論文 / 著書情報
Article / Book Information

題目(和文)	悪条件錐線形計画問題と非線形半正定値計画に対するスラック変数法の解析
Title(English)	Analysis of ill-posed conic linear programs and slack variables approach for nonlinear SDP
著者(和文)	LOURENCO Bruno F.
Author(English)	Bruno F. Lourenco
出典(和文)	学位:博士(理学), 学位授与機関:東京工業大学, 報告番号:甲第10106号, 授与年月日:2016年3月26日, 学位の種別:課程博士, 審査員:福田 光浩,水野 眞治,三好 直人,中田 和秀,山下 真,土谷 隆
Citation(English)	Degree:Doctor (Science), Conferring organization: Tokyo Institute of Technology, Report number:甲第10106号, Conferred date:2016/3/26, Degree Type:Course doctor, Examiner:,,,,,
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Analysis of ill-posed conic linear programs and slack variables approach for nonlinear SDP

Bruno Figueira Lourenço

Submitted in partial fulfillment of the requirements for the degree of
Doctor of Science

Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Japan

January, 2016

Abstract

This work is divided in two parts. In the first one, we will examine several types of ill behavior in conic linear programming such as weak infeasibility, nonattainment of optimal values and duality gaps. We will then discuss how to analyze and to, some extent, correct those issues with techniques based on facial reduction. Among our contributions we have an analysis of weak infeasibility and a description of a theoretical procedure that is able to "completely solve" an arbitrary conic linear program by means of other problems that are ensured to be well-behaved. We also show FRA-Poly, a facial reduction algorithm that exploits the presence of polyhedral faces in the underlying cone in order to finish in less reduction steps than the classical algorithm.

In the second part of this thesis, we take a look at nonlinear semidefinite programs (NSDPs). We take a look at the possibility of transforming an NSDP into a conventional nonlinear program, by using squared slack variables to remove the positive semidefiteness constraint. We then do a thorough comparison between optimality conditions for both the original problem and its reformulated version. In particular, we show that by using squared slack variables we can obtain a pair of "no-gap" second order conditions for NSDPs through elementary means. This work is an extension of an earlier work by Fukuda and Fukushima on nonlinear second order cone programs.

I dedicate this thesis to my parents, my brothers and to Ana. Although they were far away, they always supported me and gave me encouragement.

Acknowledgements

I am immensely grateful to Prof. Mituhiro Fukuda and to Prof. Takashi Tsuchiya of the Graduate Research Institute for Policy Studies. The first for giving me the opportunity to come to Japan and the second for the kind and warm guidance during research. Prof. Tsuchiya taught me the meaning of research and important lessons that I shall never forget. He was always an approachable person and I am glad for all the conversations we had on a number of topics. Their support was absolutely essential during my time at Tokyo Tech.

I am also thankful to Prof. Masakazu Muramatsu of The University of Electro-Communications, who was our frequent collaborator and with whom we spent countless hours debating research. I am glad I had the chance to exchange ideas with him.

I also had the chance of collaborating with Prof. Ellen H. Fukuda of Kyoto University and Prof. Masao Fukushima of Nanzan University. It has been an immense pleasure working with them and I am thankful for the opportunity.

I would also like to thank my colleagues from Fukuda Lab and Yamashita Lab at Tokyo Tech. I enjoyed a lot our seminars together. My special thanks also go to Ito-san from Fukuda Lab, with whom I read several interesting mathematical books and who was always open to discuss random mathematical issues.

Many professors I met at University of Brasília gave me encouragement to come to Japan and helped me with many issues and questions. They all deserve my unreserved thanks. I would like to thank, in particular, Prof. Anderson Nascimento (now at University of Washington - Tacoma) and Prof. Jacir Bordim for the many advices they gave me during critical times.

During my time at Tokyo Tech, I had the chance of meeting people from all over the world and discuss many topics, ranging from the mundane to the politically sensitive. I am thankful to all of them. Special thanks go to my Brazilian friends Bruno Ramos, João Ota and Leandro Batista. Our many politically charged discussions made lunch and dinner way more interesting. Also, I would like to send a shout-out to Alex Andr and Juan Estebán, with whom I went to several heavy metal concerts.

I also thank my friends at the Karate club, who always made me feel welcome. Training with them was always a pleasure.

Many thanks go to the Umemura family for their kindness and for the many invitations to do fun activities together.

Contents

Glossary

$\langle \cdot, \cdot \rangle$ Euclidean inner product. 19

$\|\cdot\|$ Euclidean norm. 19

$\cdot \circ \cdot$ Jordan product. 79

$\mathcal{A}$ linear map, usually from $\mathcal{E}$ to $\mathbb{R}^m$. 19

$\operatorname{aff} C$ smallest affine subpace containing C. 19

$A^\perp$ space of elements orthogonal to A. 19

$\mathcal{A}^\top$ the adjoint of $\mathcal{A}$. 19

$\operatorname{cl} \operatorname{dir}(x, C)$ tangent cone of C at x. 19

$\operatorname{cl} C$ closure of C. 19

$d(D)$ singularity degree of (D). 33

$\operatorname{dir}(x, C)$ cone of feasible directions of C at x. 19

$\operatorname{dist}(C_1, C_2)$ Euclidean distance between C_1 and C_2. 20

$d_{\mathbf{str}}(D)$ distance to strong duality. 45

$\mathcal{E}$ the default ambient space, usually $\mathbb{R}^n$ or $\mathcal{S}^n$. 19

$\mathcal{F}$ usually a face of $\mathcal{K}$. 20

$\mathcal{F}^\Delta$ conjugated face of $\mathcal{F}$, i.e., $\mathcal{K}^* \cap \mathcal{F}^\perp$. 20

$\mathcal{F}_{\mathbf{D}}$ dual feasible region $\{y \in \mathbb{R}^m \mid c - \mathcal{A}^\top y \in \mathcal{K}\}$. 19

$\mathcal{F}_{\mathbf{D}}^S$ dual feasible slacks $\{c - \mathcal{A}^\top y \mid y \in \mathcal{F}_{\mathrm{D}}\}$. 19

$\mathcal{F}_{\min}^D$ minimal face of $\mathcal{K}$ that contains $\mathcal{F}_{\mathrm{D}}^S$. 30

$\mathcal{F}(z, \mathcal{K})$ minimal face of $\mathcal{K}$ that contains z. 20

$\mathcal{F}(D, C)$ minimal face of C that contains the subset D. 20

$\mathcal{F}_{\mathbf{P}}$ primal feasible region $\{x \in \mathcal{K}^* \mid \mathcal{A}x = b\}$. 19

$\operatorname{int} C$ interior of C. 19

$\mathcal{K}$ closed convex cone. 19

$\mathcal{K}^*$ dual of $\mathcal{K}$. 19

$(\mathcal{K}, \mathcal{L}, c)$ the feasibility problem of finding $x \in \mathcal{K} \cap (\mathcal{L} + c)$. 53

L_A linear operator induced by A and the Jordan product. 81

$\operatorname{lin} \mathcal{K}$ largest subspace contained in $\mathcal{K}$, i.e., $\mathcal{K} \cap -\mathcal{K}$. 19

$\ell_{\mathcal{K}}$ longest chain of nonempty faces of $\mathcal{K}$. 32

$\ell_{\mathbf{poly}}(\mathcal{K})$ distance to polyhedrality. 43

θ_D dual optimal value. 19

$\mathcal{O}_{\mathbf{feas}}$ feasibility oracle. 71

$\mathcal{O}_{\mathbf{int}}$ interior point oracle. 72

$\mathcal{O}_{\mathbf{opt}}$ optimization oracle. 71

θ_P primal optimal value. 19

$\mathcal{Q}^n$ n-dimensional Lorentz cone. 25

$\operatorname{relbd} C$ relative boundary of C. 19

$\operatorname{rec} C$ recession cone of C. 38

$\operatorname{ri} C$ relative interior of C. 19

$\mathbb{R}^m$ m-dimensional Euclidean space. 19

$\mathcal{S}^n$ space of $n \times n$ symmetric matrices. 24

$\mathcal{S}^n_+$ cone of $n \times n$ positive semidefinite matrices. 24

$\operatorname{span} C$ smallest subspace containing C. 19

$\mathcal{T}_x C$ tangent space of C at x, i.e., $\operatorname{cl} \operatorname{dir}(x, C) \cap -\operatorname{cl} \operatorname{dir}(x, C)$. 19

Acronyms

CEA Conic Expansion Approach. 29

CLP Conic Linear Programming. 10

FP Forward Procedure. 59

FR Facial Reduction. 12, 14

FRA Facial Reduction Algorithm. 12

IPM Interior Point Method. 11

KKT Karush-Kuhn-Tucker. 10, 17, 80

LICQ Linear Independence Constraint Qualification. 84

LP Linear Programming or Linear Program. 13

MFCQ Mangasarian-Fromovitz Constraint Qualification. 83

NLP Nonlinear Programming or Nonlinear Program. 10

NSDP Nonlinear Semidefinite Programming or Nonlinear Semidefinite Programming. 10

PPS Partial Polyhedral Slater's. 34

SDFP Semidefinite Feasibility Problem. 56

SDP Semidefinite Programming or Semidefinite Program. 13

SOCFP Second Order Cone Feasibility Problem. 62

SOCP Second Order Cone Programming or Second Order Cone Program. 13

SONC Second Order Necessary Condition. 84

SOSC Second Order Sufficient Condition. 82

Chapter 1

Introduction

This thesis is focused on two kinds of conic programming. The first is Conic Linear Programming (CLP) and the second is nonlinear semidefinite programming (NSDP). In the former part, we will focus on different types of ill behavior in CLP. In the latter part, we will discuss a reformulation of NSDP using squared slack variables and the correspondence between Karush-Kuhn-Tucker (KKT) points, optimality and regularity conditions between the original problem and the reformulated version. This reformulation transforms an NSDP into a classical nonlinear program (NLP).

The motivation for the first part comes from the fact that the algorithmic development for conic linear programming was done under regularity assumptions that can fail to hold for certain problems and this make solvers misbehave. Since these are still problems we wish to solve and analyze, we need better alternatives than just giving up. For the latter part, we have both theoretical and practical motivations. The theoretical one is to find an elementary theory of optimality conditions for nonlinear semidefinite programming. From a practical perspective, since nonlinear SDP solvers are still scarce, it makes sense to check the feasibility of NSDPs via NLP solvers.

We now give some historical context behind the line of research advanced in this thesis.

The birth of conic programming most definitely happened with the original development of linear programming by Dantzig, Kantorovich and several others in the '30s, '40s and '50. See the historical notes in section 2.3 of [18]. Linear programming consists of minimizing a linear function subject to inequality/equality constraints and its importance was soon recognized as an efficient tool for solving a range of problems in economy, planning, industry and so on. A classical LP looks like this:

$$\begin{aligned} \inf_{x} \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \geq 0, \end{aligned} \tag{LP}$$

where $c \in \mathbb{R}^n$, $\mathcal{A}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, $b \in \mathbb{R}^m$. The condition "$x \geq 0$" means that it is required that all components of x be nonnegative. For LPs, the algorithm of choice was the simplex method, although LPs with special structures sometimes have their own specialized algorithms, such as the Ford-Fulkerson algorithm for max flow problems [35].

There was also a push towards "general nonlinear programming" which consisted of minimizing a general function subject to different equality and inequality functional constraints. Usually there were some assumptions of continuity or differentiability on the objective function and the constraints. Classical textbooks on the subject include the one by Fiacco and Mccormick [23], and a few by Luenberger [52, 51]. A classical nonlinear program looks like this:

$$\begin{aligned} \inf_{x} \quad & f(x) && \text{(NLP)} \\ \text{subject to} \quad & g_i(x) \le 0, i = 1, \ldots, m_1 \\ & h_j(x) = 0, j = 1, \ldots, m_2, \end{aligned}$$

where f and all the g_i and the h_j are real differentiable functions. Higher degrees of differentiability are assumed accordingly to the need. The textbooks mentioned above contain a range of different methods for NLPs.

As it often happens, researchers were eager to generalize and further extend linear/nonlinear programming to new and possibly interesting directions. In the '60s and '70s, with the advent of convex analysis, conic programming presented itself as a natural generalization in both cases. For instance, the constraint "$x \ge 0$" is the same thing as the constraint "$x \in \mathbb{R}^n_+$", where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \ge 0\}$. One the other hand, the constraints in (NLP) can be written as "$(-g(x), h(x)) \in \mathbb{R}^{m_1}_+ \times \{0\}$", where $g(x)$ is the function that maps x to $(g_1(x), \ldots, g_{m_1}(x)) \in \mathbb{R}^{m_1}$ and $h(x)$ maps x to $(h_1(x), \ldots, h_{m_2}(x)) \in \mathbb{R}^{m_2}$.

Conic programming arises precisely when we consider other choices of (possibly non-polyhedral) cones instead of $\mathbb{R}^n_+$. Here, we make the distinction between *conic linear programming* and just plain *conic programming*. In the former, the objective function is still linear and we only have linear constraints apart from the conic constraints. In the latter, we have general differentiable functions subject to general conic constraints.

There was a lot of theoretical activity on conic programming in the '70s and the '80s in fundamental aspects such as optimality conditions, existence of KKT multipliers, sensitivity, appropriate constraint qualifications and so on. This research is still carried to this day. However, back then, not all of research was translated into algorithmic achievements. At least not at the same degree that happened for LP and NLP. In fact, only recently we have seen solvers capable of handling nonlinear conic constraints, such as PENNON/PENLAB [41, 24] and the Numerical Optimizer, which was formely known as NuOPT [93].

This panorama changed drastically in the '80s and '90s with the introduction of *interior-point methods* (IPMs), first for linear programming then for conic linear programming over the so-called *symmetric cones* [22] which include: the nonnegative orthant, the positive semidefinite matrices, the Lorentz cones and direct products of all these cones. IPMs helped to solve a thorny issue in Linear Programming and Computational Complexity. Namely, it could solve LPs in polynomial time and also had performance comparable to the Simplex method. In comparison, the earlier Ellipsoid method also solves LPs in polynomial time, but in practice it is slow, see sections 7 and 9 of [8].

IPMs for the other symmetric cones are also efficient, although, for a technical reason, we can not claim that they solve, for instance, SDPs in polynomial time in the same sense that it is said that LP is solvable in polynomial time. Roughly speaking, the polynomiality of IPMs in semidefinite programmings refers to the fact that given a nicely behaved (i.e. satisfying regularity conditions) semidefinite program, we can obtain *approximate* solutions in time proportional to a polynomial of the problem size and the desired accuracy. Nevertheless, once it was understood that these types of problems admitted relatively efficient algorithms, researchers found a number of different applications for them, which can be seen in surveys such as [45, 3, 87, 83].

Today, conic linear programming (CLP) is an important tool in optimization, not only from a theoretical point of view, but from a practical one as well. However, there still are a few neglected aspects of CLP. For instance, the fact that the algorithms might breakdown when certain regularity conditions are not satisfied. Note that in the case of linear programming, at least in theory, if one uses the Simplex method it is possible to detect all the corner cases such as infeasibility, unboundedness and so on. However, this is not true for, say, infeasible methods and approaches based on the homogeneous self-dual embeddings for semidefinite programming. Worse yet, these methods might fail without giving a clear indication to the user that something is amiss, see for instance, the article by Waki, Nakata and Muramatsu [90] and also section 2.4.

Therefore, an important point is that solvers today do not perform well when faced with problems with ill behavior. In those cases, they may, for instance, output the wrong optimal value. As CLP becomes more widespread, it is important to have robust solvers that can at least identify that something is wrong. Moreover, we would like, at least in principle, to be able to detect and analyze badly behaved problems. In this sense, our line of research belongs to the bigger theme of trying to regularize a "bad" problem in a way that makes it more amenable to known algorithms and methods. It is not obvious, however, that this is indeed possible in general. Notwithstanding, one of our main contributions is showing that by applying a technique known as *Facial Reduction (FR)* [13, 12] twice, any conic linear program can be regularized into another one that satisfy the regularity conditions that common solvers usually require, see Chapter 5. Facial Reduction Algorithms (FRA) will be a central topic in this work and we will show how to use it to analyze the feasibility properties of a given CLP. In Section 1.1, we discuss in more detail the issues connected to CLP and our contributions to the literature. We should mention, however, that at this stage, the results are still very theoretical, but we hope that this will change in the not so distant future.

As for more general conic programming, the difficulties are compounded, since usually there is no assumption of convexity. And, indeed, there are currently few solvers that are capable of handling general conic constraints, even for the symmetric cones. So, for those problems, there are still lots of work to be done on the development of efficient algorithms. However, symmetric cones are also cones of squares [22], so it is possible to remove the conic constraints by using squared slack variables. This is an investigation that started with the work by Fukuda and Fukushima [29] for nonlinear second order cone programs and in this thesis, we investigate the nonlinear semidefinite programming case. At this point in time, with general conic program solvers still in their infancy, it seems reasonable to check the feasibility of this approach, since it allows us to use well-established NLPs solvers.

1.1 Conic Linear programming

Conic linear programming contains many classes of useful problems, however as we move away from linear programming the chance of encountering ill behaviour increases dramatically. In this thesis, we take a look at handful of them, such as weak infeasibility, nonattainment of optimal values and duality gaps. We will also take a deep look at a regularization technique known as *Facial Reduction*. Let us first give some context.

We will be mainly concerned with problems in the following format:

$$\begin{aligned} \inf_{x} \quad & \langle c, x\rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{K}^* \end{aligned} \tag{P}$$

$$\begin{aligned} \sup_{y} \quad & \langle b, y\rangle \\ \text{subject to} \quad & c - \mathcal{A}^\top y \in \mathcal{K}, \end{aligned} \tag{D}$$

where $\mathcal{E}$ is a finite dimensional real vector space, $\mathcal{K} \subseteq \mathcal{E}$ is a closed convex cone and $\mathcal{K}^*$ is the dual cone $\{s \in \mathcal{E} \mid \langle s, x\rangle \geq 0, \forall x \in \mathcal{K}\}$. We have that $\mathcal{A} : \mathcal{E} \to \mathbb{R}^m$ is a linear map,

$b \in \mathbb{R}^m$, $c \in \mathcal{E}$ and $\mathcal{A}^\top$ denotes the adjoint map. We also have $\mathcal{A}^\top y = \sum_{i=1}^m \mathcal{A}_i y_i$, for certain elements $\mathcal{A}_i \in \mathcal{E}$. The inner product for the corresponding spaces is denoted by $\langle .,. \rangle$. We will denote by θ_P and θ_D, the primal and dual optimal values, respectively. It is understood that $\theta_P = +\infty$ if (P) is infeasible and $\theta_D = -\infty$ if (D) is infeasible. We now mention three classes of problems that fit the framework above.

1. If $\mathcal{E} = \mathbb{R}^n$ and $\mathcal{K}$ is the nonnegative orthant $\mathbb{R}^n_+$, then the pair composed by (P) and (D) is a *Linear Program* (LP).
2. If $\mathcal{E} = \mathbb{R}^n$ and $\mathcal{K}$ is the direct product of Lorentz cones $\mathcal{Q}^{n_i} = \{x \in \mathbb{R}^{n_i} \mid x_1 \geq 0, x_1^2 \geq x_2^2 + \ldots + x_{n_i}^2\}$, then we have a *Second Order Cone Program* (SOCP).
3. Let $\mathcal{E} = \mathcal{S}^n$ be the space of $n \times n$ symmetric matrices, $\mathcal{K} = \mathcal{S}^n_+$ the cone of $n \times n$ positive semidefinite matrices and $\langle x, y \rangle = \operatorname{trace}(xy)$. In this case, we have a Semidefinite Program (SDP).

One of the standard ways of solving the problems above is to use interior point methods (IPMs) [59, 74]. However, using IPMs require that the problem at hand satisfies certain regularity conditions in order to work correctly. Typically, it is required that both (P) and (D) have relative interior feasible solutions and if this fails we may have *nonzero duality gaps* and *nonattainment*, see examples in [55, 56, 86]. A problem has zero duality gap if $\theta_D = \theta_P$. Primal nonattainment occurs when θ_P is finite but there is no primal feasible solution that achieves θ_P. Dual nonattainment is defined similarly.

Also, attached to an optimization problem we have the *feasibility problem*. Let $\mathcal{L}$ be a subspace of $\mathcal{E}$ and $c \in \mathcal{E}$. In the feasibility problem, we seek a point in the intersection of $(\mathcal{L} + c) \cap \mathcal{K}$ or some certificate that attests that no such point exists. In Linear Programming, we have the Farkas' Lemma which states that either $(\mathcal{L} + c) \cap \mathbb{R}^n_+ \neq \emptyset$ or there is some $s \in \mathcal{L}^\perp \cap \mathbb{R}^n_+$ such that $\langle s, c \rangle < 0$, where $\mathcal{L}^\perp$ denotes the orthogonal complement to $\mathcal{L}$. In this way, the *infeasibility* of a system of linear inequalities is equivalent to the *feasibility* of another system of linear inequalities. However, for nonpolyhedral cones, the situation is significantly more complicated and this is partly due to the presence of a phenomenon called *weak infeasibility*. A feasibility problem is said to be weakly infeasible if $(\mathcal{L} + c) \cap \mathcal{K} = \emptyset$ but there are points in $\mathcal{L} + c$ arbitrarily close to $\mathcal{K}$. Weak infeasibility cannot happen in linear programming, but it can happen for general linear conic programs and it complicates significantly the search for infeasibility certificates.

In order to ameliorate some of these issues, regularization procedures have been developed such as the Facial Reduction scheme by Borwein and Wolkowicz [13, 12]. However, the application of Facial Reduction to conic linear programs is more recent and there are still many outstanding issues to be solved and discussed. In this thesis we will take a look at a few of them.

1.1.1 Summary of the main results

The initial motivating question behind this thesis was the study of weakly infeasible problems. A conic linear feasibility problem is the problem of seeking for some $x \in (\mathcal{L} + c) \cap \mathcal{K}$, where $\mathcal{L} + c$ is an affine subspace and $\mathcal{K}$ is an arbitrary closed convex cone. The problem is said to be *weakly infeasible* if $(\mathcal{L} + c) \cap \mathcal{K} = \emptyset$ but the Euclidean distance between $\mathcal{L} + c$ and $\mathcal{K}$ is zero. We were concerned about the structure of weakly infeasible problems and with understanding precisely how they arise and how to explicitly construct points in $\mathcal{L} + c$ that are arbitrary close to $\mathcal{K}$. Our first contribution to the literature was as follows:

- *structural analysis of weakly infeasible problems.* In 2013, we proved that if $\mathcal{K} = \mathcal{S}^n_+$, then there is an affine subspace $\mathcal{V}$ contained in $\mathcal{L} + c$ with dimension at most $n - 1$ such that $\operatorname{dist}(\mathcal{V}, \mathcal{S}^n_+) = 0$ [47]. The meaning behind it is that starting from a special point s, we can approach $\mathcal{S}^n_+$ by using at most $n-1$ directions. Moreover, we showed that there is a "hierarchical" relaxation between those directions. Once those directions are found, we can explicitly construct points close to $\mathcal{S}^n_+$ without solving extra SDPs.

Later on, a similar result was proved for the case of feasibility problems over a direct product of Lorentz cones $\mathcal{K} = \mathcal{Q}^{n_1} \times \ldots \times \mathcal{Q}^{n_r}$, where r is the number of Lorentz cones. Similarly, we showed the existence of an affine subspace of dimension at most r such that $\operatorname{dist}(\mathcal{V}, \mathcal{K}) = 0$ [49]. Although SOCP is a particular case of SDP, this bound does not follow directly from the SDP case, so it merited a tailored analysis.

In 2015, Liu and Pataki [44] gave a generalization of sorts of our results for arbitrary closed convex cones. However, their more general result provided a slightly worse bound for the dimension of the space for the cases $\mathcal{K} = \mathcal{S}^n_+$ and $\mathcal{K} = \mathcal{Q}^{n_1} \times \ldots \times \mathcal{Q}^{n_r}$. In order to match our bound, they gave an alternative argument for the case $\mathcal{K} = \mathcal{S}^n_+$ but no special argument was given for the case $\mathcal{K} = \mathcal{Q}^{n_1} \times \ldots \times \mathcal{Q}^{n_r}$. In this thesis, we give an overarching result that includes our previous bounds as special cases and improves on Liu and Pataki's result, see also [48]. It is connected to two quantities we introduced named *distance to strong duality* and *distance to polyhedrality*, see sections 3.3.2 and 3.3.3.

One of the reasons for studying weakly infeasible problems is because they are intimately connected with optimization problems with unattained optima. Namely, if $\theta_P = \inf\{\langle c, x\rangle \mid \mathcal{A}x = b, x \in \mathcal{K}\}$ is finite but not attained then the problem of finding a point satisfying $\mathcal{A}x = b$, $x \in \mathcal{K}$ and $\langle c, x\rangle = \theta_P$ is weakly infeasible. So an approach to construct points arbitrarily close to a cone naturally produces an approach for constructing feasible solutions that are almost optimal.

When we did the analysis for weakly infeasible SDPs, one of the things that were in our minds was that in order to find the directions that compose the affine space $\mathcal{V}$ we need to solve auxiliary SDPs. We would like, at least in principle, to actually compute those directions if we are given an SDP instance. This poses the question of deciding what would be reasonable assumptions concerning our capability of solving SDPs.

A similar issue appears in the facial reduction literature and, although it was not clear to us in our initial analysis of weak infeasibility, it turned out that our approach had indeed a very strong connection to FR [13, 12]. FR aims to restore strict feasibility by finding a sequence of so-called reducing directions. And after it is finished, the final aim is to actually solve the optimization problem. It seemed to us that the situation was a bit awkward. After all, it is necessary to solve some auxiliary CLP, so some solving capability was assumed implicitly in the literature. However, if we assume that we can solve CLPs, what is really the point of doing facial reduction? We could use whatever *oracle* we have at hand to solve those auxiliary problems and directly attack the original problem instead.

An answer to that came from thinking about the nature of the most common methods for solving conic linear problems: *Interior Point Methods* (IPM). IPMs require that (P) and (D) have both relative interior feasible points, which means that they are both strictly feasible. By using infeasible methods [60] or self-dual embeddings [71, 19, 56], it is possible to relax this requirement somewhat. Still, if either (P) or (D) fails to satisfy strict feasibility, both approaches can misbehave. Moreover, even if an IPM is not used, it is still common to require primal *and* dual strict feasibility since this is a time-honored assumption that ensures simultaneously zero duality gap together with primal and dual attainment.

One of our contribution was to show that the problem of finding reducing directions in Facial Reduction can be cast as a pair of $(P_\mathcal{K})$ and $(D_\mathcal{K})$ of problems that always satisfy strict feasibility, even if (P) and (D) do not. In this sense, $(P_\mathcal{K})$ and $(D_\mathcal{K})$ are always nicely behaved. Therefore, in order to do facial reduction, we do not need to assume that we can solve arbitrary CLPs but, merely that there is an interior point oracle $\mathcal{O}_{\text{int}}$ that gives primal and dual solutions to problems that are both primal and dual strictly feasible.

It then dawned to us that we could use this kind of idea to prove a few theoretical results about the relationship between optimization and feasibility problems in conic programming. So our next two contributions were

- *completely solving CLPs with an interior point oracle.* Assuming that we have $\mathcal{O}_{\text{int}}$ at hand, we can "completely solve" *any* CLP in the following sense. We say that a scheme/method/algorithm "completely solves" (D) if is able to detect whether the problem is infeasible or feasible. If feasible, it should tell us if (D) is unbounded. If it is bounded, it must return the optimal value θ_D and an optimal solution if it exists. If no optimal solution exists, given any $\epsilon > 0$ it returns an ϵ-optimal feasible solution. When (D) is infeasible, the scheme must be able to provide a certificate of infeasibility and distinguish between strong and weak infeasibility. Finally, for weakly infeasible problems, given any $\epsilon > 0$, it returns a point in $c - \operatorname{range} \mathcal{A}^\top$ whose distance to $\mathcal{K}$ is at most ϵ.

  Initially, we proved this result for SDPs [49], but in this thesis we will present a generalization for arbitrary closed convex cones. Surprisingly, we did not find much literature on how to comprehensively solve an arbitrary CLPs, except for a chapter written by de Klerk, Terlaky and Roos, see section 5.10 of [19]. However, their assumptions are more restrictive, since they assume not only the capability of solving problems that are primal and dual strictly feasible but they also assume that maximal complementary solutions are obtainable. Moreover, their analysis is for SDPs only and uses Ramana's extended dual, which is a tool that is very specific to SDP theory.

- *equivalence of optimization and feasibility for general CLPs.* We will show that conic optimization problem and the conic feasibility problem are polynomially equivalent. Namely, if there is a method to solve the feasibility problem, with polynomially many calls to that method, we may solve the optimization problem and vice-versa. Again, surprisingly, we could not find much literature on the general conic case. For Linear Programming, the result is trivial. The SDP case, on the other hand, is surprisingly involved and is a consequence of Ramana's work [72] on extended duality. In Chapter 5, we will explain some of the subtler details associated to this question.

In order to prove the two results above, we had to take a closer look at facial reduction and we end up proving a few theoretical properties of facial reduction algorithms. In addition, we developed a new algorithm that, in many cases, has a better worst case complexity. So the last two contributions in this part of this thesis are:

- *theoretical properties of FRAs.* When applying FR to (D), the dual feasible region stays the same but the primal feasible region might expand. We proved that this expansion can only affect the feasibility properties of (P) in a very limited way. We then use this result to show that applying Facial Reduction twice is enough to obtain a pair of problems that are always strictly feasible.

- *FRA-Poly, a facial reduction algorithm that takes polyhedrality into account. [48]* Facial reduction algorithms work by successively identifying what is called "reducing directions" $\{d_1, \ldots, d_\ell\}$. Starting with $\mathcal{F}_1 = \mathcal{K}$, these directions define faces of $\mathcal{K}$ by the relation $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{d_i\}^\perp$. Of course, these directions are not arbitrary and for feasible problems, d_i must be such that $\mathcal{F}_{i+1}$ is a face of $\mathcal{K}$ containing the dual feasible slacks $\mathcal{F}^S_{\mathrm{D}}$. We then obtain a sequence $\mathcal{F}_1 \supseteq \ldots \supseteq \mathcal{F}_\ell$ of faces of $\mathcal{K}$ such that $\mathcal{F}_i \supseteq \mathcal{F}^S_{\mathrm{D}}$ for every i. Usually, a FRA proceeds until $\mathcal{F}^D_{\min}$ is found.

A key observation is that as soon as we reach a polyhedral face $\mathcal{F}_i$, we can jump to the minimal face $\mathcal{F}^D_{\min}$ in a single facial reduction step. In addition, when $\mathcal{K}$ is a direct product $\mathcal{K} = \mathcal{K}^1 \times \ldots \times \mathcal{K}^r$, each $\mathcal{F}_i$ is also a direct product $\mathcal{F}_i^1 \times \ldots \times \mathcal{F}_i^r$. In this case an even weaker condition is sufficient to jump to $\mathcal{F}^D_{\min}$, namely, if every block $\mathcal{F}_i^j$ satisfy one of the two conditions: a) $\mathcal{F}_i^j$ is polyhedral b) every $s \in \mathcal{F}_i^* \cap \ker \mathcal{A} \cap \{c\}^\perp$ is such that $s^j \in (\mathcal{F}_i^j)^\perp$. When the problem is feasible, the condition b) is equivalent to the statement that the j-th block of $\mathcal{F}^D_{\min}$ is equal to $\mathcal{F}_i^j$. In summary, we can jump to the minimal face if either a block is polyhedral or it is already a piece of the minimal face.

Our proposed algorithm FRA-Poly works in two phases. In Phase 1, it proceeds until a face $\mathcal{F}_i$ satisfying the condition above is reached. To do that, we model the search for reducing directions as a pair of primal and dual problems satisfying a generalized Slater's condition that takes into account polyhedrality. In Phase 2, $\mathcal{F}^D_{\min}$ is obtained with single facial reduction step. One interesting point is that even if $\mathcal{F}_i$ is different the minimal face, it is possible to show that if we reformulate (D) as a problem over $\mathcal{F}_i$, then strong duality will hold.

In order to analyze the number of facial reduction steps in the worst case, we introduce a measure called *distance to polyhedrality* $\ell_{\text{poly}}(\mathcal{K})$. This is the length *minus one* of the longest strictly ascending chain of non-empty faces $\mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_\ell$ for which $\mathcal{F}_1$ is polyhedral and $\mathcal{F}_i$ is not polyhedral for all $i > 1$. If $\mathcal{K}$ is a direct product of arbitrary cones $\mathcal{K}^1 \times \ldots \times \mathcal{K}^r$, we prove that FRA-Poly stops in at most $1 + \sum_{i=1}^r \ell_{\text{poly}}(\mathcal{K}^i)$ steps. This quantity is no worse than the bound given by classical FRAs and, under mild conditions, it is strictly smaller.

1.2 Nonlinear SDP

In the second part of this thesis we will be concerned about problems of the following type.

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & G(x) \in \mathcal{S}^m_+, \end{array} \tag{P1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $G\colon \mathbb{R}^n \to \mathcal{S}^m$ are twice continuously differentiable functions, $\mathcal{S}^m$ is the linear space of all real symmetric matrices of dimension $m \times m$, and $\mathcal{S}^m_+$ is the cone of all positive semidefinite matrices in $\mathcal{S}^m$.

We investigate the possibility of removing the conic constraints and transforming (P1) into a conventional nonlinear program using slack variables. We denote by $\circ$ the Jordan product [22], which is the bilinear operator satisfying

$$W \circ Z \doteq \frac{WZ + ZW}{2},$$

for every $W, Z \in \mathcal{S}^m$. Because $\mathcal{S}^m_+ = \{Y \circ Y \mid Y \in \mathcal{S}^m\}$, we reformulate (P1) as

$$\begin{array}{ll} \underset{x,Y}{\text{minimize}} & f(x) \\ \text{subject to} & G(x) - Y \circ Y = 0. \end{array} \tag{P2}$$

Note that problem (P2) no longer has conic constraints and they are, instead, replaced by $\frac{n(n+1)}{2}$ equality constraints. We now give a few reasons why one might want to do that. The first reason is merely for theoretical curiosity. The optimality theory for (P1) is vastly different from the somewhat classical theory for (P2). It might be fair to say that second-order conditions for (P1) is a relatively "deep" subject and includes a discussion of topics such as metric regularity and second-order tangent sets, which are topics that only reached maturity in the mid-'80s and '90s together with the development of variational analysis. So it is, perhaps, a theoretically interesting question how the second order conditions, KKT points and regularity conditions of (P1) and (P2) are related.

One of the motivation for this study is to derive optimality conditions for (P1) through elementary means. It might be fair to say that deriving SOCs for (P1) is a more intricate affair than carrying out the same task for classical nonlinear programs. That is one of the reasons why there is some interest in alternative ways of deriving optimality conditions for (P1), see [25, 34]. In this sense, it is surprising that, under strict complementarity, second order conditions for (P1) and (P2) are essentially the same [46]. More specifically, we will show that second order conditions for (P2) generate "no-gap" conditions for (P1) that are equivalent to the more technical conditions developed by Shapiro [77]. Our approach, therefore, provides an elementary path to optimality conditions for nonlinear semidefinite programs (NSDPs).

There is also a practical side for this kind of inquiry. Conventional wisdom would say that using squared slack variables is a bad idea, but, in reality, even for linear SDPs there are good reasons to (sometimes) use such variables. See the work by Burer and Monteiro [14, 15].

Moreover, while there are a number of solvers for linear SDPs, as we move to general nonlinear programs, the situation changes drastically. In fact, as far as we known, PENLAB is the only open-source general NLP solver that can handle nonlinear semidefinite constraints. So, it seems that it makes sense to check the feasibility of solving (P1) via (P2), as it enables one to use different well-tested and high-quality solvers.

Our main contribution in this part is showing the correspondence between second order conditions, KKT points and regularity conditions of (P1) and (P2), see also [46]. We also have a few computational experiments where nonlinear SDPs are solved with both an augmented Lagrangian algorithm for NSDPs and via slack variables. Their performance are then compared. Somewhat surprisingly, the slack variables approach was competitive for many problem instances.

Part I

Linear Conic Programming - Weak Infeasibility and Facial Reduction

Chapter 2

Preliminary notions

In this chapter, we set up the notation and collect some well-known results that will be needed in the next chapters. Recall that we are interested in the following pair of problems:

$$\begin{aligned} \inf_{x} \quad & \langle c, x \rangle \\ \text{subject to} \quad & \mathcal{A}x = b \\ & x \in \mathcal{K}^* \end{aligned} \tag{P}$$

$$\begin{aligned} \sup_{y} \quad & \langle b, y \rangle \\ \text{subject to} \quad & c - \mathcal{A}^\top y \in \mathcal{K}, \end{aligned} \tag{D}$$

where $\mathcal{E}$ is a finite dimensional real vector space, $\mathcal{K} \subseteq \mathcal{E}$ is a closed convex cone and $\mathcal{K}^*$ is the *dual cone* $\{s \in \mathcal{E} \mid \langle s, x \rangle \geq 0, \forall x \in \mathcal{K}\}$. We have that $\mathcal{A} : \mathcal{E} \to \mathbb{R}^m$ is a linear map, $b \in \mathbb{R}^m$, $c \in \mathcal{E}$ and $\mathcal{A}^\top$ denotes the adjoint map. We also have $\mathcal{A}^\top y = \sum_{i=1}^m \mathcal{A}_i y_i$, for certain elements $\mathcal{A}_i \in \mathcal{E}$. The inner product is denoted by $\langle \cdot, \cdot \rangle$. The norm induced by $\langle \cdot, \cdot \rangle$ will be noted by $\|\cdot\|$. We will denote by θ_P and θ_D, the primal and dual optimal values, respectively. It is understood that $\theta_P = +\infty$ if (P) is infeasible and $\theta_D = -\infty$ if (D) is infeasible.

We denote the *dual feasible region* by $\mathcal{F}_{\mathrm{D}} = \{y \in \mathbb{R}^m \mid c - \mathcal{A}^\top y \in \mathcal{K}\}$. We also write $\mathcal{F}_{\mathrm{D}}^S$ for the *"slack space"* $\mathcal{F}_{\mathrm{D}}^S = \{c - \mathcal{A}^\top y \mid y \in \mathcal{F}_{\mathrm{D}}\}$. The *primal feasible region* is $\mathcal{F}_{\mathrm{P}} = \{x \in \mathcal{K}^* \mid \mathcal{A}x = b\}$.

2.1 Convex sets, faces and separation theorems

Let C be a closed convex set contained in $\mathcal{E}$. Its *interior*, *relative interior*, *closure* and *relative boundary* are denoted by $\operatorname{int} C$, $\operatorname{ri} C$, $\operatorname{cl} C$ and $\operatorname{relbd} C$, respectively. We also write $\operatorname{span} C$ for the smallest subspace of $\mathcal{E}$ containing C and $\operatorname{aff} C$ for the smallest affine subset of $\mathcal{E}$ containing C. Recall that the relative interior of C is the interior with respect the topology induced by $\operatorname{aff} C$. Additionally, if $\mathcal{K}$ is a closed convex cone, we denote by $\operatorname{lin} \mathcal{K}$ the *lineality space* of $\mathcal{K}$, which is the largest subspace contained in $\mathcal{K}$.

For a given $x \in C$, we write $\operatorname{dir}(x, C)$ for the *cone of feasible directions* of C at x . This is the set $\{z \in \mathbb{R}^n \mid \exists t > 0, x + tz \in C\}$. The closure of $\operatorname{dir}(x, C)$ is the *tangent cone* of C at x and is denoted by $\operatorname{cl} \operatorname{dir}(x, C)$. The *tangent space* of C at x is the set $\mathcal{T}_x C = \operatorname{cl} \operatorname{dir}(x, C) \cap -\operatorname{cl} \operatorname{dir}(x, C)$. A *supporting hyperplane* of C at x is a hyperplane H for which $x \in H$ and C is entirely contained in one of the half-spaces defined by H. If A is an arbitrary set, we denote by $A^\perp$ the subspace which contains the elements orthogonal to it.

A convex subset $\mathcal{F}$ of C is said to be a *face* if the condition $\alpha x + (1-\alpha)y \in \mathcal{F}$ with $\alpha \in (0,1)$ and $x, y \in C$ implies $x, y \in \mathcal{F}$. If $\mathcal{F}$ is a face of $\mathcal{K}$, we define the *conjugated face* of $\mathcal{F}$ as $\mathcal{F}^\Delta = \mathcal{K}^* \cap F^\perp$. If we select a point x in the relative interior of $\mathcal{F}$, we have $\mathcal{F}^\Delta = \mathcal{K}^* \cap \{x\}^\perp$. Note that the elements of $\mathcal{F}^\Delta$ define the supporting hyperplanes of $\mathcal{K}$ passing through x. Given $z \in \mathcal{K}$, we will use the notation $\mathcal{F}(z, \mathcal{K})$ to denote the minimal face of $\mathcal{K}$ which contains z.

Two convex sets are said to be *properly separated* if there is some hyperplane such that the sets are contained in opposite closed half-spaces and at least one of them is not entirely contained in the hyperplane. The following result characterizes proper separation.

Theorem 2.1. *Let C_1 and C_2 be two non-empty convex sets. Then $\operatorname{ri} C_1 \cap \operatorname{ri} C_2 = \emptyset$ if and only if C_1 and C_2 can be* properly separated.

Proof. See Theorem 11.3 in [75]. □

If one of the sets is polyhedral, we have the following separation result.

Theorem 2.2. *Let C_1 and C_2 be two non-empty convex sets, with C_1 a polyhedral set. Then $C_1 \cap \operatorname{ri} C_2 = \emptyset$ if and only if C_1 and C_2 can be* properly separated *with a hyperplane that does not contain C_2.*

Proof. See Theorem 20.2 in [75]. □

We also have the notion of *strong separation*. Two convex sets C_1 and C_2 can be strongly separated if there exists a separating hyperplane H and $\epsilon > 0$ such that $C_1 + \epsilon B$ and $C_2 + \epsilon B$ lie in opposite open half-spaces, where B is the unit ball. The following result characterizes strong separation.

Theorem 2.3. *Let C_1 and C_2 be two non-empty convex sets. Then C_1 and C_2 can be strongly separated if and only if the Euclidean distance between C_1 and C_2 satisfies* $\operatorname{dist}(C_1, C_2) = \inf\{\|x-y\| \mid x \in C_1, y \in C_2\} > 0$.

Proof. See Theorem 11.4 in [75]. □

Given a closed convex set C and D a convex subset contained in C, we denote by $\mathcal{F}(D, C)$ the *minimal face of C that contains D*. The following well-known result characterizes $\mathcal{F}(D, C)$.

Proposition 2.4. *Let $\mathcal{F}$ be a non-empty face of C and D a convex subset contained in $\mathcal{F}$. Then the following are equivalent.*

i. $\mathcal{F}(D, C) = \mathcal{F}$;

ii. $\operatorname{ri} D \cap \operatorname{ri} \mathcal{F} \neq \emptyset$;

iii. $\operatorname{ri} D \subseteq \operatorname{ri} \mathcal{F}$.

Proof. (*i.* ⇒ *ii.*) If $\operatorname{ri} D \cap \operatorname{ri} \mathcal{F} = \emptyset$, then D and $\mathcal{F}$ can be properly separated. This means that there is an hyperplane H such that D and $\mathcal{F}$ lie in opposite closed half-spaces but at least one of them is not entirely contained in H. However, since $D \subseteq \mathcal{F}$, it must be the case that D is entirely contained in H. Therefore, $\mathcal{F}$ is not entirely contained in H. Because $\mathcal{F}$ lies in one of the closed half-spaces defined by H, H must be a supporting hyperplane of $\mathcal{F}$. Gathering all these facts, we obtain that $\mathcal{F} \cap H$ is a proper face of $\mathcal{F}$ containing D. Since, a face of $\mathcal{F}$ is also a face of $\mathcal{K}$, this contradicts the minimality of $\mathcal{F}$.

(*ii.* ⇒ *iii.*) Let $x \in \operatorname{ri} D \cap \operatorname{ri} \mathcal{F}$ and $y \in \operatorname{ri} D$. Then, due to theorem Theorem 6.4 of [75] there exists $\alpha > 1$ such that $u = (1-\alpha)x + \alpha y$ and $u \in D (\subseteq \mathcal{F})$. This means that $y = \frac{1}{\alpha}u + \frac{(\alpha-1)}{\alpha}x$. So y is a non-trivial convex combination of a relative interior point of $\mathcal{F}$

and some other arbitrary point also belonging to $\mathcal{F}$. Therefore, it must be relative interior point of $\mathcal{F}$ as well, see Theorem 6.1 in [75].

($iii. \Rightarrow i.$) Suppose that $\mathcal{F}$ is not the minimal face and let $\hat{\mathcal{F}}$ be the minimal face. Then, by what we proved so far, we have $\operatorname{ri} D \subseteq \operatorname{ri} \hat{\mathcal{F}}$. This means that $\operatorname{ri} \hat{\mathcal{F}} \cap \operatorname{ri} \mathcal{F} \neq \emptyset$, which can only happen if $\hat{\mathcal{F}} = \mathcal{F}$. □

We now gather a few facts that we will need later.

Lemma 2.5. *Let $\mathcal{K}$ be a closed convex cone, $e \in \operatorname{ri} \mathcal{K}$, $x \in \mathcal{K}$ and $z \in \mathcal{K}^*$.*

i. $\mathcal{K}^{**} = \mathcal{K}$.

ii. $\operatorname{lin} \mathcal{K} = \mathcal{K}^{*\perp}$, *where $\mathcal{K}^{*\perp}$ is a short-hand for $(\mathcal{K}^*)^\perp$.*

iii. $\mathcal{K}^\perp = \operatorname{lin}(\mathcal{K}^*)$

iv. $x + e \in \operatorname{ri} \mathcal{K}$.

v. *There exists $\alpha > 1$ such that $\alpha e + (1-\alpha)x \in \mathcal{K}$.*

vi. $z \in \mathcal{K}^\perp$ *if and only if* $\langle e, z\rangle = 0$.

vii. $(\operatorname{cl} \operatorname{dir}(x, \mathcal{K}))^* = \mathcal{F}(x,\mathcal{K})^\Delta$

viii. $\mathcal{T}_x\mathcal{K} = \mathcal{F}(x,\mathcal{K})^{\Delta\perp}$. *That is, the tangent space of x at $\mathcal{K}$ is the intersection of the supporting hyperplanes of x at $\mathcal{K}$.*

ix. *If $w \in \operatorname{cl} \operatorname{dir}(x,\mathcal{K})$ then* $\lim_{t\to+\infty} \operatorname{dist}(tx + w, \mathcal{K}) = 0$.

Proof. *i.* This is the famous bipolar theorem, see Theorem 14.1 of [75].

ii. If $z \in \operatorname{lin} \mathcal{K}$, then $\langle z, y\rangle \geq 0$ and $\langle -z, y\rangle \geq 0$, for every $y \in \mathcal{K}^*$. It follows that $z \in \mathcal{K}^{*\perp}$. Reciprocally, if $z \in \mathcal{K}^{*\perp}$, then $z \in \mathcal{K}^{**} = \mathcal{K}$, by the bipolar theorem. Since $\mathcal{K}^{*\perp}$ is a subspace, we have $\mathcal{K}^{*\perp} \subseteq \operatorname{lin} \mathcal{K}$.

iii. It follows from applying *ii.* to $\mathcal{K}^*$.

iv. Since $e \in \operatorname{ri} \mathcal{K}$, for any $z \in \mathcal{K}$ we have that all points in the relative interior of the line segment connecting z and e also belong to the relative interior of $\mathcal{K}$, see Theorem 6.1 of [75]. Since $x + e = e\frac{1}{2} + (2x + e)\frac{1}{2}$, we have $x + e \in \operatorname{ri} \mathcal{K}$.

v. See Theorem 6.4 of [75].

vi. If $z \in \mathcal{K}^\perp$, it is clear that $\langle e, z\rangle$ is zero. Now, suppose that $\langle e, z\rangle$ is zero. By item *iv*, there is $\alpha > 1$ such that $\alpha e + (1-\alpha)x \in \mathcal{K}$. On one hand, since $z \in \mathcal{K}^*$, we have $\langle u, z\rangle \geq 0$. On the other, $\langle u, z\rangle = (1-\alpha)\langle x, z\rangle \leq 0$. So, we must have $\langle x, z\rangle = 0$. As x is an arbitrary element, it holds that $z \in \mathcal{K}^\perp$.

vii. First we show that $\mathcal{F}(x,\mathcal{K})^\Delta \subseteq (\operatorname{cl} \operatorname{dir}(x,\mathcal{K}))^*$. Since a set and its closure have the same dual, we have $\operatorname{dir}(x,\mathcal{K})^* = (\operatorname{cl} \operatorname{dir}(x,\mathcal{K}))^*$. If $s \in \mathcal{F}(x,\mathcal{K})^\Delta$ and $z \in \operatorname{dir}(x,\mathcal{K})$, then we have $\langle s, x + tz\rangle \geq 0$, for some $t > 0$. Because $\langle s, x\rangle = 0$, we must have $\langle s, z\rangle \geq 0$, which shows $s \in \operatorname{dir}(x, C)^*$.

Now, suppose that $s \in (\operatorname{cl} \operatorname{dir}(x,\mathcal{K}))^*$. Because $\mathcal{K} \subseteq \operatorname{cl} \operatorname{dir}(x,\mathcal{K})$, we have that $s \in \mathcal{K}^*$. In addition, since both x and $-x$ belong to $\operatorname{cl} \operatorname{dir}(x,\mathcal{K})$, we have $\langle s, x\rangle = 0$.

viii. Follows from *vii.* and *iii.*

ix. Since $\mathcal{K}$ is a closed convex cone, we have $\operatorname{dist}(a+b,\mathcal{K}) \leq \operatorname{dist}(a,\mathcal{K})+\operatorname{dist}(b,\mathcal{K})$, for all $a, b \in \mathcal{E}$. Now, for every $\epsilon > 0$, there exists $w_\epsilon \in \operatorname{dir}(x,\mathcal{K})$ such that $\operatorname{dist}(w, w_\epsilon) < \epsilon$. Moreover, there exists t_ϵ such that $t_\epsilon x + w_\epsilon \in \mathcal{K}$. It follows that

$$\begin{aligned}\operatorname{dist}(tx+w,\mathcal{K}) &\leq \operatorname{dist}(tx+w_\epsilon,\mathcal{K}) + \operatorname{dist}(w-w_\epsilon,\mathcal{K})\\ &\leq \operatorname{dist}(tx+w_\epsilon,\mathcal{K}) + \epsilon,\end{aligned}$$

where the last inequality follows from the fact that $0 \in \mathcal{K}$, so $\operatorname{dist}(w-w_\epsilon,\mathcal{K}) \leq \operatorname{dist}(w-w_\epsilon, 0)$. However, since $t_\epsilon x+w_\epsilon \in \mathcal{K}$, we must have $\lim_{t\to+\infty} \operatorname{dist}(tx+w_\epsilon,\mathcal{K}) = 0$, since for t sufficiently large we have $tx + w_\epsilon \in \mathcal{K}$. It follows that $\lim_{t\to+\infty} \operatorname{dist}(tx+w,\mathcal{K}) \leq \epsilon$. Since ϵ is arbitrary, we conclude that *ix.* must hold.

□

2.2 Feasibility statuses and Slater's condition

We can separate (D) in four different feasibility classes:

1. *strongly feasible*: if there is $y \in \mathbb{R}^m$ such that $c - \mathcal{A}^\top y \in \operatorname{ri}\mathcal{K}$;
2. *weakly feasible*: if (D) is feasible but $(c - \operatorname{range}\mathcal{A}^\top) \cap \operatorname{ri}\mathcal{K} = \emptyset$;
3. *weakly infeasible*: if $(c - \operatorname{range}\mathcal{A}^\top) \cap \mathcal{K} = \emptyset$, but the Euclidean distance between $c - \operatorname{range}\mathcal{A}^\top$ and $\mathcal{K}$ satisfies $\operatorname{dist}(c - \operatorname{range}\mathcal{A}^\top, \mathcal{K}) = 0$;
4. *strongly infeasible*: if $\operatorname{dist}(c - \operatorname{range}\mathcal{A}^\top, \mathcal{K}) > 0$.

Note that (P) admits analogous definitions, with the affine space $\mathcal{V} = \{x \in \mathcal{E} \mid \mathcal{A}x = b\}$ in place of $c - \operatorname{range}\mathcal{A}^\top$. By convention, if $\mathcal{V} = \emptyset$ then (P) is strongly infeasible. Sometimes we will group 2. and 3. together and say that a problem is in *weak status* if it is either weakly infeasible or weakly feasible.

Using separation theorems, we can characterize *not* strong feasibility as follows, see also Lemma 3.2 in [89].

Proposition 2.6. *i.* (D) *is not strongly feasible if and only if there is* $x \in \ker\mathcal{A} \cap \mathcal{K}^*$ *such that one of the following two conditions is true: i)* $x \notin \mathcal{K}^\perp$ *and* $\langle c, x\rangle = 0$*; ii)* $\langle c, x\rangle < 0$.

ii. Suppose that there exists some $\hat{x}$ *(not necessarily feasible) such that* $\mathcal{A}\hat{x} = b$. (P) *is not strongly feasible if and only if there exists* $y \in \mathbb{R}^m$ *with* $s = -\mathcal{A}^\top y \in \mathcal{K}$ *such that one of the two conditions is true: i)* $s \notin \operatorname{lin}\mathcal{K}$ *and* $\langle b, y\rangle = 0$*; ii)* $\langle b, y\rangle > 0$.

Proof. *i.* If *ii)* holds, (D) must be infeasible, so in particular, it is not strongly feasible. If *i)* holds and y is a dual feasible solution, then $\langle x, c - \mathcal{A}^\top y\rangle = \langle x, c\rangle = 0$. Since $x \notin \mathcal{K}^\perp$, we have that $c - \mathcal{A}^\top y \notin \operatorname{ri}\mathcal{K}$, by item *vi.* of Lemma 2.5.

Now, suppose that (D) is not strongly feasible, this means that $\operatorname{ri}\mathcal{K} \cap \mathcal{F}_{\mathrm{D}}^S = \emptyset$, which holds if and only if the nonempty polyhedral set $C = \{x \mid \mathcal{A}x = b\}$ and $\mathcal{K}$ can be properly separated. Proper separation implies the existence of a nonzero $x \in \mathcal{E}$ and $\gamma \in \mathbb{R}$ such that

$$\langle x, c - \mathcal{A}^\top y\rangle \leq \gamma \leq \langle x, k\rangle, \tag{2.1}$$

for every $y \in \mathbb{R}^m$ and $k \in \mathcal{K}$. The only way (2.1) can hold is if $x \in \mathcal{K}^* \cap \ker\mathcal{A}$ and $\gamma \leq 0$. Moreover, the fact that C is polyhedral and Theorem 2.2 imply that we can also assume that $\mathcal{K}$ is not contained in the hyperplane $\{z \mid \langle x, z\rangle = \gamma\}$. We then have two cases. If $\langle x, c\rangle = 0$, then $\gamma = 0$, which implies that $x \notin \mathcal{K}^\perp$, in order for the separation to be proper. This implies that *i)* holds. On the other hand, if $\langle x, c\rangle < 0$, then *ii)* holds.

ii. The dual proof is analogous. The condition $\operatorname{ri}\mathcal{K}^* \cap \mathcal{F}_{\mathrm{P}} = \emptyset$ holds if and only if the polyhedral set $C = \{x \in \mathcal{E} \mid \mathcal{A}x = b\}$ and $\mathcal{K}^*$ can be properly separated. This implies the existence of s such that $s \in (\ker \mathcal{A})^\perp$ and $s \in \mathcal{K}^{**}$. This implies that $s = -\mathcal{A}^\top y$ for some $y \in \mathbb{R}^m$. Moreover, we must have either: *i)* $s \notin \mathcal{K}^{*\perp}$ and $\langle b, y\rangle = 0$; or *ii)* $\langle b, y\rangle > 0$. We also recall that $\mathcal{K}^{**} = \mathcal{K}$ and $\mathcal{K}^{*\perp} = \operatorname{lin}\mathcal{K}$,

□

We can also characterize strong infeasibility as follows, see also Lemma 5 in [55].

Proposition 2.7. *i. The primal* (P) *is strongly infeasible if and only if there is no solution to the linear system $\mathcal{A}x = b$ or if there exists y such that*

$$\langle b, y\rangle = 1 \quad \textit{and} \quad -\mathcal{A}^\top y \in \mathcal{K}. \tag{2.2}$$

ii. The dual (D) *is strongly infeasible if and only if there exists x such that*

$$\langle c, x\rangle = -1 \quad \textit{and} \quad \mathcal{A}x = 0 \quad \textit{and} \quad x \in \mathcal{K}^* \tag{2.3}$$

Proof. *i*. ($\Leftarrow$) if there is no solution to the linear system $\mathcal{A}x = b$ then, by definition, (P) is strongly infeasible. On the other hand, if there exists y satisfying (2.2) and x is a primal feasible solution then $1 = \langle b, y\rangle = \langle \mathcal{A}x, y\rangle = \langle x, \mathcal{A}^\top y\rangle < 0$, which is a contradiction. Therefore, (P) must be infeasible. Note also that any x satisfying $\mathcal{A}x = b$ also satisfies $\langle x, \mathcal{A}^\top y\rangle = 1$. Therefore, if (P) were not strongly infeasible, there would be a sequence of $\{x^k\}$ satisfying $\mathcal{A}x^k = b$ and a corresponding sequence $\{z^k\} \subseteq \mathcal{K}^*$ such that $\lim_{k\to+\infty} \operatorname{dist}(x^k, z^k) = 0$. However, continuity would imply that $\langle z^k, \mathcal{A}^\top y\rangle > 0$ for k large enough, which is impossible since $-\mathcal{A}^\top y \in \mathcal{K}$.

($\Rightarrow$) Let $\mathcal{V} = \{x \in \mathcal{E} \mid \mathcal{A}x = b\}$. Strong infeasibility means that either $\mathcal{V} = \emptyset$ or $\operatorname{dist}(\mathcal{V}, \mathcal{K}^*) > 0$ but $\mathcal{V} \neq \emptyset$. If the former holds, we are done. Suppose that the latter holds. Then, by Theorem 2.3, $\mathcal{V}$ and $\mathcal{K}$ can be strongly separated. So, there is an $s \in \mathcal{E}$ and a hyperplane $H = \{x \in \mathcal{E} \mid \langle x, s\rangle = \lambda\}$ such that $\mathcal{V}$ and $\mathcal{K}^*$ belong to opposite closed half-spaces and both of them lie at a positive distance from H. In other words, there are λ_1 and λ_2 such that:

$$\langle s, x\rangle \leq \lambda_1 < \lambda_2 \leq \langle s, z\rangle,$$

for all $x \in \mathcal{V}$ and all $z \in \mathcal{K}^*$. As in the proof of Proposition 2.6, in order for the inequalities above to hold, we must have $s \in \mathcal{K}^{**} = \mathcal{K}$, $s \in (\ker \mathcal{A})^\top$ and $\lambda_2 \leq 0$. It follows that there exists y such that $s = -\mathcal{A}^\top y$ and $\langle s, x\rangle = -\langle y, b\rangle < \lambda_1 < 0$. Therefore, we can multiply y by some constant in order to ensure that $\langle y, b\rangle = 1$.

ii. The proof is analogous to the previous item.

□

We recall the following basic constraint qualification.

Proposition 2.8 (Slater). *i. If there exists $x \in \operatorname{ri}\mathcal{K}^* \cap \mathcal{F}_P$, then $\theta_P = \theta_D$. In addition, if $\theta_P > -\infty$ holds as well then the dual optimal value is attained.*

ii. If there exists $s \in \mathcal{F}_D^S \cap \operatorname{ri}\mathcal{K}$, then $\theta_P = \theta_D$. In addition, if $\theta_D < +\infty$ holds as well, the primal optimal value is attained.

Suppose that x and y are optimal solutions for (P) and (D) and that there is no duality gap. Let s be the corresponding optimal slack $c - \mathcal{A}^\top y$. Then the equation $\langle x, s\rangle = 0$ holds. This implies that $x \in \mathcal{F}(s, \mathcal{K})^\Delta$ and $s \in \mathcal{F}(x, \mathcal{K}^*)^\Delta$ We have then competing definitions of strict complementarity. The next proposition shows the relation between them. Recall that a face of a cone $\mathcal{K}$ is exposed if it arises as the intersection of a supporting hyperplane of $\mathcal{K}$ and the cone itself. Then, a cone is said to be facially exposed if every face is exposed.

Proposition 2.9. *Let $s \in \mathcal{K}$ and $x \in \mathcal{K}^*$. Consider the following statements:*

i. *(a)* $\mathcal{F}(s, \mathcal{K})^\Delta = \mathcal{F}(x, \mathcal{K}^*)$;
(b) $\mathcal{F}(s, \mathcal{K}) = \mathcal{F}(x, \mathcal{K}^*)^\Delta$.
(Pataki [63], see Remark 3.6 therein).

ii. *(a) there exists a face* $\mathcal{F} \subseteq \mathcal{K}$ *such that* $s \in \operatorname{ri} \mathcal{F}$ *and* $x \in \operatorname{ri} \mathcal{F}^\Delta$;
(b) there exists face $\hat{\mathcal{F}} \subseteq \mathcal{K}^*$ *such that* $x \in \operatorname{ri} \hat{\mathcal{F}}$ *and* $s \in \operatorname{ri} \hat{\mathcal{F}}^\Delta$.
(see Section 2 in Tunçel and Wolkowicz [86]);

iii. *(a)* $x \in \operatorname{ri} \mathcal{F}(s, \mathcal{K})^\Delta$;
(b) $s \in \operatorname{ri} \mathcal{F}(x, \mathcal{K}^*)^\Delta$.

Items $i.(a), ii.(a), iii.(a)$ are equivalent. Items $i.(b), ii.(b), iii.(b)$ are also equivalent. If $\mathcal{K}$ and $\mathcal{K}^*$ are facially exposed, then $iii.(a)$ and $iii.(b)$ are also equivalent.

Proof. We will only prove one set of equivalences since the other has a similar proof.

($i.(a) \Rightarrow ii.(a)$) Suppose that $\mathcal{F}(s, \mathcal{K})^\Delta = \mathcal{F}(x, \mathcal{K}^*)$ holds. Take $\mathcal{F} = \mathcal{F}(s, \mathcal{K})$ and recall that the minimal face which contains a point x is characterized by the fact that x belongs to the relative interior of that face. Then, we must have $x \in \operatorname{ri} \mathcal{F}(x, \mathcal{K}^*)$, which implies $x \in \operatorname{ri} \mathcal{F}^\Delta$.

($ii.(a) \Rightarrow iii.(a)$) We have $\mathcal{F}(s, \mathcal{K}) = \mathcal{F}$, so that $x \in \operatorname{ri} \mathcal{F}(s, \mathcal{K})^\Delta$.

($iii.(a) \Rightarrow i.(a)$) Suppose $x \in \operatorname{ri} \mathcal{F}(s, \mathcal{K})^\Delta$. Because $\mathcal{F}(s, \mathcal{K})^\Delta$ is a face of $\mathcal{K}^*$, we have $\mathcal{F}(s, \mathcal{K})^\Delta = \mathcal{F}(x, \mathcal{K}^*)$.

($iii.(a) \Leftrightarrow iii.(b)$, under facial exposedness) Suppose that both $\mathcal{K}$ and $\mathcal{K}^*$ are facially exposed. Then $\mathcal{F}(s, \mathcal{K})^\Delta = \mathcal{F}(x, \mathcal{K}^*)$ implies $\mathcal{F}(s, \mathcal{K})^{\Delta\Delta} = \mathcal{F}(x, \mathcal{K}^*)^\Delta$. Since $\mathcal{K}$ is facially exposed, we have $\mathcal{F}(s, \mathcal{K}) = \mathcal{F}(s, \mathcal{K})^{\Delta\Delta}$. If the second alternative holds, we have that facial exposedness of $\mathcal{K}^*$ implies $\mathcal{F}(x, \mathcal{K}^*)^{\Delta\Delta} = \mathcal{F}(x, \mathcal{K}^*) = \mathcal{F}(s, \mathcal{K})$.

□

If items $i.(a), ii.(a), iii.(a)$ hold, then the pair (x, s) is said to be *primal strict complementary.* If $i.(b), ii.(b), iii.(b)$ hold, then we have *dual strict complementarity.* This distinction only matters when $\mathcal{K}$ or $\mathcal{K}^*$ is not facially exposed. We remark that when $\mathcal{K} = \mathbb{R}^n_+$, the notion of strict complementarity above is equivalent to $x + s > 0$. When $\mathcal{K} = \mathcal{S}^n_+$, it is equivalent to $x+s$ being positive definite. More generally, if $\mathcal{K}$ is a symmetric cone, then strict complementarity is equivalent to $x + s \in \operatorname{ri} \mathcal{K}$.

2.3 Facial structure of the positive semidefinite cone and the Lorentz cone

When $\mathcal{E}$ is the space of $n \times n$ symmetric matrices $\mathcal{S}^n$ and $\mathcal{S}^n_+ \subseteq \mathcal{E}$ is the cone of positive semidefinite matrices $\mathcal{S}^n_+$, (D) is called a *semidefinite program.* A good thing about $\mathcal{S}^n_+$ is that its faces are very well understood. In particular, there is a correspondence between subspaces of $\mathbb{R}^n$ and faces of $\mathcal{S}^n_+$. Moreover, each face looks like a smaller positive semidefinite cone.

Proposition 2.10. *Let $\mathcal{F}$ be a nonempty face of $\mathcal{S}^n_+$. Then:*

i. For all $x \in \operatorname{ri} \mathcal{F}$ and $y \in \mathcal{F}$, we have $\ker x \subseteq \ker y$.

ii. There exists $r \leq n$ and an orthogonal $n \times n$ matrix q such that

$$q^\top \mathcal{F} q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathcal{S}^r_+ \right\} \tag{2.4}$$

Proof. *i.* Since x is a relative interior feasible point, by item *v.* of Lemma 2.5, there is $\alpha > 1$ such that $z = \alpha x + (1-\alpha)y \in \mathcal{F}$. Let $d \in \ker x$. Then, $\langle zd, d\rangle = (1-\alpha)\langle yd, d\rangle \leq 0$. However, z is positive semidefinite, so we must have $\langle yd, d\rangle = 0$ which implies $d \in \ker y$.

ii. Let $x \in \operatorname{ri} \mathcal{F}$ and $r = n - \dim \ker x$. Let q be an orthogonal matrix such that the last $\dim \ker x$ columns span the kernel of x. Using the previous item and the fact that the matrices in $\mathcal{F}$ are positive semidefinite, it is possible to conclude that Equation (2.4) holds.

□

It follows that if $\mathcal{F}$ is a nonempty face, the rank of the elements in its relative interior is constant and they share the same kernel. We will call this quantity the *rank* of $\mathcal{F}$. Using the fact that q is orthogonal, we have that the dual of $\mathcal{F}$ satisfies

$$q^\top \mathcal{F}^* q = (q^\top \mathcal{F} q)^* = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathcal{S}^n \mid a \in \mathcal{S}^r_+ \right\}. \tag{2.5}$$

We now turn our attention to the case where $\mathcal{E} = \mathbb{R}^n$ and $\mathcal{K} = \mathcal{Q}^n$. We need some extra notation. For $d \in \mathbb{R}^n$, we define the closed half-space $H^n_d = \{x \in \mathbb{R}^n \mid d^\top x \geq 0\}$ and the ray $\mathrm{h}^n_d = \{\alpha d \in \mathbb{R}^n \mid \alpha \geq 0\}$. We also write $x = (x_0, \ldots, x_{n-1})$ for the components of x. We use the notation $\overline{x}$ to denote the last $(n-1)$ components of x, i.e., $\overline{x} = (x_1, \ldots, x_{(n-1)})$. The Lorentz cone in $\mathbb{R}^n$ is denoted by $\mathcal{Q}^n$, i.e., $\mathcal{Q}^n = \{x \in \mathbb{R}^n \mid x_0 \geq \|\overline{x}\|\}$, where $\|\cdot\|$ is the usual Euclidean norm. We remark that $\mathcal{Q}^1 = \{x \in \mathbb{R} \mid x \geq 0\}$, so the non-negative orthant in $\mathbb{R}^n$ can be written as a direct product of one-dimensional Lorentz cones. If $x \in \mathcal{Q}^n$, we write x' for the reflection of x with respect to $\mathcal{Q}^n$, i.e., $x' = (x_0, -\overline{x})$.

If $\mathcal{F}$ is a face of $\mathcal{Q}^n$, then either $\mathcal{F} = \{0\}$, $\mathcal{F} = \mathcal{Q}^n$ or $\mathcal{F} = \mathrm{h}^n_d$, for d a nonzero point in the boundary of $\mathcal{Q}^n$. This can be seen by observing that $\mathcal{Q}^n$ is the cone generated by the set $1 \times C$, where $C = \{x_1, \ldots, x_{n-1} \mid \|(x_1, \ldots, x_{n-1})\| = 1\}$.

2.4 Examples of nasty problems

We will now give a few examples of nasty problems. More examples can be found in the work by Luo, Sturm and Zhang [55]. Our first example is a weakly infeasible problem.

$$\begin{aligned} \sup_{t,s} \quad & 0 \\ \text{subject to} \quad & \begin{pmatrix} t \\ t \\ s \end{pmatrix} \times \begin{pmatrix} s \\ s \\ 1 \end{pmatrix} \in \mathcal{Q}^3 \times \mathcal{Q}^3. \end{aligned} \tag{SOCP-WI}$$

This problem is clearly infeasible, however, it might not be immediately obvious that it is *weakly infeasible*. To see that, first observe that as $s \to +\infty$ the difference between s and $\sqrt{s^2+1}$ decreases so the point $(s, s, 1)$ approaches the Lorentz cone. More explicitly, for any given $\epsilon > 0$, if $s \geq \frac{1}{4\epsilon}$ then $z_\epsilon = (s+\epsilon, s-\epsilon, 1)$ lies in the Lorentz cone. For such an s, we have $\operatorname{dist}((s,s,1), \mathcal{Q}^3) \leq \epsilon\sqrt{2}$. The same argument shows that if t is sufficiently large, say $t > \frac{s}{4\epsilon}$, then $\operatorname{dist}((t,t,s), \mathcal{Q}^3) \leq \epsilon\sqrt{2}$ as well. This shows that by appropriately increasing s and t we can make $(t,t,s) \times (s,s,1)$ as close as we want to $\mathcal{Q}^3 \times \mathcal{Q}^3$. Note that this does not generate an accumulation point, since the norm of the point goes to infinity as it approaches the cone. In Section 4.4.1, we will revisit this example and prove through other means that this problem is weakly infeasible.

We also have an example of a weakly infeasible SDP.

$$\begin{aligned} \sup_{t,s} \quad & 0 \\ \text{subject to} \quad & \begin{pmatrix} t & 1 & s \\ 1 & s & 1 \\ s & 1 & 0 \end{pmatrix} \in \mathcal{S}^3_+. \end{aligned} \tag{SDP-WI}$$

If we let s sufficiently large then the minimum eigenvalue of the lower 2×2 matrix gets very close to zero. This will make the $(1,3)$ and $(3,1)$ elements large. But we can let t much larger than s. Then, the minimum eigenvalue of the submatrix $\left(\begin{smallmatrix} t & s \\ s & 0 \end{smallmatrix}\right)$ is close to zero. Intuitively, this neutralize the effect of big off-diagonal elements, and we obtain points arbitrarily close to $\mathcal{S}^3_+$, by taking s to be large and t to be much larger than s. We will also revisit this example in Section 4.3.3.

Closely related to weakly infeasible problems are problems with unattained finite optima, as in the next example.

$$\begin{aligned} \sup_{t,s} \quad & -s \\ \text{subject to} \quad & \begin{pmatrix} t & 1 \\ 1 & s \end{pmatrix} \in \mathcal{S}^2_+. \end{aligned} \tag{SDP-UN}$$

Clearly, the optimal value is 0 but as s goes to 0, in order to keep the matrix positive semidefinite we have to take t very large. So, in this example, the optimal value is not attained. Note that this is problem is relatively well-behaved, since it has an interior point. However, the corresponding primal problem is not strongly feasible and this is a source of trouble.

The following problem has a nonzero duality gap.

$$\begin{aligned} \sup_{t,s} \quad & -s \\ \text{subject to} \quad & \begin{pmatrix} t & 1 & s-1 \\ 1 & s & 0 \\ s-1 & 0 & 0 \end{pmatrix} \in \mathcal{S}^3_+ \end{aligned} \tag{GAP-D}$$

$$\begin{aligned} \inf_{x} \quad & 2x_{12} - 2x_{13} \\ \text{subject to} \quad & x_{11} = 0 \\ & -x_{22} - 2x_{13} = -1 \\ & x \in \mathcal{S}^3_+. \end{aligned} \tag{GAP-P}$$

In order for a matrix in the dual problem be feasible, we must impose $s = 1$. So $\theta_D = 1$. On the other hand, for a matrix in the primal problem be feasible, we must have $x_{11} = 0$ which implies that $x_{12} = x_{13} = 0$. Therefore, $\theta_P = 0$. So there is a duality gap between (GAP-D) and (GAP-P).

To end this section, we include a particularly nasty example. This is a problem that has a duality gap and is not attained at *both* sides.

$$
\begin{aligned}
\sup_{y \in \mathbb{R}^8} \quad & -y_4 - 2y_6 - 2y_7 && \text{(NASTY-D)} \\
\text{subject to} \quad & \begin{pmatrix}
y_1 & & & & & & & y_3 - 1 \\
& y_1 & & & & & & y_5 - 1 \\
& & y_2 & y_3 & & & & \\
& & y_3 & y_4 - y_5 & & & & \\
& & & & 0 & y_6 & y_7 & \\
& & & & y_6 & y_4 & -0.5y_8 + 0.5 & \\
& & & & y_7 & -0.5y_8 + 0.5 & y_8 & \\
y_3 - 1 & y_5 - 1 & & & & & & 0
\end{pmatrix} \in \mathcal{S}^8_+
\end{aligned}
$$

$$
\begin{aligned}
\inf_{x} \quad & -2x_{18} - 2x_{28} + x_{67} && \text{(NASTY-P)} \\
\text{subject to} \quad & -x_{11} - x_{22} = 0 \\
& -x_{33} = 0 \\
& -2x_{18} - 2x_{34} = 0 \\
& -x_{44} - x_{66} = -1 \\
& -2x_{28} + x_{44} = 0 \\
& -2x_{56} = -2 \\
& -2x_{57} = -2 \\
& x_{67} - x_{77} = 0 \\
& x \in \mathcal{S}^8_+.
\end{aligned}
$$

Note that the dual optimal value satisfies $\theta_D = -1$. This is because the $(8,8)$ entry is 0, which forces $y_3 = y_5 = 1$. Moreover, the $(5,5)$ entry is zero too, which forces $y_6 = y_7 = 0$. Now, for feasible y, we have $y_4 - y_5 = y_4 - 1 \geq 0$, due to the fact that the $(4,4)$ entry must be nonnegative. In addition, $y_8 \geq 0$, for a similar reason. It follows that $\theta_D \leq -1$. However, for every $\epsilon > 0$, $y_\epsilon = (0, 1/\epsilon, 1, 1+\epsilon, 1, 0, 0, 0)$ is a feasible point that has value equal to $-1-\epsilon$. This shows that $\theta_D = -1$. However, θ_D is not attained because for feasible y, we have $y_4 > 1$.

The primal optimal value is zero. This is because the first and the second constraints, force $x_{11} = x_{22} = x_{33} = 0$, which implies that $x_{18} = x_{28} = 0$. Then, the fifth constraint implies that $x_{44} = 0$. Therefore, $x_{66} = 1$ by the fourth constraint. The last constraint forces $x_{67} = x_{77}$, which implies that $\theta_P \geq 0$. Note that because $x_{57} = 1$, we can never assign zero to x_{77}. However, if x_{77} is small but positive, we can construct feasible points by taking x_{55} very large, $x_{67} = x_{77}$, $x_{56} = x_{57} = x_{66} = 1$ and all the other entries equal to zero. This shows that $\theta_P = 0$ but is not attained.

We tried to solve the pair (NASTY-D) and (NASTY-P) with SeDuMi [78], SDPA [28], SDPT3 [84] and PENLAB [24]. Note that we have no hope that these solvers will be able to handle correctly the problem, since it does not satisfy any of the regularity assumptions that are usually required. Nevertheless, we were curious to see how good were the indicators that something could be wrong. First of all, we tried PENLAB, but it exceeds the maximum number of outer iterations, which by default is 100.

SeDuMi took 30 iterations before stopping and declaring that the duality gap was `6.28E-17`. Moreover, the flag `info.numerr` was set to 0, which indicates that SeDuMi believes the solutions obtained were accurate. No particular error message is given, but there are a few signs that something could be amiss. The indicador `info.feasratio` was close to zero, which suggests that there could be some kind of ill-behavior. The final objective values obtained were `-7.8962602692e-01` and `-7.0148515512e-01` for the primal and dual problems, respectively. This somewhat contradicts the duality gap indicators. So, although SeDuMi did not explicitly give an error, the user might figure that something is strange by taking a look at these anormalities.

SDPT3 took 70 iterations before stopping and declaring that the duality gap was `2.31e-08`. The termination code was zero, which is successful termination. The primal objective value was `-1.00000005e00`+ and the dual objective value was `-9.99999946e-01`. So the latter was very close to the correct answer, while the former was completely wrong. The output of SDPT3 was consistent with what one would expect under normal circumstances and the only information that helps the user to identify that something could be wrong is the fact that the solutions have large norm and the relatively high number of iterations.

When formulating the problem for SDPA, we used a slightly different formulation where (NASTY-D) is maximization problem and (NASTY-P) is a minimization problem. In this case, we should have $\theta_D = 1$ and $\theta_P = 0$. However, the output of SDPA indicated that the problem has a gap of `-5.7955238234042028e-06`. The primal and dual objective values were `9.9998558253126402e-01` and `9.9999137805508742e-01`, respectively. SDPA stops with status `pdfINF` which is an abnormal stopping code. Looking at the manual [28] for the meaning of `pdfINF`, we found out that this means that at least one of (NASTY-D) and (NASTY-P) is expected to be infeasible, which is not the case here. However, at page 18, they also give the precise meaning of `pdfINF` which seems to suggest that the more accurate interpretation of `pdfINF` is that "optimal solutions affording zero duality gap must have large norms", which is a technically accurate statement for this case, although it does not pinpoint the fact that the problem actually has a duality gap.

It would be very surprising if the solvers were able to detect the presence of duality gap so, of course, there was no such expectation. However, it is worrisome that the solvers are failing in opaque ways, without a clear indication that something is wrong. So it seems that the construction of robust SDP solvers is still an open issue.

Chapter 3

Facial Reduction

Facial reduction was originally developed by Borwein and Wolkowicz [12, 13] as a technique to regularize convex programs. The original setting in their approach was fairly general since they dealt with minimizing an arbitrary convex function subject to convex-cone constraints.

Interest in the technique seems to have increased after the '90s, and it followed the surge in popularity of conic programming. A key work of that period was the article *"An Exact duality Theory for Semidefinite Programming and its Complexity Implications"* by Ramana [72]. As the name indicates, Ramana proposed a new dual for SDPs which was the different from the usual Lagrangian dual and corrected many of its flaws. In our notation, Ramana's dual is a substitute for the problem (P) and has the following features: *i.* it always affords zero duality gap without assuming any constraint qualifications; *ii.* if θ_D is finite then Ramana's dual is attained; *iii.* it can be written as an SDP with polynomially many constraints and it could be written down explicitly in terms of the original problem data.

Later, a paper by Ramana, Tunçel and Wolkowickz [73] clarified that there was a strong connection between Ramana's dual and Facial Reduction. Ramana's dual seems to express implicitly the equations and constraints associated to performing facial reduction and due to a few clever tricks, they are casted as semidefinite constraints.

Pataki wrote a technical report [64] where he gave a simplified description of facial reduction for a class of cones known as *nice cones.* Moreover, he also showed how to construct extended duals for nice cones that share similar properties with Ramana's dual. One caveat is that Pataki's approach for extended duality might produce an extended dual that might leave the problem class in question. For instance, if we apply Pataki's approach to a second order cone program, it is not currently known whether the resulting dual problem can be expressed via second order constraints. Later on, Pataki updated his report [66].

Waki and Muramatsu [89] also gave a description of a facial reduction procedure and removed the niceness assumption. Moreover, their approach can also detect infeasibility. They also showed a few applications for the SDP case. Cheung, Schurr and Wolkowicz discuss a facial reduction algorithm for SDPs in [16] and they developed an algorithm that is backward stable.

There is also the Conic Expansion Approach (CEA), due to Luo, Sturm and Zhang [55], which can be seen as a dual version of the facial reduction algorithm. Waki and Muramatsu discuss in detail the connection between conic expansion and facial reduction in section 4 of [89].

In this chapter, we will first review the basic facial reduction technique and then we will discuss some of our contributions. First of all, we show how facial reduction can be carried out by solving problems that are ensured to be both primal and dual strongly feasible, see problems $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ below. This is important because Facial Reduction requires solutions to certain auxiliary problems. If those problems were also ill behaved, it would be harder to justify applying the technique, since we would be substituting a bad problem for a sequence of, perhaps, equally bad problems. We remark that this issue seldom appears in literature. One of the exceptions is [16], but their approach is only for SDPs.

Afterwards, we discuss versions of classical theorems for the case where there is partial polyhedrality. In particular, we prove partial polyhedral versions of the Gordan-Stiemke theorem, a proper separation theorem and a closedness result.

These results are then used to develop *FRA-poly*, which is a facial reduction algorithm divided in two phases that takes into account the presence of polyhedrality in the lattice of faces of $\mathcal{K}$. We will show that, under mild conditions, FRA-poly requires less iterations than the classical FRA. Key to the analysis of FRA and surrounding theoretical points are the notions of *distance to polyhedrality* (Definition 3.20) and *distance to strong duality* (Definition 3.26), which are also novel concepts. We will also discuss an example of a mixed second order cone program/semidefinite program that achieves the worst possible case for facial reduction. Finally, we will show how FRA-poly can be used to show that the singularity degree of problems over the doubly nonnegative cone is at most n.

3.1 The basic technique

Here, we will suppose that our main interest is in the dual problem (D). The idea is that if (D) is feasible but not strongly feasible, then substituting $\mathcal{K}$ by the minimal face $\mathcal{F}^D_{\min}$ of $\mathcal{K}$ containing $\mathcal{F}^S_{\mathrm{D}}$ is enough to restore strong feasibility, due to Proposition 2.4. In that case, we reformulate the problem over $\mathcal{F}^D_{\min}$ and $\mathcal{K}^*$ is also changed to $(\mathcal{F}^D_{\min})^*$. Therefore, the primal feasible region might get larger. The meaning of that is that we are closing the duality gap by making the primal optimal value reach the dual optimal value.

To algorithmically find $\mathcal{F}^D_{\min}$, we proceed as follows. Whenever (D) lacks a relative interior solution, we may reformulate (D) over a lower dimensional face of $\mathcal{K}$. The basis for this idea is contained in Proposition 2.6: the absence of a dual relative interior solution is equivalent to the existence of an x satisfying $\langle c, x\rangle \leq 0$ and $x \in \ker \mathcal{A} \cap \mathcal{K}^*$. There are then two possibilities. The first is $\langle c, x\rangle = 0$ and $x \notin \mathcal{K}^\perp$, in which case $\mathcal{F}^S_{\mathrm{D}} \subseteq \mathcal{K} \cap \{x\}^\perp \subsetneq \mathcal{K}$, because $\langle c, x\rangle = \langle c - \mathcal{A}^\top y, x\rangle = 0$, for all $y \in \mathcal{F}_{\mathrm{D}}$. Note that $\mathcal{F}_2 = \mathcal{K} \cap \{x\}^\perp$ is a proper face of $\mathcal{K}$, since $x \notin \mathcal{K}^\perp$. The second alternative is $\langle c, x\rangle < 0$, in which case (D) is infeasible and we can stop.

If the first alternative holds, we can then reformulate (D) as a problem over $\mathcal{F}_2$, i.e., we consider the problem $\sup\{\langle b, y\rangle \mid c - \mathcal{A}^\top y \in \mathcal{F}_2\}$. It is clear that as long as $(\operatorname{ri} \mathcal{F}_2) \cap (\mathcal{F}^S_{\mathrm{D}}) = \emptyset$, we can repeat this process and either descend to a smaller face of $\mathcal{K}$ or declare infeasibility. After a few iterations, we will end with either some face $\mathcal{F}_\ell$ such that $(\operatorname{ri} \mathcal{F}_\ell) \cap \mathcal{F}^S_{\mathrm{D}} \neq \emptyset$ or we will eventually find out that the problem is infeasible, in this case $\mathcal{F}_\ell = \emptyset$. Note that $\mathcal{F}_\ell$ must be the smallest face of $\mathcal{K}$ which contains $\mathcal{F}^S_{\mathrm{D}}$, which we will denote by $\mathcal{F}^D_{\min}$. This process is called *facial reduction* and it aims at finding $\mathcal{F}^D_{\min}$. We write below a generic facial reduction algorithm similar to the one described in [89].

[**Generic Facial Reduction**]
Input: (D)
Output: A set of reducing directions $\{d_1, \ldots, d_\ell\}$ and $\mathcal{F}^D_{\min}$.

1) $\mathcal{F}_1 \leftarrow \mathcal{K}, i \leftarrow 1$

2) Let d_i be an element in $\mathcal{F}_i^* \cap \ker \mathcal{A}$ such that either: $i) d_i \notin \mathcal{F}_i^\perp$ and $\langle c, d_i\rangle = 0$; or $ii)\langle c, d_i\rangle < 0$. If no such d_i exists, let $\mathcal{F}^D_{\min} \leftarrow \mathcal{F}_i$ and stop.

3) If $\langle c, d_i\rangle < 0$, let $\mathcal{F}^D_{\min} \leftarrow \emptyset$ and stop.

4) If $\langle c, d_i\rangle = 0$, let $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{d_i\}^\perp$, $i \leftarrow i+1$ and return to 2).

Facial reduction is a very powerful procedure and it can be used to solve feasibility problems over arbitrary closed convex cones. The main difficulty is, of course, finding the d_i. Henceforth, we will call the points $\{d_1, \ldots, d_\ell\}$ *reducing directions.* There is some degree of freedom on the choice of d_i. However, as we need to solve a conic feasibility problem in order to obtain the reducing directions, it seems that searching for the d_i could be, in principle, as hard as solving the original problem itself.

However, one important difference is that we can always search for the d_i by using a pair of well-behaved problems that are always strongly feasible, so they are ensured to not suffer from the same problems that (D) might have. We remark that in [16], Cheung, Schurr and Wolkowicz also discuss an auxiliary problem that is primal and dual strongly feasible, see the problem (AP) therein. However, their approach is for SDPs and (AP) uses an additional second order cone constraint. Because of that, (AP) is not readily generalizable to other families of cones that are not able to express those kinds of constraints.

Consider the following pair of problems, which first appeared in [49].

$$
\begin{aligned}
&\underset{x,t,w}{\text{minimize}} && t && && (P_{\mathcal{K}})\\
&\text{subject to} && -\langle c, x - te^*\rangle + t - w && = 0 && (3.1)\\
& && \langle e, x\rangle + w && = 1 && (3.2)\\
& && \mathcal{A}x - t\mathcal{A}e^* && = 0 &&\\
& && (x,t,w) \in \mathcal{K}^* \times \mathbb{R}_+ \times \mathbb{R}_+ && &&
\end{aligned}
$$

$$
\begin{aligned}
&\underset{y_1,y_2,y_3}{\text{maximize}} && y_2 && (D_{\mathcal{K}})\\
&\text{subject to} && cy_1 - ey_2 - \mathcal{A}^\top y_3 \in \mathcal{K} && (3.3)\\
& && 1 - y_1(1 + \langle c, e^*\rangle) + \langle e^*, \mathcal{A}^\top y_3\rangle \geq 0 && (3.4)\\
& && y_1 - y_2 \geq 0 && (3.5)
\end{aligned}
$$

The pair $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ are parametrized by $\mathcal{A}$, c, $\mathcal{K}$ and by fixed elements $e \in \mathcal{K}$ and $e^* \in \mathcal{K}^*$. We have the following lemma, which suggests a good choice of parameters e and e^*.

Lemma 3.1. *Consider the pair $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ with e and e^* such that $e \in \operatorname{ri}\mathcal{K}$ and $e^* \in \operatorname{ri}\mathcal{K}^*$. The following properties hold.*

i. Both $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ have relative interior feasible points.

ii. Let (x^*, t^*, w^*) be a primal optimal solution. The optimal value is zero if and only if $\mathcal{F}^D_{\min} \subsetneq \mathcal{K}$. Moreover, if the optimal value is zero, we have $\langle c, x^*\rangle < 0$ and $\mathcal{F}^S_D = \mathcal{F}^D_{\min} = \emptyset$, or $\langle c, x^*\rangle = 0$ and $\mathcal{F}^S_D \subseteq \mathcal{K} \cap \{x^*\}^\perp \subsetneq \mathcal{K}$.

iii. Let (y_1^*, y_2^*, y_3^*) be a dual optimal solution. If the common optimal value is nonzero, then $\mathcal{F}^D_{\min} = \mathcal{K}$ and $s = c - \mathcal{A}^\top \frac{y_3^*}{y_1^*}$ is a dual optimal solution satisfying $s \in (\operatorname{ri}\mathcal{F}^S_D) \cap \operatorname{ri}\mathcal{K}$.

Proof. i. Let $t = \frac{1}{\langle e, e^*\rangle + 1}$, $w = \frac{1}{\langle e, e^*\rangle + 1}$ and $x = \frac{e^*}{\langle e, e^*\rangle + 1}$. Then (x, t, w) is an interior solution to $(P_{\mathcal{K}})$. To show that $(D_{\mathcal{K}})$ has a relative interior solution, we use just observe that $(0, -1, 0)$ is a dual feasible solution which corresponds to relative interior slack.

ii. Now, let (x^*, t^*, w^*) be a primal optimal solution. If $t^* = 0$, then $x^* \in \ker \mathcal{A} \cap \mathcal{F}^*$ and Equation (3.1) implies that $\langle c, x^* \rangle \le 0$. If y is a dual feasible solution for (D), we have $c - \mathcal{A}^\top y \in \mathcal{F}$, so that $\langle c - \mathcal{A}^\top y, x^* \rangle = \langle c, x^* \rangle \le 0$. If $\langle c, x^* \rangle < 0$, then it must be the case that $\mathcal{F}_{\mathrm{D}}^S = \emptyset$. If $\langle c, x^* \rangle = 0$, then $c - \mathcal{A}^\top y \in \{x^*\}^\perp$. Moreover, Equation (3.1) implies that $w^* = 0$ as well. Using Equation (3.2), we obtain $\langle e, x^* \rangle = 1$. In view of the fact that $e \in \mathcal{F}$, it must be the case that $\mathcal{F} \cap \{x^*\}^\perp \subsetneq \mathcal{F}$. In either case we have $\mathcal{F}_{\min}^D \subsetneq \mathcal{F}$.

Reciprocally, if $\mathcal{F}_{\min}^D \subsetneq \mathcal{F}$, item *i*. of Proposition 2.6 shows that there exists some $x \in \mathcal{F}^* \cap \ker \mathcal{A}$ such that either: *i*) $\langle c, x \rangle = 0$ and $x \notin \mathcal{F}^\perp$ or *ii*) $\langle c, x \rangle < 0$. Suppose first that *i*) holds, then the condition $x \notin \mathcal{F}^\perp$ readily implies that $\langle e, x \rangle > 0$, by item *vi*. of Lemma 2.5. So let $\alpha = \frac{1}{\langle e,x \rangle}$. Then $(x\alpha, 0, 0)$ is an optimal solution for $(P_\mathcal{K})$, which shows that the optimal value is zero. Now suppose that *ii*) holds. We take $\alpha = \frac{1}{\langle e,x \rangle - \langle c,x \rangle}$ and this is well-defined because $-\langle c, x \rangle > 0$. Then $(x\alpha, 0, -\alpha\langle c, x \rangle)$ is an optimal solution for $(P_\mathcal{K})$, which also shows that the optimal value is zero

iii. If the common optimal value is nonzero, we must have $y_2^* > 0$ and $ey_2^* \in \operatorname{ri} \mathcal{F}$. This fact, together with Equation (3.3) and item *iv*. of Lemma 2.5, implies that $cy_1^* - \mathcal{A}^\top y_3^* \in \operatorname{ri} \mathcal{F}$ as well. Finally, $y_1^* \ge y_2^* > 0$, by Equation (3.5). Using Proposition 2.4, we can then conclude that $c - \mathcal{A}^\top \frac{y_3^*}{y_1^*} \in \operatorname{ri} \mathcal{F}_{\mathrm{D}}^S$ as claimed.

□

Lemma 3.1 shows how to implement one step of facial reduction. In case the optimal value of $(P_\mathcal{K})$ is zero, we have two scenarios. In the first, we have $\langle c, x^* \rangle < 0$, we can then stop the procedure and declare that (D) is infeasible. In the second, the dual feasible region is contained in the face $\hat{\mathcal{F}} = \mathcal{F} \cap \{x^*\}^\perp$ and we have $\hat{\mathcal{F}} \subsetneq \mathcal{F}$.

We may then reformulate (D) as a problem in a lower-dimensional space and apply Lemma 3.1 again. We will then stop either at the minimal face $\mathcal{F}_{\min}^D$ or find out that $\mathcal{F}_{\mathrm{D}}^S$ is actually empty. When facial reduction finally stops, we have one more pleasant surprise: item *iii*. of Lemma 3.1 shows that we can extract a relative interior solution from the dual problem $(D_\mathcal{K})$.

We write below a FRA variant, which was first described in [49] for SDPs.

[**Facial Reduction**]

Input: (D)

Output: $\mathcal{F}_{\min}^D$ and $s \in \operatorname{ri} \mathcal{F}_{\mathrm{D}}^S$ (if $\mathcal{F}_{\min}^D \neq \emptyset$), or a certificate of infeasibility (if $\mathcal{F}_{\min}^D = \emptyset$)

1) $\mathcal{F}_1 \leftarrow \mathcal{K}, i \leftarrow 1$

2) Solve $(P_\mathcal{K})$ and $(D_\mathcal{K})$ with $\mathcal{F}_i$ in place of $\mathcal{K}_i$ to obtain primal dual pairs of optimal solutions (x^*, t^*, w^*) and (y_1^*, y_2^*, y_3^*).

3) If $t^* = 0$ and $\langle c, x^* \rangle < 0$, let $\mathcal{F}_{\min}^D \leftarrow \emptyset$ and stop. (D) is infeasible.

4) If $t^* = 0$ and $\langle c, x^* \rangle = 0$, let $d_i \leftarrow x^*, \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{d_i\}^\perp, i \leftarrow i + 1$ and return to 2).

5) If $t^* > 0$, let $\mathcal{F}_{\min}^D \leftarrow \mathcal{F}_i$, $s \leftarrow c - \mathcal{A}^\top \frac{y_3^*}{y_1^*}$ and stop.

In order to describe the number of iterations needed in the worst case, we need the following definition. In what follows, if we have a chain of faces $\mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_\ell$, the length of the chain is defined to be ℓ.

Definition 3.2 (Longest chain of faces)**.** *Let $\mathcal{K}$ be a closed convex cone, we will denote by $\ell_\mathcal{K}$ the length of the longest chain of strictly ascending nonempty faces of $\mathcal{K}$.*

We then have the following complexity results.

Proposition 3.3. *i. If* (D) *is feasible, the Generic Facial Reduction algorithm stops at finding at most* $\min\{\ell_{\mathcal{K}} - 1, \dim(\ker \mathcal{A} \cap \{c\}^{\perp})\}$ *reducing directions.*

ii. If (D) *is infeasible, the Generic Facial Reduction algorithm stops at finding at most* $\min\{\ell_{\mathcal{K}} - 1, \dim(\ker \mathcal{A} \cap \{c\}^{\perp})\} + 1$ *reducing directions.*

Proof. *i.* If the problem is feasible, the algorithm will always hit the condition $d_i \in \mathcal{F}_i \setminus \mathcal{F}_i^{\perp}$, which implies that the chain of faces obtained $\mathcal{K} \supseteq \mathcal{F}_2 \ldots \subseteq \mathcal{F}_\ell$ is such that every inclusion is strict. As the longest chain of faces has length $\ell_{\mathcal{K}}$, we cannot obtain more than $\ell_{\mathcal{K}} - 1$ directions.

In this case, it can be shown that all the directions are linearly independent. This is because if d_i is in the span of the the first $i-1$ directions, then $\mathcal{K} \cap \{d_1\}^{\perp} \cap \ldots \cap \{d_{i-1}\}^{\perp} = \mathcal{K} \cap \{d_1\}^{\perp} \cap \ldots \cap \{d_{i-1}\}^{\perp} \cap \{d_i\}^{\perp}$, which would contradict the fact that the chain of faces is strict.

Because all directions are contained in $\ker \mathcal{A} \cap \{c\}^{\perp}$, we cannot obtain more than $\dim(\ker \mathcal{A} \cap \{c\}^{\perp})$ as well.

ii. If (D) is infeasible, then at some iteration ℓ the algorithm will hit Step 3 and stop. However, at all iterations up to $\ell - 1$ it holds true that $d_i \notin \mathcal{F}_i^{\perp}$ and $\langle c, d_i \rangle = 0$. Therefore the first $\ell - 1$ directions define a strictly descending chain of faces of $\mathcal{K}$, so $\ell - 1 \leq \ell_{\mathcal{K}} - 1$. Moreover, as in item *i.*, they must linearly independent and must be contained in $\ker \mathcal{A} \cap \{c\}^{\perp}$. So we also have the bound $\ell - 1 \leq \dim(\ker \mathcal{A} \cap \{c\}^{\perp})$. □

As the facial reduction algorithm described in this section is a particular case of the generic version using the auxiliary problems $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$, it is also subject to the same bounds. Of course, not all problems will require the maximum amount of steps.

We define the singularity degree $d(D)$ of (D) as the minimum number of facial reduction steps needed to find $\mathcal{F}^D_{\min}$. No matter the facial reduction strategy used, one must identify at least $d(D)$ directions before $\mathcal{F}^D_{\min}$ is reachable.

Definition 3.4 (Singularity degree)**.** *Consider the set of possible outputs* $\{d_1, \ldots, d_\ell\}$ *of the Generic Facial Reduction. The singularity degree of* (D) *is the minimum* ℓ *among all the possible outputs and is denoted by* $d(D)$*. Similarly, we will denote the singularity degree of* (P) *by* $d(P)$*.*

As far as we know, the expression "singularity degree" in this context is due to Sturm in [79], where he showed the connection between the singularity degree of a positive semidefinite program and a Hölderian error bound. In the recent work by Liu and Pataki [44], there is also a definition of singularity degree for general linear conic problems, see Definition 6 therein. One difference, however, is that Liu and Pataki only define the singularity degree for feasible problems and, indeed, when (D) is feasible, their definition matches with Definition 3.4.

3.2 Partial Polyhedrality Theorems

In Rockafellar's classic book [75], among its many notable chapters there is one on "Applications of Polyhedral Convexity". Many classical results are proved under weaker conditions if some of the objects involved are polyhedral. Borwein and Lewis also includes a discussion of the mixed Fenchel Duality Theorem in Section 5.1 of [11]. As far as we could dig in the literature, there does not seem to be many papers that deal with this subject apart from the ones by Klee such as [39], which is referenced in Rockafellar's book.

The results in this section have a similar flavour. They will be used in Section 3.3 in our discussion of FRA-Poly. Before we proceed, we need the following special version of Slater's condition which takes into account partial polyhedrality. As we could not find a precise reference for it, we will prove it. It can also be proved by invoking in an appropriate manner a version of Fenchel's duality theorem that takes into account polyhedrality, such as Theorem 31.1 in [75] or Corollary 5.1.9 in [11]. First, we need the following definition.

Definition 3.5 (Partial Polyhedral Slater's condition)**.** *Let $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$, where $\mathcal{K}^1 \subseteq \mathbb{R}^{n_1}, \mathcal{K}^2 \subseteq \mathbb{R}^{n_2}$ are closed convex cones such that $\mathcal{K}^2$ is polyhedral. We say that* (D) *satisfies the Partial Polyhedral Slater's (PPS) condition if there is a slack $(s_1, s_2) = c - \mathcal{A}^\top y$, such that $s_1 \in \operatorname{ri} \mathcal{K}^1$ and $s_2 \in \mathcal{K}^2$. Similarly, we say that* (P) *satisfies the PPS condition, if there is a primal feasible point $x = (x_1, x_2)$ for which $x_1 \in \operatorname{ri} (\mathcal{K}^1)^*$.*

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex function. We denote the domain of f by $\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$. If $\operatorname{dom} f$ is not empty and f is never $-\infty$, then f is said to be proper. Its conjugate will be denoted by f^* and it satisfies $f^*(s) = \sup_x \langle s, x\rangle - f(x)$. If the epigraph of f is a polyhedral set, then f is said to be a polyhedral function. We need to recall the following theorem on infimal convolution.

Theorem 3.6 (Rockafellar)**.** *Let $f_1, \dots f_m$ be proper convex functions and let $f_{k+1}, \dots, f_m$ be polyhedral functions. Suppose also that*

$$\operatorname{ri} (\operatorname{dom} f_1) \cap \ldots \cap \operatorname{ri} (\operatorname{dom} f_k) \cap \operatorname{dom} f_{k+1} \cap \ldots \cap \operatorname{dom} f_m \neq \emptyset.$$

Then the following holds:

$$(f_1 + \ldots f_m)^*(s) = \inf\{f_1^*(s_1) + \ldots + f_m^*(s_m) \mid s_1 + \ldots + s_m = s\},$$

where for each s the infimum is attained whenever it is finite. In other words, under the assumptions of the theorem, the conjugate of the sum is equal to the infimal convolution of the conjugates.

Proof. See Theorem 20.1 of Rockafellar [75]. □

Proposition 3.7. *Let $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$, where $\mathcal{K}^1 \subseteq \mathbb{R}^{n_1}, \mathcal{K}^2 \subseteq \mathbb{R}^{n_2}$ are closed convex cones such that $\mathcal{K}^2$ is polyhedral.*

i) If θ_P is finite and (P) *satisfies the PPS condition, then $\theta_P = \theta_D$ and the dual optimal value is attained.*

ii) If θ_D is finite and (D) *satisfies the PPS condition, then $\theta_P = \theta_D$ and the primal optimal value is attained.*

Proof. We will prove *i)* first. Let f_1 be such that $f_1(x) = \langle c, x\rangle$ if $Ax = b$ and $+\infty$ otherwise. Let f_2 be the indicator function of $\mathbb{R}^{n_1} \times (\mathcal{K}^2)^*$ and f_3 be the indicator function of $(\mathcal{K}^1)^* \times \mathbb{R}^{n_2}$. Since there is a primal feasible solution $x = (x_1, x_2)$ such that $x_1 \in \operatorname{ri} (\mathcal{K}^1)^*$, we have that $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap \operatorname{ri} (\operatorname{dom} f_3)$ is non-empty. In addition, f_1 and f_2 are polyhedral functions. Let us now observe that:

$$f_1^*(s) = \begin{cases} \langle b, y\rangle & \text{if there is } y \text{ with } s - c = \mathcal{A}^\top y \\ +\infty & \text{otherwise} \end{cases}$$

Note that, due to feasibility, for fixed s, $\langle b, y\rangle$ does not depend on the choice of y, as long as $c + \mathcal{A}^\top y = s$. This is because if $Ax = b$, then $\langle b, y\rangle = \langle x, s - c\rangle$. The conjugate f_2^* is the indicator function of $-\{0\} \times \mathcal{K}^2$ and f_3^* is the indicator function of $-\mathcal{K}^1 \times \{0\}$. Applying Theorem 3.6 with $s = 0$, we have:

$$\begin{aligned}(f_1 + f_2 + f_3)^*(0) &= \inf\left\{\langle b, y\rangle \mid c + \mathcal{A}^\top y = s_1, s_1 - (0, s_2) - (s_3, 0) = 0, s_2 \in \mathcal{K}^2, s_3 \in \mathcal{K}^1\right\} \\ &= \inf\left\{\langle b, y\rangle \mid c + \mathcal{A}^\top y = s_1, s_1 \in \mathcal{K}^1 \times \mathcal{K}^2\right\} \\ &= -\sup\left\{\langle b, y\rangle \mid c - \mathcal{A}^\top y = s_1, s_1 \in \mathcal{K}^1 \times \mathcal{K}^2\right\},\end{aligned}$$

where the sup in the last equation is attained. So, there is some dual feasible y such that $(f_1 + f_2 + f_3)^*(0) = \langle b, y\rangle$. However, using the definition of conjugate, we also have:

$$(f_1 + f_2 + f_3)^*(0) = -\inf\{\langle c, x\rangle \mid Ax = b, x \in \mathcal{K}^1 \times \mathcal{K}^2\} = -\theta_P.$$

It follows that $\theta_P = \theta_D$ and the dual is attained at y. To prove *ii*), let $g_1 = f_1^*$, and let g_2 and g_3 be the indicator functions of $\mathbb{R}^{n_1} \times \mathcal{K}^2$ and $\mathcal{K}^1 \times \mathbb{R}^{n_2}$, respectively. Again, it is enough to compute $(g_1 + g_2 + g_3)^*(0)$ using both the definition of conjugate function and using Theorem 3.6. □

We now prove a version of the Gordan-Stiemke's Theorem that takes into account partial polyhedrality. To the best of our knowledge, it is a new result.

Theorem 3.8 (Partial Polyhedral Gordan-Stiemke's Theorem)**.** *Let $\mathcal{L}$ be a subspace and $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$ be a closed convex cone, such that $\mathcal{K}^2$ is polyhedral. Then we have:*

$$\mathcal{L} \cap \mathcal{K} \subseteq (\operatorname{lin} \mathcal{K}^1) \times \mathcal{K}^2 \Leftrightarrow \mathcal{L}^\perp \cap \left((\operatorname{ri} \mathcal{K}^{1*}) \times (\mathcal{K}^2)^*\right) \neq \emptyset.$$

Proof. The "$\Leftarrow$" implication is straightforward as follows. If $s = (s_1, s_2)$ belongs to $\mathcal{L}^\perp \cap \left((\operatorname{ri} \mathcal{K}^{1*}) \times (\mathcal{K}^2)^*\right)$ and $x = (x_1, x_2)$ to $\mathcal{L} \cap \mathcal{K}$, then we must have $\langle x_1, s_1\rangle = 0$, which forces $x_1 \in \operatorname{lin} \mathcal{K}^1$, since s_1 is a relative interior point.

Next, we prove the "$\Rightarrow$" implication. Select a linear map $\mathcal{A}$ such that $\mathcal{L} = \ker \mathcal{A}$. Let $e^* \in (\operatorname{ri}(\mathcal{K}^{1*})) \times \{0\}$ and $e \in \operatorname{ri} \mathcal{K}$. Now consider the following pair of primal and dual problems:

$$\begin{array}{lll} \underset{x,t}{\text{minimize}} & t & \\ \text{subject to} & \langle e^*, x\rangle + t & = 1 \\ & \mathcal{A}x - t\mathcal{A}e & = 0 \\ & (x, t) \in \mathcal{K} \times \mathbb{R}_+ & \end{array} \tag{P_{GS}}$$

$$\begin{array}{ll} \underset{y_1,y_2}{\text{maximize}} & y_1 \\ \text{subject to} & -e^* y_1 - \mathcal{A}^\top y_2 \in \mathcal{K}^* \\ & 1 - y_1 + \langle e, \mathcal{A}^\top y_2\rangle \geq 0 \end{array} \tag{D_{GS}}$$

Taking $(\frac{e}{\langle e^*, e\rangle + 1}, \frac{1}{\langle e^*, e\rangle + 1})$, we see that the primal problem (P_{GS}) has a relative interior feasible solution. The dual problem (D_{GS}) satisfies the PPS condition and to see that, it is enough to take $y_1 = -1$ and $y_2 = 0$. This means that both (P_{GS}) and (D_{GS}) attain the optimal value and duality gap is zero. Let (x^*, t^*) be an optimal solution for (P_{GS}). If $t^* = 0$, then $\mathcal{A}x^* = 0$, $x^* \in \mathcal{K}$ and $\langle e^*, x^*\rangle = 1$. However, due to our hypothesis, $x^* \in (\operatorname{lin} \mathcal{K}^1) \times \mathcal{K}^2$, which implies that $\langle e^*, x^*\rangle = 0$, due to items *iii.* and *vi.* of Lemma 2.5. This is a contradiction, so we must have $t^* > 0$ instead. Since (D_{GS}) is attained as well, we have an optimal solution (y_1^*, y_2^*), with $y_1^* > 0$. Because $-e^* y_1 - \mathcal{A}^\top y_2 \in \mathcal{K}^*$, it readily follows that $-\mathcal{A}^\top y_2 \in (\operatorname{ri} \mathcal{K}^{1*}) \times \mathcal{K}^{2*}$.

□

For comparison, we state the classical Gordan-Stiemke's theorem. Its proof follows from Theorem 3.8. See also Corollary 2 in Luo, Sturm and Zhang [55].

Theorem 3.9 (Gordan-Stiemke's Theorem). *Let $\mathcal{L}$ be a subspace and $\mathcal{K}$ be a closed convex cone. Then we have:*

$$\mathcal{L} \cap \mathcal{K} \subseteq \operatorname{lin} \mathcal{K} \Leftrightarrow \mathcal{L}^{\perp} \cap (\operatorname{ri} \mathcal{K}^*) \neq \emptyset.$$

We now prove a theorem that dualizes the criteria in Proposition 3.7. But first, we need a lemma.

Lemma 3.10. *Let $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$ be a closed convex cone, such that $\mathcal{K}^2$ is polyhedral. Consider the pair $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ with $e \in (\operatorname{ri} \mathcal{K}^1) \times \{0\}$ and $e^* \in \operatorname{ri} \mathcal{K}^*$ The following properties hold.*

i. *$(P_{\mathcal{K}})$ has a relative interior feasible solution and $(D_{\mathcal{K}})$ satisfies the PPS condition. In particular, the duality gap is zero and both $(D_{\mathcal{K}})$ and $(P_{\mathcal{K}})$ attain the optimal value.*

ii. *Let (y_1, y_2, y_3) be a dual feasible solution satisfying $y_2 > 0$. Then, $s = c - \mathcal{A}^\top \frac{y_3}{y_1}$ satisfies $s_1 \in \operatorname{ri} \mathcal{K}^1$.*

iii. *If the optimal value is zero, we have either: a) $\langle c, x^* \rangle < 0$ and $\mathcal{F}_D^S = \mathcal{F}_{\min}^D = \emptyset$, or b) $\langle c, x^* \rangle = 0$ and $\mathcal{F}_D^S \subseteq \mathcal{K} \cap \{x^*\}^{\perp} \subsetneq \mathcal{K}$. In the latter case, we have $x_1^* \notin (\mathcal{K}^1)^{\perp} = \operatorname{lin}((\mathcal{K}^1)^*)$.*

iv. *Let (x^*, t^*, w^*) be a primal optimal solution. The optimal value is zero if and only if the PPS condition is not satisfied for* (D).

Proof. *i.* Let $t = \frac{1}{\langle e, e^* \rangle + 1}$, $w = \frac{1}{\langle e, e^* \rangle + 1}$ and $x = \frac{e^*}{\langle e, e^* \rangle + 1}$. Then (x, t, w) is a relative interior solution to $(P_{\mathcal{K}})$. Due to the choice of e, $(0, -1, 0)$ is a dual feasible solution such that the associated slack $((e, 0), 1, 1)$ belongs to $(\operatorname{ri} \mathcal{K}^1) \times \{0\} \times \operatorname{ri} \mathbb{R}_+ \times \operatorname{ri} \mathbb{R}_+$. We can then invoke Proposition 3.7, which ensures that the duality gap is zero and both problems are attained.

ii. Recall that for any closed convex cone $\mathcal{K}$ we have $\mathcal{K} + \operatorname{ri} \mathcal{K} = \operatorname{ri} \mathcal{K}$. Hence,

$$\left(c - y_1 e + \mathcal{A}^\top \frac{y_3}{y_1} \right) + y_1 e = c - \mathcal{A}^\top \frac{y_3}{y_1} \in (\operatorname{ri} \mathcal{K}^1) \times \mathcal{K}^2$$

due to the choice of e and the fact that $y_1 > 0$.

iii. Suppose that the optimal value is zero and let $(x^*, 0, w^*)$ be a primal optimal solution. We must have $\mathcal{A}x^* = 0$, $-\langle c, x^* \rangle = w^* \geq 0$ and $x^* \in \mathcal{K}^*$. If $\langle c, x^* \rangle < 0$, then we are done since this is alternative a). Note that this implies the infeasibility of (D), hence $\mathcal{F}_{\min}^D = \mathcal{F}_{\mathrm{D}}^S = \emptyset$.

On the other hand, if $-\langle c, x^* \rangle = 0$, then equation (3.1) implies $w^* = 0$. By equation (3.2), we have $\langle e, x^* \rangle = 1$, which implies that $x^* = (x_1^*, x_2^*)$ is such that $x_1^* \notin \operatorname{lin}((\mathcal{K}^1)^*) = (\mathcal{K}^1)^{\perp}$, see item *vi* of Lemma 2.5. Therefore, there is at least one element v in $\mathcal{K}$ for which $\langle v, x^* \rangle > 0$. Hence, $\mathcal{K} \cap \{x^*\}^{\perp} \subsetneq \mathcal{K}$ and the inclusion is indeed strict. Furthermore, since $\mathcal{A}x^* = 0$ and $\langle c, x^* \rangle = 0$, we also have $\mathcal{F}_{\mathrm{D}}^S \subseteq \mathcal{K} \cap \{x^*\}^{\perp}$. This is alternative *b*).

iv. Suppose that the PPS condition is not satisfied. If $t^* > 0$, then $y_2^* = t^* > 0$ for some dual optimal solution (y_1^*, y_2^*, y_3^*), because the duality gap is zero. It follows from item *ii.* that the PPS condition is satisfied, which is impossible.

Conversely, if $t^* = 0$ and $(x^*, 0, w^*)$ is an optimal solution for $(P_{\mathcal{K}})$, then either *a*) or *b*) of item *iii.* is satisfied. If *a*) is satisfied, then (D) is infeasible and we are done. If

b) is satisfied, then $\langle c, x^* \rangle = 0$, $\mathcal{A}x^* = 0$ and $x_1^* \notin (\mathcal{K}^1)^\perp$. If (s_1, s_2) is a feasible slack for (D), we have $\langle s_1, x_1 \rangle + \langle s_2, x_2 \rangle = 0$. As $x_1^* \notin (\mathcal{K}^1)^\perp$, we have that $s_1 \notin \operatorname{ri} \mathcal{K}^1$, so (D) cannot possibly satisfy the PPS condition.

□

Theorem 3.11. *Let $c \in \mathbb{R}^n$, $\mathcal{L} \subseteq \mathbb{R}^n$ be a subspace and $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2$ be a closed convex cone, such that $\mathcal{K}^2$ is polyhedral. Then $(\mathcal{L} + c) \cap ((\operatorname{ri} \mathcal{K}^1) \times \mathcal{K}^2) = \emptyset$ if and only if one of the conditions below holds:*

a) there exists $x \in \mathcal{K}^* \cap \mathcal{L}^\perp$ such that $\langle c, x \rangle < 0$;

b) there exists $x = (x_1, x_2) \in \mathcal{K}^* \cap \mathcal{L}^\perp \cap \{c\}^\perp$ such that $x_1 \notin (\mathcal{K}^1)^\perp$.

Proof. Select a linear map $\mathcal{A}$ such that $\mathcal{L} = \operatorname{range} \mathcal{A}^\top$ and consider the pair of problems (P) and (D) with this choice of $\mathcal{A}$, $b = 0$ and c, $\mathcal{K}$ as given by the current theorem. Note that $(\mathcal{L} + c) \cap ((\operatorname{ri} \mathcal{K}^1) \times \mathcal{K}^2) = \emptyset$ if and only if the PPS condition is *not* satisfied for (D).

Consider the pair of problems $(P_\mathcal{K})$ and $(D_\mathcal{K})$ with $e \in (\operatorname{ri} \mathcal{K}^1) \times \{0\}$ and $e^* \in \operatorname{ri} \mathcal{K}^*$. We are then under the setting of Lemma 3.10. Item *iv.* ensures that (D) does not satisfy the PPS condition if and only if the optimal value is zero. And, indeed, if the optimal value is zero, item *iii.* implies the existence of x^* as required by the theorem. The only thing missing is to show that if there is an x satisfying either a) or b), then the optimal value of $(P_\mathcal{K})$ is zero.

Suppose first that a) holds. We take $\alpha = \frac{1}{\langle e, x \rangle - \langle c, x \rangle}$ and this is well-defined because $-\langle c, x \rangle > 0$. Then $(x\alpha, 0, -\alpha \langle c, x \rangle)$ is an optimal solution for $(P_\mathcal{K})$, which shows that the optimal value is zero. Now suppose that b) holds, then the condition $x_1 \notin (\mathcal{K}^1)^\perp$ implies that $\langle e, x \rangle = \langle e_1, x_1 \rangle > 0$. So let $\alpha = \frac{1}{\langle e, x \rangle}$. Then $(x\alpha, 0, 0)$ is an optimal solution for $(P_\mathcal{K})$, which shows that the optimal value is zero. □

We remark that the proof of Theorem 3.11 shows that by invoking $(P_\mathcal{K})$ and $(D_\mathcal{K})$ with an appropriate choice of e and e^*, we can either prove (strong) infeasibility or obtain a point that satisfies alternative b). We note that due to item i. of Lemma 3.10, the pair $(P_\mathcal{K})$, $(D_\mathcal{K})$ can be solved by infeasible interior-point methods in the case of semidefinite and second order cone programming, even though they might fail to be strongly feasible. This is because the convergence theory relies on the existence of optimal solutions affording zero duality gap, rather than strong feasibility. See, for instance, item 2. of Theorem 11 in the work by Nesterov, Todd and Ye [60].

Ideally, we would like the condition $x_1 \notin (\mathcal{K}^1)^\perp$ to hold even if alternative a) holds. However, if $\mathcal{L} = \mathbb{R}^n \times \mathcal{L}_2$ and $c = (0, c_2)$ is such that $(\mathcal{L}_2 + c_2) \cap \mathcal{K}^2 = \emptyset$, then any $x \in \mathcal{L}^\perp$ must have $x_1 = 0$, which implies $x_1 \in (\mathcal{K}^1)^\perp$. For comparison, we also state the alternative theorem for the condition "$(\mathcal{L} + c) \cap \operatorname{ri} \mathcal{K} \neq \emptyset$", which is a consequence of Theorem 3.11. See also Lemma 3.2 in [89].

Theorem 3.12. *Let $c \in \mathbb{R}^n$, $\mathcal{L}$ be a subspace and $\mathcal{K}$ be a closed convex cone. Then $(\mathcal{L} + c) \cap \operatorname{ri} \mathcal{K} = \emptyset$ if and only if one of the conditions below holds:*

a) there exists $x \in \mathcal{K}^* \cap \mathcal{L}^\perp$ such that $\langle c, x \rangle < 0$;

b) there exists $x \in \mathcal{K}^* \cap \mathcal{L}^\perp \cap \{c\}^\perp$ such that $x \notin \operatorname{lin}(\mathcal{K}^*)$.

We remark that the proof of Theorem 3.11 shows that by invoking $(P_\mathcal{K})$ and $(D_\mathcal{K})$ with an appropriate choice of e and e^*, we can either prove (strong) infeasibility or obtain a point that satisfies alternative b).

For completeness we also include here another partial polyhedrality result we obtained in [50].

Theorem 3.13. *Let C_1 and C_2 be non-empty convex sets in $\mathbb{R}^n$ such that C_1 is polyhedral, C_2 is closed. Suppose that*

$$\operatorname{rec} C_1 \cap -\operatorname{rec} C_2 \subseteq \operatorname{lin} C_2,$$

where $\operatorname{rec} C = \{x \in \mathbb{R}^n \mid x + C \subseteq C\}$ *is the recession cone of a closed convex set C. Then $C_1 + C_2$ is closed.*

Proof. See Theorem 20.3 in [75]. □

We will show that if C_2 is the direct product of a closed convex set and a polyhedral set, we may weaken the assumptions of the Theorem 3.13.

Proposition 3.14. *Let C_1 and $C_2 \times P$ be non-empty convex sets in $\mathbb{R}^n$ such that C_1 and P are polyhedral, and C_2 is closed. Suppose that*

$$\operatorname{rec} C_1 \cap -(\operatorname{rec} C_2 \times \operatorname{rec} P) \subseteq \operatorname{lin} C_2 \times -\operatorname{rec} P. \tag{3.6}$$

Then $C_1 + (C_2 \times P)$ is closed.

Proof. We have that $C_1 + C_2 \times P = (C_1 + \{0\} \times P) + C_2 \times \{0\}$. Since C_1 and P are polyhedral sets, $(C_1 + (\{0\} \times P))$ is also polyhedral. We would like to use Theorem 3.13 with $(C_1 + \{0\} \times P)$ and $C_2 \times \{0\}$. For that purpose, we are required to check that

$$(\operatorname{rec} C_1 + (\{0\} \times \operatorname{rec} P)) \cap -(\operatorname{rec} C_2 \times \{0\}) \subseteq \operatorname{lin} C_2 \times \{0\}, \tag{3.7}$$

because, due to polyhedrality, $\operatorname{rec}(C_1 + (\{0\} \times P)) = \operatorname{rec} C_1 + (\{0\} \times \operatorname{rec} P)$. Let (x, y) be a point that belongs to the set at the left-hand side of Equation (3.7), then $x \in -\operatorname{rec} C_2$ and $y = a + p = 0$, where $p \in \operatorname{rec} P$ and $(x, a) \in \operatorname{rec} C_1$. It follows that $(x, a) \in -(\operatorname{rec} C_2 \times \operatorname{rec} P)$. Since we are under the assumption that Equation (3.6) holds, $x \in \operatorname{lin} C_2$. Hence, $(x, y) \in \operatorname{lin} C_2 \times \{0\}$ and we are done. □

The following proposition is a small modification of Corollary 20.3.1 of [75].

Proposition 3.15. *Let C_1 and $C_2 \times P$ be non-empty convex sets in $\mathbb{R}^n$ such that C_1 and P are polyhedral, and C_2 is closed. Suppose that*

$$\operatorname{rec} C_1 \cap (\operatorname{rec} C_2 \times \operatorname{rec} P) \subseteq \operatorname{lin} C_2 \times \operatorname{rec} P. \tag{3.8}$$

and that $C_1 \cap (C_2 \times P) = \emptyset$. Then C_1 and $C_2 \times P$ can be strongly separated.

Proof. Since $C_1 \cap (C_2 \times P) = \emptyset$, we have that $0 \notin C_1 - (C_2 \times P)$. Applying Proposition 3.14 to C_1 and $-(C_2 \times P)$ we find that $C_1 - (C_2 \times P)$ is closed. Therefore, both sets can be strongly separated. □

3.2.1 Existence of strict complementary solutions for polyhedral problems

When $\mathcal{K} = \mathbb{R}^n_+$ the Goldman-Tucker Theorem ensures the existence of strict complementary solutions. However, if $\mathcal{K}$ is merely an arbitrary polyhedral cone, it is not entirely obvious that strict complementary solutions exist in the sense of Proposition 2.9. At this point, we mention that extending results from Linear Programming to general polyhedral settings is not always a trivial affair and we refer to Mclinden [57] and Akgül [2] for further discussion of these issues.

Proposition 3.16. *If $\mathcal{K}$ is polyhedral, $\theta_D < +\infty$ and $\theta_P > -\infty$, then* (D) *and* (P) *admits strict complementary optimal solutions as in Proposition 2.9.*

Proof. Since $\mathcal{K}$ is a polyhedral cone, it can be written as $\mathcal{K} = \{x \in \mathcal{E} \mid \mathcal{B}x \geq 0\}$, where $\mathcal{B} : \mathcal{E} \to \mathbb{R}^m$ is a linear map. Then we have $\mathcal{K}^* = \{\mathcal{B}^\top y \mid y \geq 0\}$. We can then write (P) and (D) as

$$\begin{aligned} \inf_{z} \quad & \langle \mathcal{B}c, z\rangle \\ \text{subject to} \quad & \mathcal{A}\mathcal{B}^\top z = b \\ & z \geq 0 \end{aligned} \tag{$P_{\mathcal{B}}$}$$

$$\begin{aligned} \sup_{w} \quad & \langle b, w\rangle \\ \text{subject to} \quad & \mathcal{B}c - \mathcal{B}\mathcal{A}^\top w \geq 0. \end{aligned} \tag{$D_{\mathcal{B}}$}$$

($P_{\mathcal{B}}$) and ($D_{\mathcal{B}}$) are linear programs and admit strict complementary optimal solutions z^* and $\hat{s}^* = \mathcal{B}c - \mathcal{B}\mathcal{A}^\top w^*$ such that $z^* + \hat{s}^* > 0$, which implies $z^* \in \operatorname{ri}(\mathbb{R}^n_+ \cap \{\hat{s}^*\}^\perp)$ and $\hat{s}^* \in \operatorname{ri}(\mathbb{R}^n_+ \cap \{z^*\}^\perp)$. If we let $x^* = \mathcal{B}^\top z^*$ and $s^* = c - \mathcal{A}^\top w^*$, then it is clear that $\langle x^*, s^*\rangle = \langle z^*, \hat{s}^*\rangle$. This shows that x^* is a primal optimal solution for (P) and w^* is a dual optimal solution for (D). We will prove that $x^* \in \operatorname{ri}(\mathcal{K}^* \cap \{s^*\}^\perp)$.

For the sake of obtaining a contradiction, suppose that $x^* \notin \operatorname{ri}(\mathcal{K}^* \cap \{s^*\}^\perp)$. Then, there is a hyperplane that properly separates $\mathcal{K}^* \cap \{s^*\}^\perp$ and x^*. This means that there is $v \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$ such that $\langle x^*, v\rangle \leq \theta \leq \langle a, v\rangle$, for all $a \in \mathcal{K}^* \cap \{s^*\}^\perp$. The fact that $x^* \in \mathcal{K}^* \cap \{s^*\}^\perp$ implies that $\theta = 0$ and properness of the separation implies that $\langle a, v\rangle > 0$ for at least one $a \in \mathcal{K}^* \cap \{s^*\}^\perp$. We have the following inequality:

$$\langle z^*, \mathcal{B}v\rangle \leq \theta \leq \langle y, \mathcal{B}v\rangle,$$

for every $y \in \mathbb{R}^n_+ \cap \{\mathcal{B}s^*\}^\perp = \mathbb{R}^n_+ \cap \{\hat{s}^*\}^\perp$. Therefore, the hyperplane induced by $\mathcal{B}v$ produces proper separation between $\mathbb{R}^n_+ \cap \{\hat{s}^*\}^\perp$ and z^*, which contradicts the strict complementarity between $\hat{s}^*$ and z^*. Therefore, $x^* \in \operatorname{ri}(\mathcal{K}^* \cap \{s^*\}^\perp)$

Since a polyhedral cone is always facially exposed, Proposition 2.9 implies that $s^* \in \operatorname{ri}(\mathcal{K} \cap \{x^*\}^\perp)$ as well. □

The proof shows that strict complementary solutions to (P) and (D) can be extracted from strict complementary solutions to ($P_{\mathcal{B}}$) and ($D_{\mathcal{B}}$). We can then use the technique described in [26] to find them with a single linear program.

3.3 FRA-Poly and related notions

One of the goals of facial reduction is to close the duality gap. Here we will discuss FRA-Poly, which a modification of the facial reduction algorithm that in many cases requires fewer reduction steps than the usual FRA approach. The procedure will be divided in two phases. The first detects infeasibility and restores strong duality, while the second finds the minimal face.

The idea behind the classical FRA is that whenever strong feasibility fails, we can obtain reducing directions until strong feasibility is satisfied again. Similarly, Phase 1 of FRA-Poly is based on the fact that whenever the generalized condition in Proposition 3.7 (PPS) fails, we may also obtain reducing directions until the PPS is satisfied, thanks to Theorem 3.11. In addition, those directions can be found by using ($P_{\mathcal{K}}$) and ($D_{\mathcal{K}}$) in an appropriate manner, as indicated in Lemma 3.10. After that, a single extra facial reduction step is enough to go to the minimal face. As the PPS condition is weaker than full-on strong feasibility, FRA-poly has better worst case bounds in many cases.

We now present a disclaimer of sorts. The theoretical results presented in this section stand whether FRA-poly is doable or not for a given $\mathcal{K}$. If we wish to do facial reduction concretely (even if it is by hand!), we need to make a few assumptions on our computational capabilities and on our knowledge of the lattice of faces of $\mathcal{K}$. First of all, we must be able to solve problems over faces of $\mathcal{K}$ such that both the primal and the dual satisfy the PPS condition and we must also be able to do basic linear algebraic operations. Also, for each face $\mathcal{F}$ of $\mathcal{K}$ we must know:

1. $\operatorname{span} \mathcal{F}$,
2. at least one point $e \in \operatorname{ri} \mathcal{F}$,
3. at least one point $e^* \in \operatorname{ri} \mathcal{F}^*$,
4. whether $\mathcal{F}$ is polyhedral or not.

We remark that apart from knowledge about the polyhedral faces, our assumptions are not very different from what it is usually assumed *implicitly* in the FRA literature. For symmetric cones, which include direct products of $\mathcal{S}^n_+$, $\mathcal{Q}^n$ and $\mathbb{R}^n_+$, they are reasonable since their lattice of faces is well-understood and every face is again a symmetric cone. So, for instance, e can be taken as the identity element for the corresponding Jordan algebra. On the other hand, if $\mathcal{K}$ is, say, the copositive cone $\mathcal{C}^n$, we might have some trouble fulfilling the requirements, inasmuch as our knowledge of the faces of $\mathcal{C}^n$ is still lacking.

3.3.1 FRA-Poly

Henceforth, we will assume that $\mathcal{K}$ is the product of r cones and we will write $\mathcal{K} = \mathcal{K}^1 \times \ldots \times \mathcal{K}^r$. Consider the following FRA variant, which we call FRA-poly.

[**Facial Reduction Poly - Phase 1**]

Input: (D)

Output: A set of reducing directions $\{d_1, \ldots, d_\ell\}$. If (D) is feasible, it outputs some face $\mathcal{F} \subseteq \mathcal{K}$ for which the PPS condition holds, together with a dual slack s' for which $s'_j \in \operatorname{ri} \mathcal{F}^j$ for every j such that $\mathcal{F}^j$ is nonpolyhedral. If (D) is infeasible, the directions form a certificate of infeasibility.

1) $\mathcal{F}_1 \leftarrow \mathcal{K}, i \leftarrow 1$

2) Let e be such that $e_j = 0$ if $\mathcal{F}_i^j$ is polyhedral and $e_j \in \operatorname{ri} \mathcal{F}_i^j$ otherwise. Let $e^* \in \operatorname{ri} \mathcal{F}_i^*$. Solve $(P_\mathcal{K})$ and $(D_\mathcal{K})$ with this choice of e, e^* and with $\mathcal{F}_i$ in place of $\mathcal{K}$ to obtain primal dual pairs of optimal solutions (x^*, t^*, w^*) and (y_1^*, y_2^*, y_3^*).

3) If $t^* = 0$ and $\langle c, x^* \rangle < 0$, let $\mathcal{F}^D_{\min} \leftarrow \emptyset$ and stop. (D) is infeasible.

4) If $t^* = 0$ and $\langle c, x^* \rangle = 0$, let $d_i \leftarrow x^*, \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{d_i\}^\perp, i \leftarrow i + 1$ and return to 2).

5) If $t^* > 0$, $s' \leftarrow c - \mathcal{A}^\top \frac{y_3^*}{y_1^*}$, $\mathcal{F} \leftarrow \mathcal{F}_i$ and stop.

Note that Phase 1 of FRA-poly might not end at the minimal face. Nevertheless, Proposition 3.7 states that, in order to have zero duality gap and primal attainment, it is enough to have a feasible point s such that $s_i \in \operatorname{ri} \mathcal{K}^i$ for every i that correspond to a nonpolyhedral cone. First, we will show that the output of FRA-Poly is correct.

Proposition 3.17. *Phase 1 of FRA-Poly finishes after finding a finite number of directions and its output is correct.*

i. if (D) *is feasible, then the output face $\mathcal{F}$ contains the minimal face $\mathcal{F}^D_{\min}$ and is such that the PPS condition is satisfied if $\mathcal{K}$ is substituted by $\mathcal{F}$. Moreover, s' is a dual feasible slack such that $s'_j \in \operatorname{ri} \mathcal{F}^j$ for every j such that $\mathcal{F}^j$ is nonpolyhedral.*

ii. (D) *is infeasible if and only if Step 3. is reached.*

Proof. The correctness of FRA-poly is guaranteed by Theorem 3.11 and Lemma 3.10. In particular, due to Lemma 3.10, we will be able to find reducing directions by invoking $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ with the choice of e and e^* described in Step 2. And if the PPS condition is indeed satisfied, we will know because the optimal value will be positive. In that case, due to Equation (3.3), we have $s'_j \in \operatorname{ri} \mathcal{F}_i^j$ for all j such that $\mathcal{F}_i^j$ is nonpolyhedral.

If the optimal value is zero, we can extract a reducing direction from $(P_{\mathcal{K}})$ that satisfy alternatives *a*) or *b*) of Theorem 3.11. If it is the former, we will hit Step 3, which implies infeasibility. If it is the latter, we will have $\mathcal{F}_{i+1} \subsetneq \mathcal{F}_i$, since at least one component of x^* does not belong to the lineality space of the nonpolyhedral part of $\mathcal{F}_i^*$. As x^* is a *bona fide* reducing direction, we have $\mathcal{F}_{\min}^D \subseteq \mathcal{F}_{i+1}$. Since $\mathcal{K}$ has no infinite descending chain of faces, eventually either Step 5. will be reached or alternative *a*) of Theorem 3.11 will hold, which implies that Step 3. will be reached. □

As remarked before, Phase 1 of FRA-poly correctly detects infeasibility and if the problem is feasible, we will end up with a face $\mathcal{F}$ such that if we reformulate (D) as a problem over $\mathcal{F}$, the duality gap will be zero and the primal will be attained if the common optimal is finite. We will also obtain a solution s' such that it is almost a relative interior point, except for the polyhedral blocks. This means the output face $\mathcal{F}$ is such that $\mathcal{F}^j = (\mathcal{F}_{\min}^D)^j$ for every j such that $\mathcal{F}^j$ is nonpolyhedral. The next step is showing that we can jump directly to the minimal face in a single facial reduction step.

In Phase 2, we also perform a facial reduction step, but with an important difference. This time, when considering the problems $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$, instead of using $\mathcal{F}$, we relax the nonpolyhedral blocks of $\mathcal{F}$ to their span. By doing so, $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ become polyhedral problems thus ensuring the existence of strict complementary optimal solutions which can easily be found by solving a linear program. From $(P_{\mathcal{K}})$ we will extract a reducing direction that will allows to find $\mathcal{F}_{\min}^D$ at once. And from $(D_{\mathcal{K}})$ we will extract a slack $\tilde{s}$ for (D) such that it is a relative interior point of the polyhedral part, but may violate other nonlinear constraints. Recall that s' is a feasible slack for (D) that has the property of being a relative interior for the nonpolyhedral part. We then take a strict convex combination of $\tilde{s}$ and s' putting a larger weight to s' to tilt $\tilde{s}$ towards the relative interior of the nonpolyhedral part and, at the same time, restoring feasibility. As the convex combination will be strict, this will also shift the point towards the relative interior of polyhedral part as well. Therefore the resulting point will indeed be a point in $\operatorname{ri} \mathcal{F}_{\min}^D$.

[**Facial Reduction Poly - Phase 2**]

Input: (D), the output of Phase 1: $\mathcal{F}$ and s', with $\mathcal{F} \neq \emptyset$.

Output: $\mathcal{F}_{\min}^D$ and dual feasible slack $\hat{s} \in \operatorname{ri} \mathcal{F}_{\min}^D$. If $\mathcal{F} \neq \mathcal{F}_{\min}^D$ then the procedure outputs an extra reducing direction d.

1) Let $\hat{\mathcal{K}} = \hat{\mathcal{K}}^1 \times \ldots \times \hat{\mathcal{K}}^r$ such that $\hat{\mathcal{K}}^j = \mathcal{F}^j$ if $\mathcal{F}^j$ is polyhedral and $\hat{\mathcal{K}}^j = \operatorname{span} \mathcal{F}^j$ otherwise. Let $e \in \operatorname{ri} \hat{\mathcal{K}}$ and $e^* \in \operatorname{ri} \hat{\mathcal{K}}^*$. Build the systems $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$.

2) Solve the linear programs $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ and obtain primal dual pairs of *strictly complementary* optimal solutions (x^*, t^*, w^*) and (y_1^*, y_2^*, y_3^*).

3) If $t^* = 0$, let $d \leftarrow x^*$, $\mathcal{F}_{\min}^D \leftarrow \mathcal{F} \cap \{x^*\}^\perp$. Let $\tilde{s}$ be $c - \mathcal{A}^\top \frac{y_3^*}{y_1^*}$. Then, we let $\hat{s}$ be a convex combination of $\tilde{s}$ and s' such that $\hat{s} \in \operatorname{ri} \mathcal{F}_{\min}^D$ and stop.

4) If $t^* > 0$, $\mathcal{F}_{\min}^D \leftarrow \mathcal{F}$. Let $\tilde{s}$ be $c - \mathcal{A}^\top \frac{y_3^*}{y_1^*}$. Then, we let $\hat{s}$ be a convex combination of $\tilde{s}$ and s' such that $\hat{s} \in \operatorname{ri} \mathcal{F}_{\min}^D$ and stop.

We now prove that Phase 2 of FRA-Poly is correct. This is a consequence of the following two results.

Theorem 3.18. *Consider the pair of problems* $(P_{\mathcal{K}})$ *and* $(D_{\mathcal{K}})$*, under the setting of Lemma 3.1. If there is a pair of dual strict complementary solutions* $(x^*, t^*, w^*), (y_1^*, y_2^*, y_3^*)$ *(see comments after Proposition 2.9) and the optimal value is* 0*, then* (D) *is either strongly infeasible or* $\mathcal{F}_{\min}^D = \mathcal{K} \cap \{x^*\}^\perp$*. In the latter case, it also holds that* $c - \mathcal{A}^\top \frac{y_3^*}{y_1^*} \in \operatorname{ri} \mathcal{F}_{\min}^D$*.*

Proof. If (D) is not strongly infeasible, then we must have $t^* = w^* = 0$. Then, dual strict complementarity implies that the third inequality of $(D_{\mathcal{K}})$ must be strict, that is, $y_1^* > 0$. Also due to strict complementarity we have that $cy_1^* - \mathcal{A}^\top y_3^* \in \operatorname{ri}(\mathcal{K} \cap \{x^*\}^\perp)$. Since $y_1^* > 0$, we have $c - \mathcal{A}^\top \frac{y_3^*}{y_1^*} \in \operatorname{ri}(\mathcal{K} \cap \{x^*\}^\perp)$, which shows that $\mathcal{F}_{\min}^D = \mathcal{K} \cap \{x^*\}^\perp$. □

In particular, when $\mathcal{K}$ is polyhedral, both $(D_{\mathcal{K}})$ and $(P_{\mathcal{K}})$ are polyhedral problems. In this case, strict complementary solutions are ensured to exist thanks to Proposition 3.16. We also remark that a strict complementary solution of a polyhedral problem can be found by solving a single linear program, see, for instance, the article by Freund, Roundy and Todd [26] and the related work by Mehrotra and Ye [58].

Theorem 3.19. *The output of Phase 2 of FRA-poly is correct.*

Proof. Our assumption here is that the outputs of Phase 1 are such that $\mathcal{F} \neq \emptyset$ and s' satisfies $s'_j \in \operatorname{ri} \mathcal{F}^j$ for every j such that $\mathcal{F}^j$ is nonpolyhedral.

First, consider the case where $\mathcal{F}$ is not the minimal face, i.e., $(\operatorname{ri} \mathcal{F}) \cap (c + \operatorname{range} \mathcal{A}^\top) = \emptyset$. By Theorem 3.12, this happens if and only if $\mathcal{F}$ and $c + \operatorname{range} \mathcal{A}^\top$ can be properly separated. Therefore, there exists $x \in \mathcal{F}^*$ such that $\mathcal{A}x = 0$ and either: *i*) $\langle c, x\rangle < 0$, or *ii*) $\langle c, x\rangle = 0$ and $x \notin \mathcal{F}^\perp$. As infeasibility is detected at Phase 1, alternative *i*) cannot occur and any x inducing proper separation between $\mathcal{F}$ and $c + \operatorname{range} \mathcal{A}^\top$ must satisfy alternative *ii*). Note that x is a reducing direction as well.

However, because $\mathcal{F}$ is the output of Phase 1 of FRA-Poly, we have that neither alternative a) *nor* alternative b) of Theorem 3.11 can hold. So the possible reducing directions x must be such that $x_j \in \operatorname{lin}((\mathcal{F}^j)^*) = (\mathcal{F}^j)^\perp = (\operatorname{span} \mathcal{F}^j)^\perp$ for every j such that $\mathcal{F}^j$ is not polyhedral, lest we run afoul of Theorem 3.11. We can then conclude that the possible reducing directions are confined to the polyhedral cone $\hat{\mathcal{K}}^*$, where $\hat{\mathcal{K}} = \hat{\mathcal{K}}^1 \times \ldots \times \hat{\mathcal{K}}^r$ is such that $\mathcal{K}^j = \hat{\mathcal{F}}^j$ if $\mathcal{F}^j$ is polyhedral and $\hat{\mathcal{K}}^j = \operatorname{span} \mathcal{F}^j$ otherwise. This is precisely the cone appearing in Phase 2 of FRA-poly.

If we build the systems $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$ precisely as in Lemma 3.1 using $\hat{\mathcal{K}}$, we will obtain a pair of linear programs. Therefore, they have a pair of strictly complementary optimal solutions (x^*, t^*, w^*), (y_1^*, y_2^*, y_3^*). Because $\mathcal{F}$ is not the minimal face, we have $t^* = 0$, so we are under the hypothesis of Theorem 3.18. Let $\tilde{s} = c - \mathcal{A}^\top \frac{y_3^*}{y_1^*}$.

We will prove that $\mathcal{F}_{\min}^D = \mathcal{F} \cap \{x^*\}^\perp$ and that some convex combination of s' and $\tilde{s}$ is a relative interior feasible solution of (D). Let $z_\beta = \beta s' + (1-\beta)\tilde{s}$. For all $\beta \in (0,1)$ and all j such that $\mathcal{F}^j$ is polyhedral, we have $(z_\beta)_j \in \operatorname{ri}(\mathcal{F}^j \cap \{x_j^*\}^\perp)$, because $\tilde{s}_j \in \operatorname{ri}(\mathcal{F}^j \cap \{x_j\}^\perp)$ and s' is feasible. If $\mathcal{F}^j$ is not polyhedral, then $\mathcal{F}^j \cap \{x_j^*\}^\perp = \mathcal{F}^j$, since $x_j \in (\mathcal{F}^j)^\perp$. Because $\tilde{s}_j \in \operatorname{span} \mathcal{F}^j$ and $s'_j \in \operatorname{ri} \mathcal{F}^j$, for β sufficiently close to 1 we have $(z_\beta)_j \in \operatorname{ri} \mathcal{F}^j$. Therefore, it is possible to select $\beta \in (0,1)$ such that $(z_\beta)_j \in \operatorname{ri}(\mathcal{F}^j \cap \{x_j\}^\perp)$ for all j. This also shows that $\mathcal{F}_{\min}^D = \mathcal{F} \cap \{x^*\}^\perp$.

If $\mathcal{F}$ was already the minimal face to begin with, then $t^* > 0$. We can then proceed in a similar fashion. The only difference is that due to (3.3), we will have that $\tilde{s} = c - \mathcal{A}^\top \frac{y_3^*}{y_1^*}$ satisfies $\tilde{s}_j \in \operatorname{ri}(\mathcal{F}^j)$ for every j such that $\mathcal{F}^j$ is polyhedral. And as before, we can select a convex combination of s' and $\tilde{s}$ belonging to the relative interior of $\mathcal{F}_{\min}^D$. □

3.3.2 Distance to polyhedrality

In order to bound the number of directions obtained through FRA-poly, we introduce the notion of *distance to polyhedrality*. In what follows, if we have a chain of faces $\mathcal{F}_1 \subsetneq \ldots \subsetneq$

$\mathcal{F}_\ell$, the length of the chain is defined to be ℓ.

Definition 3.20. *Let $\mathcal{K}$ be a non-empty closed convex cone. The* distance to polyhedrality *is the length minus one of the longest strictly ascending chain of non-empty faces $\mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_\ell$ which satisfies:*

1. *$\mathcal{F}_1$ is polyhedral;*
2. *$\mathcal{F}_j$ is not polyhedral for $j > 1$.*

We will denote the distance to polyhedrality by $\ell_{poly}(\mathcal{K})$.

The distance to polyhedrality is a well-defined concept, because the lineality of $\mathcal{K}$ is always an exposed polyhedral face of $\mathcal{K}$, see item *iii.* of Lemma 2.5. Therefore it is always possible to consider the chain $\operatorname{lin}(\mathcal{K})$. Moreover, $\ell_{\text{poly}}(\mathcal{K})$ counts the maximum number of facial reduction steps that can be taken before we reach a polyhedral face. Therefore a necessary and sufficient condition for a cone to be polyhedral is that $\ell_{\text{poly}}(\mathcal{K}) = 0$.

Example 3.21. *See section 2.3 for more details on the facial structure of $\mathcal{S}^n_+$ and $\mathcal{Q}^n$. For the positive semidefinite cone $\mathcal{S}^n_+$, we have $\ell_{poly}(\mathcal{S}^n_+) = n-1$. For a single Lorentz cone $\mathcal{Q}^n = \{(x_0, \overline{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_0 \geq \|\overline{x}\|_2\}$, we have $\ell_{poly}(\mathcal{Q}^n) = 1$ if $n > 2$. This is because the Lorentz cone only has three types of faces: $\{0\}$, $\mathcal{Q}^n$ or the half-lines running along the boundary. For comparison, the longest chain of nonempty faces of $\mathcal{S}^n_+$ has length $n+1$ and the one for $\mathcal{Q}^n$ has length* 3.

Proposition 3.22. *Let $\mathcal{K} = \mathcal{K}^1 \times \ldots \times \mathcal{K}^r$. If* (D) *is feasible, Phase 1 of FRA-Poly stops after finding at most $\sum_{i=1}^r \ell_{poly}(\mathcal{K}^i)$ directions.*

If (D) *is infeasible, Phase 1 stops after finding at most $1 + \sum_{i=1}^r \ell_{poly}(\mathcal{K}^i)$ directions.*

Proof. Due to the choice made at Step 2 of Phase 1 of FRA-Poly and the analysis done in Theorem 3.11, whenever $\langle c, x^* \rangle = 0$, we have that $x^*_j \notin (\mathcal{F}^j_i)^\perp$ for at least one nonpolyhedral cone $\mathcal{F}^j_i$. This means that $\mathcal{F}^j_{i+1}$ is a proper face of $\mathcal{F}^j_i$. In other words, whenever a new proper face is found, it is because we are making progress towards a polyhedral face for at least one nonpolyhedral cone. Therefore, after finding $\ell = \sum_{i=1}^r (\ell_{\text{poly}}(\mathcal{K}^i))$ directions, $\mathcal{F}_{\ell+1}$ is polyhedral.

We now consider what happens if the algorithm has not stopped after all these directions were found. Note that system $(D_\mathcal{K})$ and $(P_\mathcal{K})$ now becomes entirely polyhedral and $e = 0$. First, suppose that (D) is feasible and let y be such that $c - \mathcal{A}^\top y \in \mathcal{F}^S_{\text{D}}$. Then $(1, 1, y)$ satisfies (3.3) and (3.5), but might fail to satisfy (3.4). This poses no problem since it is enough to multiply $(1, 1, y)$ by a sufficiently small positive constant. It follows that $(D_\mathcal{K})$ has at least one feasible solution for which $y_1 \geq y_2 > 0$, thus showing that the dual optimal value is greater than zero. This means that we will end up reaching Step 5.

Suppose that (D) is infeasible. In this case, the optimal value of $(D_\mathcal{K})$ will be zero. Since $e = 0$, equation (3.2) implies that the optimal solution of $(P_\mathcal{K})$ will be a triple $(x^*, 0, 1)$, which implies that Step 3 will be reached and a single new direction will be added. □

This result has the following immediate corollary.

Corollary 3.23. *Let $\mathcal{K} = \mathcal{K}^1 \times \ldots \times \mathcal{K}^r$. The minimum face $\mathcal{F}^D_{\min}$ that contains the feasible region of* (D) *can be found in no more than $1 + \sum_{i=1}^r \ell_{poly}(\mathcal{K}^i)$ facial reduction steps.*

Proof. If (D) is infeasible, then $\mathcal{F}^D_{\min} = \emptyset$ and the result follows from Proposition 3.22. So suppose now that (D) is feasible. Then FRA-Poly ends after finding at most $\sum_{i=1}^r \ell_{\text{poly}}(\mathcal{K}^i)$. Due to Theorem 3.19, at most one extra direction is needed to jump to the minimal face. □

The number of directions found in FRA-poly can also be bounded by a quantity that depends on $\mathcal{L} = \operatorname{range} \mathcal{A}^\top$ and c, since Proposition 3.3 still applies.

Note that if one uses the "classical" facial reduction approach, it takes no more than $\ell_{\mathcal{K}} - 1$ facial reduction steps to find the minimal face, when (D) is feasible, see Proposition 3.3. If (D) is infeasible, an extra direction might be needed, which is the one that will hit Step 3 in the Generic Facial Reduction of Section 3.1. When $\mathcal{K}$ is a direct product of several cones, we have $\ell_{\mathcal{K}} = 1 + \sum_{i=1}^{r} (\ell_{\mathcal{K}^i} - 1)$. We will end this subsection by showing that, under the relatively weak hypothesis that $\mathcal{K}^i$ is not a subspace, we have $\ell_{\text{poly}}(\mathcal{K}^i) < \ell_{\mathcal{K}^i} - 1$. This means that the number of steps needed in FRA-Poly is no worse than the classical FRA and if we have the direct product of at least two cones that are not subspaces, FRA-Poly is ensured to have a better worst case complexity.

Lemma 3.24. *Suppose that $\mathcal{K}$ is pointed, that is,* $\operatorname{lin}(\mathcal{K}) = \{0\}$ *and that its dimension is greater than zero, then $\ell_{poly}(\mathcal{K}) < \ell_{\mathcal{K}} - 1$.*

We now substitute the hypothesis of pointedness by the weaker assumption that $\mathcal{K}$ is not a subspace.

Theorem 3.25. *If $\mathcal{K}$ is not a subspace then $\ell_{poly}(\mathcal{K}) < \ell_{\mathcal{K}} - 1$. In particular, if $\mathcal{K}$ is the direct product of r closed convex cones that are not subspaces we have:*

$$r + 1 + \sum_{i=1}^{r} \ell_{poly}(\mathcal{K}^i) \leq 1 + \sum_{i=1}^{r} (\ell_{\mathcal{K}^i} - 1).$$

Proof. Let $U = \operatorname{lin} \mathcal{K}$. Then we have $\mathcal{K} = (\mathcal{K} \cap U^\perp) + U$. If we let $\hat{\mathcal{K}} = \mathcal{K} \cap (U^\perp)$, we have that $\operatorname{lin}(\hat{\mathcal{K}}) = \{0\}$. It can be shown that there is a bijection between the faces of $\mathcal{K}$ and the set $\{\mathcal{F} + U \mid \mathcal{F} \text{ is a face of } \hat{\mathcal{K}}\}$. There is also a correspondence between the polyhedral faces of $\mathcal{K}$ and the set $\{\mathcal{F} + U \mid \mathcal{F} \text{ is a polyhedral face of } \hat{\mathcal{K}}\}$. The assumption that $\mathcal{K}$ is not a subspace implies that the dimension of both $\mathcal{K}$ and $\hat{\mathcal{K}}$ is greater than zero. As $\ell_{\mathcal{K}} = \ell_{\hat{\mathcal{K}}}$, the result follows from applying Lemma 3.24 to $\hat{\mathcal{K}}$. $\square$

Proof of Lemma 3.24. Let $e^* \in \operatorname{ri} \mathcal{K}^*$, then the set $C = \{x \in \mathcal{K} \mid \langle x, e^* \rangle = 1\}$ is compact. This is because the recession cone of C consists of the elements in $\mathcal{K}$ which are orthogonal to e^*, but pointedness imply that 0 is the only element meeting these criteria. Now, the Krein-Milman Theorem implies that a nonempty compact convex set has at least one extreme point z. Then, one can verify that the half-line $h_z = \{\alpha z \mid \alpha \geq 0\}$ is an one-dimensional face[1] of $\mathcal{K}$. Let $\mathcal{F}$ be a face of $\mathcal{K}$ with dimension greater than zero. We can apply the same argument to conclude that $\mathcal{F}$ has at least one extreme ray, i.e., a face of dimension one.

So $\ell_{\mathcal{K}} \geq 2$, since we have the chain $\{0\} \subsetneq h_z$. If $\mathcal{K}$ is polyhedral, we are done, since $\ell_{\text{poly}}(\mathcal{K}) = 0$. We now move on to the nonpolyhedral case.

Let $\mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_\ell$ be a strictly ascending chain of faces such that $\mathcal{F}_1$ is polyhedral, $\mathcal{F}_j$ is not polyhedral for $j > 1$ and $\ell - 1 = \ell_{\text{poly}}(\mathcal{K})$. Due to non-polyhedrality, $\ell \geq 2$. We now consider two cases. If the dimension of $\mathcal{F}_1$ is greater or equal than one, we can augment the chain by adding the face $\{0\}$ at the beginning. In this case, we have $\ell_{\mathcal{K}} - 1 \geq \ell > \ell_{\text{poly}}(\mathcal{K})$.

On the other hand, if $\mathcal{F}_1 = \{0\}$, we have for sure that $\mathcal{F}_2$ has dimension greater or equal than two. This is because $\mathcal{F}_2$ is not polyhedral, so it cannot be an extreme ray. However, due to the previous argument, $\mathcal{F}_2$ has at least one extreme ray h_z, so we can augment the chain by inserting h_z between $\mathcal{F}_1$ and $\mathcal{F}_2$. This also shows that $\ell_{\mathcal{K}} - 1 \geq \ell > \ell_{\text{poly}}(\mathcal{K})$. $\square$

[1] To see that, first note that every nonzero element in $\mathcal{K}$ can be written as a product αx with $x \in C$ and $\alpha > 0$. It is enough to consider the case where $\beta\alpha_1 x + (1-\beta)\alpha_2 y = \gamma z$ with $\alpha_1, \alpha_2, \gamma > 0$, $x, y \in C$ and $\beta \in (0, 1)$. Then $\frac{\beta\alpha_1}{\gamma} + \frac{(1-\beta)\alpha_2}{\gamma} = 1$. Since z is an extreme point, we have $x = y = z$. Therefore, $\alpha_1 x, \alpha_2 y \in h_z$.

3.3.3 Distance to strong duality

The singularity degree of (D) is a measure that depends on $c, \mathcal{A}$ and $\mathcal{K}$. However, it is possible to give uniform bounds for $d(D)$ that do not depend on $c, \mathcal{A}$. For example, the classical facial reduction strategy gives the bounds $d(D) \leq \ell_{\mathcal{K}} - 1$ when (D) is feasible and $d(D) \leq \ell_{\mathcal{K}}$ when (D) is infeasible. Corollary 3.23 readily implies that $d(D) \leq 1 + \sum_{i=1}^{r} \ell_{\text{poly}}(\mathcal{K}^i)$, no matter whether (D) is feasible or not. Due to Theorem 3.25, this bound is likely to compare favorably to $\ell_{\mathcal{K}} - 1 = \sum_{i=1}^{r}(\ell_{\mathcal{K}^i} - 1)$.

As mentioned before, the singularity degree only depends on $c, \mathcal{A}$ and $\mathcal{K}$. Finding the minimal face $\mathcal{F}_{\min}^{D}$ ensures that no matter which b we select, as long as the problem is bounded, there will be zero duality gap and primal attainment. This suggests the following definition that also depends on b and, thus, produce a less conservative quantity.

Definition 3.26 (Distance to strong duality). *The distance to strong duality $d_{str}(D)$ is the minimum number of facial reduction steps (at* (D)*) needed to ensure $\theta_{\hat{P}} = \theta_D$, where $(\hat{P})$ is the problem $\inf\{\langle c, x\rangle \mid \mathcal{A}x = b, x \in \mathcal{F}_{\ell+1}^*\}$ and $\mathcal{F}_{\ell+1}$ is a obtained after a sequence of ℓ facial reduction steps. If $-\infty < \theta_D < +\infty$, we also require attainment of $\theta_{P'}$.*

Similarly, we define $d_{str}(P)$ as the minimum number of facial reduction steps needed to ensure that $\theta_P = \theta_{\hat{D}}$ and that $\theta_{\hat{D}}$ is attained when $-\infty < \theta_P < +\infty$. It is understood that $(\hat{D})$ is the problem in dual standard form arising after some sequence of facial reduction steps is done at (P)*.*

Clearly, we have $d_{\text{str}}(D) \leq d(D)$. However, since Phase 1 of FRA-Poly restores strong duality in the sense of Definition 3.26, we obtain the nontrivial bound $d_{\text{str}}(D) \leq \sum_{i=1}^{r} \ell_{\text{poly}}(\mathcal{K}^i)$. This will be useful to bound the length of infeasibility certificates and the dimension of certain special subspaces that appear in weakly infeasible problems in sections 4.1 and 4.2.

3.4 Worst case instance for direct products of SDPs and SOCPs

Given a dual program (D), the singularity degree $d(D)$ is defined as the minimum number of facial reduction steps needed to find $\mathcal{F}_{\min}^{D}$. As remarked before, when (D) is feasible, previous analyses of facial reduction imply that $d(D) \leq \ell_{\mathcal{K}} - 1$, where $\ell_{\mathcal{K}}$ is the longest chain of faces of $\mathcal{K}$. First, let us consider another bound for $d(D)$. We remark, however, that it is not ensured that FRA-Poly will achieve these bounds, because they depend on the algorithm being lucky enough to select certain good directions.

In what follows, we will use the following observation. Except when (D) is infeasible, doing facial reduction at (D) is the same thing as doing facial reduction for the homogenized problem $\sup_{y,y_0}\{\langle b, y\rangle \mid cy_0 - \mathcal{A}^\top y \in \mathcal{K}\}$, because in that case all the reducing directions will satisfy $\langle d_i, c\rangle = 0$.

Proposition 3.27. *Suppose that* (D) *is feasible and $\mathcal{F}_{\min}^{D} = \operatorname{lin}\mathcal{K}$, then $d(D) = 1$. That is, there is $d \in \mathcal{K}^* \cap \ker \mathcal{A} \cap \{c\}^\perp$ such that $\operatorname{lin}\mathcal{K} = \mathcal{K} \cap \{d\}^\perp$*

Proof. Let $\overline{\mathcal{L}}$ be the space spanned by range $\mathcal{A}^\top$ and c. Then, feasibility implies that the minimal face of $\mathcal{K}$ which contains $\overline{\mathcal{L}} \cap \mathcal{K}$ is $\mathcal{F}_{\min}^{D}$. We then have $\overline{\mathcal{L}} \cap \mathcal{K} \subseteq \operatorname{lin}\mathcal{K}$. By Theorem 3.9, there exists $d \in (\operatorname{ri}\mathcal{K}^*) \cap \overline{\mathcal{L}}^\perp$. Because $d \in \operatorname{ri}\mathcal{K}^*$, we have $\operatorname{lin}\mathcal{K} = \mathcal{K} \cap \{d\}^\perp$. Finally, $d \in \overline{\mathcal{L}}^\perp$ implies $d \in \ker \mathcal{A} \cap \{c\}^\perp$. $\square$

We remark that a characterization of problems with singularity degree one was recently given by Drusvyatskiy, Pataki and Wolkowicz in Theorem 4.1 of [20].

Corollary 3.28. *Suppose that every chain of faces $\mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_\ell$ of $\mathcal{K}$ satisfying $\ell = \ell_{poly}(\mathcal{K}) + 1$ is such that the dimension of $\mathcal{F}_1$ is less or equal than* $\dim(\operatorname{lin}\mathcal{K}) + 1$*. If* (D) *is feasible, then the singularity degree of* (D) *satisfies $d(D) \leq \max(\ell_{poly}(\mathcal{K}), 1)$.*

Proof. Proposition 3.22 implies the bound $\ell_{\text{poly}}(\mathcal{K})$ for the termination of Phase 1 of FRA-poly. Denote the chain of faces obtained in Phase 1 by $\mathcal{K} = \hat{\mathcal{F}}_1 \subsetneq \ldots \subsetneq \hat{\mathcal{F}}_{\hat{\ell}}$. This implies that we have found $\hat{\ell} - 1$ directions through FRA-Poly. If $\hat{\ell} - 1 < \ell_{\text{poly}}(\mathcal{K})$, then Theorem 3.19 implies that one more facial reduction step is enough to find $\mathcal{F}^D_{\min}$, so in this case we have the bound $\ell_{\text{poly}}(\mathcal{K})$.

If $\hat{\ell} = \ell_{\text{poly}}(\mathcal{K})$, then $\mathcal{F}_{\hat{\ell}}$ must have dimension less or equal to $\dim(\operatorname{lin}\mathcal{K}) + 1$. If $\mathcal{F}^D_{\min} = \mathcal{F}_{\hat{\ell}}$, we are done. Otherwise, it must be the case that $\mathcal{F}^D_{\min} = \operatorname{lin}\mathcal{K}$. However, in that case Proposition 3.27 implies that a single facial reduction step is enough. Note that, except if $\mathcal{K}$ is polyhedral, $\ell_{\text{poly}}(\mathcal{K}) \geq 1$, so the bound is also true when $\mathcal{F}^D_{\min} = \operatorname{lin}\mathcal{K}$. □

If $\mathcal{K} = \mathcal{S}^n_+$, the bound in Corollary 3.28 implies that $d(D) \leq n$, but this is not tight. This is because $\mathcal{K}$ satisfies the conditions of Corollary 3.28, so we need at most $n-1$ steps. And, in fact, there is an instance of an SDP due to Tunçel such that any facial algorithm must take at least $n-1$ steps, see Section 2.6 in [85] or the section "Worst case instance" in [16].

Here, we will consider the case where $\mathcal{K} = \mathcal{Q}^{t_1} \times \ldots \mathcal{Q}^{t_{r_1}} \times \mathcal{S}^{n_1}_+ \times \ldots \times \mathcal{S}^{n_{r_2}}_+$ is the direct product of r_1 second order (Lorentz) cones and r_2 positive semidefinite cones. Note that $\mathcal{K}$ is self-dual, so $\mathcal{K}^* = \mathcal{K}$. We will make a few assumptions to discard trivial cases. First, we will assume that we have $r_1 + r_2 > 0$. We will also assume that each Lorentz cone has dimension three or more, since a dimension two Lorentz cone must be polyhedral. We will assume that $n_j \geq 3$ for every j, since $\mathcal{S}^2_+$ is isomorphic to a single three dimensional Lorentz cone. In this case, Corollary 3.23 implies that we have the bound

$$d(D) \leq 1 + r_1 + \sum_{j=1}^{r_2}(n_j - 1)$$

for the singularity degree of (D). This bound is almost tight for direct products of SDPs and SOCPs and it matches what is achievable by running phases 1 and 2 of FRA-Poly. However, with more effort and stronger assumptions, it is possible to show that the "+1" can be removed for feasible mixed SDP-SOCPs. As far as we know, the only reference for that is a chapter written by Luo and Sturm [54]. The caveat is that for a facial reduction algorithm to achieve this bound, it must select at each step the "most interior" reducing direction possible. For instance, in the first step, instead of searching for $d \in \{c\}^\perp \cap \ker\mathcal{A} \cap (\mathcal{K}^* \setminus \mathcal{K}^\perp)$, one must, instead, obtain an element in $\operatorname{ri}(\{c\}^\perp \cap \ker\mathcal{A} \cap \mathcal{K}^*)$, which is a problem that may itself require facial reduction. Nevertheless, Luo and Sturm's analysis shows that in theory we need no more than $r_1 + \sum_{j=1}^{r_2}(n_j - 1)$ facial reduction steps to find the minimal face for a feasible problem. In this subsection, we will show that there is indeed an instance for which any facial reduction algorithm must take at least $r_1 + \sum_{j=1}^{r_2}(n_j - 1)$ steps to find $\mathcal{F}^D_{\min}$. This tells us that the uniform bound given by Luo and Sturm is tight.

We will use the following special notation. Given an element x, we will use $x^j_{i,k}$ to denote the (i,k) entry of the j-th matrix block and x^j_i to denote the i-entry of the j-th vector block. For simplicity, we will also use the same notation to single out a few special elements that have zero in almost all entries. For $j \in [1, r_2], a^j_{i,k}$ is the element of $\mathcal{K}$ such that all its blocks are zero except for the block corresponding to $\mathcal{S}^{n_j}_+$. In that block $a^j_{i,k}$ contains the $n_j \times n_j$ matrix that is zero everywhere except for the fact that the (i,k) and (k,i) entries are equal to one. In a similar fashion, for $j \in [1, r_1]$, a^j_i is the element of

$\mathcal{K}$ such that all its blocks are zero except for the block corresponding to $\mathcal{Q}^{t_j}$, where a_i^j corresponds to i-th unit vector.

The example will be constructed by considering the following special space.

[**A basis for $\mathcal{L}^\perp$**]

Let $\mathcal{L}^\perp$ be the space spanned by the following vectors:

1. $a_1^1 + a_2^1$,
2. $\{a_3^{j-1} + a_1^j + a_2^j \mid 1 < j \leq r_1\}$,
3. $a_3^{r_1} + a_{1,1}^1$ and $\{a_{i,i}^1 + a_{i-1,i+1}^1 \mid 1 < i < n_1\}$ (if $r_1 = 0$, substitute $a_3^{r_1} + a_{1,1}^1$ by $a_{1,1}^1$),
4. $a_{n_j,n_{j-1}}^{j-1} + a_{1,1}^j$ and $\{a_{i,i}^j + a_{j-1,j+1}^1 \mid 1 < i < n_j\}$, for $1 < j \leq r_2$,

where items 1. and 2. are omitted if $r_1 = 0$ and items 3. and 4. are ommited if $r_2 = 0$.

Example 3.29. *If $r_1 = r_2 = 2$, $n_1 = 3, n_2 = 4$ and $t_1 = t_2 = 3$, then following the construction above, we obtain a subspace spanned by elements having the following format:*

$$\begin{pmatrix} y_1 \\ y_1 \\ y_2 \end{pmatrix} \times \begin{pmatrix} y_2 \\ y_2 \\ y_3 \end{pmatrix} \times \begin{pmatrix} y_3 & 0 & y_4 \\ 0 & y_4 & y_5 \\ y_4 & y_5 & 0 \end{pmatrix} \times \begin{pmatrix} y_5 & 0 & y_6 & 0 \\ 0 & y_6 & 0 & y_7 \\ y_6 & 0 & y_7 & 0 \\ 0 & y_7 & 0 & 0 \end{pmatrix}.$$

In the next proposition, it will be helpful to keep this instance in mind.

Proposition 3.30. *Let $c = 0$ and $\mathcal{A}$ be such that $\mathcal{L} = \operatorname{range} \mathcal{A}^\top$, where $\mathcal{L}^\perp$ is the subspace constructed above.*

It is necessary at least

$$r_1 + \sum_{j=1}^{r_2} (n_j - 1)$$

facial reduction steps to find $\mathcal{F}_{\min}^D$.

Proof. First, suppose that $r_1 > 0$. At the first step of facial reduction, we have to find a nonzero direction in $\mathcal{K} \cap \mathcal{L}^\perp$. However, if $x \in \mathcal{K}$ and x is a linear combination of the vectors constructed above, we have $x_1^1 = x_2^1$. Then, the Lorentz cone constraint implies that $x_i^1 = 0$ for all $i \geq 3$. Therefore, the coefficient of $a_3^1 + a_1^2 + a_2^2$ appearing in x must be zero as well. By induction, it follows that all blocks of x are zero, except for the first. We have no choice but to select a positive multiple of $a_1^1 + a_2^1$ as the first reducing direction. So let $d_1 = a_1^1 + a_2^1$, we then have $\mathcal{F}_2 = \mathcal{K} \cap \{d_1\}^\perp = h_{d_1} \times \ldots \mathcal{Q}^{t_{r_1}} \times \mathcal{S}_+^{n_1} \times \ldots \times \mathcal{S}_+^{n_{r_2}}$, where h_{d_1} is contained in $\mathcal{Q}^{t_1}$ and is the half-line along the direction defined by the nonzero part $a_1^1 - a_2^1$. When it is time to perform the next step, we will find out that only the nonnegative multiples of $a_3^1 + a_1^2 + a_2^2$ belong to $\mathcal{F}_2^* \cap \mathcal{L}^\perp$. This means that facial reduction must proceed by successively selecting positive multiples of:

1. $d_1 = a_1^1 + a_2^1$,
2. $d_j = a_3^{j-1} + a_1^j + a_2^j$, for $1 < j \leq r_1$.

After r_1 steps, all the Lorentz cone blocks will be transformed to half-lines and we will have $\mathcal{F}_{r_1+1} = h_{d_1} \times \ldots h_{d_{r_1}} \times \mathcal{S}_+^{n_1} \times \ldots \times \mathcal{S}_+^{n_{r_2}}$.

If there are no positive semidefinite cones, we are done. Otherwise, we will have $\mathcal{F}_{r_1+1}^* \cap \mathcal{L}^\perp = \{t(a_3^{r_1} + a_{1,1}^1) \mid t \geq 0\}$. Again, we have no choice but to proceed "one row at the time" and select positive multiples of $a_{i,i}^1 + a_{i-1,i+1}^1$ for $1 < i < n_1$ as the reducing directions. In total, $n_1 - 1$ directions must be found before we can move to the next SDP block. Induction shows that for each block $n_j - 1$ directions will be found. In total we obtain $r_1 + \sum_{j=1}^{r_2}(n_j - 1)$. directions.

In the case where $r_1 = 0$, the argument is similar. The only difference is that $\mathcal{F}_1^* \cap \mathcal{L}^\perp = \{t(a_{1,1}^1) \mid t \geq 0\}$. We can then proceed as before. $\square$

3.5 Singularity degree of the intersection of cones

In this we section, we discuss the case where $\mathcal{K} = \mathcal{K}^1 \cap \mathcal{K}^2$. We can rewrite (D) as a problem over $\mathcal{K}^1 \times \mathcal{K}^2$ by duplicating the entries.

$$\begin{array}{rl} \sup\limits_{y} & \langle b, y \rangle \\ \text{subject to} & (c - \mathcal{A}^\top y, c - \mathcal{A}^\top y) \in \mathcal{K}^1 \times \mathcal{K}^2 \end{array} \tag{D_{dup}}$$

$$\begin{array}{rl} \inf\limits_{x} & \langle c, x^1 + x^2 \rangle \\ \text{subject to} & \mathcal{A}(x^1 + x^2) = b \\ & (x^1, x^2) \in (\mathcal{K}^1)^* \times (\mathcal{K}^2)^*. \end{array} \tag{P_{dup}}$$

While (D_{dup}) is entirely equivalent to (D), the situation for (P_{dup}) is subtler. It is true that $\theta_{P_{\text{dup}}} = \theta_P$ and that if (P_{dup}) is attained, then (P) must be attained. However, the converse is not true and (P_{dup}) might fail to be attained even if (P) is attained. This situation can happen if $(\mathcal{K}^1)^* + (\mathcal{K}^2)^* \subsetneq \mathcal{K}^*$.

Still, if we apply FRA-Poly to (D_{dup}), we will obtain a face $\mathcal{F}^1 \times \mathcal{F}^2$ of $\mathcal{K}^1 \times \mathcal{K}^2$. Doing facial reduction using the formulation (D_{dup}) might be more convenient, since we need to search for reducing directions in $(\mathcal{K}^1)^* \times (\mathcal{K}^2)^*$ instead of $\operatorname{cl}((\mathcal{K}^1)^* + (\mathcal{K}^2))^*$ and deciding membership in $(\mathcal{K}^1)^* \times (\mathcal{K}^2)^*$ could be more straightforward than doing the same for $\operatorname{cl}((\mathcal{K}^1)^* + (\mathcal{K}^2))^*$. Before we proceed we need an auxiliary result.

If $\mathcal{K} = \mathcal{K}^1 \cap \mathcal{K}^2$, it is always true that the intersection of a face of $\mathcal{K}^1$ with a face of $\mathcal{K}^2$ results in a face of $\mathcal{K} = \mathcal{K}^1 \cap \mathcal{K}^2$. However, it is not entirely obvious that every face of $\mathcal{K}$ arises as an intersection of faces of $\mathcal{K}^1$ and $\mathcal{K}^2$, so we remark that as a proposition although it is probably a well-known result.

Proposition 3.31. *Let $\mathcal{F}$ be a face of $\mathcal{K} = \mathcal{K}^1 \cap \mathcal{K}^2$. Let $\mathcal{F}^1$ and $\mathcal{F}^2$ be the minimal faces of $\mathcal{K}^1$ and $\mathcal{K}^2$, respectively, containing $\mathcal{F}$. Then $\mathcal{F} = \mathcal{F}^1 \cap \mathcal{F}^2$ and $\mathcal{F}^* = (\mathcal{F}^1)^* + (\mathcal{F}^2)^*$.*

Proof. We have $\mathcal{F}^1 \cap \mathcal{F}^2 \supseteq \mathcal{F}$ and we will prove that $\mathcal{F}^1 \cap \mathcal{F}^2 = \mathcal{F}$. A first observation is that by the choice of $\mathcal{F}^1$ and $\mathcal{F}^2$, we have $\operatorname{ri}(\mathcal{F}) \subseteq \operatorname{ri}(\mathcal{F}^1)$ and $\operatorname{ri}(\mathcal{F}) \subseteq \operatorname{ri}(\mathcal{F}^2)$. In particular, this implies that $\operatorname{ri}(\mathcal{F}^1) \cap \operatorname{ri}(\mathcal{F}^2) \neq \emptyset$. Therefore, $\operatorname{ri}(\mathcal{F}^1 \cap \mathcal{F}^2) = \operatorname{ri}(\mathcal{F}^1) \cap \operatorname{ri}(\mathcal{F}^2)$, see Theorem 6.5 in [75]. We conclude that $\operatorname{ri}(\mathcal{F}) \cap \operatorname{ri}(\mathcal{F}^1 \cap \mathcal{F}^2) = \operatorname{ri}(\mathcal{F}) \cap \operatorname{ri}(\mathcal{F}^1) \cap \operatorname{ri}(\mathcal{F}^2) \neq \emptyset$. Since $\mathcal{F}^1 \cap \mathcal{F}^2$ is a face of $\mathcal{K}$, it follows that $\mathcal{F} = \mathcal{F}^1 \cap \mathcal{F}^2$.

Because $\operatorname{ri}(\mathcal{F}^1) \cap \operatorname{ri}(\mathcal{F}^2) \neq \emptyset$, a classical closedness criteria implies that $(\mathcal{F}^1)^* + (\mathcal{F}^2)^*$ is closed (see Corollary 16.4.2 in [75]), so that $\mathcal{F}^* = \operatorname{cl}((\mathcal{F}^1)^* + (\mathcal{F}^2)^*) = (\mathcal{F}^1)^* + (\mathcal{F}^2)^*$. $\square$

Theorem 3.32. *Let $\mathcal{K} = \mathcal{K}^1 \cap \mathcal{K}^2$.*

i. Let $\hat{\mathcal{F}} = \mathcal{F}^1 \times \mathcal{F}^2$ be the minimal face of $\mathcal{K}^1 \times \mathcal{K}^2$ containing the feasible slacks of (D_{dup}). Then, $\mathcal{F}_{\min}^D = \mathcal{F}^1 \cap \mathcal{F}^2$.

ii. The singularity degree of (D) satisfies $d(D) \leq d(D_{dup}) \leq 1 + \ell_{poly}(\mathcal{K}^1) + \ell_{poly}(\mathcal{K}^2)$.

iii. The distance to strong duality satisfies $d_{str}(D) \leq d_{str}(D_{dup}) \leq \ell_{poly}(\mathcal{K}^1) + \ell_{poly}(\mathcal{K}^2)$.

Proof. *i.* Note that $\mathcal{F}^1$ must be the minimal face of $\mathcal{K}^1$ containing $\mathcal{F}^S_{\mathrm{D}} = \{c - \mathcal{A}^\top y \in \mathcal{K}\}$. Because if some proper face $\widetilde{\mathcal{F}}$ of $\mathcal{F}^1$ is minimal, then $\widetilde{\mathcal{F}} \times \mathcal{F}^2$ contains the feasible slacks of (D_{dup}), which contradicts the minimality of $\hat{\mathcal{F}}$. The same must hold for $\mathcal{F}^2$. Then Proposition 3.31 implies $\mathcal{F}^D_{\min} = \mathcal{F}^1 \cap \mathcal{F}^2$.

ii. In order to prove that the singularity degree of (D) is bounded by $d(D_{\mathrm{dup}})$, we need to check whether a sequence of reducing directions for (D_{dup}) translate into reducing directions for (D). A sequence of reducing directions $\{d_1, \ldots, d_\ell\}$ and corresponding faces for (D_{dup}) are such that:

(a) $\hat{\mathcal{F}}_1 = \mathcal{K}^1 \times \mathcal{K}^2$.

(b) $d_i = (d_i^1, d_i^2) \in \hat{\mathcal{F}}_i^*, \hat{\mathcal{F}}_{i+1} = (\hat{\mathcal{F}}_i^1) \cap \{d_i\}^\perp \times (\hat{\mathcal{F}}_i^2) \cap \{d_i\}^\perp$, for $i \in \{1, \ldots, \ell\}$,

and $\mathcal{A}(d_i^1 + d_i^2) = 0$, $\langle c, d_i^1 + d_i^2 \rangle \le 0$ for all i. We will prove that $\{d_1^1 + d_1^2, \ldots, d_\ell^1 + d_\ell^2\}$ form a valid sequence of reducing directions for (D), except that some of the directions might fail to induce a proper face. The only thing missing is to prove that $d_i^1 + d_i^2 \in \mathcal{F}_i^*$ for every i, with $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{d_i^1 + d_i^2\}^\perp$ and $\mathcal{F}_1 = \mathcal{K}$. And then, if $d_i^1 + d_i^2 \in \mathcal{F}_i^\perp$, we simply discard $d_i^1 + d_i^2$.

We will prove by induction that $\mathcal{F}_i = \hat{\mathcal{F}}_i^1 \cap \hat{\mathcal{F}}_i^2$ for every $i \in \{1 \ldots, \ell\}$ and that $d_i^1 + d_i^2 \in \mathcal{F}_i^*$ for every $i \in \{1 \ldots, \ell - 1\}$. First, note that $\mathcal{F}_1 = \hat{\mathcal{F}}_1^1 \cap \hat{\mathcal{F}}_1^2 = \mathcal{K}$. Because $\mathcal{K}^* = \mathrm{cl}\,((\mathcal{K}^1)^* + (\mathcal{K}^2)^*)$, we also have $d_1^1 + d_1^2 \in \mathcal{K}^*$. This takes care of the basis of induction.

Now, suppose that the statement holds true for some i and let us show that it holds for $i+1$. We have $\hat{\mathcal{F}}_{i+1}^1 \cap \hat{\mathcal{F}}_{i+1}^2 = \hat{\mathcal{F}}_i^1 \cap \hat{\mathcal{F}}_i^2 \cap \{d_i^1\}^\perp \cap \{d_i^2\}^\perp = \mathcal{F}_i \cap \{d_i^1\}^\perp \cap \{d_i^2\}^\perp$, by the induction hypothesis. It is clear that $\mathcal{F}_i \cap \{d_i^1\}^\perp \cap \{d_i^2\}^\perp \subseteq \mathcal{F}_i \cap \{d_i^1 + d_i^2\}^\perp = \mathcal{F}_{i+1}$. The opposite containment follows from the fact that if $x \in \mathcal{F}_i \cap \{d_i^1 + d_i^2\}^\perp$, then since $x \in \hat{\mathcal{F}}_i^1 \cap \hat{\mathcal{F}}_i^2$ and $d_i^1 \in (\hat{\mathcal{F}}_i^1)^*$, $d_i^2 \in (\hat{\mathcal{F}}_i^2)^*$, the only way that $\langle x, d_i^1 + d_i^2 \rangle$ can be zero is if $x \in \{d_i^1\}^\perp \cap \{d_i^2\}^\perp$. This shows that $\mathcal{F}_i \cap \{d_i^1\}^\perp \cap \{d_i^2\}^\perp = \mathcal{F}_i \cap \{d_i^1 + d_i^2\}^\perp = \mathcal{F}_{i+1} = \hat{\mathcal{F}}_{i+1}^1 \cap \hat{\mathcal{F}}_{i+1}^2$. As $(\hat{\mathcal{F}}_{i+1}^1)^* + (\hat{\mathcal{F}}_{i+1}^2)^* \subseteq \mathrm{cl}\,((\hat{\mathcal{F}}_{i+1}^1)^* + (\hat{\mathcal{F}}_{i+1}^2)^*) = \mathcal{F}_{i+1}^*$, we readily obtain that $d_{i+1}^1 + d_{i+1}^2 \in \mathcal{F}_{i+1}^*$, if $i + 1 < \ell$.

We have proved that the chain of faces $\mathcal{F}_1 \supseteq \ldots \supseteq \mathcal{F}_\ell$ is such that $\mathcal{F}_1 = \mathcal{K}$ and $\mathcal{F}_\ell = \mathcal{F}^D_{\min}$. However, some containments might fail to be strict. This poses no problem, since it is enough to remove the reducing directions that provide no decrease. This shows that $d(D) \le d(D_{\mathrm{dup}})$. If we apply both Phases of FRA-poly to (D_{dup}) we obtain the bound $d(D_{\mathrm{dup}}) \le 1 + \ell_{\mathrm{poly}}(\mathcal{K}^1) + \ell_{\mathrm{poly}}(\mathcal{K}^2)$.

iii. In the proof of item *ii.*, it was shown that we can obtain reducing directions for (D) from reducing directions to (D_{dup}). Suppose that $\{d_1, \ldots, d_\ell\}$ restores strong duality for (D_{dup}) in the sense of Definition 3.26. Then, it is straightforward to check that $\{d_1^1 + d_1^2, \ldots, d_\ell^1 + d_\ell^2\}$ restores strong duality for (D). This shows that $d_{\mathrm{str}}(D) \le d_{\mathrm{str}}(D_{\mathrm{dup}})$. Finally, the bound $\le \ell_{\mathrm{poly}}(\mathcal{K}^1) + \ell_{\mathrm{poly}}(\mathcal{K}^2)$, follows from applying Phase 1 of FRA-Poly to (D_{dup}).

□

We now consider the particular case where $\mathcal{K}$ is the doubly nonnegative cone $\mathcal{D}^n = \mathcal{S}^n_+ \cap \mathcal{N}^n$, where $\mathcal{N}^n$ is the cone of $n \times n$ symmetric matrices with nonnegative entries. This cone is important because it can be used as a relatively tractable relaxation for the cone of completely positive matrices, see [94, 38, 4]. The following corollary follows immediately from Theorem 3.32.

Corollary 3.33. *When $\mathcal{K} = \mathcal{D}^n$, we have $d(D) \le n$ and $d_{str}(D) \le n - 1$.*

Proof. It follows from Theorem 3.32 by recalling that $\ell_{\text{poly}}(\mathcal{S}^n_+) = n-1$ and $\ell_{\text{poly}}(\mathcal{N}^n) = 0$. □

We will compare the bound in Corollary 3.33 with the one predicted by the classical FRA. To do that, we need to compute $\ell_{\mathcal{D}^n}$.

Proposition 3.34. *The longest chain of non-empty faces in $\mathcal{D}^n$ has length $\frac{n(n+1)}{2}+1$, which is the maximum possible for a cone contained in $\mathcal{S}^n$.*

Proof. The assertion about the maximality follows from the fact that if we have two faces such that $\mathcal{F} \subsetneq \hat{\mathcal{F}}$ then we must have $\dim(\mathcal{F}) < \dim(\hat{\mathcal{F}})$. Since $\mathcal{S}^n$ has dimension $\frac{n(n+1)}{2}$ we cannot have a strictly ascending chain containing more than $\frac{n(n+1)}{2}+1$ faces.

Let $\mathcal{G}$ be any set of tuples (i,j) with $i,j \in \{1,\ldots,n\}$ and let $\mathcal{N}^n(\mathcal{G})$ be the face of $\mathcal{N}^n$ which corresponds to the matrices x such that the only entries $x_{i,j}$ that are allowed to be nonzero are the ones for which either $(i,j) \in \mathcal{G}$ or $(j,i) \in \mathcal{G}$. We will first define two chains of faces of $\mathcal{N}^n$. First, let $\mathcal{G}_0 = \emptyset$ and define $\mathcal{G}_i = \mathcal{G}_{i-1} \cup \{(i,i)\}$ for $i \in \{1,\ldots,n\}$. We now consider the following construction written in pseudocode.

$k \leftarrow 1$, $\mathcal{H}_0 \leftarrow \mathcal{G}_n$

For $i \leftarrow 1, i \leq n$ **do**

  For $j \leftarrow 1$, $j < i$ **do**
    $\mathcal{H}_k \leftarrow \mathcal{H}_{k-1} \cup \{i,j\}$
    $k \leftarrow k+1$
   $j \leftarrow j+1$.
 $i \leftarrow i+1$.

The idea is to add one non-diagonal entry per iteration, so that $\mathcal{N}^n(\mathcal{H}_k) \subsetneq \mathcal{N}^n(\mathcal{H}_{k+1})$. First $(2,1)$ will be added, then $(3,1),(3,2)$ and so on. We have

$$\mathcal{S}^n_+ \cap \mathcal{N}^n(\mathcal{G}_0) \subsetneq \ldots \subsetneq \mathcal{S}^n_+ \cap \mathcal{N}^n(\mathcal{G}_n) \subsetneq \mathcal{S}^n_+ \cap \mathcal{N}^n(\mathcal{H}_1) \subsetneq \ldots \subsetneq \mathcal{S}^n_+ \cap \mathcal{N}^n(\mathcal{H}_{\frac{n(n-1)}{2}})$$

and all inclusions are indeed strict. The first n inclusions are strict because $\mathcal{S}^n_+ \cap \mathcal{N}^n(\mathcal{G}_i) = \mathcal{N}^n(\mathcal{G}_i)$ and it is clear that $\mathcal{N}^n(\mathcal{G}_i) \subsetneq \mathcal{N}^n(\mathcal{G}_{i+1})$. Now, let $\mathcal{I}_n$ denote the $n \times n$ identity matrix. If $k > 0$ and $x \in \operatorname{ri} \mathcal{N}^n(\mathcal{H}_k)$ then $x_{i,j} > 0$ for some (i,j) entry such that neither (i,j) nor (j,i) belong to $\mathcal{H}_{k-1}$. For $\alpha > 0$ sufficiently large, we have $x + \alpha\mathcal{I}_n \in \mathcal{S}^n_+ \cap \mathcal{N}^n(\mathcal{H}_k)$ and $x + \alpha\mathcal{I}_n \notin \mathcal{S}^n_+ \cap \mathcal{N}^n(\mathcal{H}_{k-1})$. This shows the remainder of the containments and concludes the proof, since the chain has length $\frac{n(n+1)}{2}+1$. □

For feasible problems, the classical FRA analysis gives either the bound $\ell_{\mathcal{D}^n} - 1 = \frac{n(n+1)}{2}$ or, using Theorem 3.32, the bound $\ell_{\mathcal{S}^n_+} - 1 + \ell_{\mathcal{N}^n} - 1 = n + \frac{n(n+1)}{2}$. Both bounds are quadratic in n in opposition to the linear bound obtained in Corollary 3.33.

Chapter 4

Applications of Facial Reduction

In this chapter, we will discuss how to use Facial Reduction to study weak infeasibility. We will see that the reducing directions obtained by applying facial reduction to a given problem can also be used to construct points arbitrary close to the underlying cone. Once these directions are identified, there is no need to solve additional optimization problems. We will also discuss infeasibility certificates and their length.

Before we proceed, we need a theoretical result on the preservation of feasibility status. When applying Facial Reduction to (D), each d_i corresponds to a reducing direction in the sense that we can use it to confine the problem to a smaller proper face and this does not change the dual feasible region. However, if we reformulate (D) by substituting $\mathcal{K}$ by $\mathcal{F}_{i+1}$ then $\mathcal{K}^*$ gets substituted by $\mathcal{F}_{i+1}^*$ and the primal feasible region *expands*. This establishes a correspondence between Facial Reduction and the Conic Expansion Approach [55], see Section 4 of [89] for more details. One of the points of our previous discussions on SDPs and SOCPs [47, 50], was that the d_i has a few interesting properties for (P) and that feasibility properties are mostly preserved in spite of the cones getting larger. Initially, we thought that a result similar to a Schur Complement would be needed to extend our earlier results. It turns out we only need the following geometrical lemma.

Lemma 4.1. *Let e be a relative interior point of $\mathcal{K}$, $d \in \mathcal{K}$ and x a point in the tangent space of $\mathcal{K}$ at d. Then, there is a $t > 0$ such that $e + x + td \in \operatorname{ri} \mathcal{K}$.*

The intuition is as follows. If $td + x$ were a point in $\mathcal{K}$, then it would be clear that $e + td + x \in \operatorname{ri} \mathcal{K}$. Unfortunately, this does not happen in general. However, as t increases, $td + x$ gets closer and closer to $\mathcal{K}$, so adding e will eventually drag everything to the relative interior.

Proof. Let $C = \{e + x + td \mid t \geq 0\}$. To prove the assertion, it is enough to show that $\operatorname{ri} C \cap \operatorname{ri} \mathcal{K} \neq \emptyset$, that is, there is some $t > 0$ for which $e + x + td \in \operatorname{ri} \mathcal{K}$. Suppose, for the sake of obtaining a contradiction, that $\operatorname{ri} C \cap \operatorname{ri} \mathcal{K} = \emptyset$. This implies that both sets can be properly separated, which means that there is some separating hyperplane that does not contain both sets at the same time. Then, there exists z and θ such that

$$\langle e, z \rangle + \langle x, z \rangle + \langle td, z \rangle \leq \theta \leq \langle w, z \rangle$$

holds for all $t \geq 0$ and all $w \in \mathcal{K}$. For the equation above to hold, we must have $z \in \mathcal{K}^*$ and $\theta \leq 0$. Since $d \in \mathcal{K}$ and $z \in \mathcal{K}^*$ we must have $\langle d, z \rangle = 0$, since t can be taken to be any non-negative number. Because x lies in the tangent space of $\mathcal{K}$ at d, it must be contained in all supporting hyperplanes of $\mathcal{K}$ at d by item *viii.* of Lemma 2.5, therefore $\langle x, z \rangle = 0$. Because e is a relative interior point of $\mathcal{K}$ and $\theta \leq 0$ we have that $\langle e, z \rangle = 0$. This implies that $z \in \mathcal{K}^\perp$, which contradicts the fact that the separation is proper.

Therefore, $\operatorname{ri} C \cap \operatorname{ri} \mathcal{K} \neq \emptyset$ and $e + x + td \in \operatorname{ri} \mathcal{K}$ for $t > 0$ sufficiently large. $\square$

The next theorem tracks down the possible changes of feasibility status to (P) when applying facial reduction to (D). In particular, if (P) were already strongly feasible to begin with, it will stay strongly feasible. We also recall that "weak status" means either weak feasibility or weak infeasibility.

Theorem 4.2. *Suppose that $d \in \mathcal{K}^* \cap \mathcal{L}^\perp$ and $\mathcal{F} = \mathcal{K} \cap \{d\}^\perp$. Let (P') be the problem obtained by substituting $\mathcal{K}^*$ by $\mathcal{F}^* = \operatorname{cl}(\mathcal{K}^* + \mathcal{F}^\perp)$*[1] *in* (P)*. We have the following relations:*

i. (P) *is strongly feasible if and only if (P') is;*

ii. (P) *is strongly infeasible if and only if (P') is;*

iii. (P) *is in weak status if and only if (P') is.*

Proof. *i.* ($\Rightarrow$) It is enough to note that $\operatorname{ri}(\operatorname{cl}(\mathcal{K}^* + \mathcal{F}^\perp)) = \mathcal{F}^\perp + \operatorname{ri}\mathcal{K}^*$ and that $\operatorname{ri}\mathcal{K}^* \subseteq \mathcal{F}^\perp + \operatorname{ri}\mathcal{K}^*$.

($\Leftarrow$) Suppose that w is a relative interior feasible solution for (P'). This means that $\mathcal{A}w = b$ and that w can be written as $e + x$, with $e \in \operatorname{ri}\mathcal{K}^*$ and $x \in \mathcal{F}^\perp$. Recall that $\mathcal{F}^\perp = \mathcal{T}_d\mathcal{K}$ by item *viii.* of Lemma 2.5. To conclude, we use Lemma 4.1, which ensures the existence of $t > 0$ such that $x + e + td \in \operatorname{ri}\mathcal{K}^*$.

ii. ($\Rightarrow$) This part is clear, since $\mathcal{K}^* \subseteq \mathcal{F}^*$.

($\Leftarrow$). If the linear system "$\mathcal{A}x = b$" does not have a solution, then it is clear that (P') must be strongly infeasible as well. So suppose that it indeed has a solution. In this case, strong infeasibility of (P) is equivalent to the existence of y such that $-\mathcal{A}^\top y \in \mathcal{K}$ and $\langle b, y\rangle = 1$. However, $-\mathcal{A}^\top y$ is orthogonal to d, so $-\mathcal{A}^\top y \in \mathcal{F}$. By the same principle, y induces strong separation for (P') as well.

iii. Follows by elimination.

$\square$

4.1 Infeasibility certificates

In Linear Programming, a well-known tool for discussing feasibility/infeasibility is Farkas' Lemma, which states that if $\mathcal{K} = \mathbb{R}^n_+$ then either (D) is feasible or there is some $x \in \mathbb{R}^n_+$ with $\mathcal{A}x = 0$ and $\langle c, x\rangle < 0$. Due to Proposition 2.7, the existence of such a x is equivalent to the statement that (D) is strongly infeasible. Therefore, whenever (D) is infeasible, it must be strongly infeasible. This makes for some very convenient theorems of alternatives. And, in fact, entirely similar results hold when $\mathbb{R}^n_+$ is replaced by an arbitrary *polyhedral cone.*

A difficulty arises when we move on to non-polyhedral cones. In this case, we have the possibility of weak infeasibility, which complicates matters significantly. And it is not at all obvious whether (D) admits *finite* infeasibility certificates when $\mathcal{K}$ is arbitrary. And there are indeed a few descriptions in the literature of version of asymptotic Farkas' Lemma, where infeasibility is proven via sequences, see, for instance, Lemma 6 in [55] where the result is attributed to R. J. Duffin. But, in fact, one of the earliest finite infeasibility certificates for a nonpolyhedral cone was given by Ramana in [73] using his extended duality theory for SDPs. As his technique is strongly connected to facial reduction, it is no surprise that facial reduction turns out to be a helpful tool for the general case.

Currently almost all the known approaches to infeasibility certificates are connected in a way or another to facial reduction. This includes the approaches described by Sturm for SDPs in Theorem 3.5 [79], which was later generalized by Luo and Sturm to mixed

[1] An easy way to see that is to remember that $\mathcal{F} = \mathcal{K} \cap \operatorname{span}\mathcal{F}$, so that $\mathcal{F}^* = \operatorname{cl}(\mathcal{K}^* + (\operatorname{span}\mathcal{F})^\perp)$.

SDPs-SOCPs in Theorem 7.5.1 of [54]. It also includes the recent work by Liu and Pataki [44, 43]. The sole exception is the article by Klep and Schweighofer [40], where they developed infeasibility certificates for SDPs using tools from Real Algebraic Geometry.

Recall that (D) is infeasible if and only if the reducing directions $\{d_1, \ldots, d_\ell\}$ obtained through FRA or FRA-poly are such that $\langle d_\ell, c\rangle < 0$. Therefore, the reducing directions function as a finite certificate of the infeasibility of (D). The length of the certificate is then defined to be ℓ. Here, we will show that the distance to strong duality (Definition 3.26) plus one gives the length of the shortest possible infeasibility certificate arising from facial reduction.

Proposition 4.3. (D) *is infeasible if and only if there is a sequence of reducing directions* $\{d_1, \ldots, d_\ell\}$ *with* $\langle d_\ell, c\rangle < 0$ *and* $\ell = d_{str}(D') + 1$*, where* (D') *is the problem* $\sup\{0 \mid c - \mathcal{A}^\top y \in \mathcal{K}\}$.

Moreover, if (D) *is infeasible, then any sequence of reducing directions with* $\langle d_\ell, c\rangle < 0$ *must satisfy* $\ell \geq d_{str}(D') + 1$.

Proof. We first prove ($\Rightarrow$). For $i \leq \ell - 1$, the direction d_i belongs to $\mathcal{F}_i^* \cap \ker \mathcal{A} \cap \{c\}^\perp$ where $\mathcal{F}_i = \mathcal{K} \cap \{d_1\}^\perp \cap \ldots \cap \{d_{i-1}\}^\perp$. This means that $\mathcal{F}_i$ contains $(c - \operatorname{range} \mathcal{A}^\top) \cap \mathcal{K}$ for every $i \leq \ell$. The last direction d_ℓ belongs to $\mathcal{F}_\ell^* \cap \ker \mathcal{A}$, so if $s = c - \mathcal{A}^\top y$ were a feasible point, we would have $\langle s, d_\ell\rangle = \langle c, d_\ell\rangle \geq 0$. However, by hypothesis, $\langle d_\ell, c\rangle < 0$.

We now prove ($\Leftarrow$) and the rest of theorem at the same time. First note that (D) and (D') share the same facial reduction sequences. Now, let $\{d_1, \ldots, d_\ell\}$ be a facial reduction sequence for (D) such that $\langle d_\ell, c\rangle < 0$. Due to the assumption that (D) is infeasible, we have $\theta_D = -\infty$. The reducing directions define a sequence of $\ell + 1$ primal problems (P_i) with $\theta_{P_i} = \inf\{\langle c, x\rangle \mid \mathcal{A}x = 0, x \in \mathcal{F}_i^*\}$ and $\mathcal{F}_i = \mathcal{K} \cap \{d_1\}^\perp \cap \ldots \cap \{d_{i-1}\}^\perp$. Since $\langle d_\ell, c\rangle < 0$, we have $\theta_{P_\ell} = \theta_D$. Because (P_ℓ) is the problem obtained after $\ell - 1$ facial reduction steps, we have $\ell - 1 \geq d_{\text{str}}(D')$.

On the other hand, if $\{d_1, \ldots, d_\ell\}$ is a sequence of reduction directions that restores strong duality in the sense of Definition 3.26 with $\ell = d_{\text{str}}(D')$, then, using the same notation as before, we have $\theta_{P_{\ell+1}} = \inf\{\langle c, x\rangle \mid \mathcal{A}x = 0, x \in \mathcal{F}_{\ell+1}^*\} = \theta_D = -\infty$. This shows that there exists some $d_{\ell+1} \in \ker \mathcal{A} \cap \mathcal{F}_{\ell+1}^*$ with $\langle c, d_{\ell+1}\rangle < 0$. $\square$

We now discuss certificates for strong and weak infeasibility. Note that (D) is strongly infeasible if and only if it admits a certificate of length one. That is, if and only if there is $d \in \ker \mathcal{A} \cap \mathcal{K}^*$ with $\langle c, d\rangle < 0$.

For weakly infeasible problems, it is enough to recall that a problem is weakly infeasible if it is infeasible and $\operatorname{dist}(c + \operatorname{range} \mathcal{A}^\top, \mathcal{K}) = 0$. The latter is, of course, equivalent to (D) not being strongly infeasible. Therefore, to certify weak infeasibility, it is enough to give a certificate of infeasibility as in Proposition 4.3 and a certificate of not strong infeasibility, which can be obtained from item i. of Theorem 4.4. Therefore, to certify weak infeasibility we need in total $(1+d_{\text{str}}(D')) + (1+d_{\text{str}}(P'))$ vectors, which include $(1+d_{\text{str}}(D'))$ reducing directions for (D'), $d_{\text{str}}(P')$ reducing directions for (P') and an additional feasible solution to $(\hat{D})$. Note also that the problems (P') and (D') are duals of one another.

4.2 Weak infeasibility

In this section, we will take a closer look at weak infeasibility. Let $\mathcal{L} = \operatorname{range} \mathcal{A}^\top$ and let the triple $(\mathcal{K}, \mathcal{L}, c)$ denote the feasibility problem of trying to find $x \in \mathcal{K} \cap (\mathcal{L} + c)$. Sometimes we will also write $(\mathcal{K}, \mathcal{V})$ with $\mathcal{V} = \mathcal{L} + c$ to denote the same problem. Recall that $(\mathcal{K}, \mathcal{L}, c)$ is weakly infeasible if $(\mathcal{L} + c) \cap \mathcal{K} = \emptyset$ but $\operatorname{dist}(\mathcal{L} + c, \mathcal{K}) = 0$.

Many of the known characterizations of weak infeasibility involve, in a way or another, infinite sequences (see Table 1 of Luo, Sturm and Zhang [55]). As a computer cannot verify infinite sequences, it is very hard to distinguish numerically between weak infeasibility and weak feasibility, see, for instance, Pólik and Terlaky [70]. This motivates the search for ways of checking infeasibility without using sequences as in the recent work for semidefinite programming by Liu and Pataki [43], see Theorem 1 therein. See also Section 4.3 of [40] by Klep and Schweighofer. These characterizations are *finite* and no infinite sequences are needed.

We mention a few related work to weak infeasibility. The feasibility problem $(\mathcal{K}, \mathcal{L}, c)$ is weakly infeasible if and only if $c \in \mathrm{cl}\,(\mathcal{K} + \mathcal{L}) \setminus (\mathcal{K} + \mathcal{L})$. Hence, a *necessary* condition for weak infeasibility is that $\mathcal{K} + \mathcal{L}$ fails to be closed. This problem is closely related to closedness of the image of $\mathcal{K}$ by a linear map which is the problem analyzed in detail by Pataki [65]. Corollary 3.1 in [65] provides a necessary and sufficient condition for the failure of closedness of $\mathcal{K} + \mathcal{L}$, for the case when $\mathcal{K}$ is a nice cone. In that case, Pataki's result implies that $(\mathcal{K}, \mathcal{L}, c)$ is weakly infeasible if and only if $\mathcal{L}^\perp \cap (\mathrm{cl}\,\mathrm{dir}\,(x, \mathcal{K}) \setminus \mathrm{dir}\,(x, \mathcal{K})) \neq \emptyset$, where x belongs to the relative interior of $\mathcal{L} \cap \mathcal{K}$ and $\mathrm{dir}\,(x, \mathcal{K})$ is the cone of feasible directions at x. This tells us whether $\mathcal{K}$ and $\mathcal{L}$ can accommodate a weakly infeasible problem by choosing c in an appropriate manner.

Weakly infeasible problems are very hard to handle numerically, so they are interesting challenges for SDP code. In [70], Pólik and Terlaky mentioned the need for a library of infeasible problems. Later, Waki [88] showed that weakly infeasible SDPs sometimes arise from polynomial optimization problems. Bonnans and Shapiro [10] also discussed generation of weakly infeasible SDPs. As a by-product of the proof of Proposition 2.193 therein, it is shown how to construct weakly infeasible problems. More recently, Liu and Pataki [44] also discussed methods to generate weakly infeasible problems algorithmically.

In [67], Pataki introduced the notion of *well-behaved* system. $(\mathcal{K}, \mathcal{L}, c)$ is said to be well-behaved if for all $b \in \mathbb{R}^m$, the optimal value of (D) and of its dual are the same and the dual is attained whenever it is finite. A problem which is not well-behaved is said to be *badly-behaved*. Pataki showed that badly-behaved SDPs can be put into a special shape, see Theorem 6 in [67]. Then, a necessary condition for weak infeasibility is that the homogenized system $(\mathcal{S}^n_+, \widetilde{\mathcal{L}}, 0)$ be badly-behaved, where $\widetilde{\mathcal{L}}$ is spanned by $\mathcal{L}$ and c. See the comments before Section 4 in [67].

In [47], we showed that if $\mathcal{K} = \mathcal{S}^n_+$ and (D) is weakly infeasible, then there is a subaffine space $\mathcal{L}' + c'$ contained in $\mathcal{L} + c$ of dimension at most $n-1$ such that $(\mathcal{K}, \mathcal{L}', c')$ is also weakly infeasible. This can be interpreted as saying that "we need at most $n-1$ directions to approach the positive semidefinite cone". In [44], Liu and Pataki generalized this result and proved that those affine spaces always exist and $\ell_{\mathcal{K}^*} - 1$ is an upper bound for the dimension of $\mathcal{V}'$. We proved a bound of r for the direct product of r Lorentz cones [50], which is tighter than the one in [44]. Here we will refine these results.

Theorem 4.4. *i. Let (P') be the optimization problem* $\inf\{\langle c, x\rangle \mid \mathcal{A}x = 0, x \in \mathcal{K}^*\}$. *Then,* (D) *is not strongly infeasible if and only if there are:*

(a) a sequence of reducing directions $\{d_1, \ldots, d_\ell\}$ for (P') restoring strong duality in the sense of Definition 3.26 with $\ell = d_{str}(P')$;

(b) a feasible slack $\hat{s}$ to $(\hat{D})$, where $(\hat{D})$ is the problem $\sup\{0 \mid c - \mathcal{A}^\top y \in (\mathcal{K}^* \cap \{d_1\}^\perp \ldots \cap \{d_\ell\}^\perp)^*\}$.

ii. If (D) *is not strongly infeasible, there is an affine subspace $\mathcal{V}' \subseteq \mathcal{V}$ such that $(\mathcal{V}', \mathcal{K})$ is not strongly infeasible and the dimension of $\mathcal{V}'$ satisfies*

$$\dim(\mathcal{V}') \leq d_{str}(P') \leq \sum_{i=1}^{r} \ell_{poly}((\mathcal{K}^i)^*),$$

iii. If (D) *is weakly infeasible, then* $(\mathcal{V}', \mathcal{K})$ *is weakly infeasible as well, where* $\mathcal{V}'$ *is the space of item ii..*

Proof. *i.* $(\Rightarrow)$ Consider the optimization problem $\theta_{D'} = \sup\{0 \mid c - \mathcal{A}^\top y \in \mathcal{K}\}$. Its corresponding primal is $\theta_{P'} = \inf\{\langle c, x\rangle \mid \mathcal{A}x = 0, x \in \mathcal{K}^*\}$. Due to the assumption that (D) is not strongly infeasible, we have $\theta_{P'} = 0$. Now, let $\{d_1, \ldots, d_\ell\}$ be a sequence of reducing directions for (P') that restores strong duality in the sense of Definition 3.26 with $\ell = d_{\text{str}}(P')$. This time, the reducing directions define faces $\hat{\mathcal{F}}_1 \supsetneq \ldots \supsetneq \hat{\mathcal{F}}_{\ell+1}$ of $\mathcal{K}^*$ and we have $\hat{\mathcal{F}}_{\ell+1} = \mathcal{K}^* \cap \{d_1\}^\perp \cap \ldots \cap \{d_\ell\}^\perp$.

Now (P') is equivalent to $\theta_{\hat{P}} = \inf\{\langle c, x\rangle \mid \mathcal{A}x = 0, x \in \hat{\mathcal{F}}_{\ell+1}\}$ and the corresponding dual is $\theta_{\hat{D}} = \sup\{0 \mid c - \mathcal{A}^\top y \in \mathcal{F}^*_{\ell+1}\}$. Since facial reduction done at (P') preserves the primal optimal value, we have $\theta_{\hat{P}} = \theta_{P'} = 0$. Due to our assumption on the facial reduction sequence, we have $\theta_{\hat{D}} = 0$ and $\theta_{\hat{D}}$ is attained. It follows that there is $\hat{s} = c - \mathcal{A}^\top \hat{y}$ such that $\hat{s} \in \mathcal{F}^*_{\ell+1}$.

$(\Leftarrow)$ If (D) were strongly infeasible, then $\theta_{P'} = -\infty$. As $\theta_{P'} = \theta_{\hat{D}}$, it would be impossible for $(\hat{D})$ to admit a feasible solution.

ii. Let $\mathcal{V}'$ be the affine space $\hat{s} + \mathcal{L}'$, where $\mathcal{L}$ is spanned by the directions $\{d_1, \ldots, d_\ell\}$ of item *i.* and $\hat{s}$ is a feasible slack for $(\hat{D})$. Since $\ell = d_{\text{str}}(P')$, we have $\dim \mathcal{V}' = d_{\text{str}}(P')$. Suppose for the sake of contradiction that $(\mathcal{V}', \mathcal{K})$ is strongly infeasible. Then, we can use the same set $\{d_1, \ldots, d_\ell\}$ as reducing directions for $\inf\{\langle \hat{s}, x\rangle \mid x \in \mathcal{L}'^\perp, x \in \mathcal{K}^*\}$. However, item *ii.* of Theorem 4.2 implies that $\sup\{0 \mid s \in \mathcal{V}' \cap \mathcal{F}^*_{\ell+1}\}$ is strongly infeasible. But this is impossible, since $\hat{s}$ is a feasible solution.

Since the number steps required for Phase 1 of FRA-Poly gives an upper bound for $d_{\text{str}}(P')$, we obtain $d_{\text{str}}(P') \leq \sum_{i=1}^{r} \ell_{\text{poly}}((\mathcal{K}^i)^*)$.

iii. Finally, when (D) is infeasible, since $\mathcal{V}' \subseteq \mathcal{V}$ and $(\mathcal{V}', \mathcal{K})$ is not strongly infeasible, then it must be the case that $(\mathcal{V}', \mathcal{K})$ is weakly infeasible.

□

Due to Theorem 3.25, the bound in Theorem 4.4 will usually compare favorably to $\ell_{\mathcal{K}^*} - 1$. Moreover, it also recovers the bounds described in [47, 50]. Note also that the problem (P') appearing in Theorem 4.4 is such that $\theta_{P'} < 0$ if and only if (D) is strongly infeasible.

We now discuss how to use the directions in Theorem 4.4 to construct points arbitrarily close to the cone without the need of solving additional conic programs. We need a few auxiliary facts. First, if $\mathcal{K}$ is a closed convex cone, then the "distance-to-$\mathcal{K}$" function satisfies the triangle inequality, i.e., for every $a, b \in \mathcal{E}$, we have $\operatorname{dist}(a+b, \mathcal{K}) \leq \operatorname{dist}(a, \mathcal{K}) + \operatorname{dist}(b, \mathcal{K})$.

Lemma 4.5. *Let* $d \in \mathcal{K}$ *and suppose that* $s \in \mathcal{E}$ *is such that* $\operatorname{dist}(s, \operatorname{cl}(\mathcal{K} + \mathcal{T}_d\mathcal{K})) \leq \epsilon$*, then*

$$\lim_{t \to +\infty} \operatorname{dist}(s + td, \mathcal{K}) \leq \epsilon \tag{4.1}$$

Proof. Let $\delta > 0$ be arbitrary. Then, there is $x_\delta \in \mathcal{T}_d\mathcal{K}$, $z_\delta \in \mathcal{K}$ such that $\operatorname{dist}(s, x_\delta + z_\delta) \leq \epsilon + \delta$. Then

$$\begin{aligned}
\operatorname{dist}(s + td, \mathcal{K}) &\leq \operatorname{dist}(s - x_\delta - z_\delta, \mathcal{K}) + \operatorname{dist}(td + x_\delta + z_\delta, \mathcal{K}) \\
&\leq \epsilon + \delta + \operatorname{dist}(td + x_\delta + z_\delta, \mathcal{K}) \\
&\leq \epsilon + \delta + \operatorname{dist}(td + x_\delta, \mathcal{K}),
\end{aligned}$$

where the second inequality follows from the fact that $\operatorname{dist}(s - x_\delta - z_\delta, \mathcal{K}) \leq \operatorname{dist}(s - x_\delta - z_\delta, 0)$. The third inequality follows from $z_\delta \in \mathcal{K}$. Because $x_\delta \in \mathcal{T}_d\mathcal{K}$, we have $\lim_{t\to+\infty} \operatorname{dist}(x_\delta + td, \mathcal{K}) = 0$, see item *ix.* of Lemma 2.5. It follows that

$$\lim_{t\to+\infty} \operatorname{dist}(s + td, \mathcal{K}) \leq \epsilon + \delta.$$

Since δ is arbitrary, we conclude that Equation (4.1) holds. □

Suppose that (D) is weakly infeasible and that we have reducing directions $\{d_1, \ldots, d_\ell\}$ for (P') and a feasible slack $\hat{s}$ to $(\hat{D})$, as in Theorem 4.4. Suppose also that $\epsilon > 0$ is given and we wish to obtain a point in $c + \operatorname{range} \mathcal{A}^\top y$ whose distance to $\mathcal{K}$ is less or equal than ϵ.

Recall that we have $\mathcal{F}_1 = \mathcal{K}^*$ and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{d_i\}^\perp$ for every $i = 1, \ldots, \ell$. Moreover, $\mathcal{F}_{i+1}^* = \operatorname{cl}(\mathcal{F}_i^* + \mathcal{F}_{i+1}^\perp) = \operatorname{cl}(\mathcal{F}_i^* + \mathcal{T}_{d_i}\mathcal{F}_i^*)$. This is because the equality $\mathcal{F}_{i+1} = \mathcal{F}(d_i, \mathcal{F}_i^*)^\Delta$ holds and also due to item *viii.* of Lemma 2.5, where $\mathcal{F}(d_i, \mathcal{F}_i^*)$ is the minimal face of $\mathcal{F}_i^*$ which contains d_i.

Since $\hat{s}$ is a feasible slack to $(\hat{D})$, we have $\operatorname{dist}(s, \mathcal{F}_{\ell+1}^*) = \operatorname{dist}(s, \operatorname{cl}(\mathcal{F}_\ell^* + \mathcal{T}_{d_i}\mathcal{F}_\ell^*)) = 0$. By Lemma 4.5, there is $\alpha_\ell > 0$ such that $\operatorname{dist}(\hat{s} + \alpha_\ell d_\ell, \mathcal{F}_\ell^*) \leq \frac{\epsilon}{\ell}$. In a similar fashion, $\mathcal{F}_\ell^* = \operatorname{cl}(\mathcal{F}_{\ell-1}^* + \mathcal{T}_{d_i}\mathcal{F}_{\ell-1}^*)$, so we can apply Lemma 4.5 using $\mathcal{F}_{\ell-1}^*$ in place of $\mathcal{K}$ and $\hat{s} + \alpha_\ell d_\ell$ in place of s to conclude that there is $\alpha_{\ell-1} > 0$ such that

$$\operatorname{dist}(\hat{s} + \alpha_\ell d_\ell + \alpha_{\ell-1} d_{\ell-1}, \mathcal{F}_{\ell-1}^*) \leq \frac{2\epsilon}{\ell}.$$

By induction, it follows that there are positive $\alpha_\ell, \ldots, \alpha_1$ such that

$$\operatorname{dist}\left(\hat{s} + \sum_{i=1}^{\ell} \alpha_i d_i, \mathcal{F}_1^*\right) \leq \epsilon.$$

Since $\mathcal{F}_1^* = \mathcal{K}$, this shows clearly how the directions can be used to approach the cone. Note that Lemma 4.5 implies that, at each step it, is enough to pick d_i sufficiently large. So a simple strategy to compute the α_i is just to guess an initial value and keep increasing it until the distance function satisfies the required bounds.

4.3 The SDP case

In the beginning of our research we paid special attention to the case $\mathcal{K} = \mathcal{S}_+^n$. So, here we will specialize some of our results to this case, which has the advantage of giving a more concrete shape to what have been discussed so far. These results appeared originally in [47].

As before, we will use the notation $(\mathcal{S}_+^n, \mathcal{L}, c)$ to denote the following Semidefinite Feasibility Problem (SDFP).

$$\text{find } x, \text{ such that } x \in \mathcal{S}_+^n \cap (\mathcal{L} + c), \qquad \text{(SDFP)}$$

where $\mathcal{L} \subseteq \mathcal{S}^n$ is a subspace of symmetric matrices and $c \in \mathcal{S}^n$. We will take a look at how Theorem 4.4 looks like for the SDP case. First, we need the following auxiliary result.

Proposition 4.6. *If $(\mathcal{S}_+^n, \mathcal{L}, c)$ is weakly infeasible, there exists a nonzero vector in $\mathcal{S}_+^n \cap \mathcal{L}$.*

Proof. Due to weak infeasibility, there exists a sequence $\{l^k\} \subseteq \mathcal{L}$ such that

$$\lim_{k\to+\infty} \operatorname{dist}(l^k + c, \mathcal{S}_+^n) = 0.$$

Because there are no feasible solutions, the sequence $\{l^k + c\}$ can have no convergent subsequences, from which we conclude that $\lim_{k\to+\infty}\|l^k\| = +\infty$. Removing, if necessary, the l^k that are zero, we can consider the bounded sequence $\left\{\frac{l^k+c}{\|l^k\|}\right\}$. Passing to a subsequence if necessary, we may assume that it converges to some point z^*. The fact that $\mathcal{S}^n_+$ is a cone implies that $\lim_{k\to+\infty} \operatorname{dist}(\frac{l^k+c}{\|l^k\|}, K_n) = 0$, so we conclude that $z^* \in \mathcal{S}^n_+$.

Hence, $z^* = \lim_{k\to+\infty} \frac{l^k+c}{\|l^k\|} = \frac{l^k}{\|l^k\|}$ and $z^* \in L$ too. □

4.3.1 A decomposition result.

First we introduce some notation. Given $(\mathcal{S}^n_+, \mathcal{L}, c)$ and a matrix $A \in \mathcal{S}^n_+ \cap \mathcal{L}$ with rank k, we will call A a *recession direction* of rank k (a slightly abuse of the normal definition). We remark that when $(\mathcal{S}^n_+, \mathcal{L}, c)$ is feasible, A is also a recession direction of the feasible region. Moreover, A is a reducing direction for the problem $(\mathcal{S}^n_+, \mathcal{L}^\perp, 0)$.

Let $x \in \mathcal{S}^n$ and $0 \le k \le n$. We denote by $\pi_k(x)$. the upper left $k \times k$ principal submatrix of x. For instance, if

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix},$$

then,

$$\pi_2(x) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

We define the subproblem $\pi_k(\mathcal{S}^n_+, \mathcal{L}, c)$ of $(\mathcal{S}^n_+, \mathcal{L}, c)$ to be

$$\text{find } u \in \pi_k(L + c), \quad u \succeq 0.$$

In other words, it is the feasibility problem $(\pi_k(\mathcal{S}^n_+), \pi_k(\mathcal{L}), \pi_k(c))$. We denote by $\overline{\pi}_k(x)$, the lower right $(n-k)\times(n-k)$ principal submatrix. In the example above, we have $\overline{\pi}_2(x) = 6$. In a similar manner, we write $\overline{\pi}_k(\mathcal{S}^n_+, \mathcal{L}, c)$ for the feasibility problem $(\overline{\pi}_k(\mathcal{S}^n_+), \overline{\pi}_k(\mathcal{L}), \overline{\pi}_k(c))$. We remark that $\pi_n(x) = \overline{\pi}_0(x) = x$ and we define $\pi_0(x) = \overline{\pi}_n(x) = 0$.

The proposition belows summarizes the properties of the Schur Complement. For proofs, see Theorem 7.7.6 of [32].

Proposition 4.7 (Schur Complement)**.** *Suppose $M = \left(\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right)$ is a symmetric matrix divided in blocks in a way that A is positive definite, then:*

- *M is positive definite if and only if $C - B^\top A^{-1}B$ is.*
- *M is positive semidefinite if and only if $C - B^\top A^{-1}B$ is.*

We now discuss a version of Theorem 4.2 for SDPs.

Theorem 4.8. *Let $(\mathcal{S}^n_+, \mathcal{L}, c)$ be a SDFP, and consider a subproblem $\pi_k(\mathcal{S}^n_+, \mathcal{L}, c)$ for some $k > 0$. If the subproblem $\pi_k(\mathcal{S}^n_+, \mathcal{L}, c)$ admits an interior recession direction (i.e, $\operatorname{int} \pi_k(\mathcal{S}^n_+) \cap \pi_k(L) \neq \emptyset$) then:*

1. *$(\mathcal{S}^n_+, \mathcal{L}, c)$ is strongly feasible if and only if $\overline{\pi}_k(\mathcal{S}^n_+, \mathcal{L}, c)$ is.*
2. *$(\mathcal{S}^n_+, \mathcal{L}, c)$ is strongly infeasible if and only if $\overline{\pi}_k(\mathcal{S}^n_+, \mathcal{L}, c)$ is.*
3. *$(\mathcal{S}^n_+, \mathcal{L}, c)$ is in weak status if and only if $\overline{\pi}_k(\mathcal{S}^n_+, \mathcal{L}, c)$ is.*

Proof. Due to the assumption, there exists a $n \times n$ matrix

$$x = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where A is a $k \times k$ positive definite matrix.

We now prove items 1 and 2. Item 3 will follow by elimination.

(1) $\Rightarrow$) If $y \in L + c$ is positive definite, all its principal submatrices are also positive definite. Therefore, $\overline{\pi}_k(y)$ is positive definite.

(1) $\Leftarrow$) Suppose that $y \in \mathcal{L} + c$ is such that $\overline{\pi}_k(y) \in \operatorname{int} \mathcal{S}^{n-k}_+$. Then, we may write $y = \left(\begin{smallmatrix} F & E \\ E^\top & G \end{smallmatrix}\right)$, where G is $(n-k) \times (n-k)$ and positive definite. For large and positive α, $F + \alpha A$ is positive definite and the Schur complement of $y + x\alpha$ is $G - E^\top (F + \alpha A)^{-1} E$. Since G is positive definite, it is clear that, increasing α if necessary, the Schur complement is also positive definite. For such an α, $y + x\alpha \in (L + c) \cap \operatorname{int} \mathcal{S}^n_+$.

(2) $\Rightarrow$). Suppose $(\mathcal{S}^n_+, \mathcal{L}, c)$ strongly infeasible. Then there exists $s \in \mathcal{S}^n_+$ such that $s \in L^\perp$ and $\langle s, c \rangle = -1$. As $x \in \mathcal{L}$, we have $s \in \mathcal{S}^n_+ \cap \{x\}^\perp$. This means that s can be written as $\left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right)$, where D belongs to $\mathcal{S}^{n-k}_+$. It follows that $\overline{\pi}_k(s) \in \overline{\pi}_k(\mathcal{L})^\perp$ and $\langle \overline{\pi}_k(s), \overline{\pi}_k(c) \rangle = -1$. By item *ii.* of Proposition 2.7, $\overline{\pi}_k(\mathcal{S}^n_+, \mathcal{L}, c)$ is strongly infeasible.

(2) $\Leftarrow$). Now, suppose $\overline{\pi}_k(\mathcal{S}^n_+, \mathcal{L}, c)$ is strongly infeasible. Note that $\overline{\pi}_k$ is a non-expansive map, i.e, $\|\overline{\pi}_k(y) - \overline{\pi}_k(z)\| \leq \|y - z\|$ holds. In particular, if

$$\inf_{y \in L + c, z \in \mathcal{S}^n_+} \|\overline{\pi}_k(y) - \overline{\pi}_k(z)\| > 0,$$

then the same is true for $\inf_{y \in \mathcal{L} + c, z \in \mathcal{S}^n_+} \|y - z\|$. $\square$

4.3.2 Forward Procedure

Assume that $(\mathcal{S}^n_+, \mathcal{L}, c)$ admits a recession direction $\widetilde{A}_1$ of rank k_1. Theorem 4.8 might not be directly applicable but after appropriate congruence transformation by a nonsingular matrix P_1, we have that $(\mathcal{S}^n_+, P_1^\top \mathcal{L} P_1, P_1^\top c P_1)$ admits a recession direction of the form

$$A_1 = \begin{pmatrix} \widehat{A}_1 & 0 \\ 0 & 0 \end{pmatrix} = P_1^\top \widetilde{A}_1 P_1,$$

where $\widehat{A}_1$ is a $k_1 \times k_1$ positive definite matrix. The feasibility status of $(\mathcal{S}^n_+, \mathcal{L}, c)$ and

$$(\mathcal{S}^{n-k_1}_+, \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1), \overline{\pi}_{k_1}(P_1^\top c P_1))$$

are mostly the same in the sense that items $1-3$ of Theorem 4.8 hold.

Now, suppose that $(\mathcal{S}^{n-k_1}_+, \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1), \overline{\pi}_{k_1}(P_1^\top c P_1))$ admits a recession direction $\widetilde{A}_2$ of rank k_2. Then, after appropriate congruence transformation by $\widetilde{P}_2$, we obtain that

$$(\mathcal{S}^{n-k_1}_+, \widetilde{P}_2^\top \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1) \widetilde{P}_2, \widetilde{P}_2^\top \overline{\pi}_{k_1}(P_1^\top c P_1) \widetilde{P}_2)$$

admits a recession direction of the form

$$\begin{pmatrix} \widehat{A}_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\widehat{A}_2$ is $k_2 \times k_2$ positive definite matrix.

Now, the feasibility status of $(\mathcal{S}^{n-k_1}_+, \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1), \overline{\pi}_{k_1}(P_1^\top c P_1))$ and

$$(\mathcal{S}^{n-k_1-k_2}_+, \overline{\pi}_{k_2}(\widetilde{P}_2^\top \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1) \widetilde{P}_2), \overline{\pi}_{k_2}(\widetilde{P}_2^\top \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1) \widetilde{P}_2))$$

are mostly the same. Note that instead of applying a congruence transformation by $\widetilde{P}_2$ to $(\mathcal{S}^{n-k_1}_+, \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1), \overline{\pi}_{k_1}(P_1^\top c P_1))$, we can apply a congruence transformation by

$$P_2 = \begin{pmatrix} I_{k_1} & 0 \\ 0 & \widetilde{P}_2 \end{pmatrix}$$

to the original problem $(\mathcal{S}^n_+, P_1^\top \mathcal{L} P_1, P_1^\top c P_1)$, i.e., we consider

$$\left(\mathcal{S}^n_+, P_2^\top P_1^\top \mathcal{L} P_1 P_2, P_2^\top P_1^\top c P_1 P_2\right)$$

Then the subproblem defined by the $(n-k_1)\times(n-k_1)$ lower right block matrix is precisely

$$(\mathcal{S}^{n-k_1}_+, \widetilde{P}_2^\top \overline{\pi}_{k_1}(P_1^\top \mathcal{L} P_1)\widetilde{P}_2, \widetilde{P}_2^\top \overline{\pi}_{k_1}(P_1^\top c P_1)\widetilde{P}_2),$$

and we may pick $A_2 \in P_2^\top P_1^\top \mathcal{L} P_1 P_2$ such that

$$\overline{\pi}_{k_1+k_2}(A_2) = \begin{pmatrix} \widehat{A}_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that A_2 has the following shape

$$A_2 = \begin{pmatrix} * & * & * \\ * & \widehat{A}_2 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Generalizing the process outlined above, we obtain the following procedure, which we call "forward procedure". Note that the congruence matrix at each step can be taken to be orthogonal. The set of matrices $\{A_1, \ldots, A_m\}$ obtained in this way will be called a *set of reducing directions*. We note that $\{A_1, \ldots, A_m\}$ is exactly the same as the set of matrices obtained when we apply FRA to the problem (P') in Theorem 4.4. The only caveat is that we "rotate" the problem to put it into a convenient shape before finding the next direction.

After each application of Theorem 4.8, the size of the matrices is reduced at least by one. This means that after at most n iterations, a subproblem with no nonzero reducing directions is found. At this point, no further directions can be added and we will say that the set of directions is *maximal*.

[Forward Procedure (FP)]

Input: $(\mathcal{S}^n_+, \mathcal{L}, c)$

Output: an orthogonal P, a sequence $k_1, \ldots, k_m$ and a maximal set of reducing directions $\{A_1, \ldots, A_m\}$ contained in $P^\top \mathcal{L} P$. The A_i are such that $A_1 = \begin{pmatrix} \widehat{A}_1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} * & * & * \\ * & \widehat{A}_2 & 0 \\ * & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \widehat{A}_3 & 0 \\ * & * & 0 & 0 \end{pmatrix}$ and so forth, where $\widehat{A}_i$ is positive definite and lies in $\mathcal{S}^{k_i}_+$, for every i.

1. Set $i := 1$, $\widetilde{\mathcal{L}} := \mathcal{L}$, $\widetilde{c} := c$ $K := \mathcal{S}^n_+$, $P := I_n$.

2. Find (i) $\widetilde{A}_i \in \widetilde{\mathcal{L}} \cap K$, $\mathrm{tr}(\widetilde{A}_i) = 1$ or (ii) $\widetilde{B} \in \widetilde{\mathcal{L}}^\perp \cap \mathrm{int} K$, $\mathrm{tr}(\widetilde{B}) = 1$. (Exactly one of (i) and (ii) is solvable.) If (ii) is solvable, then stop. (No nonzero reducing direction exists.)

3. Compute an orthogonal $\widetilde{P}$ such that,

$$\widetilde{P}^\top \widetilde{A}_i \widetilde{P} = \begin{pmatrix} \widehat{A}_i & 0 \\ 0 & 0 \end{pmatrix}$$

   where $\hat{A}_i$ is a positive definite matrix. Let $k_i := \mathrm{rank}(\widetilde{A}_i)$.

4. Compute $M = \begin{pmatrix} I_{k_1+\ldots+k_{i-1}} & 0 \\ 0 & \widetilde{P} \end{pmatrix}$ and set $P^\top := M^\top P^\top$. (If $i = 1$, take $M = \widetilde{P}$)

5. Let A_i be any matrix in $P^\top \mathcal{L} P$ such that $\overline{\pi}_{k_1+\ldots+k_{i-1}}(A_i) = \widetilde{P}^\top \widetilde{A}_i \widetilde{P}$. For each $1 \le j < i$ exchange A_j for $M^\top A_j M$.

6. Set $\widetilde{\mathcal{L}} := \pi_{k_i}(\widetilde{P}^\top \widetilde{\mathcal{L}} \widetilde{P}), \widetilde{c} := \pi_{k_i}(\widetilde{P}^\top \widetilde{c} \widetilde{P})$, $K := \pi_{k_i}(\mathcal{S}^n_+)$, $i := i+1$ and return to Step 2. (This step is just to pick the lower-right block after the congruence transformation.)

We now show that if **FP** is applied to $(\mathcal{S}^n_+, \mathcal{L}, c)$ then the final problem obtained can never be weakly infeasible.

Proposition 4.9. *Suppose that $(\mathcal{S}^n_+, \mathcal{L}, c)$ is such that there is a nonzero element in $\mathcal{S}^n_+ \cap \mathcal{L}$. Applying **FP** to $(\mathcal{S}^n_+, \mathcal{L}, c)$ we have that:*

1. *$(\mathcal{S}^n_+, \mathcal{L}, c)$ is strongly feasible if and only if $\overline{\pi}_{k_1+\ldots+k_m}(\mathcal{S}^n_+, P^\top \mathcal{L} P, P^\top c P)$ is.*
2. *$(\mathcal{S}^n_+, \mathcal{L}, c)$ is strongly infeasible if and only if $\overline{\pi}_{k_1+\ldots+k_m}(\mathcal{S}^n_+, P^\top \mathcal{L} P, P^\top c P)$ is.*
3. *$(\mathcal{S}^n_+, \mathcal{L}, c)$ is in weak status if and only if $\overline{\pi}_{k_1+\ldots+k_m}(\mathcal{S}^n_+, P^\top \mathcal{L} P, P^\top c P)$ is weakly feasible.*

Proof. If $m = 0$, then the proposition follows because $\overline{\pi}_0$ is equal to the identity map. In the case $m = 1$, the result follows from Theorem 4.8.

Note that at the i-th iteration, if a direction A_i is found then, after applying the congruence transformation $\widetilde{P}$, $\pi_{k_i}(K, \widetilde{P}^\top \widetilde{\mathcal{L}} \widetilde{P}, \widetilde{P}^\top \widetilde{c} \widetilde{P})$ preserves feasibility properties in the sense of Theorem 1. Note that it is a SDFP over $\mathcal{S}^{n-k_1-\ldots-k_i}$. Also, due to the way M is selected, we have that equation $\pi_{k_i}(K, \widetilde{P}^\top \widetilde{\mathcal{L}} \widetilde{P}, \widetilde{P}^\top \widetilde{c} \widetilde{P}) = \overline{\pi}_{k_1+\ldots+k_i}(\mathcal{S}^n_+, P^\top \mathcal{L} P, P^\top c P)$ holds after Line 4 and before $\widetilde{\mathcal{L}}$ and K are updated. This justifies items 1. and 2..

Consider the case where $(\mathcal{S}^n_+, \mathcal{L}, c)$ is in weak status. Due to Proposition 4.6, whenever $(\mathcal{S}^n_+, \widetilde{\mathcal{L}}, \widetilde{c})$ is weakly infeasible we can always find a new direction A_i and the size of problem decreases by a positive amount, so that $(\mathcal{S}^n_+, \widetilde{\mathcal{L}}, \widetilde{c})$ cannot be weakly infeasible for all iterations. The only other possibility is weak feasibility, which justifies item 3. $\square$

The matrices $A_1, \ldots, A_m$ obtained through **FP** have the shape

$$\begin{pmatrix} \widehat{A}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * \\ * & \widehat{A}_2 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \widehat{A}_3 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \ldots$$

where $\widehat{A}_1, \widehat{A}_2, \widehat{A}_3, \ldots$ are positive definite. The matrix A_i are referred to as *reducing directions*, since the $\widehat{A}_i$ are reducing directions. The problem $\overline{\pi}_{k_1+\ldots+k_m}(\mathcal{S}^n_+, P^\top \mathcal{L} P, P^\top c P)$ will be referred to as the *last subproblem* of $(\mathcal{S}^n_+, \mathcal{L}, c)$.

We obtain the following alternative characterization of weak infeasibility based on **FP**.

Proposition 4.10. *$(K, \mathcal{L}, c)$ is weakly infeasible if and only if it is in weak status and is infeasible. Therefore, weak infeasibility is detected by executing **FP** for checking weak status and FRA for checking infeasibility.*

Example 4.11. *Let*

$$\mathcal{L} + c = \left\{ \begin{pmatrix} t & v & 1 & u \\ v & z+2 & v+1 & z+1 \\ 1 & v+1 & u-1 & s \\ u & z+1 & s & 0 \end{pmatrix} \mid t, u, v, s, z \in \mathbb{R} \right\}. \tag{4.2}$$

and let us apply **FP** *to* $(\mathcal{S}^4_+, \mathcal{L}, c)$*. The first direction can be, for instance,* $A_1 = \left(\begin{smallmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{smallmatrix}\right)$*. Then* $k_1 = 1$ *and* $\widetilde{P}$ *is the identity, at this step. At next iteration, we have* $K = \mathcal{S}^3_+$ *and* $\widetilde{\mathcal{L}} = \left\{ \left(\begin{smallmatrix} z&v&z\\v&u&s\\z&s&0 \end{smallmatrix}\right) \mid u, s, v, z \in \mathbb{R} \right\}$*. Then,* $\widetilde{A}_2$ *can be taken as* $\left(\begin{smallmatrix} 0&0&0\\0&1&0\\0&0&0 \end{smallmatrix}\right)$ *and* k_2 *is* 1*. A possible choice of* $\widetilde{P}$ *is* $\left(\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right)$*. Then* P *is* $\left(\begin{smallmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{smallmatrix}\right)$ *and we can take* $A_2 = \left(\begin{smallmatrix} 0&0&0&1\\0&1&0&0\\0&0&0&0\\1&0&0&0 \end{smallmatrix}\right)$*.* $\widetilde{\mathcal{L}}$ *is then updated and it becomes* $\{\left(\begin{smallmatrix} z&z\\z&0 \end{smallmatrix}\right) \mid z \in \mathbb{R}\}$*. The procedure stops here, because* 0 *is the only positive semidefinite matrix in* $\widetilde{\mathcal{L}}$*.*

Now, $\overline{\pi}_2(P^\top(\mathcal{L} + c)P)$ *is* $\left\{\left(\begin{smallmatrix} z+2&z+1\\z+1&0 \end{smallmatrix}\right) \mid z \in \mathbb{R}\right\}$*, so* $\overline{\pi}_2(\mathcal{S}^4_+, P^\top\mathcal{L}P, P^\top cP)$ *is a weakly feasible system. Therefore, by Proposition 4.9,* $(\mathcal{S}^4_+, \mathcal{L}, c)$ *has weak status and is either weakly infeasible or weakly feasible. The* 0 *in the lower right corner of* (4.2) *forces* $u = 0$*,* $z = -1$ *and* $s = 0$*, but this assignment produces a negative element in the diagonal. This tells us that* $(\mathcal{S}^4_+, \mathcal{L}, c)$ *is infeasible so it must be weakly infeasible.*

4.3.3 Number of directions required to approach the positive semidefinite cone

Given a weakly infeasible $(\mathcal{S}^n_+, \mathcal{L}, c)$ a natural question is whether it is always possible to select a point in $x \in \mathcal{L} + c$ and then a nonzero direction $d \in \mathcal{S}^n_+ \cap \mathcal{L}$ such that $\lim_{t\to+\infty} \operatorname{dist}(x + td, \mathcal{S}^n_+) = 0$ or not. We call weakly infeasible problems having this property *directionally weakly infeasible* (DWI). This means that we can approach the cone by walking along a single direction. The simplest instance of DWI problem is

$$\max\ 0 \text{ s.t. } \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{S}^2_+,\ t \in \mathbb{R}.$$

Unfortunately, not all weakly infeasible problems are DWI, as shown in the following instance.

Example 4.12 (A weakly infeasible problem that is not directionally weakly infeasible)**.** *Let* $(\mathcal{S}^3_+, \mathcal{L}, c)$ *be such that* $\mathcal{L} + c = \left\{ \left(\begin{smallmatrix} t&1&s\\1&s&1\\s&1&0 \end{smallmatrix}\right) \mid t, s \in \mathbb{R} \right\}$ *and let* $A_1 = \left(\begin{smallmatrix} 1&0&0\\0&0&0\\0&0&0 \end{smallmatrix}\right)$*.*

Applying Theorem 1 twice, we see that the problem is in weak status. Looking at its 2×2 *lower right block, we see this problem is infeasible and hence is weakly infeasible. But this problem is not DWI. If* $(\mathcal{S}^3_+, \mathcal{L}, c)$ *were DWI, we would have* $\lim_{t\to+\infty} \operatorname{dist}(tA_1 + c', \mathcal{S}^3_+) = 0$*, for some* $c' \in L + c$*. To show this does not hold, we fix* s*. Regardless of the value of* $t \geq 0$*, the minimum eigenvalue of the matrix is uniformly negative, since its* 2×2 *lower right block is strongly infeasible.*

In the following, we show that $n - 1$ directions are enough to approach the positive semidefinite cone. First we discuss how the set of reducing directions $\{A_1, \ldots, A_m\}$ of **FP** fits in the concept of tangent cone. We recall that for $x \in \mathcal{S}^n_+$ the cone of feasible directions is the set $\operatorname{dir}(x, \mathcal{S}^n_+) = \{d \in \mathcal{S}^n \mid \exists t > 0 \text{ s.t } x + td \in \mathcal{S}^n_+\}$. Then the tangent cone at x is the closure of $\operatorname{dir}(x, \mathcal{S}^n_+)$ and is denoted by $\operatorname{cl}\operatorname{dir}(x, \mathcal{S}^n_+)$. Recall that by item *ix.* of Lemma 2.5, we have that $d \in \operatorname{cl}\operatorname{dir}(x, \mathcal{S}^n_+)$ implies $\lim_{t\to+\infty} \operatorname{dist}(tx + d, \mathcal{S}^n_+) = 0$.

We remark that if $x = \left(\begin{smallmatrix} D&0\\0&0 \end{smallmatrix}\right)$, where D is positive definite $k\times k$ matrix, then $\operatorname{cl}\operatorname{dir}(x, \mathcal{S}^n_+)$ consists of all symmetric matrices $\left(\begin{smallmatrix} *&*\\ *&E \end{smallmatrix}\right)$, where $*$ denotes arbitrary entries and E is a positive semidefinite $(n - k) \times (n - k)$ matrix. See [63] for more details.

The output $\{A_1, \ldots, A_m\}$ of **FP** is such that $A_2 \in \operatorname{cl}\operatorname{dir}(A_1, \mathcal{S}^n_+)$. This is clear from the shape of A_1 and A_2, and from a simple argument using the Schur Complement. Now, A_3 is such that $\overline{\pi}_{k_1+k_2}(A_3)$ is positive semidefinite. We have $A_2 = \left(\begin{smallmatrix} *&*&*&*\\ *&\widehat{A}_2&0&0\\ *&0&0&0\\ *&0&0&0 \end{smallmatrix}\right)$ $A_3 = \left(\begin{smallmatrix} *&*&*&*\\ *&*&*&*\\ *&*&\widehat{A}_3&0\\ *&*&0&0 \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} *&*&*\\ *&\widehat{A}_3&0\\ *&0&0 \end{smallmatrix}\right) \in \operatorname{cl}\operatorname{dir}\left(\left(\begin{smallmatrix} \widehat{A}_2&0&0\\0&0&0\\0&0&0 \end{smallmatrix}\right), \mathcal{S}^{n-k_1}_+\right)$, i.e, $\overline{\pi}_{k_1}(A_3) \in \operatorname{cl}\operatorname{dir}\left(\overline{\pi}_{k_1}(A_2), \mathcal{S}^{n-k_1}_+\right)$. Denote

$k_1 + \ldots + k_i$ by N_i and set $N_0 = 0$. Then, for $i > 2$, we have:

$$\overline{\pi}_{N_{i-2}}(A_i) \in \operatorname{cl}\operatorname{dir}\left(\overline{\pi}_{N_{i-2}}(A_{i-1}), \mathcal{S}_+^{n-N_{i-2}}\right).$$

Moreover, if the last subproblem $\overline{\pi}_{N_m}(\mathcal{S}_+^n, \mathcal{L}, c)$ has a feasible solution, we can pick some c' such that $\overline{\pi}_{N_m}(c')$ is positive semidefinite. Then $\overline{\pi}_{N_{m-1}}(c') \in \operatorname{cl}\operatorname{dir}\left(\overline{\pi}_{N_{m-1}}(A_m), \mathcal{S}_+^{n-N_{m-1}}\right)$. Given $\epsilon > 0$, by picking $\alpha_m > 0$ sufficiently large we have $\operatorname{dist}(\overline{\pi}_{N_{m-1}}(c'+\alpha_m A_m), \mathcal{S}_+^{n-N_{m-1}}) < \epsilon$. Now, $\overline{\pi}_{N_{m-2}}(x + \alpha_m A_m)$ does not necessarily lie on the tangent cone of $\overline{\pi}_{N_{m-2}}(A_{m-1})$ at $\mathcal{S}_+^{n-N_{m-2}}$, but still it is possible to pick $\alpha_{m-1} > 0$ such that

$$\operatorname{dist}(\overline{\pi}_{N_{m-2}}(c' + \alpha_m A_m + \alpha_{m-1}A_{m-1}), \mathcal{S}_+^{n-N_{m-2}}) < 2\epsilon.$$

In order to show this, let $h \in \mathcal{S}_+^{n-N_{m-1}}$ be such that

$$\|\overline{\pi}_{N_{m-1}}(c' + \alpha_m A_m) - h\| = \operatorname{dist}(\overline{\pi}_{N_{m-1}}(c' + \alpha_m A_m), \mathcal{S}_+^{n-N_{m-1}}).$$

Now, define $\widetilde{h}$ to be the matrix $\overline{\pi}_{N_{m-2}}(c' + \alpha_m A_m)$, except that the lower right $(n - k_m) \times (n - k_m)$ block is replaced by h. It follows readily that $\widetilde{h}$ lies on the tangent cone of $\overline{\pi}_{N_{m-2}}(A_{m-1})$. Then, we may pick $\alpha_{m-1} > 0$ sufficiently large such that $\operatorname{dist}(\overline{\pi}_{N_{m-2}}(\alpha_m A_m) + h, \mathcal{S}_+^{n-N_{m-2}}) < \epsilon$. Let $y_1 = \overline{\pi}_{N_{m-2}}(c' + \alpha_m A_m)$, $y_2 = \overline{\pi}_{N_{m-2}}(\alpha_{m-1}A_{m-1})$. We then have the following implications:

$$\begin{aligned}\operatorname{dist}(y_1 + y_2, \mathcal{S}_+^{n-N_{m-2}}) &\leq \operatorname{dist}(y_1 - \widetilde{h}, \mathcal{S}_+^{n-N_{m-2}}) + \operatorname{dist}(y_2 + \widetilde{h}, \mathcal{S}_+^{n-N_{m-2}}) \\ &\leq \|\overline{\pi}_{N_{m-1}}(c' + \alpha_m A_m) - h\| + \epsilon \leq 2\epsilon.\end{aligned}$$

If we continue in this way, it becomes clear that $\alpha_1, \ldots, \alpha_m$ can be selected such that $\operatorname{dist}(c' + \alpha_m A_m + \alpha_{m-1}A_{m-1} + \ldots + \alpha_1 A_1, \mathcal{S}_+^n) < m\epsilon$. This shows how the directions $\{A_1, \ldots, A_m\}$ can be used to construct points that are arbitrarily close to $\mathcal{S}_+^n$, when the last subproblem is feasible. This leads to the next theorem, which is a special case of Theorem 4.4.

Theorem 4.13. *If $(\mathcal{S}_+^n, \mathcal{L}, c)$ is weakly infeasible then there exists an affine space of dimension at most $n - 1$ such that $\mathcal{L}' + c' \subseteq \mathcal{L} + c$ and $(\mathcal{S}_+^n, \mathcal{L}', c')$ is weakly infeasible.*

Proof. The construction above shows that if $\mathcal{L}'$ is the space spanned by $\{A_1, \ldots, A_m\}$ and c' is taken as above, then $(\mathcal{S}_+^n, \mathcal{L}', c')$ is weakly infeasible. As $(\mathcal{S}_+^n, \mathcal{L}, c)$ is weakly infeasible, we have $m > 0$. We also have $k_1 + \ldots + k_m \leq n$, which implies $m \leq n$. Notice that $\overline{\pi}_n(\mathcal{S}_+^n, P^\top \mathcal{L} P, P^\top c P)$ is strongly feasible, because it is equal to the system $(\{0\}, \{0\}, 0)$. Therefore, $k_1 + \ldots + k_m < n$, which forces $m < n$. □

Note that the discussion above is entirely analogous to what as done in Section 4.2. One important difference is that in the SDP case, if $\mathcal{F}$ is a face of $\mathcal{S}_+^n$, we have $\mathcal{F}^* = \mathcal{S}_+^n + \mathcal{F}^\perp$, so Lemma 4.5 is not needed.

4.4 The SOCP case

When $\mathcal{K}$ is a direct product of several Lorentz cones, we can also be very concrete about the analysis given so far. In this section, we present the analysis we did for SOCPs [50] based on the idea of "relaxation sequences".

4.4.1 Relaxation of SOCFPs

In this subsection, we show how Second Order Cone Feasibility Problem (SOCFP) can be relaxed in a way that the feasibility properties are mostly preserved. Consider a feasibility problem of the form $(\mathcal{K}, \mathcal{L}, c)$, where $\mathcal{K}$ is a direct product $K^{n_1} \times \ldots \times K^{n_m}$, where each K^{n_i} is: the trivial cone $\{0\}$; $\mathbb{R}^{n_i}$; a Lorentz cone $\mathcal{Q}^{n_i}$; a closed half-space defined by a supporting hyperplane to $\mathcal{Q}^{n_i}$, i.e., $H_d^{n_i}$ for $d \in \mathcal{Q}^{n_i} \setminus \{0\}$; or a half-line contained in $\mathcal{Q}^{n_i}$, i.e., $\mathrm{h}_d^{n_i}$ for $d \in \mathcal{Q}^{n_i}$.

Note that the family of cones having the format above is no more expressive than the family of products of second order cones. Still, for our purposes we need to consider this slightly more general situation because these cones will appear as byproducts of Theorem 4.15. We will call them *extended second order cones.* We remark that the dual cone $\mathcal{K}^*$ is the direct product of the duals of the cones K^{n_i} and it is also an extended second order cone. It is also clear that we have $(H_d^{n_i})^* = \mathrm{h}_d^{n_i}$.

Suppose that we have a non-zero element $a \in \mathcal{K}$, then we define: *i)* $\mathcal{H}_1(a, \mathcal{K}) = \{i \mid K^{n_i} = \mathcal{Q}^{n_i}, a_{n_i} \in \mathrm{ri}\, \mathcal{Q}^{n_i}\}$ and *ii)* $\mathcal{H}_2(a, \mathcal{K}) = \{i \mid K^{n_i} = \mathcal{Q}^{n_i}, a_{n_i} \in (\mathrm{relbd}\, \mathcal{Q}^{n_i}) \setminus \{0\}\}$. We will omit $\mathcal{K}$ when it is clear from the context. The lemma below is an easier version of Lemma 4.1.

Lemma 4.14. *Let $x \in \mathbb{R}^n$ and $a \in \mathcal{Q}^n$ be such that $x^\top a' > 0$. Then $x + ta \in \mathrm{ri}\, \mathcal{Q}^n$ for $t > 0$ sufficiently large.*

Proof. The point a must be non-zero and if it is an interior point, then the statement clearly holds. If a lies in the boundary, then

$$(x + ta)_0^2 - \|\overline{x + ta}\|^2 = 2t(a_0 x_0 - \overline{a}^T \overline{x}) + x_0^2 - \|\overline{x}\|^2.$$

However, $a_0 x_0 - \overline{a}^T \overline{x}$ is equal to $x^\top a'$. So if t is large enough we have that $(x + ta)_0^2 - \|\overline{x + ta}\|^2$ will be greater than 0. $\square$

Theorem 4.15. *Let $(\mathcal{K}, \mathcal{L}, c)$ be a feasibility problem such that $\mathcal{K} = K^{n_1} \times \ldots \times K^{n_m}$. Suppose that there is $a \in \mathcal{K} \cap L$ such that $\mathcal{H}_1(a) \cup \mathcal{H}_2(a)$ is non-empty. Define the cone $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}^{n_1} \times \ldots \times \widetilde{\mathcal{K}}^{n_m}$ such that for every i:*

- $\widetilde{\mathcal{K}}^{n_i} = \mathbb{R}^{n_i}$ *if* $i \in \mathcal{H}_1(a)$,
- $\widetilde{\mathcal{K}}^{n_i} = H_d^{n_i}$ *where* $d = a'_{n_i}$, *if* $i \in \mathcal{H}_2(a)$,
- $\widetilde{\mathcal{K}}^{n_i} = K^{n_i}$, *otherwise.*

Then

i. $(\mathcal{K}, \mathcal{L}, c)$ *is strongly feasible if and only if* $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ *is strongly feasible;*

ii. $(\mathcal{K}, \mathcal{L}, c)$ *is in weak status if and only if* $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ *is in weak status;*

iii. $(\mathcal{K}, \mathcal{L}, c)$ *is strongly infeasible if and only if* $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ *is strongly infeasible.*

Proof. (i) If $(\mathcal{K}, \mathcal{L}, c)$ is strongly feasible, then for a relative interior point $y \in \mathcal{L} + c$, we have $y_{n_i}^T a'_{n_i} > 0$, for all $i \in \mathcal{H}_2(a)$. All the other coordinate blocks of y_{n_i} stay in the relative interior of the respective cones. So, $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ is strongly feasible.

Now, if $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ is strongly feasible we pick $y \in \mathcal{L} + c$ such that y lies in the relative interior of $\widetilde{\mathcal{K}}$. For $i \in \mathcal{H}_1(a)$ we have $a_{n_i} \in \mathrm{ri}\, \mathcal{Q}^{n_i}$ and for $i \in \mathcal{H}_2(a)$ we have $y_{n_i}^T a'_{n_i} > 0$. Hence, if t is sufficiently large we have $(y + ta)_{n_i}^T a'_{n_i} \in \mathrm{int}(\mathcal{Q}^{n_i})$, for all $i \in \mathcal{H}_1(a) \cup \mathcal{H}_2(a)$, by Lemma 4.14. It is also clear that adding ta does not affect the fact that $y_{n_i} \in \mathrm{ri}(\mathcal{K}^{n_i})$ for $i \notin \mathcal{H}_1(a) \cup \mathcal{H}_2(a)$.

(*iii*) If $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ is strongly infeasible then $(\mathcal{K}, \mathcal{L}, c)$ also is because $\mathcal{K} \subseteq \widetilde{\mathcal{K}}$. Let us prove the converse now. We have that $(\mathcal{K}, \mathcal{L}, c)$ is strongly infeasible if and only if there exists s such that $s \in L^\perp \cap \mathcal{K}^*$ and $s^T c < 0$ (see Lemma 5 of [55]). In particular, $s^\top a = 0$. This means that $s_{n_i}^\top a_{n_i} = 0$ for every i, because $s \in \mathcal{K}^*$ and $a \in \mathcal{K} \cap L$. It follows that for $i \in \mathcal{H}_1(a)$ we have $s_{n_i} = 0$. Also, for $i \in \mathcal{H}_2(a)$ we have that s_{n_i} is a non-negative multiple of a'_{n_i} (including, of course, the possibility that s_{n_i} is 0)[2]. We conclude that s also produces strong separation for $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ because $s \in \widetilde{\mathcal{K}}^*$. So $(\widetilde{\mathcal{K}}, \mathcal{L}, c)$ is strongly infeasible.

Finally, (*ii*) follows by elimination.

$\square$

After applying Theorem 4.15 to $(\mathcal{K}, \mathcal{L}, c)$, it might still be possible to relax it further. This motivates the next definition.

Definition 4.16 (Relaxation sequence)**.** *A relaxation sequence for $(\mathcal{K}, \mathcal{L}, c)$ is a finite sequence of conic feasibility problems $\{(\mathcal{K}_1, \mathcal{L}, c), \ldots, (\mathcal{K}_\gamma, \mathcal{L}, c)\}$ such that $\mathcal{K}_1 = \mathcal{K}$ and:*

1. *Every $\mathcal{K}_i$ is an extended second order cone.*
2. *For $i > 1$, there is $d^{i-1} \in \mathcal{K}_{i-1} \cap L$ such that $(\mathcal{K}_i, \mathcal{L}, c)$ is obtained as a result of applying Theorem 4.15 to $\mathcal{K}_{i-1} \cap L$ and d^{i-1}. In addition, we must have $\mathcal{K}_{i-1} \subsetneq \mathcal{K}_i$ (i.e., we do not admit trivial relaxations).*

The vectors in $\{d^1, \ldots, d^{\gamma-1}\}$ are called reducing directions*, due to the fact that they came from the application of facial reduction to the dual system $(\mathcal{K}^*, \mathcal{L}^\perp, 0)$. A relaxation sequence is* maximal *if it does not admit non-trivial relaxations. The problem $(\mathcal{K}_\gamma, \mathcal{L}, c)$ is called* the last problem *of the sequence. The length of the sequence is defined to be γ.*

Since every reducing direction is responsible for relaxing at least one Lorentz cone, the maximum length of a relaxation sequence is $m+1$, where m is the number of second order cones appearing in $\mathcal{K}$. Each relaxed problem almost preserves the feasibility status of the original, in the sense of Theorem 4.15. We will prove that when the relaxation sequence is maximal, the last problem cannot be weakly infeasible. This is a version of Proposition 4.9 for SOCPs, but in this case, we need a partial polyhedrality result in order to prove it.

Proposition 4.17. *If $\{(\mathcal{K}_1, \mathcal{L}, c), \ldots, (\mathcal{K}_\gamma, \mathcal{L}, c)\}$ is a maximal relaxation sequence for $(\mathcal{K}, \mathcal{L}, c)$ then we have:*

i. $(\mathcal{K}, \mathcal{L}, c)$ *is strongly feasible if and only if* $(\mathcal{K}_\gamma, \mathcal{L}, c)$ *is strongly feasible;*

ii. $(\mathcal{K}, \mathcal{L}, c)$ *is in weak status if and only if* $(\mathcal{K}_\gamma, \mathcal{L}, c)$ *is weakly feasible;*

iii. $(\mathcal{K}, \mathcal{L}, c)$ *is strongly infeasible if and only if* $(\mathcal{K}_\gamma, \mathcal{L}, c)$ *is strongly infeasible.*

Proof. By induction and using Theorem 4.15, items (*i*) and (*iii*) follow. We can also conclude that $(\mathcal{K}, \mathcal{L}, c)$ is in weak status if and only if $(\mathcal{K}_\gamma, \mathcal{L}, c)$ is in weak status. Now, suppose that $(\mathcal{K}_\gamma, \mathcal{L}, c)$ is infeasible and that $(\mathcal{K}, \mathcal{L}, c)$ is in weak status. To finish the proof, we have to show that $(\mathcal{K}_\gamma, \mathcal{L}, c)$ cannot be weakly infeasible.

Reordering if necessary, we may assume that $\mathcal{K}_\gamma = \widetilde{\mathcal{K}} \times \widetilde{P}$, where $\widetilde{\mathcal{K}}$ is the direct product of Lorentz cones and P is a polyhedral cone. In this case, $\widetilde{P}$ is a direct product of half-spaces and vector spaces. Now, we would like to use Proposition 3.15 by setting $C_1 = \mathcal{L} + c$, $C_2 = \widetilde{\mathcal{K}}$ and $P = \widetilde{P}$. Let us check that Equation (3.8) is satisfied. We have $\operatorname{rec} C_1 \cap (\operatorname{rec} C_2 \times \operatorname{rec} P) = L \cap (\widetilde{\mathcal{K}} \times \widetilde{P})$ and $\operatorname{lin} C_2 \times \operatorname{rec} P = \{0\} \times \widetilde{P}$.

[2]Recall that if $x, y \in \mathcal{Q}^n$ satisfy $x^\top y = 0$, then $x_0 \overline{y} + y_0 \overline{x} = 0$.

Pick an element $x \in L \cap (\widetilde{\mathcal{K}} \times \widetilde{P})$. We must have $x \in \{0\} \times P$, otherwise we would be able to apply Proposition 4.15 one more time, which would contradict the assumption of maximality. Since Equation (3.8) is satisfied, it follows that if $(\mathcal{K}_\gamma, \mathcal{L}, c)$ is infeasible, it must be strongly infeasible.

□

Example 4.18. *Let $(\mathcal{K}, \mathcal{L}, c)$ be such that $\mathcal{K} = \mathcal{Q}^3 \times \mathcal{Q}^3$ and $\mathcal{L} + c = \{(t,t,s) \times (s,s,1) \mid (t,t,s) \in \mathcal{Q}^3, (s,s,1) \in \mathcal{Q}^3\}$. Then, $a = (1,1,0) \times (0,0,0) \in \mathcal{K} \cap \mathcal{L}$. Thus, we can relax the cone constraint from $\mathcal{Q}^3 \times \mathcal{Q}^3$ to $H^3_{a'_{n_1}} \times \mathcal{Q}^3$. Now, $b = (0,0,1) \times (1,1,0) \in H^3_{a'_{n_1}} \times \mathcal{Q}^3$. Thus, we can relax the problem from $H^3_{a'_{n_1}} \times \mathcal{Q}^3$ to $H^3_{a'_{n_1}} \times H^3_{b'_{n_2}}$. The problem $(H^3_{a'_{n_1}} \times H^3_{b'_{n_2}}, \mathcal{L}, c)$ is weakly feasible, because no point in $\mathcal{L} + c$ strictly satisfies the inequalities which define $H^3_{a'_{n_1}} \times H^3_{b'_{n_2}}$. This implies that $(\mathcal{K}, \mathcal{L}, c)$ is in weak status. Since it is clear that $(s,s,1)$ can never belong to $\mathcal{Q}^3$, the problem must be weakly infeasible.*

4.4.2 The maximum number of directions needed to approach $\mathcal{K}$

Let $\mathcal{K}$ be an extended second order cone, therefore it is a direct product of Lorentz cones and polyhedral cones. Suppose that there are m Lorentz cones among them. In this section, we will show that given a weakly infeasible feasibility problem $(\mathcal{K}, \mathcal{L}, c)$ there is $c' \in \mathbb{R}^n$, a subspace $\mathcal{L}'$ contained in $\mathcal{L}$ of dimension at most m such that $(\mathcal{K}, \mathcal{L}', c')$ is weakly infeasible. This means that starting at c', at most m directions are needed to approach the cone. Note that, *a priori*, the number of direction needed to approach the cone could be up to the dimension of the affine space $\mathcal{L} + c$. Theorem 4.19 states, however, it is bounded by m, regardless of the dimension of $\mathcal{L} + c$.

Theorem 4.19. *Let $(\mathcal{K}, \mathcal{L}, c)$ be a weakly infeasible problem. Then there are a subspace $L' \subseteq L$ and $c' \in \mathcal{L} + c$ such that $(\mathcal{K}, \mathcal{L}', c')$ is weakly infeasible and dimension of $\mathcal{L}' + c'$ is at most m, where m is the number of Lorentz cones.*

Proof. Let $\{(\mathcal{K}_1, \mathcal{L}, c), \ldots, (\mathcal{K}_\gamma, \mathcal{L}, c)\}$ be a maximal relaxation sequence and $\{d^1, \ldots, d^{\gamma-1}\}$ the associated set of reducing directions. Each d^i is responsible for relaxing at least one Lorentz cone. Since there are most m of them, there are at most m directions. Due to Proposition 4.17, the last problem is weakly feasible, so it admits a feasible solution c'.

If $\mathcal{L}'$ is the space spanned by $\{d^1, \ldots, d^{\gamma-1}\}$ then $(\mathcal{K}, \mathcal{L}', c')$ is weakly infeasible. After all, $(\mathcal{K}, \mathcal{L}', c')$ shares the same maximal relaxation sequence and Proposition 4.17 implies that $(\mathcal{K}, \mathcal{L}', c')$ has weak status. Also, $\mathcal{L}' + c'$ is an affine subspace of $\mathcal{L} + c$, so $(\mathcal{K}, \mathcal{L}', c')$ is an infeasible problem. □

Chapter 5

Completely solving CLPs with an interior point oracle

Interior point methods (IPMs) [59] are one of the standard methods for solving conic linear programs (CLPs). However, in order to function properly, they require that the problem at hand satisfy certain regularity conditions which may fail to be satisfied in general and this leads to numerical difficulties. The usual requirement is that both (P) and (D) be strongly feasible. In this chapter, we suppose the existence of an idealized machine that is able to solve any CLP having primal and dual relative interior feasible points and discuss whether it could be used to solve other CLPs which might not satisfy that assumption. In this chapter, we use the expression that an algorithm or a scheme *completely solves* an CLP when it works in the following way.

1. It checks whether the CLP is feasible or not.

2. When the CLP is feasible, it computes the optimal value. If the optimal value is attained, it computes a point in the relative interior of the optimal set. Moreover, if the optimal value is not attained, given arbitrary small $\epsilon > 0$, it can compute a feasible point whose objective value is within ϵ distance of the optimal value.

3. When the CLP is infeasible, it distinguishes whether it is strongly infeasible, or weakly infeasible. Whenever the CLP is strongly infeasible, it computes a certificate of infeasibility. If the CLP is weakly infeasible, then it can compute a point in the corresponding affine space whose distance to $\mathcal{K}$ is less than an arbitrary small given positive number.

We now discuss why completely solving a conic linear program is not a trivial task. We remark that most modern IPM softwares [78, 28, 84] do not require explicit knowledge of an interior feasible point beforehand. SeDuMi [78], for instance, transforms a standard form problem into the so-called homogeneous self-dual formulation, which has a trivial starting point. SDPA [28] and SDPT3 [84] use an infeasible interior point method. The fact that these methods can work without explicit knowledge of an interior feasible point, does not mean that they do *not require the existence of an interior feasible point.* Quite the opposite, the absence of interior feasible points may introduce theoretical and numerical difficulties in recovering a solution for the original problem. Also, detection of infeasibility is a complicated task. Some interior point methods, such as the one discussed in [60] by Nesterov, Todd and Ye, are able to obtain a certificate of infeasibility if the problem is dual or primal strongly infeasible, but the situation is less clear in the presence of weak infeasibility.

Even with the aid of Facial Reduction, the situation is subtle. A key difficulty is that even if we apply Facial Reduction to (D), this is only enough to restore *dual* strong feasibility and it is entirely possible that (D) is still unattained even after substituting $\mathcal{K}$ by $\mathcal{F}^D_{\min}$. Moreover, having a relative interior solution at only one of the sides of the problem might not be enough to end the numerical difficulties. For instance, in [90], Waki, Nakata and Muramatsu discussed SDP instances for which known solvers failed to obtain the correct answer and in one case, this happened even though the problem had an interior feasible point at the one of the sides of the problem but not at both.

The core of our approach is applying facial reduction two times. First at (D) then at the primal problem $\inf\{\langle c, x\rangle \mid \mathcal{A}x = b, x \in (\mathcal{F}^D_{\min})^*\}$. We then show that if θ_D is finite then this is enough to obtain a pair of problems that are both strongly feasible. We then use the tools discussed in the previous chapter to deal with the various possibilities of infeasibility, nonattainment and so on.

As a by-product of our discussion, we also show the equivalence between the general CLP optimization problem and the feasibility problem. A question that is trivial for Linear Programming, but surprisingly subtle for the nonpolyhedral case.

5.1 Double FRA

After applying facial reduction to (D), it is not guaranteed that the new primal $(\tilde{P})$ will be strongly feasible as well. Theorem 5.1 below says that if θ_D is finite then applying FRA to $(\tilde{P})$ will produce a new pair of problems that are both strongly feasible and such that the common optimal value is θ_D. Moreover, (D) is unbounded if and only if this second facial reduction ends up detecting infeasibility.

Theorem 5.1. *Suppose that* (D) *is feasible. We apply FRA to* (D) *and obtain the minimal face* $\mathcal{F}^D_{\min}$. *Now, substitute* $\mathcal{K}^*$ *by* $(\mathcal{F}^D_{\min})^*$ *in* (P) *and denote the resulting problem by (P').* *Apply FRA to (P') to obtain the minimal face* $\mathcal{F}^{P'}_{\min} \subseteq (\mathcal{F}^D_{\min})^*$. *If* $\mathcal{F}^{P'}_{\min} \neq \emptyset$ *substitute* $(\mathcal{F}^D_{\min})^*$ *by* $\mathcal{F}^{P'}_{\min}$ *and* $\mathcal{F}^D_{\min}$ *by* $(\mathcal{F}^{P'}_{\min})^*$ *and denote the resulting problems by* $(\tilde{P})$ *and* $(\tilde{D})$. *Then* $(\tilde{P})$ *and* $(\tilde{D})$ *is a pair of primal and dual conic linear programs such that:*

i. θ_D *is finite if and only if* $(\tilde{P})$ *and* $(\tilde{D})$ *are both strongly feasible. In this case the common optimal value of* $(\tilde{P})$ *and* $(\tilde{D})$ *is* θ_D.

ii. $\theta_D = +\infty$ *if and only if* $\mathcal{F}^{P'}_{\min} = \emptyset$.

Proof. *i.* ($\Rightarrow$) If θ_D is finite, then (P') attains the optimal value θ_D. In particular, (P') is feasible. When FRA is applied to (P'), it becomes strongly feasible. By item *i.* of Theorem 4.2, substituting $\mathcal{F}^D_{\min}$ by ${\mathcal{F}^{P'}_{\min}}^*$ preserves strong feasibility, so $(\tilde{P})$ and $(\tilde{D})$ are both strongly feasible.

($\Leftarrow$) The optimal value of $\tilde{D}$ is θ_D. If $(\tilde{P})$ and $(\tilde{D})$ are both strongly feasible, it must be the case that $\theta_D < +\infty$ since any feasible solution to $(\tilde{P})$ generates an upper bound to θ_D.

ii. Even though the item follows directly from the previous one, we will explain it a little bit.

($\Rightarrow$) If θ_D were finite, that would contradict the previous item.

($\Leftarrow$) If $\mathcal{F}^{P'}_{\min} \neq \emptyset$, then $(\tilde{P})$ would be strongly feasible, since it is the result of applying FRA to (P'). That would contradict the previous item as well.

□

5.2 Constructing almost optimal solutions

From the point of view of (D), the first application of facial reduction zeroes the duality gap and the second one corrects nonattainment. We now take a closer look at this second step. Our objective is to show that the directions obtained at the second facial reduction can be used to construct dual feasible points for (D) that are arbitrarily close to optimality.

In this subsection, our assumption is that (D) was reformulated over $\mathcal{F}^D_{\min}$ and a new pair of primal and dual problems (P') and (D') was obtained with $\theta_{D'} = \sup\{\langle b, y\rangle \mid c - \mathcal{A}^\top y \in \mathcal{F}^D_{\min}\}$ and $\theta_{P'} = \inf\{\langle c, x\rangle \mid \mathcal{A}x = b, x \in (\mathcal{F}^D_{\min})^*\}$. We now apply FRA to (P') and obtain problems $(\tilde{P})$ and $(\tilde{D})$ such that $\theta_{\tilde{P}} = \sup\{\langle b, y\rangle \mid c - \mathcal{A}^\top y \in \mathcal{F}^{P'}_{\min}\}$ and $\theta_{\tilde{D}} = \inf\{\langle c, x\rangle \mid \mathcal{A}x = b, x \in (\mathcal{F}^{P'}_{\min})^*\}$. Let $\{d_1, \ldots, d_\ell\}$ be the obtained reducing directions. They are "dual objects" because they lie in range $\mathcal{A}^\top$. It follows that there are $w_1, \ldots w_\ell$ such that $-\mathcal{A}^\top w_i = d_i \in \operatorname{range} \mathcal{A}^\top \cap (\mathcal{F}_i \setminus \operatorname{lin} \mathcal{F}_i)$ and $\langle b, w_i\rangle \geq 0$, where

$$\mathcal{F}_i = \begin{cases} (\mathcal{F}^*_{i-1} \cap \{d_{i-1}\}^\perp)^* & \text{if } i > 1 \\ \mathcal{F}^D_{\min} & \text{if } i = 1 \end{cases} \tag{5.1}$$

For $x \in \mathcal{K}$, recall that $\mathcal{F}(x, \mathcal{K})$ denotes the minimal face of $\mathcal{K}$ which contains x. With this notation, we have the relations $\mathcal{F}_i = \mathcal{F}(d_{i-1}, \mathcal{F}_{i-1})^{\Delta *} = \operatorname{cl}(\mathcal{F}_{i-1} + \mathcal{F}(d_{i-1}, \mathcal{F}_{i-1})^{\Delta\perp}) = \operatorname{cl}(\mathcal{F}_{i-1} + \mathcal{T}_{d_{i-1}}\mathcal{F}_{i-1})$, by item *viii.* of Lemma 2.5. We also have $\mathcal{F}_{\ell+1} = (\mathcal{F}^{P'}_{\min})^*$.

Proposition 5.2. *Suppose $\theta_D < +\infty$, so that $\mathcal{F}^{P'}_{\min} \neq \emptyset$ and $\langle b, w_i\rangle = 0$, for every i. Now suppose that y is such that $c - \mathcal{A}^\top y \in \operatorname{ri}(\mathcal{F}^{P'}_{\min})^*$, in other words, y is relative interior solution for $(\tilde{D})$. Then there are $\alpha_1, \ldots, \alpha_\ell$ such that $\alpha_i > 0$ for every i and*

$$c - \mathcal{A}^\top \tilde{y} \in \operatorname{ri} \mathcal{F}^D_{\min},$$

where

$$\tilde{y} = \left(y + \sum_{i=1}^{\ell} \alpha_i w_i\right). \tag{5.2}$$

Moreover, $\langle b, \tilde{y}\rangle = \langle b, y\rangle$.

Proof. By what we remarked before, we have the relation $(\mathcal{F}^{P'}_{\min})^* = \mathcal{F}_{\ell+1} = \operatorname{cl}(\mathcal{F}_\ell + \mathcal{T}_{d_\ell}\mathcal{F}_\ell)$. By hypothesis, we have $s = c - \mathcal{A}^\top y \in \operatorname{ri}(\mathcal{F}^{P'}_{\min})^* \in \operatorname{ri} \mathcal{F}_\ell + \mathcal{T}_{d_\ell}\mathcal{F}_\ell$. Then, Lemma 4.1 ensures the existence of $\alpha_\ell > 0$ such that $s + \alpha_\ell d_\ell \in \operatorname{ri} \mathcal{F}_\ell$. We then proceed by induction to obtain the coefficients $\alpha_{\ell-1}, \ldots, \alpha_1$ in a "top-to-bottom" fashion.

Finally, (5.2) holds because we have $\langle b, w_i\rangle = 0$ for every i. □

Since $\theta_D = \theta_{\tilde{D}}$ and $(\tilde{D})$ is strongly feasible, for every $\epsilon > 0$ there is y_ϵ such that y_ϵ is feasible for $(\tilde{D})$, $\langle b, y_\epsilon\rangle > \theta_D - \epsilon$ and the corresponding slack $s_\epsilon = c - \mathcal{A}^\top y_\epsilon$ satisfies $s_\epsilon \in \operatorname{ri}(\mathcal{F}^{P'}_{\min})^*$. We can then invoke Proposition 5.2 to obtain a feasible solution for (D) with the same objective value. The caveat is that the coefficients α_i can become arbitrary large.

Suppose that s_ϵ is obtained and we already computed the reducing directions $\{d_1, \ldots, d_\ell\}$. As long as we are able to decide membership in $\mathcal{F}_i$ and $\operatorname{ri} \mathcal{F}_i$ for every i, it is not necessary to solve any additional conic programs in order to find out a possible choice of coefficients that will generate ϵ-optimal solutions for (D). This is because $s_\epsilon + td_\ell \in \operatorname{ri} \mathcal{F}_\ell$ *for all* t sufficiently large and positive. So we may start with arbitrary positive value for α_ℓ and increase it until $s_\epsilon + \alpha_\ell d_\ell \in \operatorname{ri} \mathcal{F}_\ell$. We then do the same with $\alpha_{\ell-1}, \ldots, \alpha_1$.

Finally, note that it is also not necessary to solve a different conic problem to obtain s_ϵ every time ϵ changes. It is enough to do it *twice*. First, obtain *any* relative interior feasible slack s' for (D'). Because $(\tilde{D})$ is attained, an optimal slack s^* exists. Then for every $\beta \in [0, 1)$, we have $z_\beta = \beta s^* + (1 - \beta)s' \in \operatorname{ri}(\mathcal{F}^{P'}_{\min})^*$, since s' is a relative interior solution. It is then clear that if we wish to obtain an ϵ-optimal solution for $(\tilde{D})$ that belongs to the relative interior, it is enough to take z_β with β sufficiently close to 1. We can then select the α_i appropriately and consider the element

$$z_\beta + \sum_{i=1}^{\ell} d_i\alpha_i \in \mathcal{F}^D_{\min}.$$

5.2.1 A comparison to an earlier work by Abrams

Note that if (D) were already strongly feasible to begin with, then $\mathcal{F}^D_{\min} = \mathcal{K}$ and there is no need to do facial reduction at the dual side (D). So applying facial reduction to the primal problem (P) and considering the corresponding dual has the effect of correcting nonattainment. Here we compare this approach to an earlier work by Abrams [1], that seems to predate facial reduction by a few years. Abrams' discussion is done in the context of optimizing a convex function subject to convex constraints, but here we will restrict our attention to the conic linear programming case.

For simplicity we will write (D) in "slack format" and we will also assume that the linear system "$\mathcal{A}x = b$" has a solution x_0. Recalling that $\mathcal{L} = \operatorname{range} \mathcal{A}^\top$, we can write (D) as the following equivalent problem:

$$\begin{array}{ll} \inf\limits_{s} & \langle -x_0, s\rangle \\ \text{subject to} & s \in (c + \mathcal{L}) \cap \mathcal{K}. \end{array} \tag{$\hat{D}$}$$

If we specialize Abrams' approach to conic linear programming we obtain the following procedure:

1. Let S be the recession cone of $(\hat{D})$. This is the set $S = \mathcal{L} \cap \mathcal{K} \cap \{s \mid \langle s, x_0\rangle \geq 0\}$.

2. Let $P_{S^\perp}$ be the orthogonal projection on the subspace $S^\perp$.

3. Let $S_0 = \{(y, 0) \in \mathbb{R}^{m+1} \mid y \in S\}$ and let $P_{S_0^\perp}$ be orthogonal projection on $S_0^\perp = S^\perp \times \mathbb{R}$.

4. Let h be the convex function whose epigraph is $\operatorname{cl}(P_{S_0^\perp}(\operatorname{epi} f))$, where $f(s) = \langle s, -x_0\rangle$. We then substitute (D) by:

$$\begin{array}{ll} \inf\limits_{s\in S^\perp} & h(s) \\ \text{subject to} & s \in \operatorname{cl} P_{S^\perp}(\mathcal{F}_{\mathrm{D}}\ ), \end{array} \tag{D_S}$$

A key observation in Abrams' work is that (D) can only be unattained if $S \neq \{0\}$. It seems sensible to remove those troublesome directions by projecting the problem on $S^\perp$. Unfortunately, the new problem can still be unattained. So we repeat steps 1 through 4 until $S = \{0\}$. Of course, we still have to worry whether the optimal value will stay the same. His Theorem 3.2 states that the optimal value is preserved if the relative interior of the domain of the objective function intersects the relative interior of the feasible set. As the objective function is finite everywhere, we are covered by Theorem 3.2.

Note that when (D) is not strongly feasible, then doing facial reduction at the primal side corrects dual nonattainment, but at the cost of, perhaps, changing the dual optimal value. Therefore, we will assume that (D) is strongly feasible and will show under this hypothesis that Abram's approach can be regarded as sequence of facial reduction/conic expansion steps, where each step is followed by a projection. The first observation is that the set S contains the reducing directions that are used to do facial reduction at the primal side. Now we will take a look at h.

Proposition 5.3. *Suppose that $S \subseteq \{x_0\}^\perp$, this is the case if, for instance,* (D) *is feasible and $\theta_D < +\infty$. Then, the function whose epigraph is* $\mathrm{cl}\,(P_{S_0^\perp}(\mathrm{epi}\, f))$ *is*

$$h(s) = \begin{cases} -\langle x_0, s\rangle & s \in S^\perp \\ +\infty & s \notin S^\perp. \end{cases}$$

Proof. Note that $P_{S_0^\perp}(x,\mu) = (P_S^\perp(x),\mu)$. In addition, since f is a linear function, $\mathrm{cl}\,(P_{S_0^\perp}(\mathrm{epi}\, f)) = P_{S_0^\perp}(\mathrm{epi}\, f)$. If $(x,\mu) \in \mathrm{epi}\, h$ with $\mu < +\infty$, then $x \in S^\perp$ and $(x,\mu) \in \mathrm{epi}\, f$. As $P_{S^\perp}(x) = x$, we have $\mathrm{epi}\, h \subseteq P_{S_0^\perp}(\mathrm{epi}\, f)$.

We now show the opposite inclusion. Let $(x,\mu) \in \mathrm{epi}\, f$. We can write $x = P_S(x) + P_{S^\perp}(x)$. Since $S \subseteq \{x_0\}^\perp$, we have $\langle -x_0, x\rangle = \langle -x_0, P_{S^\perp}(x)\rangle$, which implies that $(P_{S^\perp}(x),\mu) \in \mathrm{epi}\, h$.

□

Example 5.4. *If the condition $S \subseteq \{x_0\}^\perp$ fails, then Proposition 5.3 may not hold. Consider the problem $\theta = \inf\{-t - s \mid t \geq 0, s = 0\}$. With our previous notation, we have $x_0 = (-1,-1)$ and $S = \{(t,s) \mid -t \geq 0, s = 0\}$, so that $S^\perp = \{0\} \times \mathbb{R}$ and $P_{S^\perp}(-1,-1) = (0,-1)$. If we define $\hat{h}$ as in Proposition 5.3 we have:*

$$P_{S_0^\perp}(\mathrm{epi}\, f) = \{0\} \times \mathbb{R} \times \mathbb{R}$$
$$\mathrm{epi}\, \hat{h} = \{(0,s,\mu) \mid -s \leq \mu\}.$$

However, the function h whose epigraph is $P_{S_0^\perp}(\mathrm{epi}\, f)$ is the one that maps (t,s) to $-\infty$ if $t = 0$ and to $+\infty$ if $t \neq 0$.

We will now take a more careful look at the constraints of (D_S). In general, the intersection of projections is different from the projection of intersection. Nevertheless, Abrams shows that in this specific case we have

$$\mathrm{cl}\,(P_{S^\perp}(\mathcal{F}_{\mathrm{D}})) = \mathrm{cl}\,\left(P_{S^\perp}(c - \mathrm{range}\,\mathcal{A}^\top) \cap P_{S^\perp}(\mathcal{K})\right),$$

see Lemma 3.1 therein. Under the assumption of strong feasibility, we can distribute the closure operator over the sets of an intersection. Gathering what we have discussed so far, we can conclude that when $\theta_D < +\infty$ (i.e., $\theta_{\hat{D}} > -\infty$), and (D) is strongly feasible then (D_S) can be written as:

$$\begin{aligned} &\inf_{s \in S^\perp} && \langle -x_0, s\rangle && (D_S) \\ &\text{subject to} && s \in \left(P_{S^\perp}(c - \mathrm{range}\,\mathcal{A}^\top) \cap \mathrm{cl}\, P_{S^\perp}(\mathcal{K})\right). \end{aligned}$$

The next proposition connects what we have done so far with facial reduction.

Proposition 5.5. *Suppose that C is a convex set contained in $\mathcal{K}$, let $d \in \mathrm{ri}\, C$ and let $\mathcal{F}$ be the minimal face of $\mathcal{K}$ which contains d. Then*

$$\mathrm{cl}\,(\mathcal{K} + \mathrm{span}\, C) = \mathcal{F}^{\Delta *} = \mathrm{cl}\,(\mathcal{K} + \mathcal{F}^{\Delta\perp})$$

Proof. It is enough to prove that $\mathcal{K}^* \cap \{d\}^\perp = \mathcal{K}^* \cap C^\perp$, since $(\mathcal{K}^* \cap C^\perp)^* = \mathrm{cl}\,(\mathcal{K} + \mathrm{span}\, C)$. Note that the inclusion $\mathcal{K}^* \cap \{d\}^\perp \supseteq \mathcal{K}^* \cap C^\perp$ is immediate, since $d \in C$.

Let us prove the opposite inclusion. Suppose that $z \in \mathcal{K}^* \cap \{d\}^\perp$, then z defines a supporting a hyperplane for C as well, since $C \subseteq \mathcal{K}$. Since d is a relative interior point, the equation $\langle z, d\rangle = 0$ implies that $\{z\}^\perp \supseteq C$, so that $z \in C^\perp$. □

In other words, $\mathcal{K}+\operatorname{span} S$ and $\mathcal{K}+\mathcal{F}^{\Delta\perp}$ are almost the same, since they share the same closure. It is an unfortunate fact, however, that they can be different. Still, $\operatorname{cl}(\mathcal{K}+\mathcal{F}^{\Delta\perp})$ is precisely the dual of the cone obtained through a single facial reduction step by selecting d as a reducing direction. We also remark that when $S \subseteq \{x_0\}^\perp$, $\operatorname{cl}(\mathcal{K}+\operatorname{span} S)$ correspond to $\Gamma_{\mathcal{B}}(\mathcal{K})$ where, $\mathcal{B} = \{x_0\}^\perp \cap \mathcal{L}$ and Γ is the so called "conic expansion operator", defined in Section 7 of [55]. On their way to proving the correspondence between conic expansion and facial reduction, Waki and Muramatsu proved a result essentially equivalent to Proposition 5.5, see Theorem 3.1 therein.

Example 5.6. *Let $\mathcal{K} = \mathcal{S}^2_+$ and $S = \{\left(\begin{smallmatrix} t & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid t \geq 0\}$. Note that $\mathcal{K} + \operatorname{span} S$ is not closed, because $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ does not belong to $\mathcal{K}+\operatorname{span} S$ although it belongs to its closure. On the other hand, if we pick a relative interior point of S and construct the corresponding minimal face $\mathcal{F}$ of $\mathcal{K}$, it is possible to check that $\mathcal{K} + \mathcal{F}^{\Delta\perp}$ is closed. This can also be seen as a consequence of the fact that the cone $\mathcal{S}^2_+$ is* nice *but* devious*, see [86] for more details on nice and devious cones.*

We now come to the main result of this section. Under the assumption of strong feasibility of (D), Abram's approach can be seen as a successive sequence of facial reduction/conic expansion steps followed by projections on the orthogonal complement of the recession cone of the problem.

Theorem 5.7. *Let $d \in \operatorname{ri} S$ and let $\mathcal{F} = \mathcal{K}^* \cap \{d\}^\perp$. When $\theta_{\hat{D}} > -\infty$ and* (D) *is strongly feasible then*

$$\operatorname{cl} P_{S^\perp}(\mathcal{K}) = \operatorname{cl} P_{S^\perp}(\mathcal{K} + \mathcal{F}^{\Delta\perp}).$$

In particular, (D_S) is equivalent to:

$$\begin{aligned} \inf_{s \in S^\perp} \quad & \langle -x_0, s\rangle \qquad\qquad (D_{\mathrm{FRA}}) \\ \textit{subject to} \quad & s \in \Big(P_{S^\perp}(c - \operatorname{range} \mathcal{A}^\top) \cap \operatorname{cl} P_{S^\perp}(\mathcal{K} + \mathcal{F}^{\Delta\perp})\Big). \end{aligned}$$

Proof. Recall that linear maps preserve relative interiors, so we have

$$\begin{aligned} \operatorname{ri} P_{S^\perp}(\mathcal{K}) &= \operatorname{ri} P_{S^\perp}(\mathcal{K} + \operatorname{span} S) \\ &= P_{S^\perp}(\operatorname{ri}(\operatorname{cl}(\mathcal{K} + \operatorname{span} S))) \\ &= P_{S^\perp}(\operatorname{ri}(\operatorname{cl}(\mathcal{K} + \mathcal{F}^{\Delta\perp}))) \\ &= \operatorname{ri} P_{S^\perp}(\mathcal{K} + \mathcal{F}^{\Delta\perp}), \end{aligned}$$

where the third inequality follows from Proposition 5.5. Since the cones have the same relative interior, they must have the same closure as well. ☐

5.3 Feasibility vs optimization in conic linear programming

We now discuss the equivalence between optimization and feasibility in conic linear programming. Suppose that we have an oracle $\mathcal{O}_{\mathrm{opt}}$ that receives as input (D) and returns the optimal value θ_D and an optimal solution if it exists. If the problem is unbounded it returns the symbol "$+\infty$" and if the problem is infeasible it returns the symbol "$-\infty$". Of course, any other symbol could be used in place of "$+\infty$" and "$-\infty$". Suppose also the existence of $\mathcal{O}_{\mathrm{feas}}$ that receives as input (D) and returns a feasible solution if it exists and the symbol "$-\infty$" if there is none. It is clear that we can simulate $\mathcal{O}_{\mathrm{feas}}$ by a single call to $\mathcal{O}_{\mathrm{opt}}$. The more interesting question is whether we can simulate $\mathcal{O}_{\mathrm{opt}}$ via polynomially (in m and n) many calls to $\mathcal{O}_{\mathrm{feas}}$. For linear programming, this is also easy because if θ_D is bounded, we can just feed to $\mathcal{O}_{\mathrm{feas}}$ the system $\{(x, y) \mid x \in \mathbb{R}^n_+, \mathcal{A}x = b, c - \mathcal{A}^\top y \in \mathbb{R}^n_+, \langle c, x\rangle - \langle b, y\rangle = 0\}$.

If the problem is not bounded, there is also a simple system that can be fed to $\mathcal{O}_{\text{feas}}$ to detect that. For semidefinite programming, this does not work since there could be a nonzero duality gap between (D) and (P). However, it follows from Ramana's work that a similar relation holds between optimization and feasibility in semidefinite programming, see, in particular, Section 4.3 of [72].

Here we discuss how this could be done for the general linear conic problem. A first attempt would be to do facial reduction and then feed $\{(x,y) \mid x \in \mathcal{F}_{\min}^{D}{}^*, \mathcal{A}x = b, c - \mathcal{A}^\top y \in \mathcal{F}_{\min}^{D}, \langle c, x\rangle - \langle b, y\rangle = 0\}$ to $\mathcal{O}_{\text{feas}}$. This, unfortunately, does not work. Even though the problem has zero duality gap, θ_D can be *unattained.* To fix that we have to do facial reduction twice.

The first observation is to note that FRA can be done with polynomially many calls to $\mathcal{O}_{\text{feas}}$. In fact, the FRA-Poly described in Section 3.3 must stop after finding at most $1 + \sum_{i=1}^{r} \ell_{\text{poly}}(\mathcal{K}^i)$ directions, where $\ell_{\text{poly}}(\mathcal{K}^i)$ is the distance to polyhedrality. Since $\sum_{i=1}^{r} \ell_{\text{poly}}(\mathcal{K}^i) \leq n$, this establishes our claim.

Proposition 5.8. *$\mathcal{O}_{opt}$ can be simulated with polynomially (in n and m) many calls to $\mathcal{O}_{feas}$.*

Proof. It is enough to follow the following recipe:

1. Check the feasibility of (D). If no feasible solution exists, stop. We have $\theta_D = -\infty$.

2. Do as indicated in Theorem 5.1.

3. If $\mathcal{F}_{\min}^{P'} = \emptyset$ then stop. We have $\theta_D = +\infty$.

4. Otherwise, feed the system $\{(x,y) \mid x \in \mathcal{F}_{\min}^{P'}, \mathcal{A}x = b, c - \mathcal{A}^\top y \in \mathcal{F}_{\min}^{P'}{}^*, \langle c, x\rangle - \langle b, y\rangle = 0\}$ to $\mathcal{O}_{\text{feas}}$ and let (x^*, y^*) be the output. Then $\theta_D = \langle c, x^*\rangle$.

5. Feed $\{y \mid c - \mathcal{A}^\top y \in \mathcal{K}, \langle b, y\rangle = \theta_D\}$ to $\mathcal{O}_{\text{feas}}$. If (D) is attained then $\mathcal{O}_{\text{feas}}$ will output an optimal solution y^*.

□

In practice, for many classes of cones, the only kind of problems for which we have hope of tackling directly are the ones that are strongly feasible at both the primal and the dual side, since this is the minimum necessary for running interior point methods (IPMs). So it makes sense to consider an even more restrictive oracle $\mathcal{O}_{\text{int}}$ that receives as input a pair of problems (D), (P). If (D) and (P) are both strongly feasible, then the oracle returns a pair of optimal solutions. Clearly, $\mathcal{O}_{\text{feas}}$ can simulate $\mathcal{O}_{\text{int}}$ with a single call, since it is enough to feed to it the feasibility system that correspond to the optimality conditions. This time, it works because we are assuming strong feasibility at both sides. What is, perhaps, less trivial is that we can also simulate $\mathcal{O}_{\text{feas}}$ with $\mathcal{O}_{\text{int}}$.

Theorem 5.9. *It is possible to simulate $\mathcal{O}_{feas}$ with polynomially (in m and n) many calls to $\mathcal{O}_{int}$. As a consequence, $\mathcal{O}_{opt}$ can be simulated with polynomially (in m and n) many calls to $\mathcal{O}_{int}$*

Proof. Facial reduction can solve the feasibility problem and it can be done with polynomially many calls to $\mathcal{O}_{\text{int}}$ by using the pair of problems $(P_{\mathcal{K}})$ and $(D_{\mathcal{K}})$. Note that this pair of problems have easy to obtain interior feasible points. Namely, $(\frac{e^*}{\langle e,e^*\rangle+1}, \frac{1}{\langle e,e^*\rangle+1}, \frac{1}{\langle e,e^*\rangle+1})$ for $(P_{\mathcal{K}})$ and $(0,-1,0)$ for $(D_{\mathcal{K}})$. □

5.4 Completely solving CLPs

In [49], we showed how to "completely solve" general SDPs with $\mathcal{O}_{\text{int}}$. We now discuss how to extend the results therein to general linear conic programs. First, "completely solving" includes the ability to check feasibility/infeasibility, detect unboundedness, compute the optimal value and produce an optimal solution if it exists. This is the part that is already covered by Theorem 5.9. It is still missing how to obtain points in the relative interior of the optimal set, when the problem is attained. In case of nonattainment, it would also be desirable to obtain almost optimal solution. Finally, for weakly infeasible problems, we would like to obtain points in $c + \operatorname{range} \mathcal{A}^\top$ that are arbitrarily close to $\mathcal{K}$.

Here, we summarize the steps that one may take to completely solve (D) using the $\mathcal{O}_{\text{int}}$.

1) Use facial reduction to find $\mathcal{F}^D_{\min}$ and a relative interior solution y' together with corresponding slack s'. If $\mathcal{F}^D_{\min} = \emptyset$, then (D) is infeasible.

   (a) If (D) is infeasible, solve the feasibility problem in item *ii* of Proposition 2.7, using facial reduction. If that problem is infeasible, then (D) is weakly infeasible. If it is feasible, then (D) is strongly infeasible.

   (b) If (D) turns out to be weakly infeasible, the discussion in Section 4.2 shows how to produce points arbitrarily close to $\mathcal{K}$.

2) Reformulate (D) as a dual strongly feasible CLP over the face $\mathcal{F}^D_{\min}$, denote the resulting problem by (D'). Substitute $\mathcal{K}^*$ by ${\mathcal{F}^D_{\min}}^*$ in (P) and denote the resulting problem by (P'). Apply FRA to (P') to obtain the minimal face $\mathcal{F}^{P'}_{\min} \subseteq {\mathcal{F}^D_{\min}}^*$, as in Theorem 5.1.

   (a) If $\mathcal{F}^{P'}_{\min} = \emptyset$ then $\theta_D = +\infty$.

3) If $\mathcal{F}^{P'}_{\min} \neq \emptyset$ substitute ${\mathcal{F}^D_{\min}}^*$ by $\mathcal{F}^{P'}_{\min}$ and $\mathcal{F}^D_{\min}$ by ${\mathcal{F}^{P'}_{\min}}^*$ and denote the resulting problems by $(\tilde{P})$ and $(\tilde{D})$. This case happens if and only if $-\infty < \theta_D < +\infty$

4) Note that at this point we also know relative interior feasible solutions to $(\tilde{P})$, since it is the result of applying FRA to (P'). Moreover, relative interior solutions to (D') are also relative interior solutions to $(\tilde{D})$, see the proof of Theorem 4.2. This means we can invoke $\mathcal{O}_{\text{int}}$ with $(\tilde{P})$ and $(\tilde{D})$ as inputs. At this point, we will know θ_D and an optimal solution y^* to $(\tilde{D})$. .

5) Using y', y^* and the reducing directions obtained by applying FRA to (P'), we can follow Section 5.2 and compute feasible solutions to (D) that are arbitrarily close to optimality.

6) If we wish to check whether the problem is attained or not, we can solve the feasibility problem of finding a point in $\{c - \mathcal{A}^\top y \in \mathcal{F}^D_{\min} \mid \langle b, y\rangle = \theta_D\}$. Applying FRA to this problem we will also be able to find a relative interior solution of the set of optimal solutions, if it is non-empty.

5.4.1 The SDP case

The oracles we considered in this chapter can be invoked with any choice of $\mathcal{K}$. As this is a strong assumption, we might be interested in restricting the possible $\mathcal{K}$ to certain families, which reflects the fact that we only have access to solvers for certain types of problems. Suppose that we have access to an interior point oracle for SDPs $\mathcal{O}^{\text{SDP}}_{\text{int}}$, which is the same as $\mathcal{O}_{\text{int}}$ except that we are only allowed to invoke it with $\mathcal{K} = \mathcal{S}^n_+$ for some n.

Note that the steps we described to completely solve (D) can also be used to handle an arbitrary SDP. The only technical detail is that when doing facial reduction, we must solve problems over *faces* of $\mathcal{S}^n_+$ or over their dual. As each face is linearly isomorphic to another $\mathcal{S}^k_+$ with $k \leq n$, this is not a complicated technical issue. However, for the sake of completeness, we will discuss it here.

Suppose that $\mathcal{K} = \mathcal{F}^*$, where $\mathcal{F}$ is a face of $\mathcal{S}^n_+$. In addition, we assume that (P) and (D) are both strongly feasible, so that they are solvable via $\mathcal{O}_{\text{int}}$. We will show that we can transform (P) and (D) into an equivalent problem that is solvable by $\mathcal{O}^{\text{SDP}}_{\text{int}}$.

Recall that due to Proposition 2.10 there is a $n \times n$ orthogonal matrix q and $r \leq n$ such that

$$q^\top \mathcal{F} q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}^n \mid a \in \mathcal{S}^r_+ \right\}. \tag{5.3}$$

Using the fact that q is orthogonal, we also have

$$q^\top \mathcal{F}^* q = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathcal{S}^n \mid a \in \mathcal{S}^r_+ \right\}. \tag{5.4}$$

Now, consider the map $\psi : \mathcal{S}^n \to \mathcal{S}^r$ that takes $x \in \mathcal{S}^n$ to $\pi_r(q^\top x q)$, where π_r maps an $n \times n$ matrix to its upper left $r \times r$ submatrix. Note that, restricted to $\mathcal{F}$, ψ is a bijection and maps $\mathcal{F}$ to $\mathcal{S}^r_+$. Now, let us compare $\mathcal{F}_{\text{P}} = \{x \in \mathcal{S}^n \mid \mathcal{A}x = b, x \in \mathcal{F}\}$ and $\mathcal{F}_{\text{D}} = \{y \in \mathbb{R}^m \mid c - \mathcal{A}^\top y \in \mathcal{F}^*\}$ with the feasible sets of the following pair of primal and dual problems in standard form.

$$\begin{array}{rll} \inf\limits_{x_\mathcal{F}} & \langle \psi(c), x_\mathcal{F} \rangle & \\ \text{subject to} & \langle \psi(\mathcal{A}_i), x_\mathcal{F} \rangle = b_i \quad i = 1, \ldots, m & \\ & x_\mathcal{F} \in \mathcal{S}^r_+ & \end{array} \tag{P_ψ}$$

$$\begin{array}{rl} \sup\limits_{y} & \langle b, y \rangle \\ \text{subject to} & \psi(c) - \displaystyle\sum_{i=1}^m \psi(\mathcal{A}_i) y \in \mathcal{S}^r_+, \end{array} \tag{D_ψ}$$

Note that (P_ψ) and (D_ψ) are *bona fide* SDPs, so as long as they are both strongly feasible, we may invoke $\mathcal{O}^{\text{SDP}}_{\text{int}}$ to solve it. Denote the feasible region of (P_ψ) by $\widehat{\mathcal{F}_{\text{P}}}$ and the feasible region of (D_ψ) by $\widehat{\mathcal{F}_{\text{D}}}$.

Proposition 5.10. *Consider the restriction of ψ to $\mathcal{F}$. We have $\psi(\mathcal{F}_P) = \widehat{\mathcal{F}_P}$ and $\psi^{-1}(\widehat{\mathcal{F}_P}) = \mathcal{F}_P$. Moreover, if $x \in \mathcal{F}$, then $\langle c, x \rangle = \langle \psi(c), \psi(x) \rangle$. At the dual side, we have $\widehat{\mathcal{F}_D} = \mathcal{F}_D$.*

Proof. Note that $x \in \mathcal{F}_{\text{P}}$ if and only if $x \in \mathcal{F}$ and $\langle \mathcal{A}_i, x \rangle = b_i$ for every i. However, since q is an orthogonal map, the equality $\langle \mathcal{A}_i, x \rangle = b_i$ holds if and only if $\langle q^\top \mathcal{A}_i q, q^\top x q \rangle = b_i$ holds. Since $x \in \mathcal{F}$ and (5.3) holds, we have that $\langle q^\top \mathcal{A}_i q, q^\top x q \rangle = \langle \psi(\mathcal{A}_i), \psi(x) \rangle$. This same argument also shows that $\langle c, x \rangle = \langle \psi(c), \psi(x) \rangle$. Using the fact $\psi(\mathcal{F}) = \mathcal{S}^r_+$, it follows that $\mathcal{F}_{\text{P}}$ is mapped to $\widehat{\mathcal{F}_{\text{P}}}$ by ψ. On the other hand, if we have some $x_\mathcal{F} \in \widehat{\mathcal{F}_{\text{P}}}$, we can consider the unique element $x \in \mathcal{F}$ such that $\psi(x) = x_\mathcal{F}$ and we will have $x \in \mathcal{F}_{\text{P}}$.

At the dual side, equation (5.4) readily shows that $\widehat{\mathcal{F}_{\text{D}}} = \mathcal{F}_{\text{D}}$. □

Since we are under the assumption that (P) and (D) are both primal and dual strongly feasible, it follows readily that (P_ψ) and (D_ψ) are strongly feasible as well. So we may invoke $\mathcal{O}^{\text{SDP}}_{\text{int}}$ to solve it. Now, Proposition 5.10 implies that the optimal value is the same and we may use ψ to recover solutions for (P) and (D). For example, if $\hat{x}$ is an optimal solution for (P_ψ), then $q\left(\begin{smallmatrix} \hat{x} & 0 \\ 0 & 0 \end{smallmatrix}\right)q^\top$ will be a primal optimal solution for (P).

5.4.2 Previous discussions

In the SDP case, apart from the tools described so far, we also have Ramana's extended dual. We recall that the Ramana's dual [72] is a substitute for the Lagrangian dual of (D). Remarkable features of Ramana's dual include the fact that it can be written as a SDP, always affords zero duality gap and that the dual is always attained whenever the primal optimal value is finite. However, Ramana's dual is not necessarily suitable to be used with IPMs due to the fact that it does not ensure the existence of interior feasible points at both sides. In particular, we might not be able to invoke $\mathcal{O}_{\text{int}}^{\text{SDP}}$ with a problem and its Ramana's dual.

Nevertheless, there is an approach due to de Klerk, Terlaky and Roos [19] that also aims at solving (D) in a thorough way and makes use of Ramana's dual. Their tool of choice is a self-dual embedding strategy of the original pair (P) and (D). However, in the absence of both primal and dual strong feasibility, the embedded problem might fail to reveal the optimal value of the original problem or detect infeasibility/nonattainment. To account for that, they go for a second step, where they consider an embedded problem using Ramana's dual. The Ramana's dual (P_R) is a substitute for (P) and they consider the pair formed by (P_R) and its dual (D_{cor}), which is a "corrected" version of (D). The pair (P_R, D_{cor}) can then be solved by their embedding strategy to find θ_D. As the embedded problem is both primal and dual strongly feasible, it is possible to invoke $\mathcal{O}_{\text{int}}$ to solve it. However, if the solution given by $\mathcal{O}_{\text{int}}$ is not of maximum rank at both steps, their strategy might not work. We should mention that they do show in detail how to build an interior point method suitable for their approach. Our analysis, on the other hand, is completely agnostic to the inner workings of the interior point oracle and no assumption is made on the optimal solutions returned by $\mathcal{O}_{\text{int}}$.

However, missing from their analysis is how one can recover a solution to (D) from a solution to (D_{cor}) or to check that this is not possible. Moreover, it is not clear from their approach how to obtain points close to optimality in case of nonattainment, or points close to feasibility in case of weak infeasibility.

5.5 A complete example

Here we give a complete example of our approach, as applied to the problem (NASTY-D), discussed in Section 2.4. For convenience, we restate it here. This time, however we will permute some of the rows and columns.

$$
\begin{aligned}
\sup_{y \in \mathbb{R}^8} \quad & -y_4 - 2y_6 - 2y_7 \qquad \text{(NASTY-D)}\\
\text{subject to} \quad & \begin{pmatrix}
y_1 & & & & & & & y_3 - 1\\
 & y_1 & & & & & & y_5 - 1\\
 & & y_2 & y_3 & & & & \\
 & & y_3 & y_4 - y_5 & & & & \\
 & & & & y_4 & -0.5y_8 + 0.5 & y_6 & \\
 & & & & -0.5y_8 + 0.5 & y_8 & y_7 & \\
 & & & & y_6 & y_7 & 0 & \\
y_3 - 1 & y_5 - 1 & & & & & & 0
\end{pmatrix} \in \mathcal{S}^8_+.
\end{aligned}
$$

We have $\theta_D = -1$, $\theta_P = 0$ and neither (NASTY-D) nor its primal are attained. The first step is to apply facial reduction to (D). A possible reducing direction is

$$d_1 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}.$$

Then, if $x \in \mathcal{S}^8_+ \cap \{d_1\}^\perp$, then $x_{ij} = 0$ if $i \in \{7, 8\}$ or $j \in \{7, 8\}$. We can then reformulate (NASTY-D) as the following 6×6 SDP.

$$\begin{aligned} \sup_{y \in \mathbb{R}^4} \quad & -y_4 \qquad \text{(NASTY-D')} \\ \text{subject to} \quad & \begin{pmatrix} y_1 &&&&& \\ & y_1 &&&& \\ && y_2 & 1 && \\ && 1 & y_4 - 1 && \\ &&&& y_4 & -0.5y_8 + 0.5 \\ &&&& -0.5y_8 + 0.5 & y_8 \end{pmatrix} \in \mathcal{S}^6_+. \end{aligned}$$

Note that (y_1, y_2, y_4, y_8) is feasible for (NASTY-D') if and only if $(y_1, y_2, 1, y_4, 1, 0, 0, y_8)$ is feasible for (NASTY-D). The new problem (NASTY-D') is strongly feasible, since a relative interior slack can be obtained by considering $y = (1, 2, 2, 1)$. This means that $\mathcal{F}^D_{\min} = \mathcal{S}^8_+ \cap \{d_1\}^\perp$ and facial reduction ends in one step. Note that problem is not attained. In order to fix that, we write down the corresponding primal problem.

$$\begin{aligned} \inf_{x} \quad & 2x_{34} - x_{44} + x_{56} \qquad \text{(NASTY-P')} \\ \text{subject to} \quad & -x_{11} - x_{22} = 0 \\ & -x_{33} = 0 \\ & -x_{44} - x_{55} = -1 \\ & x_{56} = 0 \\ & x \in \mathcal{S}^6_+. \end{aligned}$$

As expected, there is no duality gap and an optimal solution to (NASTY-P') is the matrix x such that $x_{44} = 1$ and all the other entries are zero. Note that (NASTY-P') is not strongly feasible. We then apply facial reduction to it. One possible reducing direction is

$$d_1' = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}.$$

The face $\mathcal{S}^6_+ \cap \{d_1'\}^\perp$ contains the feasible region of (NASTY-P') and is such that if $x \in \mathcal{S}^6_+ \cap \{d_1'\}^\perp$ then $x_{ij} = 0$ for $i \in \{1, 2, 3\}$ or $j \in \{1, 2, 3\}$. We can then reformulate (NASTY-P') as the following 3×3 SDP.

$$\begin{aligned} \inf_{x} \quad & -x_{11} + x_{23} && \text{(NASTY-P”)} \\ \text{subject to} \quad & -x_{11} - x_{22} = -1 \\ & x_{23} = 0 \\ & x \in \mathcal{S}^3_+. \end{aligned}$$

Note that the feasible solutions to (NASTY-P”) are precisely the projection of feasible solution to (NASTY-P’). More precisely, $x \in \mathcal{S}^3$ is feasible for (NASTY-P”) if and only if $\left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right) \in \mathcal{S}^6$ is feasible for (NASTY-P’). After a single facial reduction step, the new problem (NASTY-P”) is strongly feasible since $\frac{1}{2}I_3$ is a feasible solution, where I_3 is the 3×3 identity matrix.

Note that the feasible region of the dual problem can be found by “ignoring” the upper left 3×3 positive semidefinite constraint in (NASTY-D’).

$$\begin{aligned} \sup_{y \in \mathbb{R}^2} \quad & -y_4 && \text{(NASTY-D”)} \\ \text{subject to} \quad & \begin{pmatrix} y_4 - 1 & & \\ & y_4 & -0.5y_8 + 0.5 \\ & -0.5y_8 + 0.5 & y_8 \end{pmatrix} \in \mathcal{S}^3_+. \end{aligned}$$

The optimal value is (NASTY-D”) is attained, since it is enough to take $y_4 = 1$ and $y_8 = 1$. Moreover (NASTY-D”) and (NASTY-P”) are a pair of strongly feasible SDPs, so we can use $\mathcal{O}^{\text{SDP}}_{\text{int}}$ to solve them. Unfortunately from an optimal solution to (NASTY-D”) we cannot recover an optimal solution to (NASTY-D), since the latter is not attained. Nevertheless, the recipe in Section 5.2 allows us to construct almost optimal points.

For every $\epsilon > 0$, $(1 + \epsilon, 1)$ is a relative interior solution to (NASTY-D”). However, $(0, 0, 1 + \epsilon, 1)$ is not feasible for (NASTY-D’). Let s_ϵ denote the corresponding slack to $(0, 0, 1 + \epsilon, 1)$. Then, following the discussion after Proposition 5.2, if $\alpha > 0$ is large then $\alpha d'_1 + s_\epsilon$ is a feasible slack for (NASTY-D’) that has value $-1 + \epsilon$. We can then recover a feasible slack for (NASTY-D) with the same value.

Part II

Optimality conditions for nonlinear semidefinite programming via Slack Variables

Chapter 6

Slack variables, optimality conditions and numerical experiments for NSDPs

In this chapter, we will take a look at the following problem.

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & G(x) \in \mathcal{S}^m_+, \end{array} \tag{P1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $G\colon \mathbb{R}^n \to \mathcal{S}^m$ are twice continuously differentiable functions, $\mathcal{S}^m$ is the linear space of real symmetric matrices of dimension $m \times m$, and $\mathcal{S}^m_+$ is the cone of positive semidefinite matrices in $\mathcal{S}^m$.

We denote the Jordan product by $\cdot \circ \cdot$. This is the bilinear operator that takes W and Z to

$$W \circ Z \doteq \frac{WZ + ZW}{2},$$

for every $W, Z \in \mathcal{S}^m$. Recall that the positive semidefinite cone $\mathcal{S}^m_+$ is the cone of squares induced by $\circ$. That is, $\mathcal{S}^m_+ = \{Y \circ Y \mid Y \in \mathcal{S}^m\}$. This suggests that we can reformulate (P1) as

$$\begin{array}{ll} \underset{x,Y}{\text{minimize}} & f(x) \\ \text{subject to} & G(x) - Y \circ Y = 0. \end{array} \tag{P2}$$

Second Order Optimality Conditions (SOCs) for (P1) were originally derived by Shapiro in [77] and it might be fair to say that the second order analysis of (P1) is more intricated than its counterpart for classical nonlinear programs. That is one of the reasons why there is interest in alternative ways of deriving optimality conditions for (P1), see the articles by Forsgren [25] and Jarre [34]. Here, we propose a different route.

As in the nonlinear second-order cone programming case [29], if (x^*, Y^*) is a global (local) minimizer of (P2), then x^* is a global (local) minimizer of (P1). Moreover, if x^* is a global (local) minimizer of (P1), then there exists Y^* such that (x^*, Y^*) is a global (local) minimizer of (P2). Therefore, optimality conditions for (P2) induce optimality conditions for (P1) as well.

At this point we remark that SOCs for both problems are vastly different. While (P2) is a run-of-the-mill nonlinear programming problem, (P1) has nonlinear conic constraints, which are more difficult to deal with. Moreover, since the 80's, it is known that SOCs for it should include an extra term which take into account the curvature of the cone, see the papers by Kawasaki [37], Cominetti [17] and Shapiro [77].

The main contribution of this chapter is showing that, under appropriate regularity conditions, second order conditions for (P1) and (P2) are essentially the same, as we will see in Propositions 6.15, 6.16, 6.18 and 6.19. This suggests that the addition of the slack term already encapsulates most of the nonlinear structure of the cone. Of possible independent interest, we present a sharp characterization of positive semidefiniteness that takes into account the rank information, see Lemma 6.1.

We also take a look at the relation between Karush-Kuhn-Tucker (KKT) points of (P1) and (P2). For instance, if (x, Y, Λ) is a KKT point for (P2), it is not always true that (x, Λ) will be a KKT point for (P1). In this paper, we shall take a closer look at this issue and also investigate the relation between constraint qualifications for (P1) and (P2).

Finally, we present a few simple numerical experiments where NSDPs are reformulated using slack variables. Note that we are not necessarily advocating the use of slack variables and we are, instead, driven by curiosity about its computational prospects. Nevertheless, there are a few reasons why this could be interesting.

First of all, conventional wisdom would say that using squared squared slack variables is a bad idea, but, in reality, even for linear SDPs there are good reasons to (sometimes) use such variables. In [14, 15], Burer and Monteiro transform a linear SDP $\inf\{\mathrm{trace}(CX) \mid \mathcal{A}X = b, X \in \mathcal{S}^m_+\}$, with $X \in \mathcal{S}_n$ into $\inf\{\mathrm{trace}(CVV^\top) \mid \mathcal{A}VV^\top = b\}$, where V is a square matrix and trace denotes the trace map. The idea was to use a theorem, proven independently by Barvinok [5] and Pataki [62], which bounds the rank of the possible optimal solutions. By doing so, it is possible to restrict V to be a rectangular matrix instead of a square one, thus reducing the number of variables.

Furthermore, while there are a number of solvers for linear SDPs, as we move to general nonlinear programs, the situation changes drastically. Of course, there are a number algorithms proposed in the literature, such as augmented Lagrangian methods [41, 53, 81, 82], sequential semidefinite programming [21, 27, 30, 36] and interior-point methods [33, 91, 93]. See also the survey by Yamashita and Yabe [92]. Still, there is a scarcity of widely used solvers such as PENNON [41] and its open-source version PENLAB [24]. In fact, apart from PENLAB, we are not aware of any other open-source general solver that is also capable of handling NSDPs. In comparison, nonlinear programming solvers have benefited from a few more decades of polishing and they are widely available for a plethora of languages, operational systems and computer systems. So, it makes sense to check the feasibility of solving (P1) via (P2), as it enables one to use different well-tested and high-quality solvers.

6.1 Preliminaries and a sharp characterization of positive semidefiniteness

It is a well-known fact that a matrix $\Lambda \in \mathcal{S}^m$ is positive semidefinite if and only if $\langle \Lambda, W^2 \rangle \geq 0$ for all $W \in \mathcal{S}^m$. This statement is equivalent to the self-duality of the cone S^m_+. However, we get no information about the rank of Λ. In the next lemma, we give a new characterization of positive semidefinite matrices, which takes into account the rank information.

Lemma 6.1. *Let $\Lambda \in \mathcal{S}^m$. The following statements are equivalent:*

i. $\Lambda \in \mathcal{S}^m_+$,

ii. there exists $Y \in \mathcal{S}^m$ such that

$$Y \circ \Lambda = 0 \text{ and } \langle W \circ W, \Lambda \rangle > 0, \tag{6.1}$$

for every nonzero $W \in \mathcal{S}^m$ which satisfies $Y \circ W = 0$.

For any Y satisfying (6.1) *we have that* $\operatorname{rank} \Lambda = m - \operatorname{rank} Y$. *Moreover, if σ and σ' are nonzero eigenvalues of Y, then $\sigma + \sigma' \neq 0$.*

Proof. Let us prove first that *ii.* implies *i.* Since the inner product is invariant under orthogonal transformations, without loss of generality we may assume that Y is diagonal, i.e.,

$$Y = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a $k \times k$ non-singular diagonal matrix, where k is the rank of Y. We divide Λ in blocks in a similar fashion:

$$\Lambda = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix},$$

where $A \in \mathcal{S}^k, B \in \mathbb{R}^{k\times(m-k)}, C \in \mathcal{S}^{m-k}$. The Jordan product between Y and Λ can be written as

$$Y \circ \Lambda = \begin{pmatrix} D \circ A & DB/2 \\ B^\top D/2 & 0 \end{pmatrix}.$$

Since D is non-singular, $Y \circ \Lambda = 0$ implies $B = 0$. Since D is diagonal, we have

$$2(D \circ A)_{ij} = A_{ij}(D_{ii} + D_{jj}). \tag{6.2}$$

Again, because D is non-singular, it must be the case that all diagonal elements of A should be zero. We will prove that, actually, $A = 0$.

So, suppose that A_{ij} is non-zero for some i and j, with $i \neq j$. In face of (6.2), this can only happen if $D_{ii} + D_{jj} = 0$. Now, let $W \in \mathcal{S}^k$ be such that it contains two non-zero elements, $W_{ij} = 1$ and $W_{ji} = 1$. Then, $W \circ D = 0$. Moreover $W^2 = W \circ W$ is the diagonal matrix having 1 in the (i, i) entry and 1 in the (j, j) entry. Now, taking $\widetilde{W} = \left(\begin{smallmatrix} W & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we observe that $\widetilde{W} \circ Y = 0$. Moreover, $\langle \widetilde{W}^2, \Lambda\rangle = 0$, because A_{ii} and A_{jj} are zero. This contradicts our assumptions, and it follows that A must be 0. The same argument shows that $D_{ii} + D_{jj}$ is never zero, which corresponds to the statement about eigenvalues σ and σ' in the lemma.

So far, we have shown that Λ can be written as $\left(\begin{smallmatrix} 0 & 0 \\ 0 & C \end{smallmatrix}\right)$. Let us show that C is positive definite. Let $H \in \mathcal{S}^{m-k}$ be arbitrarily chosen. Then the Jordan product between $\left(\begin{smallmatrix} 0 & 0 \\ 0 & H \end{smallmatrix}\right)$ and Y is 0. It follows that $\langle H^2, C\rangle > 0$, which implies that C is positive definite. In particular, the rank of Λ is equal to the rank of C which is $m - \operatorname{rank} Y$.

Let us prove the converse. Similarly, it is enough to consider the case where Λ can be written as $\left(\begin{smallmatrix} 0 & 0 \\ 0 & C \end{smallmatrix}\right)$, with C positive definite. Then, it is enough to take $Y = \left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where E is any positive definite matrix. It follows that any matrix W which satisfies $Y \circ W = 0$, must have the shape $\left(\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right)$, for some matrix F. Since C is positive definite, it becomes clear that $\langle W \circ W, \Lambda\rangle > 0$, if W is non-zero. □

The statement about the sum of non-zero eigenvalues might seem innocuous at first, but it will be very useful in Section 6.4. For $A \in \mathcal{S}^m$, denote by $L_A : \mathcal{S}^m \to \mathcal{S}^m$ the linear operator defined by:

$$L_A(E) \doteq A \circ E,$$

for all $E \in \mathcal{S}^m$. There are many examples of invertible matrices A for which the operator L_A is non-invertible and this is essentially due to the failure of the condition on the eigenvalues[1]. The following proposition is well-known in the context of Euclidean Jordan algebras (see Proposition 1 of Sturm [80]), but we include here a short-proof nevertheless.

Proposition 6.2. *Let $A \in \mathcal{S}^m$. Then L_A is invertible if and only if $\sigma + \sigma' \neq 0$ for every pair of eigenvalues σ, σ' of A (in particular, A must be invertible).*

[1] Take $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, for example.

Proof. The statements in the proposition are all invariant under orthogonal transformations, thus before we proceed we may assume, again, that A is already diagonalized, so A_{kk} is an eigenvalue of A for every $k = 1, \ldots, m$.

Let us show that the invertibility of L_A implies the statement about the eigenvalues of A. We will do so by proving the contrapositive. Take i and j such that $A_{ii} + A_{jj} = 0$. Consider also W such that all the entries are zero except for $W_{ij} = W_{ji} = 1$. For such a W we have $A \circ W = 0$. This shows that the kernel of L_A is non-trivial and consequently, L_A is not invertible.

Reciprocally, since we assume that A is diagonal, for every $W \in \mathcal{S}^m$, we have

$$2(L_A(W))_{ij} = W_{ij}(A_{ii} + A_{jj}),$$

for all i and j. Due to the fact that $A_{ii} + A_{jj}$ is never zero, the kernel of L_A must only contain the zero matrix. Hence L_A is invertible. □

In view of Proposition 6.2, the matrix D which appears in the proof of Lemma 6.1 is such that L_D is invertible. This will play an important role when we discuss the relation between the second-order sufficient conditions (SOSCs) of problems (P1) and (P2).

Lemma 6.3. *The following statements hold.*

(a) For any matrices $A, B \in \mathbb{R}^{m\times m}$, let $\varphi\colon \mathbb{R}^{m\times m} \to \mathbb{R}$ be defined by $\varphi(Z) \doteq \operatorname{trace}(Z^\top AZB)$. Then, we have $\nabla\varphi(Z) = AZB + A^\top ZB^\top$.

(b) For any matrix $A \in \mathcal{S}^m$, let $\varphi\colon \mathcal{S}^m \to \mathbb{R}$ be defined by $\varphi(Z) \doteq \langle Z \circ Z, A\rangle$. Then, we have $\nabla\varphi(Z) = 2Z \circ A$.

(c) For any matrix $A \in \mathbb{R}^{m\times m}$ and function $\theta\colon \mathbb{R}^n \to \mathcal{S}^m$, let $\psi\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $\psi(x) = \langle\theta(x), A\rangle$. Then, we have $\nabla\psi(x) = \nabla\theta(x)^* A$.

(d) Let $A, B \in \mathcal{S}^m$. Then, they commute, i.e., $AB = BA$, if and only if A and B are simultaneously diagonalizable by an orthogonal matrix, i.e., there exists an orthogonal matrix Q such that $QAQ^\top$ and $QBQ^\top$ are diagonal.

(e) Let $A, B \in \mathcal{S}^m_+$. Then, $AB = 0$ if and only if $\langle A, B\rangle = 0$.

Proof. (a) See [6, Section 10.7].

(b) Note that $\varphi(Z) = \langle Z \circ Z, \Lambda\rangle = \frac{1}{2}\langle ZZ^\top, A\rangle + \frac{1}{2}\langle Z^\top Z, A\rangle = \frac{1}{2}\operatorname{trace}(ZZ^\top A) + \frac{1}{2}\operatorname{trace}(Z^\top ZA) = \frac{1}{2}\operatorname{trace}(Z^\top AZ) + \frac{1}{2}\operatorname{trace}(Z^\top ZA)$. Let $\varphi_1(Z) = \operatorname{trace}(Z^\top AZ)$ and $\varphi_2(Z) = \operatorname{trace}(Z^\top ZA)$. Then, from item (a), we have $\nabla\varphi_1(Z) = AZ + A^\top Z$ and $\nabla\varphi_2(Z) = ZA + ZA^\top$. Taking into account the symmetry of A, we have $\nabla\varphi_1(Z) = 2AZ$ and $\nabla\varphi_2(Z) = 2ZA$. Hence we have $\nabla\varphi(Z) = \frac{1}{2}\nabla\varphi_1(Z) + \frac{1}{2}\nabla\varphi_2(Z) = 2A \circ Z$.

(c) Observe that $\psi(x) = \langle\theta(x), A\rangle = \operatorname{trace}(\theta(x)A) = \sum_{i,j}\theta(x)_{ij}A_{ij}$ for any $x \in \mathbb{R}^n$. Then, we have

$$\nabla\psi(x) = \begin{pmatrix} \sum_{i,j}(\partial\theta(x)_{ij}/\partial x_1)A_{ij} \\ \vdots \\ \sum_{i,j}(\partial\theta(x)_{ij}/\partial x_n)A_{ij} \end{pmatrix} = \begin{pmatrix} \langle\partial\theta(x)/\partial x_1, A\rangle \\ \vdots \\ \langle\partial\theta(x)/\partial x_n, A\rangle \end{pmatrix} = \nabla\theta(x)^* A,$$

where the last equality follows from the definition of adjoint operator.

(d) See [6, Section 8.17].

(e) See [6, Section 8.12]. □

6.1.1 KKT conditions

Let the Lagrangian function $L\colon \mathbb{R}^n \times \mathcal{S}^m \to \mathbb{R}$ of problem (P1) be defined by

$$L(x, \Lambda) \doteq f(x) - \langle G(x), \Lambda \rangle.$$

We say that $(x, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m$ is a KKT pair of problem (P1) if the following conditions are satisfied:

$$\nabla_x L(x, \Lambda) = 0, \tag{P1.1}$$
$$\Lambda \succeq 0, \tag{P1.2}$$
$$G(x) \succeq 0, \tag{P1.3}$$
$$\Lambda \circ G(x) = 0, \tag{P1.4}$$

where, from Lemma 6.3(c), we have $\nabla_x L(x, \Lambda) = \nabla f(x) - \nabla G(x)^* \Lambda$. Applying the trace map at both sides of (P1.4), we see that condition (P1.4) is equivalent to $\langle \Lambda, G(x) \rangle = 0$. This result, together with the fact that $\Lambda \succeq 0$ and $G(x) \succeq 0$, shows that (P1.4) is also equivalent to $\Lambda G(x) = 0$, by Lemma 6.3(e). Moreover, the equality (P1.4) implies that Λ and $G(x)$ commute, which means, by Lemma 6.3(d), that they are simultaneously diagonalizable by an orthogonal matrix. We also have the following definition.

Definition 6.4 (Strict complementarity)**.** *If* $(x, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m$ *is a KKT pair of* (P1) *such that* $\operatorname{rank} G(x) + \operatorname{rank} \Lambda = m$*, then* (x, Λ) *is said to satisfy the* strict complementarity condition*.*

As for the equality constrained problem (P2), $(x, Y, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m$ is a KKT triple if the conditions below are satisfied:

$$\nabla_{(x,Y)} \mathcal{L}(x, Y, \Lambda) = 0,$$
$$G(x) - Y \circ Y = 0,$$

where the Lagrangian function $\mathcal{L}(x, Y, \Lambda) : \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m \to \mathbb{R}$ is given by

$$\mathcal{L}(x, Y, \Lambda) \doteq f(x) - \langle G(x) - Y \circ Y, \Lambda \rangle.$$

From Lemma 6.3(b),(c), these conditions can be written as follows:

$$\nabla f(x) - \nabla G(x)^* \Lambda = 0, \tag{P2.1}$$
$$\Lambda \circ Y = 0, \tag{P2.2}$$
$$G(x) - Y \circ Y = 0. \tag{P2.3}$$

6.1.2 Constraint Qualifications

For (P1), we say that the Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at a point x if there exists some d such that

$$G(x) + \nabla G(x) d \in \operatorname{int} \mathcal{S}^m_+,$$

where $\operatorname{int} \mathcal{S}^m_+$ denotes the interior of $\mathcal{S}^n_+$, that is, the set of symmetric positive definite matrices. If x is a local minimum for (P1), MFCQ ensures the existence of a Lagrange Multiplier Λ and that the set of multipliers is bounded. A more restrictive assumption is the nondegeneracy condition discussed in [77], where it is presented in terms of a transversality condition on the map G. However, at the end, it boils down to the following condition:

$$\mathcal{S}^m = \operatorname{lin} \operatorname{cl} \operatorname{dir} \left(G(x), \mathcal{S}^m_+ \right) + \operatorname{Im} \nabla G(x), \tag{Nondegeneracy}$$

where $\operatorname{Im} \nabla G(x)$ denotes the image of the linear map $\nabla G(x)$ and $\operatorname{lin} \operatorname{cl} \operatorname{dir} \left(G(x), \mathcal{S}^n_+\right)$ is the lineality space of the tangent cone $\operatorname{cl} \operatorname{dir} \left(G(x), \mathcal{S}^n_+\right)$, i.e, $\operatorname{lin} \operatorname{cl} \operatorname{dir} \left(G(x), \mathcal{S}^n_+\right) = \operatorname{cl} \operatorname{dir} \left(G(x), \mathcal{S}^n_+\right) \cap -\operatorname{cl} \operatorname{dir} \left(G(x), \mathcal{S}^n_+\right)$ (see, for instance, the observations on page 310 in [77]). The good thing about (Nondegeneracy) is that it ensures that Λ is unique.

For the problem (P2), a common constraint qualification is the Linear Independence Constraint Qualification (LICQ), which simply requires that the gradient of the constraints be linearly independent. In Section 6.3, we will show that LICQ and (Nondegeneracy) are essentially equivalent.

6.1.3 Second-order conditions

Since (P2) is just an ordinary equality constrained nonlinear program, second-order sufficient conditions are well-known and can be written as follows.

Proposition 6.5. *Let $(x, Y, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m$ be a KKT pair of the problem* (P2). *The second-order sufficient condition (SOSC-NLP) holds if*

$$\langle \nabla^2_x L(x, \Lambda) v, v\rangle + 2\langle W \circ W, \Lambda\rangle > 0 \tag{6.3}$$

for every non-zero $(v, W) \in \mathbb{R}^n \times \mathcal{S}^m$ such that $\nabla G(x)v - 2Y \circ W = 0$.

Proof. The second order sufficient condition for (SOSC-NLP) holds if

$$\langle \nabla^2_x \mathcal{L}(x, Y, \Lambda)(v, W), (v, W)\rangle > 0,$$

for every $(v, W) \in \mathbb{R}^n \times \mathcal{S}^m$ such that $\nabla G(x)v - 2Y \circ W = 0$, see [7, Section 3.3] or Theorem 12.6 in [61]. However, it holds that

$$\langle \nabla^2_x \mathcal{L}(x, \Lambda)(v, W), (v, W)\rangle = \langle \nabla^2_x L(x, \Lambda) v, v\rangle + 2\langle W \circ W, \Lambda\rangle.$$

□

Similarly, we have the following second order necessary condition. Note that we require LICQ to hold.

Proposition 6.6. *Let (x, Y) be a local minimum for* (P2) *and $(x, Y, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m$ be a KKT triple such that LICQ holds. Then we the following Second Order Necessary Condition (SONC) holds (SONC-NLP):*

$$\langle \nabla^2_x L(x, \Lambda) v, v\rangle + 2\langle W \circ W, \Lambda\rangle \geq 0 \tag{6.4}$$

for every $(v, W) \in \mathbb{R}^n \times \mathcal{S}^m$ such that $\nabla G(x)v - 2Y \circ W = 0$.

Proof. See Theorem 12.5 in [61]. □

SOCs for (P1) are a more delicate matter. Let (x, Λ) be KKT pair of (P1). It is true that a *sufficient* condition for optimality is that the Hessian of the Lagrangian be positive definite over the set of critical directions. However, replacing "positive definite" for "positive semidefinite" does not yield a necessary condition. Therefore, it seems that there is a gap between necessary and sufficient conditions. In order to close the gap it is necessary to add an additional term to the Lagrangian. For the theory behind this see, for instance, the papers by Kawasaki [37], Cominetti [17], and Bonnans, Cominetti and Shapiro [9]. The condition below was obtained by Shapiro in [77] and it is sufficent for (x, Λ) to be a local minimum, see Theorem 9 therein.

Proposition 6.7. *Let $(x,\Lambda) \in \mathbb{R}^n \times \mathcal{S}^m$ be a KKT pair of problem* (P1) *satisfying strict complementarity and nondegeneracy. The second-order sufficient condition (SOSC-SDP) holds if*

$$\langle (\nabla_x^2 L(x,\Lambda) + H(x,\Lambda))d, d\rangle > 0 \tag{6.5}$$

for all nonzero $d \in \mathcal{C}(x)$, where

$$\mathcal{C}(x) \doteq \left\{ d \in \mathbb{R}^n \mid \nabla G(x)d \in \operatorname{cl}\operatorname{dir}\left(G(x), \mathcal{S}_+^m\right), \langle \nabla f(x), d\rangle = 0 \right\}$$

is the critical cone at x, $\operatorname{cl}\operatorname{dir}\left(G(x), \mathcal{S}_+^m\right)$ denotes the tangent cone of $\mathcal{S}_+^m$ at $G(x)$, and

$$H(x,\Lambda)_{ij} \doteq 2\operatorname{trace}\left(\frac{\partial G(x)}{\partial x_i} G(x)^\dagger \frac{\partial G(x)}{\partial x_j}\Lambda\right), \tag{6.6}$$

for $i,j = 1,\ldots,n$. In this case, (x,Λ) is a local minimum for (P1). *Conversely, if (x,Λ) is a KKT pair satisfying strict complementarity, nondegeneracy and is a local minimum for* (P1)*, then the following second order necessary condition holds (SONC-SDP):*

$$\langle (\nabla_x^2 L(x,\Lambda) + H(x,\Lambda))d, d\rangle \geq 0 \tag{6.7}$$

for all nonzero $d \in \mathcal{C}(x)$.

6.2 Equivalence between KKT points

Proposition 6.8. *Let $(x,\Lambda) \in \mathbb{R}^n \times \mathcal{S}^m$ be a KKT pair of problem* (P1)*. Then, there exists $Y \in \mathcal{S}^m$ such that (x,Y,Λ) is a KKT triple of* (P2)*.*

Proof. Let Y be the positive semidefinite matrix satisfying $G(x) = Y \circ Y$. Let us show that (x,Y,Λ) is a KKT triple of (P2). The conditions (P2.1) and (P2.3) are immediate. We need to show that (P2.2) holds.

Recall that (P1.4) implies $G(x)\Lambda = 0$, due to Lemma 6.3(e). It means that every column of Λ lies in the kernel of $G(x)$. However, $G(x)$ and Y share exactly the same kernel, since $G(x) = Y^2$. It follows that $Y\Lambda = 0$, so that $Y \circ \Lambda = 0$. □

The converse is not always true, that is, if (x,Y,Λ) is a KKT triple of (P2) it does not follow that (x,Λ) must be a KKT pair of (P1), since Λ need not to be positive semidefinite. This, however, is the only obstacle for establishing equivalence.

Proposition 6.9. *If $(x,Y,\Lambda) \in \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m$ is a KKT triple of* (P2) *such that Λ is positive semidefinite, then (x,Λ) is a KKT tuple of* (P1)*.*

Proof. The only condition that remains to be verified is (P1.4). Due to (P2.2), we have

$$0 = \langle Y, Y \circ \Lambda\rangle = \langle Y \circ Y, \Lambda\rangle = \langle G(x), \Lambda\rangle.$$

Since $G(x)$ and Λ are both positive semidefinite, we must have $G(x) \circ \Lambda = 0$. □

The previous proposition leads us to consider conditions which ensure that Λ is positive semidefinite. It turns out that if the second order sufficient condition for (P2) is satisfied at (x,Y,Λ), then Λ is positive semidefinite. In fact, a weaker condition is enough to ensure positive semidefiniteness.

Proposition 6.10. *Suppose that $(x,Y,\Lambda) \in \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m$ is a KKT triple of* (P2) *such that Y and Λ also satisfy the assumptions of Lemma* (6.1)*, that is*

$$\langle W \circ W, \Lambda\rangle > 0, \tag{6.8}$$

for every nonzero $W \in \mathcal{S}^m$ such that $Y \circ W = 0$. Then (x,Λ) is a KKT pair of (P1) *satisfying strict complementarity (Definition 6.4).*

Proof. Due to Lemma 6.1, Λ is positive semidefinite and $\operatorname{rank} Y = m - \operatorname{rank} \Lambda$. Now, since $G(x) = Y^2$, we have that $\operatorname{rank} G(x) = \operatorname{rank} Y$. Therefore (x, Λ) must satisfy the strict complementarity condition. $\square$

Corollary 6.11. *Suppose that SOSC-NLP is satisfied at a KKT triple $(x, Y, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m$. Then (x, Λ) is a KKT pair for* (P1) *which satisfies the strict complementarity condition.*

Proof. If we take $v = 0$ in the definition of SOSC-NLP we have precisely (6.1), so the result follows from Proposition 6.10. $\square$

Proposition 6.12. *Suppose that $(x, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m$ is a KKT pair of* (P1) *which satisfies the strict complementarity condition. Then there exists some Y such that (x, Y, Λ) is a KKT triple of* (P2) *and Y, Λ satisfy* (6.1).

Proof. Without loss of generality, we may assume that $G(x)$ has the shape $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $A \in \mathcal{S}^k_+$ and $k = \operatorname{rank} G(x)$. Since $G(x)$ and Λ are both positive semidefinite, the condition $G(x) \circ \Lambda = 0$ is equivalent to $G(x)\Lambda = 0$. It follows that Λ has the shape $\left(\begin{smallmatrix} 0 & 0 \\ 0 & C \end{smallmatrix}\right)$, for some matrix $C \in \mathcal{S}^{m-k}_+$. However, the only way strict complementarity is satisfiable is if C is positive definite. Therefore, it is enough to pick Y to be the positive semidefinite matrix satisfying $Y^2 = G(x)$.

Finally, note that if $W = \left(\begin{smallmatrix} W_1 & W_2 \\ W_2^\top & W_3 \end{smallmatrix}\right)$, with $W \in \mathcal{S}^m, W_1 \in \mathcal{S}^k$, $W_2 \in \mathcal{S}^{m-k}$, then the condition $Y \circ W = 0$ together with Proposition 6.2 implies $W_1 = 0$, $W_2 = 0$. Since C is positive definite definite, we must have $\langle \Lambda, W \circ W \rangle > 0$, if $W \neq 0$. This shows that (6.1) is satisfied. $\square$

6.3 Relations between constraint qualifications

In this section, we shall show that (Nondegeneracy) is essentially equivalent to LICQ (linear independence constraint qualification) for (P2). In [77], Shapiro mentions that the nondegeneracy condition for (P1) is an analogue of sorts of LICQ, but he also states that the analogy is imperfect. For instance, when $G(x)$ is diagonal, (P1) naturally becomes an NLP, since the semidefinite constraint is reduced to nonnegativity of the diagonal. However, even in that case, LICQ and (Nondegeneracy) might not be equivalent (see page 309 of [77]). In this sense, it is interesting to see that a correspondence between the conditions can be established when (P1) is reformulated as (P2). Before that, we recall some facts about the geometry of the cone of positive semidefinite matrices.

Let $A \in \mathcal{S}^m_+$ and U be the $m \times k$ matrices whose columns form a basis for the kernel of A. Then, the tangent cone of $\mathcal{S}^m_+$ at A is the set $\operatorname{cl}\operatorname{dir}\left(A, \mathcal{S}^m_+\right) = \{E \in \mathcal{S}^m \mid U^\top E U \in \mathcal{S}^k_+\}$, see [63] or Equation 26 in [77]. For example, if A can be written as $\left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where D is positive definite, then the matrices in $\operatorname{cl}\operatorname{dir}\left(A, \mathcal{S}^m_+\right)$ have the shape $\left(\begin{smallmatrix} C & F \\ F^\top & H \end{smallmatrix}\right)$, where the only restriction is that H should be positive semidefinite.

Our first step is to notice that (Nondegeneracy) implies that the only matrix which is orthogonal to both $\operatorname{lin}\operatorname{cl}\operatorname{dir}\left(W, \mathcal{S}^n_+\right)$ and to $\operatorname{Im} \nabla G(x)$ is the trivial one. In other words:

$$W \in (\operatorname{lin}\operatorname{cl}\operatorname{dir}\left(G(x), \mathcal{S}^m_+\right))^\perp \text{ and } \nabla G(x)^* W = 0 \quad \Longrightarrow \quad W = 0. \qquad \text{(Nondegeneracy)}$$

On the other hand, the LICQ constraint qualification for (P2) holds at a feasible point (x, Y) if the linear function which maps (v, W) to $\nabla G(x)v - 2W \circ Y$ is surjective. This happens if and only if the adjoint map has trivial kernel. The adjoint map takes $W \in \mathcal{S}^m$ and maps it to $(\nabla G(x)^* W, -2W \circ Y)$. So the surjectivity assumption amounts to requiring that every W which satisfies both $\nabla G(x)^*W = 0$ and $W \circ Y = 0$ must actually be 0, that is,

$$W \circ Y = 0 \text{ and } \nabla G(x)^*W = 0 \quad \Longrightarrow \quad W = 0. \tag{LICQ}$$

The subspaces $\ker L_Y = \{W \mid Y \circ W = 0\}$ and $(\operatorname{lin} \operatorname{cl} \operatorname{dir}\left(G(x), \mathcal{S}^m_+\right))^\perp$ are closely related. The next proposition clarifies this connection.

Proposition 6.13. *Let $V = Y^2$, then $(\operatorname{lin} \operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right))^\perp \subseteq \ker L_Y$. If Y is positive semidefinite, then $\ker L_Y \subseteq (\operatorname{lin} \operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right))^\perp$ as well.*

Proof. Note that if Q is an orthogonal matrix, then $\operatorname{cl} \operatorname{dir}\left(Q^\top V Q, \mathcal{S}^m_+\right) = Q^\top \operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right) Q$. The same is true for $\ker L_Y$, i.e., $\ker L_{Q^\top Y Q} = Q^\top \ker L_Y Q$. So, without loss of generality, we may assume that Y is diagonal and that

$$Y = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is an $r \times r$ nonsingular diagonal matrix. Then, we have

$$\begin{aligned}
\operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right) &= \left\{ \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \,\middle|\, A \in \mathcal{S}^r, B \in \mathbb{R}^{r \times (m-r)}, C \in \mathcal{S}^{m-r}_+ \right\}, \\
\operatorname{lin} \operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right) &= \left\{ \begin{pmatrix} A & B \\ B^\top & 0 \end{pmatrix} \,\middle|\, A \in \mathcal{S}^r, B \in \mathbb{R}^{r \times (m-r)} \right\}, \\
(\operatorname{lin} \operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right))^\perp &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \,\middle|\, C \in \mathcal{S}^{m-r} \right\}.
\end{aligned}$$

This shows that every matrix $Z \in (\operatorname{lin} \operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right))^\perp$ satisfies $YZ = 0$ and therefore lies in $\ker L_Y$. Now, the kernel of L_Y can be described as follows:

$$\ker L_Y = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \,\middle|\, A \circ D = 0, C \in \mathcal{S}^{m-r} \right\}.$$

If Y is positive semidefinite, then D is positive definite and the operator L_D is nonsingular. Hence $A \circ D = 0$ implies $A = 0$. In this case, $\ker L_Y$ coincides with $(\operatorname{lin} \operatorname{cl} \operatorname{dir}\left(V, \mathcal{S}^m_+\right))^\perp$. □

Corollary 6.14. *If (x, Y) satisfies LICQ for (P2), then nondegeneracy is satisfied at x for (P1). On the other hand, if x satisfies nondegeneracy and if $Y = \sqrt{G(x)}$, then (x, Y) satisfies LICQ for (P2).*

6.4 Analysis of the Second Order Suficient Conditions

In this section we examine the relations that hold between KKT points of (P1) and (P2) that satisfy second order sufficient conditions.

Proposition 6.15. *Suppose that $(x, Y, \Lambda) \in \mathbb{R}^n \times \mathcal{S}^m \times \mathcal{S}^m$ is a KKT triple of (P2) satisfying SOSC-NLP. Then (x, Λ) is a KKT pair of (P1) that satisfies strict complementarity and Equation (6.5).*

If additionally, (x, Y, Λ) satisfies LICQ for (P2) or (x, Y) satisfies nondegeneracy for (P1), then SOSC-SDP is satisfied as well.

Proof. In Corollary 6.11, we have already shown that (x, Λ) is a KKT pair and strict complementarity is satisfied. Let us show that Equation (6.5) is also satisfied. So, consider an arbitrary nonzero $d \in \mathbb{R}^n$ such that $\nabla G(x)d \in \operatorname{cl}\operatorname{dir}\big(G(x), \mathcal{S}^m_+\big)$ and $\langle \nabla f(x), d\rangle = 0$. We are thus required to show that

$$\langle (\nabla^2_x L(x,\Lambda) + H(x,\Lambda))d, d\rangle > 0, \tag{6.9}$$

where $H(x,\Lambda)$ was defined in (6.6). A first observation is that due to (P1.1), we have $\langle \nabla G(x)d, \Lambda\rangle = \langle d, \nabla G(x)^*\Lambda\rangle = \langle d, \nabla f(x)\rangle = 0$, that is $\nabla G(x)d \in \{\Lambda\}^\perp$. We recall that $H(x,\Lambda)$ satisfies:

$$\langle H(x,\Lambda)d, d\rangle = 2\langle \Lambda, (\nabla G(x)d)^\top G^\dagger(x)\nabla G(x)d\rangle.$$

The strategy here is to first identify the shape and properties of the several matrices involved before showing that (6.9) holds. Without loss of generality we may assume that $G(x)$ is diagonal, i.e,

$$G(x) = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a diagonal positive definite $k \times k$ matrix. We also have

$$Y = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

with E an invertible diagonal matrix such that $E^2 = D$. Since SOSC-NLP holds, considering $v = 0$ in (6.3), we obtain $\langle W \circ W, \Lambda\rangle > 0$ for all nonzero $W \in \mathcal{S}^m$ such that $W \circ Y = 0$. From (P2.2), we also have $Y \circ \Lambda = 0$, so (6.1) is satisfied. Thus, Lemma 6.1 and Proposition 6.2 ensure that L_E is an invertible operator. Moreover, due to the strict complementarity condition, we obtain

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix},$$

where $\Gamma \in \mathcal{S}^{n-k}_+$ is positive definite. The pseudo-inverse of $G(x)$ is given by

$$G(x)^\dagger = \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

We divide $\nabla G(x)d$ in blocks in the following fashion:

$$\nabla G(x)d = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix},$$

where $A \in \mathcal{S}^k$, $B \in \mathbb{R}^{k\times(n-k)}$ and $C \in \mathcal{S}^{n-k}$. Inasmuch as $\nabla G(x)d$ lies in the tangent cone $\operatorname{cl}\operatorname{dir}\big(G(x), \mathcal{S}^m_+\big)$, C must be positive semidefinite . However, $\langle \nabla G(x)d, \Lambda\rangle = \langle C, \Gamma\rangle = 0$. Since Γ is positive definite, this implies $C = 0$. So that

$$\nabla G(x)d = \begin{pmatrix} A & B \\ B^\top & 0 \end{pmatrix}.$$

We are now ready to show that (6.9) holds and we shall do that by considering $v = d$ in (6.3) and exhibiting some W such that $\nabla G(x)d - 2Y \circ W = 0$ and $2\langle W \circ W, \Lambda\rangle = \langle H(x,\Lambda)d, d\rangle$. Then SOSC-NLP will ensure that (6.9) holds. Note that for any $Z \in \mathcal{S}^{m-k}$ we have that

$$W_Z = \begin{pmatrix} L_E^{-1}A/2 & E^{-1}B \\ B^\top E^{-1} & Z \end{pmatrix}$$

is a solution to the equation $\nabla G(x)d - 2Y \circ W = 0$. Also, any solution to that equation must have this particular shape. We will now show how to select Z in order to ensure that $2\langle W \circ W, \Lambda\rangle = \langle H(x,\Lambda)d, d\rangle$ holds.

$$\begin{aligned}
2\langle W_Z^2, \Lambda\rangle - 2\langle (\nabla G(x)d)^\top G^\dagger(x)\nabla G(x)d, \Lambda\rangle &= 2\langle Z^2 + B^\top E^{-2}B, \Gamma\rangle - 2\langle B^\top D^{-1}B, \Gamma\rangle \\
&= \langle 2Z^2 + 2B^\top D^{-1}B - 2B^\top D^{-1}B, \Gamma\rangle \\
&= \langle 2Z^2, \Gamma\rangle \\
&\geq 0 \qquad (6.10)
\end{aligned}$$

Taking Z $= 0$, we have $2\langle W_Z \circ W_Z, \Lambda\rangle = \langle H(x,\Lambda)d, d\rangle$. □

Proposition 6.16. *Suppose that (x,Λ) is a KKT point for* (P1) *satisfying Equation* (6.5) *and strict complementarity, then there is Y such that (x,Y,Λ) is a KKT triple for* (P2) *satisfying SOSC-NLP.*

Proof. Again, we assume that $G(x)$ is diagonal, so that

$$G(x) = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a diagonal positive definite $k \times k$ matrix. Take Y such that

$$Y = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

where $E^2 = G(x)$ and E is positive definite (in particular L_E is invertible). Then (x,Y,Λ) is a KKT point for (P2). Due to complementary slackness and strict complementarity, we have

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix},$$

where $\Gamma \in \mathcal{S}_+^{m-k}$ is positive definite. We are required to show that

$$\langle \nabla_x^2 L(x,\Lambda)v, v\rangle + 2\langle W \circ W, \Lambda\rangle > 0, \qquad (6.11)$$

for every non-zero (v,W) such that $\nabla G(x)v - 2Y \circ W = 0$. So, let (v,W) satisfy $\nabla G(x)v - 2Y \circ W = 0$. Let us first consider what happens when $v = 0$. Dividing W in blocks we have:

$$Y \circ \begin{pmatrix} W_1 & W_2 \\ W_2^\top & W_3 \end{pmatrix} = \begin{pmatrix} E \circ W_1 & EW_2/2 \\ W_2^\top E/2 & 0 \end{pmatrix}.$$

So $Y \circ W = 0$, implies $W_1 = 0$ (recall that L_E is invertible) and $W_2 = 0$. If W is not 0, then W_3 must be non-zero. Hence, $W_3 \circ W_3$ must also be non-zero. We have $\langle W \circ W, \Lambda\rangle = \langle W_3 \circ W_3, \Gamma\rangle$. But $\langle W_3 \circ W_3, \Gamma\rangle$ must be greater than 0, since Γ is positive definite. So, in this case, (6.11) is satisfied.

Now, we suppose that v is non-zero. First, we would like to show that $\nabla G(x)v$ lies in the tangent cone $\operatorname{cl}\operatorname{dir}\left(G(x), \mathcal{S}_+^m\right)$ and that $\nabla G(x)v$ is orthogonal to Λ. This will ensure that v lies in the critical cone $\mathcal{C}(x)$.

Note that the image of the operator L_Y only contains matrices having the lower right $(m-k) \times (m-k)$ block equal to 0. Therefore, $\nabla G(x)v = 2Y \circ W$ implies that $\nabla G(x)v$ has the shape

$$\nabla G(x)v = \begin{pmatrix} A & B \\ B^\top & 0 \end{pmatrix}.$$

Hence, $\nabla G(x)v \in \operatorname{cl}\operatorname{dir}\left(G(x), \mathcal{S}_+^m\right)$ and $\nabla G(x)v$ is orthogonal to Λ. Due to SOSC-SDP, we must have

$$\langle (\nabla_x^2 L(x,\Lambda) + H(x,\Lambda))v, v\rangle > 0,$$

so if we show that $\langle H(x,\Lambda)v,v\rangle \leq 2\langle W\circ W,\Lambda\rangle$, then this is enough to show (6.11) must hold. However, since W satisfies $\nabla G(x)v - 2Y\circ W = 0$, the same chain of equalities/inequalities finishing at (6.10) already asserts that $\langle H(x,\Lambda)v,v\rangle \leq 2\langle W\circ W,\Lambda\rangle$.

□

Here, we remark one interesting consequence of the analysis above. The second order *sufficient* condition for NSDPs in [77] is stated under the assumption that the pair (x,Λ) satisfies both strict complementarity and nondegeneracy. However, since (P1) and (P2) share the same local minima, Proposition 6.16 implies that we may remove the nondegeneracy assumption from SOSC-SDP. We now state a sufficient condition for (P1) based on the analysis above.

Proposition 6.17 (A Sufficient Condition via Slack Variables)**.** *Suppose that* (x,Λ) *is a KKT pair for* (P1) *satisfying strict complementarity. Suppose also that the following condition holds.*

$$\langle \nabla_x^2 L(x,\Lambda)v,v\rangle + 2\langle W\circ W,\Lambda\rangle > 0 \tag{6.12}$$

for every non-zero $(v,W)\in\mathbb{R}^n\times\mathcal{S}^m$ *such that* $\nabla G(x)v - 2\sqrt{G(x)}\circ W = 0$*. Then* x *is a local minimum for* (P1)*.*

Apart from the detail of requiring nondegeneracy, the condition above is equivalent to SOSC-SDP, due to Propositions 6.15 and 6.16.

6.5 Analysis of the Second Order Necessary Conditions

We now take a look at the difference between second order necessary conditions that can be derived from (P1) and (P2). Since the inequalities (6.4) and (6.7) are not strict, we need slightly stronger assumption to prove the next proposition.

Proposition 6.18. *Suppose that* $(x,Y,\Lambda)\in\mathbb{R}^n\times\mathcal{S}^m\times\mathcal{S}^m$ *is a KKT triple of* (P2) *satisfying SONC-NLP and such that* Y,Λ *are positive semidefinite. If* (x,Λ) *is a KKT pair satisfying strict complementarity for* (P1)*, then it also satisfies SONC-SDP.*

Proof. Since (x,Y,Λ) satisfies LICQ and Y is positive semidefinite, then Corollary 6.14 implies that (x,Λ) satisfies nondegeneracy. Under the assumption that (x,Λ) is strict complementary, the only thing missing is proving that (6.7) holds. To do that, we proceed as in Proposition (6.15). We divide $G(x),\Lambda$, $G(x)^\dagger$ and $\nabla G(x)d$ in blocks in the exact same way. The only difference is that since (6.4) does not hold strictly, we cannot make use of Lemma 6.1 in order to conclude that L_E is invertible. However, since we assumed that Y is positive semidefinite, all the eigenvalues of E are strictly positive anyway. So, as before, we can conclude that L_E is an invertible operator, by Proposition 6.2. Due to strict complementarity, we can also conclude that $\Gamma\in\mathcal{S}_+^{n-k}$ is positive definite and that $C=0$. All our ingredients are now in place and we can proceed exactly as in the proof of Proposition 6.15. Namely, we have to prove that given $d\in\mathcal{C}(x)$, the inequality $\langle(\nabla_x^2 L(x,\Lambda)+H(x,\Lambda))d,d\rangle\geq 0$ holds. As before, the way to go is to craft a matrix W satisfying both $\nabla G(x)d - 2Y\circ W = 0$ and $\langle H(x,\Lambda)d,d\rangle = 2\langle W\circ W,Z\rangle$. Then SONC-NLP will ensure that (6.7) holds. It is enough to take

$$W = \begin{pmatrix} L_E^{-1}(A/2) & E^{-1}B \\ B^\top E^{-1} & 0 \end{pmatrix}$$

and follow the same computations that lead to (6.10). □

Proposition 6.19. *Suppose that* (x,Λ) *is a KKT pair for* (P1) *satisfying SONC-SDP, then there is* Y *such that* (x,Y,Λ) *is a KKT triple for* (P2) *satisfying SONC-NLP.*

Proof. It is enough to choose Y to be $\sqrt{G(x)}$. If we do so, Corollary 6.14 ensures that (x, Y, Λ) satisfies LICQ. We now have to check that (6.4) holds. For this, we can follow the proof of Proposition 6.16 by considering Equation (6.7) instead of (6.5). No special considerations are needed for this case. □

Suppose that (x, Λ) is a KKT pair of (P1) satisfying nondegeneracy and strict complementarity. Then Proposition 6.18 gives an elementary route to prove that SONC-SDP holds. This is because if we select Y to be the positive semidefinite square root of $G(x)$, all the conditions of Proposition 6.18 are satisfied, which means that (6.7) must hold. Morover, if we were to derive second-order necessary conditions for (P1) from scratch, we could consider the following.

Proposition 6.20 (A Necessary Condition via Slack Variables). *Suppose that* (x, Λ) *is a KKT pair for* (P1) *satisfying strict complementarity, nondegeneracy and suppose that* x *is a local minimum. Then the following condition holds:*

$$\langle \nabla_x^2 L(x, \Lambda)v, v\rangle + 2\langle W \circ W, \Lambda\rangle \geq 0 \tag{6.13}$$

for every non-zero $(v, W) \in \mathbb{R}^n \times \mathcal{S}^m$ *such that* $\nabla G(x)v - 2\sqrt{G(x)} \circ W = 0$.

Propositions 6.18 and (6.7) ensure that the condition above is equivalent to SONC-SDP. Comparing Proposition 6.17 and 6.20, we see that the second order conditions derived through the aid of slack variables have "no-gap" in the sense that, apart from regularity conditions, the only difference between them is the change from ">" to "≥".

6.6 Computational Experiments

We tested the slack variables approach in a few simple problems. Our solver of choice was PENLAB [24], which is based on PENNON [41] and uses an algorithm based on the augmented lagrangian technique . As far as we know, PENLAB is the only open-source general nonlinear programming solver capable of handling nonlinear SDP constraints. Because of that, we have the chance of comparing the "native" approach with the slack approach using the same code. We ran PENLAB with the default parameters. All the tests were done in a notebook with the following specs: Ubuntu 14.04, CPU Intel i7-4510U with 4 cores operating at 2.0Ghz and 4GB of RAM.

In order to use an NLP solver to tackle (P1), we have to select a vectorization strategy. We decided to vectorize an $n \times n$ symmetric matrix by transforming it into an $\frac{n(n+1)}{2}$ vector, such that the columns of the lower triangular part are stacked one on top of the other. For instance, $\left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ is transformed to the column vector $(1, 2, 3)^\top$.

6.6.1 Modified Hock-Schittkowski problem 71

There is a known suite of problems for testing nonlinear optimization problems collected by Hock and Schittkowski [31, 76]. The problem below is a modification of problem 71 of [76] and it comes together with PENLAB. Both the constraints and the objective function

Table 6.1: Slack vs "native" for (HS)

	functions	gradients	Hessians	iterations	time (s)	opt. value
slack	110	57	44	13	0.54	87.7105
native	123	71	58	13	0.57	87.7105

are nonconvex. The problem has the following formulation:

$$\begin{array}{ll} \underset{x\in\mathbb{R}^6}{\text{minimize}} & x_1x_4(x_1+x_2+x_3)+x_3 \\ \text{subject to} & x_1x_2x_3x_4 - x_5 - 25 = 0, \\ & x_1^2+x_2^2+x_3^2+x_4^4 - x_6 - 40 = 0, \\ & \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ x_2 & x_4 & x_2+x_3 & 0 \\ 0 & x_2+x_3 & x_4 & x_3 \\ 0 & 0 & x_3 & x_1 \end{pmatrix} \in \mathcal{S}_+^4, \\ & 1 \le x_i \le 5, \qquad i = 1,2,3,4 \\ & x_i \ge 0, \qquad i = 5,6. \end{array} \tag{HS}$$

We reformulate the problem (HS) by removing the positive semidefiniteness constraints and adding a squared slack variable Y. We then test both formulations using PENLAB. The initial point is set to be $x = (5,5,5,5,0,0)$ and the slack variable to be the identity matrix $Y = I_4$. This produces infeasible points for both formulations. Nevertheless, PENLAB was able to solve the problem via both approaches. The results can be seen in Table 6.1. The first three columns count the numbers of evaluations of the augmented Lagrangian function, its gradients and its Hessians, respectively. The fourth column is the number of outer iterations. The "time" column indicates the time in seconds as measured by PENLAB. The last column indicates the optimal value obtained. It seems that there were no significant differences in performance between both approaches.

6.6.2 The closest correlation matrix problem - simple version

Given an $m \times m$ symmetric matrix H with diagonal entries equal to one, we want to find the element in $\mathcal{S}_+^m$ which is closest to H and has all diagonal entries also equal to one. The problem can be formulated as follows:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \langle X - H, X - H\rangle \\ \text{subject to} & X_{ii} = 1 \qquad \forall i, \\ & X \in \mathcal{S}_+^m. \end{array} \tag{Cor}$$

This problem is convex and due to its structure, we can use slack variables without increasing the number of variables. We have the following formulation.

$$\begin{array}{ll} \underset{X}{\text{minimize}} & \langle (X\circ X) - H, (X \circ X) - H\rangle_F \\ \text{subject to} & (X\circ X)_{ii} = 1 \qquad \forall i \\ & X \in S^m. \end{array} \tag{Cor-Slack}$$

In our experiments, we generated 100 symmetric matrices H such that the diagonal elements are all 1 and other elements are uniform random numbers between -1 and 1. For both Cor and Cor-Slack, we used $X = I_m$ as an initial solution in all instances. We solved problems with $m = 5, 10, 15, 20$ and the results can be found in Table 6.2. The columns "mean", "min" and "max" indicate, respectively, the mean, minimum and maximum of

Table 6.2: Comparison between Cor and Cor-Slack

	Cor-Slack			Cor		
m	mean (s)	min (s)	max (s)	mean (s)	min (s)	max (s)
5	0.090	0.060	0.140	0.201	0.130	0.250
10	0.153	0.120	0.230	0.423	0.330	0.630
15	0.287	0.210	0.430	1.306	1.020	1.950
20	0.556	0.450	1.180	3.491	2.820	4.990

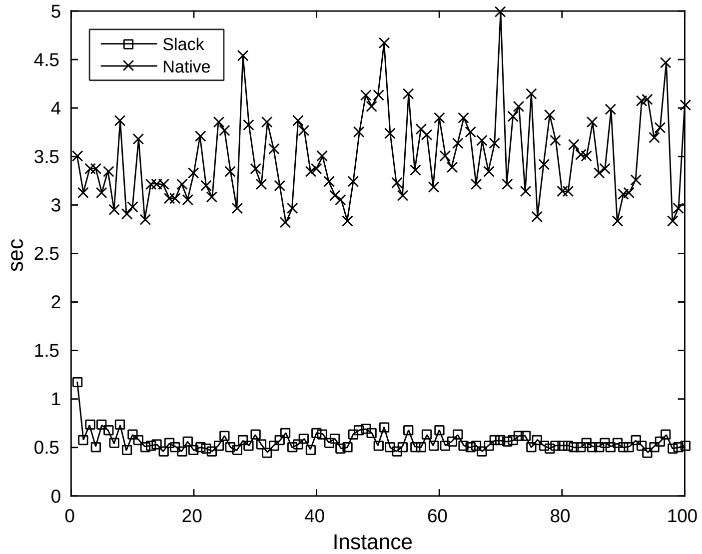

Figure 6.1: Cor vs. Cor-Slack. Instance-by-instance running times for $m = 20$.

the running times in seconds of all instances. For this problem, both formulations were able to solve all instances. We included the "mean time" column just to give an idea about the magnitude of the running time. In reality, for fixed m, the running time oscillated highly among different instances, as can be seen by the difference between the maximum and the minimum running times. We noted no significant difference between the optimal values obtained from both formulations.

We tried, as much as possible, to implement gradients and Hessians of both problems in a similar way. As Cor is an example that comes with PENLAB, we also performed some minor tweaks to conform to that goal. Performance-wise, the formulation Cor-Slack seems to be competitive for this example. In most instances, Cor-Slack had a faster running time. In Figure 6.1, we show the comparison between running times, instance-by-instance, for the case $m = 20$.

6.6.3 The closest correlation matrix problem - extended version

We consider an extended formulation for Cor as suggested in one of PENLAB's examples, with extra constraints to bound the eigenvalues of the matrices:

$$\begin{array}{ll} \underset{X,z}{\text{minimize}} & \langle zX - H, zX - H\rangle \\ \text{subject to} & zX_{ii} = 1 \qquad \forall i, \\ & I_m \preceq X \preceq \kappa I_m, \end{array} \tag{Cor-Ext}$$

where κ is some positive number greater than 1 and the notation $X \succeq \kappa I_m$ means $X - \kappa I_m \in \mathcal{S}^m_+$. This is a nonconvex problem, and using slack variables, we obtain the following

Table 6.3: Comparison between Cor-Ext and Cor-Ext-Slack

	Cor-Ext-Slack				Cor-Ext			
m	mean (s)	min (s)	max (s)	fail	mean (s)	min (s)	max (s)	fail
5	0.236	0.130	0.830	15	0.445	0.250	2.130	1
10	0.741	0.420	2.580	3	1.206	0.580	7.300	0
15	4.651	2.090	26.96	15	3.809	1.960	14.12	0
20	24.32	15.20	69.34	8	9.288	5.150	36.81	0

formulation:

$$\begin{array}{ll} \underset{X,Y_1,Y_2,z}{\text{minimize}} & \langle zX - H, zX - H\rangle \\ \text{subject to} & zX_{ii} = 1 \qquad \forall i, \\ & \kappa I_m - X = Y_1 \circ Y_1, \\ & X - I_m = Y_2 \circ Y_2. \end{array} \tag{Cor-Ext-Slack}$$

In our experiments, we set $\kappa = 10$. As before, we generated 100 symmetric matrices H whose diagonal elements are all 1 and other elements are uniform random numbers between -1 and 1. For Cor-Ext, we used $z = 1$ and $X = I_m$ as initial points. For Cor-Ext-Slack, we used an infeasible starting point $z = 1$, $X = Y_2 = I_m$ and $Y_1 = 3I_m$. We solved problems with $m = 5, 10, 15, 20$ and the results can be found in Table 6.3. The columns have the same meaning as in Section 6.6.2. This time, we saw a higher failure rate for the formulation Cor-Ext-Slack. We tried a few different initial points, but the results stayed mostly the same. The best results were obtained for the case $m = 5$ and $m = 10$, where Cor-Ext-Slack had a performance comparable to Cor-Ext, although the latter seldom failed. For $m = 15$ and $m = 20$, Cor-Ext-Slack was slower than Cor-Ext, but it was still able to solve the majority of instances. In Figure 6.2, we show the comparison of running times, instance-by-instance, for the cases $m = 10$ and $m = 20$.

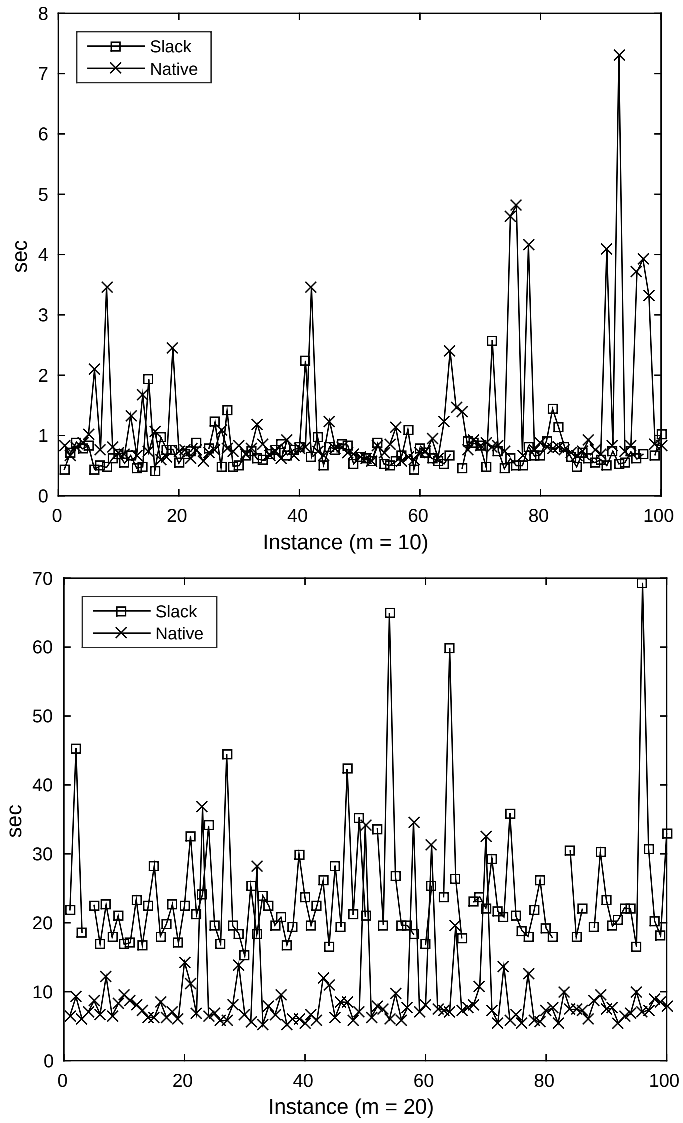

Figure 6.2: Cor-Ext vs Cor-Ext-Slack. Instance-by-instance running times for $m = 10$ and $m = 20$. Failures are represented by omitting the corresponding running time.

Part III

Epilogue

Chapter 7

Concluding remarks

In this thesis we examined topics connected to both *conic linear programming* and *nonlinear semidefinite programming*. In this chapter, we will point out directions of future inquiry for both parts.

7.1 On Part I

In this part, we discussed several examples of ill behavior in conic linear programming and ways of guarding against it. This included a discussion of FRA-poly, a facial reduction algorithm suited for directed products of cones. We also did an analysis of weak infeasibility, problems with unattained optimal value and this culminated in an approach for "completely solving" conic linear programs in Chapter 5. Here we outline two possible paths for future works, the first one is theoretical and the other one has a more practical character.

7.1.1 A few remaining theoretical issues

Although we have not discussed in detail, facial reduction can be used to obtain what is called *extended duals* for conic linear programs. The idea is to replace the usual Lagrangian dual for another one that behaves nicely even in the absence of regularity conditions. One important requirement is that in order to write the extended dual down, it should require no extra computations and the dual must be written completely in terms of the input data. This was first done by Ramana in [72] for SDPs. Ramana's dual has a number of favorable properties: i) it is written in terms of problem data; ii) has polynomial size; iii) it is always attained if the primal problem is feasible and bounded and, in that case, there is no duality gap, iv) it induces theorems of alternative similar to Farkas' lemma.

The current understanding is that Ramana's extended dual contains implicitly the constraints that define the reducing directions that are found through facial reduction. This can be seen, for instance, in the analysis made by Ramana, Tunçel and Wolkowicz [73] and the work by Pataki [66]. However, due to clever manipulations these constraints can be written as semidefinite constraints.

The extension of this approach to other cones contains a few subtle issues. Pataki [66] and, later, Liu and Pataki [44], in particular, show how to extend this approach to arbitrary closed convex cones. However, one important detail is that their extended dual might leave the problem class under consideration. A similar issue appears in the article by Pólik and Terlaky [69], where they show how to obtain extended duals for problems over *symmetric* cones. However, the obtained systems were problems over *homogeneous* cones, which properly contains the class of symmetric cones. So, again, there seems to be a need to go a more "expressive" cone in order to obtain an extended dual. In this sense, it is still a bit mysterious why there is no such need in the positive semidefinite case.

Even in the case where $\mathcal{K}$ is a single Lorentz cone $\mathcal{Q}^n$, it is not known whether it is possible to obtain something similar to Ramana's dual but written completely in terms of second order cone constraints. This is upsetting because the conventional Lagrangian dual for problems over $\mathcal{Q}^n$ almost satisfy the properties we expect from extended duals. For instance, Shapiro and Nemirovski showed that if $\mathcal{K} = \mathcal{Q}^n$ and both (D) and (P) are feasible, then there is no duality gap between (D) and (P). Note that there is no guarantee of attainment at neither side. We believe it is important to clarify these issues.

We remark that there is a notion of rank for homogeneous cone and, in particular, Lorentz cones $\mathcal{Q}^n$ have rank 2, if $n \geq 2$. In [69], Pólik and Terlaky show that if an extended dual for $\mathcal{Q}^n$ is constructed based on facial reduction, then the resulting cone $\widetilde{\mathcal{K}}$ appearing at the dual problem has rank 3. Due to a classification result for homogeneous cones, it follows that $\widetilde{\mathcal{K}}$ can neither be a *single* Lorentz cone nor a direct product of Lorentz cones with $n \geq 2$. Still, as far as we understand the situation, this result does not preclude the possibility of $\widetilde{\mathcal{K}}$ being representable as, say, the intersection of a subspace and a direct product of Lorentz cones. Therefore, there is still some hope that an extended dual for $\mathcal{Q}^n$ could be obtained by using only second order cone constraints.

Nevertheless, the prospects of obtaining an extended dual for $\mathcal{Q}^n$ that is also a second order cone problem in some sense via the usual facial reduction approach seems a bit grim. Because of that, it could be profitable to turn our attentions to other strategies. For instance, Klep and Schweighofer [40] also developed an extended dual for SDPs that can also be cast as an SDP, but they used the machinery of real algebraic geometry instead. Currently, it is not known the connection, if any, between their approach and Ramana's dual. It is also not known whether the approach in [40] can be carried out for other cones. If it is truly disconnected from facial reduction, then it might not suffer from the limitations described in [69].

7.1.2 Practical issues

Most of the results we discussed here were theoretical. There are many difficulties one must face when doing facial reduction in practice. As a consequence, since facial reduction is such a central part of our analysis, we cannot hope to do much without good facial reduction implementations.

Our assessment of the zeitgeist is that there is no question that facial reduction (FR) is a valuable theoretical tool. After all, FR can be used to prove a wealth of results, it is the theoretical backing for extended duality and it would be hard to have infeasibility certificates for general cones without it. Moreover, it is the only known way of reducing a feasible general conic linear problem to the strongly feasible case, considering that conic expansion is just "dual facial reduction" see [55] and section 4 of [89]. However, when it comes to practical aspects, there is some skepticism about its applicability. We will now discuss some of the surrounding criticism.

First of all, facial reduction is considered to be very costly. While this is definitely true, we should point out that, without facial reduction, we have no hope of solving particularly nasty problems, since they fail to satisfy important regularity conditions. A variant of this criticism is saying that facial reduction could be as costly as solving the original problem. However, there are two important points to keep in mind. The first is, again, that without FR the problem might not even be solvable in the first place. And the second point is that as shown in Chapter 3, FR can be carried out by a solving a sequence of problems that are always primal and dual strongly feasible.

In spite of that, the computation of the reducing directions is still a complicated issue. We will illustrate the difficulty with the case $\mathcal{K} = \mathcal{S}^n_+$. Suppose that we have a reducing direction $d \in \mathcal{S}^n_+ \cap \ker \mathcal{A}$ of rank k with $\langle c, d \rangle = 0$. Then, if we apply an appropriate transformation to (D), we may assume that

$$d = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where $A \in \mathcal{S}^k_+$ is positive definite. We then replace $\mathcal{S}^n_+$ by the face

$$\mathcal{S}^n_+ \cap \{d\}^\perp = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \in \mathcal{S}^n \,\middle|\, C \in \mathcal{S}^{n-k}_+ \right\}.$$

The problem is that, in practice, since d will be obtained by solving numerically an SDP, it is very likely that there will be no actual zero eigenvalues and, instead, there will be many eigenvalues with small norm. Therefore, we have to take in consideration the numerical inaccuracies when computing the rank of d and its eigenvalues. If we are not careful enough and the rank of d becomes larger than what "it should be", the face $\mathcal{S}^n_+ \cap \{d\}^\perp$ gets smaller and we might end up removing feasible solutions, which is clearly undesirable. A naive approach would be to truncate all the eigenvalues of d that are smaller than a certain ϵ. However, we do not know any bounds on ϵ that would ensure that this truncation strategy does not remove feasible solutions.

Nevertheless, this issue can be addressed in a few different ways. One is to do a very careful numerical analysis of facial reduction as done by Cheung, Schurr and Wolkowicz in [16]. Their analysis and the algorithms described therein guarantee that the reducing directions obtained are correct for some problem close to the original one. This is, of course, a very nice result but it seems that for problems that are ill-behaved, nearby problems could have widely different feasibility properties.

We can also avoid the problem altogether and try to compute the reducing directions exactly. Although this task seems hard in general, it might be possible for certain problems with additional structure, such as the sensor network localization problem in the work by Krislock and Wolkowicz [42]. See also Section 4.1 of the article by Drusvyatskiy, Pataki and Wolkowicz [20].

Another possibility is to go for partial facial reduction strategies, such as the one described by Permenter and Parrilo [68]. The idea is to substitute $\mathcal{K}$ by a larger cone $\widetilde{\mathcal{K}}$ when doing facial reduction. Since $\mathcal{K} \subseteq \widetilde{\mathcal{K}}$, we will have $\widetilde{\mathcal{K}}^* \subseteq \mathcal{K}^*$, which means that if we use $\widetilde{\mathcal{K}}$ in $(D_{\mathcal{K}})$ and $\widetilde{\mathcal{K}}^*$ in $(P_{\mathcal{K}})$, we will still obtain a valid reducing direction for (D). The advantage of doing so is that it might be easier to solve problems over $\widetilde{\mathcal{K}}$. In particular, Permenter and Parrilo suggest the usage of polyhedral cones so that the search for reducing directions can be carried out by linear programming, which means that they can be computed exactly. The caveat is that partial facial reduction may fail to find the minimal face $\mathcal{F}^D_{\min}$. In spite of that, the numerical experiments shown in [68] are encouraging and suggest that this is a viable approach for simplifying SDPs. Interestingly, in many cases where solvers fail to detect infeasibility, if the problem is first preprocessed by their technique, the solvers then become able to detect infeasibility, see Section 7.7 therein. We remark that this does not always work, as evidenced by the numerical experiments done by Liu and Pataki in Section 7 of [44].

It is still an open issue how to do facial reduction effectively for general problems and this is a topic we plan to explore in the future.

7.2 On Part II

In this part, we have shown that the optimality conditions for a nonlinear semidefinite program (P1) and its reformulation with slack variables (P2) are essentially the same. One intriguing part of this connection is the fact that the addition of squared slack variables seems to be enough to capture a great deal of the structure of $\mathcal{S}^m_+$. The natural progression from here is to expand the results to symmetric cones. In this work, we already saw some results that have a distinct Jordan-algebraic flavour, such as Lemma 6.1. It would be interesting to see how these results can be further extended and, whether clean proofs can be obtained without recoursing to the classification of simple Euclidean Jordan algebras.

As for the computational results, we found it mildly surprising that the slack variables approach was able to outperform the "native" approach in many instances. This warrants a deeper investigation of whether this could be a reliable tool for attacking NSDPs that are not linear. These are precisely the ones that are not covered by the earlier work done by Burer and Monteiro [14, 15].

Another thing that came to our minds was whether it is possible to generalize Burer and Monteiro [14, 15] approach to a nonlinear setting. Although, in general, we do not expect to find low rank optimal solutions to NSDPs, we might still find low-rank KKT multipliers. This could be explored in primal-dual approaches for NSDPs and this is a topic we are currently investigating.

Bibliography

[1] Robert A. Abrams. Projections of convex programs with unattained infima. *SIAM Journal on Control*, 13(3):706–718, 1975.

[2] Mustafa Akgül. On polyhedral extension of some LP theorems. *Mathematical Programming*, 30(1):112–120, 1984.

[3] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Mathematical Programming*, 95(1):3–51, 2003.

[4] Naohiko Arima, Sunyoung Kim, Masakazu Kojima, and Kim-Chuan Toh. Lagrangian-conic relaxations, part i: A unified framework and its applications to quadratic optimization problems. Technical Report B-475, Tokyo Institute of Technology, 2014.

[5] A.I. Barvinok. Problems of distance geometry and convex properties of quadratic maps. *Discrete & Computational Geometry*, 13(1):189–202, 1995.

[6] D. S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas.* Princeton University Press, 2nd edition, 2009.

[7] D. P. Bertsekas. *Nonlinear Programming.* Athena Scientific, 2nd edition, 1999.

[8] Robert G. Bland, Donald Goldfarb, and Michael J. Todd. The ellipsoid method: A survey. *Operations Research*, 29(6):1039–1091, 1981.

[9] J. Frédéric Bonnans, Roberto Cominetti, and Alexander Shapiro. Second order optimality conditions based on parabolic second order tangent sets. *SIAM Journal on Optimization*, 9(2):466–492, 1999.

[10] J. Frederic Bonnans and Alexander Shapiro. *Perturbation Analysis of Optimization Problems.* Springer Science & Business Media, 2000.

[11] Jonathan M. Borwein and Adrian S. Lewis. *Convex Analysis and Nonlinear Optimization: Theory and Examples.* Springer, New York, 2nd edition, November 2005.

[12] Jonathan M. Borwein and Henry Wolkowicz. Facial reduction for a cone-convex programming problem. *Journal of the Australian Mathematical Society (Series A)*, 30(03):369–380, 1981. `doi:10.1017/S1446788700017250`.

[13] Jonathan M. Borwein and Henry Wolkowicz. Regularizing the abstract convex program. *Journal of Mathematical Analysis and Applications*, 83(2):495 – 530, 1981.

[14] Samuel Burer and Renato D.C. Monteiro. A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. *Mathematical Programming*, 95(2):329–357, 2003.

[15] Samuel Burer and Renato D.C. Monteiro. Local minima and convergence in low-rank semidefinite programming. *Mathematical Programming*, 103(3):427–444, 2005.

[16] Yuen-Lam Cheung, Simon Schurr, and Henry Wolkowicz. Preprocessing and regularization for degenerate semidefinite programs. In *Computational and Analytical Mathematics*, volume 50 of *Springer Proceedings in Mathematics & Statistics*, pages 251–303. Springer New York, 2013.

[17] Roberto Cominetti. Metric regularity, tangent sets, and second-order optimality conditions. *Applied Mathematics and Optimization*, 21(1):265–287, 1990.

[18] G.B. Dantzig. *Linear Programming and Extensions.* Landmarks in Physics and Mathematics. Princeton University Press, 1959.

[19] Etienne de Klerk, Tamás Terlaky, and Kees Roos. Self-dual embeddings. In Henry Wolkowicz, Romesh Saigal, and Lieven Vandenberghe, editors, *Handbook of semidefinite programming: theory, algorithms, and applications*, chapter 5. Kluwer Academic Publishers, 2000.

[20] Dmitriy Drusvyatskiy, Gábor Pataki, and Henry Wolkowicz. Coordinate shadows of semidefinite and euclidean distance matrices. *SIAM Journal on Optimization*, 25(2):1160–1178, 2015.

[21] B. Fares, D. Noll, and P. Apkarian. Robust control via sequential semidefinite programming. *SIAM Journal on Control and Optimization*, 40(6):1791–1820, 2002.

[22] L. Faybusovich. Several Jordan-algebraic aspects of optimization. *Optimization*, 57(3):379–393, 2008.

[23] A.V. Fiacco and G.P. McCormick. *Nonlinear Programming: Sequential Unconstrained Minimization Techniques.* Classics in Applied Mathematics. SIAM, 1990.

[24] J. Fiala, M. Kočvara, and M. Stingl. PENLAB: A MATLAB solver for nonlinear semidefinite optimization. *ArXiv e-prints*, November 2013. `arXiv:1311.5240`.

[25] Anders Forsgren. Optimality conditions for nonconvex semidefinite programming. *Mathematical Programming*, 88(1):105–128, 2000.

[26] Robert M. Freund, Robin Roundy, and Michael Todd. Identifying the set of always-active constraints in a system of linear inequalities by a single linear program. Working papers 1674-85., Massachusetts Institute of Technology (MIT), Sloan School of Management, 1985. URL: `http://EconPapers.repec.org/RePEc:mit:sloanp:2111`.

[27] Roland W. Freund, Florian Jarre, and Christoph H. Vogelbusch. Nonlinear semidefinite programming: sensitivity, convergence, and an application in passive reduced-order modeling. *Mathematical Programming*, 109(2-3):581–611, 2007.

[28] Katsuki Fujisawa, Mituhiro Fukuda, Kazuhiro Kobayashi, Masakazu Kojima, Kazuhide Nakata, Maho Nakata, and Makoto Yamashita. SDPA (SemiDefinite Programming Algorithm) and SDPA-GMP User's Manual — version 7.1.1. Technical Report B-448, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2008.

[29] E. H. Fukuda and M. Fukushima. The use of squared slack variables in nonlinear second-order cone programming, 2014. Submitted.

[30] Walter Gómez and Héctor Ramírez. A filter algorithm for nonlinear semidefinite programming. *Computational & Applied Mathematics*, 29:297 – 328, 06 2010.

[31] W. Hock and K. Schittkowski. Test examples for nonlinear programming codes. *Journal of Optimization Theory and Applications*, 30(1):127–129, 1980.

[32] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990.

[33] Florian Jarre. An interior method for nonconvex semidefinite programs. *Optimization and Engineering*, 1(4):347–372, 2000.

[34] Florian Jarre. Elementary optimality conditions for nonlinear SDPs. In *Handbook on Semidefinite, Conic and Polynomial Optimization*, volume 166 of *International Series in Operations Research & Management Science*, pages 455–470. Springer, 2012.

[35] L. R. Ford Jr. and D. R. Fulkerson. Maximal flow through a network. *Canadian Journal of Mathematics*, 8:399–404, 1956.

[36] Christian Kanzow, Christian Nagel, Hirokazu Kato, and Masao Fukushima. Successive linearization methods for nonlinear semidefinite programs. *Computational Optimization and Applications*, 31(3):251–273, 2005.

[37] Hidefumi Kawasaki. An envelope-like effect of infinitely many inequality constraints on second-order necessary conditions for minimization problems. *Mathematical Programming*, 41(1-3):73–96, 1988.

[38] Sunyoung Kim, Masakazu Kojima, and Kim-Chuan Toh. A Lagrangian–DNN relaxation: a fast method for computing tight lower bounds for a class of quadratic optimization problems. *Mathematical Programming*, pages 1–27, 2015.

[39] Victor Klee. Maximal separation theorems for convex sets. *Transactions of the American Mathematical Society*, 134(1):133–147, 1968.

[40] Igor Klep and Markus Schweighofer. An exact duality theory for semidefinite programming based on sums of squares. *Math. Oper. Res.*, 38(3):569–590, August 2013. `doi:10.1287/moor.1120.0584`.

[41] Michal Kočvara and Michael Stingl. PENNON: A code for convex nonlinear and semidefinite programming. *Optimization Methods and Software*, 18(3):317–333, 2003.

[42] Nathan Krislock and Henry Wolkowicz. Explicit sensor network localization using semidefinite representations and facial reductions. *SIAM Journal on Optimization*, 20(5):2679–2708, 2010.

[43] Minghui Liu and Gábor Pataki. Exact duality in semidefinite programming based on elementary reformulations. *SIAM Journal on Optimization*, 25(3):1441–1454, 2015.

[44] Minghui Liu and Gábor Pataki. Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming. Technical report, Department of Statistics and Operations Research, University of North Carolina at Chapel Hill, 2015.

[45] Miguel Sousa Lobo, Lieven Vandenberghe, Stephen Boyd, and Hervé Lebret. Applications of second-order cone programming. *Linear Algebra and its Applications*, 284(1–3):193 – 228, 1998.

[46] Bruno F. Lourenço, Ellen H. Fukuda, and Masao Fukushima. Optimality conditions for nonlinear semidefinite programming via squared slack variables. *Optimization Online*, December 2015. URL: `http://www.optimization-online.org/DB_HTML/2015/12/5250.html`.

[47] Bruno F. Lourenço, Masakazu Muramatsu, and Takashi Tsuchiya. A structural geometrical analysis of weakly infeasible SDPs. *To Appear in the Journal of the Operations Research Society of Japan*, November 2013. URL: `http://www.optimization-online.org/DB_HTML/2013/11/4137.html`.

[48] Bruno F. Lourenço, Masakazu Muramatsu, and Takashi Tsuchiya. Facial reduction and partial polyhedrality. *arXiv e-prints*, December 2015. URL: `http://arxiv.org/abs/1512.02549`.

[49] Bruno F. Lourenço, Masakazu Muramatsu, and Takashi Tsuchiya. Solving SDP completely with an interior point oracle. *Optimization Online*, July 2015. URL: `http://www.optimization-online.org/DB_HTML/2015/06/4982.html`.

[50] Bruno F. Lourenço, Masakazu Muramatsu, and Takashi Tsuchiya. Weak infeasibility in second order cone programming. *To Appear in Optimization Letters*, September 2015. URL: `http://www.optimization-online.org/DB_HTML/2015/09/5109.html`, `doi:10.1007/s11590-015-0982-4`.

[51] D.G. Luenberger. *Linear and Nonlinear Programming*. Addison-Wesley, 1973. URL: `https://books.google.co.jp/books?id=3WhcngEACAAJ`.

[52] D.G. Luenberger. *Optimization by Vector Space Methods*. Professional Series. Wiley, 1997.

[53] H.Z. Luo, H.X. Wu, and G.T. Chen. On the convergence of augmented lagrangian methods for nonlinear semidefinite programming. *Journal of Global Optimization*, 54(3):599–618, 2012.

[54] Z. Luo and J. F. Sturm. Error analysis. In Henry Wolkowicz, Romesh Saigal, and Lieven Vandenberghe, editors, *Handbook of semidefinite programming: theory, algorithms, and applications*. Kluwer Academic Publishers, 2000.

[55] Z. Luo, J. F. Sturm, and S. Zhang. Duality results for conic convex programming. Technical report, Econometric Institute, Erasmus University Rotterdam, The Netherlands, 1997.

[56] Z. Luo, J. F. Sturm, and S. Zhang. Conic convex programming and self-dual embedding. *Optimization Methods and Software*, 14(3):169–218, 2000.

[57] L. McLinden. Polyhedral extensions of some theorems of linear programming. *Mathematical Programming*, 24(1):162–176, 1982.

[58] Sanjay Mehrotra and Yinyu Ye. Finding an interior point in the optimal face of linear programs. *Mathematical Programming*, 62(1-3):497–515, 1993.

[59] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. Society for Industrial and Applied Mathematics, 1994.

[60] Y. Nesterov, M. J. Todd, and Y. Ye. Infeasible-start primal-dual methods and infeasibility detectors for nonlinear programming problems. *Mathematical Programming*, 84(2):227–267, February 1999. `doi:10.1007/s10107980009a`.

[61] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer Verlag, New York, 1st edition, 1999.

[62] Gábor Pataki. On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues. *Mathematics of Operations Research*, 23(2):339–358, 1998.

[63] Gábor Pataki. The geometry of semidefinite programming. In Henry Wolkowicz, Romesh Saigal, and Lieven Vandenberghe, editors, *Handbook of semidefinite programming: theory, algorithms, and applications*. Kluwer Academic Publishers, online version at `http://www.unc.edu/~pataki/papers/chapter.pdf`, 2000.

[64] Gábor Pataki. A Simple Derivation of a Facial Reduction Algorithm and Extended Dual Systems. Technical report, Columbia University, 2000.

[65] Gábor Pataki. On the closedness of the linear image of a closed convex cone. *Mathematics of Operations Research*, 32(2):395–412, May 2007. `doi:10.1287/moor.1060.0242`.

[66] Gábor Pataki. Strong duality in conic linear programming: Facial reduction and extended duals. In *Computational and Analytical Mathematics*, volume 50, pages 613–634. Springer New York, 2013.

[67] Gábor Pataki. Bad semidefinite programs: they all look the same. Technical report, Department of Statistics and Operations Research, University of North Carolina at Chapel Hill, 2014.

[68] Frank Permenter and Pablo Parrilo. Partial facial reduction: simplified, equivalent SDPs via approximations of the PSD cone. *ArXiv e-prints*, 2014. `arXiv:1408.4685`.

[69] Imre Pólik and Tamás Terlaky. Exact duality for optimization over symmetric cones. AdvOL Report 2007/10, McMaster University, Advanced Optimization Lab, Hamilton, Canada, 2007.

[70] Imre Pólik and Tamás Terlaky. New stopping criteria for detecting infeasibility in conic optimization. *Optimization Letters*, 3(2):187–198, March 2009.

[71] Florian A. Potra and Rongqin Sheng. On homogeneous interior-point algorithms for semidefinite programming. *Optimization Methods and Software*, 9(1-3):161–184, 1998.

[72] Motakuri V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Mathematical Programming*, 77, 1995.

[73] Motakuri V. Ramana, Levent Tunçel, and Henry Wolkowicz. Strong duality for semidefinite programming. *SIAM Journal on Optimization*, 7(3):641–662, August 1997.

[74] J. Renegar. *A Mathematical View of Interior-Point Methods in Convex Optimization*. MPS-SIAM Series on Optimization. SIAM, 2001.

[75] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, NJ, USA, 1970.

[76] K. Schittkowski. Test Examples for Nonlinear Programming Codes - All Problems from the Hock-Schittkowski-collection. Technical report, Department of Computer Science, University of Bayreuth, 2009.

[77] Alexander Shapiro. First and second order analysis of nonlinear semidefinite programs. *Mathematical Programming*, 77(1):301–320, 1997.

[78] Jos F. Sturm. Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4):625–653, 1999.

[79] Jos F. Sturm. Error bounds for linear matrix inequalities. *SIAM Journal on Optimization*, 10(4):1228–1248, January 2000. `doi:10.1137/S1052623498338606`.

[80] Jos F. Sturm. Similarity and other spectral relations for symmetric cones. *Linear Algebra and its Applications*, 312(1-3):135 – 154, 2000.

[81] Defeng Sun, Jie Sun, and Liwei Zhang. The rate of convergence of the augmented lagrangian method for nonlinear semidefinite programming. *Mathematical Programming*, 114(2):349–391, 2008.

[82] J. Sun, L.W. Zhang, and Y. Wu. Properties of the augmented lagrangian in nonlinear semidefinite optimization. *Journal of Optimization Theory and Applications*, 129(3):437–456, 2006.

[83] M. J. Todd. Semidefinite optimization. *Acta Numerica*, 10:515–560, 5 2001. `doi:10.1017/S0962492901000071`.

[84] Kim-Chuan Toh, Michael J. Todd, and RehaH. Tütüncü. On the implementation and usage of SDPT3 – a Matlab software package for semidefinite-quadratic-linear programming, version 4.0. In Miguel F. Anjos and Jean B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, pages 715–754. Springer US, 2012.

[85] Levent Tunçel. *Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization*. American Mathematical Society, Providence, R.I. : Toronto, Ont, November 2010.

[86] Levent Tunçel and Henry Wolkowicz. Strong duality and minimal representations for cone optimization. *Computational Optimization and Applications*, 53(2):619–648, 2012.

[87] Lieven Vandenberghe and Stephen Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.

[88] Hayato Waki. How to generate weakly infeasible semidefinite programs via Lasserre's relaxations for polynomial optimization. *Optimization Letters*, 6(8):1883–1896, December 2012.

[89] Hayato Waki and Masakazu Muramatsu. Facial reduction algorithms for conic optimization problems. *Journal of Optimization Theory and Applications*, 158(1):188–215, 2013. `doi:10.1007/s10957-012-0219-y`.

[90] Hayato Waki, Maho Nakata, and Masakazu Muramatsu. Strange behaviors of interior-point methods for solving semidefinite programming problems in polynomial optimization. *Computational Optimization and Applications*, 53(3):823–844, 2012.

[91] Hiroshi Yamashita and Hiroshi Yabe. Local and superlinear convergence of a primal-dual interior point method for nonlinear semidefinite programming. *Mathematical Programming*, 132(1-2):1–30, 2012.

[92] Hiroshi Yamashita and Hiroshi Yabe. A survey of numerical methods for nonlinear semidefinite programming. *Journal of the Operations Research Society of Japan*, 58(1):24–60, 2015.

[93] Hiroshi Yamashita, Hiroshi Yabe, and Kouhei Harada. A primal–dual interior point method for nonlinear semidefinite programming. *Mathematical Programming*, 135(1-2):89–121, 2012.

[94] A. Yoshise and Y. Matsukawa. On optimization over the doubly nonnegative cone. In *Computer-Aided Control System Design (CACSD), 2010 IEEE International Symposium on*, pages 13–18, Sept 2010.

Index