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Time-dependent singularities in semilinear heat equations

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Summary

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Summary

This is a summary of the author's Ph.D thesis [6] at Tokyo Institute of Technology. In the thesis, we mainly study solutions with time-dependent singularities for heat equations with a superlinear term. Here, by time-dependent singularity, we mean a singularity with respect to the space variable whose position depends on the time variable. The thesis consists of five chapters. In Chapter 1, we recall known results and summarize our main results. Chapters 2–4 are devoted to the study of solutions with time-dependent singularities in semilinear heat equations. In Chapter 5, we consider a Cauchy problem by taking an approach similar to that of the previous chapters. Each chapter of the thesis is based on a paper or a preprint, as listed in the last part of this summary.

The study of time-dependent singularities dates back to the paper of Sato and Yanagida [7]. They constructed a positive solution with a time-dependent singularity for the following semilinear heat equation

$$\partial_t u - \Delta u = u^p \tag{1}$$

under the condition that $N \geq 3$ and $p_{sg} < p < p_* := (N+2\sqrt{N-1})/(N-4+2\sqrt{N-1})$, where the exponent p_{sg} is defined by

$$p_{sg} := \frac{N}{N-2}.$$

The exponent p_{sg} is important for the existence of singular steady states. Indeed, if $p > p_{sg}$, the equation $-\Delta u = u^p$ in $\mathbf{R}^N \setminus \{0\}$ has an explicit singular steady state

$$u_\infty(x) := L|x|^{-\frac{2}{p-1}}, \quad L := \left\{ \frac{2(N-2)}{(p-1)^2}(p - p_{sg}) \right\}^{\frac{1}{p-1}}.$$

By the perturbation of u_∞, Sato and Yanagida [7] constructed appropriate comparison functions and proved the existence of a solution with a time-dependent singularity. The exponent p_* has a relation of the stability of the singular steady state in some sense. For subsequent studies of time-dependent singularities of (1), we refer the reader to the papers of Sato and Yanagida [8, 9, 10]. We note that, in the previous research on time-dependent singularities of solutions of (1), at least the condition $p > p_{sg}$ is imposed.

In the case where $1 < p \le p_{sg}$, we cannot apply the method based on the singular steady state u_∞, because it was shown by Gidas and Spruck [4, Theorem A.1] that the equation $-\Delta u = u^p$ in $\mathbf{R}^N \setminus \{0\}$ has no nonnegative singular solutions if $1 < p \le p_{sg}$. For this reason, we need a method which is not based on the existence of singular steady states.

In Chapters 2 and 3, we study time-dependent singularities of solutions of (1) under $1 < p < p_{sg}$ by the method of external force. This method is motivated by Zhang and Zhao [11]. They considered semilinear equations involving (1) with $1 < p < p_{sg}$ and constructed solutions with a "time-independent" singularity. By their remark [11, Remark 4.1], one can see that they essentially solved the equation $\partial_t u - \Delta u = u^p + c\delta_0 \otimes \mu_L$. Here δ_0 is the Dirac measure in $\mathbf{R}^N$ concentrated at the origin, μ_L is the Lebesgue measure on the real line and $c > 0$ is a small constant. We can observe that they considered semilinear equations involving a measure as an external force term, where the support of the measure $c\delta_0 \otimes \mu_L$ is contained in a straight line in $\mathbf{R}^{N+1}$.

We sketch out the method of external force for time-dependent singularities by the following Steps 1–3. Let us consider the following problem

$$\partial_t u - \Delta u = u^p, \quad x \in \mathbf{R}^N \setminus \{\xi(t)\},\ t \in I, \tag{2}$$

where $N \ge 3$, $1 < p < p_{sg}$, $I \subset \mathbf{R}$ is an open interval and $\xi(t) : \overline{I} \to \mathbf{R}^N$ is smooth enough.

Step 1. We prove that every nonnegative solution u of (2) belongs to $L^p_{\mathrm{loc}}(\mathbf{R}^N \times I)$ and satisfies the following equation involving a measure as an external force term

$$\partial_t u - \Delta u = u^p + (\delta_0 \otimes \mu) \circ \mathcal{T} \quad \text{in } \mathcal{D}'(\mathbf{R}^N \times I). \tag{3}$$

Here $\mathcal{T} = \mathcal{T}_\xi$ is a translation operator defined by $\mathcal{T}(\varphi)(x,t) := \varphi(x + \xi(t), t)$ for a function φ on $\mathbf{R}^N \times I$ and μ is a Radon measure on I determined by the solution u, so that the support of the measure $(\delta_0 \otimes \mu) \circ \mathcal{T}$ is contained in the curve $\{(\xi(t), t) \in \mathbf{R}^{N+1}; t \in I\}$. Note that the above result is a consequence of Theorem 2.1.1. We remark that, by Theorem 2.1.2 (ii-b), we have $\mu = 0$ if $p \ge p_{sg}$.

Step 2. We show relations between the exponent p and the local growth rate of the measure μ and specify the behavior of solutions of (3). Namely, μ must satisfy that, for any open interval $I' \subset I$ with $\overline{I'} \subset I$, there exists a constant $C > 0$ such that

$$\mu((a,b)) \le \begin{cases} C & \text{if } 1 < p < p_F, \\ C\left(\log \dfrac{1}{b-a}\right)^{-\frac{N}{2}} & \text{if } p = p_F := \dfrac{N+2}{N}, \\ C(b-a)^{\frac{N}{2} - \frac{1}{p-1}} & \text{if } p_F < p < p_{sg} \end{cases} \tag{4}$$

for all $a, b \in I'$ with $0 < b - a \le 1/2$. Moreover, every nonnegative solution u of (2) satisfies, for almost all $t \in I$,

$$u(x,t) = \left\{ \frac{1}{N(N-2)\omega_N} D\mu(t) + o(1) \right\} |x - \xi(t)|^{2-N} \quad \text{as } x \to \xi(t).$$

Here ω_N is the volume of the unit ball in $\mathbf{R}^N$ and $D\mu \in L^1_{\mathrm{loc}}(I)$ is the Radon-Nikodym derivative of the absolutely continuous part of μ with respect to the Lebesgue measure. By the above result, we see that our study of time-dependent singularities for the problem (2) with $1 < p < p_{sg}$ will be done if we solve (3) under the condition (4). Note that the above result is a consequence of Theorems 2.1.2 and 2.1.3.

Step 3. We construct a solution u of (3) satisfying that u is smooth on $x \neq \xi(t)$ for all $t \in I$. For the existence of a solution of (3) with a bounded and open interval $I \neq \emptyset$, we impose the following growth condition on μ. For small constants $C > 0$ and $\varepsilon > 0$, the measure μ satisfies

$$\mu((a,b)) \le \begin{cases} C & \text{if } 1 < p < p_F, \\ C(b-a)^{\frac{N}{2}-\frac{1}{p-1}} \left(\log\left(e + \dfrac{1}{b-a}\right)\right)^{-\frac{1}{p-1}-\varepsilon} & \text{if } p_F \le p < p_{sg} \end{cases} \tag{5}$$

for any $a, b \in I$ with $a < b$. This condition is stronger than (4). However, this condition is sharp even if $p_F \le p < p_{sg}$. Indeed, for any constant $C > 0$, there exists a Radon measure μ_* on $\mathbf{R}$ satisfying

$$\mu_*((a,b)) \le C(b-a)^{\frac{N}{2}-\frac{1}{p-1}} \left(\log\left(e + \frac{1}{b-a}\right)\right)^{-\frac{1}{p-1}} \quad \text{if } p_F \le p < p_{sg} \tag{6}$$

for any $a, b \in \mathbf{R}$ with $a < b$ such that the problem (3) with μ_* does not admit any local in time and nonnegative solutions. Note that the results for existence and nonexistence are consequences of Theorems 3.1.1 and 3.1.4, respectively. The existence for $I = \mathbf{R}$ under $p_F < p < p_{sg}$ is also considered in Theorem 3.1.2.

In Chapter 4, we consider the following opposite-sign equation

$$\partial_t u - \Delta u = -u^p,$$

where $N \ge 3$ and $p > 1$. For this equation, the structure of singular steady state is different from the equation (1). More precisely, for $p < p_{sg}$, the equation $-\Delta u = -u^p$ in $\mathbf{R}^N \setminus \{0\}$ has an explicit singular steady state

$$\tilde{u}_\infty(x) := \tilde{L}|x|^{-\frac{2}{p-1}}, \quad \tilde{L} := \left\{\frac{2(N-2)}{(p-1)^2}(p_{sg} - p)\right\}^{\frac{1}{p-1}}.$$

For $p \ge p_{sg}$, it was proved by Brézis and Véron [3] that there are no singular steady states. Therefore it seems that the structure of solutions of the following problem is also different from the problem (3).

$$\partial_t u - \Delta u = -u^p, \quad x \in \mathbf{R}^N \setminus \{\xi(t)\},\ t \in (0,\infty), \tag{7}$$

where $\xi(t)$ is smooth enough.

We first show that, for the problem (7) with $p \ge p_{sg}$, there are no nonnegative solutions with time-dependent singularities. This is stated in Theorem 4.1.1. In order to prove this result, we derive a universal pointwise estimate for nonnegative solutions of (7) under $p > 1$ by using the method of Poláčik, Quittner and Souplet [5] which is based on a Liouville type theorem and a scaling argument. Note that the universal estimate are given by Corollary 4.2.3 (iii). We remark that Theorem 4.1.1 can be regarded as an indirect consequence of the general removability result by Baras and Pierre [1, THEOREME 3.1]. Their method is based on properties of some parabolic capacity.

Next, we consider the case $p < p_{sg}$ and construct two types of solution with a time-dependent singularity under the condition that ξ is smooth and ξ' is bounded. Namely, we prove that there exist solutions u_F and u_S of (7) such that, uniformly for $t \in (0, \infty)$,

$$u_F(x,t) = (w(t) + o(1))|x - \xi(t)|^{2-N} \quad \text{as } x \to \xi(t),$$
$$u_S(x,t) = (\tilde{L} + o(1))|x - \xi(t)|^{-\frac{2}{p-1}} \quad \text{as } x \to \xi(t),$$

where $w(t)$ is Hölder continuous uniformly for $t \in \mathbf{R}$ and satisfies $1/C \leq w(t) \leq C$ for all $t \in \mathbf{R}$ and a constant $C > 1$. We also prove that, under some additional assumption, every nonnegative and singular solution of (7) behaves like either u_F or u_S. These results are stated as Theorems 4.1.3 and 4.1.4.

In Chapter 5, we consider the solvability of the following Cauchy problem

$$\begin{cases} \partial_t u - \Delta u = u^p & \text{in } \mathbf{R}^N \times (0,T), \\ u(\cdot, 0) = \lambda & \text{in } \mathbf{R}^N, \end{cases} \tag{8}$$

where $N \geq 1$, $T > 0$, $p > 1$ and λ is a Radon measure on $\mathbf{R}^N$. Baras and Pierre [2, PROPOSITION 3.2] gave an explicit necessary condition for existence. They proved that if $u \geq 0$ satisfies $\partial_t u - \Delta u = u^p$ in $\mathbf{R}^N \times (0,T)$, then there exists a unique Radon measure λ on $\mathbf{R}^N$ such that $u(\cdot, t) \to \lambda$ as $t \downarrow 0$ weakly as measures on each fixed open ball. Furthermore, λ must satisfy

$$\sup_{x \in \mathbf{R}^N} \lambda(B(x;1)) < +\infty \quad \text{if } p < p_F,$$

and for any compact subset K of $\mathbf{R}^N$ there exists a constant $C > 0$ such that

$$\lambda(B(x;\rho)) \leq \begin{cases} C\left(\log \dfrac{1}{\rho}\right)^{-\frac{N}{2}} & \text{if } p = p_F, \\ C\rho^{N - \frac{2}{p-1}} & \text{if } p > p_F, \end{cases} \tag{9}$$

for any $x \in K$ and $\rho > 0$ small. Here $B(x_0; r)$ stands for the N-dimensional open ball of radius $r > 0$ centered at $x_0 \in \mathbf{R}^N$. It is also known [2, COROLLAIRE 3.4.i] that the above result under the case $p < p_F$ is also a sufficient condition for the existence of local solutions.

One can observe that Steps 1 and 2 in the context of time-dependent singularities correspond to the results of Baras and Pierre [2], especially, the exponent $N/2-1/(p-1)$

in (4) under the case $p_F < p < p_{sg}$ is a half of the exponent $N - 2/(p-1)$ in (9) under the case $p > p_F$. The correspondence between those two exponents is natural, because μ and λ are measures with respect to the time variable and the space variable, respectively, and the equation (1) is parabolic. In view of the above observation, we seem to be able to show the solvability of the problem (8) under conditions similar to (5) and (6).

We prove that (8) admits a local solution if there exist a constant $C > 0$ and a small constant $\varepsilon > 0$ such that

$$\lambda(B(x;\rho)) \le C\rho^{N-\frac{2}{p-1}} \left(\log\left(e+\frac{1}{\rho}\right)\right)^{-\frac{1}{p-1}-\varepsilon} \qquad \text{if } p \ge p_F$$

for any $x \in \mathbf{R}^N$ and $\rho > 0$. On the other hand, the problem (8) does not admit any local solutions for some initial data λ_* satisfying

$$\lambda_*(B(x;\rho)) \le C\rho^{N-\frac{2}{p-1}} \left(\log\left(e+\frac{1}{\rho}\right)\right)^{-\frac{1}{p-1}} \qquad \text{if } p \ge p_F$$

for any $x \in \mathbf{R}^N$ and $\rho > 0$ with a constant $C > 0$. Note that these conditions on λ improve the known results listed in the first part of Chapter 5 and that these conditions are stated in Theorems 5.1.2 (i) and 5.1.1.

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1. T. Kan, J. Takahashi, Time-dependent singularities in semilinear parabolic equations: behavior at the singularities, J. Differential Equations **260** (2016) 7278–7319.

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Chapters 2–5 in [6] are based on the above 1–4, respectively.

Related papers

1. T. Kan, J. Takahashi, On the profile of solutions with time-dependent singularities for the heat equation, Kodai Math. J. **37** (2014) 568–585.

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