2, 3, 4$ and $I, J = 1, 2$. Therefore, 5d $\mathcal{N} = 1$ supersymmetry has eight supercharges. In this sense, 5d $\mathcal{N} = 1$ supersymmetry is similar to 4d $\mathcal{N} = 2$ supersymmetry and 3d $\mathcal{N} = 4$ supersymmetry. This is also similar to 6d $\mathcal{N} = (1, 0)$ supersymmetry.

5d $\mathcal{N} = 1$ supersymmetry algebra consists of the following generators of transformations:

- Translational transformation $P_{\widehat{\mu}}$,
- Lorentz transformation $M_{\widehat{\mu}\widehat{\nu}}$,
- $Sp(1)_R$ transformation R_a, and
- Supersymmetry transformation $Q_{I\alpha}$.

Table 4.1: 5d $\mathcal{N}=1$ vector multiplet.

	fields	dof	$Sp(1)_R$	
bosons	gauge field	4	$\mathbf{1}$	W_μ
	scalar	1	$\mathbf{1}$	M
	auxiliary field	3	$\mathbf{3}$	Y_a
fermions	gaugino	8	$\mathbf{2}$	$\Omega_{I\alpha}$

Table 4.2: 5d $\mathcal{N}=1$ hypermultiplet.

	fields	dof	$Sp(1)_R$	$Sp(1)_F$	
bosons	scalar	4	$\mathbf{2}$	$\mathbf{2}$	q_I^A
	auxiliary field	4	$\mathbf{2}$	$\mathbf{2}$	F_I^A
fermion	symplectic Majorana	8	$\mathbf{1}$	$\mathbf{2}$	ζ_α^A

The first three of them are bosonic symmetries, while the last one is fermionic. Massless representations of this algebra are the vector multiplet and the hypermutiplet.

A vector multiplet consists of the gauge field, the real scalar field, the auxiliary field and the gaugino, as shown in Table 4.1.

A hypermultiplet consists of the fields shown in Table 4.2. An off-shell action for hypermultiplets is usually not written. The action can be written by adding a $U(1)_Z$ symmetry, which is introduced in Chapter 5. $A=1,2$ is the index of the doublet of $Sp(1)_F$ symmetry in the flavor symmetry.

The Yang-Mills action in the 5d flat spacetime is written as

$$-\frac{1}{4g_{\mathrm{YM}}^2}\mathrm{Tr}\int d^5x F_{\mu\nu}(W)F^{\mu\nu}(W), \tag{4.2}$$

where $F_{\mu\nu}(W)$ is the field strength for the gauge field W_μ. The mass dimension of the Yang-Mills coupling g_{YM} is $-1/2$. Thus the theory is non-renormalizable and weakly coupled in the IR region. One might think that the 5d theory is not worth researching. However, Seiberg [41] pointed out that there is a nontrivial dynamics in the IR region and conjectured the presence of interesting properties in the UV region. We will see these properties in the next section and later.

4.2 Low energy effective action

Given a supersymmetric gauge theory, we often consider the structure of vacua. The space of vacua, called moduli space, has some characteristic subspaces. One of them is called the Coulomb branch, in which scalar fields in vector multiplets take values. The other is called the Higgs branch, in which scalar fields in hypermultiplets take values.

For the structures of the Coulomb and Higgs branches of 5d $\mathcal{N} = 1$ supersymmetric gauge theories, we can use the arguments for 4d $\mathcal{N} = 2$ supersymmetric gauge theories. The Higgs branch is hyper-Kähler manifold and not corrected by quantum effects due to the supersymmetry. On the Coulomb branch of a 4d $\mathcal{N} = 2$ supersymmetric gauge theory, the low energy effective action is written by a holomorphic function $\mathcal{F}(\phi)$ of the scalar fields in the vector multiplets. The function $\mathcal{F}(\phi)$ is called the prepotential. For simplicity, we consider the case in which there is one vector multiplet with a gauge group G. Then ϕ takes value in the Cartan subalgebra of G. Using the prepotential, the Kähler potential for 4d supersymmetric field theory can be written as[1]

$$\mathrm{Re} \int d^4\theta \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} \overline{\Phi} \supset \mathrm{Re} \left(\frac{\partial^2 \mathcal{F}(\phi)}{\partial \phi^2} \right) \partial_\mu \phi \partial^\mu \overline{\phi}. \tag{4.3}$$

Φ on the left hand side of (4.3) is the 4d chiral superfield including ϕ.

The general form of the 5d Lagrangian on the Coulomb branch is restricted by the form of the 4d Lagrangian on the Coulomb branch. 4d $\mathcal{N} = 2$ supersymmetric gauge theory can be obtained from 5d $\mathcal{N} = 1$ supersymmetric gauge theory by the dimensional reduction. In this procedure, the complex scalar field ϕ in 4d can be identified by the real scalar field M and the fifth components of the gauge field W_5 in 5d as

$$\phi = M + iW_5. \tag{4.4}$$

Assuming that the action (4.3) can be obtained by the dimensional reduction from 5d, let us impose the condition that the action is invariant under the shift

$$W_5 \to W_5 + (\text{const.}), \tag{4.5}$$

which can be realized by the 5d gauge transformation. Then the 5d prepotential is at most cubic:

$$\mathcal{F}(M) = c_0 + c_i M^i + c_{ij} M^i M^j + c_{ijk} M^i M^j M^k, \tag{4.6}$$

where i, j, k are the indices of the gauge group. Since the constants c_0 and c_i cannot affect the Lagrangian, we set them zero. If there is only one vector multiplet, we usually write the prepotential as

$$\mathcal{F}(M) = \frac{1}{2g_{\mathrm{YM}}^2} M^2 + \frac{c}{6} M^3. \tag{4.7}$$

[1]The definition of the prepotential is different from the original one shown in (1.16) by the factor i.

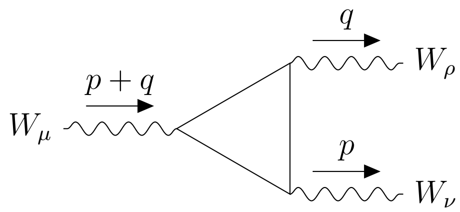

Figure 4.1: One-loop diagram generating 5d Chern-Simons term.

The first term in (4.7) gives the kinetic terms for the vector multiplet:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{g_{\text{YM}}^2}\text{Tr}\left(\frac{1}{4}F_{\mu\nu}(W)F^{\mu\nu}(W) + \frac{1}{2}D_\mu M D^\mu M + 2Y_a Y_a + 2i\Omega \not{D}\Omega - 2\Omega[M,\Omega]\right). \tag{4.8}$$

The second term in (4.7) yields

$$\mathcal{L}_{\text{CS}} = c\text{Tr}\left(\frac{1}{24}\epsilon^{\lambda\mu\nu\rho\sigma}W_\lambda F_{\mu\nu}(W)F_{\rho\sigma}(W) - \frac{1}{4}MF_{\mu\nu}(W)F^{\mu\nu}(W) - \frac{1}{2}MD_\mu M D^\mu M \right.$$
$$\left. - 2MY_aY_a + 2iM\Omega\not{D}\Omega + \Omega\not{F}(W)\Omega + 2\Omega Y\Omega - \Omega\left[M^2,\Omega\right]\right). \tag{4.9}$$

This action is a 5d supersymmetric version of a Chern-Simons action. The constant c is often written as

$$c = \frac{k}{4\pi^2}, \tag{4.10}$$

where k is called the Chern-Simons level. For gauge invariance, k should be an integer. (4.8) and (4.9) are invariant under the following supersymmetric transformation

$$\delta_Q W_\mu = 2i\xi\gamma_\mu\Omega, \tag{4.11}$$
$$\delta_Q M = 2\xi\Omega, \tag{4.12}$$
$$\delta_Q\Omega = \frac{1}{2}\not{F}(W)\Omega - \frac{i}{2}(\not{D}M)\xi - Y\xi, \tag{4.13}$$
$$\delta_Q Y_a = i\xi\tau_a\not{D}\Omega - \xi\tau_a[M,\Omega]. \tag{4.14}$$

Even if $c = 0$ in the original Lagrangian, c becomes nonzero by the loop corrections [87]. In order to show that, let us consider the one-loop WWW amplitude generated by the fermion loop. We assign the momenta $p+q$, p and q and the polarizations μ, ν and ρ for the gauge particles, as shown in Figure 4.1.

The contribution of a fermion with mass m to the amplitude is written by

$$\frac{1}{(2\pi)^5}\text{tr}\int d^5k\left(\gamma^\mu\frac{\not{k}+\not{p}-im}{(k+p)^2+m^2-i\epsilon}\gamma^\nu\frac{\not{k}-im}{k^2+m^2-i\epsilon}\gamma^\rho\frac{\not{k}-\not{q}-im}{(k-q)^2+m^2-i\epsilon}\right). \tag{4.15}$$

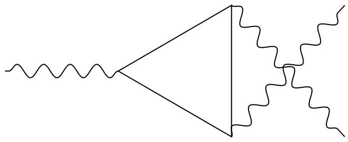

Figure 4.2: Another one-loop diagram we should consider.

The term relevant to the Chern-Simons term $\epsilon^{\mu\nu\rho\sigma\lambda} W_\mu \partial_\nu W_\rho \partial_\sigma W_\lambda$ is

$$\frac{im}{8\pi^5}\epsilon^{\mu\nu\rho\sigma\lambda} p_\sigma q_\lambda \int d^5k \frac{1}{(k^2+m^2-i\epsilon)^3}. \tag{4.16}$$

By the Wick rotation $k_0 \to ik_0$, we can compute the integral and obtain the amplitude

$$-\frac{1}{16\pi^2}\frac{m}{|m|}\epsilon^{\mu\nu\rho\sigma\lambda} p^\sigma q^\lambda. \tag{4.17}$$

By considering another diagram shown in Figure 4.2, which yields the same contribution, the Chern-Simons term generated by these diagrams is

$$-\frac{\text{sign}(m)}{48\pi^2}\int d^5x \epsilon^{\mu\nu\rho\sigma\lambda} W_\mu \partial_\nu W_\rho \partial_\sigma W_\lambda, \tag{4.18}$$

where

$$\text{sign}(m) = \frac{m}{|m|}. \tag{4.19}$$

The combinatorial factor 1/3! is taken into account in the denominator in (4.18). Therefore, the Chern-Simons level is effectively shifted by

$$\Delta k = -\frac{1}{2}\text{sign}(m) \tag{4.20}$$

due to a fermion loop. As simple examples, let us consider the following two cases:

1. $U(1)$ gauge theory with N_f hypermultiplets with charge one.

2. $SU(2)$ gauge theory with N_f hypermultiplets in the fundamental representation.

Let m_i ($i = 1, \dots, N_f$) be the mass of i-th hypermultiplet. In the $U(1)$ theory, the Coulomb branch is parametrized by the scalar field M in the vector multiplet. Hence the Coulomb branch of this theory is $\mathbb{R}$. On a general point of the Coulomb

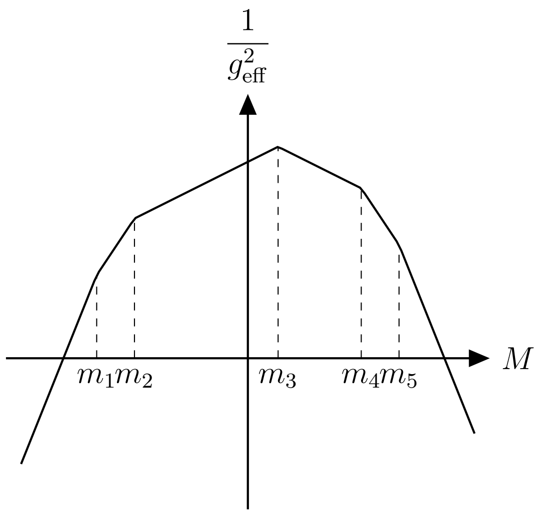

Figure 4.3: Effective Yang-Mills coupling g_{eff}^{-2} in the $U(1)$ theory.

branch, the scalar field M contributes to the masses of the hypermultiplets. Thus the effective Chern-Simons level is

$$k_{\text{eff}} = -\frac{1}{2}\sum_{i=1}^{N_f} \text{sign}\,(m_i + M)\,. \tag{4.21}$$

Because $c_{\text{eff}} = k_{\text{eff}}/4\pi^2$ is the derivative of $\mathcal{F}''(M) \sim g_{\text{eff}}^{-2}$, we obtain the effective Yang-Mills coupling g_{eff} by integrating (4.21) over M as

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g_{\text{YM}}^2} - \frac{1}{8\pi^2}\sum_{i=1}^{N_f} |m_i + M|\,. \tag{4.22}$$

It turns out that for any g_{YM} and m_i the right hand side becomes negative for sufficiently large M, as shown in Figure 4.3. This fact implies that this theory is non-renormalizable and a UV completion is needed.

In the $SU(2)$ case, the Coulomb branch can be parametrized by the scalar field in the vector multiplet

$$M = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}. \tag{4.23}$$

Because of $SU(2)$, the point of the Coulomb branch represented by $-M$ is identified with M by an $SU(2)$ gauge transformation. Thus the Coulomb branch of the $SU(2)$ theory is $\mathbb{R}/\mathbb{Z}_2 = \mathbb{R}_{\geq 0}$ and it is sufficient to consider the case $M \geq 0$. In the fermion loop, the adjoint fermion in the vector multiplet can also arise.

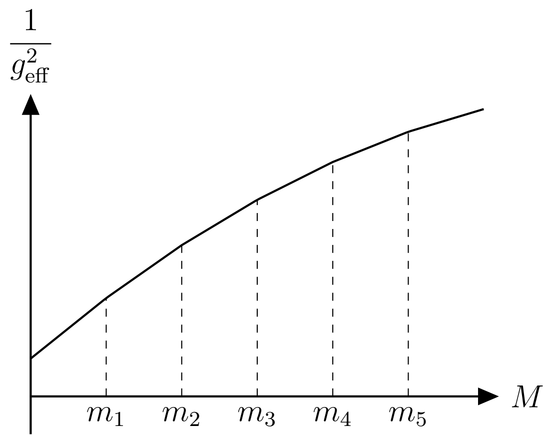

Figure 4.4: Effective Yang-Mills coupling $g_{\rm eff}^{-2}$ in the $SU(2)$ theory for $N_f = 5$.

The effective Chern-Simons level is

$$k_{\rm eff} = 8 - \frac{1}{2}\sum_{i=1}^{N_f} {\rm sign}\,(m_i + M) - \frac{1}{2}\sum_{i=1}^{N_f} {\rm sign}\,(m_i - M)\,. \qquad (4.24)$$

Integrating (4.24), the effective coupling can be written as

$$\frac{1}{g^2_{\rm eff}} = \frac{1}{g^2_{\rm YM}} + \frac{2}{\pi^2}M - \frac{1}{8\pi^2}\sum_{i=1}^{N_f}|m_i + M| - \frac{1}{8\pi^2}\sum_{i=1}^{N_f}|m_i - M|\,. \qquad (4.25)$$

It turns out that for an appropriate choice of $g_{\rm YM}$ and $N_f \leq 8$ we can take $g_{\rm eff}^{-2} \geq 0$ in the whole Coulomb branch, as shown in Figures 4.4 and 4.5. In such cases, therefore, there is a nontrivial dynamics in the IR region. For $N_f > 8$, the theory is non-renormalizable and a UV completion is needed.

4.3 Instanton in 5d

In 5d gauge theories, there is a global $U(1)_I$ symmetry, whose current is written as

$$j_\mu = \epsilon_{\mu\nu\rho\sigma\lambda}{\rm Tr}F^{\nu\rho}(W)F^{\sigma\lambda}(W). \qquad (4.26)$$

We can consider a background vector multiplet associated with the $U(1)_I$ symmetry and turn on a nonzero value m_0 for the background scalar field. We can identify the scalar field as the Yang-Mills coupling $m_0 \sim g_{\rm YM}^{-2}$. Charged objects for the $U(1)_I$ are particles, called instantons with the codimension four. Their BPS masses are related with m_0 by the BPS formula.

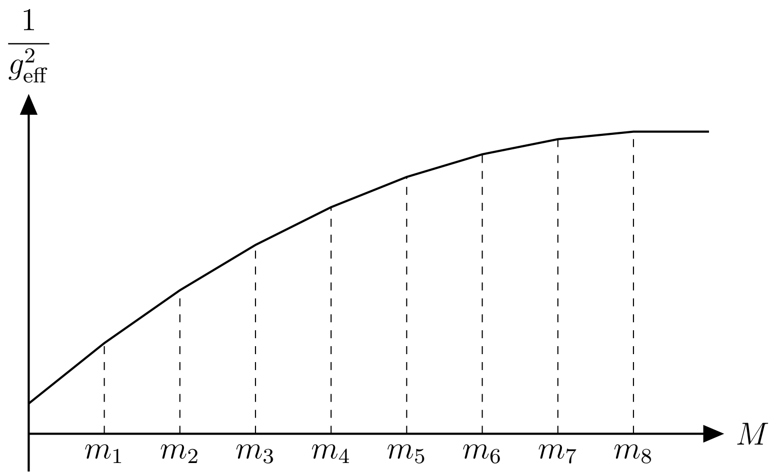

Figure 4.5: Effective Yang-Mills coupling g_{eff}^{-2} in the $SU(2)$ theory for $N_f = 8$. For $M > m_8$, g_{eff} becomes constant.

In particular, we consider a particle charged only with $U(1)_I$. Its mass m_{inst} can be computed from the central charge Z_{inst} by using the BPS formula as [41]

$$m_{\text{inst}} = \sqrt{2} Z_{\text{inst}} = \sqrt{2} m_0, \tag{4.27}$$

in the case of the $U(1)_I$ charge one[2]. Since the loop effect shifts the Yang-Mills coupling, the quantum correction affects as

$$m_{\text{inst}} = \sqrt{2}\,(m_0 + cM)\,. \tag{4.28}$$

In $M \neq 0$, the gauge symmetry breaks to $U(1)$ from $SU(2)$. In such case the instanton with a finite size cannot exist and shrinks. Its behavior depends on a UV completion.

4.4 5d $\mathcal{N} = 2$ supersymmetry

There is another class of supersymmetry, called 5d $\mathcal{N} = 2$ supersymmetry. This supersymmetry has sixteen preserved supercharges, or twice of 5d $\mathcal{N} = 1$ supersymmetry. The supersymmetry transformation parameter can be written as $\xi_{I\alpha}$, for $\alpha = 1, \ldots, 4$ and $I = 1, \ldots, 4$. I is an index of the spinor representation of the $Sp(2)_R \sim SO(5)_R$ R-symmetry, while α is a usual spinor index. Similar to 5d $\mathcal{N} = 1$ supersymmetry, spinors $\xi_{I\alpha}$ can be restricted by the symplectic-Majorana condition. This condition is expressed as in (4.1), by replacing ϵ^{IJ} to the $Sp(2)_R$

[2]For the computation for 5d $\mathcal{N} = 2$ theory, see [88].

charge conjugation matrix. Because of sixteen supercharges, 5d $\mathcal{N}=2$ supersymmetry is similar to 4d $\mathcal{N}=4$ supersymmetry and 3d $\mathcal{N}=8$ supersymmetry. This is also similar to 6d $\mathcal{N}=(2,0)$ supersymmetry.

The only supermultiplet with spin lower than two is a vector multiplet, which consists of the $\mathcal{N}=1$ vector multiplet and the $\mathcal{N}=1$ hypermultiplet in the adjoint representation. For vector multiplets, only an on-shell formalism is known. This can be obtained by the dimensional reduction from the on-shell action of 10d $\mathcal{N}=1$ supersymmetric Yang-Mills theory.

4.5 Global symmetry enhancement

One can see that a class of brane configurations in superstring theory realizes 5d $\mathcal{N}=1$ supersymmetric field theories. Let us consider the $SU(2)$ theory with $N_f<8$ fundamental hypermultiplets mentioned above. Then there is a corresponding brane configuration. There are $SO\left(2N_f\right)$ flavor symmetry and $U(1)_I$ symmetry as the obvious symmetries. If the masses of all hypermultiplets are zero and taking the strong coupling limit $g_{\text{YM}}\to\infty$ we can show, by string dualities, that the global symmetry of the $SU(2)$ gauge theory enhances to E_{N_f+1} symmetry [41, 89], where $E_5=Spin(10)$,[3] $E_4=SU(5)$, $E_3=SU(3)\times SU(2)$, $E_2=SU(2)\times U(1)$ and $E_1=SU(2)$. The obvious $SO\left(2N_f\right)\times U(1)_I$ symmetry is included in E_{N_f+1} as a subgroup.

We can check this by computing a superconformal index by using the localization technique [89]. For the definition of the superconformal index, we choose a certain supercharge Q. Then the superconformal index I essentially counts BPS states invariant under the supersymmetry transformation Q. The superconformal index is defined by

$$I\left(x,y,m_i,q\right)=\operatorname{tr}\left[(-1)^F e^{-\beta\left\{Q,Q^\dagger\right\}}x^{2(j_1+R)}y^{2j_2}e^{-i\sum_i H_i m_i}q^k\right], \tag{4.29}$$

where F is the fermion number operator, j_1 and j_2 are the Cartan generators of $Sp(2)_L\sim SO(5)_L$ Lorentz symmetry, R is the Cartan generator of $Sp(1)_R\sim SU(2)_R$, H_i are the Cartan generators of the flavor symmetry, and k is the instanton number. They commute with the supercharge Q. For these symmetries, we introduce the chemical potentials $e^{-\beta}$, $x=e^{-\gamma_1}$, $y=e^{-\gamma_2}$, e^{-im_i} and q, respectively. The trace is taken over the Hilbert space on S^4 after the radial quantization.

By the Wick rotation $x^0=-i\tau$, this superconformal index can be written in terms of a path integral as

$$I\left(x,y,m_i,q\right)=\int\mathcal{D}\Phi\exp\left(-S_E[\Phi]\right), \tag{4.30}$$

[3] The D-dimensional spin group $Spin(D)$ is the rotational group for spinors in D-dimensional space. Thus $Spin(D)$ is a double cover of $SO(D)$.

where S_E is the Euclidean action on $S^1 \times S^4$ with the S^1 radius β and unit S^4 radius. The chemical potentials induce a nontrivial twisted boundary condition.

By the localization computation for superconformal gauge theories on $S^4 \times S^1$, the superconformal index can be obtained as the following form:

$$I\left(x, y, m_i, q\right) = \int [d\alpha] I_{\text{south}}\left(\alpha, x, y, m_i, q\right) I_{\text{north}}\left(\alpha, x, y, m_i, q^{-1}\right). \tag{4.31}$$

α is the holonomy along S^1, which take values in the Cartan subalgebra of the gauge group. $[d\alpha]$ is the Haar measure, defined by the gauge group. The localized configurations are such that instantons are on the south pole of S^4 and anti-instantons are on the north pole of S^4. I_{south} and I_{north} are decomposed as

$$I_{\text{south}}\left(\alpha, x, y, m_i, q\right) = I^{\text{1-loop}}_{\text{south}} I^{\text{inst}}_{\text{south}}, \tag{4.32}$$

$$I_{\text{north}}\left(\alpha, x, y, m_i, q^{-1}\right) = I^{\text{1-loop}}_{\text{north}} I^{\text{inst}}_{\text{north}}. \tag{4.33}$$

The one-loop perturbative part is determined by the matter content of the theory. A vector multiplet contributes to the one-loop part as

$$\begin{aligned} I^{\text{1-loop}}_{\text{vec}} &= I^{\text{1-loop}}_{\text{vec,south}} I^{\text{1-loop}}_{\text{vec,north}} \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f_{\text{vec}}\left(x^n, y^n, n\alpha\right)\right], \end{aligned} \tag{4.34}$$

$$f_{\text{vec}}(x, y, \alpha) = -\frac{x\left(y + y^{-1}\right)}{\left(1 - xy\right)\left(1 - xy^{-1}\right)} \sum_{\boldsymbol{R}} e^{-i\boldsymbol{R}\cdot\alpha}, \tag{4.35}$$

where $\boldsymbol{R}$ is the roots of the gauge group. A hypermultiplet in a representation $\boldsymbol{W}$ of the gauge group contributes to the one-loop part as

$$\begin{aligned} I^{\text{1-loop}}_{\text{hyp}} &= I^{\text{1-loop}}_{\text{hyp,south}} I^{\text{1-loop}}_{\text{hyp,north}} \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f_{\text{hyp}}\left(x^n, y^n, n\alpha, nm\right)\right], \end{aligned} \tag{4.36}$$

$$f_{\text{hyp}}(x, y, \alpha, m) = \frac{x}{\left(1 - xy\right)\left(1 - xy^{-1}\right)} \sum_{\boldsymbol{w}\in\boldsymbol{W}} \left(e^{-i\boldsymbol{w}\cdot\alpha - im_i} + e^{i\boldsymbol{w}\cdot\alpha + im_i}\right), \tag{4.37}$$

where $\boldsymbol{w}$ runs over all components of the representation $\boldsymbol{W}$. The instanton contribution can be expanded in terms of q as

$$I^{\text{inst}}_{\text{south}}\left(\gamma_1, \gamma_2, \alpha, m_i, q\right) = \sum_{k=0}^{\infty} q^k I_k\left(\gamma_1, \gamma_2, \alpha, m_i\right), \tag{4.38}$$

$$I^{\text{inst}}_{\text{north}}\left(\gamma_1, \gamma_2, \alpha, m_i, q\right) = \sum_{k=0}^{\infty} q^{-k} I_k\left(\gamma_1, \gamma_2, -\alpha, -m_i\right), \tag{4.39}$$

where I_k is called the instanton index, defined by the gauge group and $I_0 = 1$.

Let us compute the superconformal index (4.31) for $Sp(1) = SU(2)$ gauge theory with N_f fundamental hypermultiplets. In this case, the superconformal index can be written as

$$
\begin{aligned}
I^{N_f}_{Sp(1)}(x, y, m_i, q) &= \int [d\alpha] \exp\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(f_{\text{vec}}\left(x^n, y^n, e^{in\alpha}\right) + f_{\text{hyp}}\left(x^n, y^n, e^{in\alpha}, e^{inm}\right)\right)\right] \\
&\quad \times \left|I_{\text{inst}}\left(x, y, e^{i\alpha}, e^{im}, q\right)\right|^2, && (4.40)\\
[d\alpha] &= \frac{1}{\pi} d\alpha \sin^2\alpha, && (4.41)\\
f_{\text{vec}} &= -\frac{x\left(y + y^{-1}\right)}{(1 - xy)\left(1 - xy^{-1}\right)}\left(e^{2i\alpha} + e^{-2i\alpha} + 1\right), && (4.42)\\
f_{\text{hyp}} &= \frac{x}{(1 - xy)\left(1 - xy^{-1}\right)} \sum_{i=1}^{N_f}\left(e^{i\alpha + im_i} + e^{-i\alpha + im_i} + e^{i\alpha - im_i} + e^{-i\alpha - im_i}\right). && (4.43)
\end{aligned}
$$

The instanton contribution is

$$
I_{\text{inst}} = \sum_{k=0}^{\infty} q^k I_k, \quad I_k = \frac{1}{2}\left(I_k^+ + I_k^-\right). \tag{4.44}
$$

We define $k = 2n + \chi$ ($\chi = 0$ or 1). Then I_k^+ and I_k^- are given by

$$
\begin{aligned}
I_k^+ &= (2i)^{k\left(N_f-4\right)-n} i^{n+2\chi} \oint [d\phi]^+ \left[\frac{\prod_{i=1}^{N_f}\sin\frac{m_i}{2}}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sin\frac{i\gamma_1\pm\alpha}{2}}\prod_{I=1}^{n}\frac{\sin\frac{\phi_I\pm 2i\gamma_1}{2}}{\sin\frac{\phi_I\pm i\gamma_1\pm i\gamma_2}{2}}\right]^{\chi} \\
&\times\prod_{I=1}^{n}\left[\frac{\sinh\gamma_1}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sin\frac{2\phi_I\pm i\gamma_1\pm i\gamma_2}{2}}\frac{\prod_{i=1}^{N_f}\sin\frac{m_i\pm\phi_I}{2}}{\sin\frac{\phi_I\pm\alpha\pm i\gamma_1}{2}}\right]\prod_{I<J}^{n}\left[\frac{\sin\frac{\phi_I\pm\phi_J\pm 2i\gamma_1}{2}}{\sin\frac{\phi_I\pm\phi_J\pm i\gamma_1\pm\gamma_2}{2}}\right],
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
I_{k:\mathrm{odd}}^- &= \frac{(2i)^{k\left(N_f-4\right)-n}}{i^{N_f-n-4}} \oint [d\phi]^- \left[\frac{\prod_{i=1}^{N_f}\cos\frac{m_i}{2}}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\cos\frac{i\gamma_1\pm\alpha}{2}}\prod_{I=1}^{n}\frac{\cos\frac{\phi_I\pm 2i\gamma_1}{2}}{\cos\frac{\phi_I\pm i\gamma_1\pm i\gamma_2}{2}}\right] \\
&\times\prod_{I=1}^{n}\left[\frac{\sinh\gamma_1}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sin\frac{2\phi_I\pm i\gamma_1\pm i\gamma_2}{2}}\frac{\prod_{i=1}^{N_f}\sin\frac{m_i\pm\phi_I}{2}}{\sin\frac{\phi_I\pm\alpha\pm i\gamma_1}{2}}\right]\prod_{I<J}^{n}\left[\frac{\sin\frac{\phi_I\pm\phi_J\pm 2i\gamma_1}{2}}{\sin\frac{\phi_I\pm\phi_J\pm i\gamma_1\pm i\gamma_2}{2}}\right],
\end{aligned}
\tag{4.46}
$$

$$
\begin{aligned}
I_{k:\mathrm{even}}^- &= (2i)^{(k-1)\left(N_f-2\right)-\frac{5}{2}k} i^{n+4} \oint [d\phi]^- \left[\frac{\cosh\gamma_1}{\cosh\frac{\gamma_1\pm\gamma_2}{2}\sinh^2\frac{\gamma_1\pm\gamma_2}{2}}\frac{\prod_{i=1}^{N_f}\sin m_i}{\sin\left(i\gamma_1\pm\alpha\right)}\right] \\
&\times\prod_{I=1}^{n-1}\left[\frac{\sinh\gamma_1\sin\left(\phi_I\pm 2i\gamma_1\right)}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\sinh\frac{2\phi_I\pm i\gamma_1\pm i\gamma_2}{2}\sin\left(\phi_I\pm i\gamma_1\pm i\gamma_2\right)}\frac{\prod_{i=1}^{N_f}\sin\frac{m_i+\phi_I}{2}}{\sin\frac{\phi_I\pm\alpha\pm i\gamma_1}{2}}\right] \\
&\times\prod_{I<J}^{n-1}\left[\frac{\sin\frac{\phi_I\pm\phi_J\pm 2i\gamma_1}{2}}{\sin\frac{\phi_I\pm\phi_J\pm i\gamma_1\pm i\gamma_2}{2}}\right].
\end{aligned}
\tag{4.47}
$$

The notation "±" is understood by taking product as

$$
\sin\frac{\phi_I\pm 2i\gamma_1}{2} = \sin\frac{\phi_I+2i\gamma_1}{2}\sin\frac{\phi_I-2i\gamma_1}{2}, \tag{4.48}
$$

$$
\begin{aligned}
\sin\frac{\phi_I\pm i\gamma_1\pm i\gamma_2}{2} &= \sin\frac{\phi_I+i\gamma_1+i\gamma_2}{2}\sin\frac{\phi_I+i\gamma_1-i\gamma_2}{2} \\
&\quad\times\sin\frac{\phi_I-i\gamma_1+i\gamma_2}{2}\sin\frac{\phi_I-i\gamma_1-i\gamma_2}{2},
\end{aligned}
\tag{4.49}
$$

and so on. The Haar measure $[d\phi]^+$ and $[d\phi]^-$ are defined by

$$[d\phi]^+ = \begin{cases} \dfrac{1}{2^{n-1}n!}\left[\prod_{I=1}^{n}\dfrac{d\phi_I}{2\pi}\right]\prod_{I<J}^{n}\left(2\sin\dfrac{\phi_I-\phi_J}{2}\right)^2\left(2\sin\dfrac{\phi_I+\phi_J}{2}\right)^2 & (\chi=0) \\ \dfrac{2^n}{n!}\left[\prod_{I=1}^{n}\dfrac{d\phi_I}{2\pi}\sin^2\dfrac{\phi_I}{2}\right]\prod_{I<J}^{n}\left(2\sin\dfrac{\phi_I-\phi_J}{2}\right)^2\left(2\sin\dfrac{\phi_I+\phi_J}{2}\right)^2 & (\chi=1) \end{cases}, \tag{4.50}$$

$$[d\phi]^- = \begin{cases} \dfrac{2^{n-1}}{(n-1)!}\left[\prod_{I=1}^{n-1}\dfrac{d\phi_I}{2\pi}\sin^2\phi_I\right]\prod_{I<J}^{n-1}\left(2\sin\dfrac{\phi_I-\phi_J}{2}\right)^2\left(2\sin\dfrac{\phi_I+\phi_J}{2}\right)^2 & (\chi=0) \\ \dfrac{2^n}{n!}\left[\prod_{I=1}^{n}\dfrac{d\phi_I}{2\pi}\cos^2\dfrac{\phi_I}{2}\right]\prod_{I<J}^{n}\left(2\sin\dfrac{\phi_I-\phi_J}{2}\right)^2\left(2\sin\dfrac{\phi_I+\phi_J}{2}\right)^2 & (\chi=1) \end{cases}. \tag{4.51}$$

For the contour integration, we define $z_I = e^{i\phi_I}$ and take a unit circle as the integration contour on the z_I-plane. By picking up residues, we can compute the superconformal index.

Up to $k=1$, there is no integration over ϕ_I. Expanding in terms of x, we can perform the integration over α from 0 to 2π. If $N_f = 3$, for example, the superconformal index can be written in terms of $SO(6)$ characters $\chi_{\boldsymbol{r}}^{SO(6)}$ of $\boldsymbol{r}$ representations as

$$\begin{aligned} I_{Sp(1)}^{N_f=3} &= 1 + \left[1 + \left(e^{-im_1-im_2} + \cdots + e^{im_2+im_3}\right)\right. \\ &\quad \left. + \left(q+q^{-1}\right)\left(e^{-im_1/2-im_2/2-im_3/2} + \cdots + e^{im_1/2+im_2/2+im_3/2}\right)\right] x^2 + \mathcal{O}\left(x^3\right) \end{aligned} \tag{4.52}$$

$$= 1 + \left[\chi_{\mathbf{1}}^{SO(6)} + \chi_{\mathbf{15}}^{SO(6)} + q\chi_{\mathbf{4}}^{SO(6)} + q^{-1}\chi_{\overline{\mathbf{4}}}^{SO(6)}\right] x^2 + \mathcal{O}\left(x^3\right). \tag{4.53}$$

From the definition of the superconformal index (4.29), the exponents of q are $U(1)_I$ charges of corresponding states. Therefore, q can be seen as one of the fugacities of the Cartan generators of $E_4 = SU(5) \supset SO(6) \times U(1)_I$. Indeed, the **24** representation of $E_4 = SU(5)$ is decomposed as

$$SU(5) \supset SO(6) \times U(1)_I \tag{4.54}$$

$$\mathbf{24} = \mathbf{1}_0 + \mathbf{15}_0 + \mathbf{4}_1 + \overline{\mathbf{4}}_{-1}, \tag{4.55}$$

and the superconformal index can be written by the $E_4 = SU(5)$ character as

$$I_{Sp(1)}^{N_f=3} = 1 + \chi_{\mathbf{24}}^{E_4} x^2 + \mathcal{O}\left(x^3\right). \tag{4.56}$$

We can perform the similar analyses for other N_f and higher x and observe that the superconformal index can be written by E_{N_f+1} characters. This suggests that the global symmetry enhances to E_{N_f+1}, by the nonperturbative corrections.

4.6 Relation to 6d $\mathcal{N} = (2,0)$ theory

Another interesting property of 5d supersymmetric field theory is that it seems to be related with a mysterious 6d $\mathcal{N} = (2,0)$ theory realized on a stack of M5-branes [42, 43], which has no known Lagrangian description. 5d $\mathcal{N} = 2$ supersymmetric field theory can be realized on a stack of D4-branes, which can be lifted to M5-branes in M-theory. [42, 43] proposed that Kaluza-Klein modes of the 6d $\mathcal{N} = (2,0)$ theory compactified on S^1 are realized as instantons in the 5d theory.

Although a 6d effective theory on multiple M5-branes is mysterious, an effective theory on a single M5-brane is known. It is a theory of a tensor multiplet, which includes a two-form gauge field $B_{\mu\nu}$ satisfying a self-dual condition $*H = H$, where $H = dB$ is the three-form field strength of the two-form gauge field. Therefore, fundamental degrees of freedom in this theory can be thought as strings coupled with the self-dual two-form gauge field. It is natural to interpret them as M2-branes which end on the M5-brane.

By compactifying x^6 direction to S^1 with radius R_6, we can obtain a 5d theory. The strings wrapping the S^1 lead to particle-like states, while the strings unwrapping the S^1 leads to string-like states.

In addition, there is the Kaluza-Klein momentum along S^1. From the 6d and 5d supersymmetry algebras, we can identify the momentum along x^6 with the central charge in 5d. Moreover, we can identify the radius R_6 and the Yang-Mills coupling in 5d as

$$g_{\mathrm{YM}}^2 = 8\pi^2 R_6. \tag{4.57}$$

Under this identification, it can be shown that the Kaluza-Klein spectrum for self-dual strings can be identified with instantons [43].

We can explain a duality in 4d $\mathcal{N} = 4$ theory by assuming (4.57). Let us additionally compactify the x^5 direction to S^1 with radius R_5. From 5d $\mathcal{N} = 2$ theory with the coupling constant (4.57), we can obtain a 4d $\mathcal{N} = 4$ theory with the coupling constant

$$g_{\mathrm{4d}}^2 = \frac{4\pi R_6}{R_5}. \tag{4.58}$$

For 4d $\mathcal{N} = 4$ theories, there is the Montonen-Olive duality [90], which says that $\tau \to \tau + 1$ and $\tau \to -1/\tau$ are the duality transformations, where

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{4d}}^2} \tag{4.59}$$

with so called the theta angle θ, which we now take $\theta = 0$. By taking $\tau \to -1/\tau$, the new 4d coupling is $g_{\mathrm{4d}}^2 = 4\pi R_5/R_6$. This can be understood straightforwardly as interchanging of radii $R_5 \leftrightarrow R_6$.

The analyses [42, 43] suggest that 5d $\mathcal{N} = 2$ theory includes all degrees of freedom of Kaluza-Klein modes of 6d $\mathcal{N} = (2,0)$ theory on S^1, and thus is equivalent to the 6d $\mathcal{N} = (2,0)$ theory on S^1 with the identification (4.57). If it was true, since the 6d $\mathcal{N} = (2,0)$ theory is finite, the 5d $\mathcal{N} = 2$ theory was also finite theory. However, it was shown in [91] that the 5d $\mathcal{N} = 2$ theory diverges at six loops. Hence they are not equal. The alternative conjecture can be proposed: the UV completion of the 5d $\mathcal{N} = 2$ theory is the 6d $\mathcal{N} = (2,0)$ theory on a circle.

One of the main results for checking this conjecture is the N^3 behavior of the free energy of 5d $\mathcal{N} = 2$ theory. The partition function for 5d $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory on S^5 with radius r was computed in [92] for the perturbative sector as[4]

$$Z = \int [d\phi] e^{-\frac{8\pi^3 r}{g_{\rm YM}^2}\mathrm{Tr}\phi^2} \prod_{\boldsymbol{R}} \Big[\left(\sin(i\pi\boldsymbol{R}\cdot\phi)\cos(i\pi\boldsymbol{R}\cdot\phi)\right)^{\frac{1}{4}} \times e^{\frac{1}{2}f(i\boldsymbol{R}\cdot\phi) - \frac{1}{4}f\left(\frac{1}{2} - i\boldsymbol{R}\cdot\phi\right) - \frac{1}{4}f\left(\frac{1}{2}+\boldsymbol{R}\cdot\phi\right)} \Big] + \mathcal{O}\left(e^{-\frac{16\pi^3 r}{g_{\rm YM}^2}}\right), \tag{4.60}$$

where $\mathcal{O}\left(e^{-\frac{16\pi^3 r}{g_{\rm YM}^2}}\right)$ includes nonperturbative contributions. The function $f(x)$ is defined by

$$f(x) = \frac{i\pi x^3}{3} + x^2 \log\left(1 - e^{-2\pi i x}\right) + \frac{ix}{\pi}\mathrm{Li}_2\left(e^{-2\pi i x}\right) + \frac{1}{2\pi^2}\mathrm{Li}_3\left(e^{-2\pi i x}\right) - \frac{\zeta(3)}{2\pi^2}, \tag{4.61}$$

with the polylogarithm $\mathrm{Li}_s(x)$ and the zeta function $\zeta(x)$. $f(x)$ satisfies

$$\frac{df(x)}{dx} = \pi x^2 \cot(\pi x). \tag{4.62}$$

For $U(N)$ case, $[d\phi] = d\phi_1 \cdots d\phi_N$ and ϕ_i corresponds to the i-th Cartan generator of $U(N)$.

We would like to compare (4.60) with the gravity dual of the 6d $\mathcal{N} = (2,0)$ theory on $S^5 \times S^1$. Let us compute the large N behavior of Z [93]. Due to the large N limit, the nonperturbative corrections $\mathcal{O}\left(e^{-\frac{16\pi^3 r}{g_{\rm YM}^2}}\right)$ becomes subdominant, by keeping the 't Hooft coupling constant

$$\lambda = \frac{g_{\rm YM}^2 N}{r} \tag{4.63}$$

[4] In the following, we use the notation of the parameters of [93], which may be different with the above by overall factors.

fixed. The partition function is written as

$$Z \sim \int \prod_{i=1}^{N} d\phi_i \exp\left(-\frac{8\pi^3 r}{g_{\rm YM}^2} \sum_{i=1}^{N} \phi_i^2 + \sum_{i\neq j} \left[\log\sinh(\pi\phi_{ij}) + \frac{1}{4}\log\cosh(\pi\phi_{ij}) \right.\right.$$
$$\left.\left. + \frac{1}{2} f(i\phi_{ij}) - \frac{1}{4} f\left(\frac{1}{2} + i\phi_{ij}\right) - \frac{1}{4} f\left(\frac{1}{2} - i\phi_{ij}\right)\right]\right), \tag{4.64}$$

where $\phi_{ij} = \phi_i - \phi_j$. Configurations of ϕ_i which become dominant in the integral are the saddle points, where the ϕ_i-derivatives of the exponent vanish:

$$\frac{16\pi^3 N}{\lambda}\phi_i = \pi \sum_{j\neq i} \left[(2 - \phi_{ij}^2)\coth(\pi\phi_{ij}) + \left(\frac{1}{4} + \phi_{ij}^2\right)\tanh(\pi\phi_{ij}) \right] \tag{4.65}$$

In the strong coupling limit $\lambda \to \infty$, $|\phi_{ij}|$ become large. Then we approximate

$$\coth(\pi\phi_{ij}) \sim \tanh(\pi\phi_{ij}) \sim \text{sign}(\phi_{ij}) \tag{4.66}$$

and obtain

$$\frac{16\pi^2 N}{\lambda}\phi_i = \frac{9}{4}\sum_{j\neq i} \text{sign}(\phi_{ij}). \tag{4.67}$$

Assuming $\phi_1 < \phi_2 < \cdots < \phi_N$, we obtain the solution

$$\phi_i = \frac{9\lambda}{64\pi^2 N}(2i - N). \tag{4.68}$$

By using this solution, we can evaluate the leading term of the free energy as

$$\begin{aligned} F &= -\log Z \\ &\approx \frac{8\pi^2 N}{\lambda}\sum_{i=1}^{N}\phi_i^2 - \frac{9\pi}{8}\sum_{i\neq j}|\phi_{ij}| \propto \frac{g_{\rm YM}^2}{r}N^3, \end{aligned} \tag{4.69}$$

whose behavior coincides with the on-shell action of supergravity on $\text{AdS}_7 \times S^4$, as required for the M5-branes [84].

The nonperturbative correction for the partition function was computed in [94]. For the $U(1)$ gauge theory, the partition function on S^5 coincides with the 6d superconformal index, which can be computed only in the Abelian case [95]. This is another check of the relation between the 5d $\mathcal{N} = 2$ supersymmetric gauge theory and the 6d $\mathcal{N} = (2, 0)$ theory.

Moreover, the partition function for 5d $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{CP}^2 \times S^1$ is computed in [96, 97]. Since S^5 is the S^1 fibration over $\mathbb{CP}^2$, $\mathbb{CP}^2 \times S^1$ is another S^1 reduction of $S^5 \times S^1$. As expected, this 5d partition function for $U(1)$ gauge theory also coincides with the 6d superconformal index. Therefore, we can expect that the S^5 partition function and the $\mathbb{CP}^2 \times S^1$ partition function are equal even if the gauge group is non-Abelian, as shown in Figure 4.6. However, the check of this is technically difficult because the coupling and the radius in 5d are exchanged from one to another.

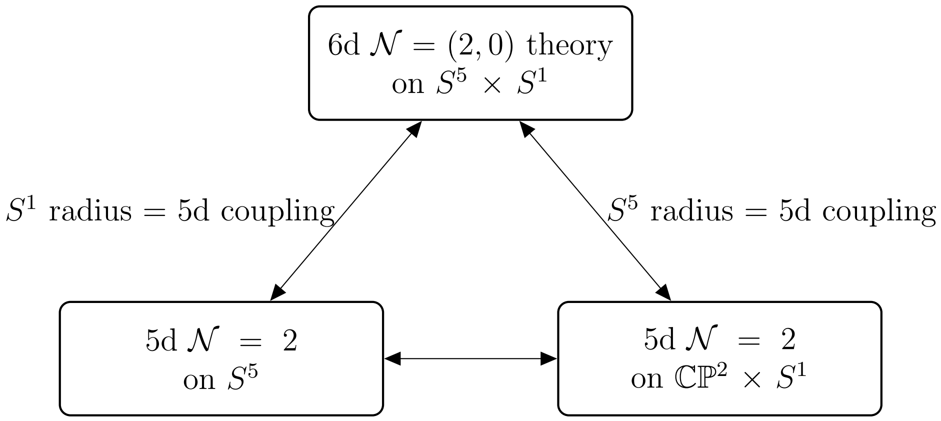

Figure 4.6: Relation between 6d $\mathcal{N} = (2,0)$ theory and 5d $\mathcal{N} = 2$ supersymmetric gauge theories. For the Abelian case, where the 6d superconformal index can be computed, their partition functions coincide.

Chapter 5

5d $\mathcal{N} = 1$ supersymmetric backgrounds

This chapter is the most central part of the thesis. This chapter is based on [45]. The goals of this chapter are

1. Construction of general supersymmetric backgrounds by solving certain spinor equations obtained from a 5d $\mathcal{N} = 1$ supergravity,

2. Showing whether each deformation of a supersymmetric background gives a Q-exact deformation or not, and

3. Construction of supersymmetric field theories on some simple manifolds as examples.

For these purposes, we firstly introduce the 5d $\mathcal{N} = 1$ Poincaré supergravity. Next, we would like to attack these problems.

We solve the condition that the background Weyl and vector multiplets preserve at least one supercharge. The solution has functional degrees of freedom. However, we find that every small deformation in a single local coordinate patch does not affect the partition function. We show that known supersymmetric field theories on S^5 or $S^3 \times \Sigma$, where Σ is a Riemann surface, can be realized from the solution. By using the solution, we cannot construct the supersymmetric field theories on $S^4 \times \mathbb{R}$ which can be obtained from the flat space by the Weyl transformation.

5.1 5d $\mathcal{N} = 1$ Poincaré supergravity

A 5d off-shell $\mathcal{N} = 1$ Poincaré supergravity was constructed in [98, 99, 100, 101]. In particular, in [100, 101], the authors started with a 6d $\mathcal{N} = (1, 0)$ conformal supergravity [102] and obtained the 5d Poincaré supergravity by the dimensional reduction and gauge fixing.

Table 5.1: The component fields of 5d $\mathcal{N}=1$ Weyl multiplet.

	fields	dof	$Sp(1)_R$	
bosons	vielbein	10	**1**	$e_\mu{}^{\widehat{\nu}}$
	$U(1)_Z$ gauge field	4	**1**	A_μ
	antisym. tensor	10	**1**	$v^{\mu\nu}$
	$Sp(1)_R$ triplet scalars	3	**3**	t_a
	$Sp(1)_R$ gauge field	12	**3**	V_μ^a
	scalar	1	**1**	C
fermions	gravitino	32	**2**	$\psi_{I\mu\alpha}$
	fermion	8	**2**	$\chi_{I\alpha}$

The bosonic symmetries included in it are the following:

- $Sp(2)_L \sim SO(5)_L$ local Lorentz symmetry,
- $Sp(1)_R \sim SU(2)_R$ local R-symmetry, and
- $U(1)_Z$ gauge symmetry which associated with the central charge.

From the 6d point of view, this $U(1)_Z$ symmetry can be identified with the translational symmetry along the reduced sixth dimension. In addition to these symmetries, the formulation in [100, 101] has the local dilatation symmetry. The corresponding gauge field is pure-gauge $b_\mu = \alpha^{-1}\partial_\mu \alpha$, thus we fix the gauge by the condition $b_\mu = 0$ in this thesis. α is a scalar field in [100, 101]. We rescale $A_\mu \to \alpha^{-1} A_\mu$ so that α disappears.

The Weyl multiplet consists of the fields shown in Table 5.1. There are two fermions in the Weyl multiplet. For two Grassmann-even spinors ξ_1 and ξ_2, the supersymmetry algebra is

$$\begin{aligned}\{\delta_Q(\xi_1), \delta_Q(\xi_2)\} = {} & 2i\left(\xi_1\gamma^{\widehat{\mu}}\xi_2\right) D_{\widehat{\mu}} + \delta_Z(2\xi_1\xi_2) \\ & + \delta_M\left(-2(\xi_1\xi_2)F_{\widehat{\mu}\widehat{\nu}}(A) + 2(\xi_1\gamma_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}}\xi_2)v^{\widehat{\rho}\widehat{\sigma}} + 4(\xi_1\gamma_{\widehat{\mu}\widehat{\nu}}\tau_a\xi_2)t_a\right) \\ & + \delta_U\left(-6(\xi_1\xi_2)t_a - 2(\xi_1\gamma_{\widehat{\mu}\widehat{\nu}}\tau_a\xi_2)\left(v^{\widehat{\mu}\widehat{\nu}} + \frac{1}{4}F^{\widehat{\mu}\widehat{\nu}}(A)\right)\right) \\ & + (\text{terms with } \eta \text{ or } \psi_\mu). \end{aligned} \tag{5.1}$$

$F_{\mu\nu}(A)$ is the field strength for the $U(1)_Z$ gauge field A_μ:

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.2}$$

D_μ is the covariant derivative defined by

$$D_\mu = \partial_\mu + \delta_M(\omega_{\mu\widehat{\rho}\widehat{\sigma}}) + \delta_U\left(V_\mu^a\right) + \delta_Z(A_\mu) + \delta_G(W_\mu), \tag{5.3}$$

where δ_M, δ_U, δ_Z and δ_G denote the Lorentz, $Sp(1)_R$, $U(1)_Z$ and gauge transformations, respectively. For example,

$$D_\mu\xi = \partial_\mu\xi + \frac{1}{4}\omega_{\mu\widehat{\rho}\widehat{\sigma}}\gamma^{\widehat{\rho}\widehat{\sigma}}\xi + V_\mu\xi. \tag{5.4}$$

The supersymmetry transformation for the fermions in the Weyl multiplet is written as

$$\delta_Q\psi_\mu = D_\mu\xi + \frac{i}{2}F_{\mu\widehat{\nu}}(A)\gamma^{\widehat{\nu}}\xi + \frac{i}{2}\gamma_{\mu\widehat{\rho}\widehat{\sigma}}v^{\widehat{\rho}\widehat{\sigma}}\xi + i\gamma_\mu t\xi, \tag{5.5}$$

$$\begin{aligned}\delta_Q\chi = &-\frac{i}{2}\gamma_{\widehat{\nu}}\xi D_{\widehat{\mu}}v^{\widehat{\mu}\widehat{\nu}} + \frac{1}{2}\xi C - \frac{i}{2}(\not{D}t)\xi - 2\left(\not{v} + \frac{1}{4}\not{F}(A)\right)t\xi \\ &- \frac{1}{32}\gamma^{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}}\xi F_{\widehat{\mu}\widehat{\nu}}(A)F_{\widehat{\rho}\widehat{\sigma}}(A),\end{aligned} \tag{5.6}$$

For notations of indices, see Appendix A.1.

5.2 5d supersymmetric backgrounds

5.2.1 Spinor bilinears and orthonormal frame

We would like to derive the condition for bosonic background fields so that there is at least one supersymmetry transformation parameter ξ satisfying

$$\delta_Q\psi_\mu = \delta_Q\chi = 0. \tag{5.7}$$

We restrict ourselves to the case in which ξ satisfies the symplectic Majorana condition (4.1). That condition is necessary for the reality of an action in the Minkowski signature, but it is not necessary in the Euclidean signature. Hence, this condition is just for simplicity of the analysis.

By using ξ, we can define the bilinears

$$S = \xi\xi, \tag{5.8}$$

$$R^\mu = \xi\gamma^\mu\xi, \tag{5.9}$$

$$J^a_{\mu\nu} = \frac{1}{S}\left(\xi\gamma_{\mu\nu}\tau^a\xi\right). \tag{5.10}$$

By the Fierz identity,

$$\gamma_\mu\xi R^\mu = \xi S. \tag{5.11}$$

The following equations are easily derived from this:

$$R_\mu R^\mu = S^2, \tag{5.12}$$

$$J^a_{\mu\nu}R^\nu = 0, \tag{5.13}$$

$$-\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma\lambda}R_\rho J^a_{\sigma\lambda} = SJ^a_{\mu\nu}. \tag{5.14}$$

Because ξ is a solution to the homogeneous first order differential equation $\delta_Q\psi_\mu = 0$, it is nowhere vanishing and so are the bilinears. From the symplectic Majorana condition, $S>0$ everywhere. We assume the vielbein $e_\mu{}^{\hat{\nu}}$ is real, and then R^μ is also real. Note that the other background fields can be complex in general. The existence of the non-vanishing real vector field R^μ enables us to treat the background manifold $\mathcal{M}_5$ as a fibration over a base manifold $\mathcal{B}$ at least locally. Here, we will not discuss global issues and focus only on a single coordinate patch. Let us define the fifth coordinate x^5 by

$$R^\mu\partial_\mu = \partial_5 \tag{5.15}$$

and use a local frame with

$$e^{\hat{m}} = e_n{}^{\hat{m}}dx^n, \quad e^{\hat{5}} = S\left(dx^5 + U_m dx^m\right). \tag{5.16}$$

With this frame R^μ has the local components

$$R_{\hat{m}} = 0, \tag{5.17}$$
$$R_{\hat{5}} = S. \tag{5.18}$$

Then (5.13) and (5.14) can be written as

$$J^a_{\hat{m}\hat{5}} = 0, \tag{5.19}$$
$$-\frac{1}{2}\epsilon^{(4)}_{\hat{m}\hat{n}\widehat{kl}}J^a_{\widehat{kl}} = J^a_{\hat{m}\hat{n}}, \tag{5.20}$$

where

$$\epsilon^{(4)}_{\hat{m}\hat{n}\widehat{kl}} = \epsilon_{\hat{m}\hat{n}\widehat{kl}\hat{5}}. \tag{5.21}$$

The equation (5.11) means that ξ has positive chirality with respect to $\gamma_{\hat{5}} = S^{-1}R^\mu\gamma_\mu$

$$\gamma_{\hat{5}}\xi = +\xi. \tag{5.22}$$

A symplectic Majorana spinor χ belongs to the $(\mathbf{4},\mathbf{2})$ representation of $Sp(2)_L\times Sp(1)_R$. Because $Sp(k)=U(k,\mathbb{H})$,[1] we can treat χ as a vector with two quaternionic components. If we use the matrix representation of quaternions, we can

[1] $\mathbb{H}$ denotes the set of quaternions defined as follows. Let $(1,\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$ be the basis of quaternions, satisfying

$$\boldsymbol{i}\cdot\boldsymbol{j} = -\boldsymbol{j}\cdot\boldsymbol{i} = -\boldsymbol{k}, \quad \boldsymbol{j}\cdot\boldsymbol{k} = -\boldsymbol{k}\cdot\boldsymbol{j} = -\boldsymbol{i}, \quad \boldsymbol{k}\cdot\boldsymbol{i} = -\boldsymbol{i}\cdot\boldsymbol{k} = -\boldsymbol{j}, \quad \boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{k}^2=-1. \tag{5.23}$$

Then $\mathbb{H}$ can be defined by

$$\mathbb{H} = \{a+\boldsymbol{i}b+\boldsymbol{j}c+\boldsymbol{k}d \mid a,b,c,d\in\mathbb{R}\}. \tag{5.24}$$

$\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ can be represented in terms of the Pauli matrices as $\boldsymbol{i}=i\tau_1$, $\boldsymbol{j}=i\tau_2$ and $\boldsymbol{k}=i\tau_3$.

represent χ as a 4×2 matrix in the form

$$\chi_\alpha{}^I = \begin{pmatrix} U \\ D \end{pmatrix}, \quad U = U_0 \mathbf{1}_2 + iU_a \tau_a, \quad D = D_0 \mathbf{1}_2 + iD_a \tau_a, \quad U_a, D_a \in \mathbb{R}. \quad (5.25)$$

The vector $R^{\widehat{\mu}}$ breaks the local Lorentz symmetry $Sp(2)_L$ to its subgroup $Sp(1)_l \times Sp(1)_r \sim SO(4)$, where $Sp(1)_l$ and $Sp(1)_r$ act on the upper and lower blocks of the matrix (5.25), respectively.

The chirality condition (5.22) implies that only the upper components of the spinor ξ can be non-zero. Furthermore, by using an $Sp(1)_l \times Sp(1)_R$ gauge transformation, we can choose a gauge such that $U \propto \mathbf{1}_2$. In this gauge, ξ is written as

$$\xi_\alpha{}^I = \sqrt{\frac{S}{2}} \begin{pmatrix} \mathbf{1}_2 \\ \mathbf{0} \end{pmatrix}, \quad (5.26)$$

where the normalization is fixed by $\xi\xi = S$. This gauge choice breaks $Sp(1)_l \times Sp(1)_R$ into its diagonal subgroup $Sp(1)_D$. $Sp(1)_r$ symmetry acting the lower block of the matrix (5.25) also remains. It is obvious in this frame that the following eight spinors form a basis of the space of symplectic spinors:

$$\xi_\alpha{}^I, \quad (\gamma_{\widehat{m}})_\alpha{}^\beta \xi_\beta{}^I, \quad \xi_\alpha{}^J (\tau_a)_J{}^I. \quad (5.27)$$

An arbitrary spinor can be expanded by this basis. For example, $\gamma_{\widehat{m}\widehat{n}}\xi$ is related to $\xi\tau_a$ by

$$\gamma_{\widehat{m}\widehat{n}}\xi = -\xi\tau_a J^a_{\widehat{m}\widehat{n}}, \quad (5.28)$$

$$\xi\tau_a = \frac{1}{4} J^a_{\widehat{m}\widehat{n}} \gamma^{\widehat{m}\widehat{n}} \xi. \quad (5.29)$$

(5.29) implies that the three matrices J^a satisfy the same algebra with the Pauli matrices:

$$J^a_{\widehat{m}\widehat{k}} J^b_{\widehat{k}\widehat{n}} = \delta_{ab}\delta_{\widehat{m}\widehat{n}} + i\epsilon_{abc} J^c_{\widehat{m}\widehat{n}}. \quad (5.30)$$

Namely, J^a enjoys the quaternion algebra.

5.2.2 $\delta_Q \psi_\mu = 0$

Let us solve the condition $\delta_Q \psi_\mu = 0$, which is investigated in [44] for the first time. Using the basis $(\xi, \gamma_{\widehat{m}}\xi, \tau_a\xi) = (\gamma_{\widehat{\mu}}\xi, \tau_a\xi)$ in (5.27), we decompose $\delta_Q\psi_\mu = 0$ into the following conditions:

$$0 = \left(\xi\gamma_{\widehat{\lambda}}\delta\psi_{\widehat{\mu}}\right) = \frac{1}{2} D_{\widehat{\mu}} R_{\widehat{\lambda}} + \frac{i}{2} S F_{\widehat{\mu}\widehat{\lambda}}(A) - \frac{i}{2} S \epsilon_{\widehat{5}\widehat{\mu}\widehat{\lambda}\widehat{\rho}\widehat{\sigma}} v^{\widehat{\rho}\widehat{\sigma}} - iSt_a J^a_{\widehat{\mu}\widehat{\lambda}}, \quad (5.31)$$

$$0 = \left(\xi\tau_a\delta\psi_{\widehat{\mu}}\right) = \xi\tau_a D_{\widehat{\mu}}\xi + \frac{i}{2}\left(\xi\tau_a\gamma_{\widehat{\mu}\widehat{\rho}\widehat{\sigma}}\xi\right) v^{\widehat{\rho}\widehat{\sigma}} + iR_{\widehat{\mu}} t_a. \quad (5.32)$$

The symmetric part of (5.31), $D_{\{\widehat{\mu}}R_{\widehat{\lambda}\}}=0$, means that R^μ is a Killing vector. We can take an $Sp(1)_D\times Sp(1)_r$ gauge such that

$$\partial_5 e_m{}^{\widehat{n}}=\partial_5 S=\partial_5 U_m=0, \tag{5.33}$$

and then $e_n{}^{\widehat{m}}$, S, and U_m can be treated as fields on the base manifold $\mathcal{B}$. The $(\widehat{\lambda},\widehat{\mu})=(\widehat{5},\widehat{m})$ components of (5.31) give

$$F_{m5}(A)=i\partial_m S. \tag{5.34}$$

From the integrability condition

$$\partial_n F_{m5}(A)=\partial_m F_{n5}(A) \tag{5.35}$$

and the Bianchi identity for $F_{\mu\nu}(A)$, we obtain

$$\partial_5 F_{mn}(A)=0. \tag{5.36}$$

This means that the $U(1)_Z$ gauge field A_μ is essentially a gauge field on $\mathcal{B}$. The condition (5.34) can be solved, up to the $U(1)_Z$ gauge transformation, by

$$A=A_m dx^m+iSdx^5,\quad \partial_5 A_m=0. \tag{5.37}$$

For later use, we give the non-vanishing components of the spin connection.

$$\omega_{\widehat{k}\widehat{m}\widehat{n}}=\omega^{(4)}_{\widehat{k}\widehat{m}\widehat{n}}, \tag{5.38}$$

$$\omega_{\widehat{m}\widehat{n}\widehat{5}}=\omega_{\widehat{5}\widehat{n}\widehat{m}}=\frac{S}{2}F_{\widehat{m}\widehat{n}}(U)=\frac{1}{S}D_{\widehat{m}}R_{\widehat{n}}, \tag{5.39}$$

$$\omega_{\widehat{5}\widehat{5}\widehat{m}}=\frac{1}{S}\partial_{\widehat{m}}S=-iF_{\widehat{m}\widehat{5}}(A), \tag{5.40}$$

where $\omega^{(4)}_{\widehat{k}\widehat{m}\widehat{n}}$ is the spin connection in the base manifold $\mathcal{B}$ defined with the vielbein $e_m{}^{\widehat{n}}$.

The anti-symmetric part of (5.31) can be used to represent the horizontal part of $v^{\mu\nu}$ in terms of other fields:

$$v_{\widehat{p}\widehat{q}}=-\frac{1}{2}\epsilon^{(4)}_{\widehat{p}\widehat{q}\widehat{m}\widehat{n}}\left(\frac{i}{4}SF_{\widehat{m}\widehat{n}}(U)-\frac{1}{2}F_{\widehat{m}\widehat{n}}(A)+t_aJ^a_{\widehat{m}\widehat{n}}\right). \tag{5.41}$$

By using (5.4), we obtain

$$\xi\tau_a D_\mu\xi=\frac{S}{4}\omega_{\widehat{\mu}\widehat{p}\widehat{q}}J^a_{\widehat{p}\widehat{q}}+V_\mu^a S. \tag{5.42}$$

Using this, we can solve (5.32) with respect to V_μ^a and obtain

$$V^a_{\widehat{m}}=-\frac{1}{4}\omega_{\widehat{m}\widehat{p}\widehat{q}}J^a_{\widehat{p}\widehat{q}}-iJ^a_{\widehat{m}\widehat{p}}v^{\widehat{p}\widehat{5}}, \tag{5.43}$$

$$V^a_{\widehat{5}}=-\frac{1}{4}\omega_{\widehat{5}\widehat{p}\widehat{q}}J^a_{\widehat{p}\widehat{q}}-\frac{i}{2}J^a_{\widehat{p}\widehat{q}}v^{\widehat{p}\widehat{q}}-it_a. \tag{5.44}$$

5.2.3 $\delta_Q \chi = 0$

By using the spinor basis (5.27), we decompose $\delta_Q \chi = 0$ into the following equations.

$$0 = S^{-1}\left(\xi \delta_Q \chi\right) = -\frac{i}{2} D_\mu v^{\mu \hat{5}} + \frac{1}{2} C - t_a J^a_{\widehat{m}\widehat{n}} \left(v_{\widehat{m}\widehat{n}} + \frac{1}{4} F_{\widehat{m}\widehat{n}}(A) \right) - \frac{1}{32} \epsilon^{(4)}_{\widehat{m}\widehat{n}\widehat{p}\widehat{q}} F^{\widehat{m}\widehat{n}}(A) F^{\widehat{p}\widehat{q}}(A), \tag{5.45}$$

$$0 = S^{-1}\left(\xi \gamma_{\widehat{m}} \delta_Q \chi\right) = -\frac{i}{2} D^\lambda v_{\lambda \widehat{m}} - \frac{i}{2} J^a_{\widehat{m}\widehat{n}} D^{\widehat{n}} t_a - 2 t_a J^a_{\widehat{m}\widehat{p}} \left(v^{\widehat{p}\hat{5}} + \frac{1}{4} F^{\widehat{p}\hat{5}}(A) \right) - \frac{1}{8} \epsilon^{(4)}_{\widehat{m}\widehat{p}\widehat{q}\widehat{r}} F^{\widehat{p}\widehat{q}}(A) F^{\widehat{r}\hat{5}}(A), \tag{5.46}$$

$$0 = S^{-1}\left(\xi \tau_a \delta_Q \chi\right) = -\frac{i}{2} D_{\hat{5}} t_a - i \epsilon_{abc} t_b J^c_{\widehat{m}\widehat{n}} \left(v^{\widehat{m}\widehat{n}} + \frac{1}{4} F^{\widehat{m}\widehat{n}}(A) \right). \tag{5.47}$$

(5.45) is the only condition including C, and can be used to determine C. The conditions (5.46) and (5.47) are drastically simplified if we substitute the solution of $\delta_Q \psi_\mu = 0$;

$$0 = S^{-1}\left(\xi \gamma_{\widehat{m}} \delta_Q \chi\right) = \frac{i}{2} \partial_{\hat{5}} v_{\widehat{m}\hat{5}}, \tag{5.48}$$

$$0 = S^{-1}\left(\xi \tau_a \delta_Q \chi\right) = -\frac{i}{2} \partial_{\hat{5}} t_a. \tag{5.49}$$

Namely, $v_{\widehat{m}\widehat{5}}$ and t_a are x^5-independent. After all, we have obtained the following solution:

$$\xi_\alpha{}^I = \sqrt{\frac{S}{2}} \begin{pmatrix} \mathbf{1}_2 \\ \mathbf{0} \end{pmatrix}, \tag{5.50}$$

$$e_m{}^{\widehat{n}} = \text{(indep.)}, \tag{5.51}$$

$$e_5{}^{\widehat{5}} = S \text{ (indep.)}, \tag{5.52}$$

$$U_m = \text{(indep.)}, \tag{5.53}$$

$$A_{\widehat{m}} = \text{(indep.)}, \tag{5.54}$$

$$A_5 = iS, \tag{5.55}$$

$$v_{\widehat{p}\widehat{q}} = -\frac{1}{2}\epsilon^{(4)}_{\widehat{p}\widehat{q}\widehat{m}\widehat{n}} \left(\frac{i}{4} S F_{\widehat{m}\widehat{n}}(U) - \frac{1}{2} F_{\widehat{m}\widehat{n}}(A) + t_a J^a_{\widehat{m}\widehat{n}} \right), \tag{5.56}$$

$$v^{\widehat{m}\widehat{5}} = \text{(indep.)}, \tag{5.57}$$

$$t_a = \text{(indep.)}, \tag{5.58}$$

$$V^a_{\widehat{m}} = -\frac{1}{4}\omega_{\widehat{m}\widehat{p}\widehat{q}} J^a_{\widehat{p}\widehat{q}} - i J^a_{\widehat{m}\widehat{p}} v^{\widehat{p}\widehat{5}}, \tag{5.59}$$

$$V^a_{\widehat{5}} = \frac{i}{4} J^a_{\widehat{m}\widehat{n}} \left(F_{\widehat{m}\widehat{n}}(A) - iS F_{\widehat{m}\widehat{n}}(U) \right) + i t_a, \tag{5.60}$$

$$\begin{aligned} C &= i D^{(4)}_{\widehat{m}} v_{\widehat{m}\widehat{5}} - \frac{1}{2} t_a J^a_{\widehat{m}\widehat{n}} F_{\widehat{m}\widehat{n}}(A) - 8 t_a t_a \\ &\quad + \frac{1}{16}\epsilon^{(4)}_{\widehat{m}\widehat{n}\widehat{p}\widehat{q}} \left(F^{\widehat{m}\widehat{n}}(A) - iS F^{\widehat{m}\widehat{n}}(U) \right) \left(F^{\widehat{p}\widehat{q}}(A) - iS F^{\widehat{p}\widehat{q}}(U) \right). \end{aligned} \tag{5.61}$$

"(indep.)" means that the field is an independent field. We can freely choose them. All the fields are x^5-independent. This is in fact a direct consequence of the algebra. From the commutation relation (5.1), we obtain

$$\begin{aligned} \delta_Q(\xi)^2 &= i R^\mu D_\mu + \delta_M \left(-S F_{\widehat{\mu}\widehat{\nu}}(A) + \epsilon_{\widehat{\mu}\widehat{\nu}\rho\sigma\lambda} R^\rho v^{\sigma\lambda} + 2 S J^a_{\widehat{\mu}\widehat{\nu}} t_a \right) \\ &\quad + \delta_Z(S) + \delta_U \left(-3 S t_a - S J^a_{\widehat{m}\widehat{n}} \left(v^{\widehat{m}\widehat{n}} + \frac{1}{4} F^{\widehat{m}\widehat{n}}(A) \right) \right) \\ &\quad + (\text{terms with } \eta \text{ or } \psi_\mu). \end{aligned} \tag{5.62}$$

In the resulting background, the right hand side reduces to the x^5 derivative:

$$\delta_Q(\xi)^2 = i R^\mu D_\mu - i \delta_M \left(R^\lambda \omega_{\lambda\widehat{\mu}\widehat{\nu}} \right) - i \delta_Z \left(R^\mu A_\mu \right) - i \delta_U \left(R^\mu V^a_\mu \right) = i \partial_5. \tag{5.63}$$

Therefore, a $\delta_Q(\xi)$-invariant background is also invariant under the isometry ∂_5.

5.3 Q-exact deformation

As mentioned in Section 1.3, a Q-exact deformation of an action

$$S \to S + \delta_Q(\cdots) \tag{5.64}$$

does not affect the partition function. Therefore, if a deformation of the supersymmetric background gives a Q-exact deformation, it does not change the partition function. Let us see whether deformations of supersymmetric backgrounds give Q-exact deformations or not.

A small deformation of the Weyl multiplet around a supersymmetric background induces the change of the action

$$S_1 = \int d^5x\sqrt{g}\left[\delta e_\mu{}^{\widehat{\nu}} T^\mu_{\widehat{\nu}} + \delta V^a_\mu R^\mu_a + \delta\psi_\mu S^\mu + \delta A_\mu J^\mu \right. \\ \left. + \delta v^{\mu\nu} M_{\mu\nu} + \delta C\Phi + \delta\widetilde{\chi}\eta + \delta t_a X_a\right], \tag{5.65}$$

where

$$R^\mu_a, \quad S^\mu_{I\alpha}, \quad T^{\mu\nu}, \quad J^\mu, \quad M^{\mu\nu}, \quad \Phi, \quad \eta_{I\alpha}, \quad X_a \tag{5.66}$$

form the supercurrent multiplet associated with the Weyl multiplet.

A Q-exact deformation that is regarded as a change of the bosonic background fields in general has the form

$$\delta_Q(\xi)\int d^5x\sqrt{g}\left[H_\mu S^\mu + K\eta\right], \tag{5.67}$$

where H_μ and K are vectorial-spinor and spinor coefficient functions. Both H_μ and K are Grassmann-even. Because $\delta_Q(\xi)^2 = i\partial_5$ for the action (5.67) to be Q-invariant the functions H_μ and K should be x^5-independent.

$\delta_Q S^\mu$ and $\delta_Q\eta$ are determined as follows. For an arbitrary deformation of the Weyl multiplet that may not preserve rigid supersymmetry, S_1 is invariant under the supersymmetry if we transform both the Weyl multiplet and matter fields. The supersymmetry transformation of the bosonic components of the Weyl multiplet are [100, 101]

$$\delta_Q e_\mu{}^{\widehat{\nu}} = -2i\xi\gamma^{\widehat{\nu}}\psi_\mu, \tag{5.68}$$

$$\delta_Q A_\mu = 2\xi\psi_\mu, \tag{5.69}$$

$$\delta V^a_\mu = 2i\xi\gamma_\mu\tau_a\widetilde{\chi} - i\xi\gamma^{\widehat{\nu}}\tau_a R_{\widehat{\nu}\mu}(Q) - \xi\not{F}(A)\tau_a\psi_\mu - 4\xi\not{v}\tau_a\psi_\mu - 6\xi\psi_\mu t_a, \tag{5.70}$$

$$\delta t_a = 2\xi\tau_a\widetilde{\chi}, \tag{5.71}$$

$$\delta v_{\widehat{\mu}\widehat{\nu}} = \frac{1}{4}\xi\gamma_{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}} R^{\widehat{\rho}\widehat{\sigma}}(Q) + 2\xi\gamma_{\widehat{\mu}\widehat{\nu}}\widetilde{\chi}, \tag{5.72}$$

$$\delta C = -2i\xi\widehat{\not{D}}\widetilde{\chi} - 22\xi t\widetilde{\chi} + 6\xi\not{v}\widetilde{\chi} - \xi\gamma^{\widehat{\mu}\widehat{\nu}} t R_{\widehat{\mu}\widehat{\nu}}(Q), \tag{5.73}$$

where $R_{\mu\nu}(Q)$ and $\widehat{D}_\mu\widetilde{\chi}$ are defined by

$$R_{\mu\nu}(Q) = 2D_{[\mu}\psi_{\nu]} + i\gamma_{\rho\sigma[\mu}\psi_{\nu]}v^{\rho\sigma} - i\gamma^\rho\psi_{[\mu}F_{\nu]\rho}(A) - 2i\gamma_{[\mu}t\psi_{\nu]}, \tag{5.74}$$

$$\widehat{D}_\mu\widetilde{\chi} = D_\mu\widetilde{\chi} - \delta_Q(\psi_\mu)\widetilde{\chi}. \tag{5.75}$$

By requiring the δ_Q-invariance we can determine the supersymmetry transformation of the supercurrent multiplet. For example, $\delta_Q(\xi)\eta$ is written by

$$\begin{aligned}\delta_Q(\xi)\eta &= 2i\gamma_\mu\tau_a\xi R^\mu_a + 2\gamma_{\mu\nu}\xi M^{\mu\nu} - 2i\gamma^\mu\xi D_\mu\Phi - 2i\not{D}\xi\Phi - 22t\xi\Phi + 6\not{v}\xi\Phi + 2\tau_a\xi X_a \\ &= 2i\gamma_\mu\tau_a\xi R^\mu_a + 2\gamma_{\mu\nu}\xi M^{\mu\nu} - 2i\gamma^\mu\xi D_\mu\Phi - 2\not{F}(A)\xi\Phi - 32t\xi\Phi + 2\tau_a\xi X_a,\end{aligned} \tag{5.76}$$

where we neglected total derivatives in the action and used the Killing spinor equation (5.5)

$$D_\mu\xi = -\frac{i}{2}F_{\mu\nu}(A)\gamma^\nu\xi - \frac{i}{2}\gamma_{\mu\rho\sigma}v^{\rho\sigma}\xi - i\gamma_\mu t\xi. \tag{5.77}$$

Let us consider the second term in the Q-exact action (5.67). It is convenient to decompose the spinor function K by the spinor basis (5.27) as

$$K = k\xi + \frac{1}{2S}k^a\xi\tau_a + \frac{1}{2S}k^{\widehat{m}}\xi\gamma_{\widehat{m}}. \tag{5.78}$$

By using (5.76), the first term $k\xi$ in (5.78) gives the action

$$\delta_Q\int d^5x\sqrt{g}k(\xi\eta) = -2i\int d^5x\sqrt{g}k\partial_5\Phi. \tag{5.79}$$

This is a total derivative and does not give a nontrivial deformation of the theory.

The second term in (5.78) gives

$$\begin{aligned}&\delta_Q\int d^5x\sqrt{g}\frac{1}{2S}k^a\left(\xi\tau_a\eta\right) \\ &= \int d^5x\sqrt{g}\left(ik^aR^{\widehat{5}}_a + k^aJ^a_{\widehat{m}\widehat{n}}M^{\widehat{m}\widehat{n}} - \frac{1}{2}k^aJ^a_{\widehat{m}\widehat{n}}F^{\widehat{m}\widehat{n}}(A)\Phi - 16k^at_a\Phi + k^aX_a\right).\end{aligned} \tag{5.80}$$

Comparing this to (5.65), we find that the addition of the Q-exact term (5.80) is equivalent to the deformation of the background as

$$\begin{aligned}&\delta V^a_{\widehat{5}} = ik^a, \quad \delta v_{\widehat{m}\widehat{n}} = k^aJ^a_{\widehat{m}\widehat{n}}, \quad \delta C = -\frac{1}{2}k^aJ^a_{\widehat{m}\widehat{n}}F^{\widehat{m}\widehat{n}}(A) - 16k^at_a, \\ &\delta t_a = k^a, \quad \delta(\text{others}) = 0.\end{aligned} \tag{5.81}$$

These deformations are consistent to the solution of the supersymmetric backgrounds (5.50)-(5.61). Therefore, the small shifting

$$t_a \to t_a + k^a \tag{5.82}$$

does not change the partition function of the theory if we keep the supersymmetry realized by the solution (5.50)-(5.61).

Similarly, the third term in (5.78) gives the Q-exact deformation

$$\delta_Q \int d^5x\sqrt{g}\left(\frac{1}{2S}k^{\widehat{m}}\left(\xi\gamma_{\widehat{m}}\eta\right)\right)$$
$$= \int d^5x\sqrt{g}\left(ik^{\widehat{m}}J^a_{\widehat{m}\widehat{n}}R^{\widehat{n}}_a + 2k^{\widehat{m}}M^{\widehat{m}\widehat{5}} + i\left(D^{(4)}_{\widehat{m}}k^{\widehat{m}}\right)\Phi\right). \tag{5.83}$$

This corresponds to the deformation of the background fields

$$\delta V^a_{\widehat{m}} = -iJ^a_{\widehat{m}\widehat{n}}k^{\widehat{n}}, \quad \delta v^{\widehat{m}\widehat{5}} = k^{\widehat{m}}, \quad \delta C = iD^{(4)}_{\widehat{m}}k^{\widehat{m}}, \quad \delta(\text{others}) = 0. \tag{5.84}$$

These variations are again consistent to the solution (5.50)-(5.61), and generated by the shift of the independent field $v^{\widehat{m}\widehat{5}}$ by

$$v^{\widehat{m}\widehat{5}} \to v^{\widehat{m}\widehat{5}} + k^{\widehat{m}}. \tag{5.85}$$

The same procedure is available for the Q-exact deformation originated from S^μ, but it is rather complicated. Thus we introduce a simpler way. The change of the action introduced by the small deformation of the background Weyl multiplet can be symbolically written as

$$S_1 = A^B_i J^B_i + A^F_i J^F_i, \tag{5.86}$$

where we neglected the integration over the space. A^B_i and A^F_i are the deformation of the bosonic and fermionic components in the Weyl multiplet, respectively. J^B_i and J^F_i are the bosonic and fermionic components of the supercurrent multiplet. The supersymmetry transformation of the fermionic components in the supercurrent multiplet can be defined by

$$0 = \delta_Q A^B_i J^B_i - A^F_i \delta_Q J^F_i. \tag{5.87}$$

It is sufficient to consider linear terms with respect to fermions because we are considering the small deformations. Thus the supersymmetry transformations for the bosonic background fields can be written by using a matrix M_{ij}:

$$\delta_Q A^B_i = A^F_j M_{ji}. \tag{5.88}$$

Then the supersymmetry transformation for the fermionic components in the supercurrent multiplet is

$$\delta_Q J^F_i = M_{ij} J^B_j. \tag{5.89}$$

By using this formulation, the Q-exact deformation (5.67) is written as

$$\delta_Q\left(f_i J^F_i\right) = f_i M_{ij} J^B_j, \tag{5.90}$$

where f_i corresponds to H_μ and K in (5.67). It is found that the Q-exact deformation (5.90) is equivalent to the deformation of the background fields as

$$A_i^B = f_j M_{ji}. \tag{5.91}$$

It is nothing but the supersymmetric transformation (5.88) of the bosonic background fields with parameters f_i instead of the fermions A_i^F. Therefore, Q-exact deformations are realized by the substitution of parameters f_i into the fermions A_i^F in the supersymmetry transformations for the bosonic background fields. Indeed, the substitutions

$$(\psi_\mu, \widetilde{\chi}) \to \left(0, \frac{1}{2S} k^a \tau_a \xi\right), \quad (\psi_\mu, \widetilde{\chi}) \to \left(0, -\frac{1}{2S} k^{\widehat{m}} \gamma_{\widehat{m}} \xi\right) \tag{5.92}$$

into (5.68)-(5.73) reproduce the Q-exact deformations of the background fields (5.81) and (5.84).

Now let us consider the Q-exact deformations originated from S^μ, by using this method. The corresponding background deformation can be obtained from the transformation laws (5.68)-(5.73) by the substitution

$$(\psi_\mu, \widetilde{\chi}) \to (H_\mu, 0)\,. \tag{5.93}$$

We expand the function H_μ by the spinor basis as

$$H_\mu = \frac{i}{2S} h_\mu \xi + \frac{1}{S} h_\mu^a \tau_a \xi + \frac{i}{2S} h_\mu^{\widehat{m}} \gamma_{\widehat{m}} \xi. \tag{5.94}$$

The deformation parameters h_μ, h_μ^a and $h_\mu^{\widehat{m}}$ are arbitrary functions on the base manifold $\mathcal{B}$.

The deformation from h_μ yields the variation of the independent fields

$$\delta S = h_5, \quad \delta U_{\widehat{m}} = \frac{1}{S} h_{\widehat{m}}, \quad \delta A_m = i h_m, \quad \delta v_{\widehat{m}\widehat{5}} = J^a_{\widehat{m}\widehat{n}} h_{\widehat{n}} t_a, \quad \delta e_m{}^{\widehat{n}} = \delta t_a = 0. \tag{5.95}$$

The variation of the dependent fields are obtained from the solution (5.50)-(5.61). We already know that the change of $v_{\widehat{m}\widehat{5}}$ does not affect the partition function. Thus it is sufficient to focus on the change of S, $U_{\widehat{m}}$ and A_m.

The deformation from h_μ^a does not give any variations of the independent fields:

$$\delta e_m{}^{\widehat{n}} = \delta S = \delta U_{\widehat{m}} = \delta A_m = \delta v^{\widehat{m}\widehat{5}} = \delta t_a = 0. \tag{5.96}$$

The deformation from $h_\mu{}^{\widehat{m}}$ gives the variations of the independent fields:

$$\delta e_\mu{}^{\widehat{m}} = h_\mu{}^{\widehat{m}}, \quad \delta v_{\widehat{m}\widehat{5}} = -\frac{i}{4} \epsilon^{(4)}_{\widehat{m}\widehat{p}\widehat{q}\widehat{r}} D^{(4)}_{\widehat{p}} h_{\widehat{q}}{}^{\widehat{r}} - 2 h_{[\widehat{p}}{}^{\widehat{p}} v_{\widehat{m}]\widehat{5}}, \tag{5.97}$$

$$\delta S = \delta U_{\widehat{m}} = \delta A_m = \delta t_a = 0. \tag{5.98}$$

Table 5.2: The component fields of 5d $\mathcal{N} = 1$ vector multiplet.

	fields	dof	$Sp(1)_R$	
bosons	gauge field	4	**1**	W_μ
	scalar	1	**1**	M
	auxiliary field	3	**3**	Y_a
fermions	gaugino	8	**2**	$\Omega_{I\alpha}$

Because of the reason mentioned above, we are not interested in the variation of $v_{\widehat{m}\widehat{5}}$. Let us focus on the change of ${e_\mu}^{\widehat{m}}$. The variation caused by ${h_5}^{\widehat{m}}$

$$\delta {e_5}^{\widehat{m}} = {h_5}^{\widehat{m}} \tag{5.99}$$

breaks our gauge choice (5.16) for the vielbein. To recover ${e_5}^{\widehat{m}} = 0$, we should perform the compensating local Lorentz transformation

$$\delta_M \left(\lambda_{\widehat{\mu}\widehat{\nu}}\right) = -{h_5}^{\widehat{m}}, \quad \lambda_{\widehat{m}\widehat{5}} = -\frac{1}{S}{h_5}^{\widehat{m}}, \quad \lambda_{\widehat{m}\widehat{n}} = 0. \tag{5.100}$$

This transformation, in turn, changes the vector field $A_{\widehat{m}}$ by

$$\delta_M \left(\lambda_{\widehat{\mu}\widehat{\nu}}\right) A_{\widehat{m}} = -\frac{i}{S}{h_5}^{\widehat{m}}. \tag{5.101}$$

Therefore, together with (5.95) caused by h_μ, we can freely change S, $U_{\widehat{m}}$, A_m and ${e_\mu}^{\widehat{m}}$.

After all, by using Q-exact deformations and gauge transformations, we find that we can freely change all the independent background fields, at least locally. Of course this does not mean that the partition function does not depend on the background at all. To clarify the complete background dependence of the partition function, careful analysis of the global structure of the background is needed.

5.4 Background vector multiplets

We can also turn on background vector multiplets. By similar analysis as the previous sections, let us derive the condition for background vector multiplets with a preserved supercharge and their independence of the partition function, in the presence of background vector multiplets.

A vector multiplet consists of fields listed in Table 5.2[2]. In [100], the super-

[2]For the auxiliary field Y_a, we use the different convention from the one in [100]. By replacing $Y_a \to Y_a - Mt_a$, we can recover the formulas in [100].

symmetry transformation law is written as follows:

$$\delta_Q W_\mu = 2i\xi\gamma_\mu\Omega + 2\xi\psi_\mu M, \tag{5.102}$$
$$\delta_Q M = 2\xi\Omega, \tag{5.103}$$
$$\delta_Q\Omega = \frac{1}{2}\not{F}(W)\xi - \frac{i}{2}(\not{D}M)\xi - Y\xi - M\left(t + \frac{1}{2}\not{F}(A)\right)\xi, \tag{5.104}$$
$$\delta_Q Y_a = i\xi\tau_a\widehat{\not{D}}\Omega - \xi\tau_a[M,\Omega] + \xi\tau_a\not{v}\Omega + \xi t\tau_a\Omega. \tag{5.105}$$

Let us make $\psi_\mu = \widetilde{\chi} = 0$ and solve $\delta_Q\Omega = 0$ in a similar manner as the previous analysis. For simplicity, we consider a $U(1)$ vector multiplet. We decompose the condition into the following two.

$$0 = (\xi\gamma_\mu\delta_Q\Omega) = \frac{1}{2}F_{\mu\nu}(W)R^\nu - \frac{iS}{2}D_\mu M - \frac{1}{2}MF_{\mu\nu}(A)R^\nu, \tag{5.106}$$
$$0 = (\xi\tau_a\delta_Q\Omega) = \frac{S}{4}J^a_{\mu\nu}F^{\mu\nu}(W) - SY_a - SMt_a - \frac{S}{4}J^a_{\mu\nu}MF^{\mu\nu}(A). \tag{5.107}$$

From (5.106), we obtain

$$D_5 M = 0, \quad F_{m5}(W) = iD_m(SM). \tag{5.108}$$

(5.107) can be used to determine Y_a. After all, the solution of $\delta_Q\Omega = 0$ is written as

$$M = \text{(indep.)}, \tag{5.109}$$
$$W_m = \text{(indep.)}, \tag{5.110}$$
$$W_5 = iSM, \tag{5.111}$$
$$Y_a = -Mt_a + \frac{S}{4}J^a_{\widehat{m}\widehat{n}}\left(F_{\widehat{m}\widehat{n}}(W) - MF_{\widehat{m}\widehat{n}}(A)\right). \tag{5.112}$$

"(indep.)" again means that the field is an independent field. Similar to the result for the Weyl multiplet, all the fields are x^5-independent.

Next, let us consider the background (in)dependence of the partition function. As mentioned in the above section, we can obtain a Q-exact deformation by replacing fermions in the background multiplet into a spinor parameter in the supersymmetric transformation law for the background bosonic fields. The replacement

$$\Omega \to -\frac{i}{2S}f_\mu\gamma^\mu\xi \tag{5.113}$$

gives the variations of the independent fields

$$\delta W_\mu = f_\mu, \quad \delta M = -\frac{i}{S}f_5. \tag{5.114}$$

The replacement

$$\Omega \to f_a\tau_a\xi \tag{5.115}$$

does not give any nontrivial deformation. Therefore, we can freely change the independent fields M and W_m, at least locally.

5.5 Examples

In the previous sections, we construct the general form of the supersymmetric background. In this section, we will check whether our solution includes known backgrounds.

First of all, let us review the action of vector multiplets and hypermultiplets in the 5d $\mathcal{N}=1$ Poincaré supergravity [100, 101]. For our purpose, we set the fermions in the Weyl multiplet to be zero.

An action for vector multiplets can be written by a gauge invariant cubic polynomial of M^i, called prepotential $\mathcal{N}(M)$. We write its derivatives with respect to M^i as

$$\mathcal{N}_i = \frac{\partial \mathcal{N}}{\partial M^i}, \quad \mathcal{N}_{ij} = \frac{\partial^2 \mathcal{N}}{\partial M^i \partial M^j}, \quad \mathcal{N}_{ijk} = \frac{\partial^3 \mathcal{N}}{\partial M^i \partial M^j \partial M^k}. \tag{5.116}$$

i, j, k label adjoint indices of the gauge group and also the central charge vector multiplet

$$(W_\mu, M, Y_a, \Omega)^{i=0} = (A_\mu, 1, -t_a, 0)\,. \tag{5.117}$$

Using the prepotential $\mathcal{N}$, we can write down the Lagrangian for vector multiplets as

$$\begin{aligned} e^{-1}\mathcal{L}_V = {}& \mathcal{N}P + \mathcal{N}_i \left(F^i_{\mu\nu}(W) \left(v^{\mu\nu} - \frac{1}{2} F^{\mu\nu}(A) \right) - [\Omega, \Omega]^i \right) \\ & - \frac{1}{2}\mathcal{N}_{ij} \left(-\frac{1}{4} F^i_{\mu\nu}(W) F^{j\mu\nu}(W) - \frac{1}{2} D_\mu M^i D^\mu M^j - 2Y^i_a Y^j_b \right. \\ & \left. + 2i\Omega^i \not{D} \Omega^j + 2\Omega^i \left(\not{v} - \frac{1}{2}\not{F}(A) \right) \Omega^j \right) \\ & - \mathcal{N}_{ijk} \left(\frac{1}{2} \Omega^i \not{F}(W)^j \Omega^k + \Omega^i Y^j \Omega^k + \frac{e^{-1}}{12} [\mathrm{CS}]^{ijk}_5 \right), \end{aligned} \tag{5.118}$$

where

$$P = 2C + 10 t_a t_a - 2F_{\mu\nu}(A) v^{\mu\nu} + \frac{3}{4} F_{\mu\nu}(A) F^{\mu\nu}(A). \tag{5.119}$$

In the background (5.50)-(5.61), P is written by the independent fields as

$$\begin{aligned} P = {}& -6 \left(t_a + \frac{1}{4} J^a_{\widehat{m}\widehat{n}} F_{\widehat{m}\widehat{n}} F_{\widehat{m}\widehat{n}}(A) \right)^2 - \frac{S^2}{8} \epsilon_{\widehat{m}\widehat{n}\widehat{p}\widehat{q}} F_{\widehat{m}\widehat{n}}(U) F_{\widehat{p}\widehat{q}}(U) \\ & + 2iD^{(4)}_{\widehat{m}} v^{\widehat{m}\widehat{5}} - \frac{4i}{S} \left(\partial_{\widehat{m}} S \right) v^{\widehat{m}\widehat{5}} - \frac{3}{2S^2} \left(\partial_{\widehat{m}} S \right)^2. \end{aligned} \tag{5.120}$$

$[\mathrm{CS}]^{ijk}_5$ is the 5d Chern-Simons term defined by

$$[\mathrm{CS}]^{ijk}_5 = \epsilon^{\lambda\mu\nu\rho\sigma} W^i_\lambda \partial_\mu W^j_\nu \partial_\rho W^k_\sigma \tag{5.121}$$

Table 5.3: 5d hypermultiplet ($A=1,\ldots,2r$).

	fields	dof	$Sp(1)_R$	
bosons	scalar	$4r$	**2**	q_I^A
	auxiliary field	$4r$	**2**	F_I^A
fermion	fermion	$8r$	**1**	ζ_α^A

for Abelian gauge fields. For non-Abelian gauge groups we should add appropriate terms W^3dW and W^5 to make it gauge invariant. If $\mathcal{N}$ includes only dynamical scalar fields, the Lagrangian becomes the supersymmetric version of the 5d Chern-Simons action, which is conformal invariant. $\mathcal{N}$ including the scalar in the central charge vector multiplet M^0 gives supersymmetric Yang-Mills action or Fayet-Iliopoulos term, which are not conformal invariant.

A hypermultiplet consists of the fields shown in Table 5.3. $A=1,\ldots,2r$ is an index of a flavor symmetry, which can be gauged and coupled with vector multiplets. The flavor indices are raised and lowered by an antisymmetric invariant tensor ρ_{AB} or ρ^{AB}

$$q_A = q^B\rho_{BA}, \quad q^A = \rho^{AB}q_B. \tag{5.122}$$

The bosonic fields satisfy the reality condition:

$$q_{AI} = \left(q^{AI}\right)^*, \quad F_{AI} = \left(F^{AI}\right)^*. \tag{5.123}$$

The fermionic field satisfies

$$(\zeta_{IA})^* = \zeta^{IA} = \epsilon^{IJ}\rho^{AB}\zeta_{JB}. \tag{5.124}$$

We often omit the flavor indices, similar to spinor indices and $SU(2)_R$ doublet indices. For example,

$$qq = q^{IA}q_{IA}. \tag{5.125}$$

The action for hypermultiplets is

$$\begin{aligned} e^{-1}\mathcal{L}_H = {} & D^\mu q D_\mu q - qM^2q + 2qYq \\ & - \left(C - \frac{1}{4}R + \frac{1}{8}F_{\mu\nu}(A)F^{\mu\nu}(A) - v_{\mu\nu}v^{\mu\nu} + 5t_at_a\right)qq \\ & - 2i\zeta \not{D}\zeta + 2\zeta M\zeta - \zeta \not{F}(A)\zeta + 2\zeta\not{v}\zeta \\ & + 8q\Omega\zeta - (1 + A^\mu A_\mu)FF. \end{aligned} \tag{5.126}$$

This action is invariant under the following supersymmetry transformation:

$$\delta_Q q = 2\xi\zeta, \tag{5.127}$$

$$\delta_Q \zeta = -i(\not{D}q)\xi + F\xi - 3qt\xi - Mq\xi + \frac{1}{2}\not{F}(A)q\xi + 2\not{v}q\xi, \tag{5.128}$$

$$\delta_Q F = -2\xi\left(i\not{D}\zeta + \frac{1}{2}\not{F}(A)\zeta - \not{v}\zeta - M\zeta + 2\Omega q\right). \tag{5.129}$$

The covariant derivative D_μ includes the $U(1)_Z$ transformation $\delta_Z(A_\mu)$. Hypermultiplets transform non-trivially under the $U(1)_Z$ transformation as

$$\delta_Z(\theta)q = \theta F, \tag{5.130}$$

$$\delta_Z(\theta)\zeta = -\theta\left(i\not{D}\zeta + \frac{1}{2}\not{F}(A)\zeta - \not{v}\zeta - M\zeta + 2\Omega q\right), \tag{5.131}$$

$$\begin{aligned}\delta_Z(\theta)F = \theta\bigg(&-\left(D^{\widehat{\mu}}D_{\widehat{\mu}} + C + \frac{1}{4}R + \frac{1}{8}F_{\mu\nu}(A)F^{\mu\nu}(A) - v_{\mu\nu}v^{\mu\nu} + 4t_a t_a\right)q \\ &+ 4\Omega\zeta + 2Yq - ttq - M^2q + 2tq + 2MF\bigg).\end{aligned} \tag{5.132}$$

Right hand sides of (5.131) and (5.132) also include δ_Z in the covariant derivative. Hence $\delta_Z\zeta$ and $\delta_Z F$ is defined recursively.

For simplicity we consider only on-shell neutral hypermultiplets below.

5.5.1 Conformally flat backgrounds

Given a superconformal field theory on a flat space, we can obtain the theory on a conformally flat background by a Weyl transformation. Then the superconformal transformation parameter ξ on the conformally flat background satisfies the Killing spinor equation

$$D_\mu\xi = \gamma_\mu\kappa \tag{5.133}$$

where κ is a spinor. By the Weyl transformation, the Lagrangian for vector multiplets is covariantized with respect to the local symmetries $Sp(2)_L$, $Sp(1)_R$ and $U(1)_Z$ and also added the term

$$\frac{R}{8}\mathcal{N}, \tag{5.134}$$

where R is the Riemann curvature of the background. For hypermultiplets, the Lagrangian is covariantized and added

$$\frac{3R}{16}qq. \tag{5.135}$$

We would like to reproduce the theory from the supergravity. In the 5d $\mathcal{N}=1$ supergravity, the Killing spinor equation corresponds to $\delta_Q\psi_\mu = 0$, where $\delta_Q\psi_\mu$ is defined in (5.5). For this condition to have only the covariant derivative and terms proportional to γ_μ, we impose

$$V^a_\mu = 0, \quad v_{\mu\nu} - \frac{1}{2}F_{\mu\nu}(A) = 0, \tag{5.136}$$

where the second condition arises because

$$F_{\mu\nu}(A)\gamma^\nu + \gamma_{\mu\rho\sigma}v^{\rho\sigma} = \gamma_\mu \not{F}(A) + \gamma_{\mu\rho\sigma}\left(v^{\rho\sigma} - \frac{1}{2}F^{\rho\sigma}(A)\right). \tag{5.137}$$

Then the condition $\delta_Q\psi_\mu = 0$ becomes the Killing spinor equation with

$$\kappa = -i\left(\frac{1}{2}\not{F}(A) + t\right)\xi. \tag{5.138}$$

Due to (5.136), extra terms which do not arise in conformal theory vanish in the action (5.118). Then what we need to show is that the combination of the background fields P yields the Riemann curvature. This is easily shown by using the condition $\delta_Q\widetilde{\chi} = 0$. If (5.136) holds, we can rewrite $\delta_Q\widetilde{\chi}$ in (5.6) as

$$\begin{aligned}\delta_Q\widetilde{\chi} = &-\frac{i}{4}[\not{D}(\not{F}(A)+2t)]\xi - \frac{1}{8}\gamma_\mu(\not{F}(A)+2t)\gamma^\mu(\not{F}(A)+2t)\xi \\ &+\left(\frac{1}{2}C + \frac{5}{2}t_a t_a - \frac{1}{16}F_{\mu\nu}(A)F^{\mu\nu}(A)\right)\xi. \end{aligned}\tag{5.139}$$

Using the Killing spinor equation (5.133) with (5.138), this equation can be rewritten further:

$$\delta_Q\widetilde{\chi} = \frac{1}{2}D_\mu D^\mu\xi + \left(\frac{1}{2}C + \frac{5}{2}t_a t_a - \frac{1}{16}F_{\mu\nu}(A)F^{\mu\nu}(A)\right)\xi = 0. \tag{5.140}$$

Using this, we obtain

$$P\xi = 4\left(\frac{1}{2}C + \frac{5}{2}t_a t_a - \frac{1}{16}F_{\mu\nu}(A)F^{\mu\nu}(A)\right)\xi = -2D_\mu D^\mu\xi = \frac{R}{8}\xi. \tag{5.141}$$

The third equality is shown by using (5.133) as follows:

$$\begin{aligned}\frac{1}{8}R\xi &= -\frac{1}{16}\gamma^{\mu\nu}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\xi \\ &= -\frac{1}{2}\gamma^{\mu\nu}D_\mu D_\nu\xi \\ &= -\frac{1}{2}\not{D}\not{D}\xi + \frac{1}{2}D_\mu D^\mu\xi \\ &= -\frac{5}{2}\not{D}\kappa + \frac{1}{2}D_\mu D^\mu\xi \\ &= -\frac{5}{2}D_\mu D^\mu\xi + \frac{1}{2}D_\mu D^\mu\xi \\ &= -2D_\mu D^\mu\xi. \end{aligned}\tag{5.142}$$

We used the flatness of the $Sp(1)_R$ connection between the first and the second lines. (5.141) shows that $\mathcal{N}P$ is precisely the same as the curvature coupling $\mathcal{N}R/8$.

For hypermultiplets, the coefficients of qq in the action of hypermultiplets (5.126) becomes

$$\begin{aligned} &-\left(C-\frac{1}{4}R+\frac{1}{8}F_{\mu\nu}(A)F^{\mu\nu}(A)-v_{\mu\nu}v^{\mu\nu}+5t_at_a\right)\\ &=\frac{R}{4}-\frac{P}{2}+\left(v_{\mu\nu}-\frac{1}{2}F_{\mu\nu}(A)\right)\left(v^{\mu\nu}-\frac{1}{2}F^{\mu\nu}(A)\right)=\frac{3R}{16}, \end{aligned} \quad (5.143)$$

which is consistent with the Weyl transformation for a 5d superconformal field theory on a flat space.

5.5.2 S^5

The supersymmetric gauge theory on a round five-sphere S^5 is constructed in [103] by using the 5d $\mathcal{N}=1$ supergravity. Let us confirm that our solution (5.50)-(5.61) includes it.

S^5 has the rotational $SO(6)$ symmetry. Let us impose the $SO(6)$ symmetry to the action (5.118). The action depends on the tensor field $v_{\mu\nu}$ and $F_{\mu\nu}(A)$ through the combination

$$v'_{\mu\nu}=v_{\mu\nu}-\frac{1}{2}F_{\mu\nu}(A). \quad (5.144)$$

Thus if $\mathcal{N}$ includes only dynamical scalar fields, we impose

$$v'_{\mu\nu}=0. \quad (5.145)$$

If $\mathcal{N}$ includes the scalar field in the central charge vector multiplet, the action also depends on $F_{\mu\nu}(A)$. Thus we impose

$$v_{\mu\nu}=F_{\mu\nu}(A)=0 \quad (5.146)$$

in this case.

In order to solve these conditions, first of all let us give the coordinates on S^5. S^5 is S^1 fibration over $\mathbb{CP}^2$. Thus we make the fifth direction as the S^1 fiber direction and the metric as

$$ds^2=ds^2_{\mathbb{CP}^2}+e^{\hat{\hat{5}}}e^{\hat{\hat{5}}},\quad ds^2_{\mathbb{CP}^2}=e^{\widehat{m}}e^{\widehat{m}},\quad e^{\hat{\hat{5}}}=r\left(dx^5+U\right), \quad (5.147)$$

where r is the radius of S^5. We take a local frame such that J^3 is the complex structure of $\mathbb{CP}^2$, and then the following relations hold.

$$S=r,\quad F(U)=\frac{2i}{r^2}J^3. \quad (5.148)$$

Due to the Kählerity, the holonomy of $\mathbb{CP}^2$ is $U(2) = Sp(1)_r \times U(1)_l$ where $U(1)_l \subset Sp(1)_l$ is the stabilizer subgroup of the complex structure J^3. The spin connection of $\mathbb{CP}^2$ commutes with J^3, and take the form

$$\omega^{\mathbb{CP}^2}_{\widehat{m}\widehat{n}} = \frac{3i}{2} U J^3_{\widehat{m}\widehat{n}} + (Sp(1)_r \text{ part}) . \tag{5.149}$$

Now we fixed the metric and the spin connection of the S^5. Then let us solve the conditions (5.145) or (5.146). If we impose the condition (5.145), the background fields satisfy

$$v_{\widehat{m}\widehat{5}} = 0, \quad F_{\widehat{m}\widehat{n}}(A) J^a_{\widehat{m}\widehat{n}} + 4t_a = \frac{2}{r}\delta^{a3}, \quad P = \frac{5}{2r^2}, \tag{5.150}$$

$$V^a_{\widehat{m}} = \frac{3i}{2} U_{\widehat{m}} \delta^{a3}, \quad V^a_{\widehat{5}} = -\frac{3i}{2r}\delta^{a3}. \tag{5.151}$$

The $Sp(1)_R$ gauge field is the flat connection and can be gauged away:

$$V^a = \frac{3i}{2}\delta^{a3} dx^5. \tag{5.152}$$

Although the condition (5.145) does not completely fix all of the background fields, the ambiguity does not affect the Lagrangians for vector and hypermultiplet.

If we impose the condition (5.146), or we consider a mass deformed theory, the background fields satisfy

$$F_{\widehat{m}\widehat{n}}(A) = 0, \quad t_a = \frac{1}{2r}\delta^{a3} \tag{5.153}$$

in addition to (5.150) and (5.151). This agrees with the background fields given in [103].

Although a superconformal theory on S^5 has sixteen supercharges, the supergravity formulation reproduces only a part of them. For the background specified by (5.150), (5.151) and (5.153), $\delta_Q \widetilde{\chi} = 0$ automatically holds and $\delta_Q \psi_\mu = 0$ gives

$$D_\mu \xi = -\frac{i}{2r}\gamma_\mu \tau_3 \xi. \tag{5.154}$$

This has eight supercharges belonging to the real representation of $(\mathbf{4},\mathbf{2}) + (\overline{\mathbf{4}},\mathbf{2})$ of $SO(6) \times Sp(1)_R$. The supersymmetry algebra can be obtained from (5.1) as

$$\begin{aligned}\{\delta_Q(\xi_1), \delta_Q(\xi_2)\} = {}& 2i\left(\xi_1 \gamma^{\widehat{\mu}} \xi_2\right) D_{\widehat{\mu}} + \delta_Z(\xi_1\xi_2) \\ & + \delta_M\left(\frac{2}{r}(\xi_1 \gamma_{\widehat{\mu}\widehat{\nu}} \tau_a \xi_2)\right) + \delta_U\left(-\frac{3}{r}(\xi_1\xi_2)\right)\end{aligned} \tag{5.155}$$

for two Grassmann-even spinor parameters ξ_1 and ξ_2 satisfying (5.154).

If we choose another background satisfying (5.150) and (5.151) we obtain a different Killing spinor equation. Although different backgrounds give the same superconformal Lagrangians, the number of supercharges which are realized by the supergravity in general depends on the choice of the background fields.

5.5.3 $S^4 \times \mathbb{R}$

The metric of $S^4 \times \mathbb{R}$ can be written as

$$ds^2 = ds^2_{S^4} + e^{\hat{5}} e^{\hat{5}}, \quad ds_{S^4} = e^{\widehat{m}} e^{\widehat{m}}, \quad e^{\hat{5}} = dx^5, \tag{5.156}$$

where we identify the fifth direction as $\mathbb{R}$ direction. From this metric, we read

$$S = 1, \quad U = 0. \tag{5.157}$$

Similarly to the previous subsection, let us impose the rotational symmetry $SO(5)$ of S^4 to the action. In the case of the prepotential including only dynamical scalar fields, the condition is $v'_{\mu\nu} = 0$. Solving this, the background fields satisfy

$$F_{\widehat{m}\widehat{n}}(A) J^a_{\widehat{m}\widehat{n}} + 4t_a = 0, \quad v^{\widehat{m}\hat{5}} = 0, \quad P = 0, \quad V^a = -\frac{1}{4}\omega^{(S^4)}_{\widehat{p}\widehat{q}} J^a_{\widehat{p}\widehat{q}}. \tag{5.158}$$

In the case of the prepotential including the non-dynamical scalar field M^0, by solving $v_{\mu\nu} = F_{\mu\nu}(A) = 0$, we obtain the conditions

$$F_{\widehat{m}\widehat{n}}(A) = t_a = 0 \tag{5.159}$$

in addition to (5.158). The latter background is given in [44].

These backgrounds are different from the one obtained by the Weyl transformation. The Weyl-transformed theory should have $P = R/4 = 3/r^2$ and flat V^a_μ. Actually it is impossible to realize a flat $Sp(1)_R$ connection in our solution because S^4 does not admit an almost complex structure. It is necessary to turn on a nontrivial $Sp(1)_R$ flux for the existence of $J^a_{\widehat{m}\widehat{n}}$. This result does not change even if we take a different direction as x^5 direction. Because an arbitrary rotation for S^4 has fixed points and R^μ is nowhere vanishing, we cannot take the x^5 direction within S^4 and R^μ necessarily has the component along $\mathbb{R}$. Thus if the topology of $\mathcal{B}$ is S^4 and if there exists $J^a_{\widehat{m}\widehat{n}}$, a nontrivial $Sp(1)_R$ flux is required. Therefore, we cannot realize the Weyl-transformed theory on $S^4 \times \mathbb{R}$ as a special case of our solution.

The reason for this impossibility may be the symplectic Majorana condition imposed on ξ. We have imposed this condition only for simplicity of the analysis, and it may be possible to realize $S^4 \times \mathbb{R}$ background without the $Sp(1)_R$ flux by relaxing this condition. In the 3d case, it is shown in [40] that $S^2 \times S^1$ backgrounds with and without $U(1)_R$ flux can be both realized in the framework of the 3d new minimal supergravity [104]. The nontrivial $U(1)_R$ flux is realized in the case where the Killing vector constructed by the Killing spinor has the direction within the S^1.

The other method for realizing the Weyl-transformed theory on $S^4 \times \mathbb{R}$ is using other supergravities. In [105], the solution of supersymmetric backgrounds are constructed by using 5d $\mathcal{N} = 1$ conformal supergravity [106, 107], while the symplectic Majorana condition for the Killing spinor is imposed. In this formalism, the Yang-Mills action is realized by turning on a scalar field in a background vector multiplet. In the case of $S^4 \times \mathbb{R}$, it is shown that the background scalar field cannot be constant. Therefore the supersymmetric Yang-Mills action with the constant coupling g_{YM} on $S^4 \times \mathbb{R}$ cannot be realized in this formalism.

Another supergravity is 5d $\mathcal{N}=2$ supergravity, which can be obtained from 6d $\mathcal{N}=(2,0)$ conformal supergravity by the dimensional reduction and gauge fixing [108]. By using this supergravity, the supersymmetric Yang-Mills theory on $S^4 \times \mathbb{R}$ is constructed in [109]. This gauge theory has eight supercharges. By the dimensional reduction, this theory yields so called $\mathcal{N}=2^*$ supersymmetric Yang-Mills theory on S^4 with a tuned mass parameter of an adjoint hypermultiplet.

5.5.4 $\Sigma \times S^3$

The last example we consider is $\Sigma \times S^3$, the direct product of a Riemann surface Σ and a three-sphere S^3 with radius r. A supersymmetric theory on this background is constructed in [110] for $\Sigma = \mathbb{R}^2$ and [111] for a general Σ. It can be reproduced by our solution as shown below.

We treat S^3 as the Hopf fibration over S^2, and identify the Hopf fiber direction with x^5. The metric of $\Sigma \times S^3$ is

$$ds^2 = ds^2_{\Sigma} + ds^2_{S^2} + e^{\widehat{5}} e^{\widehat{5}}, \tag{5.160}$$

$$ds^2_{\Sigma} = e^{\widehat{1}} e^{\widehat{1}} + e^{\widehat{2}} e^{\widehat{2}}, \quad ds^2_{S^2} = e^{\widehat{3}} e^{\widehat{3}} + e^{\widehat{4}} e^{\widehat{4}}, \quad e^{\widehat{5}} = r\left(dx^5 + U\right), \tag{5.161}$$

where U is a one-form on S^2. The following equations hold.

$$S = r, \quad \omega^{S^2}_{\widehat{34}} = 2U, \quad F(U) = \frac{2}{r^2} e^{\widehat{3}} \wedge e^{\widehat{4}}. \tag{5.162}$$

We can take a local frame so that J^3 is the complex structure of $\Sigma \times S^2$, which is the summation of the complex structure of Σ and S^2.

Let us impose the condition that the Lagrangian is invariant under the $SO(4)$ isometry of S^3. As in previous subsections, all components of $v'_{\mu\nu}$ should vanish except for $v'_{\widehat{12}}$ because of the $SO(4)$ invariance. This requires that the independent fields satisfy

$$v^{\widehat{m}\widehat{5}} = F_{\widehat{m}\widehat{n}}(A) J^a_{\widehat{m}\widehat{n}} + 4t_a = 0, \tag{5.163}$$

and then the non-vanishing component of $v'_{\mu\nu}$ is

$$v'_{\widehat{12}} = -\frac{i}{2r}. \tag{5.164}$$

The $Sp(1)_R$ connection is

$$V^a_{\widehat{m}=\widehat{1},\widehat{2}} = \frac{i}{2} \omega^{(\Sigma)}_{\widehat{m}\widehat{12}} \delta^{a3}, \quad V^a_{\widehat{m}=\widehat{3},\widehat{4}} = -\frac{i}{2} \omega^{(S^2)}_{\widehat{m}\widehat{34}} \delta^{a3}, \quad V^a_{\widehat{5}} = \frac{i}{r} \delta^{a3}. \tag{5.165}$$

The S^3 part of the connection (5.165)

$$V^{(S^3)a} = V^a_{\widehat{3}} e^{\widehat{3}} + V^a_{\widehat{4}} e^{\widehat{4}} + V^a_{\widehat{5}} e^{\widehat{5}} = i\delta^{a3} dx^5 \tag{5.166}$$

is flat and can be gauged away. This guarantees the $SO(4)$ invariance of the Lagrangian. The $Sp(1)_R$ connection on Σ is topologically twisted in such a way that a covariantly constant spinor on Σ exists.

In the case of the prepotential including the background scalar field M^0, the $SO(4)$ invariance further restricts $F_{\mu\nu}(A)$. Only non-vanishing components of $F_{\mu\nu}(A)$ should be $F_{\widehat{1}\widehat{2}}(A)$, which is related to t_3 by

$$t_a = \frac{i}{2} F_{\widehat{1}\widehat{2}}(A) \delta^{a3}. \tag{5.167}$$

If we take the prepotential $\mathcal{N} = M^0 \mathrm{tr}\,(M^2)\,/g^2_{\mathrm{YM}}$ with $M^0 = 1$, the Lagrangian (5.118) is given by

$$\begin{aligned} e^{-1}\mathcal{L}_V = \frac{1}{g^2_{\mathrm{YM}}} \mathrm{tr} \Bigg[& \frac{1}{4} F_{\mu\nu}(W) F^{\mu\nu} + \frac{1}{2} D_\mu M D^\mu M - 2i\Omega \not{D} \Omega + 2\Omega[M,\Omega] \\ & - \frac{i}{r} F_{\widehat{1}\widehat{2}}(A) M^2 + \left(F_{\widehat{1}\widehat{2}}(A) - \frac{2i}{r} \right) M F_{\widehat{1}\widehat{2}}(W) - 2i F_{\widehat{1}\widehat{2}}(A) M Y_3 \\ & + \left(\frac{i}{r} - F_{\widehat{1}\widehat{2}}(A) \right) \Omega \gamma_{\widehat{1}\widehat{2}} \Omega + i F_{\widehat{1}\widehat{2}}(A) \Omega \tau_3 \Omega - F_{\widehat{1}\widehat{2}}(A) [\mathrm{CS}]_3 \Bigg], \end{aligned} \tag{5.168}$$

where $[\mathrm{CS}]_3$ is the Chern-Simons term on S^3

$$[\mathrm{CS}]_3 = \epsilon^{\widehat{1}\widehat{2}\mu\nu\rho} \left(W_\mu \partial_\nu W_\rho + \left(W^3 \text{ term} \right) \right). \tag{5.169}$$

(5.168) gives a family of the supersymmetric Yang-Mills Lagrangian parametrized by $F_{\widehat{1}\widehat{2}}(A)$, which is a function on Σ. For the gauge invariance of the Chern-Simons term, the $U(1)_Z$ flux on Σ should be quantized as

$$\frac{1}{g^2_{\mathrm{YM}}} \int_\Sigma F(A) \in \frac{i}{4\pi} \mathbb{Z}. \tag{5.170}$$

The supersymmetric Yang-Mills Lagrangian in [111] is obtained by setting

$$F_{\widehat{1}\widehat{2}}(A) = -2it_3 = \frac{i}{r}. \tag{5.171}$$

Chapter 6

Conclusions

Supersymmetric field theories on curved spaces play important roles in the exact computations of physical quantities and tests of dualities. A systematic construction of the theories can be performed by using supergravities. Starting from each supergravity, we can obtain conditions for background fields preserving some supercharges and study parameter (in)dependence of the partition function. By these results,

- we can easily construct supersymmetric field theories on curved manifolds only by using the solutions of supersymmetric backgrounds, and
- from the parameter (in)dependence of the partition function, we can find which backgrounds give different exact results from known ones.

In this thesis, we reviewed the analysis for the 4d $\mathcal{N}=1$ and 3d $\mathcal{N}=2$ supersymmetric backgrounds and studied the 5d $\mathcal{N}=1$ supersymmetric backgrounds.

In Chapter 2 we reviewed the 4d $\mathcal{N}=1$ supersymmetric backgrounds. Starting from the 4d new minimal supergravity, the requirement that at least one supercharge is preserved in the rigid limit yields Hermitian $\mathcal{M}_4$. Background fields can be determined, up to some functional degrees of freedom. Moreover, we reviewed that only a small number of degrees of freedom contributes to the partition function, by seeing whether the deformations of the backgrounds give Q-exact deformation of the action.

In Chapter 3 we reviewed similar analysis for the 3d $\mathcal{N}=2$ supersymmetric backgrounds. $\mathcal{M}_3$ must be equipped with a mathematical structure called the transversely holomorphic foliation. Similarly to the 4d case, only a small number of degrees of freedom contributes to the partition function.

In Chapter 4, we reviewed various properties of 5d supersymmetric field theories from the viewpoints of quantum field theories and superstring/M-theory. These properties can be checked by partition functions for 5d supersymmetric field theories on curved space.

We studied 5d supersymmetric backgrounds in Chapter 5. We solved two spinorial equations and obtained the condition for the background fields. As a result, we found that we can construct 5d supersymmetric field theories on a curved manifold only if the manifold has at least one isometry. After that, we showed that all local degrees of freedom of background fields do not affect the partition function due to the Q-exactness and gauge invariance. Finally we constructed 5d supersymmetric field theories on a few simple manifolds.

In the analysis of Q-exact deformations in 5d, we focused only on the local arguments. To obtain a full background dependence of the partition function, we should carefully discuss global deformations, in which some classes of cohomology may appear, similarly to 4d and 3d.

An important feature of the solution of 5d supersymmetric backgrounds is the existence of the isometry. This suggests a close relation to 4d $\mathcal{N} = 2$ supersymmetric backgrounds. It would be interesting to study such backgrounds from the viewpoint of 4d $\mathcal{N} = 2$ supergravity [112, 113].

The solution obtained in this thesis does not contain $S^4 \times \mathbb{R}$ without the R-symmetry flux. It may be possible to include such background by relaxing the symplectic Majorana condition, which is imposed in this thesis for simplicity.

Another interesting direction is to study a relation between 6d $\mathcal{N} = (2, 0)$ theories and 5d $\mathcal{N} = 2$ theories, as mentioned in Section 4.6. In [108], a method to obtain 5d $\mathcal{N} = 2$ theories on curved backgrounds from 6d $\mathcal{N} = (2, 0)$ conformal supergravity [114] is given. This method consists of the following procedures: considering an Abelian tensor multiplet on an S^1 fibration over a 5d manifold, the dimensional reduction, non-Abelianization and an extension to an off-shell formulation. When we consider a T^2 fibration over a 4d manifold M_4, we have two S^1 direction to reduce to obtain 5d theories. If the conjectured relation between the 6d $\mathcal{N} = (2, 0)$ theory and the 5d $\mathcal{N} = 2$ theory is true, the partition functions for two 5d theories should be same, as shown in Figure 6.1. It would be interesting to compute and compare the 5d partition functions on such two manifolds.

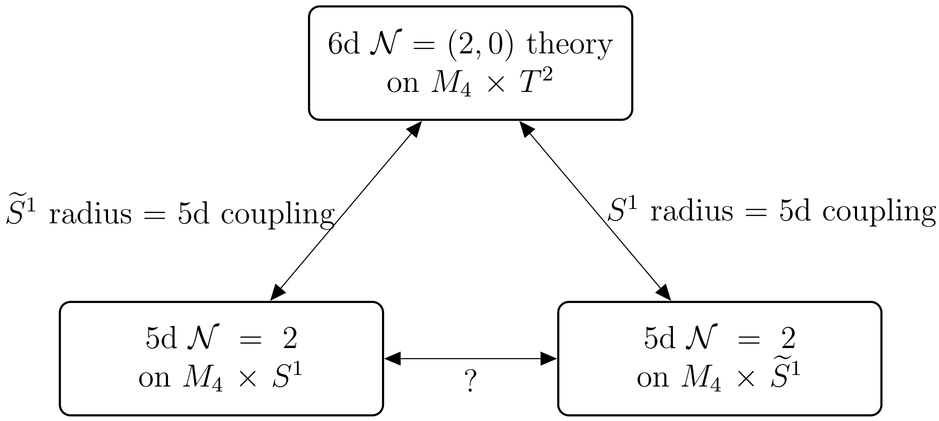

Figure 6.1: Relation between 6d $\mathcal{N} = (2,0)$ theory and 5d $\mathcal{N} = 2$ supersymmetric gauge theories. "$\times$" can be a nontrivial fibrations. This is a generalization of Figure 4.6.

Appendix A

Notations and conventions

A.1 Indices in 5d

Notation of indices is as follows:

- $\mu, \nu, \ldots = 1, \ldots, 5$: 5d spacetime indices,
- $\widehat{\mu}, \widehat{\nu}, \ldots, \widehat{1}, \ldots, \widehat{5}$: 5d local orthonormal indices,
- $\alpha, \beta, \ldots = 1, \ldots, 4$: $Sp(2)_L \sim SO(5)_L$ spinor indices
- $I, J, \ldots = 1, 2$: $Sp(1)_R \sim SU(2)_R$ doublet indices, and
- $a, b, \ldots = 1, 2, 3$: $Sp(1)_R \sim SU(2)_R$ triplet indices.

The 5d antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma\lambda}$ is defined by

$$\gamma^{\mu\nu\rho\sigma\lambda} = \epsilon^{\mu\nu\rho\sigma\lambda}\mathbf{1}_4. \tag{A.1}$$

$\alpha, \beta, \ldots$ and $I, J, \ldots$ are raised or lowered by invariant antisymmetric tensors $C_{\alpha\beta} = C^{\alpha\beta}$ and $\epsilon_{IJ} = \epsilon^{IJ}$, respectively. They satisfy

$$C^{\alpha\gamma}C_{\beta\gamma} = \delta^\alpha_\beta, \quad \epsilon^{IK}\epsilon_{JK} = \delta^I_J. \tag{A.2}$$

We often omit the contractions for these indices. In the case, we adopt the NW-SE convention. For example,

$$\eta\chi \equiv \eta^{\alpha I}\chi_{\alpha I} = C^{\alpha\beta}\epsilon^{IJ}\eta_{\beta J}\chi_{\alpha I}. \tag{A.3}$$

For a rank n antisymmetric tensor $A_{\mu_1\cdots\mu_n}$, we define

$$\not{A} \equiv \frac{1}{n!}A_{\mu_1\cdots\mu_n}\gamma^{\mu_1\cdots\mu_n}. \tag{A.4}$$

For $Sp(1)_R$ triplet fields we use the matrix notation

$$t_I{}^J \equiv t_a\left(\tau_a\right)_I{}^J, \tag{A.5}$$

where τ_a is the Pauli matrix.

A.2 Our convention from Kugo-Ohashi convention

Kugo-Ohashi's papers [100, 101] adopt the mostly-minus metric in Lorenzian spacetime: $\eta_{\widehat{\mu}\widehat{\nu}} = \mathrm{diag}(+----)$. For our purpose of the localization computation, we have to obtain the formulation based on the Euclidean signature $\eta_{\widehat{\mu}\widehat{\nu}} = \mathrm{diag}(+++++)$. For this, we should flip the overall sign of $\eta_{\widehat{\mu}\widehat{\nu}}$ and Wick rotate. In this section, we show how to move [100, 101] to our convention.

We do not change the definition of all fields except for the spin connection

$$\omega_{\mu\widehat{\nu}\widehat{\rho}} \to -\omega_{\mu\widehat{\nu}\widehat{\rho}}. \tag{A.6}$$

For the Dirac matrices, satisfying $\{\gamma_{\widehat{\mu}}, \gamma_{\widehat{\nu}}\} = 2\eta_{\widehat{\mu}\widehat{\nu}}$, we change them as following:

$$\gamma^{\widehat{\mu}} \to i\gamma^{\widehat{\mu}}, \quad \gamma_{\widehat{\mu}} \to -i\gamma_{\widehat{\mu}}. \tag{A.7}$$

In both conventions, an antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma\lambda}$ is defined by (A.1). In order to maintain (A.1) after the flip of $\eta_{\widehat{\mu}\widehat{\nu}}$, we should replace $\epsilon^{\mu\nu\rho\sigma\lambda} \to i\epsilon^{\mu\nu\rho\sigma\lambda}$. However, by the Wick rotation, we also replace $\epsilon^{\mu\nu\rho\sigma\lambda} \to -i\epsilon^{\mu\nu\rho\sigma\lambda}$. After all, we do not change $\epsilon^{\mu\nu\rho\sigma\lambda}$.

For each spinor product, we replace $i\overline{\psi}\chi \to \psi\chi$. This is simply the change of the notation.

Appendix B

Useful identities and spinor computations

B.1 Fierz identities

Let ψ and ϕ be Grassmann-even spinors in each dimension. If both spinors are Grassmann-odd, we flip the all signs in the right-hand sides.

4d Fierz identity: for the left-handed spinors ψ and ϕ,

$$\psi_\alpha \phi^\beta = \frac{1}{2}(\phi\psi)\delta_\alpha{}^\beta - \frac{1}{2}\left(\phi\sigma_{\mu\nu}\psi\right)\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta . \tag{B.1}$$

3d Fierz identity:

$$\psi_\alpha \phi^\beta = \frac{1}{2}(\phi\psi)\delta_\alpha{}^\beta + \frac{1}{2}\left(\phi\gamma_\mu\psi\right)\left(\gamma^\mu\right)_\alpha{}^\beta . \tag{B.2}$$

5d Fierz identity:

$$\begin{aligned}\psi_{I\alpha}\phi^{J\beta} = &\frac{1}{8}\phi\psi\delta_I{}^J\delta_\alpha{}^\beta + \frac{1}{8}\phi\gamma_\mu\psi\delta_I{}^J\left(\gamma^\mu\right)_\alpha{}^\beta - \frac{1}{16}\phi\gamma_{\mu\nu}\psi\delta_I{}^J\left(\gamma^{\mu\nu}\right)_\alpha{}^\beta \\ &+ \frac{1}{8}\phi\tau^a\psi\left(\tau^a\right)_I{}^J\delta_\alpha{}^\beta + \frac{1}{8}\phi\tau^a\gamma_\mu\psi\left(\tau^a\right)_I{}^J\left(\gamma^\mu\right)_\alpha{}^\beta - \frac{1}{16}\phi\tau^a\gamma_{\mu\nu}\psi\left(\tau^a\right)_I{}^J\left(\gamma^{\mu\nu}\right)_\alpha{}^\beta .\end{aligned} \tag{B.3}$$

B.2 Computation of (2.26)

Here we show the computation of (2.26), as an example of spinor computations in 4d. By Fierz identity (B.1),

$$\begin{aligned}J^\mu{}_\nu J^\nu{}_\rho &= -\frac{4}{|\xi|^4}\left(\xi^\dagger\sigma^\mu{}_\nu\xi\right)\left(\xi^\dagger\sigma^\nu{}_\rho\xi\right) \\ &= -\frac{2}{|\xi|^4}\left[\left(\xi^\dagger\xi\right)\left(\xi^\dagger\sigma^\mu{}_\nu\sigma^\nu{}_\rho\xi\right) + \left(\xi^\dagger\sigma^\tau{}_\lambda\xi\right)\left(\xi^\dagger\sigma^\mu{}_\nu\sigma^\lambda{}_\tau\sigma^\nu{}_\rho\xi\right)\right] .\end{aligned} \tag{B.4}$$

By the computation of the sigma matrices, we obtain

$$\sigma_{\mu\nu}\sigma_{\rho\lambda} = -\frac{1}{4}\epsilon_{\mu\nu\rho\lambda} + 2\delta_{[\mu[\rho}\sigma_{\nu]\lambda]} - \frac{1}{2}\delta_{\mu[\rho}\delta_{\nu\lambda]}, \tag{B.5}$$

$$\sigma_{\mu\nu}\sigma^{\nu}{}_{\rho} = \frac{3}{4}\delta_{\mu\rho} - \sigma_{\mu\rho}, \tag{B.6}$$

$$\sigma_{\mu\nu}\sigma_{\lambda\tau}\sigma^{\nu}{}_{\rho} = -\frac{1}{4}\epsilon_{\mu\rho\tau\lambda} - \frac{1}{4}g_{\mu\rho}\sigma_{\tau\lambda} - \frac{1}{2}g_{\mu[\tau}g_{\lambda]\rho}. \tag{B.7}$$

In the second term of the right hand side in (B.5), the pairs (μ, ν) and (ρ, λ) are anti-symmetrized, respectively. By using these formulas,

$$J^{\mu}{}_{\nu}J^{\nu}{}_{\rho} = -\frac{3}{2}\delta^{\mu}{}_{\rho} - \frac{i}{2}J^{\mu}{}_{\rho} + \frac{i}{4}\epsilon^{\mu}{}_{\rho\tau\lambda}J^{\tau\lambda} - \frac{1}{8}\delta^{\mu}{}_{\rho}J^{\tau}{}_{\lambda}J^{\lambda}{}_{\tau}. \tag{B.8}$$

The second and third terms cancel due to the self-duality of $J_{\mu\nu}$. Substituting ρ into μ and summing over μ, we obtain $J^{\mu}{}_{\nu}J^{\nu}{}_{\mu} = -4$. Therefore (2.26) holds.

Appendix C

Mathematical facts

C.1 Differential forms and de Rham cohomology

A space spanned by differential one-forms is defined as the dual vector space of a tangent space, spanned by tangent vectors $X = X^\mu \partial_\mu$. The tangent space on a point p in a manifold $\mathcal{M}$ is denoted by $T_p\mathcal{M}$, while the dual vector space, called the cotangent space on $p \in \mathcal{M}$ is denoted by $T_p^*\mathcal{M}$. A one-form ω^1 is represented as

$$\omega^1 = \omega_\mu dx^\mu. \tag{C.1}$$

A (q,r)-tensor is defined as a map

$$T : \otimes^q T_p^*\mathcal{M} \otimes^r T_p\mathcal{M} \to \mathbb{R} \tag{C.2}$$

as

$$T = T^{\mu_1 \cdots \mu_q}{}_{\nu_1 \cdots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_q}} dx^{\nu_1} \cdots dx^{\nu_r}. \tag{C.3}$$

An r-form is defined as a completely antisymmetric $(0,r)$-tensor. The outer product of r one-forms is defined by

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \equiv \sum_{P \in S_r} \mathrm{sgn}(P) dx^{\mu_{P(1)}} \otimes \cdots \otimes dx^{\mu_{P(r)}}, \tag{C.4}$$

where S_r is the r-th permutation group and

$$\mathrm{sgn}(P) = \begin{cases} +1 & (P : \text{even}) \\ -1 & (P : \text{odd}) \end{cases}. \tag{C.5}$$

An r-form ω^r is represented as

$$\omega^r = \frac{1}{r!}\omega_{\mu_1\cdots\mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}. \tag{C.6}$$

The Hodge dual $*$ maps an r-form to a $(D-r)$-form as

$$*\omega^r = \frac{1}{r!(D-r)!}\omega_{\mu_1\cdots\mu_r} \epsilon^{\mu_1\cdots\mu_r}{}_{\nu_{r+1}\cdots\nu_D} dx^{\nu_{r+1}} \wedge \cdots \wedge dx^{\nu_D}, \tag{C.7}$$

where D is the dimension of the manifold $\mathcal{M}$.

The exterior derivative d maps an r-form to a $(r+1)$-form as

$$d\omega^r = \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\cdots\mu_r}\right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}. \tag{C.8}$$

Because every differential form is an antisymmetric tensor, $d^2 = 0$.

If $\omega^r = d\omega^{r-1}$ with a globally-defined $(r-1)$-form ω^{r-1}, trivially $d\omega^r = 0$. However, if $d\omega^r = 0$ we cannot conclude $\omega^r = d\omega^{r-1}$ in general. In order to measure this nontriviality, we define the r-th de Rham cohomology as

$$H^r(\mathcal{M}) = \left\{\omega^r | d\omega^r = 0,\ \omega^r \sim \omega^r + d\omega^{r-1}\right\}, \tag{C.9}$$

where $\omega^r \sim \omega^r + d\omega^{r-1}$ means the identification between ω^r and $\omega^r + d\omega^{r-1}$.

C.2 Almost complex structure

Almost complex manifold is defined as follows. If there exists a $(1,1)$-tensor J satisfying $J^2 = -\mathbf{1}$ on each $p \in \mathcal{M}$, $\mathcal{M}$ is an almost complex manifold. The $(1,1)$-tensor J is called the almost complex structure.

A tangent space and a cotangent space can be decomposed into two subspaces by eigenvalues of J, respectively. A vector field X or a one-form ω^1 is holomorphic if

$$J^\mu{}_\nu X^\nu = iX^\mu, \quad \omega_\mu J^\mu{}_\nu = i\omega_\nu, \tag{C.10}$$

while X or ω^1 is anti-holomorphic if

$$J^\mu{}_\nu X^\nu = -iX^\mu, \quad \omega_\mu J^\mu{}_\nu = -i\omega_\nu. \tag{C.11}$$

Let X and Y be arbitrary holomorphic vector fields on an almost complex manifold $\mathcal{M}$. If the Lie commutator

$$[X, Y]^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu \tag{C.12}$$

is also holomorphic, the almost complex structure J is called integrable.

For an almost complex manifold, we define the Nijenhuis tensor

$$N^{\mu}{}_{\nu\rho} = J^{\lambda}{}_{\nu}\nabla_{\lambda}J^{\mu}{}_{\rho} - J^{\lambda}{}_{\rho}\nabla_{\lambda}J^{\mu}{}_{\nu} - J^{\mu}{}_{\lambda}\nabla_{\nu}J^{\lambda}{}_{\rho} + J^{\mu}{}_{\lambda}\nabla_{\rho}J^{\lambda}{}_{\nu}. \tag{C.13}$$

The almost complex structure J is integrable if and only if the Nijenhuis tensor vanishes.

If there exists a globally-defined $(1,1)$-tensor J satisfying $J^2 = -\mathbf{1}$, $\mathcal{M}$ is a complex manifold with an almost complex structure J. If an almost complex manifold $\mathcal{M}$ with an integrable almost complex structure J, $\mathcal{M}$ is a complex manifold with an almost complex structure J.

For a complex manifold, we can locally take holomorphic coordinates $\left(z^i, \overline{z}^{\overline{i}}\right)$ such that

$$J = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}. \tag{C.14}$$

An (r,s)-form is defined as an $(r+s)$-form which belongs to a direct product of r holomorphic and s anti-holomorphic cotangent spaces. A (r,s)-form $\omega^{r,s}$ is represented as

$$\omega^{r,s} = \frac{1}{r!s!}\omega_{i_1\cdots i_s\overline{j}_1\cdots\overline{j}_s} dz^{i_1}\wedge\cdots\wedge dz^{i_r}\wedge d\overline{z}^{\overline{j}_1}\wedge\cdots\wedge d\overline{z}^{\overline{j}_s}. \tag{C.15}$$

The exterior derivative $d\omega^{r,s}$ includes both $(r+1,s)$-forms and $(r,s+1)$-forms. Hence we can define new operators which act on $\omega^{r,s}$ as

$$\partial\omega^{r,s} = d\omega^{r,s}\big|_{(r+1,s)}, \quad \overline{\partial}\omega^{r,s} = d\omega^{r,s}\big|_{(r,s+1)}, \quad d = \partial + \overline{\partial}. \tag{C.16}$$

These operators ∂ and $\overline{\partial}$ are called the Dolbeault operators. Because $d^2 = 0$, $\partial^2 = \overline{\partial}^2 = 0$ holds. Therefore we can define

$$H^{r,s}_{\partial}(\mathcal{M}) = \left\{\omega^{r,s}|\partial\omega^{r,s} = 0,\ \omega^{r,s} \sim \omega^{r,s} + \partial\omega^{r-1,s}\right\}, \tag{C.17}$$

$$H^{r,s}_{\overline{\partial}}(\mathcal{M}) = \left\{\omega^{r,s}|\overline{\partial}\omega^{r,s} = 0,\ \omega^{r,s} \sim \omega^{r,s} + \overline{\partial}\omega^{r,s-1}\right\}. \tag{C.18}$$

These are called the Dolbeault cohomology.

C.3 Almost contact metric structure

C.3.1 Basic definitions

Almost contact structure is defined on an oriented 3d manifold $\mathcal{M}$ as a triple (η, ζ, Φ) with a nowhere vanishing one-form η_μ, a vector field ζ^μ and a $(1,1)$-tensor $\Phi^{\mu}{}_{\nu}$ satisfying

$$\eta_\mu\zeta^\mu = 1, \quad \Phi^{\mu}{}_{\rho}\Phi^{\rho}{}_{\nu} = -\delta^{\mu}{}_{\nu} + \zeta^\mu\eta_\nu. \tag{C.19}$$

For a vector field X^μ satisfying $\Phi^\mu{}_\nu X^\nu = 0$, then X^μ is proportional to ζ^μ. Similarly, a one-form Ω_μ satisfying $\Omega_\mu \Phi^\mu{}_\nu = 0$ is proportional to η_μ.

Vector fields orthogonal to ζ^μ define a subspace $\mathcal{D}$ of the tangent space of $\mathcal{M}$. Because $\Phi^2|_{\mathcal{D}} = -1$, $\Phi|_{\mathcal{D}}$ defines an almost complex structure on $\mathcal{D}$. Therefore Φ is a 3d analogue of an almost complex structure. Similar to an almost complex structure, we can define holomorphicity and anti-holomorphicity for vector fields and one-forms. A vector field X or a one-form ω^1 is holomorphic if

$$\Phi^\mu{}_\nu X^\nu = iX^\mu, \quad \omega_\mu \Phi^\mu{}_\nu = i\omega_\nu, \tag{C.20}$$

while X or ω^1 is anti-holomorphic if

$$\Phi^\mu{}_\nu X^\nu = -iX^\mu, \quad \omega_\mu \Phi^\mu{}_\nu = -i\omega_\nu. \tag{C.21}$$

If a manifold $\mathcal{M}$ is equipped with a Riemann metric $g_{\mu\nu}$ and the following equations hold, the almost contact structure (η, ζ, Φ) is compatible with $g_{\mu\nu}$:

$$\zeta^\mu = \eta^\mu, \quad g_{\rho\lambda}\Phi^\rho{}_\mu \Phi^\lambda{}_\nu = g_{\mu\nu} - \eta_\mu \eta_\nu. \tag{C.22}$$

These define an almost contact metric structure on $\mathcal{M}$. For an oriented manifold, we can take an almost contact structure by taking $\zeta^\mu = \eta^\mu$ and $\Phi^\mu{}_\nu = \epsilon^\mu{}_{\nu\rho}\eta^\rho$. Therefore we can always take an almost contact metric structure on an oriented 3d Riemannian manifold. This is characterized by a metric $g_{\mu\nu}$, a nowhere vanishing one-form η_μ and an orientation.

C.3.2 Integrability condition (3.30)

Here we analyze the integrability condition (3.30)

$$\Phi^\mu{}_\nu \mathcal{L}_\zeta \Phi^\nu{}_\rho = 0. \tag{C.23}$$

Let us show that for a vector ζ^μ defining an almost contact structure and a holomorphic one-form Ω_μ, there exists a local coordinates $(\tau, z, \overline{z})$ such that $\zeta = \partial_\tau$ and $\Omega = \Omega_z dz$. Since ζ is nowhere vanishing, we can take coordinates $x^1 = \tau$, x^2, x^3 such that $\zeta = \partial_\tau$. Because Ω is holomorphic, $\zeta^\mu \Omega_\mu = 0$ and Ω can written by $\Omega = \Omega_2 dx^2 + \Omega_3 dx^3$. From the holomorphicity of Ω and (C.23), $\mathcal{L}_\zeta \Omega_\mu$ is also holomorphic one-form:

$$\left(\mathcal{L}_\zeta \Omega_\mu\right) \Phi^\mu{}_\nu = i\mathcal{L}_\zeta \Omega_\nu. \tag{C.24}$$

Let us define $\rho = \Omega_2/\Omega_3$. Since holomorphic one-forms span a 1d space, ρ is a function depending only on a choice of $\Phi^\mu{}_\nu$. The holomorphicity of $\mathcal{L}_\zeta \Omega_\mu$ shows $\partial_\tau \rho = 0$. Taking new coordinates $(\tau, z, \overline{z})$ with complex z which are related with x^2 and x^3 by $x^2 = f(z, \overline{z})$ and $x^3 = g(z, \overline{z})$,

$$\begin{aligned} \Omega &= \Omega_3 \left(\rho dx^2 + dx^3\right) \\ &= \Omega_3 \left(\left(\rho \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \right) dz + \left(\rho \frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}} \right) d\overline{z} \right). \end{aligned} \tag{C.25}$$

Hence, in order to take $\Omega = \Omega_z dz$, we choose functions f and g satisfying

$$\rho \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} = \lambda (z, \overline{z}), \quad \rho \frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}} = 0 \tag{C.26}$$

for a nonzero function $\lambda (z, \overline{z})$. This differential equations (C.26) always have a solution. Therefore we can take $\Omega = \Omega_z dz$. The relation between two overlapping local coordinates $(\tau, z, \overline{z})$ and $(\tau', z', \overline{z}')$ is obtained from

$$\zeta = \partial_\tau = \partial_{\tau'}, \quad \Omega = \Omega_z dz = \Omega_{z'} dz' \tag{C.27}$$

as

$$\tau' = \tau + t (z, \overline{z}), \quad z' = f(z), \tag{C.28}$$

where $t (z, \overline{z})$ is a real function and $f(z)$ is a holomorphic function.

A compatible metric with an almost contact metric structure can be written by a complex function $h (\tau, z, \overline{z})$ and a real function $c (\tau, z, \overline{z})$ as

$$ds^2 = \left(d\tau + h (\tau, z, \overline{z}) \, dz + \overline{h} (\tau, z, \overline{z}) \, d\overline{z}\right)^2 + c (\tau, z, \overline{z})^2 \, dz d\overline{z}. \tag{C.29}$$

Appendix D

Supersymmetric backgrounds from 4d $\mathcal{N} = 1$ old minimal supergravity

There is another formulation of 4d $\mathcal{N} = 1$ off-shell supergravity, called the old minimal supergravity [56, 57]. In this theory, a supercurrent multiplet coupled with this supergravity is called the Ferrara-Zumino supercurrent multiplet [116], which consists of

$$T_{\mu\nu}, \quad S_{\mu\alpha}, \quad \overline{S}^{\dot{\alpha}}_{\mu}, \quad j_{\mu}, \quad x, \quad \overline{x}, \tag{D.1}$$

where first three of them are similar notations as the $\mathcal{R}$-multiplet in the new minimal supergravity. j_μ is an axial vector current and x and $\overline{x}$ are complex scalars. Note that there is no $U(1)_R$ symmetry current.

The corresponding supergravity multiplet consists of

$$g_{\mu\nu}, \quad \psi_{\mu\alpha}, \quad \overline{\psi}^{\dot{\alpha}}_{\mu}, \quad b^{\mu}, \quad M, \quad \overline{M}, \tag{D.2}$$

where b^μ is a vector field and M and $\overline{M}$ are scalar fields. The supersymmetric transformations for the gravitinos are

$$\delta_Q \psi_\mu = -2\nabla_\mu \xi + \frac{i}{3} M \sigma_\mu \overline{\xi} + \frac{2i}{3} b_\mu \xi + \frac{2i}{3} b^\nu \sigma_{\mu\nu} \xi, \tag{D.3}$$

$$\delta_Q \overline{\psi}_\mu = -2\nabla_\mu \overline{\xi} + \frac{i}{3} \overline{M} \overline{\sigma}_\mu \xi - \frac{2i}{3} b_\mu \overline{\xi} - \frac{2i}{3} b^\nu \overline{\sigma}_{\mu\nu} \overline{\xi}. \tag{D.4}$$

From the algebra of the supergravity transformations, the following relations hold:

$$\delta_Q^2 = 2i\delta_K, \quad K^\mu = \xi\sigma^\mu\overline{\xi}, \quad [\delta_K, \delta_Q] = 0, \tag{D.5}$$

where $\delta_K = \mathcal{L}_K$ is the Lie derivative along K_μ.

Similarly to Section 2.3, we would like to consider the condition such that there is as least one nontrivial solution of $\delta_Q\psi_\mu = \delta_Q\overline{\psi}_\mu = 0$ [38]. Before that, we should note a significant difference from the case of the new minimal supergravity. In the supersymmetric transformation for the gravitinos (2.21) and (2.22) in the new minimal supergravity, the left-handed and right-handed components of spinors are completely separated. Thus a combination of supersymmetry transformation parameters $\left(\xi,\overline{\xi}\right)$ is a solution of $\delta_Q\psi_\mu = \delta_Q\overline{\psi}_\mu = 0$ if and only if both $(\xi, 0)$ and $\left(0,\overline{\xi}\right)$ are solutions. In the old minimal supergravity, on each right hand side of (D.3) and (D.4), there are both ξ and $\overline{\xi}$. Therefore, it can be happen that $\left(\xi,\overline{\xi}\right)$ is a solution even if $(\xi, 0)$ and $\left(0,\overline{\xi}\right)$ are not solutions. In order to obtain a condition that there is at least one supersymmetry, we should consider the following two cases:

1. Only a left(right)-handed supersymmetric transforamtion parameter ξ ($\overline{\xi}$) has a nontrivial value while another parameter $\overline{\xi}$ (ξ) vanishes, and
2. A nontrivial combination $\left(\xi,\overline{\xi}\right)$ is a solution of $\delta_Q\psi_\mu = \delta_Q\overline{\psi}_\mu = 0$.

D.1 The case of vanishing $\overline{\xi}$

If $\overline{\xi}$ vanishes, $\delta_Q\psi_\mu = \delta_Q\overline{\psi}_\mu = 0$ becomes

$$\nabla_\mu\xi = \frac{i}{3}b_\mu\xi + \frac{i}{3}b^\nu\sigma_{\mu\nu}\xi, \tag{D.6}$$

$$0 = \overline{M}. \tag{D.7}$$

Hence M can take an arbitrary value. Since (D.6) is a homogeneous first-order differential equation, the solution ξ is nowhere vanishing. Similar to the case of the new minimal supergravity, we can define nowhere vanishing bilinears $|\xi|^2$, $J_{\mu\nu}$ and $P_{\mu\nu}$. $|\xi|^2$ is a positive scalar, $J^\mu{}_\nu$ is an almost complex structure and $P_{\mu\nu}$ is an anti-holomorphic two-form with respect to $J^\mu{}_\nu$.

Similar to the previous analysis, by using (D.6), we can show that $J^\mu{}_\nu$ is integrable and the 4d manifold is Hermitian. We can introduce local holomorphic coordinates z^i and take the complex structure as (2.38). Differentiating the complex structure and using (D.6),

$$\nabla_\mu J^\mu{}_\nu = \frac{1}{3}\left(b_\nu + b^*_\nu\right) - \frac{i}{3}\left(b_\mu - b^*_\mu\right) J^\mu{}_\nu. \tag{D.8}$$

From this equation, the holomorphic part of b_μ is determined as

$$b_\mu = \frac{1}{2}\left(2g_{\mu\nu} + iJ_{\mu\nu}\right)\nabla_\rho J^{\rho\nu} + b^c_\mu, \quad J_\mu{}^\nu b^c_\nu = ib^c_\mu. \tag{D.9}$$

By using the Chern connection defined in Section 2.3, (D.6) can be rewritten as

$$\left(\nabla^c_\mu - \frac{i}{2}b^c_\mu\right)\xi = 0. \tag{D.10}$$

From this, $p = P_{\overline{12}}$ satisfies

$$\left(\nabla^c_\mu - ib^c_\mu\right)p = 0. \tag{D.11}$$

This defines b^c_μ as

$$b^c_\mu = -i\nabla^c_\mu \log p. \tag{D.12}$$

Because b^c_μ is anti-holomorphic,

$$\partial_i p = 0. \tag{D.13}$$

By using (2.52), $b^c_{\bar{i}}$ is expressed as

$$b^c_{\bar{i}} = -i\partial_{\bar{i}} \log\left(pg^{-1/2}\right). \tag{D.14}$$

To summarize, the existence of the solution $\delta_Q\psi_\mu = \delta_Q\overline{\psi}_\mu = 0$ with vanishing $\overline{\xi}$ yields the existence of an integrable complex structure. The 4d manifold is Hermitian. In this situation, M is arbitrary, $\overline{M} = 0$ and b_μ takes the form as in (D.9) and (D.14).

Conversely, we can show that there is at least one solution ξ when the above background fields are given. If we take the vielbein as (2.56) and (2.57), we can find that (2.58) is a solution.

The global structure of the solution should also be considered [38]. For example, let us consider $\mathbb{R}^3 \times S^1$, which can be constructed from the flat space $\mathbb{C}^2$ by the identification $z \sim z + 2\pi i$ for one of the holomorphic coordinates. If we take $p = 1$, the spinor ξ is constant and the periodic boundary condition is realized. On the other hand, if we take $p = e^{\overline{z}}$, ξ changes its sign by $z \to z + 2\pi i$ because the solution (2.58) includes $\sqrt{p}$. Thus the anti-periodic boundary condition is realized and b_μ has a nontrivial behavior.

If the 4d manifold is compact, the existence of the nowhere vanishing anti-holomorphic two-form p gives restriction for the 4d manifold. In the Enriques-Kodaira classification, only tori, K3 surfaces and primary Kodaira surfaces have such property [115]. These do not include the Hopf surface $S^3 \times S^1$, while its non-compact version $S^3 \times \mathbb{R}$ is included.

D.2 Nontrivial ξ and $\overline{\xi}$

Let us consider the case that a nontrivial combination $\left(\xi, \overline{\xi}\right)$ is a solution of $\delta_Q\psi_\mu = \delta_Q\overline{\psi}_\mu = 0$. Because these equations are first order differential equations, both ξ and $\overline{\xi}$ cannot vanish at a point simultaneously. At first we introduce spinor bilinears, in addition to the ones (2.25)-(2.27) only from ξ. Bilinears which consists of the only $\overline{\xi}$ are

$$\left|\overline{\xi}\right|^2 = \overline{\xi}^\dagger \overline{\xi}, \tag{D.15}$$

$$\overline{J}_{\mu\nu} = \frac{2i}{\left|\overline{\xi}\right|^2} \overline{\xi}^\dagger \overline{\sigma}_{\mu\nu} \overline{\xi}, \tag{D.16}$$

$$\overline{P}_{\mu\nu} = \overline{\xi} \overline{\sigma}_{\mu\nu} \overline{\xi}. \tag{D.17}$$

$\left|\overline{\xi}\right|^2$ is a non-negative scalar. $\overline{J}^\mu{}_\nu$ is anti-self-dual and another almost complex structure, if $\overline{\xi}$ is nowhere vanishing:

$$\overline{J}^\mu{}_\nu \overline{J}^\nu{}_\rho = -\delta^\mu{}_\rho. \tag{D.18}$$

A vector U^μ is holomorphic with respect to $\overline{J}^\mu{}_\nu$ if and only if $U^\mu \sigma_\mu \overline{\xi} = 0$. $\overline{P}_{\mu\nu}$ is anti-holomorphic two-form with respect to $\overline{J}^\mu{}_\nu$:

$$\overline{J}_\mu{}^\nu \overline{P}_{\nu\rho} = i \overline{P}_{\mu\rho}. \tag{D.19}$$

We can also construct the complex vectors by using both ξ and $\overline{\xi}$ as

$$K^\mu = \xi \sigma^\mu \overline{\xi}, \quad X^\mu = \xi \sigma^\mu \overline{\xi}^\dagger. \tag{D.20}$$

When $J_{\mu\nu}$ and $\overline{J}_{\mu\nu}$ is well-defined, the following equations hold:

$$J^\mu{}_\nu K^\nu = \overline{J}^\mu{}_\nu K^\nu = iK^\mu, \tag{D.21}$$

$$J^\mu{}_\nu X^\nu = -\overline{J}^\mu{}_\nu X^\nu = iX^\mu. \tag{D.22}$$

Therefore K^μ is holomorphic with respect to both $J^\mu{}_\nu$ and $\overline{J}^\mu{}_\nu$, while X^μ is holomorphic with respect to $J^\mu{}_\nu$ and anti-holomorphic with respect to $\overline{J}^\mu{}_\nu$. Nonzero inner products of two of K^μ, X^μ and their complex conjugates are only

$$K^{*\mu} K_\mu = X^{*\mu} X_\mu = 2|\xi|^2 \left|\overline{\xi}\right|^2. \tag{D.23}$$

Hence we can construct a complete basis from these vectors. The metric can also be written as

$$g_{\mu\nu} = \frac{1}{2|\xi|^2 \left|\overline{\xi}\right|^2} \left(K_\mu K^*_\nu + K_\nu K^*_\mu + X_\mu X^*_\nu + X_\nu X^*_\mu \right). \tag{D.24}$$

Multiplying $J^{\nu}{}_{\rho}$, $\overline{J}^{\nu}{}_{\rho}$, $P^{\nu}{}_{\rho}$ and $\overline{P}^{\nu}{}_{\rho}$, we find that these tensors are represented as

$$J_{\mu\nu} = \frac{i}{2|\xi|^2\,|\overline{\xi}|^2}\left(K_\mu K^*_\nu - K_\nu K^*_\mu + X_\mu X^*_\nu - X_\nu X^*_\mu\right), \tag{D.25}$$

$$\overline{J}_{\mu\nu} = \frac{i}{2|\xi|^2\,|\overline{\xi}|^2}\left(K_\mu K^*_\nu - K_\nu K^*_\mu - X_\mu X^*_\nu + X_\nu X^*_\mu\right), \tag{D.26}$$

$$P_{\mu\nu} = \frac{1}{2\,|\overline{\xi}|^2}\left(K_\mu X_\nu - K_\nu X_\mu\right), \tag{D.27}$$

$$\overline{P}_{\mu\nu} = -\frac{1}{2|\xi|^2}\left(K_\mu \overline{X}_\nu - K_\nu \overline{X}_\mu\right). \tag{D.28}$$

Because $J_{\mu\nu}$ is self-dual and $\overline{J}_{\mu\nu}$ is anti-self-dual, they can be written as

$$J_{\mu\nu} = I_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}I^{\rho\lambda}, \quad \overline{J}_{\mu\nu} = I_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}I^{\rho\lambda}, \tag{D.29}$$

$$I_{\mu\nu} = \frac{i}{K^{*\lambda}K_\lambda}\left(K_\mu K^*_\nu - K_\nu K^*_\mu\right). \tag{D.30}$$

Thus $J_{\mu\nu}$ and $\overline{J}_{\mu\nu}$ can be defined if K_μ is nowhere vanishing.

$\delta_Q\psi_\mu = \delta_Q\overline{\psi}_\mu = 0$ can be written as

$$\nabla_\mu\xi = \frac{i}{6}M\sigma_\mu\overline{\xi} + \frac{i}{3}b_\mu\xi + \frac{i}{3}b^\nu\sigma_{\mu\nu}\xi, \tag{D.31}$$

$$\nabla_\mu\overline{\xi} = \frac{i}{6}\overline{M}\overline{\sigma}_\mu\xi - \frac{i}{3}b_\mu\overline{\xi} - \frac{i}{3}b^\nu\overline{\sigma}_{\mu\nu}\overline{\xi}. \tag{D.32}$$

The almost complex structure $J^{\mu}{}_{\nu}$ is integrable if and only if $U^{[\mu}V^{\nu]}\overline{\sigma}_\mu\nabla_\nu\xi = 0$ for arbitrary holomorphic vectors U^μ and V^μ. By using (D.31) and $U^\mu V_\mu = 0$, which is because both U^μ and V^μ are holomorphic, this condition is rewritten as

$$MU^\mu V^\nu\sigma_\mu\overline{\sigma}_\nu\overline{\xi} = 0. \tag{D.33}$$

It is sufficient to consider the case $\overline{\xi} \neq 0$. Then there are two nonzero holomorphic vectors K^μ and X^μ and we can use the set of these vectors as a basis. Thus we can take $U^\mu = X^\mu$ and $V^\mu = K^\mu$. We can show $K^\mu\sigma_\mu\overline{\xi} = 0$ by the Fierz identity and we conclude that $J^{\mu}{}_{\nu}$ is integrable. Similar analysis also shows that $\overline{J}^{\mu}{}_{\nu}$ is integrable, whenever it exists.

Using (D.31) and (D.27), $P_{\mu\nu}$ and $\overline{P}_{\mu\nu}$ are invariant along K^μ:

$$\mathcal{L}_K P_{\mu\nu} = \mathcal{L}_K\overline{P}_{\mu\nu} = 0, \tag{D.34}$$

and we can show that K_μ is a Killing vector:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{D.35}$$

The complex conjugate K^*_μ is also Killing because the metric is real. Their commutator generates another real Killing vector L^μ:

$$[K, K^*] = 4iL. \tag{D.36}$$

By using (D.31) and (D.32), we found that

$$L^\mu = \lambda X^\mu + \lambda^* X^{*\mu}, \tag{D.37}$$

$$\lambda = \frac{1}{12}\left(\overline{M}|\xi|^2 - M^* \left|\overline{\xi}\right|^2 + (b_\nu - b^*_\nu) X^{*\nu}\right). \tag{D.38}$$

Thus we consider two cases: $L = 0$ and $L \neq 0$.

D.3 Nontrivial ξ and $\overline{\xi}$: $[K, K^*] = 0$

Let us consider the case in which the commutator of K^μ and its complex conjugate vanishes. In such case, we can show that ξ is identically zero or nowhere vanishing, while $\overline{\xi}$ has the same property. For showing this, let us assume $\xi(x) = 0$ at a point x on $\mathcal{M}_4$.

We can firstly show that the covariant derivative is also zero $\nabla_\mu \xi(x) = 0$ at x. $\xi(x) = 0$ yields $X^\mu(x) = 0$ and $\overline{\xi}(x) \neq 0$, for nontrivial solutions. We can take a sufficient small neighborhood around x such that $\overline{\xi} \neq 0$ holds. If ξ vanishes identically in such neighborhood, we obtain $\nabla_\mu \xi(x) = 0$. If not, there is a point in this neighborhood such that $\xi \neq 0$, $\overline{\xi} \neq 0$, and thus $X^\mu \neq 0$ hold. Since we restrict ourselves to the case of $L^\mu = 0$, $\lambda = 0$ holds on such point. We are considering smooth backgrounds, thus $\lambda(x) = 0$. From the expression of λ (D.38), we obtain $M(x) = 0$. Therefore, from (D.31), we obtain $\nabla_\mu \xi(x) = 0$.

From $\xi(x) = \nabla_\mu \xi(x) = 0$, we can find $K_\mu(x) = \nabla_\mu K_\nu(x) = 0$. Because K^μ is Killing, $K^\mu = 0$ everywhere on $\mathcal{M}_4$. Therefore we conclude ξ identically vanishes.

We now focus on the case in which a nontrivial combination $\left(\xi, \overline{\xi}\right)$ is a solution of $\delta_Q \psi_\mu = \delta_Q \overline{\psi}_\mu = 0$. By the above discussion, it is sufficient to consider that both ξ and $\overline{\xi}$ are nowhere vanishing. In the following we consider such situation.

By using the complex structure $J^\mu{}_\nu$, we can introduce the holomorphic coordinates w and z. Because there is a Killing vector K_μ, we take $K = \partial_w$ and then the metric can be written as

$$ds^2 = \Omega\left(z, \overline{z}\right)^2 \left[\left(dw + h\left(z, \overline{z}\right) dz\right)\left(d\overline{w} + \overline{h}\left(z, \overline{z}\right) d\overline{z}\right) + c\left(z, \overline{z}\right)^2 dz d\overline{z}\right]. \tag{D.39}$$

This metric represents a T^2 fibration over a Riemann surface Σ, whose metric is given by

$$ds^2_\Sigma = \Omega\left(z, \overline{z}\right)^2 c\left(z, \overline{z}\right)^2 dz d\overline{z}. \tag{D.40}$$

The coefficient of $dwd\overline{w}$ is defined by K_μ as

$$\Omega^2 = 2K^{*\mu}K_\mu = 4|\xi|^2 \left|\overline{\xi}\right|^2. \tag{D.41}$$

Let us derive expressions for the background fields b_μ, M and $\overline{M}$. By using (D.31), the differential of $J^\mu{}_\nu$ gives

$$\nabla_\mu J^\mu{}_\nu = \frac{1}{3}\left(b_\nu + b^*_\nu\right) - \frac{i}{3}\left(b_\mu - b^*_\mu\right) J^\mu{}_\nu + \frac{1}{3|\xi|^2}\left(M^* X_\nu + M X^*_\nu\right). \tag{D.42}$$

This restricts the anti-holomorphic part of b_μ as

$$b_\mu = \frac{3}{2}\nabla_\nu J^\nu{}_\mu - \frac{1}{2|\xi|^2\left|\overline{\xi}\right|^2}\left(\overline{M}|\xi|^2 X_\mu + M\left|\overline{\xi}\right|^2 X^*_\mu\right) + B_\mu, \quad J_\mu{}^\nu B_\nu = iB_\mu. \tag{D.43}$$

Substituting this into (D.31), we obtain the equation (D.10) with

$$b^c_\mu = b_\mu - \frac{1}{2}\left(2g_{\mu\nu} + iJ_{\mu\nu}\right)\nabla_\rho J^{\rho\nu}. \tag{D.44}$$

In the previous case b^c_μ is holomorphic, but it is not the case now. $p = P_{\overline{w}\overline{z}}$ satisfies (D.11), and thus b^c_μ is represented by p as (D.12). Therefore, b^c_μ is

$$b^c_w = 0, \quad b^c_z = -i\partial_z p, \quad b^c_{\overline{i}} = -i\partial_{\overline{i}}\log\left(pg^{-1/2}\right), \tag{D.45}$$

where the first equation is because $P_{\mu\nu}$ and $K = \partial_w$ satisfy (D.34).

Let us determine the remaining background fields, M and $\overline{M}$. To do this, we differentiate $P_{\mu\nu}$ and $\overline{P}_{\mu\nu}$ by using (D.31) and (D.32):

$$\nabla^\mu P_{\mu\nu} = \frac{i}{2}MK_\nu, \quad \nabla^\mu\overline{P}_{\mu\nu} = \frac{i}{2}\overline{M}K_\nu. \tag{D.46}$$

By multiplying $K^{*\nu}$, we obtain

$$M = -\frac{2i}{K^{*\rho}K_\rho}K^{*\nu}\nabla^\mu P_{\mu\nu}, \quad \overline{M} = -\frac{2i}{K^{*\rho}K_\rho}K^{*\nu}\nabla^\mu\overline{P}_{\mu\nu}. \tag{D.47}$$

By using (D.27), (D.28) and $[K, K^*] = 0$, the above equations can be rewritten as

$$M = -i\nabla^\mu\left(\frac{X^*_\mu}{|\xi|^2}\right), \quad \overline{M} = i\nabla^\mu\left(\frac{X_\mu}{\left|\overline{\xi}\right|^2}\right). \tag{D.48}$$

Furthermore, using the expression of the metric (D.39) in terms of w and z,

$$M = \frac{2ip}{\Omega^4 c^2}\partial_z\log p, \quad \overline{M} = -\frac{2i\Omega^2}{p}\left(\partial_{\overline{z}}\log\frac{\Omega^6 c^2}{p} + \overline{h}\partial_{\overline{w}}\log p\right). \tag{D.49}$$

It turns out that M and $\overline{M}$ can be completely determined by the Hermitian metric and p.

To summarize, we obtained the explicit form of the background fields b_μ, M and $\overline{M}$, assuming the existence of a solution of $\delta_Q \psi_\mu = \delta_Q \overline{\psi}_\mu = 0$ as a nontrivial combination $\left(\xi, \overline{\xi}\right)$ such that $[K, K^*] = 0$. For the cases which do not reduce to $(\xi, 0)$ or $\left(0, \overline{\xi}\right)$, it is sufficient to consider nowhere vanishing ξ and $\overline{\xi}$. The background fields can be expressed in terms of the Hermitian metric, the complex structure and the nowhere vanishing anti-holomorphic two-form p, which is invariant under the translation along $K = \partial_w$. Of course, we can use $\overline{J}^\mu{}_\nu$ and $\overline{P}_{\mu\nu}$ to obtain the above results, instead of $J^\mu{}_\nu$ and $P_{\mu\nu}$.

Conversely, we can show that there is a solution $\left(\xi, \overline{\xi}\right)$ for a general Hermitian manifold which admits a nowhere vanishing complex Killing vector K^μ satisfying $K^\mu K_\mu = 0$ and $[K, K^*] = 0$. As shown in (D.29) and (D.30), the complex structure $J^\mu{}_\nu$ and $\overline{J}^\mu{}_\nu$ can be defined by using only K^μ. Given a nowhere vanishing anti-holomorphic two-form p with respect to $J^\mu{}_\nu$ satisfying $\mathcal{L}_K p = 0$, we can obtain the explicit solution of $\delta_Q \psi_\mu = \delta_Q \overline{\psi}_\mu = 0$. We choose the vielbein as

$$e^{\hat{1}} = \Omega(dw + hdz), \quad e^{\hat{2}} = \Omega c dz. \tag{D.50}$$

and other background fields as obtained above. Then we can find that

$$\xi_\alpha = \frac{\sqrt{s}}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \overline{\xi}^{\dot{\alpha}} = \frac{\Omega}{\sqrt{s}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad s = pg^{-1/4} \tag{D.51}$$

is a solution.

Finally, let us comment about the existence of the Killing vectors and symmetries. As shown in (D.5), the square of the supersymmetry transformation gives the translation $\mathcal{L}_K$. Thus $\mathcal{L}_K$ is a symmetry of a theory, which leaves b_μ, M and $\overline{M}$ invariant. However, $\mathcal{L}_{K^*}$ is not included in (D.5), so this transformation need not be a symmetry of a theory and may change the background fields, even though K^* is a Killing vector.

If we would like to treat $\mathcal{L}_{K^*}$ to be a symmetry, we must impose the condition that $\mathcal{L}_{K^*}$ leaves the background fields invariant. $\mathcal{L}_{K^*} b^\mu = 0$ restricts the $\overline{w}$ dependence of p as

$$p\left(\overline{w}, z, \overline{z}\right) = e^{\alpha \overline{w}} \widehat{p}\left(z, \overline{z}\right), \quad \alpha \in \mathbb{C}. \tag{D.52}$$

By using this, we obtain

$$\mathcal{L}_{K^*} M = \alpha M, \quad \mathcal{L}_{K^*} \overline{M} = -\alpha \overline{M}. \tag{D.53}$$

Therefore there are two cases for $\mathcal{L}_{K^*} M = \mathcal{L}_{K^*} \overline{M} = 0$: $\alpha \neq 0$ and $\alpha = 0$. If $\alpha \neq 0$, we obtain $M = \overline{M} = 0$ from (D.53). This reduces to the previous case and thus there are at least two supercharges parametrized by $(\xi, 0)$ and $\left(0, \overline{\xi}\right)$. If $\alpha = 0$, the supersymmetry algebra is extended as

$$\delta_Q^2 = 2i\delta_K, \tag{D.54}$$
$$[\delta_K, \delta_Q] = [\delta_{K^*}, \delta_Q] = 0, \tag{D.55}$$
$$[\delta_K, \delta_{K^*}] = 0. \tag{D.56}$$

D.4 Nontrivial ξ and $\overline{\xi}$: $[K, K^*] \neq 0$

Next let us consider the case in which a nontrivial combination $\left(\xi, \overline{\xi}\right)$ satisfying $[K, K^*] \neq 0$ is a solution of (D.31) and (D.32). Then there is another real Killing vector L^μ as in (D.36). Thus there are three real Killing vectors

$$\ell_1 = \frac{1}{2}\left(K + K^*\right), \quad \ell_2 = -\frac{i}{2}\left(K - K^*\right), \quad \ell_3 = L. \tag{D.57}$$

We can show that these are orthogonal each other. Vanishing of the inner product between first two vectors can be shown by using $K^\mu K_\mu = 0$. For the inner products between L and other ones, we consider the action of $\mathcal{L}_{K^*}$ to $K^\mu K_\mu = 0$:

$$0 = \mathcal{L}_{K^*}\left(K^\mu K_\mu\right) = -8iL^\mu K_\mu. \tag{D.58}$$

Therefore we obtain $L^\mu K_\mu = 0$. Similarly, we can find $L^\mu K^*_\mu = 0$.

By using these facts, we can show that there is $SU(2) \times SU(2)$ or $SU(2) \times U(1)$ symmetry and the metric can be written as

$$ds^2 = d\tau^2 + r(\tau)^2 d\Omega_3, \tag{D.59}$$

where $d\Omega_3$ is the metric of the round S^3 with unit radius. From the fact that ℓ_1, ℓ_2 and ℓ_3 are orthogonal each other, the commutators $[\ell_a, \ell_b]$ is also orthogonal with respect to ℓ_a and ℓ_b. If all $[\ell_a, \ell_b]$ are parallel to $\epsilon_{abc}\ell_c$ respectively, the algebra is $SU(2)$. Then the metric can be expressed in terms of the $SU(2)$ invariant one-form $\widetilde{\mu}^a$ as

$$ds^2 = d\tau^2 + h_{ab}(\tau)\widetilde{\mu}^a\widetilde{\mu}^b. \tag{D.60}$$

Since the Killing vectors ℓ_1, ℓ_2 and ℓ_3 are orthogonal each other, we obtain $h_{ab}(\tau) = r(\tau)^2\delta_{ab}$. This metric gives the expression (D.59) and the symmetry is enhanced to $SU(2)_l \times SU(2)_r$. $\ell_{a=1,2,3}$ generate either $SU(2)_l$ or $SU(2)_r$.

If $[\ell_a, \ell_b]$ additionally generates another vector T, which is orthogonal to $\epsilon_{abc}\ell_c$, T is another real Killing vector. Therefore the isometry is $SU(2) \times U(1)$, where $U(1)$ is generated by the translation along T, that is τ direction. Then the metric is given as in (D.59) and r is independent of τ.

By using the metric (D.59) and the Killing vectors, let us determine the backgroud fields b^μ, M and $\overline{M}$. We assume that $\ell_{a=1,2,3}$ generate $SU(2)_l$ isometry of the unit S^3. Their dual one-form $\mu^{\widehat{a}}$ $(a = 1, 2, 3)$ is $SU(2)_r$ invariant one-form and the metric can be represented as

$$d\Omega_3 = \left(\mu^{\widehat{1}}\right)^2 + \left(\mu^{\widehat{2}}\right)^2 + \left(\mu^{\widehat{3}}\right)^2. \tag{D.61}$$

By using $\mu^{\widehat{a}}$, we can take the vielbein as

$$e^{\widehat{a}} = r(\tau)\mu^{\widehat{a}}, \quad e^{\widehat{4}} = d\tau. \tag{D.62}$$

In this frame, $K^{*\mu}K_\mu = 2r(\tau)^2$, thus $|\xi|^2 \left|\overline{\xi}\right|^2 = r(\tau)^2$. We can introduce the fourth real vector orthogonal to K, K^* and L as

$$T^\mu = -\frac{i}{K^{*\lambda}K_\lambda}\epsilon^{\mu\nu\rho\sigma}L_\nu K_\rho K^*_\sigma = r(\tau)\delta^\mu_\tau. \tag{D.63}$$

The vector X^μ is orthogonal to K^μ and $K^{*\mu}$ and can be expressed in terms of L^μ and T^μ as

$$X^\mu = \alpha\left(L^\mu + \beta T^\mu\right). \tag{D.64}$$

The fact that X^μ satisfies $X^\mu X_\mu = 0$ and $X^{*\mu}X_\mu = 2r(\tau)^2$ constrains α and β as

$$|\alpha|^2 = 1, \quad \beta^2 = -1. \tag{D.65}$$

We take $\beta = 1$ by choosing the sign of T^μ as in (D.63). For α, let us define the complex scalar function s satisfying

$$|s| = \frac{|\xi|^2}{r(\tau)} = \frac{r(\tau)}{\left|\overline{\xi}\right|^2}, \quad X^\mu = \frac{s}{|s|}\left(L^\mu + iT^\mu\right). \tag{D.66}$$

By using (D.27), we can write $P_{\mu\nu}$ as a function of s. As shown in (D.34), $P_{\mu\nu}$ is invariant along K, thus s is invariant under the translation along K as

$$K^\mu\partial_\mu s = 0. \tag{D.67}$$

Similar to the previous analysis, we can determine the background fields in terms of the geometric quantities and the function s. By differentiating $P_{\mu\nu}$, M and $\overline{M}$ are written as (D.47). Using (D.66) and $\nabla_\mu K_\nu = -\nabla_\nu K_\mu$, they are written as follows:

$$M = i\nabla^\mu\left(\frac{s}{r(\tau)}\left(L_\mu + iT_\mu\right)\right) - \frac{2s}{r(\tau)}, \tag{D.68}$$

$$\overline{M} = -i\nabla^\mu\left(\frac{1}{sr(\tau)}\left(L_\mu - iT_\mu\right)\right) + \frac{2}{sr(\tau)}. \tag{D.69}$$

Let us determine b_μ. For this, we compute the commutator $[X, X^*]$. By using (D.31) and (D.32),

$$\begin{aligned}[X, X^*] &= -\frac{2i}{3}\left(K^{*\mu} b^*_\mu\right) K - \frac{i}{2}\left(X^{*\mu}\left(b_\mu + b^*_\mu\right)\right) X - (\text{c.c.}) \\ &= -\frac{2i}{3}\left(K^{*\mu} b^*_\mu\right) K - \frac{2i}{3}\left(K^\mu b_\mu\right) K^* - iL^\mu\left(b_\mu + b^*_\mu\right) L - iT^\mu\left(b_\mu + b^*_\mu\right) T.\end{aligned} \tag{D.70}$$

On the other hand, $[X, X^*]$ can be also computed by using (D.66) as

$$[X, X^*] = -2\left(L^\mu \partial_\mu \log \frac{s}{|s|}\right) L - 2\left(T^\mu \partial_\mu \log \frac{s}{|s|}\right) T. \tag{D.71}$$

By comparing (D.70) and (D.71), we obtain

$$K^\mu b_\mu = 0, \quad L^\mu\left(b_\mu + b^*_\mu\right) = -2iL^\mu \partial_\mu \log \frac{s}{|s|}, \quad T^\mu\left(b_\mu + b^*_\mu\right) = -2iT^\mu \partial_\mu \log \frac{s}{|s|}. \tag{D.72}$$

The differential of $|s|^2 = |\xi|^2 \left|\overline{\xi}\right|^2$ can be computed by using (D.31), (D.32), (D.68) and (D.69) as

$$\partial_\mu \log |s|^2 = i\left(b_\mu - b^*_\mu\right) + \frac{2}{r(\tau)} \delta^\tau_\mu. \tag{D.73}$$

Therefore, we can solve b_μ as

$$b_\mu = -i\partial_\mu \log s + \frac{i}{r(\tau)} \delta^\tau_\mu. \tag{D.74}$$

From (D.67) and (D.74), the background fields are invariant under the translation along K, but need not to be invariant along K^* or L.

Conversely, when we take the metric, the function s and the background fields as obtained, we can obtain the solution $\left(\xi, \overline{\xi}\right)$. In the frame (D.59), we can find that

$$\xi_\alpha = \sqrt{s r(\tau)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \overline{\xi}^{\dot{\alpha}} = \sqrt{\frac{r(\tau)}{s}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{D.75}$$

is a solution.

If we take $\tau_- \leq \tau \leq \tau_+$ satisfying $r\left(\tau_\pm\right) = 0$, we can construct S^4 and its some deformations. In such case, the spinors ξ and $\overline{\xi}$ vanishes at some points, which is consistent with the fact S^4 does not admit the almost complex structures.

Bibliography

[1] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge University Press (1998).

[2] C. V. Johnson, "D-branes," Cambridge University Press (2003).

[3] K. Becker, M. Becker and J. H. Schwarz, "String theory and M-theory: A modern introduction," Cambridge University Press (2007).

[4] J. Polchinski, "Combinatorics of boundaries in string theory," Phys. Rev. D **50** (1994) 6041 [hep-th/9407031].

[5] E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. B **443** (1995) 85 [hep-th/9503124].

[6] J. Bagger and N. Lambert, "Modeling Multiple M2's," Phys. Rev. D **75** (2007) 045020 [hep-th/0611108].

[7] A. Gustavsson, "Algebraic structures on parallel M2-branes," Nucl. Phys. B **811** (2009) 66 [arXiv:0709.1260 [hep-th]].

[8] J. Bagger and N. Lambert, "Gauge symmetry and supersymmetry of multiple M2-branes," Phys. Rev. D **77** (2008) 065008 [arXiv:0711.0955 [hep-th]].

[9] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **0810** (2008) 091 [arXiv:0806.1218 [hep-th]].

[10] P. Pasti, D. P. Sorokin and M. Tonin, "On Lorentz invariant actions for chiral p forms," Phys. Rev. D **55** (1997) 6292 [hep-th/9611100].

[11] P. Pasti, D. P. Sorokin and M. Tonin, "Covariant action for a D = 11 five-brane with the chiral field," Phys. Lett. B **398** (1997) 41 [hep-th/9701037].

[12] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426** (1994) 19 [Nucl. Phys. B **430** (1994) 485] [hep-th/9407087].

[13] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431** (1994) 484 [hep-th/9408099].

[14] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," Adv. Theor. Math. Phys. **7** (2004) 831 [hep-th/0206161].

[15] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] [hep-th/9711200].

[16] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428** (1998) 105 [hep-th/9802109].

[17] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[18] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824 [hep-th]].

[19] J. G. Russo and K. Zarembo, "Large N Limit of N=2 SU(N) Gauge Theories from Localization," JHEP **1210** (2012) 082 [arXiv:1207.3806 [hep-th]].

[20] R. Emparan, C. V. Johnson and R. C. Myers, "Surface terms as counterterms in the AdS / CFT correspondence," Phys. Rev. D **60** (1999) 104001 [hep-th/9903238].

[21] A. W. Peet and J. Polchinski, "UV / IR relations in AdS dynamics," Phys. Rev. D **59** (1999) 065011 [hep-th/9809022].

[22] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," Nucl. Phys. B **435** (1995) 129 [hep-th/9411149].

[23] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, "An Index for 4 dimensional super conformal theories," Commun. Math. Phys. **275** (2007) 209 [hep-th/0510251].

[24] C. Romelsberger, "Counting chiral primaries in N = 1, d=4 superconformal field theories," Nucl. Phys. B **747** (2006) 329 [hep-th/0510060].

[25] S. Franco, A. Hanany, K. D. Kennaway, D. Vegh and B. Wecht, "Brane dimers and quiver gauge theories," JHEP **0601** (2006) 096 [hep-th/0504110].

[26] L. F. Alday, D. Gaiotto and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," Lett. Math. Phys. **91** (2010) 167 [arXiv:0906.3219 [hep-th]].

[27] D. Gaiotto, "N=2 dualities," JHEP **1208** (2012) 034 [arXiv:0904.2715 [hep-th]].

[28] N. Hama, K. Hosomichi and S. Lee, "SUSY Gauge Theories on Squashed Three-Spheres," JHEP **1105** (2011) 014 [arXiv:1102.4716 [hep-th]].

[29] J. Gomis and S. Lee, "Exact Kahler Potential from Gauge Theory and Mirror Symmetry," JHEP **1304** (2013) 019 [arXiv:1210.6022 [hep-th]].

[30] H. Kim, S. Lee and P. Yi, "Exact partition functions on $\mathbb{RP}^2$ and orientifolds," JHEP **1402** (2014) 103 [arXiv:1310.4505 [hep-th]].

[31] L. F. Alday, D. Martelli, P. Richmond and J. Sparks, "Localization on Three-Manifolds," JHEP **1310** (2013) 095 [arXiv:1307.6848 [hep-th]].

[32] M. J. Duff, "Observations on Conformal Anomalies," Nucl. Phys. B **125** (1977) 334.

[33] J. L. Cardy, "Is There a c Theorem in Four-Dimensions?," Phys. Lett. B **215** (1988) 749.

[34] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP **1112** (2011) 099 [arXiv:1107.3987 [hep-th]].

[35] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," JHEP **1106** (2011) 114 [arXiv:1105.0689 [hep-th]].

[36] C. Klare, A. Tomasiello and A. Zaffaroni, "Supersymmetry on Curved Spaces and Holography," JHEP **1208** (2012) 061 [arXiv:1205.1062 [hep-th]].

[37] T. T. Dumitrescu, G. Festuccia and N. Seiberg, "Exploring Curved Superspace," JHEP **1208** (2012) 141 [arXiv:1205.1115 [hep-th]].

[38] T. T. Dumitrescu and G. Festuccia, "Exploring Curved Superspace (II)," JHEP **1301** (2013) 072 [arXiv:1209.5408 [hep-th]].

[39] C. Closset, T. T. Dumitrescu, G. Festuccia and Z. Komargodski, "Supersymmetric Field Theories on Three-Manifolds," JHEP **1305** (2013) 017 [arXiv:1212.3388 [hep-th]].

[40] C. Closset, T. T. Dumitrescu, G. Festuccia and Z. Komargodski, "The Geometry of Supersymmetric Partition Functions," JHEP **1401** (2014) 124 [arXiv:1309.5876 [hep-th]].

[41] N. Seiberg, "Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics," Phys. Lett. B **388** (1996) 753 [hep-th/9608111].

[42] M. R. Douglas, "On D=5 super Yang-Mills theory and (2,0) theory," JHEP **1102** (2011) 011 [arXiv:1012.2880 [hep-th]].

[43] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, "M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills," JHEP **1101** (2011) 083 [arXiv:1012.2882 [hep-th]].

[44] Y. Pan, "Rigid Supersymmetry on 5-dimensional Riemannian Manifolds and Contact Geometry," JHEP **1405** (2014) 041 [arXiv:1308.1567 [hep-th]].

[45] Y. Imamura and H. Matsuno, "Supersymmetric backgrounds from 5d $\mathcal{N}=1$ supergravity," JHEP **1407** (2014) 055 [arXiv:1404.0210 [hep-th]].

[46] C. Romelsberger, "Calculating the Superconformal Index and Seiberg Duality," arXiv:0707.3702 [hep-th].

[47] W. Peelaers, "Higgs branch localization of $\mathcal{N}=1$ theories on S^3 x S^1," JHEP **1408** (2014) 060 [arXiv:1403.2711 [hep-th]].

[48] T. Nishioka and I. Yaakov, "Generalized indices for $\mathcal{N}=1$ theories in four-dimensions," JHEP **1412** (2014) 150 [arXiv:1407.8520 [hep-th]].

[49] C. Closset and I. Shamir, "The $\mathcal{N}=1$ Chiral Multiplet on $T^2 \times S^2$ and Supersymmetric Localization," JHEP **1403** (2014) 040 [arXiv:1311.2430 [hep-th]].

[50] M. Honda and Y. Yoshida, "Supersymmetric index on $T^2 \times S^2$ and elliptic genus," arXiv:1504.04355 [hep-th].

[51] N. Hama and K. Hosomichi, "Seiberg-Witten Theories on Ellipsoids," JHEP **1209** (2012) 033 [JHEP **1210** (2012) 051] [arXiv:1206.6359 [hep-th]].

[52] T. Nosaka and S. Terashima, "Supersymmetric Gauge Theories on a Squashed Four-Sphere," JHEP **1312** (2013) 001 [arXiv:1310.5939 [hep-th]].

[53] J. Wess and J. Bagger, "Supersymmetry and supergravity," Princeton University Press (1992).

[54] P. Breitenlohner, "A Geometric Interpretation of Local Supersymmetry," Phys. Lett. B **67** (1977) 49.

[55] P. Breitenlohner, "Some Invariant Lagrangians for Local Supersymmetry," Nucl. Phys. B **124**, 500 (1977).

[56] K. S. Stelle and P. C. West, “Minimal Auxiliary Fields for Supergravity,” Phys. Lett. B **74** (1978) 330.

[57] S. Ferrara and P. van Nieuwenhuizen, “The Auxiliary Fields of Supergravity,” Phys. Lett. B **74** (1978) 333.

[58] B. de Wit and P. van Nieuwenhuizen, “The Auxiliary Field Structure in Chirally Extended Supergravity,” Nucl. Phys. B **139** (1978) 216.

[59] M. F. Sohnius and P. C. West, “An Alternative Minimal Off-Shell Version of N=1 Supergravity,” Phys. Lett. B **105** (1981) 353.

[60] V. O. Rivelles and J. G. Taylor, “On the Nonminimal Versions of Linearized $N = 1$ Supergravity,” Phys. Lett. B **113** (1982) 467.

[61] M. Sohnius and P. C. West, “The Tensor Calculus and Matter Coupling of the Alternative Minimal Auxiliary Field Formulation of $N = 1$ Supergravity,” Nucl. Phys. B **198** (1982) 493.

[62] V. O. Rivelles and J. G. Taylor, “All Sets of Lower Spin Auxiliary Fields for $N = 1$ Supergravity,” Nucl. Phys. B **212** (1983) 173.

[63] T. Kugo and S. Uehara, “Conformal and Poincare Tensor Calculi in $N = 1$ Supergravity,” Nucl. Phys. B **226** (1983) 49.

[64] K. Kodaira, “Complex Manifolds and Deformation of Complex Structures,” Springer (2005).

[65] S. Kobayashi, “Differential Geometry of Complex Vector Bundles,” Princeton University Press (1987).

[66] Z. Komargodski and N. Seiberg, “Comments on the Fayet-Iliopoulos Term in Field Theory and Supergravity,” JHEP **0906** (2009) 007 [arXiv:0904.1159 [hep-th]].

[67] Z. Komargodski and N. Seiberg, “Comments on Supercurrent Multiplets, Supersymmetric Field Theories and Supergravity,” JHEP **1007** (2010) 017 [arXiv:1002.2228 [hep-th]].

[68] O. Aharony, “IR duality in d = 3 N=2 supersymmetric USp(2N(c)) and U(N(c)) gauge theories,” Phys. Lett. B **404** (1997) 71 [hep-th/9703215].

[69] T. Dimofte, D. Gaiotto and S. Gukov, “Gauge Theories Labelled by Three-Manifolds,” Commun. Math. Phys. **325** (2014) 367 [arXiv:1108.4389 [hep-th]].

[70] T. Dimofte, D. Gaiotto and S. Gukov, “3-Manifolds and 3d Indices,” Adv. Theor. Math. Phys. **17** (2013) 5, 975 [arXiv:1112.5179 [hep-th]].

[71] A. Kapustin, B. Willett and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," JHEP **1003** (2010) 089 [arXiv:0909.4559 [hep-th]].

[72] N. Hama, K. Hosomichi and S. Lee, "Notes on SUSY Gauge Theories on Three-Sphere," JHEP **1103** (2011) 127 [arXiv:1012.3512 [hep-th]].

[73] D. L. Jafferis, "The Exact Superconformal R-Symmetry Extremizes Z," JHEP **1205** (2012) 159 [arXiv:1012.3210 [hep-th]].

[74] F. A. H. Dolan, V. P. Spiridonov and G. S. Vartanov, "From 4d superconformal indices to 3d partition functions," Phys. Lett. B **704** (2011) 234 [arXiv:1104.1787 [hep-th]].

[75] A. Gadde and W. Yan, "Reducing the 4d Index to the S^3 Partition Function," JHEP **1212** (2012) 003 [arXiv:1104.2592 [hep-th]].

[76] Y. Imamura, "Relation between the 4d superconformal index and the S^3 partition function," JHEP **1109** (2011) 133 [arXiv:1104.4482 [hep-th]].

[77] Y. Imamura and D. Yokoyama, "N=2 supersymmetric theories on squashed three-sphere," Phys. Rev. D **85** (2012) 025015 [arXiv:1109.4734 [hep-th]].

[78] J. Nian, "Localization of Supersymmetric Chern-Simons-Matter Theory on a Squashed S^3 with $SU(2) \times U(1)$ Isometry," JHEP **1407** (2014) 126 [arXiv:1309.3266 [hep-th]].

[79] A. Tanaka, "Localization on round sphere revisited," JHEP **1311** (2013) 103 [arXiv:1309.4992 [hep-th]].

[80] Y. Imamura and D. Yokoyama, "$S^3/\mathbb{Z}_n$ partition function and dualities," JHEP **1211** (2012) 122 [arXiv:1208.1404 [hep-th]].

[81] Y. Imamura, H. Matsuno and D. Yokoyama, "Factorization of the $S^3/\mathbb{Z}_n$ partition function," Phys. Rev. D **89** (2014) 8, 085003 [arXiv:1311.2371 [hep-th]].

[82] Y. Imamura and S. Yokoyama, "Index for three dimensional superconformal field theories with general R-charge assignments," JHEP **1104** (2011) 007 [arXiv:1101.0557 [hep-th]].

[83] M. Marino and P. Putrov, "ABJM theory as a Fermi gas," J. Stat. Mech. **1203** (2012) P03001 [arXiv:1110.4066 [hep-th]].

[84] I. R. Klebanov and A. A. Tseytlin, "Entropy of near extremal black p-branes," Nucl. Phys. B **475** (1996) 164 [hep-th/9604089].

[85] S. M. Kuzenko, U. Lindstrom, M. Rocek, I. Sachs and G. Tartaglino-Mazzucchelli, "Three-dimensional $\mathcal{N} = 2$ supergravity theories: From superspace to components," Phys. Rev. D **89** (2014) 8, 085028 [arXiv:1312.4267 [hep-th]].

[86] X. Gomez-Mont, "Transversal Holomorphic Structures," J. Differential Geometry **15** 161 (1980).

[87] E. Witten, "Phase transitions in M theory and F theory," Nucl. Phys. B **471** (1996) 195 [hep-th/9603150].

[88] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, "Instanton Operators in Five-Dimensional Gauge Theories," JHEP **1503** (2015) 019 [arXiv:1412.2789 [hep-th]].

[89] H. C. Kim, S. S. Kim and K. Lee, "5-dim Superconformal Index with Enhanced En Global Symmetry," JHEP **1210** (2012) 142 [arXiv:1206.6781 [hep-th]].

[90] C. Montonen and D. I. Olive, "Magnetic Monopoles as Gauge Particles?," Phys. Lett. B **72** (1977) 117.

[91] Z. Bern, J. J. Carrasco, L. J. Dixon, M. R. Douglas, M. von Hippel and H. Johansson, "D=5 maximally supersymmetric Yang-Mills theory diverges at six loops," Phys. Rev. D **87** (2013) 2, 025018 [arXiv:1210.7709 [hep-th]].

[92] J. Källén, J. Qiu and M. Zabzine, "The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere," JHEP **1208** (2012) 157 [arXiv:1206.6008 [hep-th]].

[93] J. Kallen, J. A. Minahan, A. Nedelin and M. Zabzine, "N^3-behavior from 5D Yang-Mills theory," JHEP **1210** (2012) 184 [arXiv:1207.3763 [hep-th]].

[94] H. C. Kim, J. Kim and S. Kim, "Instantons on the 5-sphere and M5-branes," arXiv:1211.0144 [hep-th].

[95] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and S. Raju, "Indices for Superconformal Field Theories in 3,5 and 6 Dimensions," JHEP **0802** (2008) 064 [arXiv:0801.1435 [hep-th]].

[96] H. C. Kim and K. Lee, "Supersymmetric M5 Brane Theories on R x CP2," JHEP **1307** (2013) 072 [arXiv:1210.0853 [hep-th]].

[97] H. C. Kim, S. Kim, S. S. Kim and K. Lee, "The general M5-brane superconformal index," arXiv:1307.7660.

[98] M. Zucker, "Minimal off-shell supergravity in five-dimensions," Nucl. Phys. B **570** (2000) 267 [hep-th/9907082].

[99] M. Zucker, "Gauged N=2 off-shell supergravity in five-dimensions," JHEP **0008** (2000) 016 [hep-th/9909144].

[100] T. Kugo and K. Ohashi, "Supergravity tensor calculus in 5-D from 6-D," Prog. Theor. Phys. **104** (2000) 835 [hep-ph/0006231].

[101] T. Kugo and K. Ohashi, "Off-shell D = 5 supergravity coupled to matter Yang-Mills system," Prog. Theor. Phys. **105** (2001) 323 [hep-ph/0010288].

[102] E. Bergshoeff, E. Sezgin and A. Van Proeyen, "Superconformal Tensor Calculus and Matter Couplings in Six-dimensions," Nucl. Phys. B **264** (1986) 653 [Nucl. Phys. B **598** (2001) 667].

[103] K. Hosomichi, R. K. Seong and S. Terashima, "Supersymmetric Gauge Theories on the Five-Sphere," Nucl. Phys. B **865** (2012) 376 [arXiv:1203.0371 [hep-th]].

[104] S. M. Kuzenko, U. Lindstrom and G. Tartaglino-Mazzucchelli, "Off-shell supergravity-matter couplings in three dimensions," JHEP **1103** (2011) 120 [arXiv:1101.4013 [hep-th]].

[105] A. Pini, D. Rodriguez-Gomez and J. Schmude, "Rigid Supersymmetry from Conformal Supergravity in Five Dimensions," JHEP **1509** (2015) 118 [arXiv:1504.04340 [hep-th]].

[106] E. Bergshoeff, S. Cucu, M. Derix, T. de Wit, R. Halbersma and A. Van Proeyen, "Weyl multiplets of N=2 conformal supergravity in five-dimensions," JHEP **0106** (2001) 051 [hep-th/0104113].

[107] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, "N = 2 supergravity in five-dimensions revisited," Class. Quant. Grav. **21** (2004) 3015 [Class. Quant. Grav. **23** (2006) 7149] [hep-th/0403045].

[108] C. Cordova and D. L. Jafferis, "Five-Dimensional Maximally Supersymmetric Yang-Mills in Supergravity Backgrounds," arXiv:1305.2886 [hep-th].

[109] J. Kim, S. Kim, K. Lee and J. Park, "Super-Yang-Mills theories on $S^4 \times \mathbb{R}$," JHEP **1408** (2014) 167 [arXiv:1405.2488 [hep-th]].

[110] T. Kawano and N. Matsumiya, "5D SYM on 3D Sphere and 2D YM," Phys. Lett. B **716** (2012) 450 [arXiv:1206.5966 [hep-th]].

[111] Y. Fukuda, T. Kawano and N. Matsumiya, "5D SYM and 2D q-Deformed YM," Nucl. Phys. B **869** (2013) 493 [arXiv:1210.2855 [hep-th]].

[112] B. de Wit, J. W. van Holten and A. Van Proeyen, "Transformation Rules of N=2 Supergravity Multiplets," Nucl. Phys. B **167** (1980) 186.

[113] B. de Wit, J. W. van Holten and A. Van Proeyen, "Structure of N=2 Supergravity," Nucl. Phys. B **184** (1981) 77 [Nucl. Phys. B **222** (1983) 516].

[114] E. Bergshoeff, E. Sezgin and A. Van Proeyen, "(2,0) tensor multiplets and conformal supergravity in D = 6," Class. Quant. Grav. **16** (1999) 3193 [hep-th/9904085].

[115] W. Barth, K. Hulek, C. Peters, A. van de Ven, "Compact Complex Surfaces," Springer (1995).

[116] S. Ferrara and B. Zumino, "Transformation Properties of the Supercurrent," Nucl. Phys. B **87** (1975) 207.