Geometric structures of favorite points and late points of a two-dimensional simple random walk

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Abstract

We consider the problem of late points and nearly favorite points of simple random walk in mainly two dimension.

We determine the power exponents for the numbers of pairs of α-favorite points as Dembo et al. conjectured. It has been shown that the exponents for the numbers of pairs of late points coincide with those of nearly favorite points and high points in the Gaussian free field, whose exact values are known. In addition, we estimate the exponents for the numbers of a multipoint set of late points in average and in probability. While there have been observed certain similarities between among three classes of points, our results exhibit a difference.

In addition, we consider the question: how many times does a simple random walk revisit the most frequently visited site among the inner boundary points? It is known that in $\mathbb{Z}^2$, the number of visits to the most frequently visited site among all of the points of the random walk range up to time n is asymptotic to $\pi^{-1}(\log n)^2$, while in $\mathbb{Z}^d$ $(d \geq 3)$, it is of order $\log n$. We prove that the corresponding number for the inner boundary is asymptotic to $\beta_d \log n$ for any $d \geq 2$, where β_d is a certain constant having a simple probabilistic expression.

Contents

1 Introduction

In this paper, we study the local time of a random walk and special sites among the random walk range, called late points and favorite points. In particular, we will solve some problems posed in Open problem 4 in [6] and Open problem 4.3 in [7], which correspond to an extension of the results in [11], and the open problem posed in [11].

The local time is a functional of a random walk with time and space parameter. It describes how many times the random walk visits a specified site until a specified time. The notion of local time is very basic and it appears in various fields. For example, Feynman-Kac formula provides us a representation of the solution of (discrete) Schrödinger equation by using a functional of the local time. Indeed, this is a fundamental tool to study Schrödinger equations by means of probability theory, and a detailed study of local time is required to investigate it. Though the research on the local time is a classical topic, many questions related to it are recently propounded and investigated and many of them remaining to be cleared up. In particular, problems concerning the local time in two dimension are most difficult than in other dimensions. Since the simple random walk in two dimension is recurrent, estimates such as asymptotic behavior of hitting probabilities are complex and often exhibits a different nature. The simple random walk is also recurrent in one dimension, but the structure of the state space is much simpler than $\mathbb{Z}^2$ and hence many special techniques are available there.

Before entering the description of our main results, we briefly review known results concerning late points and favorite points. About sixty years ago, Erdős and Taylor [18] proposed a problem on asymptotic behavior of the spatial maximum of the local time of a simple random walk in $\mathbb{Z}^2$. Over forty years later, Dembo, Peres, Rosen and Zeitouni [8, 9, 10] solved it and some related problems by developing innovative proofs. Moreover, these methods have been applied also to the study of late points in two dimensional discrete torus $\mathbb{Z}_n^2$ of size n. As one of those results, in [11], they showed that the number of pairs of late points in various sized clusters have a variety of power growth exponents. In the same paper, they gave a conjecture on the exact asymptotic form of a corresponding quantity for favorite points. [11] to j-tuple of late points with $j \geq 3$ is also suggested as an open problem in [6]. These results, as well as conjectures, concern with geometry of the configuration of (nearly) extremal points where the local time takes singular values. As another question, the position of the site where the local time attains its maximum is also studied in the literature. Lifshits and Shi [22] verified that the favorite point of 1-dimensional random walk tends to go far from the origin. Additional open problems concerning geometric structure of the favorite points are raised by Erdős and Révész [16, 17] and Shi and Tóth [29] but almost no definite solution to them is known for multi-dimensional walks.

Based on the state of the art as above, we explain three main results in this paper. The first one is on the number of pair of favorite points. We define the set of α-favorite points $\Psi_n(\alpha)$ for $0 < \alpha < 1$ as in [11] (see (2.3) in the next section) and obtain certain asymptotic estimates, say (A) (in average) and (B) (in probability), of

$$|\{(x, x') \in \Psi_n(\alpha)^2 : d(x, x') \leq n^\beta\}|.$$

This verifies the conjecture raised in [11]. The second one is on the number of j-tuples of late points. We define the set of α-late points in $\mathbb{Z}_n^2$ as $\mathcal{L}_n(\alpha)$ for $0 < \alpha < 1$ as in [11] (see (2.3) in the next section). We will obtain certain asymptotic estimates, say (A) (in average) and (B) (in probability), of

$$|\{(x_1, ..., x_j) \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \leq n^\beta \text{ for any } 1 \leq i, l \leq j\}|$$

for any $0 < \alpha, \beta < 1$ and $j \in \mathbb{N}$. This gives an answer to the open question in [6]. The third one concerns with the location of favorite points. We obtain the limit theorems for the spatial maximum of the local time over the boundary of random walk range up to time n. As a corollary, this implies that the favorite points for the whole range of the random walk does not appear on the boundary of the range asymptotically. The third results deals with a different subject from the first and second ones. Thus, our results can be categorized into two content according to its subject.

Now we state motivations for the present study of entire research of this paper in more details. We wish to understand the structure of the random field of a local time as a configuration of late points and favorite points. In particular, we want to determine the relationship between the special points of a random walk and those of a Gaussian free field (GFF). In fact, there are several results for Gaussian free fields corresponding to ones for favorite points and late points. For $0 < \alpha < 1$, the known results study the set of α-high points of the GFF in $\mathbb{Z}_n^2$ (sites where GFF takes high values). For $j = 2$, [11] obtained the result corresponding to (A), and Brummelhuis and Hilhorst [4] obtained that corresponding to (B). Daviaud [5] estimated the number of pairs of α-high points of the GFF in the same forms as (A) and (B). In each case, the exponent coincides with that of (A) or (B). We will discuss this similarity in more details in Section 2.2. It is worth mentioning that some other functional of a random field exhibits a different behavior between favorite points and late points (See Section 5.5). Note that any result on asymptotic behavior of the location of high points of GFF as $n \to \infty$ is not obtained yet as far as the author knows.

There is another motivation for first and second main results. We wish to employ detailed information of the geometric structures of late points and favorite points to obtain the diverse large deviation estimates of the local time or high moment of functionals of the local time. Of course, there are corresponding questions for high points also. In general, the limit theorem for random variables has a connection with its large deviation estimates or high moments. We believe that our estimates of the local time helps the study of asymptotic behavior of various functionals of the local time. In fact, there are already some results in this direction as an application of our results (See Proposition 5.10). Actually, (A) of our main results (with $j = 2$) are almost equivalent to an estimate of the second moments of $|\mathcal{L}_n(\alpha)|$ or $|\Psi_n(\alpha)|$.

We introduce the contents of this paper. In Section 2, we will introduce six main theorems (Theorem 2.1–2.6). We also provide a more detailed observations on our main results as well as an outline of the proof there. After stating our framework precisely in Section 2.1, we will provide the results concerning the geometric structures of $\mathcal{L}_n(\alpha)$ and $\Psi_n(\alpha)$ in Section 2.2. The proof of theorems in Section 2.2 and some related results are given in Section 3–5. In Section 2.3, we will argue the results of favorite point among the inner boundary. In Section 3, we will collect preliminary results concerning a simple random walk and a finite Markov chain that will be used in later Sections 4 and 5. In Section 4, we will provide the proof of the results concerning the number of pair points of favorite points. In Section 5, we will discuss the number of multipoints of late points and some related results. First, in section 5.1, we will give the basic matrix's arguments as preliminaries to the proof of the second main results. Second, we will give the proof of main results in Sections 5.2 and 5.3. The rest of Section 5 is devoted to some problems. In Section 5.4, we will estimate to what extent the probability that a certain geometric structure of multipoints which are late points remains when some spatial element is changed. In Section 5.5, we will argue the difference between $\mathcal{L}_n(\alpha)$ and $\Psi_n(\alpha)$(or $\mathcal{V}_n(\alpha)$). As mentioned above, from [4, 11] and from our forthcoming paper, it is known that there exist similarities between $\mathcal{L}_n(\alpha)$, $\Psi_n(\alpha)$ and $\mathcal{V}_n(\alpha)$, but there are also differences. We will obtain the complex distribution of pair points there. Section 5.6 is an appendix. There we show a sort of monotonicity of exponents which appear in the description of the second main theorems. We will use it in Sections 5.2 and 5.3. It is elementary but complicated. Finally, in Section 6, we will provide the proof of the results concerning favorite point among the inner boundary.

To close the introduction, we mention that this thesis is based on three papers [24, 25, 26]. Theorems 2.1 and 2.2, Theorems 2.3 and 2.4 and Theorems 2.5 and 2.6 are established in [25], [25] and [24] respectively.

2 Known results and main results

2.1 Framework

To state our main results, we will introduce notations as follows. Let d be the Euclidean distance and $\mathbb{N} := \{1, 2, \cdots\}$. For $n \in \mathbb{N}$, let $D(x, r) := \{y \in \mathbb{Z}_n^2 : d(x, y) < r\}$ and for any $G \subset \mathbb{Z}_n^2$, $\partial G := \{y \in G^c : d(x, y) = 1 \text{ for some } x \in G\}$. For $x \in \mathbb{Z}_n^2$, we sometimes omit $\{\}$ in writing the one-point set $\{x\}$. Let $\{S_k\}_{k=1}^{\infty}$ be a simple random walk on a 2-dimensional square lattice. Let P^x denote the probability of a simple random walk starting at x. We will simply write P for P^0. Let $K(n, x)$ be the number of times the simple random walk visits x up to time n, that is, $K(n, x) = \sum_{i=0}^{n} 1_{\{S_i = x\}}$. For any $D \subset \mathbb{Z}_n^2$, let $T_D := \inf\{m \geq 1 : S_m \in D\}$. For simplicity, we will write $T_{x_1,\dots,x_j}$ for $T_{\{x_1,\dots,x_j\}}$. Let $\tau_n := \inf\{m \geq 0 : S_m \in \partial D(0, n)\}$. We will use the same notations for a simple random walk in $\mathbb{Z}^2$. In addition, $\lceil a \rceil$ denotes the smallest integer n with $n \geq a$.

Now, we will recall details of some known results of favorite points which we mentioned in Section 1 to define α-favorite points. Erdős and Taylor [18] showed that for a simple random walk in $\mathbb{Z}^d$ with $d \geq 3$

$$\lim_{n\to\infty} \frac{\max_{x\in\mathbb{Z}^d} K(n, x)}{\log n} = \frac{1}{-\log P(T_0 < \infty)} \quad \text{a.s.} \tag{2.1}$$

For $\mathbb{Z}^2$, they obtained

$$\frac{1}{4\pi} \leq \liminf_{n\to\infty} \frac{\max_{x\in\mathbb{Z}^2} K(n, x)}{(\log n)^2} \leq \limsup_{n\to\infty} \frac{\max_{x\in\mathbb{Z}^2} K(n, x)}{(\log n)^2} \leq \frac{1}{\pi} \quad \text{a.s.},$$

and conjectured that the limit exists and is almost surely (a.s.) to. Forty years later, Dembo et al. [9] showed that for a simple random walk in $\mathbb{Z}^2$,

$$\lim_{n\to\infty} \frac{\max_{x\in\mathbb{Z}^2} K(\tau_n, x)}{(\log n)^2} = \lim_{n\to\infty} \frac{4\max_{x\in\mathbb{Z}^2} K(n, x)}{(\log n)^2} = \frac{4}{\pi} \quad \text{a.s.} \tag{2.2}$$

Subsequently, Rosen [27] provided another proof of this. Based on this result, as [9, 11], for $0 < \alpha < 1$, we define the set of α-favorite points in $\mathbb{Z}^2$ by

$$\Psi_n(\alpha) := \left\{ x \in D(0,n) : K(\tau_n, x) \geq \left\lceil \frac{4\alpha}{\pi} (\log n)^2 \right\rceil \right\}.$$

Similarly, we will recall some known results for the cover time in $\mathbb{Z}_n^2$ to define α-late points. Originally, [8] estimated the cover time of a simple random walk in $\mathbb{Z}_n^2$, that is,

$$\lim_{n\to\infty} \frac{\max_{x\in\mathbb{Z}_n^2} T_x}{(n\log n)^2} = \frac{4}{\pi} \quad \text{in probability.}$$

For $0 < \alpha < 1$, we will define the set of α-late points in $\mathbb{Z}_n^2$ by

$$\mathcal{L}_n(\alpha) := \left\{ x \in \mathbb{Z}_n^2 : \frac{T_x}{(n\log n)^2} \geq \frac{4\alpha}{\pi} \right\}. \tag{2.3}$$

2.2 Results of geometric structures of favorite points and late points

Now, we introduce main results. We clear up the geometric structure of $\Psi_n(\alpha)$ and attend the problem in [11] as follows.

Theorem 2.1. *For any $0 < \alpha, \beta < 1$ with probability one*

$$\lim_{n\to\infty} \frac{\log |\{(x,x') \in \Psi_n(\alpha)^2 : d(x,x') \leq n^\beta\}|}{\log n} = \rho_2(\alpha,\beta),$$

where

$$\rho_2(\alpha,\beta) := \begin{cases} 2 + 2\beta - \frac{4\alpha}{2-\beta} & (\beta \leq 2(1-\sqrt{\alpha})), \\ 8(1-\sqrt{\alpha}) - 4(1-\sqrt{\alpha})^2/\beta & (\beta \geq 2(1-\sqrt{\alpha})). \end{cases}$$

Theorem 2.2. *For any $0 < \alpha, \beta < 1$*

$$\lim_{n\to\infty} \frac{\log E[|\{(x,x') \in \Psi_n(\alpha)^2 : d(x,x') \leq n^\beta\}|]}{\log n} = \hat{\rho}_2(\alpha,\beta),$$

where

$$\hat{\rho}_2(\alpha,\beta) := \begin{cases} 2 + 2\beta - \frac{4\alpha}{2-\beta} & (\beta \leq 2 - \sqrt{2\alpha}), \\ 6 - 4\sqrt{2\alpha} & (\beta \geq 2 - \sqrt{2\alpha}). \end{cases}$$

Theorem 2.3. *For any $0 < \alpha, \beta < 1$ and $j \in \mathbb{N}$*

$$\lim_{n\to\infty} \frac{\log E[|\{(x_1,...,x_j) \in \mathcal{L}_n(\alpha)^j : d(x_i,x_l) \leq n^\beta \text{ for any } 1 \leq i,l \leq j\}|]}{\log n}$$
$$= \hat{\rho}_j(\alpha,\beta),$$

where

$$\hat{\rho}_j(\alpha,\beta) := \begin{cases} 2 + 2(j-1)\beta - \frac{2j\alpha}{(1-\beta)(j-1)+1} & (\beta \leq 1 + \frac{1-\sqrt{j\alpha}}{j-1}), \\ 2(j+1-2\sqrt{j\alpha}) & (\beta \geq 1 + \frac{1-\sqrt{j\alpha}}{j-1}). \end{cases}$$

We will extend the condition $d(x_i,x_l) \leq n^\beta$ to be more general in Section 5.4.

Theorem 2.4. *For any $0 < \alpha, \beta < 1$ and $j \in \mathbb{N}$ in probability*

$$\lim_{n\to\infty} \frac{\log |\{(x_1, ..., x_j) \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \le n^\beta \text{ for any } 1 \le i, l \le j\}|}{\log n}$$
$$= \rho_j(\alpha, \beta),$$

where

$$\rho_j(\alpha,\beta) := \begin{cases} 2 + 2(j-1)\beta - \frac{2j\alpha}{(1-\beta)(j-1)+1} & (\beta \le \frac{j}{j-1}(1-\sqrt{\alpha})), \\ 4j(1-\sqrt{\alpha}) - 2j(1-\sqrt{\alpha})^2/\beta & (\beta \ge \frac{j}{j-1}(1-\sqrt{\alpha})). \end{cases}$$

Therefore, the mean and median numbers of multipoints have different orders of magnitude. If late points or favorite points were spread out uniformly in $\mathbb{Z}_n^2$ or $\mathbb{Z}^2$, the limit in main results would be $2 + 2(j-1)\beta - 2j\alpha$ for $j \ge 2$. However, we obtain much larger exponents in Theorems 2.1, 2.2, 2.3 and 2.4.

To state the relationship between $\hat{\rho}_j(\alpha, \beta)$ and $\rho_j(\alpha, \beta)$, for $0 < \beta < 1$, γ and $h > 0$, respectively, let

$$F_{h,\beta}(\gamma) := \frac{1-\gamma\beta}{1-\beta} + h\gamma^2\beta.$$

It can easily be checked that

$$\rho_j(\alpha, \beta) = 2 + 2(j-1)\beta - 2\alpha \inf_{\gamma \in \Gamma_{\alpha,\beta}} F_{j,\beta}(\gamma),$$

where

$$\Gamma_{\alpha,\beta} = \{\gamma \ge 0 : 2 - 2\beta - 2\alpha F_{0,\beta}(\gamma) \ge 0\}.$$

It is also easy to verify that

$$\hat{\rho}_j(\alpha, \beta) = \sup_{\beta' \le \beta} \sup_{\gamma \ge 0} \{2 + 2(j-1)\beta' - 2\alpha F_{j,\beta'}(\gamma)\}, \tag{2.4}$$

and so the difference between $\rho_j(\alpha, \beta)$ and $\hat{\rho}_j(\alpha, \beta)$ is that the supremum in (2.4) is not subjected to the constraint $\gamma \in \Gamma_{\alpha,\beta}$. We will define

$$\hat{F}_{h,\beta}(\gamma) := \gamma^2(1-\beta) + \frac{h}{\beta}(1-\gamma(1-\beta))^2$$

in Theorems 2.1 and 2.2. Note that $\hat{F}_{h,\beta}(\gamma) = F_{h,\beta}((1-\gamma(1-\beta))/\beta)$. Then, the same explanation holds by using $\hat{F}_{h,\beta}(\gamma)$. We will now explain why we need to restrict the domain of a function $F_{j,\beta}(\gamma)$ in $\Gamma_{\alpha,\beta}$ to have $\rho_j(\alpha, \beta)$. As [6, 7] mentioned, if we estimate Theorem 2.4, we are tempted to consider the following wrong heuristic: if we set $A(n^\beta, a_n) := \{z \in (\lceil n^\beta \rceil \mathbb{Z}_n^2) \cap \mathbb{Z}_n^2$: a simple random walk goes and returns to $\partial D(z, n^\beta)$ via some closed circle a_n times up to time $4\alpha(n \log n)^2/\pi$ $\}$, in probability

$$\begin{aligned}
&E[|\text{set of multipoints which are } \alpha\text{-late points within distance } n^\beta \text{ of each other}|] \\
&= n^{2+2(j-1)\beta} \\
&\times P(x_1, ..., x_j \text{ are } \alpha\text{-late points when } d(x_i, x_l) \text{ is close to } n^\beta (\forall i, l \le j)) \\
&\approx (\text{Typical value of such number of multipoints}) \\
&\approx \max_{a_n \in \mathbb{N}} E[|A(n^\beta, a_n)|] \times \left(\frac{E[|\mathcal{L}_n(\alpha) \cap D(z, n^\beta)|; z \in A(n^\beta, a_n)]}{P(z \in A(n^\beta, a_n))} \right)^j.
\end{aligned}$$

This approximation is discussed in [4]. In fact, we only have to consider $\{a_n\}_{n\in\mathbb{N}}$ with

$$P(|\{z \in (\lceil n^\beta \rceil \mathbb{Z}_n^2) \cap \mathbb{Z}_n^2 : z \in A(n^\beta, a_n)\}| \geq 1) \to 1.$$

This restriction corresponds to (5.39) and, hence, that of $F_{j,\beta}(\gamma)$ restricted in $\Gamma_{\alpha,\beta}$. Therefore, we obtain different quantities for the exponent in probability from that in average.

Now we introduce the known results concerning late points and favorite points. [4] showed (B) for $j = 2$, and, [11] showed (A) for $j = 2$. Theorem 1.4 of [11] estimated the power growth exponent of pairs of α-late points within distance n^β of each other. In addition, As mentioned in Section 1, it is known that there exist some relationships between late points, favorite points and high points. We will explain the high points of the GFF in $\mathbb{Z}_n^2$. In general, it is known that GFF is a central model of random surfaces. Sheffield [28] provided a mathematical survey of the GFF. In addition, Bolthausen et al. [3] showed that in probability,

$$\lim_{n\to\infty} \frac{\max_{x\in\mathbb{Z}_n^2} \phi_n(x)}{\log n} = 2\sqrt{\frac{2}{\pi}},$$

where $\{\phi_n(x)\}_{x\in\mathbb{Z}_n^2}$ is the GFF whose definition is the same as that in [3, 5]. In what follows, for $0 < \alpha < 1$, we will define the set of α-high points of the GFF such that

$$\mathcal{V}_n(\alpha) := \left\{ x \in \mathbb{Z}_n^2 : \frac{\phi_n(x)^2}{2} \geq \frac{4\alpha}{\pi}(\log n)^2 \right\}.$$

[5] showed that if $\mathcal{L}_n(\alpha)$ in Theorems 2.1 and 2.2 is changed into $\mathcal{V}_n(\alpha)$, each exponent is the same as that of Theorems 2.1 and 2.2. In addition, there are some interesting results concerning a relationship between the local time of a random walk and a GFF. Eisenbaum et al. [14] provided a powerful equivalence law called the generalized second Ray-Knight theorem. Ding et al. [12, 13] showed a strong connection between the expected maximum of the GFF and the expected cover time.

We explain the outline of the proof of Theorems 2.1 and 2.2. The proof of the lower bound in Theorem 2.1 relies on the moment method which is used by [8, 10, 11]. Now, we explain one of the reason why the lower bound in Theorem 1.1 in [9] has been unsolved for a long time and the method of the proof in [9, 27], which is essentially the same as that of [8, 9, 10, 11]. If we use the Paley-Zygmund inequality for $|\Psi_n(\alpha)|$, we need to estimate the second moment of it. However, it is much larger than the square of the first moment and, hence, it is impossible to use directly the moment method for $|\Psi_n(\alpha)|$. Then, to estimate the lower bound, they consider the set of points $\Psi'_n(\alpha)$ called successful points which is a slightly smaller set than $\Psi_n(\alpha)$. (For example, see (3.7) in [27].) In fact, the second moment of $|\Psi'_n(\alpha)|$ is of almost same order as the square of the first moment of $|\Psi'_n(\alpha)|$ and the first moment of $|\Psi'_n(\alpha)|$ is same order as that of $|\Psi_n(\alpha)|$. Then, they use the moment method for $|\Psi'_n(\alpha)|$, and, hence, the definition of $\Psi'_n(\alpha)$ naturally yields the lower bound of $|\Psi_n(\alpha)|$. In this paper, to use this method, we choose a proper set corresponding to $|\Psi'_n(\alpha)|$.

Next, we will explain the summary of proofs for Theorems 2.3 and 2.4. We uniquely develop three main ideas, (I), (II) and (III).

(I) Matrix's arguments

For use in the proof of the upper bound, we will consider the special set of the matrices $\mathcal{M}_j$ ($\tilde{\mathcal{M}}_j$) in Section 5.1. Now, we will explain the difficulty of the proof of the upper bound. For example, if we consider the value in Theorem 2.2, we can write

$$\begin{aligned} &E[|\{(x_1, ..., x_j) \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \leq n^\beta \text{ for any } 1 \leq i, l \leq j\}|] \\ &= \sum_{\substack{d(x_i,x_l)\leq n^\beta, \\ x_i\in\mathbb{Z}_n^2}} P(x_1, ..., x_j \in \mathcal{L}_n(\alpha)). \end{aligned}$$

Consider the sequences $\{x_i^{(n)}\}_{n\in\mathbb{N}}$ with $x_i^{(n)} \in \mathbb{Z}_n^2$ for $1 \le i \le 3$. Hence, if we estimate $\hat{\rho}_3(\alpha,\beta)$, from the symmetry of $x_1^{(n)}$, $x_2^{(n)}$ and $x_3^{(n)}$, it is plausible that we need to consider the following two positions:

$$(1) d(x_1^{(n)}, x_2^{(n)}) \approx d(x_2^{(n)}, x_3^{(n)}) \approx d(x_3^{(n)}, x_1^{(n)}),$$
$$(2) d(x_1^{(n)}, x_2^{(n)}) << d(x_1^{(n)}, x_3^{(n)}), d(x_2^{(n)}, x_3^{(n)}),$$

where $a_n \approx b_n$ means $\log a_n/\log b_n \to 1$ as $n \to \infty$ for any sequence. In general $j \in \mathbb{N}$, there are various positions of $(x_1, ..., x_j)$ which we need to compute and, hence, we need some original computation for multipoints. Thus, we need to observe properties of $\mathcal{M}_j$ for $j \in \mathbb{N}$, which correspond to properties of a variety of positions. For example, we will divide $\mathcal{M}_j$ into the value of some function Ξ (as introduced later) and argue the relationship with the summation of all the elements of a certain matrix in $\mathcal{M}_j$. In particular, Proposition 5.5 (or Lemma 5.7) is interesting and important since the proof depends on remarkable properties of $\mathcal{M}_j$ and corresponds to key estimates for main results.

(II) The information of Green's function

We will estimate hitting probabilities for some subset in $\mathbb{Z}_n^2$ until some stopping time in Section 3.1.1, and apply these arguments to Lemma 5.13. To obtain Lemma 5.13, we need to compute hitting probabilities to a variety of subsets and stopping times, which can't be simply computed. Our results imply that there exists a strong connection between such hitting probabilities and a matrix of Green's function of a simple random walk.

(III) Enumeration of a Markov chain's path

We will use the certain general Markov chain rule studied in Section 3.1.3. In fact, we will estimate the probabilities under the condition on a local time of some edges in the graph.

Now, we will explain how they lead to our main result with the above three ideas. The proof of the upper bound in Theorem 2.4 partially relies on the proof in [11], in which, they chose many small circles and considered the probability that pair points in the circle were not covered by a random walk under the condition of a crossing number around the circle. We will extend this result by changing pair points to multipoints. The result of Lemma 5.11 corresponds to that of Lemma 6.1 in [11]. However, the method of the proof of Lemma 5.11 is completely different and uses the above ideas (II) and (III). In particular, we will replace the information regarding the combination of paths conditioned by the excursions of the random walk to that of the Markov chain's path which we obtained using idea (III). In addition, with Lemma 5.11, we will produce the proof of the upper bound in Theorems 2.3 and 2.4 using idea (I). For example, Lemma 5.8 implies that for $\{x_i^{(n)}\}_{n\in\mathbb{N}}$ with $x_i^{(n)} \in \mathbb{Z}_n^2$ $(1 \le i \le j)$, we only need to consider the position that is close to $d(x_i^{(n)}, x_l^{(n)}) \approx n^{1-a_{i,l}}$ for $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$. In Lemma 6.4, we provide the method of the enumeration of points included in a certain subset. Therefore, we obtain Proposition 5.10 by using Propositions 5.3 and 5.5 and complete the desired results with the aid of the upper bound in (ii) of Lemma 5.11.

The proof of the lower bound in Theorem 2.4 relies on the moment method used in the proof of the lower bound in Theorem 2.2. In this paper, to show the lower bound in Theorem 2.4, we only need minor modifications to [11].

2.3 Main Results of favorite point among the inner boundary

In Section 2.3, we focus on the most frequently visited site among the inner boundary points of the random walk range, rather than among all of the points of the range, and propose the question: how many times does a random walk revisit the most frequently visited site among the inner boundary points?

To state main results, we will introduce notations as follows. For $n \in \mathbb{N}$, set $R(n) = \{S_0, S_1, \ldots, S_n\}$ as the random walk range up to the time n. We call $z \in \mathbb{Z}^d$ a neighbor of $a \in \mathbb{Z}^d$ if $|a - z| = 1$. Let $\mathcal{N}(a)$ be the set of all neighbors of a defined as

$$\mathcal{N}(a) = \{z \in \mathbb{Z}^d : d(a, z) = 1\}.$$

Note that the inner boundary of $A \subset \mathbb{Z}^d$, which is denoted by ∂A in Section 2.1, can alternatively be represented by

$$\{x \in A : \mathcal{N}(x) \not\subset A\}.$$

For $l, n \in \mathbb{N}$ let

$$R[l,n] = \{S_l, S_{l+1}, ..., S_n\}$$

if $n \geq l$ and $R[l,n] = \emptyset$ if $l > n$ and $R[0,-1] = \emptyset$. The inner boundary of the random walk range $R[l,n]$ is denoted simply by $\partial R[l,n]$ as it is for $R(n)$. In addition, we set

$$M(n) := \max_{x \in \partial R(n)} K(n,x).$$

Clearly, this is the maximal number of visits of the random walk of length n to $\partial R(n)$, the inner boundary of $R(n)$. We are now ready to state our main theorems. The first theorem provides us with sharp asymptotic behavior of $M(n)$.

Theorem 2.5. *For $d \geq 2$*

$$\lim_{n \to \infty} \frac{M(n)}{\log n} = \beta_d \qquad a.s.,$$

where

$$\beta_d = \frac{1}{-\log P(T_0 < T_b)}$$

for any $b \in \mathcal{N}(0)$. Note that $P(T_0 < T_b)$ does not depend on the choice of $b \in \mathcal{N}(0)$, but rather depends only on the dimension.

Leading to the second main theorem, we first define $\Theta_n(\delta)$ for $n \in \mathbb{N}$ and $0 < \delta < 1$ as

$$\Theta_n(\delta) := \left| \left\{ x \in \partial R(n) : \frac{K(n,x)}{\log n} \geq \beta_d \delta \right\} \right|.$$

This is the cardinality of points in $\partial R(n)$ whose number of visits is comparable to the maximal order appearing in Theorem 2.5, with a ratio greater than or equal to δ. Our second main theorem exhibits the sharp logarithmic asymptotic behavior of $\Theta_n(\delta)$ as $n \to \infty$.

Theorem 2.6. *For $d \geq 2$ and $0 < \delta < 1$,*

$$\lim_{n \to \infty} \frac{\log \Theta_n(\delta)}{\log n} = 1 - \delta \quad a.s.$$

We compare our main results to the corresponding results for the whole random walk range $R(n)$. We denote the quantity corresponding to $M(n)$ by $M_0(n)$. That is, $M_0(n) = \max_{x \in \mathbb{Z}^d} K(n,x)$ where $M_0(n)$ represents the maximal number of visits of the random walk to a single site in the entire random walk range until time n. Recall results of [9], which are introduced in Section 2.1. For $0 < a < 1$, we define

$$\Theta_{0,n} = \left| \left\{ x \in \mathbb{Z}^2 : \frac{K(n,x)}{(\log n)^2} \geq \frac{a}{\pi} \right\} \right|.$$

Then, [9] yields

$$\lim_{n \to \infty} \frac{\log \Theta_{0,n}}{\log n} = 1 - a \qquad \text{a.s.}$$

In view of these results, Theorem 2.5 entails the following corollary.

Corollary 2.7. *For $d \geq 2$, the favorite point does not appear in the inner boundary from some time onwards a.s. In other words, $M_0(n) > M(n)$ for all but finitely many n with probability one.*

Next, we explain proofs. The upper bounds for both Theorem 2.5 and Theorem 2.6 are obtained using the idea in [15]. The Chebyshev inequality and the Borel-Cantelli lemma are also used in the same way as in [15]. On the other hand, $M(n)$ is not monotone, while $M_0(n)$ is monotone. We work with the walk and its trajectory at the times 2^k and find a process that is monotone and a bit larger than $M(n)$, but with the desired asymptotics.

On the other hand, the idea for the proof of the lower bound is different from that for the known results. In [9], a Brownian occupation measure was used in the proof. Rosen [27] provided another proof to the result of [9], in which he computed a crossing number instead of the number of the frequently visited site. In this paper, we use the Chebyshev inequality and the Borel-Cantelli lemma as in [27] but for the number of the frequently visited sites among the inner boundary points. In addition, as the proof of the upper bound, we estimate a number slightly smaller than the number of the frequently visited site among the inner boundary points.

These results illuminate the geometric structure of the random walk range as well as the nature of recurrence or transience of random walks. We are able to infer that the favorite point is outside the inner boundary from some time onwards with probability one. This may appear intuitively clear; it seems improbable for the favorite point to continue to be an inner boundary point since it must be visited many times, but our result further shows that there are many inner boundary points that are visited many times, with amounts comparable to that of the favorite point for $d \geq 3$. In addition, the growth order of $M_0(n)$ is the same for all $d \geq 2$, meaning the phase transition which occurs between $d = 2$ and $d \geq 3$ for $M_0(n)$ does not occur for $M(n)$.

In Theorem 2.6, which is a strong claim in comparison to Theorem 2.5, we will provide an explicit answer to the question of how many frequently visited sites among the inner boundary points exist.

We conclude this introduction by mentioning some known results about the inner boundary points of the random walk range that are closely related to the present subject. Let L_n be the number of the inner boundary points up to time n. In [2], it is noticed that the entropy of a random walk is essentially governed by the asymptotic of L_n. In [23], the law of large numbers for L_n is shown and $\lim L_n/n$ is identified for a simple random walk on $\mathbb{Z}^d$ for every $d \geq 1$. Let $J_n^{(p)}$ denote the number of p-multiplicity points in the inner boundary and be defined as

$$J_n^{(p)} = |\{S_i \in \partial R(n) : |\{m : 0 \leq m \leq n, S_m = S_i\}| = p\}|,$$

where $\partial R(n)$ is the set of the inner boundary of $\{S_0, S_1, ..., S_n\}$. In [23], it is also shown that for a simple random walk in $\mathbb{Z}^2$, with $p \geq 1$,

$$\begin{aligned} \frac{\pi^2}{2} &\leq \lim_{n\to\infty} EL_n \times \frac{(\log n)^2}{n} \leq 2\pi^2, \\ \frac{\tilde{c}^{p-1}\pi^2}{4} &\leq \lim_{n\to\infty} EJ_n^{(p)} \times \frac{(\log n)^2}{n} \leq \tilde{c}^{p-1}\pi^2, \end{aligned} \tag{2.5}$$

where $\tilde{c} = P(T_0 < T_b)$ for any/some neighbor site b of the origin. These may be compared with the results for the entire range; according to [19], $|\{S_i : 0 \leq i \leq n\}|$ in $\mathbb{Z}^2$ is asymptotic to $\pi n/\log n$ and the asymptotic form of the number of p-multiplicity points in it is independent of p.

In the proofs given in the remainder of this paper we denote contextual constants by C or c. In addition, $\lceil a \rceil$ denotes the smallest integer n with $n \geq a$, and A^c denotes a complementary set of A.

3 Preliminary for main results

3.1 Basic properties

3.1.1 Hitting probabilities

First, we compute probabilities that a simple random walk in $\mathbb{Z}_n^2$ doesn't hit a multipoint set until a certain random time. Given j distinct points $x_1,\ x_2, ..., x_j$ of $\mathbb{Z}_n^2$ and a non-empty subset $\tilde{D}$ of $\mathbb{Z}_n^2$ which is disjoint of $\{x_1, ..., x_j\}$, let $\tilde{\tau}$ denote the first time/enters $\tilde{D}$ and define for $1 \leq i, l \leq j$ and $y \notin \{x_1, ..., x_j\}$,

$$Q_{i,l} := \sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < \tilde{\tau} \wedge T_y).$$

By decomposing the probability $P^{x_i}(T_y = \tilde{\tau} \wedge T_y)$ by means of the last time the walk leaves the set $\{x_1, ..., x_j\}$ before $\tilde{\tau} \wedge T_y$ we obtain

$$P^{x_i}(T_y = \tilde{\tau} \wedge T_y) = \sum_{l=1}^{j} Q_{i,l} P^{x_l}(T_y = T_{x_1,\dots,x_j} \wedge \tilde{\tau} \wedge T_y),$$

for $1 \leq i \leq j$. The matrix $Q = (Q_{i,l})_{1 \leq i,l \leq j}$ is regarded as the Green kernel for the Markov chain on $\{x_1, ..., x_j\}$ with the substochastic transition matrix $q = (q_{i,l})_{1 \leq i,l \leq j}$ given by

$$q_{i,l} = P^{x_i}(T_{x_l} = T_{x_1,\dots,x_j} < \tilde{\tau} \wedge T_y),$$

so that $qQ = Q - E$, where E denotes the unit matrix. Accordingly, Q is regular and

$$Q^{-1} := E - q. \tag{3.1}$$

For any regular matrix A, let $\chi(A)$ (or χA) be the summation over all the elements of A^{-1}. Note that for $1 \leq u \leq j$,

$$P^{x_u}(T_y = T_{x_1,\dots,x_j} \wedge \tilde{\tau} \wedge T_y) = P^{x_u}(T_y = \tilde{\tau} \wedge T_y) \frac{\sum_{i=1}^{j} (\text{cofactor of } Q_{u,i})}{\det((Q_{i,l})_{1 \leq i,l \leq j})}. \tag{3.2}$$

Hence, we have

$$\begin{aligned} &\min_{1 \leq u \leq j} P^{x_u}(T_y = \tilde{\tau} \wedge T_y) \chi((Q_{i,l})_{1 \leq i,l \leq j}) \\ &\leq \sum_{l=1}^{j} P^{x_l}(T_y = T_{x_1,\dots,x_j} \wedge \tilde{\tau} \wedge T_y) \\ &\leq \max_{1 \leq u \leq j} P^{x_u}(T_y = \tilde{\tau} \wedge T_y) \chi((Q_{i,l})_{1 \leq i,l \leq j}). \end{aligned} \tag{3.3}$$

In particular, we consider the summation of (3.2) and (3.3) over $y \in \tilde{D}$. Note that $\sum_{y \in \tilde{D}} P^{x_u}(T_y = \tilde{\tau} \wedge T_y) = 1$ holds for any $1 \leq u \leq j$. Then, if we let

$$W_{i,l} := \sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < \tilde{\tau}),$$

we have for $1 \leq l \leq j$,

$$P^{x_l}(\tilde{\tau} < T_{x_1,\dots,x_j}) = \frac{\sum_{i=1}^{j} (\text{cofactor of } W_{l,i})}{\det((W_{i,l})_{1 \leq i,l \leq j})}, \tag{3.4}$$

$$\sum_{l=1}^{j} P^{x_l}(\tilde{\tau} < T_{x_1,\dots,x_j}) = \chi((W_{i,l})_{1 \leq i,l \leq j}). \tag{3.5}$$

In addition, we consider the case that $j=2$, the definition of $(\tilde{Q}_{i,l})_{1\le i,l\le j}$ yields that for $i\neq l\in\{1,2\}$,

$$\begin{aligned}P^{x_i}(T_{x_i}=T_{x_1,x_2}\wedge\tilde{\tau}) =&1-\frac{W_{l,l}}{W_{1,1}W_{2,2}-W_{1,2}W_{2,1}},\\ P^{x_i}(T_{x_l}=T_{x_1,x_2}\wedge\tilde{\tau}) =&\frac{W_{l,i}}{W_{1,1}W_{2,2}-W_{1,2}W_{2,1}}.\end{aligned}\tag{3.6}$$

Next, we introduce estimates of the hitting probabilities for a simple random walk in $\mathbb{Z}^2$ since we need only estimates for "$\mathbb{Z}^2$" in this paper. By Exercise 1.6.8 of [20] or (4.1) and (4.3) in [27], we have that, uniformly in $0<r<|x|<R$,

$$\begin{aligned}P^x(T_0<\tau_R) &= \frac{\log(R/|x|)+O(|x|^{-1})}{\log R}(1+O((\log|x|)^{-1})),\\ P^x(\tau_r<\tau_R) &= \frac{\log(R/|x|)+O(r^{-1})}{\log(R/r)}.\end{aligned}\tag{3.7}$$

For $x,\,y\in D(0,n)$ let $G_n(x,y):=\sum_{m=0}^{\infty}P^x(S_m=y,m<\tau_n)$. In addition, by Proposition 1.6.7 in [20] or (2.1) in [27], we have that for any $x\in D(0,n)$

$$\begin{aligned}G_n(x,0) &= \sum_{m=0}^{\infty}P^x(S_m=0,m<\tau_n)=\frac{2}{\pi}\log\left(\frac{n}{|x|}\right)+O(|x|^{-1}+n^{-1}),\\ G_n(0,0) &= \frac{2}{\pi}\log n+O(1).\end{aligned}\tag{3.8}$$

Therefore, we have that for $x,y\in D(0,n/3)$

$$G_n(x,y)=\frac{2}{\pi}\log\left(\frac{n}{d(x,y)}\right)+O(d(x,y)^{-1}+n^{-1}+1).\tag{3.9}$$

Indeed,

$$\begin{aligned}G_n(x,y)\le&\sum_{m=0}^{\infty}P^{x-y}(S_m=0,m<\tau_{4n/3})\\ =&\frac{2}{\pi}\log\left(\frac{n}{d(x,y)}\right)+O(d(x,y)^{-1}+n^{-1}+1),\\ G_n(x,y)\ge&\sum_{m=0}^{\infty}P^{x-y}(S_m=0,m<\tau_{2n/3})\\ =&\frac{2}{\pi}\log\left(\frac{n}{d(x,y)}\right)+O(d(x,y)^{-1}+n^{-1}+1).\end{aligned}$$

In addition, by (3.8) the strong Markov property yields

$$P(\tau_n<T_0)=(\sum_{m=0}^{\infty}P(S_m=0,m<\tau_n))^{-1}=\frac{\pi}{2\log n}(1+o(1)).\tag{3.10}$$

3.1.2 Probabilities conditioned by the local time

Now, we prepare some expression of the probability of events associated with a Markov chain on $\{1,2,3\}$. It will be used when considering transitions of the random walk in $\mathbb{Z}^2$ from a given set to another one. We can reduce such a problem to that for Markov chain by considering a sequence of hitting times.

Consider the general Markov chain $\{X_m\}_{m=0}^{\infty}$ of discrete time on the state space $\{1,2,3\}$. Let $d_{i,l}$ be the transition probability of $\{X_m\}_{m=0}^{\infty}$. We claim that for any $n_1,\ n_2 \in \mathbb{N}$,

$$\begin{aligned}
&P^1\Big(\sum_{m=1}^{n_1+n_2} 1_{\{X_m=1\}} = n_1,\ \sum_{m=1}^{n_1+n_2} 1_{\{X_m=2\}} = n_2, X_{n_1+n_2} = 2\Big)\\
&= \sum_{0\le i\le n_1\wedge(n_2-1)} d_{1,1}^{n_1-i} d_{1,2}^{i+1} \frac{n_1!}{(n_1-i)!i!} d_{2,1}^{i} d_{2,2}^{n_2-i-1} \frac{(n_2-1)!}{(n_2-i-1)!i!}
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&P^1\Big(\sum_{m=1}^{n_1+n_2} 1_{\{X_m=1\}} = n_1,\ \sum_{m=1}^{n_1+n_2} 1_{\{X_m=2\}} = n_2, X_{n_1+n_2} = 1\Big)\\
&= \sum_{1\le i\le n_1\wedge n_2} d_{1,1}^{n_1-i} d_{1,2}^{i} \frac{n_1!}{(n_1-i)!i!} d_{2,1}^{i} d_{2,2}^{n_2-i} \frac{(n_2-1)!}{(n_2-i)!(i-1)!},
\end{aligned} \tag{3.12}$$

where $0! = 1$. We will show (3.13) and (3.14) by induction on n_1+n_2. It is trivial that (3.13) and (3.14) hold for $n_1+n_2 = 2$, that is, $n_1 = n_2 = 1$. Fix $n \ge 2$ and assume that (3.13) and (3.14) hold for any $n_1,\ n_2 \in \mathbb{N}$ with $n_1+n_2 = n$. First, we will show that (3.13) holds for any $\tilde{n}_1,\ \tilde{n}_2 \in \mathbb{N}$ with $\tilde{n}_1+\tilde{n}_2 = n+1$. When $\tilde{n}_2 = 1$,

$$\begin{aligned}
&P^1\Big(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=1\}} = \tilde{n}_1,\ \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=2\}} = \tilde{n}_2, X_{\tilde{n}_1+\tilde{n}_2} = 2\Big)\\
=&P^1\Big(X_m = 1 \text{ for } 1 \le m \le \tilde{n}_1+\tilde{n}_2-1, X_{\tilde{n}_1+\tilde{n}_2} = 2\Big)\\
=&d_{1,1}^{\tilde{n}_1+\tilde{n}_2-1} d_{1,2},
\end{aligned}$$

and, hence, the claim holds. In addition, the Markov property yields that for any $\tilde{n}_2 \ge 2$ and $\tilde{n}_1 \in \mathbb{N}$,

$$\begin{aligned}
&P^1\Big(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=1\}} = \tilde{n}_1,\ \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2} 1_{\{X_m=2\}} = \tilde{n}_2, X_{\tilde{n}_1+\tilde{n}_2} = 2\Big)\\
=&d_{1,1}P^1\Big(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=1\}} = \tilde{n}_1-1,\ \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=2\}} = \tilde{n}_2, X_{\tilde{n}_1+\tilde{n}_2-1} = 2\Big)\\
&+ d_{1,2}P^2\Big(\sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=1\}} = \tilde{n}_1,\ \sum_{m=1}^{\tilde{n}_1+\tilde{n}_2-1} 1_{\{X_m=2\}} = \tilde{n}_2-1, X_{\tilde{n}_1+\tilde{n}_2-1} = 2\Big)\\
=&d_{1,1} \sum_{0\le i\le(\tilde{n}_1-1)\wedge(\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-1-i} d_{1,2}^{i+1} \frac{(\tilde{n}_1-1)!}{(\tilde{n}_1-1-i)!i!} d_{2,1}^{i} d_{2,2}^{\tilde{n}_2-1-i} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-1-i)!i!}\\
&+ d_{1,2} \sum_{1\le i\le \tilde{n}_1\wedge(\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-i} d_{1,2}^{i} \frac{(\tilde{n}_1-1)!}{(\tilde{n}_1-i)!(i-1)!} d_{2,1}^{i} d_{2,2}^{\tilde{n}_2-1-i} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-1-i)!i!}\\
=& \sum_{0\le i\le \tilde{n}_1\wedge(\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-i} d_{1,2}^{i+1} \frac{\tilde{n}_1!}{(\tilde{n}_1-i)!i!} d_{2,1}^{i} d_{2,2}^{\tilde{n}_2-i-1} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-i-1)!i!}.
\end{aligned}$$

The second equality comes from the assumption and the symmetry of sites 1 and 2. Therefore, we obtain (3.11). In addition, the same argument yields the corresponding result for (3.12), and, hence, we obtain the desired result.

3.1.3 Probabilities conditioned by the local time of edge

Now, we give some expression of the probability of events associated with a Markov chain. It will be used when considering transitions of the random walk in $\mathbb{Z}_n^2$ from a given set to another one. Consider the general Markov chain $\{X_m\}_{m=0}^{\infty}$ of discrete time on the state space $\mathbb{Z}$ such that $P^4(T_3 < \infty) = 1$. Let us extend the definition of $K(n, x)$ as follows: for $n \in \mathbb{N}$ and $i, l \in \mathbb{Z}$, set $K(n, e_{i,l}) := \sum_{m=0}^{n-1} 1_{\{X_m=i, X_{m+1}=l\}}$ and in addition $\mathcal{Q}(n, e_{i,l}) := \inf\{m > 0 : K(m, e_{i,l}) = n\}$. Let $\tilde{T}_2 := \inf\{m > T_1 : X_m = 2\}$ and $d_{i,l}$ as follows:

$$
\begin{aligned}
d_{1,1} :=& P^2(T_1 < T_4, \tilde{T}_2 < T_0), \quad d_{1,2} := P^2(T_4 < T_1),\\
d_{2,1} :=& P^3(T_1 < T_4, \tilde{T}_2 < T_0), \quad d_{2,2} := P^3(T_4 < T_1).
\end{aligned}
$$

We claim that for any n_1, $n_2 \in \mathbb{N}$,

$$
\begin{aligned}
&P^2(K(\mathcal{Q}(n_2, e_{3,4}), e_{1,2}) = n_1, \mathcal{Q}(n_2, e_{3,4}) < T_0)\\
&= \sum_{0 \le i \le n_1 \wedge (n_2-1)} d_{1,1}^{n_1-i} d_{1,2}^{i+1} \frac{n_1!}{(n_1-i)!i!} d_{2,1}^{i} d_{2,2}^{n_2-i-1} \frac{(n_2-1)!}{(n_2-i-1)!i!}
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
&P^3(K(\mathcal{Q}(n_2, e_{3,4}), e_{1,2}) = n_1, \mathcal{Q}(n_2, e_{3,4}) < T_0)\\
&= \sum_{1 \le i \le n_1 \wedge n_2} d_{1,1}^{n_1-i} d_{1,2}^{i} \frac{(n_1-1)!}{(n_1-i)!(i-1)!} d_{2,1}^{i} d_{2,2}^{n_2-i} \frac{n_2!}{(n_2-i)!i!},
\end{aligned}
\tag{3.14}
$$

where $0! = 1$. We will show (3.13) and (3.14) by induction on $n_1 + n_2$. It is trivial that (3.13) and (3.14) hold for $n_1 + n_2 = 2$, that is, $n_1 = n_2 = 1$. Fix $n \ge 2$ and assume that (3.13) and (3.14) hold for any n_1, $n_2 \in \mathbb{N}$ with $n_1 + n_2 = n$. First, we will show that (3.13) holds for any $\tilde{n}_1$, $\tilde{n}_2 \in \mathbb{N}$ with $\tilde{n}_1 + \tilde{n}_2 = n + 1$. When $\tilde{n}_2 = 1$,

$$
P^2(K(\mathcal{Q}(\tilde{n}_2, e_{3,4}), e_{1,2}) = \tilde{n}_1, \mathcal{Q}(\tilde{n}_2, e_{3,4}) < T_0) = d_{1,1}^{\tilde{n}_1+\tilde{n}_2-1} d_{1,2},
$$

and, hence, the claim holds. In addition, the strong Markov property yields that for any $\tilde{n}_2 \ge 2$ and $\tilde{n}_1 \in \mathbb{N}$,

$$
\begin{aligned}
&P^2(K(\mathcal{Q}(\tilde{n}_2, e_{3,4}), e_{1,2}) = \tilde{n}_1, \mathcal{Q}(\tilde{n}_2, e_{3,4}) < T_0)\\
=& d_{1,1} P^2(K(\mathcal{Q}(\tilde{n}_2, e_{3,4}), e_{1,2}) = \tilde{n}_1 - 1, \mathcal{Q}(\tilde{n}_2, e_{3,4}) < T_0)\\
&+ d_{1,2} P^3(K(\mathcal{Q}(\tilde{n}_2 - 1, e_{3,4}), e_{1,2}) = \tilde{n}_1, \mathcal{Q}(\tilde{n}_2 - 1, e_{3,4}) < T_0)\\
=& d_{1,1} \sum_{0 \le i \le (\tilde{n}_1-1) \wedge (\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-1-i} d_{1,2}^{i+1} \frac{(\tilde{n}_1-1)!}{(\tilde{n}_1-1-i)!i!} d_{2,1}^{i} d_{2,2}^{\tilde{n}_2-i-1} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-1-i)!i!}\\
&+ d_{1,2} \sum_{1 \le i \le \tilde{n}_1 \wedge (\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-i} d_{1,2}^{i} \frac{(\tilde{n}_1-1)!}{(\tilde{n}_1-i)!(i-1)!} d_{2,1}^{i} d_{2,2}^{\tilde{n}_2-1-i} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-1-i)!i!}\\
=& \sum_{0 \le i \le \tilde{n}_1 \wedge (\tilde{n}_2-1)} d_{1,1}^{\tilde{n}_1-i} d_{1,2}^{i+1} \frac{\tilde{n}_1!}{(\tilde{n}_1-i)!i!} d_{2,1}^{i} d_{2,2}^{\tilde{n}_2-i-1} \frac{(\tilde{n}_2-1)!}{(\tilde{n}_2-i-1)!i!}.
\end{aligned}
$$

The second equality comes from the assumption and the fact that $P^4(T_3 < \infty) = 1$. Therefore, we obtain (3.13). In addition, the same argument yields the corresponding result for (3.14), and, hence, we obtain the desired result.

4 Proof of Theorems 2.1 and 2.2

4.1 Proof of Theorem 2.2

4.1.1 Proof of the lower bound in Theorem 2.2

To show Theorem 2.2, we prepare some notations to make a correspondence of the framework of Section 3.2 to that of the present one. Let $\tilde{\alpha} := \lceil \frac{4\alpha}{\pi}(\log n)^2 \rceil$, and for x, x' which are two distinct points of $D(0,n)$, set

$$(U_1, U_2, U_3) := (x, x', \partial D(0,n))$$

and

$$b_{i,l} := P^{U_i}(\min_{s\in\{1,2,3\}} T_{U_s} = T_{U_l}),$$

for $i \in \{1,2\}$ and $l \in \{1,2,3\}$. As a first step, we will estimate $P(x,x' \in \Psi_n(\alpha))$. By the symmetry of x and x', we may assume that $b_{1,1} \leq b_{2,2}$ and

$$P(T_{x'} < T_x \wedge \tau_n) \leq P(T_x < T_{x'} \wedge \tau_n) \tag{4.1}$$

all the other three cases can be discussed in the same way as we will see below. Note that the time-reversal of simple random walk yields that $b_{1,2} = b_{2,1}$. Then, (3.13) yields

$$\begin{aligned}
P(x,x' \in \Psi_n(\alpha)) \geq & P(T_x < T_{x'} \wedge \tau_n) P^x(x,x' \in \Psi_n(\alpha), T_x^{\tilde{\alpha}} > T_{x'}^{\tilde{\alpha}}) \\
\geq & P(T_x < T_{x'} \wedge \tau_n) \\
& \times \sum_{0\leq i\leq \tilde{\alpha}-1} b_{1,1}^{\tilde{\alpha}-1-i} b_{1,2}^{i+1} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} b_{2,1}^{i} b_{2,2}^{\tilde{\alpha}-1-i} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} \\
= & P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \\
& \times \sum_{0\leq i\leq \tilde{\alpha}-1} b_{1,1}^{\tilde{\alpha}-1-i} b_{1,2}^{i} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} b_{2,1}^{i} b_{2,2}^{\tilde{\alpha}-1-i} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} \\
\geq & P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \left(\max_{0\leq i\leq \tilde{\alpha}-1} b_{1,1}^{\tilde{\alpha}-1-i} b_{1,2}^{i} \frac{(\tilde{\alpha}-1)!}{(\tilde{\alpha}-1-i)!i!} \right)^2 \\
\geq & P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \frac{1}{\tilde{\alpha}^2} (b_{1,1} + b_{1,2})^{2\tilde{\alpha}-2} \\
= & P(T_x < T_{x'} \wedge \tau_n) b_{1,2} \frac{1}{\tilde{\alpha}^2} P^x(T_{x,x'} < \tau_n)^{2\tilde{\alpha}-2}.
\end{aligned} \tag{4.2}$$

The right most number of (4.2), we estimate each of probabilities involved in it. Set $s := \log d(x,x')/\log n$ and consider $0 < \epsilon < 1$. Then, by (3.9), we obtain that for sufficiently large $n \in \mathbb{N}$, uniformly in x, $x' \in D(0, n^{1-\epsilon})$,

$$\sum_{m=0}^{\infty} P^{U_i}(S_m \in U_l, m < \tau_n) = \begin{cases} \frac{2}{\pi}(1+o(1))\log n & \text{if } i = l, \\ \frac{2(1-s)}{\pi}(1+o(1))\log n & \text{if } i \neq l, \end{cases} \tag{4.3}$$

and, hence, if we set $\tilde{\tau} = \tau_n$ in (3.5), we have

$$b_{1,3} = \frac{\pi(1+o(1))}{2(2-s)\log n}. \tag{4.4}$$

Then, we obtain that, uniformly in x, x',

$$\begin{aligned}
P^x(T_{x,x'} < \tau_n)^{2\tilde{\alpha}} = & \exp(-2\tilde{\alpha} b_{1,3} + o(1)\log n) \\
= & \exp\left(-\frac{\tilde{\alpha}\pi}{(2-s)\log n} + o(1)\log n \right).
\end{aligned} \tag{4.5}$$

In addition, (3.10) yields that, uniformly in x,

$$P^x(\tau_n < T_x) = \frac{\pi}{2\log n}(1+o(1)) \tag{4.6}$$

and (3.7) yields that for any $\epsilon > 0$ there exists $c > 0$ such that for all sufficiently large $n \in \mathbb{N}$

$$P^x(T_{x'} < \tau_n) \geq P^x(T_{x'} < T_{\partial D(x',n/2)}) \geq \frac{\log(n/2n^{1-\epsilon})}{\log n/2}(1+O((\log n)^{-1})) \geq c, \tag{4.7}$$

$$P(T_x < \tau_n) = P^x(T_0 < \tau_n) \geq \frac{\log(n/n^{1-\epsilon})}{\log n}(1+O((\log n)^{-1})) \geq c, \tag{4.8}$$

where the equality comes from the time-reversal of simple random walk. Note that

$$P^x(T_{x'} < \tau_n) = \sum_{i=0}^{\infty} b_{1,1}^i b_{1,2} = \frac{1}{1-b_{1,1}} b_{1,2}$$

and, hence, by (4.6) and (4.7), for all sufficiently large $n \in \mathbb{N}$,

$$b_{1,2} \geq P^x(\tau_n < T_x) P^x(T_{x'} < \tau_n) \geq \frac{c}{\log n}.$$

In addition, (4.1) and (4.8) yield that there exists $c > 0$ such that for any $n \in \mathbb{N}$,

$$P(T_x < T_{x'} \wedge \tau_n) \geq \frac{1}{2} P(T_{x,x'} < \tau_n) \geq \frac{1}{2} P(T_x < \tau_n) \geq c.$$

We now go back to (4.2). By applying all these estimates, for any $\delta > 0$ and $\epsilon > 0$ there exists such that $n_0 \in \mathbb{N}$ for $n \geq n_0$ and x, $x' \in D(0, n^{1-\epsilon})$,

$$P(x, x' \in \Psi_n(\alpha)) \geq \exp\left(- \frac{\tilde{\alpha}\pi}{(2-s)\log n} - \delta \log n \right). \tag{4.9}$$

Now we turn to the desired lower bound. By (4.9), for any $\delta > 0$, $\epsilon > 0$, $\beta' \in [\epsilon, \beta]$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} E[|\{(x,x') \in \Psi_n(\alpha)^2 : d(x,x') \leq n^\beta\}|] &\geq \sum_{\substack{x,x' \in D(0,n^{1-\epsilon}), \\ 0 < d(x,x') \leq n^{\beta'}}} P(x,x' \in \Psi_n(\alpha)) \\ &\geq \sum_{\substack{x,x' \in D(0,n^{1-\epsilon}), \\ 0 < d(x,x') \leq n^{\beta'}}} \exp\left(- \frac{\tilde{\alpha}\pi}{(2-\beta')\log n} - \delta \log n \right) \\ &\geq n^{2+2\beta' - 2\epsilon - 4\alpha/(2-\beta') - 2\delta}. \end{aligned}$$

Note that $2 + 2\beta' - 4\alpha/(2-\beta')|_{\beta' = \beta \wedge (2-\sqrt{2\alpha})} = \hat{\rho}_2(\alpha, \beta)$. Then, since $\delta > 0$ and $\epsilon > 0$ are arbitrary, we obtain the desired result.

4.1.2 Proof of the upper bound in Theorem 2.2

The idea of the upper bound in Theorem 2.2 is similar to that of the lower bound, while we do not require a combinatorial argument in Section 3.2. Indeed, the strong Markov property yields that

$$\begin{aligned}
&P(x,x' \in \Psi_n(\alpha)) \\
\le &P(|\{l < \tau_n : S_l \in \{x',x\}\}| \ge 2\tilde{\alpha}) \\
= &P(T_x < T_{x'} \wedge \tau_n) P^x(|\{l < \tau_n : S_l \in \{x',x\}\}| \ge 2\tilde{\alpha} - 1) \\
&\quad + P(T_{x'} < T_x \wedge \tau_n) P^{x'}(|\{l < \tau_n : S_l \in \{x',x\}\}| \ge 2\tilde{\alpha} - 1) \\
\le &\max_{y \in \{x,x'\}} P^y(|\{l < \tau_n : S_l \in \{x',x\}\}| \ge 2\tilde{\alpha} - 1) \\
\le &\max_{y' \in \{x,x'\}} P^{y'}(T_{x,x'} < \tau_n) \max_{y \in \{x,x'\}} P^y(|\{l < \tau_n : S_l \in \{x',x\}\}| \ge 2\tilde{\alpha} - 2) \\
\le &\cdots \\
\le &\max_{y' \in \{x,x'\}} P^{y'}(T_{x,x'} < \tau_n)^{2\tilde{\alpha}-1} \\
\le &\max_{y' \in \{x,x'\}} P^{y'}(T_{x,x'} < \tau_{2n})^{2\tilde{\alpha}-1}.
\end{aligned} \tag{4.10}$$

Then, letting $\tilde{\tau} = \tau_{2n}$ in (3.5), and arguing similarly as we obtained (4.3) and (4.4), we obtain

$$1 - \max_{y' \in \{x,x'\}} P^{y'}(T_{x,x'} < \tau_{2n}) = \frac{\pi(1+o(1))}{2(2-s)\log n},$$

uniformly in x, $x' \in D(0,n)$. Hence, by (4.10), for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ and x, $x' \in D(0,n)$,

$$P(x,x' \in \Psi_n(\alpha)) \le \exp\left(- \frac{\tilde{\alpha}\pi}{(2-s)\log n} + \delta \log n \right).$$

Unlike the lower bound, we need more estimate to bound the summation of $P(x,x' \in \Psi_n(\alpha))$ in (x,x'). Since $\min\{1/\sqrt{\alpha}, 2/(2-\beta)\} \ge 1$,

$$\hat{\rho}_2(\alpha,\beta) \ge \rho_2(\alpha,\beta) \ge 2 + 2\beta - 2\alpha F_{2,\beta}(1) = 2(1-\alpha) + 2\beta(1-\alpha). \tag{4.11}$$

In addition, (3.10) yields that for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$,

$$\begin{aligned}
E[|\Psi_n(\alpha)|] = &\sum_{x \in D(0,n)} P(x \in \Psi_n(\alpha)) \\
\le &\sum_{x \in D(0,n)} P^x(T_x < \tau_n)^{\tilde{\alpha}-1} \\
\le &\sum_{x \in D(0,n)} P^x(T_x < T_{\partial D(x,2n)})^{\tilde{\alpha}-1} \\
\le &4n^2 \times \left(1 - \frac{\pi}{2\log n} + o\left(\frac{1}{\log n}\right)\right)^{\tilde{\alpha}-1} \le n^{2(1-\alpha)+\delta}.
\end{aligned} \tag{4.12}$$

Now we are ready to enter the main part of the proof. By (4.12), we obtain

$$\begin{aligned}
&E[|\{(x,x') \in \Psi_n(\alpha)^2 : d(x,x') \le n^{\beta(1-\alpha)}\}|] \\
\le &E[|\Psi_n(\alpha)|] \times Cn^{2\beta(1-\alpha)} \le n^{\hat{\rho}_2(\alpha,\beta)+\delta}.
\end{aligned} \tag{4.13}$$

In addition, for any $\delta > 0$, $0 < h < 1$, $\beta' \in [\beta(1-\alpha)h^{-1}, \beta]$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}
&E[|\{(x,x') \in \Psi_n(\alpha)^2 : n^{\beta' h} < d(x,x') \le n^{\beta'}\}|] \\
&= \sum_{\substack{x,x' \in D(0,n), \\ n^{\beta' h} < d(x,x') \le n^{\beta'}}} P(x,x' \in \Psi_n(\alpha)) \\
&\le \sum_{\substack{x,x' \in D(0,n), \\ n^{\beta' h} < d(x,x') \le n^{\beta'}}} \exp\left(- \frac{\tilde{\alpha}\pi}{(2-\beta' h)\log n} + 2\delta \log n \right) \\
&\le n^{2+2\beta' - 4\alpha/(2-\beta' h) + 3\delta}.
\end{aligned} \tag{4.14}$$

Note that $\max_{(1-\alpha)\beta \le \beta' \le \beta} 2 + 2\beta' - 4\alpha/(2-\beta') = \hat{\rho}_2(\alpha,\beta)$. Then, since $\delta > 0$ and $0 < h < 1$ are arbitrary, (4.13) and (4.14) yield the desired result.

4.2 Proof of Theorem 2.1

4.2.1 Key lemma for the proof of the upper bound in Theorem 2.1

this section provides large deviation estimates that are keys to the proof of the upper bound in Theorem 2.1. In addition, we start by defining the several sequence. Let $r_0 := 0$ and $r_k := (k!)^3$ for $k \in \mathbb{N}$. Set $K_n := n^3 r_n$ for $n \in \mathbb{N}$. Now fix $0 < \alpha, \beta < 1$. Set $n_k = 6\alpha(n-k)^2 \log k$. For $z \in D(0, K_n)$, let $N^z_{n,k}$, denote the number of excursions from $\partial D(z, r_k)$ to $\partial D(z, r_{k-1})$ until time τ_{K_n}. To state the assertion, we define the domain $\mathbf{Go}^{\beta}$ and $\mathbf{Go}^{h,\beta}$ by

$$\begin{aligned}
\mathbf{Go}^{\beta} &:= \{(z,x) : z \in D(0,K_n), x \in D(z, r_{\beta n - 2}) \cap D(0,K_n)\}, \\
\mathbf{Go}^{h,\beta} &:= \{(z,x,x') : (z,x),(z,x') \in \mathbf{Go}^{\beta}, d(x,x') \ge r_{\beta h n/2 - 3}\}.
\end{aligned}$$

Let $\tilde{\Psi}_n(\alpha) := \{x : K(\tau_n, x) \in [4\alpha/\pi(\log n)^2, 4/\pi(\log n)^2]\}$. Recall that we set $\hat{F}_{h,\beta}(\gamma) := \gamma^2(1-\beta) + h(1-\gamma(1-\beta))^2/\beta$.

Lemma 4.1.

(1) *For any $\delta > 0$, there exists $C > 0$ such that for any $\gamma \ge 0$,*

$$\max_{z \in D(0,K_n)} P\left(\frac{N^z_{n,\beta n}}{n_{\beta n}} \ge \gamma^2 \right) \le C K_n^{-2\alpha \hat{F}_{0,\beta}(\gamma) + \delta}. \tag{4.15}$$

(2) *For any $\delta, \delta' > 0$, there exist $C > 0$ and $0 < h < 2$ (with h close to 2) such that for any $\gamma \in [0, 2/(2-\beta)]$,*

$$\max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha), \gamma^2 \le \frac{N^z_{n,\beta n}}{n_{\beta n}} < \gamma^2 + \delta' \right) \le C K_n^{-2\alpha \hat{F}_{2,\beta}(\gamma) + \delta}. \tag{4.16}$$

From (4.16), we obtain the following bound, which we will use in the sequel in a crucial way.

Corollary 4.2. *For any $\delta > 0$, there exist C, $\delta_0 > 0$ such that*

$$\max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P\left(x, x' \in \tilde{\Psi}_{K_n}(\alpha), \frac{N^z_{n,\beta n}}{n_{\beta n}} < \left(\frac{1}{\sqrt{\alpha}} + \delta_0 \right)^2 \right) \le C K_n^{-2\alpha \hat{F}_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\}) + 2\delta}.$$

Proof. Take $\delta_0 > 0$ satisfying

$$2\alpha |\hat{F}_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\} + \delta_0) - \hat{F}_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\})| \le \delta. \tag{4.17}$$

If we set $\delta' = \delta_0$ and $C' = \lceil((1/\sqrt{\alpha} + \delta_0)^2 + \delta_0)/\delta_0\rceil$, (4.16) yields that

$$\begin{aligned}
&\max_{(z,x,x')\in\mathbf{Go}^{h,\beta}} P\bigg(x, x' \in \tilde{\Psi}_{K_n}(\alpha), \frac{N^z_{n,\beta n}}{n_{\beta n}} < \Big(\frac{1}{\sqrt{\alpha}} + \delta_0\Big)^2\bigg) \\
\le& C' \max_{\gamma' \le (1/\sqrt{\alpha}+\delta_0)^2} K_n^{-2\alpha\hat{F}_{2,\beta}(\sqrt{\gamma'})+\delta} \\
=& C' K_n^{-2\alpha\hat{F}_{2,\beta}(\min\{1/\sqrt{\alpha}+\delta_0, 2/(2-\beta)\})+\delta} \\
\le& C' K_n^{-2\alpha\hat{F}_{2,\beta}(\min\{1/\sqrt{\alpha}, 2/(2-\beta)\})+2\delta}.
\end{aligned}$$

Here the equality comes from the fact that $\hat{F}_{2,\beta}$ is minimized at $2/(2-\beta)$ and the last inequality is a consequence of the choice of δ_0 in (4.17). Hence, we obtain the desired result. □

Let us enter the proof of Lemma 4.1. The proof of the first assertion (4.15) is rather simple.

Proof of Lemma 4.1 (1). Substituting $R = K_n$ and $r = r_{\beta n-1}$ for (3.7), the strong Markov property provides

$$\begin{aligned}
\max_{z\in D(0,K_n)} P\bigg(\frac{N^z_{n,\beta n}}{n_{\beta n}} \ge \gamma^2\bigg) \le& \max_{z\in D(0,K_n), y\in\partial D(z,r_{\beta n})} P^y(T_{\partial D(z,r_{\beta n-1})} < T_{\partial D(z,2K_n)})^{\gamma^2 n_{\beta n}} \\
\le& \max_{y\in\partial D(0,r_{\beta n})} P^y(\tau_{r_{\beta n-1}} < \tau_{2K_n})^{\gamma^2 n_{\beta n}} \\
\le& \bigg(\frac{\log(2K_n/r_{\beta n}) + O(r^{-1}_{\beta n-1})}{\log(2K_n/r_{\beta n-1})}\bigg)^{\gamma^2 n_{\beta n}} \\
=& \bigg(1 - \frac{1+o(1)}{n-\beta n}\bigg)^{\gamma^2 n_{\beta n}} \\
=& \exp(-6\alpha\gamma^2(n-\beta n)\log n + o(n\log n)) \\
=& C K_n^{-2\alpha F_{0,\beta}(\gamma)+o(1)}
\end{aligned}$$

and, hence, the desired result holds. □

To show the second assertion (4.16), we define and estimate the following probabilities. For any $(z, x, x') \in \mathbf{Go}^{h,\beta}$, let

$$(V_1, V_2) := (\partial D(z, r_{\beta n}), \{x, x'\})$$

and for $l = 1, 2$,

$$\begin{aligned}
a_{1,l} :=& \max_{y\in\partial D(z,r_{\beta n-1})} P^y(\min_{s\in\{1,2\}} T_{V_s} = T_{V_l}), \\
a_{2,l} :=& \max_{y\in\{x,x'\}} P^y(\min_{s\in\{1,2\}} T_{V_s} = T_{V_l}).
\end{aligned}$$

Lemma 4.3. *For any $\epsilon > 0$ there exists $0 < h < 2$ such that for any $(z, x, x') \in \mathbf{Go}^{h,\beta}$ and all sufficiently large $n \in \mathbb{N}$, the following hold:*

$$a_{1,1} \le 1 - \frac{2-\epsilon}{\beta n}, \quad a_{1,2} \le \frac{2+\epsilon}{\beta n}, \tag{4.18}$$

$$a_{2,1} \le \frac{\pi+\epsilon}{6\beta n\log n}, \quad a_{2,2} \le 1 - \frac{\pi-\epsilon}{6\beta n\log n}. \tag{4.19}$$

Proof. Fix $\epsilon > 0$. First, we will show (4.18). By (6.16) in [11], we obtain for all sufficiently large $n \in \mathbb{N}$,

$$a_{1,1} \leq 1 - \frac{2-\epsilon}{\beta n}.$$

In addition, since $D(x, r_{\beta n-1}/2) \subset D(z, r_{\beta n-1})$ and $D(z, r_{\beta n}) \subset D(x, 2r_{\beta n})$ hold for $(z,x) \in \mathbf{Go}^{\beta}$, (3.7) yields that for $y' \in \{x, x'\}$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}
\max_{y \in \partial D(z, r_{\beta n-1})} P^y(T_{y'} < T_{\partial D(z, r_{\beta n})}) &\leq \max_{y \in D(y', r_{\beta n-1}/2)^c} P^y(T_{y'} < T_{\partial D(y', 2r_{\beta n})}) \\
&= \max_{y \in \partial D(y', r_{\beta n-1}/2)} P^y(T_{y'} < T_{\partial D(y', 2r_{\beta n})}) \\
&\leq \frac{\log(4r_{\beta n}/r_{\beta n-1}) + O(r_{\beta n-1}^{-1})}{\log r_{\beta n}}(1+o(1)) \\
&\leq \frac{1+\epsilon}{\beta n}.
\end{aligned}$$

Since $a_{1,2} \leq \max_{y \in \partial D(z, r_{\beta n-1})} P^y(T_x < T_{\partial D(z, r_{\beta n})}) + \max_{y \in \partial D(z, r_{\beta n-1})} P^y(T_{x'} < T_{\partial D(z, r_{\beta n})})$, we have (4.18). Next, we will estimate (4.19). By (3.9) and (3.10), for any $(z, x, x') \in \mathbf{Go}^{h,\beta}$ and $y, y' \in \{x, x'\}$,

$$\sum_{m=0}^{\infty} P^y(S_m = y', m < T_{\partial D(z, r_{\beta n})}) \begin{cases} = \frac{2+o(1)}{\pi} 3\beta n \log n & \text{if } y = y', \\ \leq \frac{2+o(1)}{\pi} \frac{3\beta(2-h)}{2} n \log n & \text{if } y \neq y'. \end{cases} \tag{4.20}$$

Hence, if we substitute $T_{\partial D(z, r_{\beta n})}$ into $\tilde{\tau}$ in (3.5), (4.20) and picking $h < 2$ close to 2 yield (4.19) for all sufficiently large $n \in \mathbb{N}$. $\square$

Proof of Lemma 4.1 (2). Let

$$\tilde{T} := \inf\{m > T_{\partial D(z, r_{\beta n-1})} : S_m \in \partial D(z, r_{\beta n})\}$$

and

$$\begin{aligned}
a'_{1,1} :&= \max_{y \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < \tau_{K_n}, \tilde{T} < T_{x_1, x_2}), \\
a'_{1,2} :&= \max_{y \in \partial D(z, r_{\beta n})} P^y(T_{\partial D(z, r_{\beta n-1})} < \tau_{K_n}, \tilde{T} > T_{x_1, x_2}).
\end{aligned}$$

Then, the strong Markov property yields for any c_1, c_2, $c_3 \in \mathbb{N}$,

$$\begin{aligned}
&\max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} P(K(\tau_{K_n}, x) = c_2, K(\tau_{K_n}, x') = c_3, N^z_{n,\beta n} = c_1) \\
\leq &\max_{(z,x,x') \in \mathbf{Go}^{h,\beta}} \max_{y \in \{x,x'\}} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N^z_{n,\beta n} = c_1) \\
+ &\max_{\substack{(z,x,x') \in \mathbf{Go}^{h,\beta}, \\ y \in \partial D(z, r_{\beta n})}} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N^z_{n,\beta n} = c_1).
\end{aligned} \tag{4.21}$$

Now we use weight $a_{2,l}$ (or $a'_{1,l}$) instead of $d_{i,l}$ in (3.11) or (3.12). For $z \in D(0, K_n)$, let $M^z_{n,k}$ be the number of excursions from $\partial D(z, r_{k-1})$ to $\partial D(z, r_k)$ until time τ_{K_n}. Note that if we consider the simple random walk starting at x or x', it holds that

$$K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N^z_{n,\beta n} = c_1) \subset \{M^z_{n,\beta n} = c_1 + 1\},$$

then we can consider the path corresponding to (3.11) as $n_1 = c_2 + c_3 + 1$ and $n_2 = c_1 + 1$. In addition, if we consider the simple random walk starting at $\partial D(z, r_{\beta n})$, it holds that

$$K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N^z_{n,\beta n} = c_1) \subset \{M^z_{n,\beta n} = c_1\},$$

then we can consider the path corresponding to (3.11) as $n_1 = c_2 + c_3$ and $n_2 = c_1$. Therefore, for $c_1 \le c_2 + c_3 - 1$, (3.11) and (3.12) yield

$$
\begin{aligned}
&\max_{y\in\{x,x'\}} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N^z_{n,\beta n} = c_1) \\
&\le \sum_{0\le i\le c_1} {a'}^{c_1-i}_{1,1} {a'}^{i}_{1,2} \frac{c_1!}{(c_1-i)!i!} a^{i+1}_{2,1} a^{c_2+c_3-1-i}_{2,2} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!i!}, \\
&\max_{y\in\partial D(z,r_{\beta n})} P^y(K(\tau_{K_n}, x) + K(\tau_{K_n}, x') = c_2 + c_3, N^z_{n,\beta n} = c_1) \\
&\le \sum_{1\le i\le c_1} {a'}^{c_1-i}_{1,1} {a'}^{i}_{1,2} \frac{c_1!}{(c_1-i)!i!} a^{i}_{2,1} a^{c_2+c_3-i}_{2,2} \frac{(c_2+c_3-1)!}{(c_2+c_3-i)!(i-1)!}.
\end{aligned}
$$

Let us denote the probability in the left hand side of (4.21) by A. Then the above estimate implies

$$
A \le 2(c_1+1) \max_{0\le i\le c_1} {a'}^{c_1-i}_{1,1} {a'}^{i}_{1,2} \frac{c_1!}{(c_1-i)!i!} a^{i}_{2,1} a^{c_2+c_3-1-i}_{2,2} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!(i-1)!}. \tag{4.22}
$$

Now, the strong Markov property yields that for any $i = 1, 2$

$$
a'_{1,i} \le \max_{y\in\partial D(z,r_{\beta n})} P^y(T_{\partial D(z,r_{\beta n-1})} < \tau_{K_n}) a_{1,i}.
$$

Note that for any $0 \le i \le c_1$,

$$
{a'}^{c_1-i}_{1,1} {a'}^{i}_{1,2} \le (\max_{y\in\partial D(z,r_{\beta n})} P^y(T_{\partial D(z,r_{\beta n-1})} < \tau_{K_n}))^{c_1} a^{c_1-i}_{1,1} a^{i}_{1,2}. \tag{4.23}
$$

Now, we combine the last estimate with our assertion. For $\gamma \in [0, 2/(2-\beta)]$ and $\delta > 0$, we take c_1, c_2 and c_3 so that $\gamma^2 \le c_1/n_{\beta n} < \gamma^2 + \delta_0$, $c_2, c_3 \in [4\alpha(\log K_n)^2/\pi, 4(\log K_n)^2/\pi]$. It holds that $c_1 \le c_2 + c_3 - 1$ for sufficiently large $n \in \mathbb{N}$. Then, (3.7) yields that there exists $C > 0$ such that for all sufficiently large $n \in \mathbb{N}$,

$$
(\max_{y\in\partial D(z,r_{\beta n})} P^y(T_{\partial D(z,r_{\beta n-1})} < \tau_{K_n}))^{c_1} \le C K_n^{-2\alpha \hat{F}_{0,\beta}(\gamma)+\delta/4}. \tag{4.24}
$$

Therefore, by applying (4.23) and (4.24) to (4.22), we obtain

$$
A \le C K_n^{-2\alpha \hat{F}_{0,\beta}(\gamma)+\delta/4} \max_{0\le i\le c_1} a^{c_1-i}_{1,1} a^{i}_{1,2} \frac{c_1!}{(c_1-i)!i!} a^{i}_{2,1} a^{c_2+c_3-1-i}_{2,2} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!(i-1)!}. \tag{4.25}
$$

With the aid of Lemma 4.3 we will estimate the maximal term in (4.25). Let us define $g(i)$ by

$$
\begin{aligned}
g(i) :=& \left(1 - \frac{2-\epsilon}{\beta n}\right)^{c_1-i} \left(\frac{2+\epsilon}{\beta n}\right)^{i} \frac{c_1!}{(c_1-i)!i!} \\
&\times \left(\frac{\pi+\epsilon}{6\beta n \log n}\right)^{i} \left(1 - \frac{\pi-\epsilon}{6\beta n \log n}\right)^{c_2+c_3-1-i} \frac{(c_2+c_3-1)!}{(c_2+c_3-1-i)!(i-1)!}.
\end{aligned}
$$

Then, by Lemma 4.3 and (4.25), we have $A \le C K_n^{-2\alpha \hat{F}_{0,\beta}(\gamma)+\delta/4} \max_{0\le i\le c_1} g(i)$. Note that for any $1 \le i \le c_1$,

$$
\frac{g(i-1)}{g(i)} = \frac{(1 - \frac{\pi-\epsilon}{6\beta n \log n})(1 - \frac{2-\epsilon}{\beta n}) i(i-1)}{(\frac{\pi+\epsilon}{6\beta n \log n})(\frac{2+\epsilon}{\beta n})(c_2+c_3-i)(c_1+1-i)}.
$$

By taking the growth order of c_1, c_2 and c_3 into account, the maximum of $g(i)$ among $0 \le i \le c_1$ is attained at

$$
i = \left\lceil (1+o(1)) \sqrt{(c_2+c_3) c_1 \frac{2+\epsilon}{\beta n} \times \frac{\pi+\epsilon}{6\beta n \log n}} \right\rceil - 1.
$$

In addition, the Stirling formula yields

$$
\begin{aligned}
&g\left(\left\lceil (1+o(1))\sqrt{(c_2+c_3)c_1\frac{2+\epsilon}{\beta n}\times\frac{\pi+\epsilon}{6\beta n\log n}}\right\rceil-1\right)\\
\leq & K_n^{-4\alpha(\sqrt{(c_2'+c_3')/2}-\gamma(1-\beta))^2/\beta+f(\epsilon)+\delta/8}\\
\leq & K_n^{-4\alpha(1-\gamma(1-\beta))^2/\beta+f(\epsilon)+\delta/8},
\end{aligned}
$$

where $f(\epsilon)\to 0$ as $\epsilon\to 0$, $c_2'=c_2\pi/(4\alpha(\log K_n)^2)$ and $c_3'=c_3\pi/(4\alpha(\log K_n)^2)$. The last inequality comes from $\gamma\leq 2/(2-\beta)\leq 1/(1-\beta)$ for $\gamma\in[0,2/(2-\beta)]$. Hence, we obtain

$$
\begin{aligned}
&\max_{(z,x,x')\in\mathbf{Go}^{h,\beta}} P(K(\tau_{K_n},x)=c_2,K(\tau_{K_n},x')=c_3,N^z_{n,\beta n}=c_1)\\
\leq & CK_n^{-2\alpha\hat{F}_{0,\beta}(\gamma)+\delta/4}K_n^{-4\alpha(1-\gamma(1-\beta))^2/\beta+\delta/4}\\
= & CK_n^{-2\alpha\hat{F}_{2,\beta}(\gamma)+\delta/2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\max_{(z,x,x')\in\mathbf{Go}^{h,\beta}} P\left(x,x'\in\tilde{\Psi}_{K_n}(\alpha),\gamma^2\leq\frac{N^z_{n,\beta n}}{n_{\beta n}}<\gamma^2+\delta_0\right)\\
\leq & \sum_{\substack{\gamma^2\leq c_1/n_{\beta n}<\gamma^2+\delta_0,\\ c_2,c_3\in[4\alpha/\pi(\log K_n)^2,4/\pi(\log K_n)^2]}} \max_{(z,x,x')\in\mathbf{Go}^{h,\beta}} P(K(\tau_{K_n},x)=c_2,K(\tau_{K_n},x')=c_3,N^z_{n,\beta n}=c_1)\\
\leq & CK_n^{-2\alpha\hat{F}_{2,\beta}(\gamma)+\delta},
\end{aligned}
$$

and, hence, the desired result holds. □

4.2.2 Proof of the upper bound in Theorem 2.1

We keep using notations introduced at the beginning of the last subsection. We will prove in this section that for any $0<\alpha,\beta,\delta<1$, there exist $C>0$ and $\epsilon>0$ such that

$$
P(|\tilde{\Theta}_{\alpha,0,\beta,n}|\geq K_n^{\rho_2(\alpha,\beta)+4\delta})\leq CK_n^{-\epsilon}, \tag{4.26}
$$

where

$$
\tilde{\Theta}_{\alpha,\beta_2,\beta_1,n}:=\{(x,x')\in\tilde{\Psi}_{K_n}(\alpha)^2 : r_{(\beta_2 n-3)\vee 0}\leq d(x,x')\leq r_{(\beta_1 n-3)\vee 0}\}.
$$

Indeed, (4.26) implies the upper bound in Theorem 2.1. We begin with observing this.

Note that Theorem 1.1 in [9] provides that for all sufficiently large $n\in\mathbb{N}$,

$$
|\Psi_{K_n}(\alpha)\cap\tilde{\Psi}_{K_n}(\alpha)^c|\leq K_n^{\delta}\quad \text{a.s.} \tag{4.27}
$$

Again, Theorem 1.1 in [9] yields $|\Psi_{K_n}(\alpha)|\leq K_n^{2(1-\alpha)+\delta}$ for all sufficiently large $n\in\mathbb{N}$ a.s. and, hence, (4.11) yields

$$
|\Psi_{K_n}(\alpha)|\leq K_n^{\rho_2(\alpha,\beta)+\delta}\quad \text{a.s.} \tag{4.28}
$$

Then, (4.27) and (4.28) yield that for all sufficiently large $n\in\mathbb{N}$,

$$
|\{(x,x') : x\in\Psi_{K_n}(\alpha),x'\in\Psi_{K_n}(\alpha)\cap\tilde{\Psi}_{K_n}(\alpha)^c,d(x,x')\leq r_{\beta n-3}\}|\leq K_n^{\rho_2(\alpha,\beta)+2\delta}\quad \text{a.s.} \tag{4.29}
$$

Note that $\log r_{\beta n-3}/\log K_{n+1}\to\beta$ and $\beta\mapsto\rho_2(\alpha,\beta)$ is continuous. Hence, if we obtain (4.26), the symmetry of x and x', Borel-Cantelli lemma and (4.29) yield the upper bound in Theorem 2.1.

To show (4.26), we divide the assertion into two corresponding estimate for $\tilde{\Theta}_{\alpha,0,\beta(1-\alpha),n}$ and $\tilde{\Theta}_{\alpha,\beta(1-\alpha),\beta,n}$. For the former one, note

$$|\tilde{\Theta}_{\alpha,0,\beta(1-\alpha),n}| \le 4r_{\beta n-3}^2 |\tilde{\Psi}_{K_n}(\alpha)|.$$

Therefore, by (2.18) in [27], (4.11) yields

$$\begin{aligned} P(|\tilde{\Theta}_{\alpha,0,\beta(1-\alpha),n}| \ge K_n^{\rho_2(\alpha,\beta)+4\delta}) \le& P(|\tilde{\Psi}_{K_n}(\alpha)| \ge K_n^{2(1-\alpha)+3\delta}) \\ \le& P(|\Psi_{K_n}(\alpha)| \ge K_n^{2(1-\alpha)+3\delta}) \le CK_n^{-\epsilon}. \end{aligned} \tag{4.30}$$

For the latter one, we use the following lemma.

Lemma 4.4. *There exist $h < 2$, $C > 0$ and $\epsilon > 0$ such that for any $\beta' \in [\beta(1-\alpha), \beta]$*

$$q_{n,\beta'} := P(|\tilde{\Theta}_{\alpha,\beta' h/2,\beta',n}| \ge K_n^{\rho_2(\alpha,\beta')+4\delta}) \le CK_n^{-\epsilon}.$$

We observe that the assertion for $\tilde{\Theta}_{\alpha,\beta(1-\alpha),\beta,n}$ indeed follows from this lemma. Set $\beta_j = \beta(h/2)^j$ and $l = \min\{j : \beta_j \le \beta(1-\alpha)\}$. By Lemma 4.4 we have $q_{n,\beta_j} \to 0$ as $n \to \infty$ for $j = 0,..,l-1$. Combining this with the monotonicity of $\beta \mapsto \rho_2(\alpha,\beta)$, we obtain the desired assertion. By virtue of this and (4.30), we establish (4.26). Thus, the proof of the upper bound is now reduced to show Lemma 4.4.

Proof of Lemma 4.4. Let $\hat{Z}_{n,\beta'} := 4r_{\beta' n-4}\mathbb{Z}^2 \cap D(0,K_n)$ and $z_{\beta'}(x)$ the point closest to x in $\hat{Z}_{n,\beta'}$. Let δ_0 be as in Corollary 4.2. By the simple argument, we have

$$q_{n,\beta'} \le I_1 + I_2,$$

where

$$\begin{aligned} I_1 :=& P\Bigg(\max_{z\in \hat{Z}_{n,\beta'}} \frac{N^z_{n,\beta' n}}{n_{\beta' n}} \ge \Big(\frac{1}{\sqrt{\alpha}} + \delta_0\Big)^2 \Bigg), \\ I_2 :=& P\Bigg(|\tilde{\Theta}_{\alpha,\beta' h/2,\beta',n}| \ge K_n^{\rho_2(\alpha,\beta')+4\delta}; \max_{z\in \hat{Z}_{n,\beta'}} \frac{N^z_{n,\beta' n}}{n_{\beta' n}} < \Big(\frac{1}{\sqrt{\alpha}} + \delta_0\Big)^2 \Bigg). \end{aligned}$$

Then,

$$\begin{aligned} I_2 \le& P\Bigg(\Bigg| \Bigg\{ (x,x') \in \tilde{\Psi}_{K_n}(\alpha)^2 : r_{\beta' hn/2-3} \le d(x,x') \le r_{\beta' n-3}, \frac{N^{z_{\beta'}(x)}_{n,\beta' n}}{n_{\beta' n}} < \Big(\frac{1}{\sqrt{\alpha}} + \delta_0\Big)^2 \Bigg\} \Bigg| \ge K_n^{\rho_2(\alpha,\beta')+4\delta} \Bigg) \\ \le& K_n^{-\rho_2(\alpha,\beta')-4\delta} E\Bigg[\Bigg| \{ (x,x') \in \tilde{\Psi}_{K_n}(\alpha)^2 : r_{\beta' hn/2-3} \le d(x,x') \le r_{\beta' n-3}, \frac{N^{z_{\beta'}(x)}_{n,\beta' n}}{n_{\beta' n}} < \Big(\frac{1}{\sqrt{\alpha}} + \delta_0\Big)^2 \Bigg\} \Bigg| \Bigg] \\ \le& K_n^{-\rho_2(\alpha,\beta')-4\delta} \sum_{\substack{x\in D(0,K_n), \\ x' \in D(x, r_{\beta' n-3}) \cap D(x, r_{\beta' hn/2-3})^c}} P\Bigg(x,x' \in \tilde{\Psi}_{K_n}(\alpha); \frac{N^{z_{\beta'}(x)}_{n,\beta' n}}{n_{\beta' n}} < \Big(\frac{1}{\sqrt{\alpha}} + \delta_0\Big)^2 \Bigg) \\ \le& CK_n^{-2\alpha F_{2,\beta'}(\min\{1/\sqrt{\alpha}, 2/(2-\beta')\})-3\delta} \max_{\substack{x\in D(0,K_n), \\ x' \in D(x, r_{\beta' n-3}) \cap D(x, r_{\beta' hn/2-3})^c}} P\Bigg(x,x' \in \tilde{\Psi}_{K_n}(\alpha); \frac{N^{z_{\beta'}(x)}_{n,\beta' n}}{n_{\beta' n}} < \Big(\frac{1}{\sqrt{\alpha}} + \delta_0\Big)^2 \Bigg) \\ \le& CK_n^{-\delta}. \end{aligned}$$

The last inequality comes from Corollary 4.2. Set $\epsilon := \alpha \hat{F}_{0,\beta'}(1/\sqrt{\alpha} + \delta_0) - (1-\beta')$. Note that $\epsilon > 0$. Then, there exists $C > 0$ such that

$$\begin{aligned} I_1 \le& |\hat{Z}_{n,\beta'}| CK_n^{-2\alpha \hat{F}_{0,\beta'}(1/\sqrt{\alpha}+\delta_0)+\epsilon/2} \\ \le& CK_n^{2-2\beta'-2\alpha \hat{F}_{0,\beta'}(1/\sqrt{\alpha}+\delta_0)+\epsilon} \le CK_n^{-\epsilon}. \end{aligned}$$

This completes the proof of Lemma 4.4. □

4.2.3 Proof of the lower bound in Theorem 2.1

To prove the lower bound in Theorem 2.1, let $r_{n,k} := r_n/r_k$ and

$$\Theta_{\alpha,\beta,n,n'} := \left\{(x,x') \in \Psi_{K_{n'}}(\alpha)^2 : d(x,x') \leq \frac{r_{n,(1-\beta)n}}{2}\right\}.$$

Fix $\delta > 0$, $0 < \alpha, \beta < 1$ and $\gamma > 0$ with $\delta < \alpha$ and

$$2 - 2\beta - 2\alpha\hat{F}_{0,\beta}(\gamma) > 2\delta. \tag{4.31}$$

Our main goal in this section is to show that there exist $\epsilon > 0$ and $C > 0$ such that for any $n \in \mathbb{N}$

$$P(|\Theta_{\alpha-\delta,\beta,n,n+1}| \leq K_n^{2+2\beta-2\alpha\hat{F}_{2,\beta}(\gamma)-5\delta}) \leq Ce^{-\epsilon n}. \tag{4.32}$$

Note that $\log r_{n,(1-\beta)n+1}/\log K_{n+1} \to \beta$ and $(\alpha,\beta) \mapsto \rho_2(\alpha,\beta)$ is continuous. Hence, if we obtain (4.32), the Borel-Cantelli lemma yields the lower bound in Theorem 2.1.

Let $\mathcal{A}_n$ be a maximal set of points in $D(0,4r_n) \cap D(0,3r_n)^c$ which are $4r_{n,(1-\beta)n-4}$ separated. The primary idea of the proof of (4.32) is to consider $\Theta_{\alpha-\delta,\beta,n,n+1} \cap D(z, r_{n,(1-\beta)n}/2)^2$ for each $z \in \mathcal{A}_n$ instead of the whole $\Theta_{\alpha-\delta,\beta,n,n+1}$ and to count the number of elements in these sets. By the choice of the radius, if both x, $x' \in D(z, r_{n,(1-\beta)n}/2)$ are $(\alpha-\delta)$-favorite points, then $(x,x') \in \Theta_{\alpha-\delta,\beta,n,n+1}$ automatically holds.

As the secondary idea, we restrict random sets to smaller one in order to make the variance of the number of those sets smaller, as explained in introduction. The restriction we introduce here is to control the number of visits to $D(z, r_{n,(1-\beta)n}/2)$ by the number of excursions passing through annuli around $D(z, r_{n,(1-\beta)n}/2)$.

For $z \in \mathcal{A}_n$, $x \in D(z, r_{n,(1-\beta)n}/2)$ and $1 \leq l \leq (1-\beta)n$, let $\hat{N}^z_{n,l}$ denote the number of excursions from $\partial D(z, r_{n,l})$ to $\partial D(z, r_{n,l-1})$ until time $T_{\partial D(z,r_n)}$. Let $\hat{\mathcal{R}}^z_{k,m} = \hat{\mathcal{R}}^z_k(m)$ be the time until completion of the first m excursions from $\partial D(z, r_{n,k})$ to $\partial D(z, r_{n,k-1})$. Let

$$\tilde{W}^z_m = \tilde{W}^z(m) := \left|\left\{y \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : K(m,y) \geq \frac{4\alpha}{\pi}(\log K_n)^2\right\}\right|$$

and

$$\hat{H}^z_{k,m} := \{\tilde{W}^z_{\hat{\mathcal{R}}^z_{k,m}} \geq K_n^{2\beta-2\alpha(1-\gamma(1-\beta))^2/\beta-2\delta}\}.$$

Especially, we let

$$H^z_k := \{\tilde{W}^z(\hat{\mathcal{R}}^z_k(\hat{N}^z_{n,k})) \geq K_n^{2\beta-2\alpha(1-\gamma(1-\beta))^2/\beta-2\delta}\}. \tag{4.33}$$

Roughly speaking, the event H^z_k describes the situation that sufficiently many points as $(\alpha-\delta)$-favorite in a neighborhood of z when all the visit to the neighborhood before leaving $D(z,r_n)$ is finished. We will say that a point $z \in \mathcal{A}_n$ is (n,β)-successful if $|\hat{N}^z_{n,k} - \tilde{n}_k| \leq k$ for all $3 \leq k \leq (1-\beta)n$ and $H^z_{n-\beta n}$ occurs. That is, the first condition gives a restriction on the number of excursions to a "typical" one. Indeed, definition comes from the fact which is written in Remark 3.2 in [1]. Note that, this definition is slightly different from that of [8].

By using the notion of successful points, we can reduce our problem to that for the number of successful points. Indeed, by the definition of these notations,

$$\begin{aligned}|\Theta_{\alpha-\delta,\beta,n,n}| \geq &\sum_{z\in\mathcal{A}_n} (\tilde{W}^z(\hat{\mathcal{R}}^z_{n-\beta n}(\hat{N}^z_{n,(1-\beta)n})))^2 \\ \geq &|\{z \in \mathcal{A}_n : z \text{ is } (n,\beta)-\text{successful}\}|K_n^{4\beta-4\alpha(1-\gamma(1-\beta))^2/\beta-4\delta}.\end{aligned}$$

Then, if there exists $C \in (0,1)$ such that

$$P(|\{z \in \mathcal{A}_n : z \text{ is } (n,\beta) - \text{successful}\}| \geq K_n^{2(1-\beta)-2\gamma^2\alpha(1-\beta)-\delta}) \geq C, \tag{4.34}$$

we have

$$P(|\Theta_{\alpha-\delta,\beta,n,n}| \leq K_n^{2+2\beta-2\alpha\hat{F}_{2,\beta}(\gamma)-5\delta}) \leq 1 - C.$$

Since we have $K_{n+1}/K_n \geq n^3$ for $n \in \mathbb{N}$, by the same argument as that after (3.4) in [27] we have

$$P(|\Theta_{\alpha-\delta,\beta,n,n+1}| \leq K_n^{2+2\beta-2\alpha\hat{F}_{2,\beta}(\gamma)-5\delta}) \leq P(|\Theta_{\alpha-\delta,\beta,n,n}| \leq K_n^{2+2\beta-2\alpha\hat{F}_{2,\beta}(\gamma)-5\delta})^{n^3/2},$$

and, hence, we obtain (4.32). Thus it suffices to show (4.34) to conclude the lower bound.

According to the definition of successful point, we divide the major part of the problem into an estimate of the probability of the event $H^z_{(1-\beta)n}$ and study of the effect of restriction on the number of excursions. Lemma 4.5 concerns with the former one. The latter one will be treated in Lemma 4.10 below in combination with Lemma 4.5. Set $\tilde{n}_k := 6\gamma^2\alpha k^2 \log k$.

Lemma 4.5. *It holds that, uniformly in $z \in \mathcal{A}_n$*

$$P(\hat{H}^z_{n-\beta n,\tilde{n}_{(1-\beta)n}-(1-\beta)n}) = 1 - o(1_n).$$

To prove the Lemma 4.5, we show the following three propositions. To state them we prepare some notation. For $z \in \mathcal{A}_n$, $x \in D(z, r_{n,(1-\beta)n}/2)$ and $(1-\beta)n + 2 \leq l \leq n$, $\hat{N}^{z,x}_{n,l}$ the number of excursions from $\partial D(x, r_{n,l})$ to $\partial D(x, r_{n,l-1})$ until time $\hat{\mathcal{R}}^z_{(1-\beta)n,\tilde{n}_{(1-\beta)n}-(1-\beta)n}$. For $(1-\beta)n \leq l \leq n$, set

$$\hat{n}_l := 6\alpha\left(\frac{l-(1-\beta)n}{\beta} + \left(n - \frac{l-(1-\beta)n}{\beta}\right)\gamma(1-\beta)\right)^2 \log l.$$

For $z \in \mathcal{A}_n$ and $\eta \in \mathbb{N} \cap \{1\}^c$ we say that $x \in D(z, r_{n,(1-\beta)n}/2)$ is (n,β)-qualified if

$$|\hat{N}^{z,x}_{n,l} - \hat{n}_l| \leq n \text{ for } (1-\beta)n + \eta \leq l \leq n.$$

After this, we consider $n \in \mathbb{N}$ with $(1-\beta)n \geq \eta/2$.

Proposition 4.6. *It holds that as $n \to \infty$,*

$$P\Bigg(\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \textit{ is } (n,\beta)\textit{-qualified}\right\}$$
$$\subset \left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : \frac{K(\hat{\mathcal{R}}^z_{(1-\beta)n,\tilde{n}_{(1-\beta)n}-(1-\beta)n}, x)}{(\log K_n)^2} \geq \frac{4\gamma^2\alpha}{\pi} - \frac{2}{\log\log n}\right\}\Bigg) \to 1.$$

Proposition 4.7. *For $x \in D(z, r_{n,(1-\beta)n}/2)$,*

$$P(x \textit{ is } (n,\beta)\textit{-qualified}) = (1 + o(1_n))q_n,$$

where q_n satisfies the following: there exist $c_1(\gamma,\alpha,\beta,\eta)$ and $c_2(\gamma,\alpha,\beta,\eta) > 0$ such that for all sufficiently large $n \in \mathbb{N}$,

$$e^{-c_1 n \log\log n} K_n^{-2\alpha(1-\gamma(1-\beta))^2/\beta} \leq q_n \leq e^{-c_2 n \log\log n} K_n^{-2\alpha(1-\gamma(1-\beta))^2/\beta}.$$

Proposition 4.8. *For $x, y \in D(z, r_{n,(1-\beta)n}/2)$, set*

$$l(x,y) := \min\{l : D(x, r_{n,l}+1) \cap D(y, r_{n,l}+1) = \emptyset\} \wedge n.$$

(i) There exist $c_3(\gamma,\alpha,\beta)$ and $c_4(\gamma,\alpha,\beta) > 0$ such that for sufficiently large $\eta \in \mathbb{N}$, all $x, y \in D(z, r_{n,(1-\beta)n}/2)$ with $(1-\beta)n + \eta \le l(x,y) \le n$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}&P(x \text{ and } y \text{ are } (n,\beta)\text{-qualified})\\ \le & n^{c_3} e^{c_4(l-n+\beta n-\eta+1)\log\log n} q_n^2 \exp\left(\frac{6\alpha}{\beta}(1-\gamma(1-\beta))^2(l-n+\beta n-\eta+1)\log n\right).\end{aligned}$$

(ii) For all $x, y \in D(z, r_{n,(1-\beta)n}/2)$ with $l(x,y) \le (1-\beta)n + \eta - 1$

$$P(x \text{ and } y \text{ are } (n,\beta)\text{-qualified}) = (1+o(1_n))q_n^2.$$

Remark 1. *Note that the choice of sufficiently large $\eta \in \mathbb{N}$ is used instead of the fact that sufficiently large $\overline{\gamma} > 0$ is picked in [1] or [11].*

Roughly speaking, Proposition 4.6 ensures that the set of qualified point is essentially a restriction of $\tilde{W}^z_{\tilde{n}_{(1-\beta)n}-(1-\beta)n}$. Propositions 4.7 and 4.8 will be used to apply the moment method based on the Paley-Zygmund inequality.

Proof of Proposition 4.6. We refer to the proof of Lemma 3.1 in [27]. (See also Proposition 3.3 in [1].) Note that for any $x \in D(z, r_{n,(1-\beta)n}/2)$

$$\begin{aligned}&\Big\{x \text{ is } (n,\beta)\text{-qualified}\Big\} \cap \left\{\frac{K(\hat{\mathcal{R}}^z_{(1-\beta)n,\tilde{n}_{(1-\beta)n}-(1-\beta)n}, x)}{(\log K_n)^2} \ge \frac{4\gamma^2\alpha}{\pi} - \frac{2}{\log\log n}\right\}^c\\ \subset &\left\{K(\hat{\mathcal{R}}^x_{n,\tilde{n}_n-n}, x) \le \left(\frac{4\gamma^2\alpha}{\pi} - \frac{1}{\log\log n}\right)(\log K_n)^2\right\}.\end{aligned}$$

Since there exists $c > 0$ $|D(0,K_n)| \le e^{cn\log n}$ for all sufficienly large $n \in \mathbb{N}$, it is sufficient to prove that there exists $c > 0$ such that

$$P\left(K(\hat{\mathcal{R}}^x_{n,\tilde{n}_n-n}, x) \le \left(\frac{4\gamma^2\alpha}{\pi} - \frac{1}{\log\log n}\right)(\log K_n)^2\right) \le \exp(-cn(\log n/\log_2 n)^2).$$

This proof is same as that of Lemma 3.1 in [27]. $\square$

Proof of Proposition 4.7. Fix $x \in D(z, r_{n,(1-\beta)n}/2)$. Note that by (3.7), we have for $(1-\beta)n + \eta \le l \le n-1$, $x_0 \in \partial D(x, r_{n,l})$,

$$P^{x_0}(T_{\partial D(x,r_{n,l+1})} < T_{\partial D(x,r_{n,l-1})}) = \frac{1}{2}(1+O(n^{-8}))$$

as well as (4.14) in [27]. Then, for $x_0 \in \partial D(x, r_{n,(1-\beta)n+\eta-1})$,

$$P^{x_0}(T_{\partial D(z,r_{n,(1-\beta)n-1})} < T_{\partial D(x,r_{n,(1-\beta)n+\eta})}) = (1+O(n^{-8}))\frac{1}{\eta+1}.$$

In addition, for $m_{(1-\beta)n+\eta}$ with $|m_{(1-\beta)n+\eta} - \hat{n}_{(1-\beta)n+\eta}| \le n$, let

$$\begin{aligned}\hat{q}_n := \sum_{1\le i\le m_{(1-\beta)n+\eta}\wedge(\tilde{n}_{(1-\beta)n}-(1-\beta)n-1)} &\left(\frac{\eta}{\eta+1}\right)^{\tilde{n}_{(1-\beta)n}-(1-\beta)n-1-i}\left(\frac{1}{\eta+1}\right)^i \frac{(\tilde{n}_{(1-\beta)n}-(1-\beta)n-1)!}{(\tilde{n}_{(1-\beta)n}-(1-\beta)n-1-i)!i!}\\ &\left(\frac{1}{\eta+1}\right)^i\left(\frac{\eta}{\eta+1}\right)^{m_{(1-\beta)n+\eta}-i}\frac{(m_{(1-\beta)n+\eta}-1)!}{(m_{(1-\beta)n+\eta}-1-i)!i!}.\end{aligned}$$

Then, (3.12) and the same argument as Proposition 3.4 in [1], (4.13) in [27] or (5.9) in [11] yield that

$$P(x \text{ is } (n,\beta)-\text{qualified}) = \sum P(|\hat{N}^{z,x}_{n,l} - \hat{n}_l| \le n \text{ for } (1-\beta)n+\eta \le l \le n) = (1+o(1_n))q_n, \quad (4.35)$$

where

$$q_n := \sum \hat{q}_n \times \prod_{l=(1-\beta)n+\eta+1}^{n} \binom{m_l + m_{l-1} - 1}{m_l} \left(\frac{1}{2}\right)^{m_l+m_{l-1}-1}. \quad (4.36)$$

Here, the summations in (4.35) and (4.36) are over all $m_{(1-\beta)n+\eta}, ..., m_n$ with $|m_i - \hat{n}_i| \le n$ for $(1-\beta)n + \eta \le l \le n$. Hence, the Stirling formula yields that for $m_{(1-\beta)n+\eta}$ with $|m_{(1-\beta)n+\eta} - \hat{n}_{(1-\beta)n+\eta}| \le n$,

$$\begin{aligned}\hat{q}_n \ge &\left(\frac{\eta}{\eta+1}\right)^{\tilde{n}_{\beta n}-\beta n-1-i} \left(\frac{1}{\eta+1}\right)^{i} \frac{(\tilde{n}_{\beta n}-\beta n-1)!}{(\tilde{n}_{\beta n}-\beta n-1-i)!i!} \left(\frac{1}{\eta+1}\right)^{i} \left(\frac{\eta}{\eta+1}\right)^{m_{(1-\beta)n+\eta}-i} \\ &\left.\frac{(m_{(1-\beta)n+\eta}-1)!}{(m_{(1-\beta)n+\eta}-1-i)!i!}\right|_{i=(2\eta\sqrt{\tilde{n}_{\beta n} m_{(1-\beta)n+\eta}}-(m_{(1-\beta)n+\eta}+\tilde{n}_{\beta n}-\beta n))/2(\eta^2-1)} \\ =&n^{o(1)}. \end{aligned} \quad (4.37)$$

By the same estimate as in the proof of Proposition 3.4 in [1], we have for all m_i with $|m_i - \hat{n}_i| \le n$ and $(1-\beta)n + \eta + 1 \le l \le n$,

$$\begin{aligned}\binom{m_l + m_{l-1} - 1}{m_l} \left(\frac{1}{2}\right)^{m_l+m_{l-1}-1} &\ge c(m_l)^{-1/2} \exp\left\{ - m_{l-1}\left(\frac{1}{4}\left(\frac{m_l}{m_{l-1}}-1\right)^2 + c'\left|\frac{m_l}{m_{l-1}}-1\right|^3\right)\right\} \\ &\ge cn^{-1}(\log n)^{-1/2} \exp\left\{ -\frac{6\alpha}{\beta^2}(1-\gamma(1-\beta))^2 \log n\right\}. \end{aligned} \quad (4.38)$$

Therefore, we obtain

$$q_n \ge c(\log n)^{-c'n} K_n^{-2\alpha(1-\gamma(1-\beta))^2/\beta},$$

and, hence, the desired lower bound holds. Note that by (3.12) it is trivial that

$$\sum_{|m_{(1-\beta)n+\eta} - \hat{n}_{(1-\beta)n+\eta}| \le n} \hat{q}_n \le 1.$$

If we use it instead of (4.37) and the similar computation to (4.38), we have the upper bound. □

In order to prove Proposition 4.8, we set the following sigma field as [1, 11]. First, define a sequence of stopping times. Fix $z \in \mathcal{A}_n$ and $x \in D(z, r_{n,(1-\beta)n}/2)$. For $(1-\beta)n + \eta \le l \le n$, let $\sigma_x^{(0)}[l] := 0$ and

$$\begin{aligned}\tau_x^{(0)}[l] &:= \inf\{m \ge 0 : S_m \in \partial D(x, r_{n,l})\}, \\ \sigma_x^{(i)}[l] &:= \inf\{m \ge \tau_x^{(i-1)}[l] : S_m \in \partial D(x, r_{n,l-1})\} \text{ for } i \ge 1, \\ \tau_x^{(i)}[l] &:= \inf\{m \ge \sigma_x^{(i)}[l] : S_m \in \partial D(x, r_{n,l})\} \text{ for } i \ge 1.\end{aligned}$$

In addition, let $\mathcal{G}_l^x := \sigma(\bigcup_{i\ge 0}\{S_m : \sigma_x^{(i)}[l] \le m \le \tau_x^{(i)}[l]\})$. Then, we will obtain the following lemma.

Lemma 4.9. *There exists ϵ_n with $\lim_{n\to\infty} \epsilon_n = 0$ such that the following holds for all $n - \beta n + \eta \le l \le n-1$, with $|m_l - \hat{n}_l| \le n$ and $x \in D(z, r_{n,(1-\beta)n}/2)$,*

$$\begin{aligned}&P(\hat{N}^{z,x}_{n,i} = m_i \text{ for all } i = l, ..., n | \mathcal{G}_l^x) \\ =&(1+\epsilon_n) P(\hat{N}^{z,x}_{n,i} = m_i \text{ for all } i = l+1, ..., n | \hat{N}^{z,x}_{n,l} = m_l) 1_{\{\hat{N}^{z,x}_{n,l} = m_l\}}.\end{aligned}$$

Proof. The proof is the same as that of Corollary 5.1 in [11] since Lemma 2.4 in [11] holds even for the simple random walk in $\mathbb{Z}^2$ instead of the one in 2-dimensional torus. (See also Lemma 3.6 in [1].) $\square$

Proof of Proposition 4.8. The proof is almost same as that of Proposition 3.5 in [1] since the same argument as [1] holds even for the discrete-time simple random walk in $\mathbb{Z}^2$ instead of the continuos-time one with Lemma 4.9. (See also Lemma 3.2 in [27] or Lemma 4.2 in [11].) $\square$

Proof of Lemma 4.5. As mentioned in Remark 1, we consider the same argument as that of proof of the lower bound in Theorem 1.2 (i) in [1](or [11]). If we pick η with $c_3 < 6\eta - 7$ in Proposition 4.8, Propositions 4.7 and 4.8 yield that for all sufficiently large $n \in \mathbb{N}$,

$$E\left[\left|\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \text{ is } (n,\beta)\text{-qualified}\right\}\right|\right] \geq (1+o(1))\left|D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right)\right| q_n,$$

$$\begin{aligned}
&E\left[\left|\left\{(x,y) \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right)^2 : x, y \text{ are } (n,\beta)\text{-qualified, } l(x,y) \geq (1-\beta)n + \eta\right\}\right|\right] \\
\leq &\left|D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right)\right|^2 (1+o(1)) n^{c_3} q_n^2 \sum_{l=(1-\beta)n+\eta}^{n} n^{-6\eta+7-6(l-n+\beta n-\eta+1)} e^{c_4(l-n+\beta n-\eta+1)\log\log n} \\
\leq &o(1) E\left[\left|\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \text{ is } (n,\beta)\text{-qualified}\right\}\right|\right]^2
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\left|\left\{(x,y) \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right)^2 : x, y \text{ are } (n,\beta)\text{-qualified, } l(x,y) \leq (1-\beta)n + \eta - 1\right\}\right|\right] \\
\leq &(1+o(1))\left|D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right)\right|^2 q_n^2.
\end{aligned}$$

Then, uniformly in $z \in \mathcal{A}_n$,

$$\begin{aligned}
&E\left[\left|\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \text{ is } (n,\beta)\text{-qualified}\right\}\right|^2\right] \\
\leq &(1+o(1)) E\left[\left|\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \text{ is } (n,\beta)\text{-qualified}\right\}\right|\right]^2.
\end{aligned}$$

In addition, Proposition 4.7 yields that for all sufficiently large $n \in \mathbb{N}$,

$$E\left[\left|\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \text{ is } (n,\beta)\text{-qualified}\right\}\right|\right] \geq K_n^{2\beta - 2\alpha(1-\gamma(1-\beta))^2/\beta - \delta}.$$

Then, by the Paley-Zygmund inequality, we obtain that as $n \to \infty$,

$$P\left(\left|\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : x \text{ is } (n,\beta)\text{-qualified}\right\}\right| \geq K_n^{2\beta - 2\alpha(1-\gamma(1-\beta))^2/\beta - 2\delta}\right) \to 1.$$

Therefore, by Proposition 4.6, as $n \to \infty$,

$$\begin{aligned}
P\Bigg(&\left|\left\{x \in D\left(z, \frac{r_{n,(1-\beta)n}}{2}\right) : K(\hat{\mathcal{R}}^z_{(1-\beta)n, \tilde{n}_{(1-\beta)n} - (1-\beta)n}, x) \geq \left(\frac{4\gamma^2\alpha}{\pi} - \frac{1}{\log\log n}\right)(\log K_n)^2\right\}\right| \\
&\geq K_n^{2\beta - 2\alpha(1-\gamma(1-\beta))^2/\beta - 2\delta}\Bigg) \to 1,
\end{aligned}$$

and, hence, Lemma 4.5 holds for all sufficiently large $n \in \mathbb{N}$ with $1/\log_2 n < \delta$. $\square$

As mentioned above, we will introduce the following lemma in order to prove (4.34).

Lemma 4.10. *There exists $\delta_n > 0$ with $\lim_{n\to\infty} \delta_n = 0$ such that*

$$\overline{q}_n := \inf_{x\in\mathcal{A}_n} P(x \textit{ is } (n,\beta)\textit{-successful}) \geq r_{(1-\beta)n}^{-(2\gamma^2\alpha+\delta_n)}. \tag{4.39}$$

Further, there exists $c > 0$ such that for all $x \neq y \in \mathcal{A}_n$,

$$P(x \textit{ and } y \textit{ are } (n,\beta)\textit{-successful}) \leq c\overline{q}_n^2 r_{l(x,y)}^{2\gamma^2\alpha+\delta_{l(x,y)}}. \tag{4.40}$$

Proof. We refer to the proof of Lemma 2.1 in [8] or Lemma 10.1 in [11]. Substitute n, $1-\beta$, $l(x,y)$ $2\gamma^2\alpha$ into m, β, $k(x,y)$, a in Lemma 2.1 of [8] respectively and consider our event $H^x_{(1-\beta)n}$ and $\sup_{x\in\mathcal{A}_n} P(H^z_{(1-\beta)n,\tilde{n}_{(1-\beta)n}-(1-\beta)n})$ in (6.23) instead of $H^x_{\beta m}$ and ξ_m in [8]. Now, we will say that a point $z \in \mathcal{A}_n$ is (n,β)-pre-successful if $|\hat{N}^z_{n,k} - \tilde{n}_k| \leq k$ for all $3 \leq k \leq n - \beta n$. As mentioned in [8], if we replace "(n,β)-successful" in (4.39) and (4.40) by (n,β)-pre-succesful, the corresponding assertion is essentially established in Lemma 3.2 in [27], while he considered the case $\beta = 0$ and his $r_{n,k}$ is different from ours. Since (n,β)-successful point is also (n,β)-pre-successful point, this yields (4.40). In addition, we obtain that, uniformly in $x \in \mathcal{A}_n$,

$$P(x \text{ is } (n,\beta)\text{-successful}) \geq (1+o(1_n))P(x \text{ is } (n,\beta)\text{-pre-successful}).$$

by using Lemma 6.46 instead of (2.13) in [8]. Therefore, (4.39) holds. □

Proof of (4.34). By the same argument as (2.11) in [8], we have that there exists $C > 0$ such that

$$E[|\{z \in \mathcal{A}_n : z \text{ is } (n,\beta)\text{-successful}\}|^2] \leq CE[|\{z \in \mathcal{A}_n : z \text{ is } (n,\beta)\text{-successful}\}|]^2,$$

by using Lemma 4.10 instead of Lemma 2.1 in [8]. By the Paley-Zygmund inequality, we have (4.34). □

5 Proof of Theorems 2.3 and 2.4 and corollaries

5.1 Matrix's argument

5.1.1 Basic matrix's argument

Our argument in later is independent of the arrangement of the elements included in $\Lambda \subset \mathbb{N}$. Then, we consider $(a_{i,l})_{i,l\in\Lambda}$ as the matrix for some arrangement. Given $\eta > 0$ and for $j \in \mathbb{N}$ let

$$\begin{aligned}\mathcal{M}_j =&\mathcal{M}_j(\eta)\\ :=&\{(a_{i,l})_{i,l\in\Lambda} : |\Lambda| = j, \Lambda \subset \mathbb{N}, \text{ symmetric},\\ &\quad \text{for any } 1 \leq i,l,p \leq j \text{ with } i \neq l, l \neq p, p \neq i, a_{i,i} = 1, 0 \leq a_{i,l} \leq 1-\eta,\\ &\quad a_{i,l} < a_{i,p} \Rightarrow a_{l,p} = a_{i,l} \text{ and } a_{i,l} = a_{i,p} \Rightarrow a_{l,p} \geq a_{i,l}\},\\ \tilde{\mathcal{M}}_1 =&\tilde{\mathcal{M}}_1(\eta) := \{(a_{i,i})_{i\in\Lambda} : |\Lambda| = 1, \Lambda \subset \mathbb{N}, a_{i,i} = 1\}\end{aligned}$$

and for $j \geq 2$

$$\begin{aligned}\tilde{\mathcal{M}}_j =&\tilde{\mathcal{M}}_j(\eta)\\ :=&\{(a_{i,l})_{i,l\in\Lambda} : |\Lambda| = j, \Lambda \subset \mathbb{N}, \text{ symmetric},\\ &\quad \text{for any } 1 \leq i \neq l \leq j, a_{i,i} = 1, 0 \leq a_{i,l} \leq 1-\eta,\\ &\quad \exists G, H \subset \mathbb{N}, g, h \in \mathbb{N} \text{ s.t. } |G| = g, |H| = h, G \cup H = \Lambda, g + h = j,\\ &\quad (a_{i,l})_{i,l\in G} \in \tilde{\mathcal{M}}_g, (a_{i,l})_{i,l\in H} \in \tilde{\mathcal{M}}_h, \text{ for any } i \in G, l \in H, a_{i,l} = \min_{i,l\in\Lambda} a_{i,l}\}.\end{aligned}$$

Proposition 5.1. *It holds that $\tilde{\mathcal{M}}_j = \mathcal{M}_j$ for any $j \in \mathbb{N}$.*

Proof. To prove that $\mathcal{M}_j \subset \tilde{\mathcal{M}}_j$, we show the result by induction on $j \in \mathbb{N}$. If $j = 1$, it is trivial that the claim holds. We assume that the claim holds for $1, ..., j-1$ and show the claim for j. Consider any $(a_{i,l})_{i,l\in\Lambda} \in \mathcal{M}_j$ and $s := \min_{i,l\in\Lambda} a_{i,l}$. Pick $u \in \Lambda$ and set $\tilde{\Lambda} := \Lambda \cap \{u\}^c$. Note that $(a_{i,l})_{i,l\in\tilde{\Lambda}} \in \mathcal{M}_{j-1}$, and, hence, $(a_{i,l})_{i,l\in\tilde{\Lambda}} \in \tilde{\mathcal{M}}_{j-1}$ holds under assumption. Then, set $\tilde{s} = \min_{i,l\in\tilde{\Lambda}} a_{i,l}$, $\tilde{g}$, $\tilde{h}$ with $\tilde{g}+\tilde{h} = j-1$ and $\tilde{G}$, $\tilde{H}$ which is satisfied with the conditions of $\tilde{\mathcal{M}}_{j-1}$. We will show $(a_{i,l})_{i,l\in\Lambda} \in \tilde{\mathcal{M}}_j$. It is trivial that all the condition of $\tilde{\mathcal{M}}_j$ except for last two conditions holds. Then, we will set G, H, g, $h \in \mathbb{N}$ such that $|G| = g$, $|H| = h$, $G \cup H = \Lambda$, $g + h = j$ and observe that this division is satisfied with the condition of $\tilde{\mathcal{M}}_j$. It is trivial that $(a_{i,l})_{i,l\in G} \in \mathcal{M}_g$ and $(a_{i,l})_{i,l\in H} \in \mathcal{M}_h$ yield $(a_{i,l})_{i,l\in G} \in \tilde{\mathcal{M}}_g$ and $(a_{i,l})_{i,l\in H} \in \tilde{\mathcal{M}}_h$ under assumption. Hence, it suffices to show that $a_{i,l} = s$ for any $i \in G$, $l \in H$.

(i) Case: $s = \tilde{s}$

First, we consider the case that there exists $m \in \tilde{\Lambda}$ such that $a_{m,u} > s$.

If $a_{m,u} > s$ for some $m \in \tilde{G}$, we should set $G = \tilde{G} \cup \{u\}$, $H = \tilde{H}$, $g = \tilde{g} + 1$, $h = \tilde{h}$. Then for any $l \in H$, $a_{m,l} = \tilde{s} = s < a_{m,u}$ holds and, hence, the definition of $\mathcal{M}_j$ yields that $a_{l,u} = s$ for any $l \in H$. Therefore, $a_{i,l} = s$ for any $i \in G$, $l \in H$.

If $a_{m,u} > s$ for some $m \in \tilde{H}$, the same argument as above yields $a_{i,l} = s$ for any $i \in H$, $l \in G$ where $G = \tilde{G}$, $H = \tilde{H} \cup \{u\}$, $g = \tilde{g}$ and $h = \tilde{h} + 1$.

Second, we consider the case that for any $m \in \tilde{\Lambda}$, $a_{m,u} = s$. Then, we should set $g = \tilde{g} + \tilde{h}$, $h = 1$, $G = \tilde{G} \cup \tilde{H}$ and $H = \{u\}$. Then, we obtain for any $l \in G$, $a_{l,u} = s$.

(ii) Case: $s \neq \tilde{s}$

We should set $g = \tilde{g} + \tilde{h}$, $h = 1$, $G = \tilde{G} \cup \tilde{H}$ and $H = \{u\}$. Note that there exists $m \in G$ such that $a_{m,u} = s$. Then, since $a_{m,u} < a_{m,l}$ for any $l \in G$, the definition of $\mathcal{M}_j$ yields that $a_{u,l} = s$. Therefore, we obtain $\mathcal{M}_j \subset \tilde{\mathcal{M}}_j$.

Next, we will prove $\tilde{\mathcal{M}}_j \subset \mathcal{M}_j$. If $j = 1$ or 2, it is trivial that the claim holds. For $j \geq 3$, we consider any $(a_{i,l})_{i,l\in\Lambda} \in \tilde{\mathcal{M}}_j$. It is trivial that $(a_{i,l})_{i,l\in\{u_1,u_2,u_3\}} \in \tilde{\mathcal{M}}_3$ holds for any $\{u_1, u_2, u_3\} \subset \Lambda$. Then, it suffices to prove the case that $j = 3$. Consider the case that $G = \{u_1, u_2\}$, $H = \{u_3\}$ and observe the conditions (i) $a_{i,l} < a_{i,p} \Rightarrow a_{l,p} = a_{i,l}$ and (ii) $a_{i,l} = a_{i,p} \Rightarrow a_{l,p} \geq a_{i,l}$.

1. $(i, l, p) = (u_1, u_2, u_3)$: (i) The assumption $a_{u_1,u_2} < a_{u_1,u_3}$ contradicts the condition of $a_{u_1,u_3} = \min_{i,l\in\{u_1,u_2,u_3\}} a_{i,l}$.

   (ii) If $a_{u_1,u_2} = a_{u_1,u_3}$ holds, the fact that $a_{u_2,u_3} = a_{u_1,u_3}$ yields the result.

2. $(i, l, p) = (u_2, u_1, u_3)$: The proof is same as above by the symmetry of u_1 and u_2.

3. $(i, l, p) = (u_3, u_1, u_2)$: (i) The assumption $a_{u_3,u_1} < a_{u_3,u_2}$ contradicts the condition of $a_{u_3,u_1} = a_{u_3,u_2}$.

   (ii) The result is trivial since $a_{u_1,u_3} = \min_{i,l\in\{u_1,u_2,u_3\}} a_{i,l}$ yields $a_{u_1,u_2} \geq a_{u_3,u_1}$.

4. $(i, l, p) = (u_3, u_2, u_1)$: The proof is same as above by the symmetry of u_1 and u_2.

5. $(i, l, p) = (u_2, u_3, u_1)$: (i) (ii) The condition of $\tilde{\mathcal{M}}_3$ yields $a_{u_3,u_1} = a_{u_2,u_3}$ and hence the results holds in any case.

6. $(i, l, p) = (u_1, u_3, u_2)$: The proof is same as above by the symmetry of u_1 and u_2.

Therefore, the symmetry of u_1, u_2 and u_3 yields the desired result. □

Remark 2. *Now, we explain why we set $\mathcal{M}_j$. We will show that*

$$\left(\frac{\pi G_n(x_i, x_l)}{2\log n}\right)_{1\leq i,l\leq j}$$

for $x_i \in D(0,n/3)$ with $1 \le i \le j$ is asymptotically close to some element in $\mathcal{M}_j$ in some way later. All the conditions in the definition of $\mathcal{M}_j$ except for last two conditions are imaginable. Hence, we will observe that $(G_n(x_i,x_l))_{1\le i,l\le j}$ for $x_i \in D(0,n/3)$ with $1 \le i \le j$ has the property which asymptotically is close to last two conditions of $\mathcal{M}_j$. Consider some sequence $\{x_k^{(n)}\}_{n\in\mathbb{N}}$ with $x_k^{(n)} \in D(0,n/3)$ for $k \in \{i,l,p\}$ such that $d(x_i^{(n)},x_l^{(n)}) = n^{\beta+o(1)}$ and $d(x_i^{(n)},x_p^{(n)}) = n^{\beta-\epsilon+o(1)}$ with $0<\epsilon<\beta$. Then, (3.9) yields that

$$\begin{aligned}\frac{2(1-\beta)+o(1)}{\pi}\log n =& G_n(x_i^{(n)},x_l^{(n)})\\ <& G_n(x_i^{(n)},x_p^{(n)}) = \frac{2(1-\beta+\epsilon)+o(1)}{\pi}\log n\end{aligned}$$

for all sufficiently large $n\in\mathbb{N}$. In addition, it is trivial that $d(x_l^{(n)},x_p^{(n)}) = n^{\beta+o(1)}$ and, hence,

$$\frac{2(1-\beta)+o(1)}{\pi}\log n = G_n(x_l^{(n)},x_p^{(n)}) \sim G_n(x_i^{(n)},x_l^{(n)}),$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n\to\infty$ for any sequence. This fact corresponds to "$a_{i,l} < a_{i,p} \Rightarrow a_{l,p} = a_{i,l}$" in the definition of $\mathcal{M}_j$. To observe other condition in the definition of $\mathcal{M}_j$, consider some sequence $\{x_k^{(n)}\}_{n\in\mathbb{N}}$ with $x_k^{(n)} \in D(0,n/3)$ for $k\in\{i,l,p\}$ such that $d(x_i^{(n)},x_l^{(n)}) = n^{\beta+o(1)}$ and $d(x_i^{(n)},x_p^{(n)}) = n^{\beta+o(1)}$. Then, (3.9) yields that

$$G_n(x_i^{(n)},x_l^{(n)}) = \frac{2(1-\beta)+o(1)}{\pi}\log n, G_n(x_i^{(n)},x_p^{(n)}) = \frac{2(1-\beta)+o(1)}{\pi}\log n.$$

In addition, it is trivial that $d(x_l^{(n)},x_p^{(n)}) \le n^{\beta+o(1)}$ holds and, hence,

$$G_n(x_l^{(n)},x_p^{(n)}) \ge \frac{2(1-\beta)+o(1)}{\pi}\log n \sim G_n(x_i^{(n)},x_l^{(n)}).$$

This fact corresponds to "$a_{i,l} = a_{i,p} \Rightarrow a_{l,p} \ge a_{i,l}$" in the definition of $\mathcal{M}_j$.

Hereafter, in Sections 5.1 and 5.1.2, fix $j\in\mathbb{N}$ and if we mention no thing, we consider $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$ as the represented element of $\mathcal{M}_j$ and its division $G=\{1,...,g\}$, $H=\{g+1,...,g+h(=j)\}$ as above. In addition, let $s=\min_{1\le i,l\le j} a_{i,l}$.

Proposition 5.2. *Any element included in $\mathcal{M}_j$ is regular matrix. In other words, for any $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$, there exists a unique solution $y_1,...,y_j$ such that*

$$(a_{i,l})_{1\le i,l\le j}(y_1,...,y_j)^T = (1,...,1)^T.$$

Proof. We will show the claim and in addition the following conditions (A) and (B): when $j=2$,

$$\begin{aligned}&(A)\quad 1 - s\sum_{i=1}^{j} y_i > 0,\\ &(B)\quad y_i > 0 \text{ for any } 1\le i\le j.\end{aligned}$$

When $j=1$, we change (A) to $1-y_1\ge 0$. We will show the results by induction on $j\in\mathbb{N}$. It is trivial that the claim holds for $j=1$ since $y_1=1$. Assume that the claim, (A) and (B) hold for $1,...,j-1$. We will show that the claim, (A) and (B) hold for j. The assumption yields that $(a_{i,l})_{1\le i,l\le g} \in \mathcal{M}_g$ determines a unique solution $(z_1,...,z_g)^T$ such that $(a_{i,l})_{1\le i,l\le g}(z_1,...,z_g)^T = (1,...,1)^T$ and $(a_{i,l})_{g+1\le i,l\le j} \in \mathcal{M}_h$ determines a unique solution $(z'_1,...,z'_h)^T$ such that $(a_{i,l})_{g+1\le i,l\le j}(z'_1,...,z'_h)^T = (1,...,1)^T$. In addition, the assumption of (B) yields $z_i>0$ for any $1\le i\le g$ and $z'_i>0$ for any $1\le i\le h$. Therefore, it holds that $z_i + s\sum_{1\le l\le g, l\neq i} z_l \le 1$ for any $1\le i\le g$ and $z_i + s\sum_{1\le l\le h, l\neq i} z'_l \le 1$ for any $1\le i\le h$, and, hence,

$$\sum_{i=1}^{g} z_i \le \frac{g}{1+(g-1)s}, \quad \sum_{i=1}^{h} z'_i \le \frac{h}{1+(h-1)s}.$$

Then, there exists $c>0$ such that

$$1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i \ge c. \tag{5.1}$$

It comes from $s \le 1-\eta$. If we set

$$\begin{aligned} y_l =& \frac{(1-s\sum_{i=1}^{h} z'_i) z_l}{1-s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} \qquad \text{for any } 1\le l\le g,\\ y_l =& \frac{(1-s\sum_{i=1}^{g} z_i) z'_{l-g}}{1-s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} \qquad \text{for any } g+1\le l\le j, \end{aligned}$$

it is the solution such that $(a_{i,l})_{1\le i,l\le j}(y_1,...,y_j)^T = (1,...,1)^T$. Hence, we have proved the existence of the solution. Hereafter, we will observe the properties of $y_1,...,y_j$ under assuming the existence.

First, we will show (B). By Proposition 5.1, it holds that

$$\sum_{i=1}^{g}(a_{l,i}y_i) + s\sum_{i=g+1}^{j} y_i = 1 \qquad \text{for any } 1\le l\le g, \tag{5.2}$$

$$s\sum_{i=1}^{g} y_i + \sum_{i=g+1}^{j}(a_{l,i}y_i) = 1 \qquad \text{for any } g+1\le l\le j. \tag{5.3}$$

Hence, by the definition of $z_1,...,z_g,\ z'_1,...,z'_h$, we obtain

$$\sum_{i=1}^{g} y_i = \left(1 - s\sum_{i=g+1}^{j} y_i\right)\sum_{i=1}^{g} z_i, \quad \sum_{i=g+1}^{j} y_i = \left(1 - s\sum_{i=1}^{g} y_i\right)\sum_{i=1}^{h} z'_i.$$

Then, the simple computation and (5.1) yield

$$\begin{aligned} \sum_{i=1}^{g} y_i &= \frac{\sum_{i=1}^{g} z_i - s\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}{1-s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i},\\ \sum_{i=g+1}^{j} y_i &= \frac{\sum_{i=1}^{h} z'_i - s\sum_{i=1}^{g} z'_i \sum_{i=1}^{g} z_i}{1-s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}, \end{aligned} \tag{5.4}$$

and, hence,

$$\sum_{i=1}^{j} y_i = \frac{\sum_{i=1}^{g} z_i + \sum_{i=1}^{h} z'_i - 2s\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}{1-s^2\sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i}. \tag{5.5}$$

Now, if we let $\tilde{s} = \min_{1\le i,l\le g} a_{i,l}$, under assumption of (A) we have that for $g\ge 2$,

$$1 - \tilde{s}\sum_{i=1}^{g} z_i > 0. \tag{5.6}$$

Since $\tilde{s} \geq s$ for $g \geq 2$, (5.4) and (5.6) yield

$$1 - s\sum_{i=1}^{g} y_i = \frac{1 - s\sum_{i=1}^{g} z_i}{1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} \geq \frac{1 - \tilde{s}\sum_{i=1}^{g} z_i}{1 - s^2 \sum_{i=1}^{g} z_i \sum_{i=1}^{h} z'_i} > 0.$$

In addition, since $\tilde{s} > s$ for $g = 1$, (5.4) and (5.6) yield

$$1 - sy_1 = \frac{1 - sz_1}{1 - s^2 z_1 \sum_{i=1}^{h} z'_i} > \frac{1 - \tilde{s}z_1}{1 - s^2 z_1 \sum_{i=1}^{h} z'_i} \geq 0.$$

By the definition of $y_{g+1}, ..., y_j$, $z'_1, ..., z'_h$ and (5.3), we have $(y_{g+1}, ..., y_j) = (1 - s\sum_{i=1}^{g} y_i)(z'_1, ...z'_h)$. Then, since we assume the solution $z'_1, ..., z'_h$ is satisfied with $z'_i > 0$ for any $1 \leq i \leq h$, it holds that any solution $y_{g+1}, ..., y_j$ is satisfied with $y_i > 0$ for any $g + 1 \leq i \leq j$. In addition, by the same argument as above, the definition of $y_1, ..., y_g$, $z_1, ..., z_g$ and (5.2) yield $y_i > 0$ for any $1 \leq i \leq g$, and, hence, (B) holds.

Second, we will show (A). The fact that $y_i > 0$ for any $1 \leq i \leq j$ and $\sum_{i=1}^{j} a_{l,i} y_i = 1$ for any $1 \leq l \leq j$ yields $s\sum_{i=1}^{j} y_i < 1$ and hence we obtain the desired result.

Now, we turn to prove uniqueness of the solution by using the result of (B) we have already obtained. In general, it is known that

$$V := \{(y_1, ..., y_j) : (a_{i,l})_{1\leq i,l\leq j}(y_1, ..., y_j)^T = (1, ..., 1)^T\} = x_0 + \mathrm{Ker}A,$$

where x_0 is a characteristic solution for $(a_{i,l})_{1\leq i,l\leq j}(y_1, ..., y_j)^T = (1, ..., 1)^T$. By the result of (B), it holds that $\{v = (v_1, ..., v_j)^T : v_j > 0\} \supset V$. Since V is a linear space, that $\mathrm{Ker}(a_{i,l})_{1\leq i,l\leq j} \neq \{0\}$ contradicts it. Then $\mathrm{Ker}(a_{i,l})_{1\leq i,l\leq j} = \{0\}$ and, hence, we have the desired claim. □

Recall that $\chi(A)$ is the summation over all the elements of A^{-1}. Note that if we consider $(a_{i,l})_{1\leq i,l\leq j}$ and $(y_1, ..., y_j)$ such that

$$(a_{i,l})_{1\leq i,l\leq j}(y_1, ..., y_j)^T = (1, ..., 1)^T,$$

$\chi((a_{i,l})_{1\leq i,l\leq j}) = \sum_{i=1}^{j} y_i$ holds.

Proposition 5.3. *For any $\epsilon > 0$ there exists $\delta > 0$ such that for any $(a_{i,l})_{1\leq i,l\leq j} \in \mathcal{M}_j$, $(\tilde{a}_{i,l})_{1\leq i,l\leq j}$ with $\max_{1\leq i,l\leq j} |a_{i,l} - \tilde{a}_{i,l}| \leq \delta$ the following hold: $(\tilde{a}_{i,l})_{1\leq i,l\leq j}$ is regular matrix and*

$$\left|\chi((a_{i,l})_{1\leq i,l\leq j}) - \chi((\tilde{a}_{i,l})_{1\leq i,l\leq j})\right| \leq \epsilon.$$

Remark 3. *It is trivial that Proposition 5.3 yields the following: for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}$ and $(a_{i,l})_{1\leq i,l\leq j} \in \mathcal{M}_j$, $(\tilde{a}_{i,l})_{1\leq i,l\leq j}$ with $\max_{1\leq i,l\leq j} |a_{i,l}n - \tilde{a}_{i,l}| \leq \delta n$ the following holds: $(\tilde{a}_{i,l})_{1\leq i,l\leq j}$ is regular matrix and*

$$\left|\frac{\chi((a_{i,l})_{1\leq i,l\leq j})}{n} - \chi((\tilde{a}_{i,l})_{1\leq i,l\leq j})\right| \leq \frac{\epsilon}{n}.$$

Proof. First, we will prove that $\mathcal{M}_j$ is the closed set for each $j \in \mathbb{N}$. We consider $\max_{1\leq i,l\leq j} |a_{i,l} - a'_{i,l}|$ for $(a_{i,l})_{1\leq i,l\leq j}$, $(a'_{i,l})_{1\leq i,l\leq j} \in \mathcal{M}_j$ as the metric on $\mathcal{M}_j$. Consider $(a^m_{i,l})_{1\leq i,l\leq j} \in \mathcal{M}_j$ and $(a^\infty_{i,l})_{1\leq i,l\leq j}$ such that $(a^m_{i,l})_{1\leq i,l\leq j} \to (a^\infty_{i,l})_{1\leq i,l\leq j}$ as $m \to \infty$. Set for $k \in \mathbb{N}$, $m_k := \inf\{m > m_{k-1} : a^m_{i,l} > a^m_{i,p}\}$ with $m_1 = 1$ and $\inf \emptyset = 0$. First, we assume that $a^\infty_{i,l} < a^\infty_{i,p}$ holds for some $1 \leq i, l, p \leq j$ with $i \neq l, l \neq p, p \neq i$. Then, $\sup m_k < \infty$ and hence by the definition of $\mathcal{M}_j$ there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$, $a^m_{l,p} = a^m_{i,l}$. Therefore, $a^\infty_{l,p} = a^\infty_{i,l}$ holds.

Next, we assume that $a^\infty_{i,l} = a^\infty_{i,p}$ holds for some $1 \leq i, l, p \leq j$ with $i \neq l, l \neq p, p \neq i$. If $\sup m_k < \infty$, by the definition of $\mathcal{M}_j$ there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$, $a^m_{l,p} \geq a^m_{i,l}$, and, hence, $a^\infty_{l,p} \geq a^\infty_{i,l}$. If $\sup m_k = \infty$, by the definition of $\mathcal{M}_j$ we obtain $a^\infty_{l,p} = a^\infty_{i,p}$, and, hence, the assumption of $a^\infty_{i,l} = a^\infty_{i,p}$ yields $a^\infty_{l,p} = a^\infty_{i,l}$. Therefore, $\mathcal{M}_j$ is the closed set and hence compact.

Now, we turn to prove the desired result. Note that

$$\max_{1\le i,l\le j} |a_{i,l}| \le 1. \tag{5.7}$$

In addition, Proposition 5.2 yields that $\det(a_{i,l})_{1\le i,l\le j} \neq 0$ holds for $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$. Then, by compactness of $\mathcal{M}_j$, we have $\inf_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{M}_j} |\det(a_{i,l})_{1\le i,l\le j}| \neq 0$. Hence, (5.7) yields that there exists $\delta > 0$ such that for any $\delta_{i,l}$ with $|\delta_{i,l}| \le \delta$,

$$\inf_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{M}_j} |\det(a_{i,l}+\delta_{i,l})_{1\le i,l\le j}| \neq 0. \tag{5.8}$$

Therefore, the first claim holds. Finally, it is trivial that (5.7) and (5.8) again yield the second claim, and, hence, we obtain the desired result. $\square$

Remark 4. *By the same argument as Remark 3, (5.7) and (5.8) yield the following: for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}$ and $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$, $(\tilde{a}_{i,l})_{1\le i,l\le j}$ with $\max_{1\le i,l\le j} |a_{i,l}n - \tilde{a}_{i,l}| \le \delta n$, it holds that for any $1 \le i \le j$*

$$|y_i - \tilde{y}_i| \le \frac{\epsilon}{n}, \tag{5.9}$$

where

$$y_i := \frac{\sum_{i=1}^{j} \textit{(cofactor of } a_{l,i}n)}{\det((a_{i,l}n)_{1\le i,l\le j})}, \quad \tilde{y}_i := \frac{\sum_{i=1}^{j} \textit{(cofactor of } \tilde{a}_{l,i})}{\det((\tilde{a}_{i,l})_{1\le i,l\le j})}.$$

In addition, (B) in Proposition 5.2 and compactness of $\mathcal{M}_j$ yield that there exists $c > 0$ such that for any $n \in \mathbb{N}$ and $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$

$$y_i > \frac{c}{n}. \tag{5.10}$$

Therefore, (5.9) and (5.10) yield that there exist $\delta > 0$ and $c > 0$ such that for any $n \in \mathbb{N}$ and $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$, $(\tilde{a}_{i,l})_{1\le i,l\le j}$ with $\max_{1\le i,l\le j} |a_{i,l}n - \tilde{a}_{i,l}| \le \delta n$, it holds that for any $1 \le i \le j$

$$\tilde{y}_i > \frac{c}{n}. \tag{5.11}$$

We write $(a_{i,l})_{i,l\in\Lambda}$ which is satisfied with conditions of $\tilde{\mathcal{M}_j}$ as

$$M\{(a_{i,l})_{i,l\in G}, (a_{i,l})_{i,l\in H}, s\}.$$

However, we conveniently use the following meaning of this symbol. If we write

$$(a_{i,l})_{1\le i,l\le j} = M\{(b_{i,l})_{1\le i,l\le g}, (\hat{b}_{i,l})_{1\le i,l\le h}, s\}$$

for $g + h = j$ and $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$, we consider the case that

$$\begin{cases} a_{i,l} = \min_{1\le i,l\le j} a_{i,l} = s & \text{for } 1 \le i \le g, g+1 \le l \le j, \\ a_{i,l} = b_{i,l} & \text{for } 1 \le i, l \le g, \\ a_{i,l} = \hat{b}_{i-g,l-g} & \text{for } g+1 \le i, l \le j. \end{cases}$$

To divide $\mathcal{M}_j$, we define $\Xi : \cup_{j\in\mathbb{N}}\mathcal{M}_j \to R$ as follows:

$$\begin{aligned} \Xi((a_{i,l})_{i,l\in\Lambda}) :=& \Xi((a_{i,l})_{i,l\in G}) + \Xi((a_{i,l})_{i,l\in H}) + 1 - s, \\ \Xi((a_{i,i})_{i\in\Lambda}) :=& 0 \quad \text{for any } \Lambda \text{ with } |\Lambda| = 1. \end{aligned}$$

Lemma 5.4. *Ξ is independent of the division we select, and hence Ξ is well defined.*

Proof. We simply write this proof by induction on $j \in \mathbb{N}$. It is trivial that the claim holds for $j = 1$. We assume that the claim holds for $1, ..., j-1$ and show the claim for j. Assume that there exist two different divisions G, H and $\tilde{G}$, $\tilde{H}$ which are satisfied with conditions of $\tilde{\mathcal{M}}_j$. If we consider the case that $H \cap \tilde{H} \neq \emptyset$, $H \cap \tilde{G} \neq \emptyset$, $G \cap \tilde{H} \neq \emptyset$ and $G \cap \tilde{G} \neq \emptyset$, it holds that

$$\min_{i,l \in G} a_{i,l} = \min_{i,l \in H} a_{i,l} = \min_{i,l \in \tilde{G}} a_{i,l} = \min_{i,l \in \tilde{H}} a_{i,l} = s$$

and

$$\begin{aligned}
(a_{i,l})_{i,l \in G} &= M\{(a_{i,l})_{i,l \in G \cap \tilde{G}}, (a_{i,l})_{i,l \in G \cap \tilde{H}}, s\}, \\
(a_{i,l})_{i,l \in H} &= M\{(a_{i,l})_{i,l \in H \cap \tilde{G}}, (a_{i,l})_{i,l \in H \cap \tilde{H}}, s\}, \\
(a_{i,l})_{i,l \in \tilde{G}} &= M\{(a_{i,l})_{i,l \in \tilde{G} \cap G}, (a_{i,l})_{i,l \in \tilde{G} \cap H}, s\}, \\
(a_{i,l})_{i,l \in \tilde{H}} &= M\{(a_{i,l})_{i,l \in \tilde{H} \cap G}, (a_{i,l})_{i,l \in \tilde{H} \cap H}, s\}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\Xi((a_{i,l})_{i,l \in G}) + \Xi((a_{i,l})_{i,l \in H}) \\
=&\Xi((a_{i,l})_{i,l \in G \cap \tilde{G}}) + \Xi((a_{i,l})_{i,l \in G \cap \tilde{H}}) \\
&+\Xi((a_{i,l})_{i,l \in H \cap \tilde{G}}) + \Xi((a_{i,l})_{i,l \in H \cap \tilde{H}}) + 2(1-s), \\
&\Xi((a_{i,l})_{i,l \in \tilde{G}}) + \Xi((a_{i,l})_{i,l \in \tilde{H}}) \\
=&\Xi((a_{i,l})_{i,l \in \tilde{G} \cap G}) + \Xi((a_{i,l})_{i,l \in \tilde{G} \cap H}) \\
&+\Xi((a_{i,l})_{i,l \in \tilde{H} \cap G}) + \Xi((a_{i,l})_{i,l \in \tilde{H} \cap H}) + 2(1-s),
\end{aligned}$$

and, hence, $\Xi((a_{i,l})_{i,l \in G}) + \Xi((a_{i,l})_{i,l \in H}) = \Xi((a_{i,l})_{i,l \in \tilde{G}}) + \Xi((a_{i,l})_{i,l \in \tilde{H}})$ holds. In addition, if we consider the case that $H \cap \tilde{H} = \emptyset$, it holds that

$$\min_{i,l \in G} a_{i,l} = \min_{i,l \in \tilde{G}} a_{i,l} = s,$$

and

$$\begin{aligned}
(a_{i,l})_{i,l \in G} &= M\{(a_{i,l})_{i,l \in G \cap \tilde{G}}, (a_{i,l})_{i,l \in G \cap \tilde{H}}, s\}, \\
(a_{i,l})_{i,l \in \tilde{G}} &= M\{(a_{i,l})_{i,l \in \tilde{G} \cap G}, (a_{i,l})_{i,l \in \tilde{G} \cap H}, s\}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\Xi((a_{i,l})_{i,l \in G}) + \Xi((a_{i,l})_{i,l \in H}) \\
=&\Xi((a_{i,l})_{i,l \in G \cap \tilde{H}}) + \Xi((a_{i,l})_{i,l \in G \cap \tilde{G}}) + \Xi((a_{i,l})_{i,l \in H}) + 1 - s \\
=&\Xi((a_{i,l})_{i,l \in \tilde{H}}) + \Xi((a_{i,l})_{i,l \in G \cap \tilde{G}}) + \Xi((a_{i,l})_{i,l \in H}) + 1 - s, \\
&\Xi((a_{i,l})_{i,l \in \tilde{G}}) + \Xi((a_{i,l})_{i,l \in \tilde{H}}) \\
=&\Xi((a_{i,l})_{i,l \in \tilde{G} \cap G}) + \Xi((a_{i,l})_{i,l \in \tilde{G} \cap H}) + \Xi((a_{i,l})_{i,l \in \tilde{H}}) + 1 - s \\
=&\Xi((a_{i,l})_{i,l \in \tilde{G} \cap G}) + \Xi((a_{i,l})_{i,l \in H}) + \Xi((a_{i,l})_{i,l \in \tilde{H}}) + 1 - s,
\end{aligned}$$

and, hence, $\Xi((a_{i,l})_{i,l \in G}) + \Xi((a_{i,l})_{i,l \in H}) = \Xi((a_{i,l})_{i,l \in \tilde{G}}) + \Xi((a_{i,l})_{i,l \in \tilde{H}})$ holds. Since the same proof as second case holds for each case that $H \cap \tilde{G} = \emptyset$, $G \cap \tilde{H} = \emptyset$, or $G \cap \tilde{G} = \emptyset$, the desired result holds. □

Then, let

$$\mathcal{G}_r := \{(a_{i,l})_{i,l \in \Lambda} \in \mathcal{M}_j : \Xi((a_{i,l})_{i,l \in \Lambda}) = r\}$$

and for $j \in \mathbb{N}$ and $r \leq 1 - \eta$, $A_r^{(j)} := (a_{i,l})_{1 \leq i,l \leq j}$ which is satisfied with $(a_{i,l})_{1 \leq i,l \leq j} \in \mathcal{M}_j$ and $a_{i,l} = r$ for $1 \leq i \neq l \leq j$. Note that $A_r^{(1)}$ is independent of r. We sometimes write it as $A^{(1)}$. Next, we will observe the additional property of the matrix included in $\mathcal{M}_j$.

Proposition 5.5. *For any $r \le j-1$*

$$\min_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{G}_r} \chi((a_{i,l})_{1\le i,l\le j}) = \chi(A^{(j)}_{1-r/(j-1)}) = \frac{j}{j-r}.$$

To show Proposition 5.5, we use the following two lemmas.

Lemma 5.6. *Consider any $(a_{i,l})_{1\le i,l\le j}$, $(\overline{a}_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j$ with $(a_{i,l})_{1\le i,l\le j} = M\{(a_{i,l})_{i,l\in G}, (a_{i,l})_{i,l\in H}, s\}$, $(\overline{a}_{i,l})_{1\le i,l\le j} = M\{(\overline{a}_{i,l})_{i,l\in G}, (\overline{a}_{i,l})_{i,l\in H}, s\}$. Then, if*

$$\chi((a_{i,l})_{i,l\in G}) \ge \chi((\overline{a}_{i,l})_{i,l\in G}) \text{ and } \chi((a_{i,l})_{i,l\in H}) \ge \chi((\overline{a}_{i,l})_{i,l\in H}),$$

it holds that

$$\chi((a_{i,l})_{1\le i,l\le j}) \ge \chi((\overline{a}_{i,l})_{1\le i,l\le j}).$$

Proof. Note that it suffices to prove the case that $(a_{i,l})_{i,l\in G} = (\overline{a}_{i,l})_{i,l\in G}$ since we can prove the claim by repeating the same proof. Let

$$g(t,b,c) := \frac{b+c-2tbc}{1-t^2bc},$$

where $1-t^2bc > 0$. Then g is monotonically increasing in c since the simple computation yields

$$\frac{\partial g}{\partial c} = \frac{(1-tb)^2}{(1-t^2bc)^2} \ge 0.$$

Note that (5.5) yields

$$\begin{aligned}\chi((a_{i,l})_{1\le i,l\le j}) =& \frac{\chi((a_{i,l})_{i,l\in G}) + \chi((a_{i,l})_{i,l\in H}) - 2s\chi((a_{i,l})_{i,l\in G})\chi((a_{i,l})_{i,l\in H})}{1-s^2\chi((a_{i,l})_{i,l\in G})\chi((a_{i,l})_{i,l\in H})}\\ =& g(s, \chi((a_{i,l})_{i,l\in G}), \chi((a_{i,l})_{i,l\in H})).\end{aligned}$$

Since $g(t,b,c)$ is monotonically increasing in c, the assumption yields the desired result. ☐

Lemma 5.7. *Consider $r \le j-1$, $0 \le \gamma \le \gamma_1 \le \gamma_2 \le 1$ with $r = (g-1)(1-\gamma_2)+(h-1)(1-\gamma_1)+1-\gamma$ and $M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\}$, which is satisfied with $r = \Xi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\})$. Then, fixing the values γ_2 and r, $\chi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\})$ is minimized at $\gamma = \gamma_1$.*

Proof. Fix the values γ_2 and r. Note that $r = \Xi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\}) = (g-1)(1-\gamma_2)+(h-1)(1-\gamma_1)+1-\gamma$ is constant and hence $(g-1)\gamma_2+(h-1)\gamma_1+\gamma$ is. Then, if we set $p = (g-1)\gamma_2+1$ and $q = (h-1)\gamma_1+\gamma+1$, p and q are constants. In addition,

$$\chi(A^{(g)}_{\gamma_2}) = \frac{g}{(g-1)\gamma_2+1}, \quad \chi(A^{(h)}_{\gamma_1}) = \frac{h}{(h-1)\gamma_1+1}.$$

Then by (5.5) it holds that

$$\begin{aligned}f(\gamma) :=& \chi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\})\\ =& \frac{\chi(A^{(g)}_{\gamma_2}) + \chi(A^{(h)}_{\gamma_1}) - 2\gamma\chi(A^{(g)}_{\gamma_2})\chi(A^{(h)}_{\gamma_1})}{1-\gamma^2\chi(A^{(g)}_{\gamma_2})\chi(A^{(h)}_{\gamma_1})}\\ =& \frac{g((h-1)\gamma_1+1) + h((g-1)\gamma_2+1) - 2\gamma gh}{((g-1)\gamma_2+1)((h-1)\gamma_1+1) - \gamma^2 gh}\\ =& \frac{g(q-\gamma) + hp - 2\gamma gh}{(q-\gamma)p - \gamma^2 gh}.\end{aligned}$$

Note that we consider $0 \leq \gamma \leq (q-1)/h$ and show that f is monotonically decreasing in $0 \leq \gamma \leq (q-1)/h$. The simple computation yields

$$\begin{aligned}\frac{\partial f(\gamma)}{\partial \gamma} =& \frac{((q-\gamma)p - \gamma^2 gh)(-g-2gh) - (g(q-\gamma)+hp-2\gamma gh)(-p-2\gamma gh)}{((q-\gamma)p-\gamma^2 gh)^2} \\ =& \frac{-g^2h(1+2h)(\gamma - \frac{gq+hp}{g+2gh})^2 + \frac{h(gq+hp)^2}{(1+2h)} - 2qpgh + hp^2}{((q-\gamma)p-\gamma^2 gh)^2}\end{aligned}$$

and set

$$\tilde{f} := -g^2h(1+2h)(\gamma - \frac{gq+hp}{g+2gh})^2 + \frac{h(gq+hp)^2}{(1+2h)} - 2qpgh + hp^2. \tag{5.12}$$

Note that it holds that

$$\frac{q-1}{h} \leq \text{the apex (summit) of } \tilde{f} = \frac{(gq+hp)}{g(1+2h)}. \tag{5.13}$$

Because we can compute

$$\begin{aligned}h(gq+hp) - (q-1)g(1+2h) =& h^2p + g(h+1) + gh - gq(h+1) \\ =& h^2p - g(h+1)(q-1) + gh \\ \geq& h^2((g-1)\gamma_2 + 1) - g(h+1)h\gamma_2 + gh \\ =& h(g+h)(1-\gamma_2) \geq 0.\end{aligned}$$

The first inequality comes from $-(q-1) \geq -((h-1)\gamma_2 + \gamma_2) = -h\gamma_2$ and second one $\gamma_2 \leq 1$. Therefore, we obtain (5.13). In addition, we claim

$$\tilde{f}\left(\frac{q-1}{h}\right) \leq 0. \tag{5.14}$$

Because we can compute

$$\begin{aligned}h\tilde{f}\left(\frac{q-1}{h}\right) =& -g(g+2gh)(q-1)^2 + 2gh(gq+hp)(q-1) + h^2p(p-2gq) \\ =& -g^2(q-1)^2 - 2g^2h(q-1)^2 + 2g^2h(q-1)^2 \\ &+ 2g^2h(q-1) + 2gh^2p(q-1) + h^2p(p-2gq) \\ =& -g^2(q-1)^2 + 2g^2h(q-1) - 2gh^2p + h^2p^2 \\ =& -(ph - g(q-1))(2gh - g(q-1) - hp) \leq 0.\end{aligned}$$

The inequality comes from $ph - g(q-1) \geq gh\gamma_2 \geq 0$ and $2gh - g(q-1) - hp \geq 2gh - gh - hp \geq 0$. Therefore, by (5.12), (5.13) and (5.14), we have that for $0 \leq \gamma \leq (q-1)/h$, $\tilde{f}(\gamma) \leq 0$. Then, we obtain the desired result. □

Proof of Proposition 5.5. It is trivial that $\chi(A^{(j)}_{1-r/(j-1)}) = j/(j-r)$ holds, and, hence, we will show only $\min_{(a_{i,l})_{1\leq i,l\leq j} \in \mathcal{G}_r} \chi((a_{i,l})_{1\leq i,l\leq j}) = \chi(A^{(j)}_{1-r/(j-1)})$. We will prove by induction on $j \in \mathbb{N}$. If $j = 1$ or 2, it is obvious that the claim holds. Assume that the claim holds for $1, 2, ..., j-1$ and show the claim for j. If we show the case that $g \wedge h = 1$, Lemma 5.7 yields the desired result. Only the case that $j \geq 4$ and $g, h \geq 2$ is remained since $g \wedge h = 1$ holds for $j = 3$.

For any $r \leq j-1$, consider some $(a_{i,l})_{1\leq i,l\leq j} \in \mathcal{G}_r$. For $(a_{i,l})_{1\leq i,l\leq j} = M\{(a_{i,l})_{i,l\in G}, (a_{i,l})_{i,l\in H}, s\}$, we pick $M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\}$ which is satisfied with $\gamma = s$, $\Xi((a_{i,l})_{i,l\in G}) = \Xi(A^{(g)}_{\gamma_2}) = (g-1)(1-\gamma_2)$ and

$\Xi((a_{i,l})_{i,l\in H}) = \Xi(A^{(h)}_{\gamma_1}) = (h-1)(1-\gamma_1)$. Without loss of generality, we can assume that $\gamma_1 \le \gamma_2$. Note that $M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\} \in \mathcal{G}_r$. By Lemma 5.6 and the assumption, we obtain

$$\chi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\}) \le \chi((a_{i,l})_{1\le i,l\le j}). \tag{5.15}$$

In addition, consider $M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\tilde{\gamma}_1}, \tilde{\gamma}_1\}$, which is satisfied with $(h-1)\gamma_1 + \gamma = h\tilde{\gamma}_1$. Note that $M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\tilde{\gamma}_1}, \tilde{\gamma}_1\} \in \mathcal{G}_r$. By Lemma 5.7, we obtain

$$\chi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\tilde{\gamma}_1}, \tilde{\gamma}_1\}) \le \chi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\gamma_1}, \gamma\}). \tag{5.16}$$

Next, we consider $M\{A^{(j-1)}_{\tilde{\gamma}_2}, A^{(1)}, \tilde{\gamma}_1\}$, which is satisfied with $(g-1)\gamma_2 + h\tilde{\gamma}_1 = (j-2)\tilde{\gamma}_2 + \tilde{\gamma}_1$. Note that $M\{A^{(j-1)}_{\tilde{\gamma}_2}, A^{(1)}, \tilde{\gamma}_1\} \in \mathcal{G}_r$. In addition,

$$M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\tilde{\gamma}_1}, \tilde{\gamma}_1\} = M\{M\{A^{(g)}_{\gamma_2}, A^{(h-1)}_{\tilde{\gamma}_1}, \tilde{\gamma}_1\}, A^{(1)}, \tilde{\gamma}_1\},$$

since $\gamma_2 \ge \tilde{\gamma}_1$ holds. Then, by Lemma 5.6 and the assumption, we obtain

$$\chi(M\{A^{(j-1)}_{\tilde{\gamma}_2}, A^{(1)}, \tilde{\gamma}_1\}) \le \chi(M\{A^{(g)}_{\gamma_2}, A^{(h)}_{\tilde{\gamma}_1}, \tilde{\gamma}_1\}). \tag{5.17}$$

Finally, Lemma 5.7 yields

$$\chi(A^{(j)}_{1-r/(j-1)}) \le \chi(M\{A^{(j-1)}_{\tilde{\gamma}_2}, A^{(1)}, \tilde{\gamma}_1\}). \tag{5.18}$$

Note that $A^{(j)}_{1-r/(j-1)} \in \mathcal{G}_r$. Therefore, by (5.15), (5.16), (5.17) and (5.18) we obtain the desired result. $\square$

Remark 5. *We write the example of above matrices. Consider the case that $j=5$, $g=3$, $h=2$ and some $(a_{i,l})_{1\le i,l\le 5} = M\{(a_{i,l})_{1\le i,l\le 3}, (a_{i,l})_{i,l\in\{4,5\}}, s\}$. Then, (5.15), (5.16), (5.17) and (5.18) imply*

$$\begin{aligned}
\chi\begin{pmatrix} 1 & a_{1,2} & a_{1,3} & s & s \\ a_{2,1} & 1 & a_{2,3} & s & s \\ a_{3,1} & a_{3,2} & 1 & s & s \\ s & s & s & 1 & a_{4,5} \\ s & s & s & a_{5,4} & 1 \end{pmatrix}
\ge& \chi\begin{pmatrix} 1 & \gamma_2 & \gamma_2 & \gamma & \gamma \\ \gamma_2 & 1 & \gamma_2 & \gamma & \gamma \\ \gamma_2 & \gamma_2 & 1 & \gamma & \gamma \\ \gamma & \gamma & \gamma & 1 & \gamma_1 \\ \gamma & \gamma & \gamma & \gamma_1 & 1 \end{pmatrix}
\ge \chi\begin{pmatrix} 1 & \gamma_2 & \gamma_2 & \tilde{\gamma}_1 & \tilde{\gamma}_1 \\ \gamma_2 & 1 & \gamma_2 & \tilde{\gamma}_1 & \tilde{\gamma}_1 \\ \gamma_2 & \gamma_2 & 1 & \tilde{\gamma}_1 & \tilde{\gamma}_1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_1 & \tilde{\gamma}_1 & 1 & \tilde{\gamma}_1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_1 & \tilde{\gamma}_1 & \tilde{\gamma}_1 & 1 \end{pmatrix} \\
\ge& \chi\begin{pmatrix} 1 & \tilde{\gamma}_2 & \tilde{\gamma}_2 & \tilde{\gamma}_2 & \tilde{\gamma}_1 \\ \tilde{\gamma}_2 & 1 & \tilde{\gamma}_2 & \tilde{\gamma}_2 & \tilde{\gamma}_1 \\ \tilde{\gamma}_2 & \tilde{\gamma}_2 & 1 & \tilde{\gamma}_2 & \tilde{\gamma}_1 \\ \tilde{\gamma}_2 & \tilde{\gamma}_2 & \tilde{\gamma}_2 & 1 & \tilde{\gamma}_1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_1 & \tilde{\gamma}_1 & \tilde{\gamma}_1 & 1 \end{pmatrix}
\ge \chi\begin{pmatrix} 1 & \tilde{\gamma} & \tilde{\gamma} & \tilde{\gamma} & \tilde{\gamma} \\ \tilde{\gamma} & 1 & \tilde{\gamma} & \tilde{\gamma} & \tilde{\gamma} \\ \tilde{\gamma} & \tilde{\gamma} & 1 & \tilde{\gamma} & \tilde{\gamma} \\ \tilde{\gamma} & \tilde{\gamma} & \tilde{\gamma} & 1 & \tilde{\gamma} \\ \tilde{\gamma} & \tilde{\gamma} & \tilde{\gamma} & \tilde{\gamma} & 1 \end{pmatrix},
\end{aligned}$$

where $\tilde{\gamma} := 1 - r/(j-1)$.

5.1.2 Preliminary for main results

In this section, our goal is to show Proposition 5.10, which is used in the upper bound in Theorems 2.3 and 2.4. We will estimate this by using matrix's arguments in Section 5.1. To show this, we prepare additional notations. Given real valued $j \times j$ matrices $(m_{i,l})_{1\le i,l\le j}$, $(m'_{i,l})_{1\le i,l\le j}$, let

$$\begin{aligned}
&\mathcal{E}[(m_{i,l})_{1\le i,l\le j}, (m'_{i,l})_{1\le i,l\le j}] \\
=&\mathcal{E}[(m_{i,l})_{1\le i,l\le j}, (m'_{i,l})_{1\le i,l\le j}](j,n) \\
:=&\{(x_1, ..., x_j) \in (\mathbb{Z}^2_n)^j : m_{i,l} \le d(x_i, x_l) \le m'_{i,l} \text{ for any } 1 \le i \ne l \le j\}.
\end{aligned}$$

Note that this set is independent of diagonal element of matrix. When $m_{i,l}$ for $1 \le i,l \le j$ is constant and $m'_{i,l}$ be, we simply write $\mathcal{E}[(m_{i,l}),(m'_{i,l})]$. Fix $0<\beta<1$. For $0<\eta<1$ with $\eta \le (1-\beta)\wedge\beta$, let

$$\mathcal{M}_j^{\beta,\eta} := \mathcal{M}_j \cap \mathcal{E}[(1-\beta),(1-\eta)]$$

and for $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j^{\beta,\eta}$ and $\delta'>0$ let

$$\begin{aligned}
&\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}] = \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j,n) \\
:=&\mathcal{E}\left[(\frac{1}{2^j}n^{1-a_{i,l}})_{1\le i,l\le j},(2^j n^{1-a_{i,l}+\delta'})_{1\le i,l\le j}\right], \\
&\mathcal{H}^{\delta'} = \mathcal{H}^{\delta'}(j,n) := \{\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j,n) : (a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j^{\beta,\eta}\}.
\end{aligned}$$

To show Proposition 5.10, we use the following two lemmas.

Lemma 5.8. *For any $0<\delta',\eta<1$ with $\eta\le(1-\beta)\wedge\beta$ there exists $n_0\in\mathbb{N}$ such that for any $n\ge n_0$ and $(x_1,...,x_j)\in\mathcal{E}[(n^\eta),(n^\beta)]$ there exists $V\in\mathcal{H}^{\delta'}(j,n)$ such that $(x_1,...,x_j)\in V$ holds.*

Proof. We will show by induction on $j\in\mathbb{N}$. It is obvious that the claim holds for $j=1,2$. Assume that the claim holds for $j-1$ and consider any $(x_1,...,x_j)\in\mathcal{E}[(n^\eta),(n^\beta)]$. Put $1\le i_0,l_0\le j$ such that $d(x_{i_0},x_{l_0}) = \min_{1\le i\ne l\le j} d(x_i,x_l)$. Then, by the assumption, for $(x_1,...,x_{l_0-1},x_{l_0+1},...,x_j)\in\mathcal{E}[(n^\eta),(n^\beta)](j-1,n)$ there exists

$$(a_{i,l})_{i,l\in\{1,...,l_0-1,l_0+1,...,j\}} \in \mathcal{M}_{j-1}^{\beta,\eta} \tag{5.19}$$

such that for any $i,l\in\{1,...,l_0-1,l_0+1,...,j\}$

$$\frac{1}{2^{j-1}}n^{1-a_{i,l}} \le d(x_i,x_l) \le 2^{j-1}n^{1-a_{i,l}+\delta'}.$$

Set $(\tilde{a}_{i,l})_{1\le i,l\le j}$ as follows:

$$\begin{cases}
\tilde{a}_{i,l} &= \tilde{a}_{l,i} = a_{i,l} \quad \text{for any } 1\le i,l\ne l_0\le j,\\
\tilde{a}_{l_0,l} &= \tilde{a}_{l,l_0} = a_{i_0,l} \quad \text{for any } 1\le l\ne i_0,l_0\le j,\\
\tilde{a}_{l_0,i_0} &= \tilde{a}_{i_0,l_0} = (1-\eta)\wedge\left(\left(\frac{(j-1)\log 2-\log d(x_{i_0},x_{l_0})}{\log n}+1+\delta'\right)\vee(1-\beta)\right),\\
\tilde{a}_{l_0,l_0} &= 1.
\end{cases}$$

In addition, $\tilde{a}_{i_0,l_0} = \max_{1\le i\ne l\le j}\tilde{a}_{i,l}$ holds since $\hat{g}(s) := \max\{b\in[1-\beta,1-\eta] : 2^{-j+1}n^{1-b}\le s\le 2^{j-1}n^{1-b+\delta'}\}$ is monotonically decreasing and $\hat{g}(d(x_{i_0},x_{l_0})) = \tilde{a}_{i_0,l_0}$ and $\hat{g}(d(x_i,x_l))\ge\tilde{a}_{i,l}$ hold for any $1\le i\ne l\le j$.

Now, we will prove that $(\tilde{a}_{i,l})_{1\le i,l\le j}\in\mathcal{M}_j^{\beta,\eta}$ as follows. It is obvious that the definition of $(\tilde{a}_{i,l})_{1\le i,l\le j}$ yields that $(\tilde{a}_{i,l})_{1\le i,l\le j}$ is symmetric and $(\tilde{a}_{i,l})_{1\le i,l\le j}\in\mathcal{E}[(1-\beta),(1-\eta)]$. Then, we will prove that for any $1\le i,l,p\le j$ with $i\ne l, l\ne p, p\ne i$, (i) $\tilde{a}_{i,l}<\tilde{a}_{i,p}\Rightarrow\tilde{a}_{l,p}=\tilde{a}_{i,l}$ and (ii) $\tilde{a}_{i,l}=\tilde{a}_{i,p}\Rightarrow\tilde{a}_{l,p}\ge\tilde{a}_{i,l}$ as follows:

1. $i,l,p\ne l_0$: (5.19) yields $(\tilde{a}_{i,l})_{1\le i,l\le j}\in\mathcal{M}_j$. Hence, we obtain (i) and (ii).
2. $i=l_0$ and $p,l\ne i_0$: (i) If $\tilde{a}_{l_0,l}<\tilde{a}_{l_0,p}$, $a_{i_0,l}<a_{i_0,p}$ holds. Therefore, (5.19) yields $\tilde{a}_{l,p}=a_{l,p}=a_{i_0,l}=\tilde{a}_{l_0,l}$ holds. (ii) The proof is almost same as (i).
3. ($p=l_0$ and $i,l\ne i_0$) or ($l=l_0$ and $l,i\ne i_0$) : The proof is almost same as above.

Then since $(\tilde{a}_{i,l})_{1\le i,l\le j}$ is symmetric for l_0 and i_0, it is remained to prove (i) and (ii) for the following cases.

1. $i = l_0$, $l = i_0$: (i) The assumption contradicts. (ii) Since $\tilde{a}_{l_0,p} = \tilde{a}_{i_0,p}$, $\tilde{a}_{l_0,i_0} = \tilde{a}_{l_0,p}$ yields the result.

2. $i = l_0$, $p = i_0$: (i) The result is trivial. (ii) The proof is almost same as above.

3. $l = l_0$, $p = i_0$: (i) The assumption contradicts. (ii) The result is trivial.

Therefore, we obtain $(\tilde{a}_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j^{\beta,\eta}$. In addition, we will prove $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(\tilde{a}_{i,l})_{1\le i,l\le j}]$. Note that the triangle inequality yields that for any $1 \le l \le j$ with $l \ne i_0, l_0$,

$$\begin{aligned} d(x_{l_0}, x_l) \le& d(x_{l_0}, x_{i_0}) + d(x_{i_0}, x_l) \\ \le& 2^{j-1} n^{1-\tilde{a}_{i_0,l_0}+\delta'} + 2^{j-1} n^{1-a_{i_0,l}+\delta'} \le 2^j n^{1-\tilde{a}_{l_0,l}+\delta'}, \end{aligned}$$

and since $d(x_{l_0}, x_l) + d(x_{l_0}, x_{i_0}) \ge d(x_{i_0}, x_l)$ and $d(x_{l_0}, x_l) \ge d(x_{l_0}, x_{i_0})$,

$$d(x_{l_0}, x_l) \ge \frac{1}{2} d(x_{i_0}, x_l) \ge \frac{1}{2} \frac{1}{2^{j-1}} n^{1-\tilde{a}_{i_0,l}} = \frac{1}{2^j} n^{1-\tilde{a}_{l_0,l}}.$$

Therefore, for any $1 \le i \ne l \le j$

$$\frac{1}{2^j} n^{1-\tilde{a}_{i,l}} \le d(x_i, x_l) \le 2^j n^{1-\tilde{a}_{i,l}+\delta'},$$

and, hence, we obtain the desired result. $\square$

Note that $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j^{\beta,\eta}$ is satisfied with $0 < (j-1)\eta \le \Xi((a_{i,l})_{1\le i,l\le j}) \le (j-1)\beta$. In addition, to introduce the next lemma, for $0 < t \le (j-1)\beta$ and $\delta' > 0$ set

$$\mathcal{H}_t^{\delta'} := \{\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, n) : (a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j^{\beta,\eta}, \Xi((a_{i,l})_{1\le i,l\le j}) = t\}.$$

Lemma 5.9. *For any $\epsilon > 0$ there exist $\delta_0 > 0$ and $C > 0$ such that for any $0 < t \le (j-1)\beta$, $0 < \delta' < \delta_0$, $\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, n) \in \mathcal{H}_t^{\delta'}$ and any $n \in \mathbb{N}$,*

$$|\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]| \le C n^{2t+2+\epsilon/2}.$$

Proof. To prove the claim, for $0 < t \le (j-1)\beta$, $\delta' > 0$, $x \in \mathbb{Z}_n^2$ and $|\Lambda| = 1$ set

$$D_x^{\delta'}((a_{i,i})_{i\in\Lambda}) := \emptyset,$$

and for $j \ge 2$ and $m \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} &D_x^{\delta'}((a_{i,l})_{m\le i,l\le m+j}) \\ :=& \{(x_2, ..., x_j) \in \mathcal{E}[(0), (L n^{1-a_{i,l}+\delta'})_{m+2\le i,l\le m+j}](j-1, n) : \\ & d(x, x_l) \le L n^{1-a_{m+1,l}+\delta'} \text{ for any } m+2 \le l \le m+j\}, \end{aligned} \tag{5.20}$$

and in addition

$$\tilde{\mathcal{H}}_{t,M,x}^{\delta'}(j) := \{D_x^{\delta'}((a_{i,l})_{1\le i,l\le j}) : (a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j^{\beta,\eta}, \Xi((a_{i,l})_{1\le i,l\le j}) = t\}.$$

Then, it suffices to prove that for any $\epsilon > 0$ and $L < \infty$ there exist $\delta_0 > 0$ and $C > 0$ such that for any $0 < \delta' < \delta_0$, $x \in \mathbb{Z}_n^2$, $t \le (j-1)\beta$, $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}_j^{\beta,\eta}$ and any $n \in \mathbb{N}$

$$|D_x^{\delta'}((a_{i,l})_{1\le i,l\le j})| \le C n^{2t+\epsilon/2}.$$

We prove this by induction on $j \in \mathbb{N}$. It is obvious that the claim holds for $j = 1, 2$. Assume that the claim holds for $1, 2, ..., j-1$. To show that the claim holds for j, consider $D_{x_1}^{\delta'}((a_{i,l})_{1\le i,l\le j}) \in \tilde{\mathcal{H}}_{t,M,x}^{\delta'}(j)$.

Set $\Xi((a_{i,l})_{i,l\in G}) = t_1$ and $\Xi((a_{i,l})_{i,l\in H}) = t_2$, and hence $t = \Xi((a_{i,l})_{1\le i,l\le j}) = t_1 + t_2 + 1 - s$ holds. In addition, note that if we let $x' := x_1$ and $x'' := x_{g+1}$, it holds that

$$D^{\delta'}_{x'}((a_{i,l})_{i,l\in G}) \in \tilde{\mathcal{H}}^{\delta'}_{t_1,M,x'}(g), \quad D^{\delta'}_{x''}((a_{i,l})_{i,l\in H}) \in \tilde{\mathcal{H}}^{\delta'}_{t_2,M,x''}(h). \tag{5.21}$$

Then, by the assumption we can obtain that there exist $0 < \delta_0 < \epsilon/12$ and $C > 0$ such that for any $x', x'' \in \mathbb{Z}^2_n$, $0 < \delta' < \delta_0$, $t_1 \le (g-1)\beta$, $t_2 \le (h-1)\beta$, $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}^{\beta,\eta}_j$ and any $n \in \mathbb{N}$, it holds that $|D^{\delta'}_{x'}((a_{i,l})_{i,l\in G})| \le Cn^{2t_1+\epsilon/6}$, $|D^{\delta'}_{x''}((a_{i,l})_{i,l\in H})| \le Cn^{2t_2+\epsilon/6}$ and, hence,

$$\begin{aligned}
&|D^{\delta'}_{x'}((a_{i,l})_{1\le i,l\le j})| \\
\le &\sum_{x''\in\mathbb{Z}^2_n : d(x',x'')\le Ln^{1-s+\delta'}} |\{(x', x_2, ..., x_g, x'', x_{g+2}, ..., x_j) : \\
&(x_2, ..., x_g) \in D^{\delta'}_{x'}((a_{i,l})_{i,l\in G}), (x_{g+2}, ..., x_j) \in D^{\delta'}_{x''}((a_{i,l})_{i,l\in H})\}| \\
\le &Cn^{2-2s+2\delta'}|D^{\delta'}_{x'}((a_{i,l})_{i,l\in G})| \times |D^{\delta'}_{x''}((a_{i,l})_{i,l\in H})| \\
\le &Cn^{2t_1+2t_2+2-2s+\epsilon/2} = Cn^{2t+\epsilon/2}.
\end{aligned}$$

The last inequality comes from $\delta' < \epsilon/12$. Therefore, we can obtain the desired result. □

Remark 6. *The reason why we set "L" in (5.20) instead of "2^j" is to ensure (5.21).*

Note that by Lemma 5.8, it holds that for any $\delta' > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ and $(x_1, ..., x_j) \in \mathcal{E}[(n^\eta), (n^\beta)]$

$$\{(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}^{\beta,\eta}_j : (x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]\} \ne \emptyset.$$

Then, we can define the function $\tilde{K} : \mathcal{E}[(n^\eta), (n^\beta)] \to \mathbb{R}$ for all sufficiently large $n \in \mathbb{N}$ as follows.

$$\tilde{K}(x_1, ..., x_j) := \inf\{\chi((b_{i,l})_{1\le i,l\le j}) : (x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(b_{i,l})_{1\le i,l\le j}]\}.$$

To state the following proposition, we prepare notations. Recall $F_{h,\beta}(\gamma) = F(h, \beta, \gamma) = h\gamma^2\beta + (1 - \gamma\beta)^2/(1-\beta)$ and let $\gamma(h) := 1/(h(1-\beta) + \beta)$. Note that for any $(a_{i,l})_{1\le i,l\le j} \in \mathcal{M}^{\beta,\eta}_j$, $((\beta - 1 + a_{i,l})/\beta)_{1\le i,l\le j} \in \mathcal{M}_j$ holds. Then, we can define the following: for $(x_1, ..., x_j) \in \mathcal{E}[(n^\eta), (n^\beta)]$ with $\tilde{K}(x_1, ..., x_j) = \chi((a_{i,l})_{1\le i,l\le j})$ and $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]$

$$\begin{aligned}
\tilde{A}^{(j)}_\beta &:= \left(\frac{\beta - 1 + a_{i,l}}{\beta}\right)_{1\le i,l\le j}, \quad \gamma_0 := \beta^{-1}(1 - \sqrt{\alpha}^{-1}(1-\beta)), \\
B_1 = B_{1,\delta'} &:= F(\chi(\tilde{A}^{(j)}_\beta), \beta, \gamma(\chi(\tilde{A}^{(j)}_\beta)) \vee \gamma_0), \quad B_2 = B_{2,\delta'} := \chi((a_{i,l})_{1\le i,l\le j}).
\end{aligned}$$

Note that definitions of $\tilde{A}^{(j)}_\beta$, B_1 and B_2 depend on $(a_{i,l})_{1\le i,l\le j}$ and hence $(x_1, ..., x_j)$.

Proposition 5.10. *For any $\epsilon > 0$ there exists $C > 0$ such that for any $n \in \mathbb{N}$ and $\delta' < \delta_0$*

$$\begin{aligned}
&(A) \sum_{(x_1,...,x_j)\in\mathcal{E}[(n^\eta),(n^\beta)]} n^{-2\alpha B_1} \le Cn^{\rho_j(\alpha,\beta)+\epsilon}, \\
&(B) \sum_{(x_1,...,x_j)\in\mathcal{E}[(n^\eta),(n^\beta)]} n^{-2\alpha B_2} \le Cn^{\hat{\rho}_j(\alpha,\beta)+\epsilon}.
\end{aligned} \tag{5.22}$$

Proof. Note that for any $b \in \mathbb{R}$

$$\chi((a_{i,l})_{1\le i,l\le j}) = \frac{\chi((b + a_{i,l})_{1\le i,l\le j})}{1 - b\chi((b + a_{i,l})_{1\le i,l\le j})}. \tag{5.23}$$

Indeed,

$$\begin{aligned}&\{(y_1,...,y_j):(a_{i,l})_{1\le i,l\le j}(y_1,...,y_j)^T=(1,...,1)^T\}\\ =&\left\{\left(\frac{y_1}{1-b\chi((b+a_{i,l})_{1\le i,l\le j})},...,\frac{y_j}{1-b\chi((b+a_{i,l})_{1\le i,l\le j})}\right):\right.\\ &\left.(b+a_{i,l})_{1\le i,l\le j}(y_1,...,y_j)^T=(1,...,1)^T\right\}.\end{aligned}$$

Thus, by Proposition 5.5,

$$\min_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{G}_t}\chi(\tilde{A}^{(j)}_\beta)=\min_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{G}_t}\frac{\beta\chi((a_{i,l})_{1\le i,l\le j})}{1-\chi((a_{i,l})_{1\le i,l\le j})(1-\beta)}=\frac{j\beta}{j\beta-t},$$

and, hence,

$$\min_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{G}_t}F(\chi(\tilde{A}^{(j)}_\beta),\beta,\gamma_0)=\frac{1-\beta}{\alpha}+\frac{j}{j\beta-t}(1-\sqrt{\alpha}^{-1}(1-\beta))^2.$$

By (5.23), we have

$$F(\chi(\tilde{A}^{(j)}_\beta),\beta,\gamma(\chi(\tilde{A}^{(j)}_\beta)))=\frac{\chi(\tilde{A}^{(j)}_\beta)}{\chi(\tilde{A}^{(j)}_\beta)(1-\beta)+\beta}=\chi((a_{i,l})_{1\le i,l\le j}).\tag{5.24}$$

Note that Proposition 5.5 implies $\min_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{G}_t}\chi((a_{i,l})_{1\le i,l\le j})=j/(j-t)$. Therefore, we obtain

$$\begin{aligned}\tilde{B}_1(t)&:=\min_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{G}_t}B_1=\begin{cases}\frac{j}{j-t} & \text{if } \gamma(\chi(\tilde{A}^{(j)}_\beta))\ge\gamma_0,\\ \frac{1-\beta}{\alpha}+\frac{j}{j\beta-t}(1-\sqrt{\alpha}^{-1}(1-\beta))^2 & \text{otherwise,}\end{cases}\\ \tilde{B}_2(t)&:=\min_{(a_{i,l})_{1\le i,l\le j}\in\mathcal{G}_t}B_2=\frac{j}{j-t}.\end{aligned}$$

Consider any $0<\delta'<\delta_0$ and $N_0\in\mathbb{N}$ with $\delta_0N_0=\beta-\eta$. Then, by above, for any $\delta'>0$ and $\epsilon>0$ there exists $C>0$ such that for any $n\in\mathbb{N}$, the right hand side of (5.22) is bounded by

$$\begin{aligned}&N_0^j\max_{0\le t\le(j-1)\beta}\max_{\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]\in\mathcal{H}^{\delta'}_t}\sum_{(x_1,...,x_j)\in\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]}n^{-2\alpha B_i}\\ \le&C\max_{0\le t\le(j-1)\beta}n^{2t+2+\epsilon/2}\max_{\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]\in\mathcal{H}^{\delta'}_t}n^{-2\alpha B_i}\\ \le&C\max_{0\le t\le(j-1)\beta}n^{2t+2+\epsilon}n^{-2\alpha\tilde{B}_i(t)},\end{aligned}$$

for $i=1,2$. The first inequality comes from Lemma 6.4 and the last one comes from Proposition 5.5. Therefore, since

$$\begin{aligned}\max_{0\le t\le(j-1)\beta}2t+2-2\alpha\tilde{B}_1(t)&=\rho_j(\alpha,\beta),\\ \max_{0\le t\le(j-1)\beta}2t+2-2\alpha\tilde{B}_2(t)&=\hat{\rho}_j(\alpha,\beta),\end{aligned}$$

we obtain the desired result. □

5.2 Proof of Theorem 2.4

5.2.1 Preliminary of the proof of Theorem 2.4

Fix $0 < \alpha, \beta < 1$. Let $\mathcal{R}_n^z$ be the time until completion of the first n_n excursions from $\partial D(z, r_{n-1})$ to $\partial D(z, r_n)$. For $z \in \mathbb{Z}_{K_n}^2$, let $N_{n,k}^z$ denote the number of excursions from $\partial D(z, r_{k-1})$ to $\partial D(z, r_k)$ until time $\mathcal{R}_n^z$. Let

$$I(h,\gamma) := \begin{cases} [0,\gamma^2] & \text{for } \gamma < \gamma_h, \\ [0,\infty] & \text{for } \gamma = \gamma_h, \\ [\gamma^2,\infty] & \text{for } \gamma > \gamma_h. \end{cases}$$

To state the following lemma, we define the domain $\mathbf{Go}^\beta$ such that

$$\mathbf{Go}^\beta := \{(z, x_1, ..., x_j) : z \in \mathbb{Z}_{K_n}^2, x_1, ..., x_j \in D(z, r_{\beta n-2})\}.$$

Lemma 5.11.

(i) For any $\delta > 0$ there exists $C > 0$ such that for any $z \in \mathbb{Z}_{K_n}^2$

$$P\Big(\frac{N_{n,\beta n}^z}{n_{\beta n}} \in I(0,\gamma)\Big) \le C K_n^{-2\alpha F_{0,\beta}(\gamma)+\delta}.$$

(ii) For any $\delta > 0$ there exist $\delta_1 > 0$ and $C > 0$ such that for any $\delta' < \delta_1$, $\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n) \in \mathcal{H}^{\delta'}$, $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$ and $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$

$$\begin{aligned} &C^{-1} K_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)-\delta} \\ \le & P\Big(x_1, ..., x_j \in \mathcal{L}_{K_n}(\alpha), \frac{N_{n,\beta n}^z}{n_{\beta n}} \in I(\chi(\tilde{A}_\beta^{(j)}),\gamma)\Big) \\ \le & C K_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)+\delta}. \end{aligned}$$

Remark 7. (6.3) *in [11] yields* (i) *in Theorem 5.11.*

Corollary 5.12. *Recall that $B_{1,\delta'} = F(\chi(\tilde{A}_\beta^{(j)}), \beta, \gamma(\chi(\tilde{A}_\beta^{(j)})) \vee \gamma_0)$. For any $\delta > 0$ there exists δ_1 and $C > 0$ such that for any $\delta' < \delta_1$, $\hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n) \in \mathcal{H}^{\delta'}$, $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$ and $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$*

$$P\Big(x_1, ..., x_j \in \mathcal{L}_{K_n}(\alpha), \frac{N_{n,\beta n}^z}{n_{\beta n}} \le \gamma_0^2\Big) \le C K_n^{-2\alpha B_{1,\delta'}+\delta}.$$

Proof. Note that the minimum of $F_{h,\beta}(\gamma)$ is $h/(h(1-\beta)+\beta)$ at $\gamma_h(\beta)$. Then, Lemma 5.11 and the simple computation yield the desired result. $\square$

To prove (ii) in Lemma 5.11, we will estimate the following probabilities. For any $z \in \mathbb{Z}_{K_n}^2$ let $\tilde{T}_{\partial D(z,r_{\beta n})} := \inf\{m > T_{\partial D(z,r_{\beta n-1})} : S_m \in \partial D(z, r_{\beta n})\}$ and for any $y \in \partial D(z, r_{\beta n})$ and $y' \in \partial D(z, r_{n-1})$

$$\begin{aligned} q_{1,1} &:= P^y(T_{\partial D(z,r_{\beta n-1})} < T_{\partial D(z,r_n)}, \tilde{T}_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j}), \\ q_{1,2} &:= P^y(T_{\partial D(z,r_n)} < T_{\partial D(z,r_{\beta n-1})}), \\ q_{2,2} &:= P^{y'}(T_{\partial D(z,r_n)} < T_{\partial D(z,r_{\beta n-1})}), \\ q_{2,1} &:= P^{y'}(T_{\partial D(z,r_{\beta n-1})} < T_{\partial D(z,r_n)}, \tilde{T}_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j}). \end{aligned}$$

Lemma 5.13.

(i) It holds that, uniformly in $0 < \delta' < \delta_2$, $y \in \partial D(z, r_{\beta n})$, $y' \in \partial D(z, r_{n-1})$, $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$ with $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$

$$q_{1,1} = \left(1 - \frac{1 + o(1_{\delta_2,n})}{(1-\beta)n}\right)\left(1 - \frac{\pi + o(1_{\delta_2,n})}{\beta n}\chi(\tilde{A}^{(j)}_\beta)\right), \tag{5.25}$$

$$q_{1,2} = \frac{1 + o(1_{\delta_2,n})}{(1-\beta)n}, \tag{5.26}$$

$$q_{2,2} = \left(1 - \frac{1 + o(1_{\delta_2,n})}{(1-\beta)n}\right), \tag{5.27}$$

$$q_{2,1} = \frac{1 + o(1_{\delta_2,n})}{(1-\beta)n}\left(1 - \frac{\pi + o(1_{\delta_2,n})}{\beta n}\chi(\tilde{A}^{(j)}_\beta)\right), \tag{5.28}$$

where $d_{\delta_2,n} = o(1_{\delta_2,n})$ means $\lim_{\delta_2\to 0}\lim_{n\to\infty} d_{\delta_2,n} \to 0$.

(ii) For any $\delta > 0$ there exist $c > 0$ and $\delta_2 > 0$ such that for any $0 < \delta' < \delta_2$ and $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$ with $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$,

$$P(T_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j}) \ge cK_n^{-\delta/8}.$$

Proof of (i) in Lemma 5.13. (3.7) yields that, uniformly in $y \in \partial D(z, r_{\beta n})$ and $y' \in \partial D(z, r_{n-1})$,

$$P^y(T_{\partial D(z,r_{\beta n-1})} < T_{\partial D(z,r_n)}) = 1 - \frac{1 + o(1)}{(1-\beta)n},$$

$$P^{y'}(T_{\partial D(z,r_{\beta n-1})} < T_{\partial D(z,r_n)}) = \frac{1 + o(1)}{(1-\beta)n}.$$

Therefore, we obtain (5.26) and (5.27). By the strong Markov property, we have

$$q_{1,1} = E^y[P^{y_0}(T_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j} | \mathcal{F}(T_{\partial D(z,r_{\beta n-1})}))1_{\{T_{\partial D(z,r_{\beta n-1})} < T_{\partial D(z,r_n)}\}}],$$

$$q_{2,1} = E^{y'}[P^{y_0}(T_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j} | \mathcal{F}(T_{\partial D(z,r_{\beta n-1})}))1_{\{T_{\partial D(z,r_{\beta n-1})} < T_{\partial D(z,r_n)}\}}].$$

Hence, it suffices to estimate

$$P^{y_0}(T_{x_1,...,x_j} < T_{\partial D(z,r_{\beta n})}) = \frac{\pi + o(1_{\delta_2,n})}{\beta n}\chi(\tilde{A}^{(j)}_\beta) \tag{5.29}$$

for $y_0 \in \partial D(z, r_{\beta n-1})$ to show (5.25) and (5.28). Note that for $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$ and $y_0 \in \partial D(z, r_{\beta n-1})$

$$\sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < T_{D(z,r_{\beta n})} \wedge T_{y_0}) \le G_{r_{\beta n}}(x_i - z, x_l - z),$$

$$\sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < T_{D(z,r_{\beta n})} \wedge T_{y_0}) \ge G_{r_{\beta n-1}}(x_i - z, x_l - z).$$

Hence, since $r_{\beta n-1}/r_{\beta n-2} \ge 3$ for all sufficiently large $n \in \mathbb{N}$, (3.9) yields

$$\sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < T_{D(z,r_{\beta n})} \wedge T_{y_0}) = \frac{2}{\pi}(\log r_{\beta n} - \log d(x_i, x_l)^+ + O(1)),$$

where $(a)^+ = a \vee 1$. Then, (3.7) yields that, uniformly in $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$,

$$P^{x_i}(T_{y_0} < T_{D(z,r_{\beta n})}) = \frac{1 + o(1)}{\beta n}.$$

Then, for all sufficiently large $n \in \mathbb{N}$, $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$ and $\delta' < \delta_2$,

$$\max \left| \frac{2}{\pi}(\log r_{\beta n} - \log d(x_i, x_l)^+ + o(1)) - \frac{\beta - 1 + a_{i,l}}{\beta} \frac{6\beta n \log n}{\pi} \right|$$
$$\le \max \frac{6\beta n \log n}{\pi} \left| \frac{\beta - 1 + b_{i,l}}{\beta} - \frac{\beta - 1 + a_{i,l}}{\beta} \right| = o(1_{\delta_2,n}) n \log n,$$

where the above maximum is over $b_{i,l} = a_{i,l} + o(1_{\delta_2,n})$ with $1 \le i, l \le j$. Hence, by Remark 3 for all sufficiently large $n \in \mathbb{N}$, any $\delta' < \delta_2$ and $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$,

$$\left| \chi\left(\left(\frac{2}{\pi}(\log r_{\beta n} - \log d(x_i, x_l)^+ + O(1)) \right)_{1\le i,l\le j} \right) - \frac{\pi}{6\beta n \log n} \chi\left(\tilde{A}_\beta^{(j)} \right) \right|$$
$$= \frac{o(1_{\delta_2,n})}{n \log n}. \tag{5.30}$$

Therefore, if we substitute $T_{D(z,r_{\beta n})}$ and y for $\tilde{\tau}$ and y_0 in (3.3), (3.9) yields

$$\sum_{l=1}^{j} P^{x_l}(T_{y_0} < T_{x_1,...,x_j} \wedge T_{\partial D(z,r_{\beta n})}) = \frac{1 + o(1_{\delta_2,n})}{\beta n} \frac{\pi}{6\beta n \log n} \chi(\tilde{A}_\beta^{(j)}). \tag{5.31}$$

Note that by (3.10) we obtain

$$P^{y_0}(T_{y_0} < T_{x_1,...,x_j} \wedge T_{\partial D(z,r_{\beta n})}) = 1 - \frac{\pi + o(1_{\delta_2,n})}{6\beta n \log n}.$$

Then,

$$\begin{aligned}
&P^{y_0}(T_{x_1,...,x_j} < T_{\partial D(z,r_{\beta n})}) \\
=&\sum_{i=0}^{\infty} P^{y_0}(T_{y_0} < T_{x_1,...,x_j} \wedge T_{\partial D(z,r_{\beta n})})^i P^{y_0}(T_{x_l} = T_{y_0} \wedge T_{x_1,...,x_j} \wedge T_{\partial D(z,r_{\beta n})}) \\
=&\frac{1}{1 - P^{y_0}(T_{y_0} < T_{x_1,...,x_j} \wedge T_{\partial D(z,r_{\beta n})})} P^{y_0}(T_{x_1,...,x_j} < T_{y_0} \wedge T_{\partial D(z,r_{\beta n})}) \\
=&\frac{6(1 + o(1_{\delta_2,n}))\beta n \log n}{\pi} P^{y_0}(T_{x_1,...,x_j} < T_{y_0} \wedge T_{\partial D(z,r_{\beta n})}) \\
=&\frac{6(1 + o(1_{\delta_2,n}))\beta n \log n}{\pi} \sum_{l=1}^{j} P^{x_l}(T_{y_0} < T_{x_1,...,x_j} \wedge T_{\partial D(z,r_{\beta n})}).
\end{aligned}$$

The last equality comes from the time-reversal of random walk. Therefore, in view of (5.31), (5.29) is completed. □

Proof of (ii) in Lemma 5.13. If $z \notin D(0, r_{\beta n}) \cup \partial D(0, r_{\beta n})$, $P(T_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j}) = 1$ holds, and, hence, the result holds.

Next, we consider the case that $z \in D(0, r_{\beta n}) \cup \partial D(0, r_{\beta n})$. Fix $\delta > 0$. First, we will show that there exist $c > 0$ and $\delta_2 > 0$ such that for any $0 < \delta' < \delta_2$, $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$ with $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$ and $x_{j+1} \in \cup_{i=1}^{j} \partial D(x_i, 2^{-j-1} {K_n}^\eta)$,

$$P^{x_{j+1}}(T_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_{j+1}}) \ge c K_n^{-\delta/16}. \tag{5.32}$$

For any $\delta' > 0$, consider $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}](j, K_n)$. For $1 \le i_0 \le j$, set $(\tilde{b}_{i,l}(i_0))_{1\le i,l\le j+1} = (\tilde{b}_{i,l})_{1\le i,l\le j+1}$ as follows:

$$\begin{cases} \tilde{b}_{i,l} = \tilde{b}_{l,i} = a_{i,l} & \text{for any } 1 \le i, l \le j, \\ \tilde{b}_{j+1,l} = \tilde{b}_{l,j+1} = a_{i_0,l} & \text{for any } 1 \le l \ne i_0 \le j, \\ \tilde{b}_{j+1,i_0} = \tilde{b}_{i_0,j+1} = 1 - \eta, \\ \tilde{b}_{j+1,j+1} = 1. \end{cases}$$

We have $(\tilde{b}_{i,l})_{1\le i,l\le j+1} \in \mathcal{M}_{j+1}^{\beta,\eta}$ with modification of the proof of Lemma 5.8. In addition, note that for $x_j \in \partial D(x_{i_0}, 2^{-j-1}K_n{}^\eta)$,

$$(x_1, ..., x_{j+1}) \in \hat{\mathcal{E}}_{\delta'}[(\tilde{b}_{i,l})_{1\le i,l\le j+1}](j+1, K_n) \tag{5.33}$$

with modification of the proof of Lemma 5.8. Since $r_{\beta n}/(r_{\beta_n-2}+K_n^\eta) \ge 3$ for all sufficiently large $n \in \mathbb{N}$ and small $\eta > 0$, (3.9) yields that for $1 \le i, l \le j+1$

$$\sum_{m=0}^{\infty} P^{x_i}(S_m = x_l, m < T_{D(z,r_{\beta n})}) = \frac{2}{\pi}(\log r_{\beta n} - \log d(x_i, x_l)^+ + O(1)).$$

Then, for all sufficiently large $n \in \mathbb{N}$ and $(x_1, ..., x_{j+1}) \in \hat{\mathcal{E}}_{\delta'}[(\tilde{b}_{i,l})_{1\le i,l\le j+1}](j+1, K_n)$ with $\delta' < \delta_2$,

$$\begin{aligned}
&\max \left| \frac{2}{\pi}(\log r_{\beta n} - \log d(x_i, x_l)^+ + O(1)) - \frac{\beta - 1 + \tilde{b}_{i,l}}{\beta} \frac{6\beta n \log n}{\pi} \right| \\
\le &\max \frac{6\beta n \log n}{\pi} \left| \frac{\beta - 1 + b_{i,l}}{\beta} - \frac{\beta - 1 + \tilde{b}_{i,l}}{\beta} \right| = o(1_{\delta_2,n}) n \log n,
\end{aligned}$$

where the above maximum is over $b_{i,l} = \tilde{b}_{i,l} + o(1_{\delta_2,n})$ with $1 \le i, l \le j+1$. Therefore, (5.11) and (5.23) yield that there exists $c > 0$ such that for any $n \in \mathbb{N}$ and $x_j \in \partial D(x_{i_0}, 2^{-j-1}K_n{}^\eta)$,

$$P^{x_{j+1}}(T_{\partial D(z,r_{\beta n})} < T_{x_1,\dots,x_{j+1}}) = \frac{\sum_{i=1}^{j+1} (\text{cofactor of } d_{j+1,i})}{\det((d_{i,l})_{1\le i,l\le j+1})} \ge \frac{c}{n \log n},$$

where $(d_{i,l})_{1\le i,l\le j+1} := 2/\pi(\log r_{\beta n} - \log d(x_i, x_l)^+ + O(1))_{1\le i,l\le j+1}$. Therefore, with the aid of the symmetry of $x_1, ..., x_j$ we have (6.2.3).

Now we turn to the desired result. Note that (3.7) and (3.10) yield that there exists $c > 0$ such that for $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$

$$P(\min_{1\le i\le j} T_{\partial D(x_i, 2^{-j-1}K_n{}^\eta)} < T_{x_1,\dots,x_j}) \ge cK_n^{-\delta/16}.$$

Therefore, with the aid of (6.2.3) the strong Markov property yields

$$\begin{aligned}
&P(T_{\partial D(z,r_{\beta n})} < T_{x_1,\dots,x_j}) \\
\ge &P(\min_{1\le i\le j} T_{\partial D(x_i, 2^{-j-1}K_n{}^\eta)} < T_{x_1,\dots,x_j}) \\
\times &\min_{x_{j+1}\in\cup_{i=1}^{j}\partial D(x_i, 2^{-j-1}K_n{}^\eta)} P^{x_{j+1}}(T_{\partial D(z,r_{\beta n})} < T_{x_1,\dots,x_{j+1}}) \ge cK_n^{-\delta/8},
\end{aligned}$$

and, hence, the desired result is completed. □

Proof of (ii) in Lemma 5.11. For $l \in \{1, 2\}$ let

$$\begin{aligned}
\hat{q}_{1,l} &:= \max_{y\in\partial D(z,r_{\beta n})} q_{1,l}, \quad \hat{q}_{2,l} := \max_{y'\in\partial D(z,r_{n-1})} q_{2,l}, \\
\tilde{q}_{1,l} &:= \min_{y\in\partial D(z,r_{\beta n})} q_{1,l}, \quad \tilde{q}_{2,l} := \min_{y'\in\partial D(z,r_{n-1})} q_{2,l}.
\end{aligned}$$

For $\delta_2 > 0$, we consider any $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$ with $(x_1, ..., x_j) \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]$ and $0 < \delta' < \delta_2$. First, we will prove the desired upper bound. By the strong Markov property, we obtain that for $c_0 \in \mathbb{N}$

$$\begin{aligned}
&P(T_{x_1,\dots,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0) \\
\le &\max_{y_0\in\partial D(z,r_{\beta n-1})} P^{y_0}(T_{x_1,\dots,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0) \\
+ &\max_{y'\in\partial D(z,r_{n-1})} P^{y'}(T_{x_1,\dots,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0).
\end{aligned} \tag{5.34}$$

Note that $P^{\tilde{y}}(T_{\partial D(z,r_{n-1})} < \infty) = 1$ for any $\tilde{y} \in \partial D(z,r_n)$. Then, if we substitute $\{x_1,...,x_j\}$, $\partial D(z,r_{\beta n-1})$, $\partial D(z,r_{\beta n})$, $\partial D(z,r_{n-1})$, $\partial D(z,r_{\beta n-1})$, c_0-1, n_n into $0,1,2,3,4,n_1,n_2$ in (3.13),

$$
\begin{aligned}
&\max_{y_0\in\partial D(z,r_{\beta n-1})} P^{y_0}(T_{x_1,...,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0) \\
\leq &\max_{y_0\in\partial D(z,r_{\beta n-1})} P^{y_0}(T_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j}) \\
&\times \max_{y\in\partial D(z,r_{\beta n})} P^{y}(T_{x_1,...,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0-1) \\
\leq &\sum_{0\leq i\leq (c_0-1)\wedge(n_n-1)} \hat{q}_{1,1}^{c_0-1-i}\hat{q}_{1,2}^{i+1}\frac{(c_0-1)!}{(c_0-1-i)!i!}\hat{q}_{2,1}^{i}\hat{q}_{2,2}^{n_n-i-1}\frac{(n_n-1)!}{(n_n-i-1)!i!}
\end{aligned}
$$

In addition, if we substitute $\{x_1,...,x_j\}$, $\partial D(z,r_{\beta n-1})$, $\partial D(z,r_{\beta n})$, $\partial D(z,r_{n-1})$, $\partial D(z,r_{\beta n-1})$, c_0, n_n into $0,1,2,3,4,n_1,n_2$ in (3.14),

$$
\begin{aligned}
&\max_{y'\in\partial D(z,r_{n-1})} P^{y'}(T_{x_1,...,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0) \\
\leq &\sum_{1\leq i\leq c_0\wedge n_n} \hat{q}_{1,1}^{c_0-i}\hat{q}_{1,2}^{i}\frac{(c_0-1)!}{(c_0-i)!(i-1)!}\hat{q}_{2,1}^{i}\hat{q}_{2,2}^{n_n-i}\frac{n_n!}{(n_n-i)!i!}.
\end{aligned}
$$

Let us denote the probability in the left hand side of (5.34) by A. Then, the above estimate implies

$$
\begin{aligned}
A \leq &2((c_0\wedge n_n)+1) \\
&\times \max_{0\leq i\leq c_0\wedge n_n} \hat{q}_{1,1}^{c_0-1-i}\hat{q}_{1,2}^{i}\frac{c_0!}{(c_0-1-i)!(i-1)!}\hat{q}_{2,1}^{i}\hat{q}_{2,2}^{n_n-i}\frac{n_n!}{(n_n-i-1)!i!}.
\end{aligned}
$$

With the aid of (ii) in Lemma 5.13 we will estimate the maximal term. Let us define $h(i)$ by

$$
\begin{aligned}
h(i) := &\left(1-\frac{\pi+o(1_{\delta_2,n})}{\beta n}\chi(\tilde{A}_\beta^{(j)})\right)^{c_0-1}\left(1-\frac{1+o(1_{\delta_2,n})}{(1-\beta)n}\right)^{c_0-1-i} \\
&\times\left(\frac{1+o(1_{\delta_2,n})}{(1-\beta)n}\right)^{i}\frac{c_0!}{(c_0-1-i)!(i-1)!} \\
&\times\left(\frac{1+o(1_{\delta_2,n})}{(1-\beta)n}\right)^{i}\left(1-\frac{1+o(1_{\delta_2,n})}{(1-\beta)n}\right)^{n_n-i}\frac{n_n!}{(n_n-i-1)!i!}.
\end{aligned}
$$

Then, by Lemma 5.13, we have $A \leq 2((c_0\wedge n_n)+1)\max_{0\leq i\leq c_0\wedge n_n} h(i)$. Note that for $0\leq i\leq c_0\wedge n_n$,

$$
\frac{h(i-1)}{h(i)} = \frac{\left(1-\frac{1+o(1_{\delta_2,n})}{(1-\beta)n}\right)^2 i(i-1)}{\left(\frac{1+o(1_{\delta_2,n})}{(1-\beta)n}\right)^2 (n_n-i)(c_0-i)}.
$$

By taking the growth order of c_0 and n_n into account, the maximum of $h(i)$ among $0\leq i\leq c_0\wedge n_n$ is attained at

$$
i_0 = \left\lceil (1+o(1_{\delta_2,n}))\frac{\sqrt{n_n c_0}}{(1-\beta)n}\right\rceil + o((\log n)^2).
$$

In addition, the Stirling formula yields that picking appropriate $\delta_2>0$ for any $\delta>0$, it holds that

$$
h(i_0) \leq CK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\sqrt{c_0/n_{\beta n}})+\delta/2}
$$

and, hence, we have

$$
A \leq CK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\sqrt{c_0/n_{\beta n}})+\delta}.
$$

Note that by (3.19) in [11] for any $0 < \delta_3 < 1-\alpha$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $(z, x_1, ..., x_j) \in \mathbf{Go}^\beta$,

$$P\bigg(\frac{4(\alpha+\delta_3)}{\pi}(K_n \log K_n)^2 < \mathcal{R}_n^z\bigg) \leq c^{-1}\exp(-cn^2\log n).$$

Then, we obtain that for any $0 < \delta_3 < 1-\alpha$, all sufficiently large $n \in \mathbb{N}$ and $\gamma \leq \gamma(\chi(\tilde{A}_\beta^{(j)}))$

$$\begin{aligned}
&P\bigg(x_1, ..., x_j \in \mathcal{L}_{K_n}(\alpha+\delta_3), \frac{N_{n,\beta n}^z}{n_{\beta n}} \leq \gamma^2\bigg)\\
\leq &\sum_{c_0 \leq \gamma^2 n_{\beta n}} P(T_{x_1,....,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0) + P\bigg(\frac{4(\alpha+\delta_3)}{\pi}(K_n \log K_n)^2 < \mathcal{R}_n^z\bigg)\\
\leq &\sum_{c_0 \leq \gamma^2 n_{\beta n}} K_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\sqrt{c_0/n_{\beta n}})+\delta} + c^{-1}\exp(-cn^2\log n)\\
\leq &CK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)+\delta}
\end{aligned}$$

and for any $\gamma(\chi(\tilde{A}_\beta^{(j)})) \leq \gamma$

$$\begin{aligned}
&P\bigg(x_1, ..., x_j \in \mathcal{L}_{K_n}(\alpha+\delta_3), \frac{N_{n,\beta n}^z}{n_{\beta n}} \geq \gamma^2\bigg)\\
\leq &\sum_{c_0 \leq \gamma^2 n_{\beta n}} P(T_{x_1,....,x_j} > \mathcal{R}_n^z, N_{n,\beta n}^z = c_0) + P\bigg(\frac{4(\alpha+\delta_3)}{\pi}(K_n \log K_n)^2 < \mathcal{R}_n^z\bigg)\\
\leq &\sum_{c_0 \leq \gamma^2 n_{\beta n}} K_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\sqrt{c_0/n_{\beta n}})+\delta} + c^{-1}\exp(-cn^2\log n)\\
\leq &CK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)+\delta}.
\end{aligned}$$

Hence, if we pick $\delta_1 < \delta_2 \wedge \delta_3$, the desired upper bound is completed.

Next, we will prove the desired lower bound. Let $\hat{T}_{\partial D(z,r_{\beta n-1})} := \inf\{m > T_{\partial D(z,r_{\beta n})} : S_m \in T_{\partial D(z,r_{\beta n-1})}\}$. Note that by (i) and (ii) in Lemma 5.13, the strong Markov property yields

$$\begin{aligned}
&P(T_{\partial D(z,r_{n-1})} < T_{x_1,...,x_j} \wedge \hat{T}_{\partial D(z,r_{\beta n-1})})\\
\geq &P(T_{\partial D(z,r_{\beta n})} < T_{x_1,...,x_j})\tilde{q}_{1,2} \geq cK_n^{-\delta/4}.
\end{aligned}$$

In addition, for $z \in D(0, r_{\beta n-1})$

$$\begin{aligned}
&P(T_{x_1,....,x_j} > \mathcal{R}_n, N_{n,\beta n}^z = c_0+1)\\
\geq &P(T_{\partial D(z,r_{n-1})} < T_{x_1,...,x_j} \wedge \hat{T}_{\partial D(z,r_{\beta n-1})})\\
&\times \min_{y' \in \partial D(z,r_{n-1})} P^{y'}(T_{x_1,....,x_j} > \mathcal{R}_n, N_{n,\beta n}^z = c_0),
\end{aligned}$$

for $z \in D(0, r_{\beta n-1})^c \cap D(0, r_{\beta n})$

$$\begin{aligned}
&P(T_{x_1,....,x_j} > \mathcal{R}_n, N_{n,\beta n}^z \in \{c_0, c_0+1\})\\
\geq &P(T_{\partial D(z,r_{n-1})} < T_{x_1,...,x_j} \wedge \hat{T}_{\partial D(z,r_{\beta n-1})})\\
\times &\min_{y' \in \partial D(z,r_{n-1})} P^{y'}(T_{x_1,....,x_j} > \mathcal{R}_n, N_{n,\beta n}^z = c_0),
\end{aligned}$$

and otherwise

$$
\begin{aligned}
&P(T_{x_1,\dots,x_j} > \mathcal{R}_n, N^z_{n,\beta n} = c_0)\\
\geq &P(T_{\partial D(z,r_{n-1})} < T_{x_1,\dots,x_j} \wedge \hat{T}_{\partial D(z,r_{\beta n-1})})\\
\times &\min_{y'\in \partial D(z,r_{n-1})} P^{y'}(T_{x_1,\dots,x_j} > \mathcal{R}_n, N^z_{n,\beta n} = c_0).
\end{aligned}
$$

Therefore, by (3.14),

$$
\begin{aligned}
&P(T_{x_1,\dots,x_j} > \mathcal{R}_n, N^z_{n,\beta n} \in \{c_0, c_0+1\})\\
\geq &P(T_{\partial D(z,r_{n-1})} < T_{x_1,\dots,x_j} \wedge \hat{T}_{\partial D(z,r_{\beta n-1})})\\
\times &\sum_{0\leq i\leq (c_0-1)\wedge(n_n-1)} \tilde{q}_{1,1}^{c_0-1-i}\tilde{q}_{1,2}^{i+1}\frac{(c_0-1)!}{(c_0-1-i)!i!}\tilde{q}_{2,1}^{i}\tilde{q}_{2,2}^{n_n-i-1}\frac{(n_n-1)!}{(n_n-i-1)!i!}\\
\geq &cK_n^{-\delta/4}\tilde{q}_{1,1}^{c_0-1-i_0}\tilde{q}_{1,2}^{i_0+1}\frac{(c_0-1)!}{(c_0-1-i_0)!i_0!}\tilde{q}_{2,1}^{i_0}\tilde{q}_{2,2}^{n_n-i_0-1}\frac{(n_n-1)!}{(n_n-i_0-1)!i!}\\
\geq &cK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\sqrt{c_0/n_{\beta n}})-\delta/2},
\end{aligned}
$$

picking appropriate $\delta_2 > 0$ in (ii) of Lemma 5.13. Note that by Lemma 4.1 in [11] for any $0 < \delta_3 < \alpha$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $(z, x_1, \dots, x_j) \in \mathbf{Go}^\beta$,

$$
P\left(\frac{4(\alpha-\delta_3)}{\pi}(K_n\log K_n)^2 > \mathcal{R}^z_n\right) \leq c^{-1}\exp(-cn^2\log n).
$$

Thus, we obtain that for all sufficiently large $n \in \mathbb{N}$, $(z, x_1, \dots, x_j) \in \mathbf{Go}^\beta$ and $\gamma \leq \gamma(\chi(\tilde{A}_\beta^{(j)}))$,

$$
\begin{aligned}
&P\left(x_1,\dots,x_j \in \mathcal{L}_{K_n}(\alpha-\delta_3), \frac{N^z_{n,\beta n}}{n_{\beta n}} \leq \gamma^2\right)\\
\geq &P(T_{x_1,\dots,x_j} > \mathcal{R}^z_n, N^z_{n,\beta n} \in \{\lceil \gamma^2 n_{\beta n}\rceil, \lceil \gamma^2 n_{\beta n}\rceil - 1\})\\
-&P\left(\frac{4(\alpha-\delta_3)}{\pi}(K_n\log K_n)^2 > \mathcal{R}^z_n\right)\\
\geq &cK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)-\delta/2} - c^{-1}\exp(-cn^2\log n)\\
\geq &cK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)-\delta},
\end{aligned}
$$

and for $\gamma \geq \gamma(\chi(\tilde{A}_\beta^{(j)}))$

$$
\begin{aligned}
&P\left(x_1,\dots,x_j \in \mathcal{L}_{K_n}(\alpha-\delta_3), \frac{N^z_{n,\beta n}}{n_{\beta n}} \geq \gamma^2\right)\\
\geq &P(T_{x_1,\dots,x_j} > \mathcal{R}^z_n, N^z_{n,\beta n} \in \{\lceil \gamma^2 n_{\beta n}\rceil, \lceil \gamma^2 n_{\beta n}\rceil + 1\})\\
-&P\left(\frac{4(\alpha-\delta_3)}{\pi}(K_n\log K_n)^2 > \mathcal{R}^z_n\right)\\
\geq &cK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)-\delta/2} - c^{-1}\exp(-cn^2\log n)\\
\geq &cK_n^{-2\alpha F(\chi(\tilde{A}_\beta^{(j)}),\beta,\gamma)-\delta}.
\end{aligned}
$$

Therefore, if we pick $\delta_1 < \delta_2 \wedge \delta_3$, the desired result is completed. □

5.2.2 Proof of the upper bound in Theorem 2.4

We will prove in this section that for any $0 < \alpha, \beta, \delta < 1$, there exist $C > 0$ and $\epsilon > 0$ such that

$$P(|\tilde{\Theta}_{\alpha,0,\beta,n}| \geq K_n^{\rho_j(\alpha,\beta)+(4+j)\delta}) \leq CK_n^{-\epsilon}, \tag{5.35}$$

where

$$\tilde{\Theta}_{\alpha,\beta_2,\beta_1,n} = \tilde{\Theta}_{\alpha,\beta_2,\beta_1,n}(j) := \{(x_1,...,x_j) \in \mathcal{L}_{K_n}(\alpha)^j : K_n^{\beta_2} \leq d(x_i,x_l) \leq K_n^{\beta_1}\}.$$

Indeed, (5.35) implies the upper bound in Theorem 2.4 since $\log K_n^{\beta}/\log K_{n+1} \to \beta$ and $\beta \mapsto \rho_j(\alpha,\beta)$ is continuous.

Now we show (5.35) by induction on $j \in \mathbb{N}$. Note that [11] showed (5.35) for $j = 1, 2$. We assume that (5.35) holds for $j-1$. We divide the assertion into two corresponding estimate for $\tilde{\Theta}_{\alpha,\eta,\beta,n}$ and $\tilde{\Theta}_{\alpha,0,\beta,n} \cap \tilde{\Theta}^c_{\alpha,\eta,\beta,n}$ for $0 < \eta < \beta$. Note that

$$\begin{aligned}
&\mathcal{E}[(0),({K_n}^{\beta})](j) \cap \mathcal{E}[({K_n}^{\eta}),({K_n}^{\beta})](j)^c \\
&\subset \cup_{i=1}^j \{(x_1,...,x_{i-1},x_{i+1},...,x_j) \in \mathcal{E}[(0),({K_n}^{\beta})](j-1) : \\
&\quad \exists l \in \{1,...,i-1,i+1,...,j\} \text{ such that } d(x_i,x_l) \leq n^{\eta}\},
\end{aligned}$$

and, hence, there exists $C > 0$ such that for any $n \in \mathbb{N}$,

$$\begin{aligned}
&|\mathcal{E}[(0),({K_n}^{\beta})](j) \cap \mathcal{E}[({K_n}^{\eta}),({K_n}^{\beta})](j)^c| \\
\leq &|\mathcal{E}[(0),({K_n}^{\beta})](j-1)| \times j(j-1)Cn^{2\eta}.
\end{aligned} \tag{5.36}$$

Hence, for the former one, note

$$|\tilde{\Theta}_{\alpha,0,\beta,n}(j) \cap \tilde{\Theta}^c_{\alpha,\eta,\beta,n}(j)| \leq CK_n^{\delta}|\tilde{\Theta}_{\alpha,0,\beta,n}(j-1)|,$$

for $2\eta < \delta$ and all sufficiently large $n \in \mathbb{N}$. Therefore, (B) in Lemma 5.25 yields

$$\begin{aligned}
&P(|\tilde{\Theta}_{\alpha,0,\beta,n}(j) \cap \tilde{\Theta}^c_{\alpha,\eta,\beta,n}(j)| \geq K_n^{\rho_j(\alpha,\beta)+(4+j)\delta}) \\
\leq &P(|\tilde{\Theta}_{\alpha,0,\beta,n}(j-1)| \geq K_n^{\rho_{j-1}(\alpha,\beta)+(3+j)\delta}) \leq CK_n^{-\epsilon}.
\end{aligned} \tag{5.37}$$

For the latter one, we use the following lemma.

Lemma 5.14. *There exist $h < 2$, $C > 0$ and $\epsilon > 0$ such that for any $\beta' \in [\eta, \beta]$*

$$q_{n,\beta'} := P(|\tilde{\Theta}_{\alpha,\beta' h/2,\beta',n}| \geq K_n^{\rho_j(\alpha,\beta')+3\delta}) \leq CK_n^{-\epsilon}.$$

We observe that the assertion for $\tilde{\Theta}_{\alpha,\eta,\beta,n}$ indeed follows from this lemma. Set $\beta_j = \beta(h/2)^j$ and $l = \min\{j : \beta_j \leq \eta\}$. By Lemma 6.6 we have $q_{n,\beta_j} \to 0$ as $n \to \infty$ for $j = 0,...,l-1$. Combining this with the monotonicity of $\beta \mapsto \rho_j(\alpha,\beta)$, we obtain the desired assertion. By virtue of this and (5.37), we establish (5.35). Thus, the proof of the upper bound is now reduced to show Lemma 6.6.

Proof of Lemma 6.6. Let $\hat{Z}_{n,\beta'} := (4r_{\beta' n-4}\mathbb{Z}_n^2) \cap \mathbb{Z}_n^2$ and $z_{\beta'}(x)$ the point closest to x in $\hat{Z}_{n,\beta'}$. Pick δ_1 which is satisfied with the result of Corollary 5.12. In addition, we change β in the definition of γ_0 into β'. By the simple argument, we have

$$q_{n,\beta'} \leq I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &:= P\left(\min_{z \in \hat{Z}_{n,\beta'}} \frac{N^z_{n,\beta' n}}{n_{\beta' n}} \leq \gamma_0^2\right), \\
I_2 &:= P\left(|\tilde{\Theta}_{\alpha,\beta' h/2,\beta',n}| \geq K_n^{\rho_j(\alpha,\beta')+3\delta};\ \min_{z \in \hat{Z}_{n,\beta'}} \frac{N^z_{n,\beta' n}}{n_{\beta' n}} > \gamma_0^2\right).
\end{aligned}$$

Set $\epsilon := (\alpha F_{0,\beta'}(\gamma_0^2 - \delta_0) - (1-\beta')) \wedge \delta$. Note that $\epsilon > 0$. Then, if we consider $\delta' < \delta_0 \wedge \delta_1$ for any $\delta > 0$, we obtain

$$
\begin{aligned}
I_2 \le & P\Bigg(\Bigg| \Bigg\{ (x_1, ..., x_j) \in \mathcal{L}_{K_n}(\alpha)^j \cap \mathcal{E}[(K_n^{\eta}), (K_n^{\beta'})] : \\
& \frac{N_{n,\beta' n}^{z_{\beta'}(x_1)}}{n_{\beta' n}} > \gamma_0^2 \Bigg\} \Bigg| \ge K_n^{\rho_j(\alpha,\beta')+3\delta} \Bigg) \\
\le & K_n^{-\rho_j(\alpha,\beta')-3\delta} \\
& \times E\Bigg[\Bigg| \Bigg\{ (x_1, ..., x_j) \in \mathcal{L}_{K_n}(\alpha)^j \cap \mathcal{E}[(K_n^{\eta}), (K_n^{\beta'})] : \frac{N_{n,\beta' n}^{z_{\beta'}(x_1)}}{n_{\beta' n}} > \gamma_0^2 \Bigg\} \Bigg| \Bigg] \\
= & K_n^{-\rho_j(\alpha,\beta')-3\delta} \sum_{(x_1,...x_j) \in \mathcal{E}[(K_n^{\eta}),(K_n^{\beta'})]} P\Bigg(x_1, ..., x_j \in \mathcal{L}_{K_n}(\alpha); \frac{N_{n,\beta' n}^{z_{\beta'}(x_1)}}{n_{\beta' n}} > \gamma_0^2 \Bigg) \\
\le & C K_n^{-\rho_j(\alpha,\beta')-3\delta} \sum_{(x_1,...x_j) \in \mathcal{E}[(K_n^{\eta}),(K_n^{\beta'})]} K_n^{-2\alpha B_{1,\delta'}+\delta} \\
\le & C K_n^{-\delta}.
\end{aligned}
$$

The last inequality comes from (A) in Proposition 5.10. Then, by (i) in Lemma 5.11, there exists $C > 0$ such that for all sufficiently large $n \in \mathbb{N}$,

$$
I_1 \le |\hat{Z}_{n,\beta'}| C K_n^{-2\alpha F_{0,\beta'}(\gamma_0^2 - \delta_0)+\epsilon/2} \le C K_n^{2-2\beta'-2\alpha F_{0,\beta'}(\gamma_0^2-\delta_0)+\epsilon} \le C K_n^{-\epsilon}.
$$

This completes the proof of Lemma 6.6. □

5.2.3 Proof of the lower bound in Theorem 2.4

Proof of the lower bound in Theorem 2.4. In this section, we will prove for $2 - 2\beta - 2\alpha F_{0,\beta}(\gamma) \ge 0$,

$$
\lim_{n\to\infty} P(|\tilde{\Theta}_{\alpha,0,\beta,n}| \ge K_n^{-2+2\beta-\alpha j F_{2,\beta}(\gamma)-(2j+1)\delta}) = 1. \tag{5.38}
$$

Since $r_{\beta n-3} \le K_n^{\beta}$ and $\beta \mapsto \rho_j(\alpha, \beta)$ is continuous, it suffices to show this to obtain the desired lower bound. Let $\mathcal{Z}_n \subset \mathbb{Z}_n^2$ be a maximal set which are $4r_{\beta n+4}$ separated. We will say that a point $z \in \mathcal{Z}_n$ is (n,β)-qualified if $|N_{n,k}^z - n_k\gamma^2| \le k$ for $\beta n \le k \le n-1$ and in addition

$$
\tilde{W}^z := |\{y \in D(z, r_{\beta n-4}), y \in \mathcal{L}_{K_n}(\alpha)\}| \ge K_n^{2\beta(1-\alpha\gamma^2)-2\delta}.
$$

So it holds that

$$
|\tilde{\Theta}_{\alpha,0,\beta,n}| \ge \sum_{z\in\mathcal{Z}_n} (\tilde{W}^z)^j \ge |\{z \in \mathcal{Z}_n, z \text{ is } (n,\beta)\text{-qualified}\}| K_n^{2j\beta(1-\alpha\gamma^2)-2j\delta}.
$$

Therefore, we obtain (5.38) as soon as we show that

$$
\lim_{n\to\infty} P(|\{z \in \mathcal{Z}_n, z \text{ is } (n,\beta)\text{-qualified}\}| \ge K_n^{(2-2\beta)(1-\alpha(1-\gamma\beta)^2/(1-\beta)^2)-\delta}) = 1. \tag{5.39}
$$

(5.39) has already been shown by (10.3) in [11]. Indeed, while the definition of K_n is slightly different from that of [11], (5.39) comes from the fact that $\delta > 0$ is arbitrary. Therefore, we obtain the desired result. □

5.3 Proof of Theorem 2.3

5.3.1 Proof of the upper bound in Theorem 2.3

Proposition 5.15. *For any $\delta > 0$ there exist $C > 0$ and $\delta' > 0$ such that for any $n \in \mathbb{N}$ and uniformly in $x_1, ..., x_j \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\le i,l\le j}]$*

$$C^{-1}n^{-2\alpha B_{2,\delta'}-\delta} \le P(x_1, ..., x_j \in \mathcal{L}_n(\alpha)) \le Cn^{-2\alpha B_{2,\delta'}+\delta}.$$

Proof. Then, if we substitute γ in (ii) of Lemma 5.11 into $\gamma(\chi(\tilde{A}^{(j)}_\beta))$, (5.24) completes the desired result. □

Proof of the upper bound in Theorem 2.3. (B) in Proposition 5.10 and Proposition 5.15 yield that for any $\epsilon > 0$ there exists $C > 0$ such that for any $n \in \mathbb{N}$,

$$\sum_{(x_1,...,x_j)\in\mathcal{E}[(n^\eta),(n^\beta)]} P(x_1, ..., x_j \in \mathcal{L}_n(\alpha)) \le Cn^{\hat{\rho}_j(\alpha,\beta)+\epsilon}.$$

Now we will extend the condition of "$\mathcal{E}[(n^\eta),(n^\beta)]$" to "$\mathcal{E}[(0),(n^\beta)]$" by induction on $j \in \mathbb{N}$. It suffices to show for any $\epsilon > 0$ there exists $C > 0$ such that for any $n \in \mathbb{N}$,

$$\sum_{(x_1,...,x_{j-1})\in\mathcal{E}[(0),(n^\beta)]} P(x_1, ..., x_{j-1} \in \mathcal{L}_n(\alpha)) \le Cn^{\hat{\rho}_j(\alpha,\beta)+j\epsilon}.$$

For $j = 1$, by (B) in Proposition 5.10 and Proposition 5.15 it is trivial that

$$\sum_{x\in\mathbb{Z}_n^2} P(x \in \mathcal{L}_n(\alpha)) \le Cn^{\hat{\rho}_1(\alpha,\beta)+\epsilon}.$$

Assume the claim holds for $j-1$ with $j \ge 2$. We will show that the claim holds for j. Pick $\eta > 0$ with $2\eta < \epsilon$. Therefore, by (B) in Proposition 5.10, Lemma 5.25 and Proposition 5.15, the same argument as (5.36) yields

$$\begin{aligned}
&E[|\{(x_1, ..., x_j) \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \le n^\beta \text{ for any } 1 \le i, l \le j\}|] \\
=&\sum_{(x_1,...,x_j)\in\mathcal{E}[(0),(n^\beta)](j)} P(x_1, ..., x_j \in \mathcal{L}_n(\alpha)) \\
\le&\sum_{(x_1,...x_j)\in\mathcal{E}[(n^\eta),(n^\beta)](j)} P(x_1, ..., x_j \in \mathcal{L}_n(\alpha)) \\
&+\sum_{(x_1,...x_{j-1})\in\mathcal{E}[(0),(n^\beta)](j-1)} P(x_1, ..., x_{j-i+1} \in \mathcal{L}_n(\alpha))Cn^{2\eta} \\
\le&Cn^{\hat{\rho}_j(\alpha,\beta)+\epsilon} + Cn^{\hat{\rho}_{j-1}(\alpha,\beta)+(j-1)\epsilon+2\eta} \le Cn^{\hat{\rho}_j(\alpha,\beta)+j\epsilon},
\end{aligned}$$

and, hence, the desired result is completed. □

5.3.2 Proof of the lower bound in Theorem 2.3

Proof of the lower bound in Theorem 2.3. It is trivial that $\chi(A^{(j)}_{1-s}) = j/(1 + (j-1)(1-s))$. Hence, if we consider $(x_1, ..., x_j) \in \mathcal{E}[(2^{-j}n^s),(2^jn^s)]$ for $\eta < s < 1$, then Proposition 5.15 yields

$$P(x_1, ..., x_j \in \mathcal{L}_n(\alpha)) \ge \exp\Big(-\frac{2j\alpha\log n}{1 + (j-1)(1-s)} + o(\log n)\Big).$$

Let

$$R_s := \Big\{(x_1, ..., x_j) : x_1 \in \mathbb{Z}_n^2,$$
$$x_i \in (0, 4^{-j}n^s) + D(x_{i-1}, (2^{-j} + 2^{-2j+1})n^s) \text{ for any } 2 \le i \le j\Big\}.$$

Note that there exists $c > 0$ such that for any $n \in \mathbb{N}$,

$$|R_s| \ge cn^{2+2(j-1)s}.$$

In addition,

$$\mathcal{E}[(2^{-j}n^s), (2^j n^s)] \supset R_s.$$

Therefore, (B) in Proposition 5.10 and the simple computation yield for $\eta < s < 1$

$$\begin{aligned}
&\sum_{(x_1,...,x_j)\in\mathcal{E}[(2^{-j}n^s),(2^j n^s)]} P(x_1, ..., x_j \in \mathcal{L}_n(\alpha)) \\
\ge &\sum_{(x_1,...,x_j)\in R_s} \exp\Big(-\frac{2j\alpha \log n}{1 + (j-1)(1-s)} + o(\log n)\Big) \\
\ge &cn^{2+2(j-1)s} \times \exp\Big(-\frac{2j\alpha \log n}{1 + (j-1)(1-s)} + o(\log n)\Big).
\end{aligned}$$

Since $2 + 2(j-1)s - 2\alpha j/(1 + (j-1)(1-s))|_{s=(1+(1-\sqrt{j\alpha})/(j-1))\wedge\beta} = \hat{\rho}_j(\alpha, \beta)$ and η is arbitrary, we obtain the result. □

5.4 Corollaries

In this section, we will argue variety results which correspond to the extension of main results. By the arrange of the proof of Theorem 2.2 we obtain the following corollaries. The first result implies that the property "the probability that multipoint are late points is high" is preserved when other points are added.

Corollary 5.16. *(additivity for the probability as the geometric structures of late points) Fix* $0 < \alpha, \eta < 1$, $\epsilon > 0$, h, $g \in \mathbb{N}$. *There exists* $c > 0$ *such that for any* $n \in \mathbb{N}$ *and* $x_1(n), ..., x_h(n)$, $w_1(n), ..., w_g(n)$, $z_1(n), ..., z_g(n) \in \mathcal{E}[(n^\eta), (n^{1-\eta})]$ *with*

$$\begin{aligned}
&P(w_1, ..., w_g \in \mathcal{L}_n(\alpha)) \ge P(z_1, ..., z_g \in \mathcal{L}_n(\alpha)), \\
&d(x_1, w_1) = d(x_1, z_1) \ge 4 \max_{2\le i\le g} d(w_1, w_i) \vee d(z_1, z_i),
\end{aligned}$$

it holds that

$$P(x_1, ..., x_h, w_1, ..., w_g \in \mathcal{L}_n(\alpha)) \ge cn^{-\epsilon} P(x_1, ..., x_h, z_1, ..., z_g \in \mathcal{L}_n(\alpha)).$$

The second result implies that the property "the probability that multipoint are late points is high" is preserved when the space state is changed.

Corollary 5.17. *Fix* $0 < \alpha, \eta < 1$, $\xi > 1$, $\epsilon > 0$ *and* $j \in \mathbb{N}$. *There exists* $c > 0$ *such that for any* $n \in \mathbb{N}$ *and* $w_1(n), ..., w_g(n)$, $z_1(n), ..., z_g(n) \in \mathcal{E}[(n^\eta), (n^{1-\eta})]$ *and*

$$P(w_1, ..., w_j \in \mathcal{L}_n(\alpha)) \ge P(z_1, ..., z_j \in \mathcal{L}_n(\alpha)),$$

it holds that

$$P(w_1, ..., w_j \in \mathcal{L}_{n^\xi}(\alpha)) \ge cn^{-\epsilon} P(z_1, ..., z_j \in \mathcal{L}_{n^\xi}(\alpha)).$$

Proof of Corollary 5.16. By Lemma 5.8, for any $\delta' > 0$ there exists $n_0 \in \mathbb{N}$ for any $n \geq n_0$ and $w_1, ..., w_j$, $z_1, ..., z_g \in D(0, n/3)$ the following holds: there exist $(a^1_{i,l})_{1\leq i,l\leq g}$, $(a^2_{i,l})_{1\leq i,l\leq g}$ and $(a^3_{i,l})_{1\leq i,l\leq h}$ such that $(w_1, ..., w_g) \in \hat{\mathcal{E}}_{\delta'}[(a^1_{i,l})_{1\leq i,l\leq g}](g,n)$, $(z_1, ..., z_g) \in \hat{\mathcal{E}}_{\delta'}[(a^2_{i,l})_{1\leq i,l\leq g}](g,n)$ and $(x_1, ..., x_h) \in \hat{\mathcal{E}}_{\delta'}[(a^3_{i,l})_{1\leq i,l\leq h}](h,n)$. Proposition 5.15 yields that for any $\delta > 0$ there exists $\delta' > 0$ such that, uniformly in $(x_1, ..., x_j) \in \mathbb{Z}^2_n$ and for all sufficiently large $n \in \mathbb{N}$

$$\begin{aligned} &P(w_1, ..., w_g \in \mathcal{L}_n(\alpha)) \leq n^{-2\alpha\chi((a^1_{i,l})_{1\leq i,l\leq g})+\delta}, \\ &P(z_1, ..., z_g \in \mathcal{L}_n(\alpha)) \geq n^{-2\alpha\chi((a^2_{i,l})_{1\leq i,l\leq g})-\delta}. \end{aligned} \tag{5.40}$$

Hence, if we assume

$$P(w_1, ..., w_g \in \mathcal{L}_n(\alpha)) \geq P(z_1, ..., z_g \in \mathcal{L}_n(\alpha)),$$

for any $\epsilon' > 0$, there exists $\delta' > 0$ and $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$\chi((a^1_{i,l})_{1\leq i,l\leq g}) \leq \chi((a^2_{i,l})_{1\leq i,l\leq g}) + \epsilon'. \tag{5.41}$$

In addition, let $s = s(n) := 1 - \log d(x_1, w_1)/\log n = 1 - \log d(x_1, z_1)/\log n$. Let $(\hat{a}^v_{i,l})_{1\leq i,l\leq j} := M\{(a^3_{i,l})_{1\leq i,l\leq h}, (a^v_{i,l})_{1\leq i,l\leq g}, s\}$ for $v = 1, 2$. Note that $d(x_i, w_l) \leq d(x_1, w_1)/4$ and, hence, simple triangle inequality yields that for any $1 \leq i \leq h$ and $1 \leq l \leq g$,

$$\begin{aligned} d(x_i, w_l) \geq& -d(x_i, x_1) + d(x_1, w_1) - d(w_1, w_l) \\ \geq& -\frac{n^{1-s}}{4} + n^{1-s} - \frac{n^{1-s}}{4} \geq \frac{1}{2}n^{1-s}, \\ d(x_i, w_l) \leq& d(x_i, x_1) + d(x_1, w_1) + d(w_1, w_l) \\ \leq& \frac{n^{1-s}}{4} + n^{1-s} + \frac{n^{1-s}}{4} \leq 2n^{1-s+\delta'}. \end{aligned}$$

In addition, the same argument yields that $n^{1-s}/2 \leq d(x_i, z_l) \leq 2n^{1-s+\delta'}$. Then,

$$\begin{aligned} &(x_1, ..., x_h, w_1, ..., w_g) \in \hat{\mathcal{E}}_{\delta'}[(\hat{a}^1_{i,l})_{1\leq i,l\leq g+h}](g+h, n), \\ &(x_1, ..., x_h, z_1, ..., z_g) \in \hat{\mathcal{E}}_{\delta'}[(\hat{a}^2_{i,l})_{1\leq i,l\leq g+h}](g+h, n). \end{aligned}$$

Again, by the same argument as (5.40), we obtain that for any $\delta > 0$ there exists $\delta' > 0$ such that, uniformly in $(x_1, ..., x_j) \in \mathbb{Z}^2_n$ and for all sufficiently large $n \in \mathbb{N}$

$$\begin{aligned} P(x_1, ..., x_h, w_1, ..., w_g \in \mathcal{L}_n(\alpha)) \geq& n^{-2\alpha\chi((\hat{a}^1_{i,l})_{1\leq i,l\leq j})-\delta} \\ \geq& n^{-2\alpha g(s, \chi(a^3_{i,l})_{1\leq i,l\leq j}), \chi((a^1_{i,l})_{1\leq i,l\leq j}))-\delta}, \\ P(x_1, ..., x_h, z_1, ..., z_g \in \mathcal{L}_n(\alpha)) \leq& n^{-2\alpha\chi((\hat{a}^2_{i,l})_{1\leq i,l\leq j})+\delta} \\ \leq& n^{-2\alpha g(s, \chi(a^3_{i,l})_{1\leq i,l\leq j}), \chi((a^2_{i,l})_{1\leq i,l\leq j}))+\delta}, \end{aligned}$$

where $g(t, b, c)$ is the same function as Lemma 5.7. Since $g(t, b, c)$ is continuous and monotonically increasing in c, we obtain the desired result. $\square$

Proof of Corollary 5.17. We consider $(a^1_{i,l})_{1\leq i,l\leq j}$, $(a^2_{i,l})_{1\leq i,l\leq j}$ for $w_1, ..., w_j$, $z_1, ..., z_j \in \mathcal{E}[(n^\eta), (n^{1-\eta})]$ as well as Corollary 5.16. Then, we also obtain the result which corresponds to (5.41). Note that for $v = 1, 2$

$$\hat{\mathcal{E}}_{\delta'}[(a^v_{i,l})_{1\leq i,l\leq j}](j, n) = \hat{\mathcal{E}}_{\delta'/\xi}\left[\left(\frac{\xi - 1 + a^v_{i,l}}{\xi}\right)_{1\leq i,l\leq j}\right](j, n^\xi).$$

Hence, we obtain that for any $\delta > 0$ there exists $\delta' > 0$ such that, uniformly in $(x_1, ..., x_j) \in \mathbb{Z}_n^2$ and for all sufficiently large $n \in \mathbb{N}$

$$\begin{aligned} P(w_1, ..., w_j \in \mathcal{L}_{n^\xi}(\alpha)) &\geq n^{-2\alpha\chi(((\xi-1+a_{i,l}^1)/\xi)_{1\leq i,l\leq j})-\delta}, \\ P(z_1, ..., z_j \in \mathcal{L}_{n^\xi}(\alpha)) &\leq n^{-2\alpha\chi(((\xi-1+a_{i,l}^2)/\xi)_{1\leq i,l\leq j})+\delta}. \end{aligned} \tag{5.42}$$

Note that (5.23) and (5.41) yield that for any $\epsilon' > 0$ there exists $\delta' > 0$ such that for all sufficiently large $n \in \mathbb{N}$,

$$\chi\left(\left(\frac{\xi - 1 + a_{i,l}^1}{\xi}\right)_{1\leq i,l\leq j}\right) \leq \chi\left(\left(\frac{\xi - 1 + a_{i,l}^2}{\xi}\right)_{1\leq i,l\leq j}\right) + \epsilon'. \tag{5.43}$$

Therefore, (5.42) and (5.43) complete the desired result. $\square$

Next, we verify general results as Theorem 2.3. By Lemma 5.8, we have that any $(x_1, ..., x_j)$ included in some $V \in \hat{\mathcal{E}}_{\delta'}[(a_{i,l})_{1\leq i,l\leq j}](j, n)$ for some $(a_{i,l})_{1\leq i,l\leq j} \in \mathcal{M}_j^{\beta,0}$ and $0 < \beta < 1$. Then we extend the condition "$d(x_i, x_l) \leq n^\beta$" to $d(x_i, x_l) \leq n^{1-a_{i,l}}$ for some $(a_{i,l})_{1\leq i,l\leq j} \in \mathcal{M}_j^{\beta,0}$ as follows.

Corollary 5.18. *For any $0 < \alpha < 1$ and $(a_{i,l})_{1\leq i,l\leq j} \in \mathcal{M}_j^{\beta,0}$ and $0 < \beta < 1$,*

$$\begin{aligned} &\lim_{n\to\infty} \frac{\log E[|\{(x_1, ..., x_j) \in \mathcal{L}_n(\alpha)^j : (x_1, ..., x_j) \leq \mathcal{E}[(0), (n^{1-a_{i,l}})_{1\leq i,l\leq j}]\}|]}{\log n} \\ &= \max_{\substack{(a'_{i,l})\in\mathcal{M}_j^{\beta,0}, \\ a'_{i,l}\geq a_{i,l}}} 2 + 2\Xi((a'_{i,l})_{1\leq i,l\leq j}) - 2\alpha\chi((a'_{i,l})_{1\leq i,l\leq j}). \end{aligned}$$

Proof. Therefore, if we again use the same method as that of Theorem 2.3 by using Proposition 5.15, the desired result is completed with the minor modification. $\square$

5.5 Difference from favorite points and high points of a Gaussian free field

In this section, we discuss the difference between late points and favorite points or high points of a GFF in two dimensions.

5.5.1 Arguments

As mentioned in Section 1, we have obtained correspondence of $\mathcal{L}_n(\alpha)$, $\Psi_n(\alpha)$ and $\mathcal{V}_n(\alpha)$. Then, we have the question: do late points and favorite points or high points of a GFF have the difference? The answer is yes and in this section, we will discuss it. We clear up the geometric structure of $\mathcal{L}_n(\alpha)$ and attend some problem as follows.

Theorem 5.19. *For any $0 < \alpha_1 \leq \alpha_2 < 1$, $0 < \beta < 1$*

$$\lim_{n\to\infty} \frac{\log E[|\{(x_1, x_2) \in \mathcal{L}_n(\alpha_1) \times \mathcal{L}_n(\alpha_2) : d(x_1, x_2) \leq n^\beta\}|]}{\log n} = \hat{\nu}_2(\alpha_1, \alpha_2, \beta),$$

where

$$\begin{aligned} \hat{\nu}_2(\alpha_1, \alpha_2, \beta) &:= \max_{\beta'\leq\beta} 2 + 2\beta' - \frac{4\alpha_1}{2-\beta'} - 2(1 - \alpha_2 + \alpha_1) \\ &\begin{cases} 2 + 2\beta - \frac{4\alpha_1}{2-\beta} - 2(1 - \alpha_2 + \alpha_1) & (\beta \leq 2 - \sqrt{2\alpha_1}), \\ 6 - 4\sqrt{2\alpha_1} - 2(1 - \alpha_2 + \alpha_1) & (\beta \geq 2 - \sqrt{2\alpha_1}). \end{cases} \end{aligned}$$

Next, we study the geometric structure of $\Psi_n(\alpha)$ and solve the problem as follows.

Theorem 5.20. *For any* $0 < \alpha_1 \le \alpha_2 < 1$, $0 < \beta < 1$

$$\lim_{n\to\infty} \frac{\log E[|\{(x_1,x_2) \in \Psi_n(\alpha_1) \times \Psi_n(\alpha_2) : d(x_1,x_2) \le n^\beta\}|]}{\log n} = \hat{\mu}_2(\alpha_1,\alpha_2,\beta),$$

where

$$\hat{\mu}_2(\alpha_1,\alpha_2,\beta) := \max_{\substack{\beta' \le \beta, \\ \alpha_1 \le \alpha_1'}} 2 + 2\beta' - \frac{2(\sqrt{\alpha_2} - \sqrt{\alpha_1'})^2}{\beta'(2-\beta')} - \frac{4\sqrt{\alpha_1'\alpha_2}}{2-\beta'}.$$

Theorem 5.21. *For any* $0 < \alpha_1 \le \alpha_2 < 1$, $0 < \beta < 1$

$$\lim_{n\to\infty} \frac{\log E[|\{(x_1,x_2) \in \mathcal{V}_n(\alpha_1) \times \mathcal{V}_n(\alpha_2) : d(x_1,x_2) \le n^\beta\}|]}{\log n} = \hat{\mu}_2(\alpha_1,\alpha_2,\beta).$$

Note that it is easy to compute that

$$\hat{\nu}_2(\alpha_1,\alpha_2,\beta) < \hat{\mu}_2(\alpha_1,\alpha_2,\beta).$$

The essential reason of the difference is that $\mathcal{L}_n(\alpha)$ is satisfied with the following: for $0 < \alpha_1 \le \alpha_2 < 1$, $0 < \beta < 1$ and $\{x_1^{(n)}\}_{n\in\mathbb{N}}$, $\{x_2^{(n)}\}_{n\in\mathbb{N}}$ with $d(x_1^{(n)}, x_2^{(n)}) \approx n^\beta$

$$\lim_{n\to\infty} \frac{\log P\left(x_1^{(n)} \in \mathcal{L}_n(\alpha_1) \middle| x_2^{(n)} \in \mathcal{L}_n(\alpha_2)\right)}{\log n}$$

is independent of the value of α_2. This property doesn't hold for $\Psi_n(\alpha_1) \times \Psi_n(\alpha_2)$ or $\mathcal{V}_n(\alpha_1) \times \mathcal{V}_n(\alpha_2)$.

Remark 8. *We conjecture these points also have the difference in probability sense. That is, we expect that in probability the value of*

$$\lim_{n\to\infty} \frac{\log |\{(x_1,x_2) \in \mathcal{L}_n(\alpha_1) \times \mathcal{L}_n(\alpha_2) : d(x_1,x_2) \le n^\beta\}|}{\log n}$$

is different from that of $\Psi_n(\alpha_1) \times \Psi_n(\alpha_2)$ *or* $\mathcal{V}_n(\alpha_1) \times \mathcal{V}_n(\alpha_2)$. *However, we omit these results since it is expected that the proofs are long and almost the same as that of [5, 11].*

5.5.2 Proof of Theorems 5.19, 5.20 and 5.21

To show Theorem 5.19, we introduce the following proposition, which corresponds to (B) in Proposition 5.15.

Proposition 5.22. *Fix* $0 < \beta < 1$. *Then, for any* $\delta > 0$ *there exist* $0 < \delta' < 1 - \beta$ *and* $C > 0$ *such that for any* x_1, $x_2 \in \mathbb{Z}_n^2$ *with* $n^\beta/4 \le d(x_1,x_2) \le 4n^{\beta+\delta'}$ *and* $n \in \mathbb{N}$,

$$\begin{aligned}
&C^{-1} \exp\left\{ -\left(\frac{4\alpha_1}{2-\beta} - 2(1-\alpha_2+\alpha_1)\right)\log n - \delta \log n\right\} \\
\le &P(x_1 \in \mathcal{L}_n(\alpha_1), x_2 \in \mathcal{L}_n(\alpha_2)) \\
\le &C \exp\left\{ -\left(\frac{4\alpha_1}{2-\beta} - 2(1-\alpha_2+\alpha_1)\right)\log n + \delta \log n\right\}.
\end{aligned}$$

Proof. The Markov property yields

$$\begin{aligned}
&P(x_1 \in \mathcal{L}_n(\alpha_1), x_2 \in \mathcal{L}_n(\alpha_2)) \\
=&E\left[P^y\left(x_2 \in \mathcal{L}_n(\alpha_2 - \alpha_1 + o(1)) \middle| \mathcal{F}\left(\frac{4\alpha_1}{\pi}(n\log n)^2\right)\right) 1_{\{x_1,x_2 \in \mathcal{L}_n(\alpha_1)\}}\right] \\
=&E\left[P^y\left(x_2 \in \mathcal{L}_n(\alpha_2 - \alpha_1 + o(1)) \middle| \mathcal{F}\left(\frac{4\alpha_1}{\pi}(n\log n)^2\right)\right)\right] \times P(x_1, x_2 \in \mathcal{L}_n(\alpha_1)).
\end{aligned}$$

Therefore, since Proposition 5.15 holds uniformly in x_1, $x_2 \in \mathbb{Z}_n^2$ with $n^\beta/4 \le d(x_1,x_2) \le 4n^{\beta+\delta'}$, we obtain the desired results. $\square$

Proof of Theorem 5.19. If we again use the same method as that that used in the Theorem 2.3, by using Proposition 5.22 instead of Proposition 5.15, with a minor modification, the desired result is completed. $\square$

To show Theorems 5.20 and 5.21, we introduce the following propositions.

Proposition 5.23. *Fix $0 < \beta < 1$. Then, for any $\delta > 0$ there exist $0 < \delta' < 1 - \beta$ and $C > 0$ such that for any x_1, $x_2 \in D(0, n)$ with $n^\beta/4 \le d(x_1, x_2) \le 4n^{\beta+\delta'}$ and $n \in \mathbb{N}$,*

$$\begin{aligned}
&\exp\Big\{ - \min_{\alpha_1 \le \alpha_1'} \Big(\frac{2(\sqrt{\alpha_2} - \sqrt{\alpha_1'})^2}{\beta(2-\beta)} + \frac{4\sqrt{\alpha_1'\alpha_2}}{2-\beta} \Big) \log n - \delta \log n \Big\} \\
\le & P(x_1 \in \Psi_n(\alpha_1), x_2 \in \Psi_n(\alpha_2)) \\
\le & \exp\Big\{ - \min_{\alpha_1 \le \alpha_1'} \Big(\frac{2(\sqrt{\alpha_2} - \sqrt{\alpha_1'})^2}{\beta(2-\beta)} + \frac{4\sqrt{\alpha_1'\alpha_2}}{2-\beta} \Big) \log n + \delta \log n \Big\}.
\end{aligned}$$

Proof. Let $\tilde{\alpha} := \lceil 4\alpha(\log n)^2/\pi \rceil$ and x_1, x_2 be two distinct points included in $D(0, n)$, and set

$$(U_1, U_2, U_3) := (x_1, x_2, \partial D(0, n))$$

and

$$b_{i,l} := P^{U_i}(\min_{s \in \{1,2,3\}} T_{U_s} = T_{U_l}),$$

for $i \in \{1, 2\}$ and $l \in \{1, 2, 3\}$. First, we will show the lower bound. Then, (3.11) yields that for $\alpha_1 \le \alpha_2$,

$$\begin{aligned}
& P(x_1 \in \Psi_n(\alpha_1), x_2 \in \Psi_n(\alpha_2), T_{x_1} < T_{x_2} \wedge \tau_n) \\
\ge & P(T_{x_1} < T_{x_2} \wedge \tau_n) \times \max_{\alpha_1 \le \alpha_1'} P^{x_1}(K(\tau_n, x_1) = \alpha_1' - 1, K(\tau_n, x_2) = \alpha_2) \\
\ge & P(T_{x_1} < T_{x_2} \wedge \tau_n) \\
& \times \max_{\alpha_1 \le \alpha_1'} \sum_{0 \le i \le (\alpha_1'-1) \wedge (\alpha_2 - 1)} b_{1,1}^{\alpha_1'-1-i} b_{1,2}^{i+1} \frac{(\alpha_1' - 1)!}{(\alpha_1' - 1 - i)! i!} b_{2,1}^{i} b_{2,2}^{\alpha_2 - 1 - i} \frac{(\alpha_2 - 1)!}{(\alpha_2 - 1 - i)! i!} \\
\ge & P(T_{x_1} < T_{x_2} \wedge \tau_n) \\
& \times \max_{\alpha_1 \le \alpha_1'} b_{1,1}^{\alpha_1'-1-i_0} b_{1,2}^{i_0+1} \frac{(\alpha_1' - 1)!}{(\alpha_1' - 1 - i_0)! i_0!} b_{2,1}^{i_0} b_{2,2}^{\alpha_2 - 1 - i_0} \frac{(\alpha_2 - 1)!}{(\alpha_2 - 1 - i_0)! i_0!},
\end{aligned} \tag{5.44}$$

where $s := \log d(x_1, x_2)/\log n$ and $i_0 = \sqrt{\alpha_1'\alpha_2}\pi(1-s)/(2s(2-s)\log n)$. Consider $0 < \epsilon < 1$. By (3.6) and (3.9), we obtain that for $i, l \in \{1, 2\}$ and uniformly in x_1, $x_2 \in D(0, n/3)$,

$$b_{i,l} = \begin{cases} 1 - \frac{\pi}{2s(2-s)\log n}(1 + o(1)) & \text{if } i = l, \\ \frac{\pi(1-s)}{2s(2-s)\log n}(1 + o(1)) & \text{if } i \neq l. \end{cases} \tag{5.45}$$

Therefore, if we apply the Stirling formula to the right most member of (5.44) with the aid of (5.45), we obtain

$$\begin{aligned}
& P(x_1 \in \Psi_n(\alpha_1), x_2 \in \Psi_n(\alpha_2), T_{x_1} < T_{x_2} \wedge \tau_n) \\
\ge & P(T_{x_1} < T_{x_2} \wedge \tau_n) \\
& \times \exp\Big\{ - \min_{\alpha_1 \le \alpha_1'} \Big(\frac{2(\sqrt{\alpha_2} - \sqrt{\alpha_1'})^2}{\beta(2-\beta)} + \frac{4\sqrt{\alpha_1'\alpha_2}}{2-\beta} \Big) \log n - \delta \log n \Big\}.
\end{aligned}$$

Note that (3.7) yields that there exists $c > 0$ such that

$$P(T_{x_1} < T_{x_2} \wedge \tau_n) + P(T_{x_2} < T_{x_1} \wedge \tau_n) \geq P(T_{x_1} < \tau_n) \geq c.$$

Therefore, if we estimate the summation over x_1, $x_2 \in D(0, n/3)$, by the symmetry of x_1 and x_2, the desired lower bound is completed.

Second, we will show the upper bound. If we change n in the definition of $b_{i.l}$ into $3n$, (3.6) and (3.9) yield (5.45). Let $\tilde{\alpha}_i = \lceil 4\alpha_i (\log n)^2/\pi \rceil$ for $i = 1, 2$. Then, (3.11) and (3.12) yield for x_1, $x_2 \in D(0, n)$

$$\begin{aligned}
&P(x_1 \in \Psi_n(\alpha_1), x_2 \in \Psi_n(\alpha_2)) \\
\leq &P(x_1 \in \Psi_{3n}(\alpha_1), x_2 \in \Psi_{3n}(\alpha_2), T_{x_1} < T_{x_2} \wedge \tau_{3n}) \\
+&P(x_1 \in \Psi_{3n}(\alpha_1), x_2 \in \Psi_{3n}(\alpha_2), T_{x_2} < T_{x_1} \wedge \tau_{3n}) \qquad (5.46) \\
\leq &\sum_{\substack{0 \leq i \leq (\tilde{\alpha}'_1 - 1) \wedge (\tilde{\alpha}_2 - 1), \\ \tilde{\alpha}_1 \leq \tilde{\alpha}'_1}} b_{1,1}^{\tilde{\alpha}'_1 - 1 - i} b_{1,2}^{i+1} \frac{(\tilde{\alpha}'_1 - 1)!}{(\tilde{\alpha}'_1 - 1 - i)! i!} b_{2,1}^{i} b_{2,2}^{\tilde{\alpha}_2 - 1 - i} \frac{(\tilde{\alpha}_2 - 1)!}{(\tilde{\alpha}_2 - 1 - i)! i!} \\
+ &\sum_{\substack{1 \leq i \leq \tilde{\alpha}'_1 \wedge (\tilde{\alpha}_2 - 1), \\ \tilde{\alpha}_1 \leq \alpha'_1}} b_{1,1}^{\tilde{\alpha}'_1 - 1 - i} b_{1,2}^{i} \frac{(\tilde{\alpha}'_1 - 1)!}{(\tilde{\alpha}'_1 - i)! (i-1)!} b_{2,1}^{i} b_{2,2}^{\tilde{\alpha}_2 - i} \frac{(\tilde{\alpha}_2 - 1)!}{(\tilde{\alpha}_2 - 1 - i)! i!}. \qquad (5.47)
\end{aligned}$$

Note that if we estimate the maximal term of below by same argument as (ii) in Lemma 5.11, (5.47) is bounded by

$$\begin{aligned}
&(\tilde{\alpha}_2 + 1) \\
\times &\max_{\substack{0 \leq i \leq (\tilde{\alpha}'_1 - 1) \wedge \tilde{\alpha}_2, \\ \tilde{\alpha}_1 \leq \tilde{\alpha}'_1}} b_{1,1}^{\tilde{\alpha}'_1 - 1 - i} b_{1,2}^{i} \frac{(\tilde{\alpha}'_1 - 1)!}{(\tilde{\alpha}'_1 - 1 - i)! i!} b_{2,1}^{i} b_{2,2}^{\tilde{\alpha}_2 - 1 - i} \frac{(\tilde{\alpha}_2 - 1)!}{(\tilde{\alpha}_2 - 1 - i)! (i-1)!} \\
\leq &\exp\left\{ - \min_{\alpha_1 \leq \alpha'_1} \left(\frac{2(\sqrt{\alpha_2} - \sqrt{\alpha'_1})^2}{\beta(2-\beta)} + \frac{4\sqrt{\alpha'_1 \alpha_2}}{2-\beta} \right) \log n + \delta \log n \right\}
\end{aligned}$$

for any $\alpha_1 \leq \alpha'_1, \alpha_2 \leq \alpha'_2$. Therefore, a simple computation yields the desired upper bound. □

Proposition 5.24. *Fix $0 < \beta < 1$. Then, for any $\delta > 0$ there exist $0 < \delta' < 1 - \beta$ and $C > 0$ such that for any x_1, $x_2 \in \mathbb{Z}_n^2$ with $n^\beta/4 \leq d(x_1, x_2) \leq 4n^{\beta + \delta'}$ and $n \in \mathbb{N}$,*

$$\begin{aligned}
&C^{-1} \exp\left\{ - \min_{\alpha_1 \leq \alpha'_1} \left(\frac{2(\sqrt{\alpha_2} - \sqrt{\alpha'_1})^2}{\beta(2-\beta)} + \frac{4\sqrt{\alpha'_1 \alpha_2}}{2-\beta} \right) \log n - \delta \log n \right\} \\
\leq &P(x_1 \in \mathcal{V}_n(\alpha_1), x_2 \in \mathcal{V}_n(\alpha_2)) \\
\leq &C \exp\left\{ - \min_{\alpha_1 \leq \alpha'_1} \left(\frac{2(\sqrt{\alpha_2} - \sqrt{\alpha'_1})^2}{\beta(2-\beta)} + \frac{4\sqrt{\alpha'_1 \alpha_2}}{2-\beta} \right) \log n + \delta \log n \right\}.
\end{aligned}$$

Proof. By the simple computation, the definition of GFF, (3.1) and (5.45) yield that for any α'_1, $\alpha'_2 > 0$

$$\begin{aligned}
&P\left(\cap_{i=1,2} \left\{ \frac{1}{2} \phi_n^2(x_i) \geq \frac{4\alpha'_i}{\pi} (\log n)^2 \right\} \right) \\
\leq &\exp\left\{ - \left(\frac{2(\sqrt{\alpha'_2} - \sqrt{\alpha'_1})^2}{\beta(2-\beta)} + \frac{4\sqrt{\alpha'_1 \alpha'_2}}{2-\beta} \right) \log n + \delta \log n \right\},
\end{aligned}$$

and in addition, the corresponding lower bound holds. Therefore, the simple computation yields the desired result. □

Proof of Theorems 5.20 and 5.21. If we again use the same method as that of Theorem 5.19, by using Proposition 5.23 or 5.24 instead of Proposition 5.22 with the minor modification, the desired results are completed. □

5.6 Computation of exponents

In this section, we will show monotonicity of exponents on $j \in \mathbb{N}$ by using only elementary computation.

Lemma 5.25. *For any $j \geq 2$ and $0 < \alpha, \beta < 1$*

$$\begin{aligned}&(A)\ \hat{\rho}_j(\alpha,\beta) - \hat{\rho}_{j-1}(\alpha,\beta) \geq 0,\\ &(B)\ \rho_j(\alpha,\beta) - \rho_{j-1}(\alpha,\beta) \geq 0.\end{aligned}$$

Remark 9. *As discussed previously, this result yields Theorems 2.3 and 2.4. In addition, (B) in above lemma is equivalent to Theorem 2.3, and (A) is equivalent to Theorem 2.4. Indeed, for any $0 < \alpha, \beta < 1$*

$$|\{(x_1, ..., x_j) \in \mathcal{L}_n(\alpha)^j : d(x_i, x_l) \leq n^\beta \text{ for any } 1 \leq i, l \leq j\}|$$

is monotonically increasing in $j \in \mathbb{N}$. Hence, Theorems 2.3 and 2.4 naturally yield above lemma.

Proof. First, we will prove (A) for $j = 2$. Consider $\beta \leq (2 - \sqrt{2\alpha}) \wedge 1$. Note that $2\beta - 4\alpha/(2-\beta)$ is monotonically increasing in $\beta \in [0, 1]$, and

$$\hat{\rho}_2(\alpha, 1) - \hat{\rho}_1(\alpha, 1) = 4(1-\alpha) - 2(1-\alpha) \geq 0.$$

Hence, we have the desired result. In addition, we consider $\beta \geq 2 - \sqrt{2\alpha}$. Since $-4\sqrt{2\alpha} + 2\alpha$ is monotonically decreasing in α,

$$\begin{aligned}\hat{\rho}_2(\alpha,\beta) - \hat{\rho}_1(\alpha,\beta) =& 6 - 4\sqrt{2\alpha} - (2-2\alpha)\\ \geq& 4 - 4\sqrt{2\alpha} + 2\alpha\\ \geq& 6 - 4\sqrt{2} \geq 0.\end{aligned}$$

Therefore, we have (A) for $j = 2$.

Second, we will prove (B) for $j = 2$. Consider $\beta \leq (2 - 2\sqrt{\alpha}) \wedge 1$. Note that $2 - 2\sqrt{\alpha} \leq 2 - \sqrt{2\alpha}$. Then, by the same argument as (A) for $\beta \leq (2 - \sqrt{2\alpha}) \wedge 1$, we have the desired result. In addition, we consider $\beta \geq 2 - 2\sqrt{\alpha}$. Since $4(1-\sqrt{\alpha})^2/\beta$ is monotonically decreasing in β,

$$\begin{aligned}\rho_2(\alpha,\beta) - \rho_1(\alpha,\beta) =& 8(1-\sqrt{\alpha}) - 4(1-\sqrt{\alpha})^2/\beta - (2-2\alpha)\\ \geq& 8(1-\sqrt{\alpha}) - 2(1-\sqrt{\alpha}) - (2-2\alpha)\\ =& 2(2-\sqrt{\alpha})(1-\sqrt{\alpha}) \geq 0.\end{aligned}$$

Therefore, we have (B).

Next we will prove (A) for $j \geq 3$. Let $\varphi(\alpha, j) := (1 - \sqrt{j\alpha})/(j-1)$, $\zeta(\alpha, \beta, j) := 2j\alpha/(1 + (j-1)(1-\beta))$.

(*i*) $\beta \leq (1 + \varphi(\alpha, j-1)) \wedge (1 + \varphi(\alpha, j))$

Note that

$$\begin{aligned}\zeta(\alpha,\beta,j) - \zeta(\alpha,\beta,j-1) =& \frac{2\alpha\beta}{(1+(j-1)(1-\beta))(1+(j-2)(1-\beta))}\\ \leq& \frac{2\alpha\beta}{(1+(j-2)(1-\beta))^2}\end{aligned}$$

and

$$\max_{\substack{0 \leq \beta \leq 1+\varphi(\alpha,j-1),\\ j \geq 3}} \frac{2\alpha}{(1+(j-2)(1-\beta))^2} = \max_{j\geq 3} \frac{2\alpha}{(j-1)\alpha} = 1.$$

Therefore,

$$\begin{aligned}
&\hat{\rho}_j(\alpha,\beta) - \hat{\rho}_{j-1}(\alpha,\beta)\\
=&2 + 2(j-1)\beta - \zeta(\alpha,\beta,j) - (2 + 2(j-2)\beta - \zeta(\alpha,\beta,j-1))\\
\geq&2\beta - \beta \geq 0.
\end{aligned}$$

(ii) $1 + \varphi(\alpha, j-1) \leq 1 + \varphi(\alpha, j)$ and $1 + \varphi(\alpha, j-1) \leq \beta \leq 1 + \varphi(\alpha, j)$

$$\begin{aligned}
&\hat{\rho}_j(\alpha,\beta) - \hat{\rho}_{j-1}(\alpha,\beta)\\
=&2 + 2(j-1)\beta - \zeta(\alpha,\beta,j) - (2j - 4\sqrt{(j-1)\alpha})\\
\geq&2 + 2(j-1)\beta - 2\sqrt{j\alpha} - (2j - 4\sqrt{(j-1)\alpha})\\
\geq&2j + \frac{2(j-1)}{j-2}(1 - \sqrt{(j-1)\alpha}) - 2\sqrt{j\alpha} - (2j - 4\sqrt{(j-1)\alpha})\\
=&\frac{2(j-1) - 2\sqrt{(j-1)\alpha}}{j-2} + 2\sqrt{\alpha}(\sqrt{j-1} - \sqrt{j})\\
\geq&\frac{2(j-1) - 2\sqrt{j-1}}{j-2} + 2(\sqrt{j-1} - \sqrt{j})\\
\geq&4 - 2\sqrt{2} + 2\sqrt{2} - 2\sqrt{3} \geq 0.
\end{aligned}$$

The first inequality comes from the fact that $1 + \varphi(\alpha, j-1) \leq \beta \leq 1 \wedge (1 + \varphi(\alpha, j))$ and the second inequality comes from $1 + \varphi(\alpha, j-1) \leq \beta$. The third inequality comes from monotonicity in α. The forth inequality comes from the fact that $(2(j-1) - 2\sqrt{j-1})/(j-2)$ and $\sqrt{j-1} - \sqrt{j}$ are monotonically increasing for j and considering the case $j = 3$.

(iii) $1 + \varphi(\alpha, j) \leq 1 + \varphi(\alpha, j-1)$ and $1 + \varphi(\alpha, j) \leq \beta \leq 1 + \varphi(\alpha, j-1)$

$$\begin{aligned}
\hat{\rho}_j(\alpha,\beta) - \hat{\rho}_{j-1}(\alpha,\beta) =&2(j+1) - 4\sqrt{j\alpha} - (2 + 2(j-2)\beta - \zeta(\alpha,\beta,j-1))\\
\geq&2(j+1) - 4\sqrt{j\alpha} - (2j - 4\sqrt{(j-1)\alpha})\\
\geq&2 - 4\sqrt{j} + 4\sqrt{j-1}\\
\geq&2 - 4\sqrt{3} + 4\sqrt{2} \geq 0.
\end{aligned}$$

The first inequality comes from the fact that $2(j-2)\beta - \zeta(\alpha, \beta, j-1)$ is monotonically increasing in β with $\beta \leq 1 + \varphi(\alpha, j-1)$. The second inequality comes from the fact that $-4\sqrt{j\alpha} + 4\sqrt{(j-1)\alpha}$ is monotonically decreasing in α.

(iv) $(1 + \varphi(\alpha, j)) \vee (1 + \varphi(\alpha, j-1)) \leq \beta$

$$\begin{aligned}
\hat{\rho}_j(\alpha,\beta) - \hat{\rho}_{j-1}(\alpha,\beta) =&2 + 2j - 4\sqrt{j\alpha} - (2j - 4\sqrt{(j-1)\alpha})\\
\geq&2 - 4\sqrt{j\alpha} + 4\sqrt{(j-1)\alpha}\\
\geq&2 - 4\sqrt{j} + 4\sqrt{j-1}\\
\geq&2 - 4\sqrt{3} + 4\sqrt{2} \geq 0.
\end{aligned}$$

The second inequality comes from the fact that $-4\sqrt{j\alpha} + 4\sqrt{(j-1)\alpha}$ is monotonically decreasing in α. Therefore, the desired result is completed.

Next we will prove (B) for $j \geq 3$. Since $j(1 - \sqrt{\alpha})/(j-1)$ is monotonically decreasing in j, it suffices to show the following three cases (i), (ii), (iii):

(i) $\beta \leq \frac{j}{j-1}(1 - \sqrt{\alpha})$

Note that $j(1 - \sqrt{\alpha})/(j-1) \leq (1 + \varphi(\alpha, j)) \wedge (1 + \varphi(\alpha, j-1))$. Then, by the same argument as that of (A) for $\beta \leq (1 + \varphi(\alpha, j)) \wedge (1 + \varphi(\alpha, j-1))$, the desired result holds.

(*ii*) $\frac{j-1}{j-2}(1-\sqrt{\alpha}) \le \beta$

$$\begin{aligned}
&\rho_j(\alpha,\beta) - \rho_{j-1}(\alpha,\beta)\\
=&4j(1-\sqrt{\alpha}) - 2j(1-\sqrt{\alpha})^2/\beta - (4(j-1)(1-\sqrt{\alpha}) - 2(j-1)(1-\sqrt{\alpha})^2/\beta)\\
\ge&4(1-\sqrt{\alpha}) - 2(1-\sqrt{\alpha})^2/\beta\\
\ge&4(1-\sqrt{\alpha}) - 2(1-\sqrt{\alpha}) \ge 0.
\end{aligned}$$

The second inequality comes from the fact that $2(1-\sqrt{\alpha})^2/\beta$ is monotonically increasing in β and considering the case $\beta = 1-\alpha$.

(*iii*) $\frac{j}{j-1}(1-\sqrt{\alpha}) \le \beta \le \frac{j-1}{j-2}(1-\sqrt{\alpha})$

$$\begin{aligned}
&\rho_j(\alpha,\beta) - \rho_{j-1}(\alpha,\beta)\\
=&4j(1-\sqrt{\alpha}) - 2j(1-\sqrt{\alpha})^2/\beta - (2 + 2(j-2)\beta - \zeta(\alpha,\beta,j-1))\\
\ge&4j(1-\sqrt{\alpha}) - 2j(1-\sqrt{\alpha})^2/\beta - (4(j-1)(1-\sqrt{\alpha}) - 2(j-1)(1-\sqrt{\alpha})^2/\beta).
\end{aligned}$$

The inequality comes from the fact that $2(j-2)\beta - \zeta(\alpha,\beta,j-1)$ is monotonically increasing in β. Then, by the same argument as (*ii*), the desired result is completed. Therefore, we obtain (B) for $j \ge 3$. □

6 Proof of Theorems 2.5 and 2.6

6.1 The upper bound in Theorem 2.5

Here, we prove the following proposition.

Proposition 6.1. *For $d \ge 2$*

$$\limsup_{n\to\infty} \frac{M(n)}{\log n} \le \beta_d \qquad a.s.$$

Unlike the proof of the lower bound below, the proof of Proposition 6.1 will be performed independently of the dimension d. As mentioned above, neither $M(n)$ nor $\Theta_n(\delta)$ is monotone in $n \in \mathbb{N}$. To mitigate this issue, we introduce the random variables. For $\beta > 0$, we set

$$\tilde{\Theta}_n(\beta) = |\{x \in \partial R(T_x^{\lceil \beta \log n/2 \rceil}) : K(n,x) \ge \lceil \beta \log \frac{n}{2} \rceil\}|$$

(T_x^p is defined by (**??**)). This is a modification of $\Theta_n(\beta/\beta_d)$ made by relaxing the constraint of being on the inner boundary. Note that $\tilde{\Theta}_n(\beta)$ vanishes for all sufficiently large $n \in \mathbb{N}$ if $\beta > \beta_d$.

Lemma 6.2. *For $\beta > 0$ there exists $C > 0$ such that for any $n \in \mathbb{N}$*

$$E[\tilde{\Theta}_n(\beta)] \le C n^{1-\frac{\beta}{\beta_d}}.$$

Proof. First we introduce the elementary property. For any intervals I_0, I_1, $I_2 \subset \mathbb{N} \cup \{0\}$ with $I_0 \subset I_1 \subset I_2$, it holds that

$$R(I_0) \cap \partial R(I_2) \subset \partial R(I_1). \tag{6.1}$$

Note that we can write

$$\tilde{\Theta}_n(\beta) = \sum_{l=0}^{n} 1_{B_{l,n}}, \tag{6.2}$$

where

$$B_{l,n} = \{S_l \in R(l-1)^c \cap \partial R(T_{S_l}^{\lceil \beta \log n/2 \rceil}), K(n,S_l) \ge \lceil \beta \log \frac{n}{2} \rceil\}.$$

Since $K(l-1, S_l) = 0$ on $\{S_l \in R(l-1)^c\}$, for $l \le n$

$$\begin{aligned}
P(B_{l,n}) =& P(\sum_{j=0}^{n} 1_{\{S_j = S_l\}} \ge \lceil \beta \log \frac{n}{2} \rceil, S_l \in R(l-1)^c \cap \partial R(T_{S_l}^{\lceil \beta \log n/2 \rceil})) \\
=& P(\sum_{j=l}^{n} 1_{\{S_j = S_l\}} \ge \lceil \beta \log \frac{n}{2} \rceil, S_l \in R(l-1)^c \cap \partial R(T_{S_l}^{\lceil \beta \log n/2 \rceil})) \\
\le& P(\sum_{j=l}^{n} 1_{\{S_j = S_l\}} \ge \lceil \beta \log \frac{n}{2} \rceil, S_l \in \partial R[l, T_{S_l}^{\lceil \beta \log n/2 \rceil}]) \\
=& P(K(n-l, 0) \ge \lceil \beta \log \frac{n}{2} \rceil, 0 \in \partial R(T_0^{\lceil \beta \log n/2 \rceil - 1})).
\end{aligned}$$

Here, the inequality comes from (6.1) with $I_0 = \{l\}$, $I_1 = [l, T_{S_l}^{\lceil \beta \log n/2 \rceil}]$ and $I_2 = [0, T_{S_l}^{\lceil \beta \log n/2 \rceil}]$. The last equality follows from the Markov property and the translation invariance for S_l. In addition, by applying the Markov property repeatedly, we obtain

$$\begin{aligned}
& P(K(n-l, 0) \ge \lceil \beta \log \frac{n}{2} \rceil, 0 \in \partial R(T_0^{\lceil \beta \log n/2 \rceil - 1})) \\
\le& P(T_0^{\lceil \beta \log n/2 \rceil - 1} < \infty, 0 \in \partial R(T_0^{\lceil \beta \log n/2 \rceil - 1})) \\
=& P(\cup_{b \in \mathcal{N}(0)} \{T_0^{\lceil \beta \log n/2 \rceil - 1} < T_b\}) \\
\le& 2d P(T_0 < T_b)^{\lceil \beta \log n/2 \rceil - 1} \le C n^{-\frac{\beta}{\beta_d}}.
\end{aligned}$$

Hence, the assertion holds by $E[\tilde{\Theta}_n(\beta)] \le n \max_{1 \le l \le n} P(B_{l,n})$ which follows from (6.2). □

Proof of Proposition 6.1. Since $M(n)$ is not monotonically increasing in n, we instead first consider $\tilde{M}(n) = \max_{l \le n} M(l)$. If $\tilde{M}(n) > m$, there exist $x_1 \in \mathbb{Z}^d$ and $n_1 \le n$ such that $\tilde{M}(n) = M(n_1) = K(n_1, x_1)$ and $x_1 \in \partial R(n_1)$. Therefore, for such n_1, m and x_1, it holds that $T_{x_1}^m \le n_1$ and hence $x_1 \in \partial R(n_1) \cap \partial R(T_{x_1}^m)$ holds. Further, $K(n_1, x_1) \le K(n, x_1)$. Accordingly, we have

$$\begin{aligned}
& P(\tilde{M}(n) \ge \lceil \beta \log \frac{n}{2} \rceil) \\
\le& P(\cup_{x \in \mathbb{Z}^2} \{x \in \partial R(T_x^{\lceil \beta \log n/2 \rceil}), K(n, x) \ge \lceil \beta \log \frac{n}{2} \rceil\}).
\end{aligned}$$

Thus, by Lemma 6.2 and the Chebyshev inequality, we obtain

$$P(\tilde{M}(n) \ge \lceil \beta \log \frac{n}{2} \rceil) \le P(\tilde{\Theta}_n(\beta) \ge 1) \le C n^{1 - \frac{\beta}{\beta_d}}.$$

By using the Borel-Cantelli lemma for any $\beta > \beta_d$, we can show that the events $\{\tilde{M}(2^k) > \beta \log 2^{k-1}\}$ happen only finitely often with probability one. Therefore, it holds that for any $\beta > \beta_d$,

$$\limsup_{k \to \infty} \frac{\tilde{M}(2^k)}{\beta \log 2^{k-1}} \le 1 \quad \text{a.s.} \tag{6.3}$$

Now we consider $M(n)$. For any k, $n \in \mathbb{N}$ with $2^{k-1} \le n < 2^k$ we have

$$\frac{M(n)}{\log n} \le \frac{\tilde{M}(2^k)}{\log 2^{k-1}},$$

and so with (6.3), for any $\beta > \beta_d$

$$\limsup_{n \to \infty} \frac{M(n)}{\beta \log n} \le 1 \quad \text{a.s.}$$

Therefore, the proof is completed. □

6.2 The lower bound in Theorem 2.5

6.2.1 Reduction of the lower bound in Theorem 2.5 to key lemmas

Our goal in this section is to prove the following Proposition 6.3.

Proposition 6.3. *For $d \geq 2$*

$$\liminf_{n\to\infty} \frac{M(n)}{\log n} \geq \beta_d \qquad a.s. \tag{6.4}$$

Unlike Sections 4.3 and 4.4, the argument of Sections 4.1 and 4.2 will be performed independently of the dimension d. In what follows, we discuss the proof for each fixed $\beta < \beta_d$. Let

$$u_n = \lceil \exp(n^2) \rceil.$$

In this section, we will reduce the proof of Proposition 6.3 to two key lemmas given below (Lemmas 6.5 and 6.6). For $b \in \mathcal{N}(0)$ and $A \subset \mathbb{Z}^d$, we define $\partial_b A$ as

$$\partial_b A := \{x \in A : x + b \notin A\}.$$

We can extend the property (6.1) in the following way: for any intervals I_0, I_1, $I_2 \subset \mathbb{N} \cup \{0\}$ with $I_0 \subset I_1 \subset I_2$, it holds that

$$R(I_0) \cap \partial_b R(I_2) \subset \partial_b R(I_1) \subset \partial R(I_1). \tag{6.5}$$

Let us define $\tilde{Q}_n$ as

$$\tilde{Q}_n := |\{x \in R(u_{n-1}) \cap \partial_b R(u_n), T_x^{\lceil \beta n^2 \rceil} \leq u_{n-1}\}|.$$

We begin by providing a sufficient condition for the inequality (6.4) asserted in Proposition 6.3 to be true by means of $\tilde{Q}_n$.

Lemma 6.4. *If*

$$P(\tilde{Q}_n \geq 1 \qquad \textit{for all but finitely many } n) = 1, \tag{6.6}$$

for any $\beta \in (0, \beta_d)$ then the inequality (6.4) holds.

Proof. Set

$$L(n) := \max_{x \in R(u_{n-1}) \cap \partial_b R(u_n)} K(u_{n-1}, x).$$

Note that $L(n) \geq \beta n^2$ if $\tilde{Q}_n \geq 1$. Hence, it holds that

$$P(L(n) \geq \beta n^2 \qquad \text{for all but finitely many } n) = 1.$$

Moreover, by (6.5), for any m, $n \in \mathbb{N}$ with $u_{m-1} \leq n < u_m$ we have

$$R(u_{m-1}) \cap \partial_b R(u_m) \subset \partial R(n) \tag{6.7}$$

and hence $L(m) \leq \max_{x \in \partial R(n)} K(u_{m-1}, x) \leq \max_{x \in \partial R(n)} K(n, x) \leq M(n)$. Therefore, we conclude that for any $\beta < \beta_d$

$$\liminf_{n\to\infty} \frac{M(n)}{\beta \log n} \geq \liminf_{m\to\infty} \frac{L(m)}{\beta \log u_m} \geq 1 \qquad \text{a.s.},$$

as per (6.4) and as desired. □

In order to verify the condition (6.6), we introduce a new quantity Q_n by modifying the definition of $\tilde{Q}_n$. To do this, we first introduce several notions. Set $T^0_{x,l} := \inf\{j \geq l : S_j = x\}$ and $T^p_{x,l} := \inf\{j > T^{p-1}_{x,l} : S_j = x\}$. Note that $T^a_{x,0} = T^a_x$ for any $a \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, and that $T^p_{S_l} = T^p_{S_l,l}$ holds for each p on the event $S_l \notin R(l-1)$. Note that $T^p_{S_l,l}$ is a stopping time while $T^p_{S_l}$ is not. Although we can state the key lemmas without using this notion, we introduce it for later use. For $k \in \mathbb{N}$, let $h_k = \beta \log P(T_0 < T_b \wedge k) + 1$. Since $\lim_{k\to\infty} h_k = 1 - \beta/\beta_d > 0$, we have $h_k > 0$ for all sufficiently large k. We fix such a k and simply denote h_k by h hereafter. Let

$$I_n := [\frac{u_{n-1}}{n^2}, u_{n-1} - k\lceil \beta_d n^2 \rceil] \cap \mathbb{N}.$$

For any $l \in I_n$, we introduce the events $E_{l,n}$ and $A_{l,n}$ defined by

$$E_{l,n} := \{T^j_{S_l,l} - T^{j-1}_{S_l,l} < k \text{ for any } 1 \leq j \leq \lceil \beta n^2 \rceil\},$$

and

$$A_{l,n} := \{S_l \in R(l-1)^c \cap \partial_b R(u_n)\} \cap E_{l,n}.$$

Then, we set

$$Q_n := \sum_{l \in I_n} 1_{A_{l,n}}. \tag{6.8}$$

Although I_n, $A_{l,n}$ and Q_n depend on the choice of parameters β and k, we do not indicate such dependence explicitly by symbols. By the definition of I_n, $I_n \subset [0, u_{n-1}] \cap (\mathbb{N} \cup \{0\})$ and $l + k\lceil \beta n^2 \rceil \leq u_{n-1}$ hold. These facts imply $Q_n \leq \tilde{Q}_n$. As we will see, the verification of condition (6.6) is reduced to the following two estimates for Q_n.

Lemma 6.5. *Let $\beta < \beta_d$ and take $k \in \mathbb{N}$ so that $h = h_k > 0$ as above. Then, there exists $c > 0$ such that for any $n \in \mathbb{N}$, the following hold:*

(i) When $d = 2$,

$$EQ_n \geq \frac{c \exp(hn^2 - 2n)}{n^4}.$$

(ii) When $d \geq 3$,

$$EQ_n \geq c \exp(hn^2 - 2n).$$

Lemma 6.6. *Let $\beta < \beta_d$ and take $k \in \mathbb{N}$ so that $h = h_k > 0$ as above. Then, there exists $C > 0$ such that for any $n \in \mathbb{N}$, the following hold:*

(i) When $d = 2$,

$$\mathrm{Var}(Q_n) \leq C \Big(\frac{\exp(hn^2 - 2n)}{n^4} \Big)^2 \frac{\log n}{n^2}.$$

(ii) When $d \geq 3$,

$$\mathrm{Var}(Q_n) \leq C \exp(2hn^2 - 4n) \times \frac{1}{n^{10}}.$$

Now we give a proof of Proposition 6.3 by assuming Lemmas 6.5 and 6.6 are true.

Deduction of Proposition 6.3 from Lemmas 6.5 and 6.6. If Lemmas 6.5 and 6.6 hold, then we only need to prove the assumption of Lemma 6.4 to obtain Proposition 6.3. Take $k \in \mathbb{N}$ and $h > 0$ as above. By the Chebyshev inequality, we have

$$P(|Q_n - EQ_n| > \frac{EQ_n}{2}) \leq \frac{4\mathrm{Var}(Q_n)}{(EQ_n)^2}. \tag{6.9}$$

By Lemmas 6.5 and 6.6, we can see that there exists $C > 0$ such that the following is true:

$$\frac{\mathrm{Var}(Q_n)}{(EQ_n)^2} \begin{cases} \leq \frac{C\log n}{n^2} & \text{if } d = 2, \\ \leq \frac{C}{n^{10}} & \text{if } d \geq 3. \end{cases}$$

As a result, the right hand side of (6.9) is summable. Since $|b - a| \leq \frac{a}{2}$ implies $b \geq \frac{a}{2}$ for $a, b \geq 0$, the Borel-Cantelli lemma yields

$$P(Q_n \geq \frac{1}{2}EQ_n \qquad \text{for all but finitely many } n) = 1. \tag{6.10}$$

Since $h > 0$, Lemma 6.5 implies $EQ_n \geq 2$ for all sufficiently large n. Hence, the assertion of Lemma 6.4 holds by combining this fact with (6.10). $\square$

6.2.2 Preparations for the proof of Lemmas 6.5 and 6.6

In this section, we estimate $P(A_{l,n})$ using the strong Markov property. For later use, we will consider more general events than $A_{l,n}$. For any $n', l, \tilde{n} \in \mathbb{N} \cup \{0\}$ with $n' \leq l \leq \tilde{n}$ and $n \in \mathbb{N}$, let

$$F_{n',l,\tilde{n},n} = \{S_l \in R[n', l-1]^c \cap \partial_b R[n', \tilde{n}]\} \cap E_{l,n},$$

which we will sometimes denote $F(n', l, \tilde{n}, n)$ for typographical reasons. Note that $F_{0,l,u_n,n} = A_{l,n}$ holds.

Lemma 6.7. *There are constants c, $C > 0$ such that for any $n', l, \tilde{n} \in \mathbb{N} \cup \{0\}$ with $n' \leq l \leq \tilde{n}$ and $n, a \in \mathbb{N}$ with $l + k\lceil \beta_d n^2 \rceil \leq a \leq \tilde{n}$*

$$P(F_{n',l,\tilde{n},n}) \leq P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} P(T_0 \wedge T_b > l - n') \times P(T_b > \tilde{n} - a)$$

and

$$P(F_{n',l,\tilde{n},n}) \geq P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} P(T_0 \wedge T_b > l - n') \times P(T_b > \tilde{n}).$$

Proof. We first remark that $l < T_{S_l,l}^{\lceil \beta n^2 \rceil}$ holds and, hence, $\mathcal{F}(l) \subset \mathcal{F}(T_{S_l,l}^{\lceil \beta n^2 \rceil})$. By taking a conditional expectation with respect to $\mathcal{F}(T_{S_l,l}^{\lceil \beta n^2 \rceil})$, we obtain

$$\begin{aligned} P(F_{n',l,\tilde{n},n}) =& E[1\{S_l \in R[n', l-1]^c \cap \partial_b R[n', T_{S_l,l}^{\lceil \beta n^2 \rceil}]\} \cap E_{l,n} \\ & \times P(S_l \in \partial_b R[T_{S_l,l}^{\lceil \beta n^2 \rceil}, \tilde{n}] | \mathcal{F}(T_{S_l,l}^{\lceil \beta n^2 \rceil}))]. \end{aligned} \tag{6.11}$$

On the event $E_{l,n}$, we have

$$0 \leq T_{S_l,l}^{\lceil \beta n^2 \rceil} \leq k\lceil \beta_d n^2 \rceil + T_{S_l,l}^0.$$

Since $l = T_{S_l,l}^0$, our choice of a and this inequality imply

$$0 \leq T_{S_l,l}^{\lceil \beta n^2 \rceil} \leq a. \tag{6.12}$$

The Markov property and the translation invariance for S_l yield

$$P(S_l \in \partial_b R[T_{S_l,l}^{\lceil \beta n^2 \rceil}, \tilde{n}] | \mathcal{F}(T_{S_l,l}^{\lceil \beta n^2 \rceil})) = P(0 \in \partial_b R(\tilde{n} - t))\big|_{t=T_{S_l,l}^{\lceil \beta n^2 \rceil}}. \tag{6.13}$$

Substituting (6.13) for (6.11) and keeping (6.12) in mind, we obtain

$$\begin{aligned} P(F_{n',l,\tilde{n},n}) \leq & P(\{S_l \in R[n', l-1]^c \cap \partial_b R[n', T_{S_l,l}^{\lceil \beta n^2 \rceil}]\} \cap E_{l,n}) \\ & \times P(0 \in \partial_b R(\tilde{n} - a)), \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} P(F_{n',l,\tilde{n},n}) \geq & P(\{S_l \in R[n', l-1]^c \cap \partial_b R[n', T_{S_l,l}^{\lceil \beta n^2 \rceil}]\} \cap E_{l,n}) \\ & \times P(0 \in \partial_b R(\tilde{n})). \end{aligned} \tag{6.15}$$

Thus, the proof is reduced to the estimate of the common first factor in the right hand side of (6.14) and (6.15). If we take a conditional expectation with respect to $\mathcal{F}(l)$, by the Markov property and the translation invariance for S_l, we obtain

$$\begin{aligned} & P(\{S_l \in R[n', l-1]^c \cap \partial_b R[n', T_{S_l,l}^{\lceil \beta n^2 \rceil}]\} \cap E_{l,n}) \\ = & P(S_l \in R[n', l-1]^c \cap \partial_b R[n', l]) \times P(\{0 \in \partial_b R(T_0^{\lceil \beta n^2 \rceil})\} \cap E_{0,n}). \end{aligned} \tag{6.16}$$

By the choice of h, it holds that

$$P(\{0 \in \partial_b R(T_0^{\lceil \beta n^2 \rceil})\} \cap E_{0,n}) = P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil}, \tag{6.17}$$

where there exist c, $C > 0$ such that for any $n \in \mathbb{N}$

$$c \exp((h-1)n^2) \leq P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \leq C \exp((h-1)n^2). \tag{6.18}$$

By considering a time-reversal, we obtain

$$P(S_l \in R[n', l-1]^c \cap \partial_b R[n', l]) = P(0 \in R[1, l-n']^c \cap \partial_b R(l-n')). \tag{6.19}$$

In addition, for $m \in \mathbb{N}$, we have $P(0 \in R[1,m]^c \cap \partial_b R(m)) = P(T_0 \wedge T_b > m)$, and $P(0 \in \partial_b R(m)) = P(T_b > m)$. Therefore, by (6.14), (6.15), (6.16), (6.17) and (6.19), the desired formulas hold. $\square$

Remark 10. *We substitute $T_{S_l,l}^{\lceil \beta n^2 \rceil}$ for $\tilde{n}$ of $F_{n',l,\tilde{n},n}$. That is, for any $n', l \in \mathbb{N} \cup \{0\}$ with $n' \leq l$, we write*

$$F(n', l, T_{S_l,l}^{\lceil \beta n^2 \rceil}, n) = \{S_l \in R[n', l-1]^c \cap \partial_b R[n', T_{S_l,l}^{\lceil \beta n^2 \rceil}]\} \cap E_{l,n}.$$

By the same argument, we obtain the following: for any $n', l \in \mathbb{N} \cup \{0\}$ with $n' \leq l$

$$P(F(n', l, T_{S_l,l}^{\lceil \beta n^2 \rceil}, n)) \leq P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} P(T_0 \wedge T_b > l - n'). \tag{6.20}$$

(See the argument after (6.16).)

Corollary 6.8. *For any $l \in I_n$ and all sufficiently large $n \in \mathbb{N}$ with $u_n/n^{11} \leq u_n - u_{n-1}$,*

$$\begin{aligned} P(A_{l,n}) \leq & P(T_0 \wedge T_b > l) \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \\ & \times P(T_b > \frac{u_n}{n^{11}}) \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} P(A_{l,n}) \geq & P(T_0 \wedge T_b > l) \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \\ & \times P(T_b > u_n). \end{aligned} \tag{6.22}$$

Proof. For $l \in I_n$

$$l = T^0_{S_l,l} \leq u_{n-1} - k\lceil \beta_d n^2 \rceil$$

holds. Therefore, by using Lemma 6.7 with $F_{0,l,u_n,n} = A_{l,n}$ and substituting u_{n-1} for a in the assumption of Lemma 6.7 we obtain (6.22) and

$$\begin{aligned} P(A_{l,n}) \leq & P(T_0 \wedge T_b > l) \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \\ & \times P(T_b > u_n - u_{n-1}). \end{aligned}$$

Therefore, we obtain (6.21) for all sufficiently large $n \in \mathbb{N}$. □

6.2.3 Proof of Lemmas 6.5 and 6.6 for $d \geq 3$

By Corollary 6.8, we obtain the following estimate of $P(A_{l,n})$.

Lemma 6.9. *There exist constants C, $c > 0$ such that for any $n \in \mathbb{N}$ and $l \in I_n$*

$$P(A_{l,n}) \leq C \exp((h-1)n^2) \tag{6.23}$$

$$P(A_{l,n}) \geq c \exp((h-1)n^2). \tag{6.24}$$

Moreover, for any $n \in \mathbb{N}$ and $l \in I_n$

$$\begin{aligned} P(A_{l,n}) =& P(T_0 \wedge T_b = \infty) \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \times P(T_b = \infty) \\ & + O(\exp((h-1)n^2 - cn^2)). \end{aligned} \tag{6.25}$$

Proof. Since (6.18) and (6.25) yield (6.23) and (6.24), we only need to prove (6.25). First, we introduce some estimates of hitting times. Since $P(S_n = 0) = O(n^{-\frac{d}{2}})$, we obtain $P(T_0 = n) = O(n^{-\frac{d}{2}})$ and, hence, for $d \geq 3$ and $b \in \mathcal{N}(0)$,

$$P(n \leq T_0 < \infty) \leq \sum_{m=n}^{\infty} O(n^{-\frac{d}{2}}) = O(n^{-\frac{d}{2}+1}). \tag{6.26}$$

In addition, by the Markov property and the translation invariance for b' we have

$$P(T_0 \geq a+1) = \frac{1}{2d} \sum_{b' \in \mathcal{N}(0)} P^{b'}(T_0 \geq a) = \frac{1}{2d} \sum_{b' \in \mathcal{N}(0)} P(T_{-b'} \geq a) = P(T_b \geq a), \tag{6.27}$$

for $a \in \mathbb{N}$. Hence, (6.26) yields

$$P(n \leq T_b < \infty) = O(n^{-\frac{d}{2}+1}). \tag{6.28}$$

Moreover, it holds that

$$\begin{aligned} & P(n \leq T_0 \wedge T_b < \infty) \leq P(\{n \leq T_0 < \infty\} \cup \{n \leq T_b < \infty\}) \\ =& P(n \leq T_0 < \infty) + P(n \leq T_b < \infty), \end{aligned}$$

and so with (6.26) and (6.28),

$$P(n \leq T_0 \wedge T_b < \infty) = O(n^{-\frac{d}{2}+1}). \tag{6.29}$$

Therefore, by (6.26), (6.28) and (6.29) there exists $c > 0$ for any $n \in \mathbb{N}$ and $l \in I_n$

$$P(T_0 \wedge T_b > l) = P(T_0 \wedge T_b = \infty) + O(\exp(-cn^2)), \tag{6.30}$$

$$P(T_b > u_n) = P(T_b = \infty) + O(\exp(-cn^2)), \tag{6.31}$$

$$P(T_b > \frac{u_n}{n^{11}}) = P(T_b = \infty) + O(\exp(-cn^2)). \tag{6.32}$$

Substituting (6.30), (6.31) and (6.32) for the right hand sides of (6.21) and (6.22), we obtain the desired formula. □

Proof of Lemma 6.5 for $d \geq 3$. Since for all sufficiently large $n \in \mathbb{N}$

$$|I_n| \geq u_{n-1} - k\lceil \beta_d n^2 \rceil - \frac{u_{n-1}}{n^2} \geq c u_{n-1} \tag{6.33}$$

holds, by (6.24) and the definition of Q_n in (6.8), we obtain

$$EQ_n \geq |I_n| \times \min_{l \in I_n} P(A_{l,n}) \geq c \exp(hn^2 - 2n),$$

as required. $\square$

To prove Lemma 6.6, we decompose $I_n \times I_n$ into three parts $J_{n,j}$, $j = 1, 2, 3$ defined by

$$\begin{aligned}
J_{n,1} &:= \{(l, l') \in I_n \times I_n : 0 \leq l' - l \leq k\lceil \beta_d n^2 \rceil\}, \\
J_{n,2} &:= \{(l, l') \in I_n \times I_n : k\lceil \beta_d n^2 \rceil < l' - l \leq 2\lceil \frac{u_{n-1}}{n^{10}} \rceil\}, \\
J_{n,3} &:= \{(l, l') \in I_n \times I_n : 2\lceil \frac{u_{n-1}}{n^{10}} \rceil < l' - l\}.
\end{aligned}$$

For all sufficiently large $n \in \mathbb{N}$, $k\lceil \beta_d n^2 \rceil < 2\lceil u_{n-1}/n^{10} \rceil$ holds and hence $J_{n,2}$ is non-empty. By a simple computation,

$$\begin{aligned}
\mathrm{Var}(Q_n) =& EQ_n^2 - (EQ_n)^2 \\
\leq& 2 \sum_{l, l' \in I_n, l \leq l'} (E[1_{A_{l,n}} 1_{A_{l',n}}] - E[1_{A_{l,n}}] E[1_{A_{l',n}}]) \\
\leq& 2 \Big(\sum_{(l,l') \in J_{n,1}} E[1_{A_{l,n}}] + \sum_{(l,l') \in J_{n,2}} E[1_{A_{l,n}} 1_{A_{l',n}}] \\
&+ \sum_{(l,l') \in J_{n,3}} (E[1_{A_{l,n}} 1_{A_{l',n}}] - E[1_{A_{l,n}}] E[1_{A_{l',n}}]) \Big).
\end{aligned} \tag{6.34}$$

Lemma 6.10. *There exists $C > 0$ such that for any $n \in \mathbb{N}$*

$$\sum_{(l,l') \in J_{n,1}} E[1_{A_{l,n}}] \leq Cn^2 \exp(hn^2 - 2n) \tag{6.35}$$

and

$$\sum_{(l,l') \in J_{n,2}} E[1_{A_{l,n}} 1_{A_{l',n}}] \leq C \exp(2hn^2 - 4n) \times \frac{1}{n^{10}}. \tag{6.36}$$

Remark 11. *This Lemma also holds for $d = 2$ by the same proof.*

Proof. First, we show (6.35). By the definition, we have $|J_{n,1}| \leq k\lceil \beta_d n^2 \rceil u_{n-1}$. Thus, (6.23) yields

$$\sum_{(l,l') \in J_{n,1}} E[1_{A_{l,n}}] \leq |J_{n,1}| \times \max_{l \in I_n} P(A_{l,n}) \leq Cn^2 \exp(hn^2 - 2n).$$

Hence, we obtain (6.35). To show (6.36), let us introduce additional notations. We define

$$\tilde{A}_{l,n} := \{S_l \in \partial_b R[l+1, T_{S_l,l}^{\lceil \beta n^2 \rceil}]\} \cap E_{l,n}. \tag{6.37}$$

Note that

$$\begin{aligned}
&\tilde{A}_{l,n} = F(l, l, T_{S_l,l}^{\lceil \beta n^2 \rceil}, n), \\
&\tilde{A}_{l,n} \cap \{S_l \in R(l-1)^c \cap \partial_b R(l)\} \cap \{S_l \in \partial_b R[T_{S_l,l}^{\lceil \beta n^2 \rceil}, u_n]\} = A_{l,n}
\end{aligned}$$

and

$$\tilde{A}_{l,n} \in \sigma\{S_j - S_l : j \in [l, T^{\lceil \beta n^2 \rceil}_{S_l,l}]\}.$$

By Remark 10, we obtain

$$P(\tilde{A}_{l,n}) = P(F(l,l,T^{\lceil \beta n^2 \rceil}_{S_l,l}, n)) \leq C\exp((h-1)n^2). \tag{6.38}$$

We obtain (6.36) as follows: by the definition of $A_{l,n}$ and $\tilde{A}_{l,n}$, we have $A_{l,n} \subset \tilde{A}_{l,n}$ and $A_{l',n} \subset \tilde{A}_{l',n}$. In addition, $\tilde{A}_{l',n}$ is independent of $\tilde{A}_{l,n}$ for $(l,l') \in J_{n,2}$. Thus,

$$E[1_{A_{l,n}} 1_{A_{l',n}}] \leq E[1_{\tilde{A}_{l,n}} 1_{\tilde{A}_{l',n}}] = E[1_{\tilde{A}_{l,n}}] E[1_{\tilde{A}_{l',n}}],$$

and so by (6.38),

$$\begin{aligned}\sum_{(l,l') \in J_{n,2}} E[1_{A_{l,n}} 1_{A_{l',n}}] =& \sharp J_{n,2} \times C(\exp((h-1)n^2))^2 \\ \leq& C\exp(2hn^2 - 4n) \times \frac{1}{n^{10}}.\end{aligned}$$

Therefore, we obtain (6.36). □

Proof of Lemma 6.6 for $d \geq 3$. We estimate the last sum appearing in (6.34). To this end, set

$$A'_{l,n} := \{S_l \in R(l-1)^c \cap \partial_b R(l + \lceil \frac{u_{n-1}}{n^{10}} \rceil)\} \cap E_{l,n} \tag{6.39}$$

$$A''_{l',n} := \{S_{l'} \in R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l'-1]^c \cap \partial_b R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, u_n]\} \cap E_{l',n}. \tag{6.40}$$

Note that

$$A'_{l,n} = F_{0,l,l+\lceil \frac{u_{n-1}}{n^{10}} \rceil, n}, \quad A''_{l',n} = F_{l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l', u_n, n}.$$

By Lemma 6.7, (6.30) and (6.32), we can estimate $P(A'_{l,n})$ and $P(A''_{l',n})$ as follows: for any $l \in I_n$ and all sufficiently large $n \in \mathbb{N}$ with $u_{n-1}/n^{11} \leq (\lceil u_{n-1}/n^{10} \rceil - k\lceil \beta n^2 \rceil) \wedge l$

$$\begin{aligned}&P(A'_{l,n}) = P(F_{0,l,l+\lceil \frac{u_{n-1}}{n^{10}} \rceil, n}) \\ \leq& P(T_0 \wedge T_b > l) \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \times P(T_b > \frac{u_{n-1}}{(n-1)^{11}}) \\ \leq& P(T_b = \infty) P(T_0 \wedge T_b = \infty) P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} + O(\exp((h-1)n^2 - cn^2)) \end{aligned} \tag{6.41}$$

and for any $l' \in I_n$ and all sufficiently large $n \in \mathbb{N}$ with $u_n/n^{11} \leq u_n - (l' + k\lceil \beta n^2 \rceil)$ and $u_{n-1}/n^{11} \leq \lceil u_{n-1}/n^{10} \rceil \leq l'$

$$\begin{aligned}&P(A''_{l',n}) = P(F_{l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l', u_n, n}) \\ \leq& P(T_0 \wedge T_b > \lceil \frac{u_{n-1}}{n^{10}} \rceil) \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \times P(T_b > \frac{u_n}{n^{11}}) \\ \leq& P(T_b = \infty) P(T_0 \wedge T_b = \infty) P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} + O(\exp((h-1)n^2 - cn^2)). \end{aligned} \tag{6.42}$$

Therefore, by (6.25), (6.41) and (6.42), we obtain

$$
\begin{aligned}
&\sum_{(l,l')\in J_{n,3}} (E[1_{A_{l,n}}1_{A_{l',n}}] - E[1_{A_{l,n}}]E[1_{A_{l',n}}]) \\
\le &\sum_{(l,l')\in J_{n,3}} (E[1_{A'_{l,n}}]E[1_{A''_{l',n}}] - E[1_{A_{l,n}}]E[1_{A_{l',n}}]) \\
\le &\sum_{(l,l')\in J_{n,3}} (P(T_b=\infty)^2 P(T_0\wedge T_b=\infty)^2 P(T_0<T_b\wedge k)^{2\lceil \beta n^2\rceil} \\
&- P(T_b=\infty)^2 P(T_0\wedge T_b=\infty)^2 P(T_0<T_b\wedge k)^{2\lceil \beta n^2\rceil} \\
&+ O(\exp(2(h-1)n^2 - cn^2))) \\
\le & C(\exp(hn^2-2n))^2 \times \exp(-cn^2). \qquad (6.43)
\end{aligned}
$$

By (6.35), (6.36) and (6.43), the right hand side of (6.34) is bounded by $\exp(2hn^2-4n)\times 1/n^{10}$. This completes the proof of Lemma 6.6 for $d\ge 3$. $\square$

6.2.4 Proof of Lemmas 6.5 and 6.6 for $d=2$

First, we state a lemma that is important for our proof of Lemma 6.6.

Lemma 6.11. *There exists $C>0$ such that for any $n\in\mathbb{N}$ and $x\in\mathbb{Z}^2$ with $0<|x|<n\sqrt{u_{n-1}}$*

$$P(T_0\wedge T_x > \lceil \frac{u_n}{n}\rceil) \le \frac{\pi}{(n+1)^2} + C\frac{\log n}{n^4}. \qquad (6.44)$$

Moreover, it holds that for $b\in\mathcal{N}(0)$

$$P(T_b\wedge T_{x+b} > \lceil \frac{u_n}{n}\rceil) \le \frac{\pi}{(n+1)^2} + C\frac{\log n}{n^4}. \qquad (6.45)$$

Proof. We prove only the first claim since the second one follows from (6.44) by a similar observation as in (6.27). Decomposing the whole event by means of the last exit time from $\{0,x\}$ by time $\lceil u_n/n\rceil$, we obtain

$$
\begin{aligned}
1 = &\sum_{k=0}^{\lceil u_n/n\rceil} P(S_k=0)P^0(0,x\notin R[1,\lceil\frac{u_n}{n}\rceil - k]) \\
&+\sum_{k=0}^{\lceil u_n/n\rceil} P(S_k=x)P^x(0,x\notin R[1,\lceil\frac{u_n}{n}\rceil - k]). \qquad (6.46)
\end{aligned}
$$

By the local central limit theorem (see Theorem 1.2.1 in [20]), there exists $c>0$ such that for any $k\in\mathbb{N}\cup\{0\}$, $x\in\mathbb{Z}^2$

$$
\begin{cases}
P(S_k=x) \ge \frac{2}{\pi k}\exp(-\frac{|x|^2}{k}) - \frac{c}{k^2} & \text{if } k \rightleftharpoons x \\
P(S_k=x) = 0 & \text{if } k+1 \rightleftharpoons x,
\end{cases}
$$

where $k\rightleftharpoons x=(x_1,x_2)(\in\mathbb{Z}^2)$ means that $k+x_1+x_2$ is even. Let

$$\gamma(n) = P(T_0\wedge T_x > \lceil\frac{u_n}{n}\rceil) = P(0,x\notin R[1,\lceil\frac{u_n}{n}\rceil]).$$

By the invariance property of S_n under an isomorphism of $\mathbb{Z}^2$, for $a\le\lceil u_n/n\rceil$

$$\gamma(n)\le P^0(0,x\notin R[1,a]) = P^0(-x,0\notin R[1,a]) = P^x(0,x\notin R[1,a]).$$

Note that $\sum_{k=1}^{\lceil u_n/n\rceil} \frac{2}{\pi k} 1_{\{k \rightleftharpoons x\}} = \sum_{m=1}^{\lceil u_n/n\rceil/2} \frac{1}{\pi m} > \frac{1}{\pi}\log(1+\lceil u_n/n\rceil/2)$ holds. Then, by (6.46) we obtain

$$
\begin{aligned}
1 \geq& \Bigg(\sum_{k=1}^{\lceil u_n/n\rceil} (\frac{2}{\pi k} - \frac{c}{k^2}) 1_{\{k \rightleftharpoons 0\}} + \sum_{k=1}^{\lceil u_n/n\rceil} \Big(\frac{2}{\pi k}\exp(-\frac{|x|^2}{k}) - \frac{c}{k^2}\Big) 1_{\{k \rightleftharpoons x\}}\Bigg)\gamma(n) \\
\geq& \Bigg(\frac{n^2}{\pi} - \frac{\log n}{\pi} - c + \sum_{k=\lceil n|x|^2\rceil}^{\lceil u_n/n\rceil} \frac{2}{\pi k}\exp(-\frac{|x|^2}{k}) 1_{\{k \rightleftharpoons x\}}\Bigg)\gamma(n) \\
\geq& \Bigg(\frac{n^2}{\pi} - \frac{\log n}{\pi} - c + \sum_{k=\lceil n|x|^2\rceil}^{\lceil u_n/n\rceil} \frac{2}{\pi k}\exp(-\frac{1}{n}) 1_{\{k \rightleftharpoons x\}}\Bigg)\gamma(n) \\
\geq& \Bigg(\frac{n^2}{\pi} - \frac{\log n}{\pi} - c + \sum_{k=n^3 u_{n-1}}^{\lceil u_n/n\rceil} \frac{2}{\pi k}\exp(-\frac{1}{n}) 1_{\{k \rightleftharpoons x\}}\Bigg)\gamma(n) \\
\geq& \frac{1}{\pi}\Bigg(n^2 - \log n - c + n^2 - (n-1)^2 - 4\log n - (1-\exp(-\frac{1}{n}))2n\Bigg)\gamma(n) \\
\geq& \frac{1}{\pi}\Bigg((n+1)^2 - 5\log n - c\Bigg)\gamma(n).
\end{aligned}
$$

Thus the assertion follows from an easy rearrangement. □

To prove Lemma 6.5, we first introduce the following lemma.

Lemma 6.12. *There exist constants C, $c > 0$ such that for any $n \in \mathbb{N}$ and $l \in I_n$*

$$
P(A_{l,n}) \leq \frac{C\exp((h-1)n^2)}{n^4}, \tag{6.47}
$$

$$
P(A_{l,n}) \geq \frac{c\exp((h-1)n^2)}{n^4}. \tag{6.48}
$$

In addition, for any $n \in \mathbb{N} \cap \{1\}^c$ and $l \in I_n$

$$
P(A_{l,n}) = \frac{\pi^2 P(T_0 < T_b \wedge k)^{\lceil \beta n^2\rceil}}{2n^2(n-1)^2} + O(\frac{\exp((h-1)n^2)\log n}{n^6}). \tag{6.49}
$$

Proof. Since (6.18) and (6.49) yield (6.47) and (6.48), we only prove (6.49). First we introduce the following estimates: for any $M > 0$ there exists $C > 0$ such that for any $n \in \mathbb{N}$

$$
\frac{\pi}{n^2} - C\frac{\log n}{n^4} \leq P(T_b > \frac{u_n}{n^M}) \leq \frac{\pi}{n^2} + C\frac{\log n}{n^4}, \tag{6.50}
$$

$$
\frac{\pi}{2n^2} - C\frac{\log n}{n^4} \leq P(T_0 \wedge T_b > \frac{u_n}{n^M}) \leq \frac{\pi}{2n^2} + C\frac{\log n}{n^4}. \tag{6.51}
$$

Since we know

$$
P(T_0 > n) = \frac{\pi}{\log n} + O\Bigg(\frac{1}{(\log n)^2}\Bigg)
$$

(see (2.5) in [15]), the assertion (6.50) follows by a simple calculation of (6.27). For the latter assertion, we already know a weaker estimate of (6.51) involving only the leading term. (See Lemma 3.3 in [23].) We can obtain the error term of (6.51) by modifying the proof in [23] along the argument in [15] in a straightforward way. Thus, we omit the proofs of (6.50) and (6.51). From (6.50) and (6.51), we already have estimates of each term in (6.21) and (6.22). Indeed, for any $l \in I_n$ and all sufficiently large $n \in \mathbb{N}$, (6.50) yields

$$
P(T_b > \frac{u_n}{n^{11}}) = \frac{\pi}{n^2} + O(\frac{\log n}{n^4}), \tag{6.52}
$$

$$
P(T_b > u_n) = \frac{\pi}{n^2} + O(\frac{\log n}{n^4}).
$$

Since $l \in I_n$, (6.51) implies

$$P(T_0 \wedge T_b > l) = \frac{\pi}{2(n-1)^2} + O(\frac{\log n}{n^4}). \tag{6.53}$$

Therefore, by substituting these estimates for the right hand sides of (6.21) and (6.22) we obtain the desired formula. □

Proof of Lemma 6.5 for $d = 2$. Recall (6.33). Then, (6.8) and (6.48) yield

$$EQ_n \geq \sum_{l \in I_n} \frac{c \exp((h-1)n^2)}{n^4} \geq \frac{c \exp(hn^2 - 2n)}{n^4}$$

for any $n \in \mathbb{N}$, as desired. □

Proof of Lemma 6.6 for $d = 2$. By the same argument for $d \geq 3$, we obtain (6.34) for $d = 2$. We consider the estimate of the right hand side of (6.34). The first term and the second term of the right hand side of (6.34) are already estimated by Lemma 6.10. (Note Remark 11.) To estimate the third term, we will give a uniform upper bound of $P(A_{l,n} \cap A_{l',n})$ for $(l, l') \in J_{n,3}$. Here, uniform means that the bound is independent of the choice of $(l, l') \in J_{n,3}$. Instead of using $A''_{l',n}$ in (6.40) as we did when $d \geq 3$, we use more complicated events. Let

$$A'''_{l,l',n} := A''_{l',n} \cap \{S_l \in \partial_b R[T^{\lceil \beta n^2 \rceil}_{S_{l'},l'}, u_n]\}.$$

Recall the definition of $A'_{l,n}$ in (6.40). By the definition of $A_{l,n}$ and $A'{}_{l,n}$, we have $A_{l,n} \subset A'_{l,n}$ and $A_{l,n} \cap A_{l',n} \subset A'''_{l,l',n}$. Note that $A'_{l,n}$ is not independent of $A'''_{l,l',n}$ for $(l, l') \in J_{n,3}$. Denote the event $0 < |S_{l'} - S_l| < n\sqrt{u_{n-1}}$ by D_1, and the event $|S_{l'} - S_l| \geq n\sqrt{u_{n-1}}$ by D_2. Since $S_{l'} \notin R(l'-1)$ on $A_{l',n}$, we have $\{S_l = S_{l'}\} \cap A_{l',n} = \emptyset$ for $l < l'$. Thus, $A_{l',n} = (A_{l',n} \cap D_1) \cup (A_{l',n} \cap D_2)$ and therefore $A_{l,n} \cap A_{l',n} \subset (A'_{l,n} \cap A'''_{l,l',n} \cap D_1) \cup (\tilde{A}_{l,n} \cap \tilde{A}_{l',n} \cap D_2)$ holds. Then, the following holds:

$$E[1_{A_{l,n}} 1_{A_{l',n}}] \leq (E[1_{A'_{l,n}} 1_{A'''_{l,l',n}} 1_{D_1}] + E[1_{\tilde{A}_{l,n}} 1_{\tilde{A}_{l',n}} 1_{D_2}]). \tag{6.54}$$

Hence, by putting (6.52) and (6.53) into the right hand side of the inequalities given in Lemma 6.7 we can see that there exists $C > 0$ such that for any $l \in I_n$ and all sufficiently large $n \in \mathbb{N}$ with $u_{n-1}/(n-1)^{11} \leq (\lceil u_{n-1}/n^{10} \rceil - k\lceil \beta n^2 \rceil) \wedge l$

$$\begin{aligned}
P(A'_{l,n}) &= P(F_{0,l,l+\lceil \frac{u_{n-1}}{n^{10}} \rceil, n}) \\
&\leq P(T_0 \wedge T_b > l) \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \times P(T_b > \frac{u_{n-1}}{(n-1)^{11}}) \\
&\leq \frac{\pi}{2(n-1)^2} \times P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil} \times \frac{\pi}{(n-1)^2} + C\frac{\exp((h-1)n^2) \log n}{n^6}.
\end{aligned} \tag{6.55}$$

Taking tha conditional probability of the event $A'_{l,n} \cap A'''_{l,l',n} \cap D_1$ on $\mathcal{F}(T^{\lceil \beta n^2 \rceil}_{S_{l'},l'})$ and using (6.55), we see that for any $l, l' \in I_n$,

$$\begin{aligned}
&E[1_{A'_{l,n}} 1_{A'''_{l,l',n}} 1_{D_1}] \\
=&E[1_{A'_{l,n}} 1_{D_1} 1\{\{S_{l'} \notin R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l'-1], \\
&S_{l'} \in \partial_b R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, T^{\lceil \beta n^2 \rceil}_{S_{l'},l'}]\} \cap E_{l'n}\} \\
&P(S_{l'}, S_l \in \partial_b R[T^{\lceil \beta n^2 \rceil}_{S_{l'},l'}, u_n] | \mathcal{F}(T^{\lceil \beta n^2 \rceil}_{S_{l'},l'}))].
\end{aligned} \tag{6.56}$$

Note that $T_{x',l}^{\lceil \beta n^2 \rceil} < u_{n-1}$ if $(l,l') \in J_{n,3}$. Hence, by (6.45), we see that for all sufficiently large $n \in \mathbb{N}$ with $u_n - u_{n-1} \geq u_n/n$ and $x, x' \in \mathbb{Z}^2$ with $0 < d(x,x') < n\sqrt{u_{n-1}}$, it holds that

$$\begin{aligned}
&P(x, x' \in \partial_b R[T_{x',l}^{\lceil \beta n^2 \rceil}, u_n] | \mathcal{F}(T_{x',l}^{\lceil \beta n^2 \rceil})) \\
=&P(0, x - x' \in \partial_b R(u_n - t))|_{t = T_{x',l}^{\lceil \beta n^2 \rceil}} \\
\leq &\max_{0 < d(x,x') < n\sqrt{u_{n-1}}} P(T_b \wedge T_{x-x'+b} > \lceil \frac{u_n}{n} \rceil) \leq \frac{\pi}{(n+1)^2} + \frac{C \log n}{n^4}.
\end{aligned}$$

By the inequalities in the last line restricted to $x = S_l$ and $x' = S_{l'}$, the right hand side of (6.56) is bounded by

$$\begin{aligned}
&E[1_{A'_{l,n}} 1\{\{S_{l'} \in R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l'-1]^c \cap \partial_b R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, T_{S_{l'},l'}^{\lceil \beta n^2 \rceil}\} \cap E_{l',n}\}] \\
&\times \left(\frac{\pi}{(n+1)^2} + \frac{C \log n}{n^4} \right). \qquad (6.57)
\end{aligned}$$

Moreover, it holds that

$$\begin{aligned}
&E[1_{A'_{l,n}} 1\{\{S_{l'} \in R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l'-1]^c \cap \partial_b R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, T_{S_{l'},l'}^{\lceil \beta n^2 \rceil}]\} \cap E_{l',n}\}] \\
=&E[1_{A'_{l,n}}] E[1\{\{S_{l'} \in R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l'-1]^c \cap \partial_b R[l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, T_{S_{l'},l'}^{\lceil \beta n^2 \rceil}]\} \cap E_{l',n}\}] \\
=&E[1_{A'_{l,n}}] E[1 F(l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l', T_{S_{l'},l'}^{\lceil \beta n^2 \rceil}, n)]. \qquad (6.58)
\end{aligned}$$

By substituting (6.20) for Remark 10, we obtain

$$\begin{aligned}
&P(F(l' - \lceil \frac{u_{n-1}}{n^{10}} \rceil, l', T_{S_{l'},l'}^{\lceil \beta n^2 \rceil}, n)) \\
\leq &\frac{\pi P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil}}{2(n-1)^2} + C \frac{\exp((h-1)n^2) \log n}{n^4}.
\end{aligned}$$

Therefore, (6.55), (6.57) and (6.58) yield

$$\begin{aligned}
&E[1_{A'_{l,n}} 1_{A'''_{l,l',n}} 1_{D_1}] \\
\leq &\frac{\pi^4 P(T_0 < T_b \wedge k)^{2\lceil \beta n^2 \rceil}}{4(n-1)^6 (n+1)^2} + C \frac{\exp(2(h-1)n^2) \log n}{n^{10}}. \qquad (6.59)
\end{aligned}$$

Now, we turn to the estimate of $E[1_{\tilde{A}_{l,n}} 1_{\tilde{A}_{l',n}} 1_{D_2}]$. From the large deviation result (see (11) in [21]), there exist C, $c > 0$ such that for any n, $m \in \mathbb{N} \cap \{1\}^c$ with $m \leq u_{n-1}$

$$P(|S_m| \geq n\sqrt{u_{n-1}}) \leq Ce^{-cn}. \qquad (6.60)$$

Thus, by the strong Markov property, we can estimate $E[1_{\tilde{A}_{l,n}} 1_{\tilde{A}_{l',n}} 1_{D_2}]$ for any $(l,l') \in J_{n,3}$ as

$$\begin{aligned}
E[1_{\tilde{A}_{l,n}} 1_{\tilde{A}_{l',n}} 1_{D_2}] =&E[1_{\tilde{A}_{l,n}} 1_{D_2}] E[1_{\tilde{A}_{l',n}}] \\
=&E[1_{\tilde{A}_{l,n}} E[1_{D_2} | \mathcal{F}(T_{S_l,l}^{\lceil \beta n^2 \rceil})]] E[1_{\tilde{A}_{l',n}}] \\
=&E[1_{\tilde{A}_{l,n}} P(|S_{l'-l-t}| \geq n\sqrt{u_{n-1}})_{t = T_{S_l,l}^{\lceil \beta n^2 \rceil}}] E[1_{\tilde{A}_{l',n}}] \\
\leq &E[1_{\tilde{A}_{l,n}} \max_{|l-l'| \leq u_{n-1}} P(|S_{l'-l}| \geq n\sqrt{u_{n-1}})] E[1_{\tilde{A}_{l',n}}] \\
=&E[1_{\tilde{A}_{l,n}}] \max_{m \leq u_{n-1}} P(|S_m| \geq n\sqrt{u_{n-1}}) E[1_{\tilde{A}_{l',n}}] \\
\leq &C \frac{\exp(2(h-1)n^2)}{e^{cn}}. \qquad (6.61)
\end{aligned}$$

The last inequality comes from (6.38) and (6.60). Finally, by (6.49), (6.59) and (6.61), we obtain the following estimate. Since $|J_{n,3}| \le (u_{n-1})^2$, for any $n \in \mathbb{N} \cap \{0\}^c$,

$$\begin{aligned}
&\sum_{(l,l')\in J_{n,3}} (E[1_{A'_{l,n}} 1_{A'''_{l,l',n}} 1_{D_1}] + E[1_{\tilde{A}_{l,n}} 1_{\tilde{A}_{l',n}} 1_{D_2}] - E[1_{A_{l,n}}]E[1_{A_{l',n}}]) \\
&\le \sum_{(l,l')\in J_{n,3}} \Big(\frac{\pi^4 P(T_0 < T_b \wedge k)^{2\lceil \beta n^2 \rceil}}{4(n-1)^6(n+1)^2} - \frac{\pi^4 P(T_0 < T_b \wedge k)^{2\lceil \beta n^2 \rceil}}{4(n-1)^4 n^4} \\
&\quad + C\frac{\exp(2(h-1)n^2)}{e^{cn}} + C\frac{\exp(2(h-1)n^2)\log n}{n^{10}} \Big) \\
&\le C \sum_{(l,l')\in J_{n,3}} \frac{\exp(2(h-1)n^2)\log n}{n^{10}} \\
&\le C \Big(\frac{\exp(hn^2 - 2n)}{n^4} \Big)^2 \times \frac{\log n}{n^2}.
\end{aligned} \tag{6.62}$$

The second inequality comes from the fact that there exists $C > 0$ such that for any $n \in \mathbb{N} \cap \{1\}^c$

$$\frac{1}{(n-1)^6(n+1)^2} - \frac{1}{(n-1)^4 n^4} \le \frac{C}{n^{10}}. \tag{6.63}$$

By (6.35), (6.36) and (6.62), the right hand side of (6.34) is bounded by $(\exp(hn^2 - 2n)/n^4)^2 \times \log n / n^2$. Therefore, we obtain the proof of Lemma 6.6 for $d = 2$. $\square$

Remark 12. *We observe what happens if we try to estimate the third term of the right hand side of (6.34) in the case $d = 2$ by the same argument as in the case $d \ge 3$. Recall the definition of $A''_{l',n}$ in (6.40). Then, by substituting (6.20) for Lemma 6.7, we can see that for any $l' \in I_n$,*

$$P(A''_{l',n}) \le \frac{\pi^2 P(T_0 < T_b \wedge k)^{\lceil \beta n^2 \rceil}}{2(n-1)^2 n^2} + O(\frac{\exp((h-1)n^2)\log n}{n^6}).$$

Hence, if we choose $A''_{l',n}$ instead of $A'''_{l,l',n}$ in (6.54),

$$\begin{aligned}
E[1_{A'_{l,n}} 1_{A''_{l',n}}] =& E[1_{A'_{l,n}}]E[1_{A''_{l',n}}] \\
&\le \frac{\pi^4 P(T_0 < T_b \wedge k)^{2\lceil \beta n^2 \rceil}}{4(n-1)^6 n^2} + O(\frac{\exp(2(h-1)n^2)\log n}{n^{10}}).
\end{aligned}$$

Based on this estimate, we obtain

$$\begin{aligned}
&\sum_{(l,l')\in J_{n,3}} (E[1_{A'_{l,n}} 1_{A''_{l',n}}] - E[1_{A_{l,n}}]E[1_{A_{l',n}}]) \\
&\le \sum_{(l,l')\in J_{n,3}} \Big(\frac{\pi^4 P(T_0 < T_b \wedge k)^{2\lceil \beta n^2 \rceil}}{4(n-1)^6 n^2} - \frac{\pi^4 P(T_0 < T_b \wedge k)^{2\lceil \beta n^2 \rceil}}{4(n-1)^4 n^4} \\
&\quad + O(\frac{\exp(2(h-1)n^2)\log n}{n^{10}}) \Big) \\
&\le C \sum_{(l,l')\in J_{n,3}} \frac{\exp(2(h-1)n^2)}{n^9} \\
&\le C (\frac{\exp(hn^2 - 2n)}{n^4})^2 \times \frac{1}{n}.
\end{aligned}$$

The second inequality comes from the fact that there exists $C > 0$ such that for any $n \in \mathbb{N} \cap \{1\}^c$

$$\frac{1}{(n-1)^6 n^2} - \frac{1}{(n-1)^4 n^4} \leq \frac{C}{n^9}.$$

(Compare with (6.63).) Consequently, with the aid of Lemma 6.5, we obtain

$$\frac{\mathrm{Var}(Q_n)}{c^2 (EQ_n)^2} \leq \frac{C}{n}.$$

This estimate is not sufficient to apply the Borel-Cantelli lemma as we did in the proof of Theorem 2.5 in Section 4.1. *Thus, we cannot use the Borel-Cantelli lemma here.*

6.3 Proof of Theorem 2.6

6.3.1 Proof of the upper bound in Theorem 2.6 for $d \geq 2$

Proof. Note that if we substitute $\beta_d \delta$ for β in Lemma 6.2, we obtain $E[\tilde{\Theta}_n(\beta_d \delta)] = O(n^{1-\delta})$. By the Chebyshev inequality, we find that for any $\epsilon > 0$, there exists $C > 0$

$$P\left(\tilde{\Theta}_n(\beta_d \delta) \geq \left(\frac{n}{2}\right)^{1-\delta+\epsilon}\right) < Cn^{-\epsilon}.$$

Using the Borel-Cantelli lemma, we see that the events $\{\tilde{\Theta}_{2^k}(\beta_d \delta) \geq 2^{(k-1)(1-\delta+\epsilon)}\}$ happen only finitely often with probability one. Hence, it holds that for any $\epsilon > 0$

$$\limsup_{k\to\infty} \frac{\log \tilde{\Theta}_{2^k}(\beta_d \delta)}{\log 2^{k-1}} \leq 1 - \delta + \epsilon \quad a.s. \tag{6.64}$$

Note that if $K(n,x) \geq \lceil \beta_d \delta \log n \rceil$, for all k, $n \in \mathbb{N}$ with $2^{k-1} \leq n < 2^k$ $T_x^{\lceil \beta_d \delta \log 2^{k-1} \rceil} \leq n$ and $K(2^k, x) \geq \lceil \beta_d \delta \log 2^{k-1} \rceil$, and hence $\Theta_n(\delta) \leq \tilde{\Theta}_{2^k}(\beta_d \delta)$ holds. Thus, for all k, $n \in \mathbb{N}$ with $2^{k-1} \leq n < 2^k$, we have

$$\frac{\log \Theta_n(\delta)}{\log n} \leq \frac{\log \tilde{\Theta}_{2^k}(\beta_d \delta)}{\log 2^{k-1}}.$$

Therefore, with (6.64) we obtain for any $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{\log \Theta_n(\delta)}{\log n} \leq 1 - \delta + \epsilon \quad a.s.$$

The desired upper bound holds by combining these bounds. □

6.3.2 Proof of the lower bound in Theorem 2.6 for $d \geq 2$

Proof. We closely follow the argument in the proof of Lemma 4.2 with $\beta = \beta_d \delta$. Take k and h_k as in section 4.1. Set

$$W_n(\beta_d \delta) = |\{x \in \partial R(u_n) \cap R(u_{n-1}) : K(u_{n-1}, x) \geq \lceil \beta_d \delta n^2 \rceil\}|.$$

Note that $W_n(\beta_d \delta) \geq Q_n$ holds for any $n \in \mathbb{N}$. Indeed, if $x \in \partial_b R(u_n)$, then $x \in \partial R(u_n)$. Moreover, if $l \in I_n$, then $l + k\lceil \beta_d \delta n^2 \rceil \leq u_{n-1}$ holds for all sufficiently large $n \in \mathbb{N}$. Therefore, since we know (6.10) and Lemma 6.5, we have

$$P(W_n(\beta_d \delta) \geq \frac{1}{2} EQ_n \geq \frac{c \exp(h_k n^2 - 2n)}{n^4} \qquad \text{for all but finitely many } n) = 1.$$

Let $u_{m-1} \leq n < u_m$. Note that $K(u_{m-1,x}) \leq K(n,x)$ holds for all $x \in \mathbb{Z}^d$ and by virtue of (6.1), $\partial R(u_m) \cap R(u_{m-1}) \in \partial R(n)$. Hence, $W_m(\beta_d \delta) \leq \Theta_n(\delta)$ holds. Therefore, it holds that

$$\liminf_{n\to\infty} \frac{\log \Theta_n(\delta)}{\log n} \geq h_k \quad \text{a.s.}$$

Since $h_k \to 1 - (\beta_d \delta)/\beta_d$ as $k \to \infty$, the desired result holds, completing the proof. □

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