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Introduction

In this thesis we study the hydrodynamic limit for a lattice gas model of non-gradient type. The study of the hydrodynamic limit is an attempt of mathematically understanding how a macroscopic evolution equation comes from microscopic dynamics of a statistically large system regulated by conservation laws. Ideal models are Hamiltonian systems where particles evolve deterministically according to Newton's equations. If one assumes an Ansatz that the equilibrium is locally established, one can derive a compressible Euler equation from a typical Hamiltonian system. Since the Ansatz has not been verified mainly due to the lack of good ergodic properties of the system, there has been traditionally introduced for simplification certain stochastic dynamics, by which the intrinsic structure of the derivation of the macroscopic equation seems to be not altered and nice ergodicity is secured. There are several models of such stochastic dynamics whose hydrodynamic limits are studied. The lattice gas models are among them.

In the problem of hydrodynamic limit, one of the most important key word is 'gradient condition'. Though classical models like the Hamiltonian system mentioned above or the system of interacting Brownian particles satisfy the 'gradient condition', lattice gas models do not in general. Hydrodynamic limit for a lattice gas satisfying 'gradient condition' is obtained in [1]. For the 'non-gradient' case, hydrodynamic limit is verified for a class of lattice gases reversible under Bernoulli measures in [2], and for that reversible under Gibbs measures with mixing condition in [8]. They introduced 'gradient replacement' or 'fluctuation dissipation equation' formula. So-called 'gradient replacement' formula is introduced and verified in [7] for 'non-gradient' Ginzberg-Landau model.

In Chapter 1 we introduce a lattice gas model which has two conservation laws and study the hydrodynamic limit of this model. In this model we consider the lattice gas model on a d-dimensional discrete torus $\mathbf{T}_N := \{1, 2, \dots N\}^d$ (N is identified with 0). Each particle carries energy whose value is a positive integer less than some prescribed constant. Each particle moves on the lattice subject to the exclusion rule (at most one particle at one site) with a rate which may depend on the energy of the particle. At the same time nearest neighboring two particles exchange their energy according to a certain stochastic rule. The process on the space of particle-energy configurations on $\mathbf{T}_N$ is a continuous time Markov process that is reversible with respect to a certain product measure. In this model not only the number of particles (as in the usual lattice gas) but also the total energy is conserved. It does not satisfy gradient condition. We consider empirical measures for the number of particles and for the energy under the usual diffusion scaling. It is proved that in the limit as $N \to \infty$ the limit densities $\rho = {}^t(\rho^E, \rho^P) \in [0,\infty)^2$ (E and P indicate energy and particle, respectively) satisfy a system of non-linear diffusion equations of the form

$$\frac{\partial}{\partial t}\rho = \nabla \left(D(\rho) \nabla \rho \right),$$

where $D(\rho)$ is a $2d \times 2d$ matrix whose elements are functions of ρ and $\nabla \rho = {}^t(\nabla \rho^E, \nabla \rho^P)$. Since we do not know of the uniqueness of a (weak) solution to the Cauchy problem for this diffusion equation, we can not speak of the convergence of the process of the empirical measures, nor of whether the limit densities ρ are deterministic or not.

In Chapter 2 we introduce the "zerorange-exclusion process", and obtain an estimate of the spectral gap for it. The process is a kind of lattice gas. It has two conserved quantities, the number of particles and the total energy. It will be proved that the spectral gap for the process confined to a cube in $\mathbf{Z}^d$ with width n is bounded below by Cn^{-2}, where C is independent of n but depends on the particle and energy densities. This estimate is motivated by the study of the hydrodynamic limit of the process; the obtained estimate is sufficient for obtaining the characterization of the space of germs of closed forms [8].

In Chapter 3 we study the gradient condition for one-dimensional symmetric exclusion processes. Given a Gibbs measure on the one dimensional lattice $\mathbf{Z}$ with translation-invariant potential of finite range, we construct an exchange rate for one-dimensional lattice gas which satisfies both the detailed balance condition relative to the Gibbs measure and the gradient condition. Based on an exchange rate which satisfies both the detailed balance condition and the gradient condition, we can prove the hydrodynamic limit for every one-dimensional lattice gas reversible under the Gibbs measure that is not necessarily of gradient type, in a way parallel to [2] and [7] with the help of the result of [5] on the spectral gap.

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Hydrodynamic limit of lattice gases with energy

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1 Introduction

In this paper we study the hydrodynamic limit for a lattice gas model on a d-dimensional discrete torus $\mathbf{B}_N := \{1, 2, \dots N\}^d$ (N is identified with 0). Each particle carries energy whose value is a positive integer less than some prescribed constant. Each particle moves on the lattice subject to the exclusion rule (at most one particle at one site) with a rate which may depend on the energy of the particle. At the same time nearest neighboring two particles exchange their energy according to a certain stochastic rule. The process on the space of particle-energy configurations on $\mathbf{B}_N$ is a continuous time Markov process that is reversible with respect to a certain product measure. In this model not only the number of particles (as in the usual lattice gas) but also the total energy is conserved. It does not satisfy gradient condition. We consider empirical measures for the number of particles and for the energy under the usual diffusion scaling. It is proved that in the limit as $N \to \infty$ the limit densities $\rho = {}^t(\rho^E, \rho^P) \in [0, \infty)^2$ (E and P indicate energy and particle, respectively) satisfy a system of non-linear diffusion equations of the form

$$\frac{\partial}{\partial t}\rho = \nabla \left(D(\rho) \nabla \rho \right),$$

where $D(\rho)$ is a $2d \times 2d$ matrix whose elements are functions of ρ and $\nabla \rho = {}^t(\nabla \rho^E, \nabla \rho^P)$. Since we do not know of the uniqueness of a (weak) solution to the Cauchy problem for this diffusion equation, for the moment we can not speak of the convergence of the process of the empirical measures, nor of whether the limit densities ρ are deterministic or not.

The study of the hydrodynamic limit is an attempt of mathematically understanding how a macroscopic evolution equation comes from microscopic dynamics of a statistically large system regulated by conservation laws. Ideal models are Hamiltonian systems where particles evolve deterministically according to Newton's equations. If one assumes an Ansatz that the equilibrium is locally established, one can derive a compressible Euler equation from a typical Hamiltonian system. Since the Ansatz has not been verified mainly due to the lack of good ergodic properties of the system, there has been traditionally introduced for simplification certain stochastic dynamics, by which the intrinsic structure of the derivation of the macroscopic equation seems to be not altered and nice ergodicity is secured. There are several models of such stochastic dynamics whose hydrodynamic limits are studied. The lattice gas models are among them.

In the problem of hydrodynamic limit, one of the most important key word is 'gradient condition'. Though classical models like the Hamiltonian system mentioned above or the system of interacting Brownian particles satisfy the 'gradient condition', lattice gas models do not in general. Hydrodynamic limit for a lattice gas satisfying 'gradient condition' is obtained in [1]. For the 'non-gradient' case, hydrodynamic limit is verified for a class of lattice gases reversible under Bernoulli measures in [2], and for that reversible under Gibbs measures with mixing condition in [7]. They introduced 'gradient replacement' or 'fluctuation dissipation equation' formula. So-called 'gradient replacement' formula is introduced and verified in [6] for 'non-gradient' Ginzberg-Landau model.

The framework of our proof is essentially the same as that developed in [7] except for the characterization of a class of closed forms. First we estimate the lower bound of the spectral gap of the generator of our Markov process uniformly in densities of particle number and of energy. We reduce the problem of proving the hydrodynamic limit to that of estimating the upper bound of the spectrum of certain operators involving currents. Using the estimate of spectral gap and 'fluctuation-dissipation equation' we can estimate the upper bound of the spectrum. To derive 'fluctuation-dissipation equation' we have to characterize a class of closed forms. We project a closed form from the class onto a space of cylinder functions of configurations on a cube centered at the origin with width $2n+1$. This projection results in a closed form on the cube and a boundary term. Taking the limit as $n \to \infty$, the boundary term converges weakly. In this model, the limit of the boundary term is a linear combination of $2d$ specific functions that are linearly independent, so that the dimension of the family of limiting boundary terms is the same as d times the number of conservation laws. This situation makes the analysis of the boundary term non trivial in our case where there are two conserved quantities.

This paper is organized as follows: In Section 2 we describe the model and state the main results. In Section 3 we give an outline of proof. We prove the spectral gap inequality and two blocks estimate in Section 4, the tightness in Section 5, and energy estimate in Section 6. The eigenvalue estimate is proved in Section 7. In Section 8 and 9, we give a fluctuation-dissipation equation. Finally, in Section 10, we give proofs for some well-known facts.

2 Model and result

Let $B_N := \{1, 2, \ldots, N\}^d$ be the discrete d-dimensional torus with width N (N is identified with 0) . Denote by $\eta = (\eta_x)_{x \in B_N}$ the configuration of a lattice gas where for each x, $\eta_x \in \{0, 1, 2, \ldots, k\}$. Let us define $\xi_x := 1_{\{\eta_x \neq 0\}}$. $\xi_x = 0$ means that the site x is vacant, and $\xi_x = 1$ means that there exists a particle at site x. $\eta_x = l$ (for $1 \leq l \leq k$) means that the particle at site x has energy l.

For a point $x \in \mathbf{Z}^d$ we use two kinds of norms:

$$|x|_1 := \sum_{i=1}^{d} |x_i|,$$
$$|x| := \max_i |x_i|.$$

Let $\eta^{(x,y)}$ and $\eta^{x \to y}$ be the configurations defined by

$$(\eta^{(x,y)})_z = \begin{cases} \eta_y, & \text{if } z = x, \\ \eta_x, & \text{if } z = y, \\ \eta_z, & \text{otherwise}, \end{cases} \tag{1}$$
$$(\eta^{x \to y})_z = \begin{cases} \eta_x - 1, & \text{if } z = x, \\ \eta_y + 1, & \text{if } z = y, \\ \eta_z, & \text{otherwise}, \end{cases} \tag{2}$$

and let us denote $T^{(x,y)}\eta := \eta^{(x,y)}$ and $T^{x \to y}\eta := \eta^{x \to y}$. Let $\pi^{(x,y)}$ and $\pi^{x \to y}$ be operators defined by

$$\pi^{(x,y)} f(\eta) := f(\eta^{(x,y)}) - f(\eta), \tag{3}$$
$$\pi^{x \to y} f(\eta) := f(\eta^{x \to y}) - f(\eta), \tag{4}$$

for any local function f. Let $c_{\text{ex}}(r), c_{\text{ge}}(r)$ be functions for $r = 0, 1, 2 \ldots k$ such that $c_{\text{ex}}(0) = 0$ and $c_{\text{ex}}(l) > 0$, for all $1 \leq l \leq k$, and $c_{\text{ge}}(0) = c_{\text{ge}}(1) = 0$ and $c_{\text{ge}}(l) > 0$, for all $2 \leq l \leq k$.

Denote by $b = (x, y)$ a directed bond; provided that $|x - y|_1 = 1$. This convention will be followed in the rest of paper unless otherwise stated. For any directed bond b, let L_b be the operator defined by

$$L_b f(\eta) := c_{\text{ex}}(\eta_x)(1 - \xi_y)\pi^{(x,y)} f(\eta) + c_{\text{ge}}(\eta_x) 1_{\{1 \leq \eta_y \leq k-1\}} \pi^{x \to y} f(\eta),$$

for any local function f. Let L_{B_N} be the Markov operator defined by

$$L_{B_N} := \sum_{b \in B_N} L_b.$$

Consider the family of product measure on the product space $\{0,1,2,\dots k\}^{B_N}$ with marginal

$$\bar{P}_{p,\rho}(\eta_x = l) := \begin{cases} 1-p & \text{if } l=0, \\ p\dfrac{1}{Z_{\alpha(p,\rho)}} & \text{if } l=1, \\ p\dfrac{1}{Z_{\alpha(p,\rho)}}\dfrac{(\alpha(p,\rho))^{l-1}}{c_{\rm ge}(2)c_{\rm ge}(3)\cdots c_{\rm ge}(l)} & \text{if } 2\le l\le k, \end{cases} \tag{5}$$

for $0\le p\le 1$ and $p\le\rho\le kp$, where $Z_{\alpha(p,\rho)}$ is the normalizing constant and $\alpha(p,\rho)$ is a positive constant depending on p and ρ and determined by the relation

$$\bar{E}_{p,\rho}[\eta_x] = \rho.$$

From the definition, we have reversibility such that

$$\begin{aligned} &c_{\rm ge}(\eta_x)1_{\{1\le\eta_w\le k-1\}}\bar{P}_{p,\rho}(\{\eta\}) = c_{\rm ge}((\eta^{x\to w})_w)1_{\{1\le(\eta^{x\to w})_x\le k-1\}}\bar{P}_{p,\rho}(\{\eta^{x\to w}\}) \\ &\bar{P}_{p,\rho}(\{\eta\}) = \bar{P}_{p,\rho}(\{\eta^{(x,w)}\}), \end{aligned} \tag{6}$$

for any p,ρ, for any $x,w\in B_N$ and $\eta\in\{0,1,2,\dots k\}^{B_N}$. It is easy to check that L_{B_N} is symmetric under $\bar{P}_{p,\rho}$.

We speed up the process by N^2, so that the generator of the process is $N^2L_{B_N}$. Let us consider the system with initial density f_0^N with respect to $\bar{P}_{p,\rho}$. We denote by $P_{M_N}^{f_0^N}$ the probability law of Markov process, generated by $N^2L_{B_N}$ and starting with initial distribution $f_0^N\bar{P}_{p,\rho}$, on the space of trajectories $\eta(\cdot)$.

Denote by $\mathbf{T}^d$ the d-dimensional torus. Let us consider the empirical measures for energy and for particle defined by

$$m_t^{E,N}(d\theta) := \operatorname*{Av}_{x\in B_N}\eta_x^N(t)\delta_{x/N}(d\theta), \tag{7}$$

$$m_t^{P,N}(d\theta) := \operatorname*{Av}_{x\in B_N}\xi_x^N(t)\delta_{x/N}(d\theta), \tag{8}$$

where $\theta\in\mathbf{T}^d$ and Av is an averaging operator defined by

$$\operatorname*{Av}_{x\in L} = \frac{1}{|L|}\sum_{x\in L}$$

for any $L\subset\mathbf{Z}^d$. The measures (7), (8) induce (from $P_{M_N}^{f_0^N}$) a distribution $Q_N^{f_0^N}$ of $(m^{E,N}(\cdot,\cdot), m^{P,N}(\cdot,\cdot))$ on the Skorohod space $X = D([0,T]\to(M[\mathbf{T}^d])^2)$, where M is the space of nonnegative finite measures on $\mathbf{T}^d$, the space which we equip with the topology of weak convergence.

Theorem 2.1 *Suppose that the initial distribution satisfies*

$$\overline{\lim_{N\to\infty}} P_{M_N^{f_0^N}}\Big[\big|\int m_0^{E,N}(d\theta)J(\theta) - \int \rho_0^E(\theta)J(\theta)d\theta\big| > \delta\Big] = 0,$$
$$\overline{\lim_{N\to\infty}} P_{M_N^{f_0^N}}\Big[\big|\int m_0^{P,N}(d\theta)J(\theta) - \int \rho_0^P(\theta)J(\theta)d\theta\big| > \delta\Big] = 0,$$

for any smooth function J and positive constant δ, then all limit points of Q^N are concentrated on trajectories $m_t^{E,N}(d\theta) = \rho^E(t,\theta)d\theta$ and $m_t^{P,N}(d\theta) = \rho^P(t,\theta)d\theta$ that are weak solution of

$$\begin{aligned}
\frac{\partial}{\partial t}\rho^E(t,\theta) &= \sum_{i=1}^d \frac{\partial}{\partial\theta_i}\Big(D_{E,E}(\rho)\frac{\partial}{\partial\theta_i}\rho^E(t,\theta) + D_{E,P}(\rho)\frac{\partial}{\partial\theta_i}\rho^P(t,\theta)\Big)\\
\frac{\partial}{\partial t}\rho^P(t,\theta) &= \sum_{i=1}^d \frac{\partial}{\partial\theta_i}\Big(D_{P,E}(\rho)\frac{\partial}{\partial\theta_i}\rho^E(t,\theta) + D_{P,P}(\rho)\frac{\partial}{\partial\theta_i}\rho^P(t,\theta)\Big)\\
\rho^E(0,\theta) &= \rho_0^E(\theta)\\
\rho^P(0,\theta) &= \rho_0^P(\theta)
\end{aligned} \tag{9}$$

where $D = (D_{p,q})_{p,q\in\{E,P\}}$ is a 2×2 matrix defined as follows.
Let $\tilde{D}$ be a 2×2 matrix defined by the variational formula:

$$\begin{aligned}
&(\alpha, \tilde{D}\alpha)\\
&= \sum_{i,j\in\{E,P\}} \alpha_i \tilde{D}_{i,j}\alpha_j\\
&:= \inf_{u,v}\Big\{\bar{E}_{\rho^P,\rho^E}[c_{\mathrm{ex}}(\eta_0)(1-\xi_e)\{\alpha_E(\eta_0 + \pi^{(0,e)}(\sum_x \tau_x u))\\
&\quad +\alpha_P(1+\pi^{(0,e)}(\sum_x \tau_x v))\}^2]\\
&\quad +\bar{E}_{\rho^P,\rho^E}[c_{\mathrm{ge}}(\eta_0)1_{\{1\le\eta_e\le k-1\}}\{\alpha_E(1+\pi^{(0\to e)}(\sum_x \tau_x u))\\
&\quad +\alpha_P(\pi^{0\to e}(\sum_x \tau_x v))\}^2]\Big\},
\end{aligned}$$

where infimum u, v are taken over all local functions and τ_x be shift operator defined by

$$\begin{aligned}
\tau_x h(\eta) &:= h(\tau_x\eta),\\
(\tau_x\eta)_z &:= \eta_{z-x}.
\end{aligned}$$

Put

$$\chi(\rho^E,\rho^P) := \mathrm{Cov}_{\rho^P,\rho^E}(\eta_0,\xi_0) = \begin{pmatrix} \bar{E}_{\rho^P,\rho^E}[\eta_0^2] - (\rho^E)^2 & (1-\rho^P)\rho^E \\ (1-\rho^P)\rho^E & \rho^P(1-\rho^P) \end{pmatrix}.$$

Then

$$D(\rho^E,\rho^P) = \tilde{D}(\rho^E,\rho^P)\chi^{-1}(\rho^E,\rho^P). \tag{10}$$

3 Outline of the proof

First we state a result on the lower bound of the spectral gap. Let $P_{L,y,E}$ be canonical measure defined by

$$P_{L,y,E}(\cdot) := \bar{P}_{p,\rho}(\cdot| \sum_{x\in L}\xi_x = y, \sum_{x\in L}\eta_x = E),$$

for any $L \subset\subset \mathbf{Z}^d$ and for any $0 \le y \le |L|$ and $y \le E \le ky$.

Proposition 3.1 *(Spectral gap inequality) There exists a constant C such that for any N, y, E, we have*

$$V_{B_N,y,E}(f) \le CN^2 D_{B_N,y,E}(f),$$

where V denotes a variance and D denotes a Dirichlet form defined by

$$D_{B_N,y,E}(f) := -E_{B_N,y,E}[fL_{B_N}f],$$

for each canonical measure.

We will give a proof of Proposition 3.1 in Section 4.

In the next step we prove the tightness of the measures $\{Q_N^{f_0^N}\}$. To prove the tightness, in view of Prohorov's theorem, we have only to show the following.

Lemma 3.2 *(Tightness) For any initial distribution $f_0^N\bar{P}_{p,\rho}$, for any smooth function J and for any $\delta > 0$, it holds that*

$$\overline{\lim_{\alpha\to 0}}\ \overline{\lim_{N\to\infty}}\ P_{M_N^{f_0^N}}\Big[\sup_{|s-t|<\alpha, 0\le s,t\le T} |m_t^{E,N}(J) - m_s^{E,N}(J)| > \delta\Big] = 0,$$
$$\overline{\lim_{\alpha\to 0}}\ \overline{\lim_{N\to\infty}}\ P_{M_N^{f_0^N}}\Big[\sup_{|s-t|<\alpha, 0\le s,t\le T} |m_t^{P,N}(J) - m_s^{P,N}(J)| > \delta\Big] = 0,$$

where

$$m_t^{E,N}(J) := \int_{\mathbf{T}^d} m_t^{E,N}(d\theta)J(\theta),$$
$$m_t^{P,N}(J) := \int_{\mathbf{T}^d} m_t^{P,N}(d\theta)J(\theta).$$

We will give a proof of Lemma 3.2 in Section 5.

Since

$$\overline{\lim_{N\to\infty}} \,|\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x| \le k||J||_1,$$
$$\overline{\lim_{N\to\infty}} \,|\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\xi_x| \le ||J||_1,$$

under all limit points of $Q_N^{f_0^N}$ trajectories are functions from $[0,T]$ to absolutely continuous measures with respect to the Lebesgue measures. Let us write $\rho^E(t,\theta)$ and $\rho^P(t,\theta)$ the densities of the measures.

Lemma 3.3 *Let Q be a limit point of $Q_N^{f_0^N}$. Then there exists a constant C such that*

$$E^Q\Big[\int_0^T dt\int_{\mathbf{T}^d} d\theta\sum_{i=1}^d(\frac{\partial}{\partial\theta_i}\rho^E(t,\theta))^2\Big] \le C,$$
$$E^Q\Big[\int_0^T dt\int_{\mathbf{T}^d} d\theta\sum_{i=1}^d(\frac{\partial}{\partial\theta_i}\rho^P(t,\theta))^2\Big] \le C.$$

We will give a proof of Lemma 3.3 in Section 6.

Finally, we prove that Q in Lemma 3.3 is supported on a set of trajectories whose densities are weak solution of (9). Denote by $\bar\eta_{x,l}$ and $\bar\xi_{x,l}$ the average energy density and average particle density in a cube of width $2l+1$ centered at x, namely

$$\bar\eta_{x,l} = \operatorname*{Av}_{y:|y-x|\le l}\eta_y,$$
$$\bar\xi_{x,l} = \operatorname*{Av}_{y:|y-x|\le l}\xi_y.$$

Theorem 3.4 *(Identification of the equation) For any $\delta>0$, let*

$$B_{a,b,\delta}^{J,E} := \Big\{\eta(\cdot),\xi(\cdot) : \sup_{0\le t\le T}|\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x(t) - \operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x(0)$$

$$
\begin{aligned}
& +\int_0^t \frac{1}{2b} \operatorname*{Av}_{x\in B_N} \sum_{i=1}^d (\frac{\partial}{\partial\theta_i} J)(\frac{x}{N}) \\
& \times\{[D_{E,E}(\bar\eta_{x,aN}(s),\bar\xi_{x,aN}(s))][\eta_{x+bNe}(s)-\eta_{x-bNe}(s)] \\
& +[D_{E,P}(\bar\eta_{x,aN}(s),\bar\xi_{x,aN}(s))][\xi_{x+bNe}(s)-\xi_{x-bNe}(s)]\}ds| > \delta\Big\},
\end{aligned}
$$

$$
\begin{aligned}
B^{J,P}_{a,b,\delta} \;:=\; & \Big\{\eta(\cdot),\xi(\cdot): \sup_{0\le t\le T} | \operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\xi_x(t) - \operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\xi_x(0) \\
& +\int_0^t \frac{1}{2b} \operatorname*{Av}_{x\in B_N} \sum_{i=1}^d (\frac{\partial}{\partial\theta_i} J)(\frac{x}{N}) \\
& \times\{[D_{P,E}(\bar\eta_{x,aN}(s),\bar\xi_{x,aN}(s))][\eta_{x+bNe}(s)-\eta_{x-bNe}(s)] \\
& +[D_{P,P}(\bar\eta_{x,aN}(s),\bar\xi_{x,aN}(s))][\xi_{x+bNe}(s)-\xi_{x-bNe}(s)]\}ds| > \delta\Big\}.
\end{aligned}
$$

Then

$$
\overline{\lim_{a\to 0}}\,\overline{\lim_{b\to 0}}\,\overline{\lim_{N\to\infty}} P_{M_N^{f_0^N}}[B^{J,E}_{a,b,\delta}] = 0,
$$

$$
\overline{\lim_{a\to 0}}\,\overline{\lim_{b\to 0}}\,\overline{\lim_{N\to\infty}} P_{M_N^{f_0^N}}[B^{J,P}_{a,b,\delta}] = 0.
$$

Outline of the proof of Theorem 3.4.
Denote by e the unit vectors of $\mathbf{Z}^d$. Denote the currents for energy and for particle by

$$
\begin{aligned}
w^E_{x,x+e}(\eta) \;&:=\; c_{\rm ex}(\eta_x)(1-\xi_{x+e})\eta_x - c_{\rm ex}(\eta_{x+e})(1-\xi_x)\eta_{x+e} \\
&\quad + c_{\rm ge}(\eta_x)1_{\{1\le\eta_{x+e}\le k-1\}} - c_{\rm ge}(\eta_{x+e})1_{\{1\le\eta_x\le k-1\}}, \\
w^P_{x,x+e}(\eta) \;&:=\; c_{\rm ex}(\eta_x)(1-\xi_{x+e}) - c_{\rm ex}(\eta_{x+e})(1-\xi_x),
\end{aligned}
$$

respectively. Let us define ∇_e by

$$
\nabla_e h(x) = h(x+e) - h(x)
$$

for all functions on $\mathbf{Z}^d$. From the definition of the currents, we have

$$
\begin{aligned}
\operatorname*{Av}_x \Big(J(\frac{x}{N})\eta_x(t)\Big) - \operatorname*{Av}_x \Big(J(\frac{x}{N})\eta_x(0)\Big) &= \int_0^t U^E(\eta(s))ds + M^E_N(t) \\
\operatorname*{Av}_x \Big(J(\frac{x}{N})\xi_x(t)\Big) - \operatorname*{Av}_x \Big(J(\frac{x}{N})\xi_x(0)\Big) &= \int_0^t U^P(\eta(s))ds + M^P_N(t)
\end{aligned}
$$

where the drift terms U^E and U^P are defined by

$$
U^E(\eta) = N \operatorname*{Av}_x \sum_{e>0} (N\nabla_e J(\frac{x}{N}))w^E_{(x,x+e)}(\eta),
$$

$$
U^P(\eta) = N \operatorname*{Av}_x \sum_{e>0} (N\nabla_e J(\frac{x}{N}))w^P_{(x,x+e)}(\eta),
$$

and M^E and M^P are martingales. Let $F_N^E(\eta) := \mathrm{Av}_x\, J(\frac{x}{N})\eta_x$. Then we have

$$(M_N^E(T))^2 = \int_0^T \Big(N^2 L_{B_N}(F_N^E(\eta(t)))^2 - 2F_N^E(\eta)N^2 L_{B_N} F_N^E(\eta)\Big)dt + \tilde{M}_T^E,$$

where $\tilde{M}$ is a martingale. Since $(F+\pi^{(x,y)}F)(\eta) = F(\eta^{(x,y)})$, $(F+\pi^{x\to y}F)(\eta) = F(\eta^{x\to y})$, we have

$$\begin{aligned}
&\Big(N^2 L_{B_N}(F_N^E(\eta))^2 - 2F_N^E(\eta)N^2 L_{B_N} F_N^E(\eta)\Big) \\
&= \sum_b \sum_{e>0} \Big\{ c_{\mathrm{ex}}(\eta_x)(1-\xi_{x+e})(\frac{1}{N^d}(N\nabla_e J(\frac{x}{N})(\eta_x - \eta_{x+1}))^2 \\
&\quad + c_{\mathrm{ge}}(\eta_x) 1_{\{1\le \eta_{x+e} \le k-1\}}(\frac{1}{N^d}(N\nabla_e J(\frac{x}{N}))^2\Big\}.
\end{aligned}$$

Hence we have

$$E_{M_N^{f_0^N}}\Big[(M_N^E(T))^2\Big] \le \frac{1}{N^d} C(J)T.$$

In the same way we also have

$$E_{M_N^{f_0^N}}\Big[(M_N^P(T))^2\Big] \le \frac{1}{N^d} C(J)T.$$

Hence we can neglect the martingale terms. Our problem is reduced to proving

$$\begin{aligned}
&\overline{\lim_{a\to 0}}\,\overline{\lim_{b\to 0}}\,\overline{\lim_{N\to\infty}}\, P_{M_N^{f_0^N}}[|\int_0^t \tilde{B}_{a,b}^{J,E} ds| > \delta] = 0, \\
&\overline{\lim_{a\to 0}}\,\overline{\lim_{b\to 0}}\,\overline{\lim_{N\to\infty}}\, P_{M_N^{f_0^N}}[|\int_0^t \tilde{B}_{a,b}^{P,E} ds| > \delta] = 0.
\end{aligned}$$

where

$$\begin{aligned}
\tilde{B}_{a,b}^{J,E} &= N\Big\{ \mathop{\mathrm{Av}}_{x\in B_N} \sum_e J(\frac{x}{N}) w_{x,x+e}^E \\
&\quad + \frac{1}{2bN} \mathop{\mathrm{Av}}_{x\in B_N} \sum_e J(\frac{x}{N}) \\
&\quad \times \{[D_{E,E}(\bar{\eta}_{x,aN}(s), \bar{\xi}_{x,aN}(s))][\eta_{x+bNe}(s) - \eta_{x-bNe}(s)] \\
&\quad + [D_{E,P}(\bar{\eta}_{x,aN}(s), \bar{\xi}_{x,aN}(s))][\xi_{x+bNe}(s) - \xi_{x-bNe}(s)]\}\Big\}, \\
\tilde{B}_{a,b}^{J,P} &= N\Big\{ \mathop{\mathrm{Av}}_{x\in B_N} \sum_e J(\frac{x}{N}) w_{x,x+e}^P
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2bN}\operatorname*{Av}_{x\in B_N}\sum_e J(\frac{x}{N}) \\
&\times\{[D_{P,E}(\bar{\eta}_{x,aN}(s),\bar{\xi}_{x,aN}(s))][\eta_{x+bNe}(s)-\eta_{x-bNe}(s)] \\
&+[D_{P,P}(\bar{\eta}_{x,aN}(s),\bar{\xi}_{x,aN}(s))][\xi_{x+bNe}(s)-\xi_{x-bNe}(s)]\}\Big\}.
\end{aligned}
$$

In our situation, we can reduce our problem to estimating the upper limit of a spectrum like (12) below;

Lemma 3.5 *Let $X_N(t)$ be a Markov process with generator $\mathcal{L}_N$ starting from initial distribution μ_N. The Markov generator $\mathcal{L}_N$ is symmetric with respect to a probability measure ν_N. Denote by $||\cdot||_{p,\nu_N}$ a L^p norm with respect to ν_N. Denote by $P^N_{\mu_N}$ the probability law of the Markov process starting with initial distribution μ_N. If the relative entropy satisfies*

$$
H(\mu_N|\nu_N)\le CN^d, \tag{11}
$$

and it holds that

$$
\overline{\lim_{N\to\infty}}\sup\operatorname{spec}\Big[V-\frac{1}{\gamma N^d}(-\mathcal{L}_N)\Big]\le 0 \tag{12}
$$

for any $\gamma>0$, where

$$
\sup\operatorname{spec}\Big[V-(-\mathcal{L}_N)\Big]=\sup_{\int f^2 d\nu_N=1}\left\{\int Vf^2d\nu_N-\int f(-\mathcal{L}_N)fd\nu_N\right\},
$$

then we have

$$
\lim_{N\to\infty}P^N_{\mu_N}\Big[\int_0^T V(X_N(s))ds>\delta\Big]=0. \tag{13}
$$

Furthermore, if it holds that

$$
||\frac{d\mu_N}{d\nu_N}||_{p,\nu_N}\le\exp[cN^d] \tag{14}
$$

for some $1<p\le\infty$ and satisfies (12) for any $\gamma>0$, then we have

$$
\lim_{N\to\infty}\frac{1}{N^d}\log P^N_{\mu_N}\Big[\int_0^T V(X_N(s))ds>\delta\Big]=-\infty. \tag{15}
$$

We will give a proof of Lemma 3.5 in Section 10.

Proposition 3.6 *Let P^N denote the grand canonical measure on $\{0,1,\ldots,k\}^{B_N}$. Then there exists a constant c depending only on k and c_{ge} such that*

$$||\frac{d\mu_N}{dP^N}||_{p,P^N} \le \exp[cN^d]$$

for all $1 < p \le \infty$ and for all probability measures μ_N on $\{0,1,\ldots,k\}^{B_N}$.

Let us define

$$\sup\mathrm{spec}\Big[V-(-\mathcal{L})\Big] := \sup_{\int f^2 dP^N=1}\Big\{\int Vf^2dP^N - \int f(-\mathcal{L})f dP^N\Big\},$$

where P^N is the grand canonical measure on $\{0,1,\ldots,k\}^{B_N}$.

In view of Lemma 3.5 and Proposition 3.6, we have only to obtain the following estimate

$$\overline{\lim_{a\to 0}}\,\overline{\lim_{b\to 0}}\,\overline{\lim_{N\to\infty}}\,\sup\mathrm{spec}[\pm\tilde{B}^{J,E}_{a,b} - \gamma N^2 \operatorname*{Av}_{b\in B_N}(-L_b)] \le 0. \tag{16}$$

To this end, we use the perturbation theory of eigenvalues.

Lemma 3.7 *Let P be a probability measure on a finite set B. Let $\mathcal{L}$ be a generator of a Markov process on B, which is symmetric with respect to P. Suppose that $-\mathcal{L}$ has a spectral gap $\delta > 0$. Let f be a bounded function on B. Let $||f||_\infty$ denote the supremum norm of f. Assume that $E[f]=0$. Then we have*

$$\sup\mathrm{spec}\Big(\mathcal{L}+f\Big) \le E[f(-\mathcal{L})^{-1}f] + 4||f||^3_\infty\frac{1}{\delta^2},$$

where

$$\sup\mathrm{spec}\Big[f-(-\mathcal{L})\Big] := \sup_{\int g^2dP=1}\Big\{\int fg^2dP - \int g(-\mathcal{L})gdP\Big\}.$$

We will give a proof of Lemma 3.7 in Section 10.

Let $\Lambda_n := \Lambda_{0,n}$ and $\Lambda_{x,n}$ be a cube of side $2n+1$ centered at x, namely $\Lambda_{x,n} := \{y \in \mathbf{Z}^d : |x-y| \le n\}$. Let us define

$$\begin{aligned}\mathcal{G} \;:=\; &\{h : h \text{ is a local function and satisfies } E_{\Lambda_n,y,L}[h]=0\\ &\text{for some } n \text{ and for all } y, L\}.\end{aligned}$$

Let τ_x be shift operator defined by

$$\tau_x h(\eta) := h(\tau_x \eta),$$
$$(\tau_x \eta)_z := \eta_{z-x}.$$

For any local function $h \in \mathcal{G}$ we can define

$$V^{(l)}(h; y, E) := E_{\Lambda_l, y, E}[\underset{x \in \Lambda_{l_1}}{\mathrm{Av}}\ \tau_x h(-L^{(l)})^{-1} \underset{x \in \Lambda_{l_1}}{\mathrm{Av}}\ \tau_x h],$$

where $l_1 = l - \sqrt{l}$, $L^{(l)} = \mathrm{Av}_{b \in \Lambda_l} L_b$. In Corollary 8.5 of Section 8 we obtain that there exists a limit

$$\lim_{l \to \infty, (p', \rho') \to (p, \rho)} \bar{E}_{p', \rho'}[V^{(l)}(h; y, E)]$$

for any $h \in \mathcal{G}$ and denote the limit by $V(h; p, \rho)$.

Denote the density gradients for energy and for particle by

$$\nabla_e \eta := \nabla_e \eta_0 = \eta_e - \eta_0.$$
$$\nabla_e \xi := \nabla_e \xi_0 = \xi_e - \xi_0,$$

respectively.

Theorem 3.8 *Fix densities ρ and p. Suppose $\{g_e^q\}_{e \in \{e_1, \dots e_d\}, q \in \{E, P\}}$ is a set of $(2d)$ local functions of η. For each direction e, let*

$$\phi_e(g) := \begin{pmatrix} \phi_e^E \\ \phi_e^P \end{pmatrix} = \begin{pmatrix} w_e^E \\ w_e^P \end{pmatrix} + L_{B_N} \begin{pmatrix} g_e^E \\ g_e^P \end{pmatrix} + D(\rho, p) \begin{pmatrix} \nabla_e \eta \\ \nabla_e \xi \end{pmatrix},$$

where D is defined by (10). Then for any $\{\alpha_e^q\}$ satisfying $\sum_{e,q} (\alpha_e^q)^2 = 1$,

$$\inf_{\{g_e^q\}} V\Big(\sum_e \{\alpha_e^E \phi_e^E(g) + \alpha_e^P \phi_e^P(g)\}; p, \rho \Big) = 0.$$

We will prove Theorem 3.8 in Section 8. In this theorem, minimizing sequence $\{g_{n,e}^q\}$ depends on the densities p and ρ.

Proposition 3.9 *For any $\{\alpha_e^q\}$ satisfying $\sum_{e,q} (\alpha_e^q)^2 = 1$ and $\delta > 0$, there exists a positive integer $n = n(\delta)$ and a vector valued function $\{g_e^q(p, \rho, \eta)\}$ satisfying following properties.*
For each p, ρ, each component depends only on $\{\eta_x : x \in \Lambda_n\}$.
For each η each component is a smooth function of p, ρ.
It holds that

$$\sup_{p, \rho} V\Big(\sum_e \{\alpha_e^E \phi_e^E(g_e^q(p, \rho, \cdot)) + \alpha_e^P \phi_e^P(g_e^q(p, \rho, \cdot))\}; p, \rho \Big) \le \delta.$$

We will prove Proposition 3.9 in section 10.

In order to apply Proposition 3.9, we have to show that Lg is negligible. By simple computation, we have

$$\begin{aligned}
&\operatorname*{Av}_x \Big(J(\frac{x}{N})\eta_x(t)\Big) - \operatorname*{Av}_x \Big(J(\frac{x}{N})\eta_x(0)\Big) + \Omega_g(\eta(t)) - \Omega_g(\eta(0)) \\
&= \int_0^t U^E(\eta(s), g)ds + M^E_{N,g}(t) \\
&\operatorname*{Av}_x J(\frac{x}{N})\xi_x(t) - \operatorname*{Av}_x J(\frac{x}{N})\xi_x(0) + \Omega_g(\eta(t)) - \Omega_g(\eta(0)) \\
&= \int_0^t U^P(\eta(s), g)ds + M^P_{N,g}(t)
\end{aligned}$$

where

$$\begin{aligned}
&\Omega_g(\eta) = \frac{1}{N} \operatorname*{Av}_x \sum_{e>0} (N\nabla_e J(\frac{x}{N}))g(\tau_x\eta), \\
&U^E(\eta, g) = N \operatorname*{Av}_x \sum_{e>0} (N\nabla_e J(\frac{x}{N}))(w^E_{(x,x+e)}(\eta) + L_{B_N} g(\tau_x\eta)), \\
&U^P(\eta, g) = N \operatorname*{Av}_x \sum_{e>0} (N\nabla_e J(\frac{x}{N}))(w^P_{(x,x+e)}(\eta) + L_{B_N} g(\tau_x\eta)),
\end{aligned}$$

and $M^E_{N,g}$ and $M^P_{N,g}$ are martingales. We can easily check that

$$\begin{aligned}
E_{M_N^{f_0^N}}\left[(M^E_{N,g}(T))^2\right] \le \frac{1}{N^d} C(J)T, \\
E_{M_N^{f_0^N}}\left[(M^P_{N,g}(T))^2\right] \le \frac{1}{N^d} C(J)T.
\end{aligned}$$

From the definition, Ω_g is of order $1/N$. Hence we can neglect both Ω_g and the martingale terms.

Let us consider the function $g = g(\eta, p, \rho)$ which is a smooth function of p and ρ, and a local function of η. Assume that g depends only on $\{\eta_x : |x| \le n\}$. Then we can easily check that for $n < l - 1$

$$L_{B_N} g(\eta, \bar{\xi}_{0,l}, \bar{\eta}_{0,l}) = \sum_{b\in\Lambda_l} (L_b) g(\eta, \bar{\xi}_{0,l}, \bar{\eta}_{0,l}) + O(\frac{1}{l}),$$

namely L does not act on densities. Since the difference is only on $b = (x, y)$ such that $x \in \Lambda_l$ and $y \notin \Lambda_l$, or $x \notin \Lambda_l$ and $y \in \Lambda_l$, The number of such b is of order l^{d-1}, but $L_b g$ is at most of order l^{-d}.

Theorem 3.10 *Suppose J is a $2d$ dimensional vector valued test function and $g = \{g_e^q(\eta, p, \rho)\}_{e,q}$ is a $2d$ dimensional vector valued function whose components are smooth functions of p, ρ for each η, and local functions of η for each p, ρ. Let us define a $2d$ dimensional vector ϕ by*

$$\phi_e(g)(x) := \begin{pmatrix} \phi_e^E(g)(x) \\ \phi_e^P(g)(x) \end{pmatrix}$$

$$= \begin{pmatrix} w^E_{(x,x+e)} \\ w^P_{(x,x+e)} \end{pmatrix} + L \begin{pmatrix} g_e^E(\tau_x\eta, \bar{\xi}_{x,l}, \bar{\eta}_{x,l}) \\ g_e^P(\tau_x\eta, \bar{\xi}_{x,l}, \bar{\eta}_{x,l}) \end{pmatrix} + D(\bar{\eta}_{x,l}, \bar{\xi}_{x,l}) \begin{pmatrix} \Psi^E_{N,c,e}(x) \\ \Psi^P_{N,c,e}(x) \end{pmatrix},$$

where Ψ^E and Ψ^P are (d- dimensional) density gradients defined by

$$\begin{aligned} \Psi^E_{N,c,e}(x) &= \frac{1}{2cN}(\eta_{x+cNe} - \eta_{x-cNe}), \\ \Psi^P_{N,c,e}(x) &= \frac{1}{2cN}(\xi_{x+cNe} - \xi_{x-cNe}). \end{aligned}$$

Then for any $\gamma > 0$, there exists a constant $C(\gamma)$ such that

$$\begin{aligned} &\overline{\lim_{a\to 0}}\,\overline{\lim_{b\to 0}}\,\overline{\lim_{N\to\infty}} \sup \operatorname{spec}[\pm(\tilde{B}^{J,E}_{a,b} + \tilde{B}^{J,P}_{a,b}) - \gamma N^2 \operatorname*{Av}_b(-L_b)] \\ &\le \overline{\lim_{l\to\infty}}\,\overline{\lim_{a\to 0}}\,\overline{\lim_{b\to 0}}\,\overline{\lim_{N\to\infty}} \sup \operatorname{spec}\Big[\pm N \operatorname*{Av}_x \sum_{e>0} (J_e^E(\frac{x}{N})\phi_e^E(g)(x) \\ &\quad + J_e^P(\frac{x}{N})\phi_e^P(g)(x)) - \gamma N^2 \operatorname*{Av}_b(-L_b)\Big] \qquad (17) \\ &\le C(\gamma)||J||^2_\infty \sup_\alpha \sup_{p,\rho} V\Big(\sum_e \Phi^\alpha_e(g); p, \rho\Big), \end{aligned}$$

where α is taken over $\sum_e (\alpha_e^E)^2 + (\alpha_e^P)^2 = 1$ and

$$\Phi^\alpha_e(g) := (\alpha_e^E, \alpha_e^P) \begin{pmatrix} w_e^E + Lg_e^E \\ w_e^E + Lg_e^P \end{pmatrix} + (\alpha_e^E, \alpha_e^P) D(\bar{\eta}_{x,l}, \bar{\xi}_{x,l}) \begin{pmatrix} \nabla_e \eta \\ \nabla_e \xi \end{pmatrix}.$$

We conclude the proof of Theorem 3.4, since we can choose g so that the right hand side of (17) is arbitrarily small for any fixed γ in view of Proposition 3.9.

4 Spectral gap and Two blocks estimate

In this section we prove Proposition 3.1. Adapting the methods found in [3] or [4], we have only to prove Lemmas 4.1 and 4.3 below. By the way we can prove two blocks estimate Lemma 4.4 by Lemma 4.1. We will also use Lemma 4.3 in Section 9. In this section we omit the parameters of grand canonical or canonical measures from the notations P, E, D, V. Estimates involving these are understood to hold uniformly in these parameters unless otherwise is stated.

Let $\gamma(x, y)$ denote the canonical path from x to y defined by

$$\gamma(x, y) := \{z(i) : 0 \le i \le |x - y|_1\}, \tag{18}$$

where $z(i)$ is defined by

$$\begin{aligned}
&z(i) = (z(i)_1, \dots, z(i)_d) \text{ such that} \\
&z(i)_j = x_j \text{ for } i \le \sum_{k=1}^{j-1} |x_k - y_k|, \\
&z(i)_j = y_j \text{ for } i \ge \sum_{k=1}^{j} |x_k - y_k|, \\
&z(i)_j = x_j + (i - \sum_{k=1}^{j-1} |x_k - y_k|) \frac{y_i - x_i}{|y_i - x_i|} \\
&\qquad\qquad \text{for } \sum_{k=1}^{j-1} |x_k - y_k| < i < \sum_{k=1}^{j} |x_k - y_k|,
\end{aligned}$$

namely $\gamma(x, y)$ denotes the nearest neighbor path that goes from x to y, moving successively as far as it has to in each of the coordinate directions, following the natural order for the different coordinate directions.

Lemma 4.1 *There exists a constant C such that for each grand canonical or canonical measure and for any local function f, we have*

$$\begin{aligned}
D^{x \to y}(f) &:= E[c_{\mathrm{ge}}(\eta_x) 1_{\{1 \le \eta_y \le k-1\}} ((\pi^{x \to y} f)(\eta))^2] \\
&\le C|x - y|_1 \sum_{b \in \gamma(x,y)} D_b(f)
\end{aligned} \tag{19}$$

$$\begin{aligned}
D^{(x,y)}(f) &:= E[c_{\mathrm{ex}}(\eta_x)(1 - \xi_y)((\pi^{(x,y)} f)(\eta))^2] \\
&\le C|x - y|_1 \sum_{b \in \gamma(x,y)} D_b(f),
\end{aligned} \tag{20}$$

where $\gamma(x, y)$ is a canonical path from x to y, and $D_b(f)$ is one site Dirichlet form defined by

$$D_b(f) = E[c_{\mathrm{ge}}(\eta_z) 1_{\{1 \le \eta_w \le k-1\}} ((\pi^{z \to w} f)(\eta))^2 + c_{\mathrm{ex}}(\eta_z)(1 - \xi_w)((\pi^{(z,w)} f)(\eta))^2],$$

for $b = (z, w)$.

In the proof of Lemma 4.1, the next lemma plays an important role.

Lemma 4.2 *There exists a constant* C *such that for each grand canonical or canonical measure and for any local function* f*, we have*

$$E[(\pi^{(0,e)} f)(\eta))^2] \le C D_{(0,e)}(f).$$

This lemma is essentially proved in [3], since this lemma is obvious except for the situation that both 0 and e are occupied by particles.

Proof of Lemma 4.2. Fix $1 \le a < b \le k$, consider the configuration η with $\eta_0 = a$ and $\eta_e = b$. For $0 \le l \le b - a$, let η^l be the configuration defined by

$$\begin{aligned} \eta^l &:= T^{e \to 0} \eta^{l-1}, \quad \text{if } 1 \le l \le b - a, \\ \eta^0 &:= \eta. \end{aligned}$$

Then using the Schwarz inequality and reversibility (6) we have

$$\begin{aligned} & E[1_{\{\eta_0 = a, \eta_e = b\}}((\pi^{(0,e)} f)(\eta))^2] \\ \le & \ (b-a) \sum_{l=1}^{b-a} \sum_{\eta} P(\eta^{l-1})[1_{\{(\eta^{l-1})_0 = a+l-1, (\eta^{l-1})_e = b-l+1\}} \\ & \times \frac{c_{\mathrm{ge}}(a+1) \cdots c_{\mathrm{ge}}(a+l-1)}{c_{\mathrm{ge}}(b) \cdots c_{\mathrm{ge}}(b-l)} (f(\eta^l) - f(\eta^{l-1}))^2]. \end{aligned}$$

Since c_{ge} and $1/c_{\mathrm{ge}}$ are bounded, and the number of c_{ge} in the last expression is at most $2k$, there exists a constant C such that

$$\begin{aligned} & E[1_{\{\eta_0 = a, \eta_e = b\}}((\pi^{(0,e)} f)(\eta))^2] \\ \le & \ C(b-a) \sum_{l=1}^{b-a} \sum_{\eta} P(\eta^{l-1})[1_{\{\eta_0 = a+l-1, \eta_e = b-l+1\}} (f(\eta^l) - f(\eta^{l-1}))^2]. \end{aligned}$$

Summing up over $1 \le a < b \le k$, we have

$$E[1_{\{0 < \eta_0 < \eta_e\}}((\pi^{(0,e)} f)(\eta))^2] \le C k^2 D_{(0,e)}(f).$$

For $0 < \eta_e < \eta_0$, we have the same estimate. Hence we conclude that

$$\begin{aligned} & E[(\pi^{(0,e)} f)(\eta))^2] \\ = & \ E[\Big((1-\xi_0)\xi_e + \xi_0(1-\xi_e) + 1_{\{0 < \eta_0 < \eta_e\}} + 1_{\{0 < \eta_e < \eta_0\}}\Big)(\pi^{(0,e)} f)(\eta))^2] \\ \le & \ C D_{(0,e)}(f). \end{aligned}$$

□

Proof of the Lemma 4.1. Let $z(i) := z_{x,y}(i)$ $(0 \le i \le |x-y|_1)$ be defined by (18), namely $z(i)$ is the i-th point from x on canonical path $\gamma(x,y)$. We can rewrite $\eta^{x\to y}$ and $\eta^{(x,y)}$ as

$$\begin{aligned}
\eta^{x\to y} &= T^{(z(0),z(1))} \circ T^{(z(1),z(2))} \circ \cdots \circ T^{(z(|x-y|_1-2),z(|x-y|_1-1))} \\
&\circ T^{z(|x-y|_1-1)\to z(|x-y|_1)} \\
&\circ T^{(z(|x-y|_1-1),z(|x-y|_1-2))} \circ \cdots \circ T^{(z(0),z(1))}\eta, \\
\eta^{(x,y)} &= T^{(z(0),z(1))} \circ T^{(z(1),z(2))} \circ \cdots \circ T^{(z(|x-y|_1-2),z(|x-y|_1-1))} \\
&\circ T^{(z(|x-y|_1-1),z(|x-y|_1))} \\
&\circ T^{(z(|x-y|_1-1),z(|x-y|_1-2))} \circ \cdots \circ T^{(z(0),z(1))}\eta.
\end{aligned}$$

From here we only consider the generalized exclusion type jump (19). Let $T_i\eta$ be defined by $T_0\eta := \eta$,

$$T_i\eta = T^{(z(i-1),z(i))}T_{i-1}\eta$$

for $1 \le i \le |x-y|_1 - 1$, and

$$T_{|x-y|_1}\eta = T^{z(|x-y|_1-1)\to z(|x-y|_1)}T_{|x-y|_1-1}\eta$$

and

$$T_i\eta = T^{(z(2|x-y|_1-i-1),z(2|x-y|_1-i))}T_{i-1}\eta$$

for $|x-y|_1 + 1 \le i \le 2|x-y|_1 - 1$. Then we have

$$\pi^{x\to y}f(\eta) = \sum_{i=1}^{2|x-y|-1} (f(T_i\eta) - f(T_{i-1}\eta)),$$

and

$$f(T_i\eta) - f(T_{i-1}\eta) = \pi^{(z(i-1),z(i))}f(T_{i-1}\eta).$$

for $1 \le i \le |x-y|_1 - 1$, and

$$f(T_{|x-y|_1}\eta) - f(T_{|x-y|_1-1}\eta) = \pi^{|x-y|_1-1\to|x-y|_1}f(T_{|x-y|_1-1}\eta),$$

and

$$f(T_i\eta) - f(T_{i-1}\eta) = \pi^{(z(2|x-y|_1-i-1),z(2|x-y|_1-i))}f(T_{i-1}\eta),$$

for $|x-y|_1+1 \le i \le 2|x-y|_1-1$. Using the Schwarz inequality and Lemma 4.2,

$$\begin{aligned} D^{x\to y}(f) &\le (2|x-y|_1-1)E[c_{\mathrm{ge}}(\eta_x)\sum_{i=1}^{2|x-y|_1-1}(f(T_i\eta)-f(T_{i-1}\eta))^2] \\ &\le C|x-y|_1\sum_{b\in\gamma(x,y)}D_b(f). \end{aligned}$$

We can estimate the exclusion type jump (20) in the same way. □

Let $\mathcal{L}_{(x,y)}$ be an operator defined by

$$\xi_x(1-\xi_y)\pi^{(x,y)}+1_{\{\eta_x\ge 2\}}1_{\{1\le\eta_y\le k-1\}}\pi^{x\to y}.$$

Let us define $\mathcal{F}_L$ for $L\subset\mathbf{Z}^d$ by

$$\mathcal{F}_L=\sigma\text{- algebra generated by }\{\eta_x : x\in L\}.$$

Lemma 4.3 *There exist finite constants C_1, C_2 such that for each grand canonical measure and for each $\mathcal{F}_{\Lambda_{3n}}$-measurable function f*

$$E[(\mathcal{L}_{z,z+e}E[f|\mathcal{F}_{\Lambda_n}])^2]\le\frac{C_1}{|\Lambda_n|}V[f]+C_2\operatorname*{Av}_{y\in\Lambda_{n,z,e}}E[(1+c_{\mathrm{ge}}(\eta_y))(\mathcal{L}_{y,z}f)^2],\quad (21)$$

where $z\in\Lambda_n$ and $z+e\notin\Lambda_n$ and $\Lambda_{n,z,e}=\tau_{z+ne}\Lambda_n$. In particular

$$E[(\mathcal{L}_{z,z+e}E[f|\mathcal{F}_{\Lambda_n}])^2]\le\frac{C_1}{|\Lambda_n|}V[f]+C_2 n\operatorname*{Av}_{y\in\Lambda_{n,z,e}}\sum_{b\in\gamma(y,z)}E[(1+c_{\mathrm{ge}}(\eta_b))(\mathcal{L}_b f)^2],\quad (22)$$

where V denotes a variance, $\gamma(y,z)$ denotes the canonical path from y to z defined by(18), and $\eta_b=\eta_i$, for $b=(i,j)$.

Proof. The inequality (22) is immediately obtained by (21) and Lemma 4.1. Denote a configuration on Λ_n by (ω,ζ_z) where ω denotes the configuration on $\Lambda_n\setminus\{z\}$ and ζ_z denotes the configuration on $\{z\}$. Then let $E[\cdot|(\omega,\zeta_z)]$ be a conditional expectation defined by

$$E[\ \cdot\ |(\omega,\zeta_z)]:=E[\ \cdot\ |\eta_{\Lambda_n}=(\omega,\zeta_z)].$$

Without risk of confusion, we simply write $P(\eta)$ for $P(\{\eta\})$ and $P(\eta_x = r)$ for $P(\{\eta : \eta_x = r\})$ and so on. From our definition

$$\begin{aligned}
&\mathcal{L}_{z,z+e}E[f|(\omega,\zeta_z)] \\
&\quad = \xi_z(1-\xi_{z+e})\{E[f|(\omega,0)] - E[f|(\omega,\zeta_z)]\} \\
&\qquad +1_{\{\eta_z\geq 2\}}1_{\{1\leq \eta_{z+e}\leq k-1\}}\{E[f|(\omega,\zeta_z-1)] - E[f|(\omega,\zeta_z)]\}.
\end{aligned} \tag{23}$$

In order to estimate these terms, we use following two general equalities (24) and (25). Denote a covariance of f, g with respect to E by

$$E[f;g] := E[fg] - E[f]E[g].$$

Then for any f, g such that $E[g] \neq 0$,

$$E[f] = \frac{1}{E[g]}\{E[fg] - E[f;g]\}. \tag{24}$$

For any f and any measurable event A,

$$E[f1_A] = P(A)E[F|A]. \tag{25}$$

Let us define $f^{(x,y)}$ and $f^{x\to y}$ by

$$\begin{aligned}
f^{(x,y)}(\eta) &:= \xi_x(1-\xi_y)f(\eta^{(x,y)}), \\
f^{x\to y}(\eta) &:= 1_{\{2\leq \eta_x\leq k\}}1_{\{1\leq \eta_x\leq k-1\}}f(\eta^{x\to y}).
\end{aligned}$$

For the first term of (23), using (24) for each $1 \leq r \leq k$ we have

$$\begin{aligned}
&E[f|(\omega,r)] \\
&\quad = \frac{1}{E[(1-\xi_{z+e})|(\omega,r)]}E[f \operatorname*{Av}_{y\in\Lambda_{n,z,e}}(1-\xi_y)|(\omega,r)] \\
&\qquad -\frac{1}{E[(1-\xi_{z+e})|(\omega,r)]}E[f; \operatorname*{Av}_{y\in\Lambda_{n,z,e}}(1-\xi_y)|(\omega,r)].
\end{aligned}$$

For the first term, using (25), we have

$$\begin{aligned}
&E[f \operatorname*{Av}_{y\in\Lambda_{n,z,e}}(1-\xi_y)|(\omega,r)] \\
&\quad = \operatorname*{Av}_{y\in\Lambda_{n,z,e}} E[(1-\xi_y)(f-f^{(z,y)})|(\omega,r)] \\
&\qquad + \operatorname*{Av}_{y\in\Lambda_{n,z,e}} P[\xi_y = 0|(\omega,r)]\frac{1}{P[\eta_y = r|(\omega,0)]}E[1_{\{\eta_y=r\}}f|(\omega,0)].
\end{aligned} \tag{26}$$

Let us define $M^{\mathrm{ex},r}_{n,z,e}(\eta)$ by

$$M^{\mathrm{ex},r}_{n,z,e}(\eta) := \frac{1}{E[1_{\{\eta_y=r\}}]} \operatorname*{Av}_{y\in\Lambda_{n,z,e}} 1_{\{\eta_y=r\}}.$$

It is obvious that $E[M^{\mathrm{ex},r}_{n,z,e}|(\omega,0)] = 1$. Then the second term of (26) is equal to

$$P[\xi_y = 0|(\omega,r)]\{E[f; M^{\mathrm{ex},r}_{n,z,e}|(\omega,0)] + E[f|(\omega,0)]\}.$$

We conclude that

$$\begin{aligned} &E[f|(\omega,r)] - E[f|(\omega,0)] \\ &= \operatorname*{Av}_{y\in\Lambda_{n,z,e}} \frac{1}{P[\xi_{z+e}=0|(\omega,r)]} E[(1-\xi_y)(f - f^{(z,y)})|(\omega,r)] \qquad (27) \\ &\quad + E[f; M^{\mathrm{ex},r}_{n,z,e}|(\omega,0)] - \frac{1}{E[(1-\xi_{z+e})|(\omega,r)]} E[f; \operatorname*{Av}_{y\in\Lambda_{n,z,e}} (1-\xi_y)|(\omega,r)]. \end{aligned}$$

We turn to estimate the second term of (23). Using (24) again, for each $1 \le r \le k-1$ we have

$$\begin{aligned} &E[f|(\omega,r+1)] \\ &= \frac{1}{E[1_{\{1\le\eta_{x+e}\le k-1\}}|(\omega,r+1)]} E[f \operatorname*{Av}_{y\in\Lambda_{n,z,e}} 1_{\{1\le\eta_y\le k-1\}}|(\omega,r+1)] \\ &\quad - \frac{1}{E[1_{\{1\le\eta_{x+e}\le k-1\}}|(\omega,r+1)]} E[f; \operatorname*{Av}_{y\in\Lambda_{n,z,e}} 1_{\{1\le\eta_y\le k-1\}}|(\omega,r+1)]. \end{aligned}$$

For the first term, using (25) and reversibility (6), we have

$$\begin{aligned} &E[f \operatorname*{Av}_{y\in\Lambda_{n,z,e}} 1_{\{1\le\eta_y\le k-1\}}|(\omega,r+1)] \\ &= \operatorname*{Av}_{y\in\Lambda_{n,z,e}} \frac{1}{c_{\mathrm{ge}}(r+1)} \frac{P[\eta_z = r|\eta_{\Lambda_n\setminus\{z\}}=\omega]}{P[\eta_z = r+1|\eta_{\Lambda_n\setminus\{z\}}=\omega]} \qquad (28) \\ &\quad \times\{E[c_{\mathrm{ge}}(\eta_y)(f^{y\to z} - f)1_{\{2\le\eta_y\le k\}}|(\omega,r)] + E[c_{\mathrm{ge}}(\eta_y) f 1_{\{2\le\eta_y\le k\}}|(\omega,r)]\}. \end{aligned}$$

Let us define $M^{\mathrm{ge},r}_{n,z,e}(\eta)$ by

$$M^{\mathrm{ge},r}_{n,z,e}(\eta) := \frac{P[\eta_y = r]}{c_{\mathrm{ge}}(r+1)P[\eta_y = r+1]} \operatorname*{Av}_{y\in\Lambda_{n,z,e}} c(\eta_y).$$

From reversibility (6), it is easy to check that

$$P(1 \le \eta_x \le k-1)c_{\mathrm{ge}}(r+1)P(\eta_y = r+1) = P(\eta_y = r)E[c_{\mathrm{ge}}],$$

for $1 \leq r \leq k-1$. Using this equality, we have

$$E[M_{n,z,e}^{\mathrm{ge},r}|(\omega,r)] = E[1_{\{1\leq\eta_y\leq k-1\}}|(\omega,r)],$$

then the second term of (28) is equal to

$$E[f;M_{n,z,e}^{\mathrm{ge},r}|(\omega,r)] + E[1_{\{1\leq\eta_y\leq k-1\}}|(\omega,r)]E[f|(\omega,r)].$$

We conclude that

$$\begin{aligned}
&E[f|(\omega,r+1)] - E[f|(\omega,r)] \\
&= \operatorname*{Av}_{y\in\Lambda_{n,z,e}} \frac{1}{E[c_{\mathrm{ge}}|\eta_{\Lambda_n\setminus\{z\}} = \omega]} E[c_{\mathrm{ge}}(\eta_y)(f^{y\to z} - f)|(\omega,r)] \qquad (29) \\
&\quad - \frac{1}{E[1_{\{1\leq\eta_{x+e}\leq k-1\}}|(\omega,r+1)]} E[f;\operatorname*{Av}_{y\in\Lambda_{n,z,e}} 1_{\{1\leq\eta_y\leq k-1\}}|(\omega,r+1)] \\
&\quad + E[f;M_{n,z,e}^{\mathrm{ge},r}|(\omega,r)].
\end{aligned}$$

We return to prove the main statement. By (23),(27),(29) and the Schwarz inequality,

$$\begin{aligned}
&E[(\mathcal{L}_{z,z+e}E[f|\mathcal{F}_{\Lambda_n}])^2] \\
&\leq CP(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k} P(\omega)P(\zeta_z = r) \\
&\quad \times \Big\{ (\operatorname*{Av}_{y\in\Lambda_{n,z,e}} \frac{1}{P[\xi_{z+e}=0|(\omega,r)]} E[(1-\xi_y)(f-f^{(z,y)})|(\omega,r)])^2 \\
&\quad + (E[f;M_{n,z,e}^{\mathrm{ex},r}|(\omega,0)])^2 \\
&\quad + (\frac{1}{E[(1-\xi_{z+e})|(\omega,r)]} E[f;\operatorname*{Av}_{y\in\Lambda_{n,z,e}}(1-\xi_y)|(\omega,r)])^2 \Big\} \\
&\quad + CP(1\leq\eta_{z+e}\leq k-1)\sum_{\omega}\sum_{r=1}^{k-1} P(\omega)P(\zeta_z = r+1) \\
&\quad \times \Big\{ (\operatorname*{Av}_{y\in\Lambda_{n,z,e}} \frac{1}{E[c_{\mathrm{ge}}|\eta_{\Lambda_n\setminus\{z\}}=\omega]} E[c_{\mathrm{ge}}(\eta_y)(f^{y\to z}-f)|(\omega,r)])^2 \\
&\quad + (\frac{1}{E[1_{\{1\leq\eta_{x+e}\leq k-1\}}|(\omega,r+1)]} E[f;\operatorname*{Av}_{y\in\Lambda_{n,z,e}} 1_{\{1\leq\eta_y\leq k-1\}}|(\omega,r+1)])^2 \\
&\quad + (E[f;M_{n,z,e}^{\mathrm{ge},r}|(\omega,r)])^2 \Big\}.
\end{aligned}$$

Using the Schwarz inequality for the first term, we have

$$
\begin{aligned}
& P(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k}P(\omega)P(\zeta_z=r) \\
& (\operatorname*{Av}_{y\in\Lambda_{n,z,e}}\frac{1}{P[\xi_{z+e}=0|(\omega,r)]}E[(1-\xi_y)(f-f^{(z,y)})|(\omega,r)])^2 \\
\leq\ & P(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k}P(\omega)P(\zeta_z=r) \\
& \operatorname*{Av}_{y\in\Lambda_{n,z,e}}(\frac{1}{P[\xi_{z+e}=0|(\omega,r)]}E[(1-\xi_y)(f-f^{(z,y)})|(\omega,r)])^2 \\
\leq\ & P(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k}P(\omega)P(\zeta_z=r)(\frac{1}{P[\xi_{z+e}=0|(\omega,r)]})^2 \\
& \times\operatorname*{Av}_{y\in\Lambda_{n,z,e}}E[(1-\xi_y)(f-f^{(z,y)})^2|(\omega,r)]E[(1-\xi_y)|(\omega,r)] \\
=\ & \operatorname*{Av}_{y\in\Lambda_{n,z,e}}E[(1-\xi_y)(f-f^{(z,y)})^2].
\end{aligned}
$$

Using the same argument and reversibility (6) for the fourth term, we have

$$
\begin{aligned}
& P(1\leq\eta_{z+e}\leq k-1)\sum_{\omega}\sum_{r=1}^{k-1}P(\omega)P(\zeta_z=r+1) \\
& (\operatorname*{Av}_{y\in\Lambda_{n,z,e}}\frac{1}{E[c_{\mathrm{ge}}|\eta_{\Lambda_n\setminus\{z\}}=\omega]}E[c_{\mathrm{ge}}(\eta_y)(f^{y\to z}-f)|(\omega,r)])^2 \\
& \leq CE[\operatorname*{Av}_{y\in\Lambda_{n,z,e}}c_{\mathrm{ge}}(\eta_y)(f^{y\to z}-f)^2].
\end{aligned}
$$

For the second term, since P is product measure, for any fixed ω we have

$$
V[M^{\mathrm{ex},r}_{n,z,e}|(\omega,0)]=\frac{1}{|\Lambda_n|}\frac{1-P(\eta_x=r)}{P(\eta_x=r)}.
$$

Using this equality and the Schwarz inequality, we have

$$
\begin{aligned}
& CP(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k}P(\omega)P(\zeta_z=r)(E[f;M^{\mathrm{ex},r}_{n,z,e}|(\omega,0)])^2 \\
\leq\ & CP(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k}P(\omega)P(\zeta_z=r)V[f|(\omega,0)]V[M^{\mathrm{ex},r}_{n,z,e}|(\omega,0)] \\
\leq\ & C\frac{1}{|\Lambda_n|}P(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k}P(\omega)V[f|(\omega,0)]
\end{aligned}
$$

$$\leq \quad C\frac{1}{|\Lambda_n|}kV[f].$$

For the last term, using reversibility (6), for any fixed ω and r we have

$$V[M^{\mathrm{ge},r}_{n,z,e}|(\omega,r)] = \frac{1}{|\Lambda_n|}\frac{V[c_{\mathrm{ge}}]}{E[c_{\mathrm{ge}}]}P(1\leq \eta_x \leq k-1).$$

From this equality and the Schwarz inequality, we have

$$\begin{aligned}
&CP(1\leq \eta_{z+e}\leq k-1)\sum_{\omega}\sum_{r=1}^{k-1}P(\omega)P(\zeta_z=r+1)\\
&\quad\times (E[f;M^{\mathrm{ge},r}_{n,z,e}|(\omega,r)])^2\\
&\leq C\frac{1}{|\Lambda_n|}V[f]V[c_{\mathrm{ge}}].
\end{aligned}$$

For the third term, it is easy to check that for any ω and r,

$$V[\operatorname*{Av}_{y\in\Lambda_{n,z,e}}(1-\xi_y)|(\omega,r)] = \frac{1}{|\Lambda_n|}P(\xi_y=0)(1-P(\xi_y=0)).$$

Then we have

$$\begin{aligned}
&CP(\xi_{z+e}=0)\sum_{\omega}\sum_{r=1}^{k}P(\omega)P(\zeta_z=r)\Big(\frac{1}{E[(1-\xi_{z+e})|(\omega,r)]}\\
&\qquad E[f;\operatorname*{Av}_{y\in\Lambda_{n,z,e}}(1-\xi_y)|(\omega,r)]\Big)^2\\
&\leq \quad CP(\xi_{z+e}=0)\Big(\frac{1}{E[(1-\xi_{z+e})|(\omega,r)]}\Big)^2\\
&\qquad \sum_{\omega}\sum_{r=1}^{k}P(\omega)P(\zeta_z=r)V[f|(\omega,r)]V[\operatorname*{Av}_{y\in\Lambda_{n,z,e}}(1-\xi_y)|(\omega,r)]\\
&\leq \quad C\frac{1}{|\Lambda_n|}V[f].
\end{aligned}$$

For the fifth term we can easily check that

$$\begin{aligned}
&CP(1\leq \eta_{z+e}\leq k-1)\sum_{\omega}\sum_{r=1}^{k-1}P(\omega)P(\zeta_z=r+1)\\
&\qquad (\frac{1}{E[1_{\{1\leq \eta_{x+e}\leq k-1\}}|(\omega,r+1)]}E[f;\operatorname*{Av}_{y\in\Lambda_{n,z,e}}1_{\{1\leq\eta_y\leq k-1\}}|(\omega,r+1)])^2\\
&\leq \quad C\frac{1}{|\Lambda_n|}V[f].
\end{aligned}$$

From these bounds we conclude that

$$
\begin{aligned}
&E[(\mathcal{L}_{z,z+e}E[f|\mathcal{F}_{\Lambda_n}])^2] \\
&\le \frac{C_1}{|\Lambda_n|}V[f] + C_2 \operatorname*{Av}_{y\in\Lambda_{n,z,e}} E[(1+c_{\text{ge}}(\eta_y))(\mathcal{L}_{y,z}f)^2].
\end{aligned}
$$

□

On using Lemma 4.1 and methods of [7], we can easily obtain the two blocks estimate.

Lemma 4.4 *(Two blocks Estimate) For any bounded continuous function h and any $\gamma > 0$,*

$$
\begin{aligned}
\overline{\lim_{l,a,N}} \sup \operatorname{spec}\Big\{ \operatorname*{Av}_{x\in B_N} \operatorname*{Av}_{y:|y-x|\le aN} [h(\bar\eta_{x,l}) - h(\bar\eta_{y,l})]^2 - \gamma N^2 \operatorname*{Av}_b(-L_b)\Big\} &= 0, \\
\overline{\lim_{k,a,N}} \sup \operatorname{spec}\Big\{ \operatorname*{Av}_{x\in B_N} \operatorname*{Av}_{y:|y-x|\le aN} [h(\bar\xi_{x,l}) - h(\bar\xi_{y,l})]^2 - \gamma N^2 \operatorname*{Av}_b(-L_b)\Big\} &= 0,
\end{aligned}
$$

where $\overline{\lim}_{l,a,N} = \overline{\lim}_{l\to\infty} \overline{\lim}_{a\to 0} \overline{\lim}_{N\to\infty}$ in this order. In particular,

$$
\begin{aligned}
\overline{\lim_{l,a,N}} \sup \operatorname{spec}\Big\{ \operatorname*{Av}_{x\in B_N} [h(\bar\eta_{x,aN}) - h(\bar\eta_{x,l})]^2 - \gamma N^2 \operatorname*{Av}_b(-L_b)\Big\} &= 0, \\
\overline{\lim_{l,a,N}} \sup \operatorname{spec}\Big\{ \operatorname*{Av}_{x\in B_N} [h(\bar\xi_{x,aN}) - h(\bar\xi_{x,l})]^2 - \gamma N^2 \operatorname*{Av}_b(-L_b)\Big\} &= 0.
\end{aligned}
$$

Proof. We only prove the relation for energy part. From Lemma 4.1, we have operator sense inequality

$$
-L_{w,z} \le C|w-z| \sum_{b\in\gamma(w,z)} (-L_b).
$$

Taking the average, we have operator sense inequality

$$
\operatorname*{Av}_{x\in B_N} \operatorname*{Av}_{y:2k<|y-x|<aN} \operatorname*{Av}_{w,z\in\Lambda_{x,l}\cup\Lambda_{y,l}} (-L_{w,z}) \le CN^2a^2 \operatorname*{Av}_b(-L_b).
$$

Then we have only to estimate the largest eigenvalue of

$$
\operatorname*{Av}_{x\in B_N} \operatorname*{Av}_{y:2l<|y-x|<aN} \sup \operatorname{spec}\Big\{[h(\bar\eta_{x,l}) - h(\bar\eta_{y,l})]^2 - Ca^{-2} \operatorname*{Av}_{w,z\in\Lambda_{x,l}\cup\Lambda_{y,l}} (-L_{w,z})\Big\}.
$$

Since the operator

$$
Ca^{-2} \operatorname*{Av}_{w,z\in\Lambda_{x,l}\cup\Lambda_{y,k}} (-L_{w,z})
$$

has a spectral gap $C(l)a^{-2}$, and h is a bounded function, on using Lemma 3.7, the largest eigenvalue is not greater than

$$\sup_{x,y:2l<|x-y|<aN} \Big\{E\Big[\{h(\bar\eta_{x,l})-h(\bar\eta_{y,l})\}^2\Big]+C'(l)a^2\Big\}.$$

Since we take limits $a\to 0$, $l\to\infty$ in this order and our measure is product measure, we conclude the proof. □

5 Tightness

The proof of the Lemma 3.2 given below is adapted from Varadhan [6]. It depends on the next theorem found in [5];

Theorem 5.1 *(Garsia Rademic and Rumsey) Let g and Φ be strictly increasing and continuous functions on $[0,\infty)$ and $g(0)=\Phi(0)=0$. Let $T>0$ be given. Let ϕ be continuous function on $[0,T]$. Assume that there exists $B<\infty$ such that*

$$\int_0^T\int_0^T \Phi\Big(\frac{|\phi(t)-\phi(s)|}{g(|t-s|)}\Big)dsdt \le B. \tag{30}$$

Then for any $0\le s<t\le T$, we have

$$|\phi(t)-\phi(s)|\le 8\int_0^{|t-s|}\Phi^{-1}\Big(\frac{4B}{u^2}\Big)g(du).$$

We will give a proof of Theorem 5.1 in Section 10.

Lemma 5.2 *For $0<\delta<1$, $\beta>0$, $T\ge\delta/2$, and any continuous function ϕ on $[0,T]$, we have*

$$\exp\Big[\frac{\beta}{\sqrt{\delta}}\sup_{t,s\in[0,T],|t-s|\le\delta}|\phi(t)-\phi(s)|\Big]\le\frac{4e^4}{\delta^2}\int_0^T\int_0^T\exp\Big[\frac{8\beta|\phi(t)-\phi(s)|}{\sqrt{|t-s|}}\Big]dtds.$$

Proof. We will apply Theorem 5.1 where we take

$$\Phi(u):=\exp[8u]-1,$$
$$g(u):=\frac{\sqrt{u}}{\beta}.$$

Using integration by parts, we have

$$
\begin{aligned}
|\phi(t)-\phi(s)| &\le \int_0^{|t-s|} \log\Big(1+\frac{4B}{u^2}\Big) g(du) \\
&= \log\Big(1+\frac{4B}{|t-s|^2}\Big)\frac{|t-s|}{\beta} + \int_0^{|t-s|} \Big(\frac{\frac{8B}{u^3}}{1+\frac{4B}{u^2}}\Big)\frac{\sqrt{u}}{\beta} du.
\end{aligned}
$$

Since the last term is not greater than $4\sqrt{|t-s|}/\beta$, we have

$$
\sup_{t,s\in[0,T],|t-s|\le\delta} |\phi(t)-\phi(s)| \le \frac{\sqrt{\delta}}{\beta}\Big\{\log\Big(1+\frac{4B}{\delta^2}\Big)+4\Big\}.
$$

From the definition of B, we have

$$
\exp\Big[\frac{\beta}{\sqrt{\delta}}\sup|\phi(t)-\phi(s)|\Big] \le e^4\Big[1+\frac{4}{\delta^2}\int_0^T\int_0^T \Big\{\exp\Big(\frac{8\beta|\phi(t)-\phi(s)|}{\sqrt{|t-s|}}\Big)-1\Big\} dtds\Big].
$$

Since $1-4T/\delta^2<0$, we have

$$
\exp\Big[\frac{\beta}{\sqrt{\delta}}\sup|\phi(t)-\phi(s)|\Big] \le \frac{4e^4}{\delta^2}\int_0^T\int_0^T \exp\Big(\frac{8\beta|\phi(t)-\phi(s)|}{\sqrt{|t-s|}}\Big) dtds.
$$

□

Proof of Lemma 3.2. Let us consider the continuous process

$$
Y_t^N := \int_0^t N^2 L_{B_N} \operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x^N(s) ds,
$$

depending on J, then we have

$$
\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x^N(t) - \operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x^N(0) = Y_t^N + M_t^N
$$

where M_t^N is a martingale. We have already proved that $E[(M_T^N)^2] \le \frac{1}{N^d}C(J)T$. By Doob's inequality, we see that for the proof of Lemma 3.2 it suffices to show that $P(\sup_{|t-s|\le\alpha}|Y_t^N - Y_s^N| > \delta) \to 0$ as $N\to\infty$ and $\alpha\to 0$ in this order. From Chebyshev's inequality and the last part of the proof of Lemma 3.5, we have only to show that

$$
\overline{\lim_{\delta\to 0}}\ \overline{\lim_{N\to\infty}} \frac{1}{\gamma|B_N|}\log E\exp\Big[\gamma|B_N|\sup_{|t-s|\le\alpha}|Y_t^N - Y_s^N|\Big] \le 0,
$$

for any γ, where E is expectation with respect to the probability law of the Markov process starting with an equilibrium distribution. By Lemma 5.2, the left hand side of last inequality is not greater than

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{N\to\infty}} \frac{1}{\gamma|B_N|}\log\frac{4e^4}{\delta^2}\int_0^T\int_0^T E\exp\Big[\gamma|B_N|\frac{\sqrt{\delta}|Y_t^N - Y_s^N|}{\sqrt{|t-s|}}\Big]dtds. \tag{31}$$

By stationarity and from definition of Y, we have

$$\begin{aligned}
&E\exp\Big[\gamma|B_N|\frac{\sqrt{\delta}|Y_t^N - Y_s^N|}{\sqrt{|t-s|}}\Big]\\
&= E\exp\Big[\gamma|B_N|\frac{\sqrt{\delta}|Y_{|t-s|}^N|}{\sqrt{|t-s|}}\Big]\\
&\le E\exp\Big[\gamma|B_N|\frac{\sqrt{\delta}Y_{|t-s|}^N}{\sqrt{|t-s|}}\Big] + E\exp\Big[-\gamma|B_N|\frac{\sqrt{\delta}-Y_{|t-s|}^N}{\sqrt{|t-s|}}\Big].
\end{aligned}$$

Using Lemma 10.1 of Section 10, we have

$$\begin{aligned}
&E\exp\Big[\gamma|B_N|\frac{\sqrt{\delta}Y_{|t-s|}^N}{\sqrt{|t-s|}}\Big]\\
&\le \exp\Big[|t-s|\sup\mathrm{spec}\Big\{\frac{\gamma|B_N|\sqrt{\delta}}{\sqrt{|t-s|}}N^2 L_{B_N}\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x^N - N^2(-L_{B_N})\Big\}\Big].
\end{aligned}$$

Since L_{B_N} is a negative operator, we have

$$\begin{aligned}
&\sup\mathrm{spec}\Big\{\frac{\gamma|B_N|\sqrt{\delta}}{\sqrt{|t-s|}}N^2 L_{B_N}\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x^N - N^2(-L_{B_N})\Big\}\\
&= \sup_{f\ge 0, Ef^2=1} E\Big[\frac{\gamma|B_N|\sqrt{\delta}}{\sqrt{|t-s|}}N^2 L_{B_N}\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x^N f^2 - N^2 f(-L_{B_N})f\Big]\\
&\le \sup_{f\ge 0, Ef^2=1} E\Big[\frac{\gamma|B_N|\sqrt{\delta}}{\sqrt{|t-s|}}N^2 L_{B_N}\operatorname*{Av}_{x\in B_N} J(\frac{x}{N})\eta_x^N f^2\Big].
\end{aligned}$$

On using symmetry of L, the Schwarz inequality and simple computation, the last term is not greater than

$$\frac{C_J\gamma\sqrt{\delta}}{\sqrt{|t-s|}}|B_N|$$

for some constant C_J depending only on J. For the minus term we have the same estimate. In view of these inequality (31) is not greater than zero. $\square$

6 Energy estimate

In this section we will prove Lemma 3.3. In this paper, we only show the energy estimate for ρ^E, since we can prove that for ρ^P in the same way. We have only to show that, for some finite C,

$$E^Q\Big[\sup_J \{ \int_0^T \int (\frac{\partial}{\partial\theta_e} J)(t,\theta)\rho^E(t,\theta)dtd\theta - \frac{C}{2}\int_0^T \int |J(t,\theta)|^2 dtd\theta\}\Big] \leq C$$

where supremum is taken over all smooth functions. At first we ignore the supremum. Since we have the entropy bound we have only to show that

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log E^{P_N^{\mathrm{eq}}}\Big[\exp\{N\int_0^T \sum_x J(t,\frac{x}{N})\tau_x \nabla_e \eta(t)\}\Big] \tag{32}$$
$$\leq \frac{C}{2}\int_0^T \int |J(t,\theta)|^2 dtd\theta,$$

where P_N^{eq} is the probability law of the Markov process starting with an equilibrium distribution. Using Lemma 3.5, we can estimate the left hand side of (32) by the largest eigenvalue. If we denote

$$\lambda_N(t) := \sup\mathrm{spec}[N\sum_x J(t,\frac{x}{N})\tau_x\nabla_e\eta - N^2\sum_b(-L_b)],$$

it is sufficient to prove

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d}\int_0^T \lambda_N(t)dt \leq \frac{C}{2}\int_0^T\int |J(t,\theta)|^2 dtd\theta.$$

Using operator sense inequality $(-L_b) \geq 0$ for any b, we have

$$\frac{1}{N^d}\lambda_N(t) \leq \frac{1}{N^d}\sum_x \sup\mathrm{spec}\Big[NJ(t,\frac{x}{N})\tau_x\nabla_e\eta - N^2\sum_{b\in\Lambda_{x,l}}(-L_b)\Big]$$

for some fixed $l > 0$. Since the spectral gap for $\sum_{b\in\Lambda_{x,l}}(-L_b)$ is greater than $C = C(l) > 0$ and $||\nabla_e\eta||_\infty \leq k$, there exists $C = C(l,k)$ by Lemma 3.7 such that

$$\sup\mathrm{spec}\Big[NJ(t,\frac{x}{N})\tau_x\nabla_e\eta - N^2\sum_{b\in\Lambda_{x,l}}(-L_b)\Big] \leq \frac{1}{C}|J(t,\frac{x}{N})|^2 + 4\frac{|J(t,\frac{x}{N})|^3}{C^2N}.$$

This shows that

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d}\int_0^T \lambda_N(t)dt \le \frac{C}{2}\int_0^T\int |J(t,\theta)|^2 dtd\theta.$$

We have proved

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log E^{P_N^{\mathrm{eq}}}[\exp[N^d F_\alpha]] \le 0,$$

but to conclude the proof, we have to prove

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log E^{P_N^{\mathrm{eq}}}[\sup_\alpha \exp[N^d F_\alpha]] \le 0.$$

It is obvious that if $G_\alpha := \exp[N^d F_\alpha]$, then

$$\sup_\alpha G_\alpha \le \sum_\alpha G_\alpha.$$

Hence

$$E[\sup G_\alpha] \le \#\{\alpha\} \sup_\alpha E[G_\alpha],$$

then take a logarithm, multiply N^{-d} and let $N \to \infty$ for each side. If $\#\{\alpha\}$ is finite, the right hand side is not greater than 0 by our assumption. Hence taking the increasing family of α and using the monotone convergence theorem, we can obtain the required estimate.

7 Eigenvalue estimate

We recall some notations. We have set $\Lambda_{x,l} := \{y \in \mathbf{Z}^d : |x-y| = l\}$,

$$\begin{aligned}\mathcal{G} \;\; := \;\; & \{h : h \text{ is a local function and satisfies } E_{\Lambda_n,y,L}[h] = 0, \\ & \text{for some } n \text{ and for any } y, L\},\end{aligned}$$

and

$$V^{(l)}(h;y,E) := E_{\Lambda_l,y,E}[\operatorname*{Av}_{x\in\Lambda_{l_1}} \tau_x h(-L^{(l)})^{-1} \operatorname*{Av}_{x\in\Lambda_{l_1}} \tau_x h],$$

for any local function $h \in \mathcal{G}$ where $l_1 = l - \sqrt{l}$, $L^{(l)} = \mathrm{Av}_{b\in\Lambda_l} L_b$.
For the proof of Theorem 3.10, we use following two lemmas.

Lemma 7.1 *For any $\gamma > 0$ for any $h \in \mathcal{G}$, we have*

$$\overline{\lim_{l\to\infty}}\;\overline{\lim_{N\to\infty}} \sup \mathrm{spec}[N \operatorname*{Av}_{x\in\Lambda_{l_1}} \tau_x h - \gamma N^2 \operatorname*{Av}_{b\in\Lambda_l}(-L_b) - \frac{1}{\gamma}V^{(l)}(h;\cdot,\cdot)] \le 0.$$

Proof. From Proposition 3.1, the spectral gap of $N^2 \operatorname{Av}_{b\in\Lambda_l}(-L_b)$ is of order $N^2 l^{-2-d}$. Using Lemma 3.7, we have

$$\sup\operatorname{spec}[N \operatorname*{Av}_{x\in\Lambda_{l_1}} h - \gamma N^2 \operatorname*{Av}_{b\in\Lambda_l}(-L_b) - \frac{1}{\gamma}V^{(l)}(h)] \le O(\frac{l^{4+2d}}{N}).$$

□

Let us define

$$\mathcal{K}_{x,l} = \sigma\text{-algebra generated by } \{\bar\eta_{x,l}, \bar\xi_{x,l}\} \cup \{\eta_y : y \notin \Lambda_{x,l}\}.$$

Note that if $l < m$ and suppose f is $\mathcal{K}_{x,m}$ measurable function, then f is automatically $\mathcal{K}_{x,l}$ measurable function. For any $h \in \mathcal{G}$, let us define $s = s_h$ the range of the function h by

$$s_h := \min\{s \in \mathbf{N} : E_{\Lambda_s,y,E}[h] = 0, \text{ for any } y, E\}. \tag{33}$$

Lemma 7.2 *Let $h \in \mathcal{G}$ and G be a bounded and $\mathcal{K}_{0,s}$ measurable function where $s = s_h$. Then there exists a constant C_h depending only on h such that*

$$\sup\operatorname{spec}[NhG - \frac{C_h}{\gamma}G^2 - \gamma N^2 \operatorname*{Av}_{b\in\Lambda_s}(-L_b)] \le 0,$$

for large enough N.

Proof. From the definition, we have

$$\begin{aligned}
&\sup\operatorname{spec}[NhG - \frac{C_h}{\gamma}G^2 - \gamma N^2 \operatorname*{Av}_{b\in\Lambda_s}(-L_b)] \\
&= \sup_{\bar E_{\Lambda_n}[f^2]=1} \bar E_{\Lambda_n}[NhGf^2 - \frac{C_h}{\gamma}G^2 f^2 - \gamma N^2 f \operatorname*{Av}_{b\in\Lambda_s}(-L_b) f] \\
&= \sup_{\bar E_{\Lambda_n}[f^2]=1} \bar E_{\Lambda_n}\Big[\bar E_{\Lambda_n}[NhGf^2 - \frac{C_h}{\gamma}G^2 f^2 - \gamma N^2 f \operatorname*{Av}_{b\in\Lambda_s}(-L_b) f | \mathcal{F}_{\Lambda_s}]\Big].
\end{aligned}$$

Since G is $\mathcal{K}_{0,s}$ measurable, by Lemma 3.7 the last line is not greater than

$$\sup_{\bar E_{\Lambda_n}[f^2]=1} \bar E_{\Lambda_n}\Big[\bar E_{\Lambda_n}[f^2|\mathcal{F}_{\Lambda_s}]\Big(G^2\Big(\bar E_{\Lambda_n}[h \operatorname*{Av}_{b\in\Lambda_s}(-L_b)h|\mathcal{F}_{\Lambda_s}]] + O(\frac{s^{4+2d}}{N})\Big) - \frac{C_h}{\gamma}G^2\Big)\Big].$$

Since spectral gap of $(\operatorname{Av}_{b\in\Lambda_s}(-L_b))$ is of order s^{-2-d} and s is defined by (33), we have

$$\sup_{y,E} E_{\Lambda_s,y,E}[h(\operatorname*{Av}_{b\in\Lambda_s}(-L_b))^{-1}h] < \infty.$$

Hence we can choose C_h depending only on h such that

$$\sup \operatorname{spec}[NhG - \frac{C_h}{\gamma} G^2 - \gamma N^2 \operatorname*{Av}_{b \in \Lambda_s} (-L_b)] \leq 0,$$

for large enough N. □

Proof of Theorem 3.10. Recall that we have to prove following:

$$\begin{aligned}
&\overline{\lim_{l\to\infty}}\, \overline{\lim_{a\to 0}}\, \overline{\lim_{c\to 0}}\, \overline{\lim_{N\to\infty}} \sup \operatorname{spec}\Big[\pm N \operatorname*{Av}_{x\in B_N} \sum_{e>0} \Big(J_e^E(\frac{x}{N}) \phi_e^E(g)(x) \\
&\quad + J_e^P(\frac{x}{N}) \phi_e^P(g)(x) \Big) - \gamma N^2 \operatorname*{Av}_b (-L_b) \Big] \qquad (34) \\
&\leq C(\gamma) ||J||_\infty^2 \sup_\alpha \sup_{p,\rho} V\Big(\sum_e \Phi_e^\alpha(g); p, \rho \Big).
\end{aligned}$$

Let us define $\Phi_1^J(g,x)$ and $\Phi_i^J(x)$ $(i = 2,3)$ by

$$\begin{aligned}
\Phi_{e,1}^J(g,x) &= (J_e^E(\frac{x}{N}), J_e^P(\frac{x}{N})) \begin{pmatrix} \tau_x(w_e^E + Lg_e^E) \\ \tau_x(w_e^E + Lg_e^P) \end{pmatrix} \\
&\quad + (\alpha_e^E, \alpha_e^P) D(\bar\eta_{x,l}, \bar\xi_{x,l}) \begin{pmatrix} \tau_x \nabla_e \eta \\ \tau_x \nabla_e \xi \end{pmatrix}, \\
\Phi_{e,2}^J(x) &= (J_e^E(\frac{x}{N}), J_e^P(\frac{x}{N}))[-D(\bar\eta_{x,l}, \bar\xi_{x,l}) + D(\bar\eta_{x,aN}, \bar\xi_{x,aN})] \begin{pmatrix} \tau_x \nabla_e \eta \\ \tau_x \nabla_e \xi \end{pmatrix}, \\
\Phi_{e,3}^J(x) &= (J_e^E(\frac{x}{N}), J_e^P(\frac{x}{N})) D(\bar\eta_{x,aN}) \begin{pmatrix} \tau_x(-\nabla_e \eta + \Psi_e^E) \\ \tau_x(-\nabla_e \xi + \Psi_e^P) \end{pmatrix}.
\end{aligned}$$

Consider the largest eigenvalues Ω_i, $(i = 1,2,3)$ represented as

$$\begin{aligned}
\Omega_1 &= \sup \operatorname{spec}[N \operatorname*{Av}_{x\in B_N} \sum_{e>0} \Phi_{e,1}^J(g,x) - \frac{\gamma}{3} N^2 \operatorname*{Av}_{b\in B_N} (-L_b)], \\
\Omega_2 &= \sup \operatorname{spec}[N \operatorname*{Av}_{x\in B_N} \sum_{e>0} \Phi_{e,2}^J(x) - \frac{\gamma}{3} N^2 \operatorname*{Av}_{b\in B_N} (-L_b)], \\
\Omega_3 &= \sup \operatorname{spec}[N \operatorname*{Av}_{x\in B_N} \sum_{e>0} \Phi_{e,3}^J(x) - \frac{\gamma}{3} N^2 \operatorname*{Av}_{b\in B_N} (-L_b)].
\end{aligned}$$

Inequality (34) is reduced to showing

$$\begin{aligned}
\overline{\lim_{l\to\infty}}\, \overline{\lim_{a\to 0}}\, \overline{\lim_{c\to 0}}\, \overline{\lim_{N\to\infty}}\, \Omega_1 &\leq C(\gamma) ||J||_\infty^2 \sup_\alpha \sup_{p,\rho} V\Big(\sum_e \Phi_{e,1}^\alpha(g,x); p, \rho \Big), \qquad (35) \\
\overline{\lim_{l\to\infty}}\, \overline{\lim_{a\to 0}}\, \overline{\lim_{c\to 0}}\, \overline{\lim_{N\to\infty}}\, \Omega_2 &\leq 0, \qquad (36) \\
\overline{\lim_{l\to\infty}}\, \overline{\lim_{a\to 0}}\, \overline{\lim_{c\to 0}}\, \overline{\lim_{N\to\infty}}\, \Omega_3 &\leq 0, \qquad (37)
\end{aligned}$$

where α varies over constant unit vectors.
To estimate Ω_1, from the property of spectrum and shift invariance of the measure, we have

$$\begin{aligned}\Omega_1 \quad \leq \quad & ||J||^2 \sup_{\alpha} \Big\{ \sup \operatorname{spec}[\operatorname*{Av}_{x\in\Lambda_{l_1}} N\Phi^{\alpha}_{e,1}(g,x) - \frac{\gamma}{6}N^2 \operatorname*{Av}_{b\in\Lambda_l}(-L_b) \\ & \qquad - \frac{6}{\gamma}V^{(l)}(\Phi^{\alpha}_{e,1}(g,x);\cdot,\cdot)] \\ & + \sup \operatorname{spec}[\operatorname*{Av}_{x\in B_N} \frac{6}{\gamma}V^{(l)}(\Phi^{\alpha}_{e,1}(g,x);\cdot,\cdot) - \frac{\gamma}{6}N^2 \operatorname*{Av}_{b\in B_N}(-L_b)]\Big\}.\end{aligned}$$

By Lemma 7.1 the first term in the braces is not positive. For $k \geq l$, let

$$U^l_{k,y,E} := E_{\Lambda_k,y,E}[\frac{6}{\gamma}V^{(l)}(\Phi^{\alpha}_{e,1}(g,x);\cdot,\cdot)].$$

Then the second term in the braces is not greater than

$$\sup \operatorname{spec}[\operatorname*{Av}_{x\in B_N}(\frac{6}{\gamma}V^{(l)}(\Phi^{\alpha}_{e,1}(g,x);\cdot,\cdot) - U^l_{k,y,E}) - \frac{\gamma}{6}N^2 \operatorname*{Av}_{b\in B_N}(-L_b)] + \sup_{y,E} U^l_{k,y,E}.$$

By Lemma 3.7 the first term is of order $1/N^2$ for each fixed l. For each fixed l, we can take the superior limit of $\sup_{y,E} U^l_{k,y,E}$. At the same time all limit point of $U^l_{k,y,E}$ is given by

$$\bar{E}_{p,\rho}[\frac{6}{\gamma}V^{(l)}(\Phi^{\alpha}_{e,1}(g,x);\cdot,\cdot)]$$

for some p, ρ. Hence we obtain (35).
For Ω_2 applying Lemma 7.2, there exists a constant $C = C(J,\gamma)$ such that

$$\begin{aligned}\Omega_2 \quad \leq \quad & \sup \operatorname{spec}[\operatorname*{Av}_{x\in B_N} \sum_{q,q'\in\{E,P\}} C[-D_{q,q'}(\bar{\eta}_{x,l},\bar{\xi}_{x,l}) + D_{q,q'}(\bar{\eta}_{x,aN},\bar{\xi}_{x,aN})]^2 \\ & -\frac{\gamma}{3}N^2 \operatorname*{Av}_{b\in\Lambda_l}(-L_b)].\end{aligned}$$

Since D is bounded and continuous,using Lemma 4.4 we have (36).
For Ω_3, we claim that

$$\Psi^E_e = \frac{1}{2cN}\sum_{x=-cN+1}^{cN} \tau_{xe}\nabla_e\eta.$$

Hence we can rewrite Ω_3 as

$$\Omega_3 = \sup \operatorname{spec}[N \operatorname*{Av}_{x\in B_N} \sum_{e>0} \tilde{\Phi}^J_{e,3}(x) - \frac{\gamma}{3} N^2 \operatorname*{Av}_{b\in B_N} (-L_b)],$$

where

$$\begin{aligned} \tilde{\Phi}^J_{e,3}(x) &= (J^E_e(\frac{x}{N}), J^P_e(\frac{x}{N}))\{ - D(\bar{\eta}_{x,aN}, \bar{\xi}_{x,aN}) \\ &\quad + \frac{1}{2cN} \sum_{y=-cN+1}^{cN} D(\bar{\eta}_{x+ye,aN}, \bar{\xi}_{x+ye,aN})\} \begin{pmatrix} \nabla_e \eta \\ \nabla_e \xi \end{pmatrix} \end{aligned}$$

Since we assume $a > c$, each component of

$$\{ - D(\bar{\eta}_{x,aN}, \bar{\xi}_{x,aN}) + \frac{1}{2cN} \sum_{y=-cN+1}^{cN} D(\bar{\eta}_{x+ye,aN}, \bar{\xi}_{x+ye,aN})\}$$

is at most $\mathcal{K}_{x,(a-c)N}$ measurable function. Then applying Lemma 7.2, 4.4 we have (37). □

8 Computation of the variance

We recall the definition of the currents,

$$\begin{aligned} w^E_{x,x+e}(\eta) &:= c_{\rm ex}(\eta_x)(1-\xi_{x+e})\eta_x - c_{\rm ex}(\eta_{x+e})(1-\xi_x)\eta_{x+e} \\ &\quad + c_{\rm ge}(\eta_x)1_{\{1\le\eta_{x+e}\le k-1\}} - c_{\rm ge}(\eta_{x+e})1_{\{1\le\eta_x\le k-1\}}, \\ w^P_{x,x+e}(\eta) &:= c_{\rm ex}(\eta_x)(1-\xi_{x+e}) - c_{\rm ex}(\eta_{x+e})(1-\xi_x). \end{aligned}$$

Recall that $l_1 := l - \sqrt{l}$ for large enough l, $h^{(l)} := \mathrm{Av}_{x\in\Lambda_{l_1}} \tau_x h$, $\mathcal{G}$ is a set of local functions such that if $g \in \mathcal{G}$ there exists n such that for any k, E $E_{\Lambda_n,k,E}[g] = 0$, and $L^{(l)} := \mathrm{Av}_{b\in\Lambda_l} L_b$. If $g \in \mathcal{G}$ then

$$V^{(l)}(h; y, E) := E_{\Lambda_l,y,E}[h^{(l)}(-L^{(l)})^{-1}h^{(l)}]$$

is well-defined for large enough l. We have to define $V(h;\cdot)$ by limit of $V^{(l)}$. In general, it is difficult but for some good $g \in \mathcal{G}$, we can take a limit of $V^{(l)}$.

For any local function h and any $g \in \mathcal{G}$, let

$$\langle g, h\rangle_0(p,\rho) := \sum_x \bar{E}_{p,\rho}[g\tau_x h]. \tag{38}$$

For any $g \in \mathcal{G}$, let

$$t_e(g;p,\rho) = \sum_x (e,x)\bar{E}_{p,\rho}[\eta_x g], \tag{39}$$

$$s_e(g;p,\rho) = \sum_x (e,x)\bar{E}_{p,\rho}[\xi_x g]. \tag{40}$$

From the definition of $\mathcal{G}$, these are well-defined.

For large enough l, for any $g, h \in \mathcal{G}$, let

$$V^{(l)}(h,g;y,E) := \frac{1}{4}\{V^{(l)}(h+g;y,E) - V^{(l)}(h-g;y,E)\}. \tag{41}$$

Since $V^{(l)}$ is a nonnegative quadratic form, for special g, h we can take a limit.

If h is written as $h = Lv$ for some local function v, then h is an element of $\mathcal{G}$. If h is written as $h = Lv$ for some local function v, and if $g \in \mathcal{G}$ then we have

$$V^{(l)}(h,g;\cdot,\cdot) = -\frac{(2l+1)^d}{(2l_1+1)^{2d}} E_{\Lambda_l,\cdot,\cdot}[\sum_{x,y\in\Lambda_{l_1}} \tau_x g \tau_y v].$$

So we can take a limit and we have

$$V(h,g;p,\rho) := \lim_{l\to\infty,(p',\rho')\to(p,\rho)} V^{(l)}(h,g;p'\rho') = -\langle g,v\rangle_0(p,\rho). \tag{42}$$

It is easy to check that

$$L_{\Lambda_l}(\sum_{x\in\Lambda_l}(x,e)\xi_x) = \sum_{x\in\Lambda_l}\tau_x w_e^P,$$

$$L_{\Lambda_l}(\sum_{x\in\Lambda_l}(x,e)\eta_x) = \sum_{x\in\Lambda_l}\tau_x w_e^E.$$

Then for any $g \in \mathcal{G}$, we have

$$V(w_e^P,g;p,\rho) = \lim_{l\to\infty,(p',\rho')\to(p,\rho)} V^{(l)}(w_e^P,g;p'\rho') = -s_e(g;p,\rho), \tag{43}$$

$$V(w_e^E,g;p,\rho) = \lim_{l\to\infty,(p',\rho')\to(p,\rho)} V^{(l)}(w_e^E,g;p'\rho') = -t_e(g;p,\rho). \tag{44}$$

Proposition 8.1 *For each fixed p and ρ, we have*

$$\bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)\eta_0]\bar{E}_{p,\rho}[1_{\{1\le\eta_e\le k-1\}}] - \bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)]\bar{E}_{p,\rho}[\eta_0 1_{\{1\le\eta_e\le k-1\}}]$$
$$= \bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)1_{\{1\le\eta_e\le k-1\}}].$$

Proof. Without risk of confusion, we simply write α for $\alpha(p,\rho)$ and Z for $Z_{\alpha(p,\rho)}$. From our definition, we have

$$\begin{aligned}
&\bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)\eta_0] \\
&= \frac{p}{Z}\sum_{l=2}^{k}\frac{\alpha^{l-1}c_{\mathrm{ge}}(l)l}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)} \\
&= \frac{p\alpha}{Z}\sum_{l=1}^{k-1}\frac{\alpha^{l-1}(l+1)}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)} \\
&= \frac{p\alpha}{Z}\{\sum_{l=1}^{k-1}\frac{\alpha^{l-1}l}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)} + \sum_{l=1}^{k-1}\frac{\alpha^{l-1}l}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)}\}.
\end{aligned}$$

For the other parts we can easily compute that

$$\begin{aligned}
\bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)] &= \frac{p\alpha}{Z}\sum_{l=1}^{k-1}\frac{\alpha^{l-1}}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)}, \\
\bar{E}_{p,\rho}[1_{\{1\le\eta_0\le k-1\}}] &= \frac{p}{Z}\sum_{l=1}^{k-1}\frac{\alpha^{l-1}}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)}, \\
\bar{E}_{p,\rho}[\eta_0 1_{\{1\le\eta_0\le k-1\}}] &= \frac{p}{Z}\sum_{l=1}^{k-1}\frac{\alpha^{l-1}l}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)}.
\end{aligned}$$

Hence we conclude that

$$\begin{aligned}
&\bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)\eta_0]\bar{E}_{p,\rho}[1_{\{1\le\eta_e\le k-1\}}] - \bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)]\bar{E}_{p,\rho}[\eta_0 1_{\{1\le\eta_e\le k-1\}}] \\
&= \frac{p\alpha}{Z}\sum_{l=1}^{k-1}\frac{\alpha^{l-1}}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)}\frac{p}{Z}\sum_{l=1}^{k-1}\frac{\alpha^{l-1}}{c_{\mathrm{ge}}(2)\dots c_{\mathrm{ge}}(l)} \\
&= \bar{E}_{p,\rho}[c_{\mathrm{ge}}(\eta_0)1_{\{1\le\eta_e\le k-1\}}].
\end{aligned}$$

□

From the definition of $s_e, t_e, \langle\cdot,\cdot\rangle_0$ and Proposition 8.1 we can easily compute following;

$$\begin{aligned}
s_e(w^P_{e'};p,\rho) &= -\delta_{e,e'}(1-p)\bar{E}_{p,\rho}[c_{\mathrm{ex}}(\eta_0)], \\
t_e(w^P_{e'};p,\rho) &= -\delta_{e,e'}(1-p)\bar{E}_{p,\rho}[c_{\mathrm{ex}}(\eta_0)\eta_0], \\
s_e(w^E_{e'};p,\rho) &= -\delta_{e,e'}(1-p)\bar{E}_{p,\rho}[c_{\mathrm{ex}}(\eta_0)\eta_0], \\
t_e(w^E_{e'};p,\rho) &= -\delta_{e,e'}\{(1-p)\bar{E}_{p,\rho}[c_{\mathrm{ex}}(\eta_0)\eta_0^2]
\end{aligned}$$

$$
\begin{aligned}
& -\bar{E}_{p,\rho}[c_{\rm ge}(\eta_0)]\bar{E}_{p,\rho}[\eta_0 1_{\{1\le\eta_0\le k-1\}}] \\
& +\bar{E}_{p,\rho}[c_{\rm ge}(\eta_0)\eta_0]\bar{E}_{p,\rho}[1_{\{1\le\eta_0\le k-1\}}]\},
\end{aligned}
$$

$$
\begin{aligned}
s_e(Lu;p,\rho) &= -\bar{E}_{p,\rho}[(c_{\rm ex}(\eta_0)(1-\xi_e))(\pi^{(0,e)}(\sum_x \tau_x u))(\xi_0-\xi_e)], \\
t_e(Lu;p,\rho) &= -\bar{E}_{p,\rho}[(c_{\rm ex}(\eta_0)(1-\xi_e))(\pi^{(0,e)}(\sum_x \tau_x u))(\eta_0-\eta_e)] \\
&\quad +\bar{E}_{p,\rho}[c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}(\pi^{0\to e}(\sum_x \tau_x u))], \\
\langle Lu,v\rangle_0(p,\rho) &= -\{\bar{E}_{p,\rho}[(c_{\rm ex}(\eta_0)(1-\xi_e))(\pi^{(0,e)}(\sum_x \tau_x u))(\pi^{(0,e)}(\sum_x \tau_x v))] \\
&\quad +\bar{E}_{p,\rho}[c_{\rm ge}(\eta_0)1_{\{1\le\eta_0\le k-1\}}(\pi^{0\to e}(\sum_x \tau_x u))(\pi^{0\to e}(\sum_x \tau_x v))]\}, \\
s_e(\nabla_{e'}\xi;p,\rho) &= -\delta_{e,e'}p(1-p), \\
t_e(\nabla_{e'}\xi;p,\rho) &= -\delta_{e,e'}(1-p)\bar{E}_{p,\rho}[\eta_0], \\
\langle u,\nabla_{e'}\xi\rangle_0(p,\rho) &= 0, \\
s_e(\nabla_{e'}\eta;p,\rho) &= -\delta_{e,e'}(1-p)\bar{E}_{p,\rho}[\eta_0], \\
t_e(\nabla_{e'}\eta;p,\rho) &= -\delta_{e,e'}(\bar{E}_{p,\rho}[\eta_0^2]-(\bar{E}_{p,\rho}[\eta_0])^2), \\
\langle u,\nabla_e\eta\rangle_0(p,\rho) &= 0.
\end{aligned}
$$

We collect these results to conclude the following lemma:

Lemma 8.2 *For any $\{\alpha\}$, any local function g and any $h\in\mathcal{G}$, we have*

$$
\begin{aligned}
& V\left(\sum_e(\alpha_e^E w_e^E+\alpha_e^P w_e^P)+Lg,h;p,\rho\right) \\
& =-\sum_e\left\{\alpha_e^E t_e(h;p,\rho)+\alpha_e^P s_e(h;p,\rho)\right\}-\langle g,h\rangle_0(p,\rho).
\end{aligned}
$$

In particular, for any $\{\alpha\}$, and any local function g

$$
\begin{aligned}
& V\left(\sum_e(\alpha_e^E w_e^E+\alpha_e^P w_e^P)+Lg,\sum_e(\alpha_e^E w_e^E+\alpha_e^P w_e^P)+Lg;p,\rho\right) \\
& =\sum_e\left\{\bar{E}_{p,\rho}[c_{\rm ex}(\eta_0)(1-\xi_e)(\alpha_e^E\eta_0+\pi^{(0,e)}(\sum_x \tau_x g))^2]\right. \\
& \quad +\bar{E}_{p,\rho}[c_{\rm ex}(\eta_0)(1-\xi_e)(\alpha_e^E 1+\pi^{(0,e)}(\sum_x \tau_x g))^2] \\
& \quad \left.+\bar{E}_{p,\rho}[c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}(\alpha_e^P 1+\pi^{0\to e}(\sum_x \tau_x g))^2]\right\}.
\end{aligned}
$$

Proof of Theorem 3.8 and the diffusion coefficient. Fix densities p and ρ. Let us define three classes of linear space of functions;

$$\begin{aligned}\mathcal{G}^{(0)} &:= \{\nabla_e \eta, \nabla_e \xi\}_e, \\ \mathcal{G}^{w} &:= \{w_e^E, w_e^P\}_e, \\ L\mathcal{G} &:= \{Lg : g \text{ is a local function}\}.\end{aligned}$$

It is obvious that a linear combination of these three elements is an element of $\mathcal{G}$. Let us consider the relation $\sim$ in $\mathcal{G}$ which is defined by

$$h \sim h' \text{ if and only if } V(h - h') = 0.$$

Let us consider the quotient set of $\mathcal{G}$ relative to the relation $\sim$. Since we only consider the quotient set, without risk of confusion we denote the quotient set by the same letter $\mathcal{G}$. We shall prove that

$$\begin{aligned}&V \text{ is an inner product on } \mathcal{G} \\ &\mathcal{G}^w + L\mathcal{G} \text{ is dense in } \bar{\mathcal{G}}\end{aligned} \tag{45}$$

where $\bar{\mathcal{G}}$ is the closure of $\mathcal{G}$ relative to the inner product V. For the moment taking these relations for granted we derive variational representation for the diffusion coefficient matrix D. From the definition of the space and the calculation of the $\langle \cdot, \cdot \rangle_0$, we have $\mathcal{G}^{(0)} \perp \overline{L\mathcal{G}}$. From the calculation of the s_e, t_e, we see that the projection of the space $\mathcal{G}^w$ onto $\mathcal{G}^{(0)}$ has rank $2d$. Since the dimension of the $\mathcal{G}^w$ is $2d$, and we assume that $\mathcal{G}^w + L\mathcal{G}$ is dense in $\bar{\mathcal{G}}$, it follows that

$$\mathcal{G}^{(0)} + L\mathcal{G} \text{ is dense in } \bar{\mathcal{G}}.$$

First we consider the case $d = 1$. Our diffusion coefficient matrix $D = (D_{p,q})_{p,q \in \{E,P\}}$ should satisfy following relation; for any $\varepsilon > 0$, there exist local functions g^E and g^P such that for any unit vector $\{\alpha\}$,

$$V\left((\alpha^E, \alpha^P) \left(\begin{pmatrix} w^E \\ w^P \end{pmatrix} + D \begin{pmatrix} \nabla \eta \\ \nabla \xi \end{pmatrix} - L \begin{pmatrix} g^E \\ g^P \end{pmatrix} \right) ; p, \rho \right) < \varepsilon.$$

On the other hands, since we assume $\mathcal{G}^{(0)} + L\mathcal{G}$ is dense in $\bar{\mathcal{G}}$ and we have $\mathcal{G}^{(0)} \perp \overline{L\mathcal{G}}$, if we project w^E and w^P onto $\mathcal{G}^{(0)}$, then there exists a matrix $\hat{D}$ and some $\zeta^E, \zeta^P \in \overline{L\mathcal{G}}$ such that

$$\begin{pmatrix} w^E \\ w^P \end{pmatrix} = \hat{D} \begin{pmatrix} \nabla \eta \\ \nabla \xi \end{pmatrix} - \begin{pmatrix} \zeta^E \\ \zeta^P \end{pmatrix}. \tag{46}$$

Comparing these two relations, $-\hat{D}$ should be our diffusion coefficient matrix. Hence we have to show that the explicit formula of the $\hat{D}$.

For each element of (46), taking inner product with w^E and w^P, we have

$$\begin{pmatrix} V(w^E+\zeta^E,w^E) & V(w^E+\zeta^E,w^P) \\ V(w^P+\zeta^P,w^E) & V(w^P+\zeta^P,w^P) \end{pmatrix} = \hat{D}\begin{pmatrix} -t_e(\nabla_e\eta) & -s_e(\nabla_e\eta) \\ -t_e(\nabla_e\xi) & -s_e(\nabla_e\xi) \end{pmatrix}.$$

Note that the last matrix is $-\chi$. From our definition of ζ^E and ζ^P, $w^E+\zeta^E$ and $w^P+\zeta^P$ are elements of $\mathcal{G}^{(0)}$, then the left hand side of last equality is equal to

$$\begin{pmatrix} V(w^E+\zeta^E,w^E+\zeta^E) & V(w^E+\zeta^E,w^P+\zeta^P) \\ V(w^P+\zeta^P,w^E+\zeta^E) & V(w^P+\zeta^P,w^P+\zeta^P) \end{pmatrix}.$$

Denote this matrix by $\check{D}$. Since we project w^E and w^P onto $\mathcal{G}^{(0)}$, applying Lemma 8.2, we have a variational formula for $\check{D}$: for ${}^t\alpha=(\alpha^E,\alpha^P)$, we have

$$\begin{aligned}(\alpha,\check{D}\alpha) \;=\; & \inf_{g^E,g^P}\Big\{\bar{E}[c_{\mathrm{ex}}(\eta_0)(1-\xi_e) \\ & \{\alpha^E(\eta_0+\pi^{(0,e)}(\sum_x\tau_x g^E))+\alpha^P(1+\pi^{(0,e)}(\sum_x\tau_x g^P))\}^2] \\ & +\bar{E}_{\rho^P,\rho^E}[c_{\mathrm{ge}}(\eta_0)1_{\{1\le\eta_e\le k-1\}} \\ & \{\alpha^E(1+\pi^{(0\to e)}(\sum_x\tau_x g^E))+\alpha^P(\pi^{0\to e}(\sum_x\tau_x g^P))\}^2]\Big\}.\end{aligned}$$

For $d\ge 2$, the diffusion coefficient matrix $D=(D_{e_i,e_j,p,q})_{1\le i,j\le d,p,q\in\{E,P\}}$ is identified by

$$\begin{pmatrix} w^E_{e_1} \\ w^P_{e_1} \\ \vdots \\ w^E_{e_d} \\ w^P_{e_d}\end{pmatrix} = D\begin{pmatrix} \nabla_{e_1}\eta \\ \nabla_{e_1}\xi \\ \vdots \\ \nabla_{e_d}\eta \\ \nabla_{e_d}\xi\end{pmatrix} - \begin{pmatrix} \zeta^E_{e_1} \\ \zeta^P_{e_1} \\ \vdots \\ \zeta^E_{e_d} \\ \zeta^P_{e_d}\end{pmatrix}.$$

Lemma 8.3 *For $d\ge 2$ if $e\ne e'$ then $D_{e,e',p,q}=0$, for all $p,q\in\{E,P\}$.*

Proof. We only show that $D_{e,e',E,E}=D_{e,e',E,P}=0$. Since we have

$$w^E_e=\sum_{e^*}D_{e,e^*,E,E}\nabla_{e^*}\eta+\sum_{e^*}D_{e,e^*,E,P}\nabla_{e^*}\xi+\zeta^E_e,$$

taking inner product with $\nabla_{e'}\eta$ and $\nabla_{e'}\xi$, we have

$$V(w_e^E, \nabla_{e'}\eta) = \sum_{e^*} D_{e,e^*,E,E} V(\nabla_{e^*}\eta, \nabla_{e'}\eta) + \sum_{e^*} D_{e,e^*,E,P} V(\nabla_{e^*}\xi, \nabla_{e'}\eta),$$
$$V(w_e^E, \nabla_{e'}\xi) = \sum_{e^*} D_{e,e^*,E,E} V(\nabla_{e^*}\eta, \nabla_{e'}\xi) + \sum_{e^*} D_{e,e^*,E,P} V(\nabla_{e^*}\xi, \nabla_{e'}\xi),$$

Since $V(w_e^E, \nabla_{e'}\eta)$ and $V(w_e^E, \nabla_{e'}\xi)$ are zero if $e \neq e'$, we have only to show that if $e^* \neq e'$ then $V(\nabla_{e^*}\eta, \nabla_{e'}\eta)$ and $V(\nabla_{e^*}\xi, \nabla_{e'}\eta)$ are zero. Denote by θ_e the reflection operator with respect to the origin along the e direction. We may extend θ_e to the configuration space naturally by $(\theta_e\eta)_x = \eta_{\theta_e x}$ and $(\theta_e f)(\eta) - f(\theta_e\eta)$. Since our model is symmetric under θ_e, we have $V(f,g) = V(\theta_e f, \theta_e g)$. Since $\theta_e \nabla_{e'}\eta$ is equal to $\nabla_{e'}\eta$ if $e \neq e'$ and $-\nabla_{e'}\eta$ if $e = e'$, we have

$$V(\nabla_e\eta, \nabla_{e'}\eta) = -V(\nabla_e\eta, \nabla_{e'}\eta) = 0,$$

for all $e \neq e'$. In the same way we have

$$V(\nabla_e\eta, \nabla_{e'}\xi) = -V(\nabla_e\eta, \nabla_{e'}\xi) = 0,$$

for all $e \neq e'$. Hence we conclude the proof of the lemma. □

Next step we show the variational formula. Let us define $V^*(h,g)$ by

$$V^*(h, g; p, \rho) := \overline{\lim_{l\to\infty,(p',\rho')\to(p,\rho)}} \bar{E}_{p',\rho'}[V^{(l)}(h; y, E)],$$

for all $h, g \in \mathcal{G}$. From now on we omit the parameter of the grand canonical measure for the notation $\bar{E}, V, V^*$. Let us recall $\mathcal{F}_L$ for $L \subset \mathbf{Z}^d$ as

$$\mathcal{F}_L = \sigma\text{- algebra generated by } \{\eta_x : x \in L\}.$$

Then for any $h \in \mathcal{G}$, there exists $s = s_h$ and $\mathcal{F}_{\Lambda_s}$ measurable local function H such that

$$h = \sum_{b \in \Lambda_s} L_b H.$$

Let us define

$$\Phi_{\mathrm{ex}}^{h,e}(\eta) := -\frac{1}{2} \sum_{x:\tau_x e \in \Lambda_s} \tau_x(\pi^{(0,e)} H) \tag{47}$$
$$\Phi_{\mathrm{ge}}^{h,e}(\eta) := -\frac{1}{2} \sum_{x:\tau_x e \in \Lambda_s} \tau_x(\pi^{0\to e} H).$$

Lemma 8.4 *For any $h \in \mathcal{G}$ we have variational formula;*

$$\begin{aligned} V^*(h,h) \;=\; & \sup_{\{\alpha\},u} 2\bar{E}\Big[\sum_e \{c_{\rm ex}(\eta_0)(1-\xi_e)\Phi^{h,e}_{\rm ex}(\eta)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u) \\ & +c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}\Phi^{h,e}_{\rm ge}(\eta)(\alpha^E_e+\pi^{0\to e}u)\}\Big] \\ & -\frac{1}{2}\bar{E}\Big[\sum_e \{c_{\rm ex}(\eta_0)(1-\xi_e)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u)^2 \\ & +c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}(\alpha^E_e+\pi^{0\to e}u)^2\}\Big]. \end{aligned} \tag{48}$$

where $\Phi^{h,e}_{\rm ex}$ and $\Phi^{h,e}_{\rm ge}$ are defined by (47) and supremum is taken over $2d$ dimensional vectors $\{\alpha\}$ and all local functions u.

Remark. In view of Lemma 8.2 and simple computation of t_e, s_e, we have

$$\begin{aligned} & 2V(h-(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu),(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu);p,\rho) \\ & \quad + V((\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu),(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu);p,\rho) \\ & = 2\bar{E}\Big[\sum_e \{c_{\rm ex}(\eta_0)(1-\xi_e)\Phi^{h,e}_{\rm ex}(\eta)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u) \\ & \quad + c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}\Phi^{h,e}_{\rm ge}(\eta)(\alpha^E_e+\pi^{0\to e}u)\}\Big] \\ & \quad - \frac{1}{2}\bar{E}\Big[\sum_e \{c_{\rm ex}(\eta_0)(1-\xi_e)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u)^2 \\ & \quad + c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}(\alpha^E_e+\pi^{0\to e}u)^2\}\Big] \end{aligned} \tag{49}$$

for all $h \in \mathcal{G}$, all $2d$ dimensional vectors $\{\alpha\}$ and all local functions u, where $\Phi^{h,e}_{\rm ex}$ and $\Phi^{h,e}_{\rm ge}$ are defined by (47).
Remark. Let us define $V_*(h,g)$ by

$$V_*(h,g;p,\rho) := \varliminf_{l\to\infty,(p',\rho')\to(p,\rho)} \bar{E}_{p',\rho'}[V^{(l)}(h,g;y,E)],$$

for all $h,g \in \mathcal{G}$. For the finite case, we have

$$\begin{aligned} & V^{(l)}(h,h;y,E) \\ & = V^{(l)}(h-(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu),h-(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu);y,E) \\ & \quad + 2V^{(l)}(h-(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu),(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu);y,E) \\ & \quad + V^{(l)}((\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu),(\alpha^E_e w^E_e+\alpha^P_e w^P_e+Lu);y,E) \end{aligned}$$

for all $2d$ dimensional vectors $\{\alpha\}$ and all $\mathcal{F}_{\Lambda_{\sqrt{l}}}$ measurable local functions u. Since the first line of the right hand side is not negative, taking the inferior limit of the both sides, we have

$$\begin{aligned}
&V_*(h,h;p,\rho)\\
&\geq 2V_*(h-(\alpha_e^E w_e^E+\alpha_e^P w_e^P+Lu),(\alpha_e^E w_e^E+\alpha_e^P w_e^P+Lu);p,\rho)\\
&\quad + V_*((\alpha_e^E w_e^E+\alpha_e^P w_e^P+Lu),(\alpha_e^E w_e^E+\alpha_e^P w_e^P+Lu);p,\rho)
\end{aligned}$$

for all $2d$ dimensional vectors $\{\alpha\}$ and all local functions u. By (49) we can rewrite the right hand side by

$$\begin{aligned}
&2\bar{E}\Big[\sum_e \{c_{\rm ex}(\eta_0)(1-\xi_e)\Phi_{\rm ex}^{h,e}(\eta)(\alpha_e^E\eta_0+\alpha_e^P+\pi^{(0,e)}u)\\
&+c_{\rm ge}(\eta_0)1_{\{1\leq\eta_e\leq k-1\}}\Phi_{\rm ge}^{h,e}(\eta)(\alpha_e^E+\pi^{0\to e}u)\}\Big]\\
&-\frac{1}{2}\bar{E}\Big[\sum_e \{c_{\rm ex}(\eta_0)(1-\xi_e)(\alpha_e^E\eta_0+\alpha_e^P+\pi^{(0,e)}u)^2\\
&+c_{\rm ge}(\eta_0)1_{\{1\leq\eta_e\leq k-1\}}(\alpha_e^E+\pi^{0\to e}u)^2\}\Big],
\end{aligned}$$

where $\Phi_{\rm ex}^{h,e}$ and $\Phi_{\rm ge}^{h,e}$ are defined by (47). By (48) we have

$$V_*(h,h;p,\rho)\geq V^*(h,h;p,\rho) \tag{50}$$

for all $h\in\mathcal{G}$. Hence there exists a limit

$$V(h;p,\rho):=\lim_{l\to\infty,(p',\rho')\to(p,\rho)}\bar{E}_{p',\rho'}[V^{(l)}(h;y,E)],$$

for all $h\in\mathcal{G}$. Using (49) and (50), we have following corollary of Lemma 8.4.

Corollary 8.5 *For any $h\in\mathcal{G}$ we have variational formula;*

$$V(h,h)=\sup_{\{\alpha\},u}\Big\{2V(h,\alpha_e^E w_e^E+\alpha_e^P w_e^P+Lu)-V(\alpha_e^E w_e^E+\alpha_e^P w_e^P+Lu)\Big\}.$$

where supremum is taken over $2d$ dimensional vectors $\{\alpha\}$ and all local functions u.

Note that the assertion (45) is immediately deduced from Corollary 8.5.
Proof of Lemma 8.4. First we prove

$$V^*(h,h)\ \geq\ \sup_{\{\alpha\},u}2\bar{E}\Big[\sum_e\{c_{\rm ex}(\eta_0)(1-\xi_e)\Phi_{\rm ex}^{h,e}(\eta)(\alpha_e^E\eta_0+\alpha_e^P+\pi^{(0,e)}u)$$

$$+c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}\Phi^{h,e}_{\rm ge}(\eta)(\alpha^E_e+\pi^{0\to e}u)\}\Big] \tag{51}$$
$$-\frac{1}{2}\bar{E}\Big[\sum_e\{c_{\rm ex}(\eta_0)(1-\xi_e)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u)^2$$
$$+c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}(\alpha^E_e+\pi^{0\to e}u)^2\}\Big].$$

From the definition $V^{(l)}$ is a nonnegative quadratic form, hence we have

$$V^{(l)}(h,h;y,E) \tag{52}$$
$$=\sup_v\Big\{2E_{\Lambda_l,y,E}[\operatorname*{Av}_{x\in\Lambda_{l_1}}\tau_x h,v]-\frac{1}{2}E_{\Lambda_l,y,E}[\operatorname*{Av}_{b=(x,y)\in\Lambda_l}\{c_{\rm ex}(\eta_x)(1-\xi_y)(\pi^{(x,y)}v)^2$$
$$+\,c_{\rm ge}(\eta_x)1_{\{1\le\eta_y\le k-1\}}(\pi^{x\to y}v)^2\}]\Big\},$$

for all y,E, where supremum is taken over $\mathcal{F}_{\Lambda_l}$ measurable functions. Take

$$v:=\Big[\sum_{x\in\Lambda_l}\{(\alpha^E,x)\eta_x+(\alpha^p,x)\xi_x\}-2\sum_{x\in\Lambda_{l_1}}\tau_x u\Big]$$

for some d dimensional vectors α^E, α^P and some local function u and take a superior limit of the both sides of (52). Then using Lemma 8.2 we get

$$V^*(h,h)$$
$$\ge\varlimsup_{l\to\infty}\Big\{2E_{\Lambda_l,y,E}[\operatorname*{Av}_{x\in\Lambda_{l_1}}\tau_x h,[\sum_{x\in\Lambda_l}\{(\alpha^E,x)\eta_x+(\alpha^p,x)\xi_x\}-2\sum_{x\in\Lambda_{l_1}}\tau_x u]]$$
$$-\frac{1}{2}E_{\Lambda_l,y,E}[\operatorname*{Av}_{b=(x,y)\in\Lambda_l}\{c_{\rm ex}(\eta_x)(1-\xi_y)$$
$$(\pi^{(x,y)}[\sum_{x\in\Lambda_l}\{(\alpha^E,x)\eta_x+(\alpha^p,x)\xi_x\}-2\sum_{x\in\Lambda_{l_1}}\tau_x u])^2$$
$$+\,c_{\rm ge}(\eta_x)1_{\{1\le\eta_y\le k-1\}}$$
$$(\pi^{x\to y}[\sum_{x\in\Lambda_l}\{(\alpha^E,x)\eta_x+(\alpha^p,x)\xi_x\}-2\sum_{x\in\Lambda_{l_1}}\tau_x u])^2\}]\Big\}$$
$$=\bar{E}\Big[\sum_e\{c_{\rm ex}(\eta_0)(1-\xi_e)\Phi^{h,e}_{\rm ex}(\eta)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u)$$
$$+\,c_{\rm ge}(\eta_0)1_{\{1\le\eta_e\le k-1\}}\Phi^{h,e}_{\rm ge}(\eta)(\alpha^E_e+\pi^{0\to e}u)\}\Big],$$

for all d dimensional vectors α^E, α^P and some local function u. Hence the inequality (51) is proved.

We turn to the proof of

$$
\begin{aligned}
V^*(h,h) \quad \le \quad & \sup_{\{\alpha\},u} 2\bar{E}\Big[\sum_e \{c_{\mathrm{ex}}(\eta_0)(1-\xi_e)\Phi^{h,e}_{\mathrm{ex}}(\eta)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u) \\
& +c_{\mathrm{ge}}(\eta_0)1_{\{1\le\eta_e\le k-1\}}\Phi^{h,e}_{\mathrm{ge}}(\eta)(\alpha^E_e+\pi^{0\to e}u)\}\Big] \qquad (53)\\
& -\frac{1}{2}\bar{E}\Big[\sum_e \{c_{\mathrm{ex}}(\eta_0)(1-\xi_e)(\alpha^E_e\eta_0+\alpha^P_e+\pi^{(0,e)}u)^2 \\
& +c_{\mathrm{ge}}(\eta_0)1_{\{1\le\eta_e\le k-1\}}(\alpha^E_e+\pi^{0\to e}u)^2\}\Big].
\end{aligned}
$$

where $\Phi^{h,e}_{\mathrm{ex}}$ and $\Phi^{h,e}_{\mathrm{ge}}$ are defined by (47) and supremum is taken over $2d$ dimensional vectors $\{\alpha\}$ and all local functions u.

Step 1. Let us define $D_b(f), D_b(f,g), \tilde{D}_b(f;L,y,E), \tilde{D}_b(f,g;L,y,E)$ by

$$
\begin{aligned}
& D_b(f) := D_b(f,f),\\
& D_b(f,g) := -\bar{E}[fL_bg],\\
& \tilde{D}_b(f;L,y,E) := \tilde{D}_b(f,f;L,y,E),\\
& \tilde{D}_b(f,g;L,y,E) := -E_{L,y,E}[fL_bg],
\end{aligned}
$$

for all local function f,g, $L\subset \mathbf{Z}^d$, and integer y,E. Let us define $V^* := \overline{\lim}\,\bar{E}[V^{(l)}(h,h)]$. Suppose that l is large enough so that, for some small $\delta>0$,

$$\bar{E}[V^{(l)}(h,h)] \ge V^* - \delta.$$

For $V^{(l)}$, we have the following variational formula

$$
\begin{aligned}
& V^{(l)}(h,h;y,E)\\
& = \sup_v \Big\{2E_{\Lambda_l,y,E}[\operatorname*{Av}_{x\in\Lambda_{l_1}} \tau_x h, v]\\
& \quad -\frac{1}{2}E_{\Lambda_l,y,E}[\operatorname*{Av}_{b=(x,y)\in\Lambda_l}\{c_{\mathrm{ex}}(\eta_x)(1-\xi_y)(\pi^{(x,y)}v)^2\\
& \quad + c_{\mathrm{ge}}(\eta_x)1_{\{1\le\eta_y\le k-1\}}(\pi^{x\to y}v)^2\}]\Big\},\\
& = \sup_v \Big\{2E_{\Lambda_l,y,E}[\operatorname*{Av}_{x\in\Lambda_{l_1}} \tau_x h, v] - \frac{1}{2}\operatorname*{Av}_{b\in\Lambda_l}\tilde{D}_b(f;\Lambda_l,y,E)\Big\}
\end{aligned}
$$

for all y,E, where supremum is taken over $\mathcal{F}_{\Lambda_l}$ measurable functions. Hence there exists $u=u_l$ such that u is a $\mathcal{F}_{\Lambda_l}$ measurable and satisfies

$$
\begin{aligned}
& V^{(l)}(h,h;y,E)\\
& \le \Big\{2E_{\Lambda_l,y,E}[\operatorname*{Av}_{x\in\Lambda_{l_1}} \tau_x h, u] - \frac{1}{2}\operatorname*{Av}_{b\in\Lambda_l}\tilde{D}_b(f;\Lambda_l,y,E)\Big\} + \delta,
\end{aligned}
$$

for all y, E. We can assume that

$$\bar{E}\Big\{2E_{\Lambda_l,y,E}[\operatorname*{Av}_{x\in\Lambda_{l_1}} \tau_x h, u] - \frac{1}{2}\operatorname*{Av}_{b\in\Lambda_l} \tilde{D}_b(f;\Lambda_l,y,E)\Big\} \geq 0, \tag{54}$$

since if this term is negative, then V^* is 0 and the variational formula holds as $V(h,h)=0$. Since $h\in\mathcal{G}$, there exists s and $\mathcal{F}_{\Lambda_s}$ measurable function H such that $h=\sum_{b\in\Lambda_s} L_b H$. Hence we have

$$\bar{E}[(\tau_x h)u] \leq \frac{1}{2\gamma}\sum_{b+x\in\Lambda_s} D_b(u) + \frac{\gamma}{2}C(h),$$

for any constant $\gamma>0$ and $x\in\Lambda_{l_1}$, where constant $C(h)$ depends only on h. Taking $\gamma = 2(2s+1)^d$, averaging both sides over $x\in\Lambda_{l_1}$ and using (54), we have following bound uniformly in l;

$$\operatorname*{Av}_{b\in\Lambda_l} D_b(u) \leq C_s C(h).$$

Step 2. From the definition of H and D_b, we have

$$\bar{E}[u(\tau_x h)] = \bar{E}[u\tau_x\sum_{b\in\Lambda_s} L_b H] = -\sum_{b+x\in\Lambda_s} D_b(H,u).$$

Averaging the leftmost side and the rightmost side over $x\in\Lambda_{l_1}$, we have

$$\begin{aligned}
&\operatorname*{Av}_{x\in\Lambda_{l_1}} E[(\tau_x h)u]\\
&= -\operatorname*{Av}_{b\in\Lambda_{l_1}} D_b(\sum_{x\in\Lambda_{l_1}} \tau_x H, u)\\
&\quad - \frac{1}{(2l_1+1)^d}\sum_{b\in\Lambda_{l_1+s+1}\setminus\Lambda_{l_1}} D_b(\sum_{x:x+b\in\Lambda_{s+1},x\in\Lambda_{l_1}} \tau_x H, u)\\
&\quad + \frac{1}{(2l_1+1)^d}\sum_{b\in\Lambda_{l_1}\setminus\Lambda_{l_1+s-1}} D_b(\sum_{x:x+b\in\Lambda_{s+1},x\in\Lambda_{l_1}} \tau_x H, u)
\end{aligned}$$

From step 1, we have uniform bound for $D_b(u)$, hence the last two terms tend to 0 as l tends to ∞. Hence we conclude this step by

$$2\operatorname*{Av}_{b\in\Lambda_{l_1}} D_b(\sum_{x\in\Lambda_{l_1}} \tau_x H, u) - \frac{1}{2}\operatorname*{Av}_{b\in\Lambda_l} D_b(u) \geq V^* - 3\delta.$$

Step 3. Let us pick $k >> 1$ and fix it. There must be a good box such that $\Lambda_{x,k} \subset \Lambda_{l_1}$ and

$$2 \operatorname*{Av}_{b\in\Lambda_{x,k}} D_b(\sum_{x\in\Lambda_{l_1}} \tau_x H, u) - \frac{1}{2} \operatorname*{Av}_{b\in\Lambda_{x,k}} D_b(u) \geq V^* - 4\delta.$$

If we replace u by $\tilde{u}^l := \tau_{-x}\bar{E}[u_l|\mathcal{F}_{\Lambda_{x,k}}]$ we still have

$$2 \operatorname*{Av}_{b\in\Lambda_k} D_b(\sum_{x\in\Lambda_{l_1}} \tau_x H, \tilde{u}^l) - \frac{1}{2} \operatorname*{Av}_{b\in\Lambda_k} D_b(\tilde{u}^l) \geq V^* - 4\delta.$$

We can take (a subsequence and take) the limit $l \to \infty$. Since $D_b(u)$ is bounded uniformly in l, we have a limit function u_k such that it is $\mathcal{F}_{\Lambda_k}$ measurable and satisfies

$$2 \operatorname*{Av}_{b\in\Lambda_{x,k}} D_b(\sum_{x\in\Lambda_{l_1}} \tau_x H, u_k) - \frac{1}{2} \operatorname*{Av}_{b\in\Lambda_{x,k}} D_b(u_k) \geq V^* - 4\delta.$$

Using Jensen's inequality, we have

$$2 \sum_{e=(0,e):|e|_1=1} D_e(\sum_{x\in\Lambda_{l_1}} \tau_x H, \operatorname*{Av}_{x\in\Lambda_k} \tau_x u_k) - \frac{1}{2} \sum_{e=(0,e):|e|_1=1} D_e(\operatorname*{Av}_{x\in\Lambda_k} \tau_x u_k) \geq V^* - 4\delta.$$

Recall that $\mathcal{L}_{(x,y)}$ is the generator defined by

$$\xi_x(1-\xi_y)\pi^{(x,y)} + 1_{\{\eta_x\geq 2\}} 1_{\{1\leq\eta_y\leq k-1\}}\pi^{x\to y}.$$

From the definition of $D_b(f,g), D_b(f), \Phi^{h,e}_{\text{ex}}, \Phi^{h,e}_{\text{ge}}, \mathcal{L}_{(x,y)}$, and

$$(1-\xi_e)1_{\{1\leq\eta_e\leq k-1\}} = 0$$

we have

$$\begin{aligned} &2\bar{E}\Big\{ \sum_{e=(0,e):|e|_1=1} \{c_{\text{ex}}(\eta_0)(1-\xi_e)\Phi^{h,e}_{\text{ex}}(\mathcal{L}_{(0,e)}(\sum_{x\in\Lambda_k} \tau_x u_k)) \\ &\quad + c_{\text{ge}}(\eta_x)1_{\{1\leq\eta_y\leq k-1\}}\Phi^{h,e}_{\text{ge}}(\mathcal{L}_{(x,y)}(\sum_{x\in\Lambda_k} u_k))\}\Big\} \\ &-\frac{1}{2}\bar{E}\Big\{ \sum_{e=(0,e):|e|_1=1} \{c_{\text{ex}}(\eta_x)(1-\xi_y)(\mathcal{L}_{(0,e)}(\sum_{x\in\Lambda_k} \tau_x u_k))^2 \\ &\quad + c_{\text{ge}}(\eta_x)1_{\{1\leq\eta_y\leq k-1\}}(\mathcal{L}_{(0,e)}(\sum_{x\in\Lambda_k} \tau_x u_k))^2\}\Big\} \\ &\geq V^* - 4\delta. \end{aligned}$$

To complete the proof of Lemma 8.4, we have only to prove that all limit point of $\mathcal{L}_{(0,e)}(\sum_{x\in\Lambda_k}\tau_x u_k)$ is an element of $\overline{\mathcal{G}_0+\mathcal{G}_E}$ where $\mathcal{G}_0$ is the linear space generated by

$$1_{\{\eta_0>0\}}1_{\{\eta_e=0\}},$$
$$\eta_0 1_{\{\eta_e=0\}}+1_{\{\eta_0\geq 2\}}1_{\{1\leq\eta_e\leq k-1\}},$$

$\mathcal{G}_E=\{\mathcal{L}_{0,e}\sum_{y\in\mathbf{Z}^d}\tau_y g : g \text{ is a local function}\}$, and $\overline{\mathcal{G}_0+\mathcal{G}_E}$ is the closure of $\mathcal{G}_0+\mathcal{G}_E$ relative to standard L^2 norm of P. On the other hands, in Lemma 9.1 we will show that all limit point of $\mathcal{L}_{(0,e)}(\sum_{x\in\Lambda_k}\tau_x u_k)$ is an element of the representation of the translation covariant closed form which is defined in Section 9. In Lemma 9.2, we will prove that a element of the representation of the translation covariant closed form is an element of $\overline{\mathcal{G}_0+\mathcal{G}_E}$. Hence we have only to show that the Lemma 9.1, 9.2 in Section 9. □

9 Structure of the space of closed forms

We have set that $\mathcal{L}_{(x,y)}$ is the generator defined by

$$\xi_x(1-\xi_y)\pi^{(x,y)}+1_{\{\eta_x\geq 2\}}1_{\{1\leq\eta_y\leq k-1\}}\pi^{x\to y}.$$

For any directed bond $b=(x,y)$, let T_b be operator defined by

$$T_b\eta=\begin{cases}\eta^{x\to y} & \text{if } \eta_x\geq 2 \text{ and } 1\leq\eta_y\leq k-1,\\ \eta^{(x,y)} & \text{if } \xi_x=1 \text{ and } \xi_y=0,\\ \eta & \text{otherwise.}\end{cases}$$

For each fixed sequence of the directed bonds $b_1,b_2,\dots,b_n$ and configuration η, let η^i for $0\leq i\leq n$ be defined by $\eta^0=\eta$ and $\eta^i=T_{b_i}\eta^{i-1}$. A set of functions $\{\Phi_b\}_b$ is closed or closed form if the next condition holds: for each fixed sequence of the directed bonds $b_1,b_2,\dots,b_n$ and configuration η, if it holds that $\eta^n=\eta$ then

$$\sum_{i=1}^n\Phi_{b_i}(\eta^{i-1})=0.$$

Let us define a set of translation covariant closed forms by

$$\{\{\Phi_b\}_b : \{\Phi_b\}_b \text{ is closed}, \bar{E}[(\Phi_b)^2]<\infty \text{ and } \Phi_b=\tau_x\Phi_{\tau_x b}\}$$

We write $\mathcal{G}_c$ for the set of the representation of translation covariant closed forms: i.e.

$$\mathcal{G}_c := \{\{\Phi_{(0,e)}\}_e : \{\Phi_b\}_b \text{ is a translation covariant closed form}\},$$

where e varies over unite vectors.

Lemma 9.1 *Suppose a sequence of functions $\{u_k\}_k$ satisfies following: For each k, u_k is $\mathcal{F}_{\Lambda_k}$ measurable. There exists a constant C such that*

$$\operatorname*{Av}_{b\in\Lambda_k} \bar{E}[(\mathcal{L}_b u_k)^2] \le C$$

for all k. Let us define a set of functions $\{\phi_b^k\}_b$ by

$$\begin{aligned}\phi^k_{(x,x+e)} &:= \tau_x \phi_{(0,e)}\\ \phi^k_{(0,e)} &:= \mathcal{L}_{(0,e)} \operatorname*{Av}_{y\in\Lambda_k}[\tau_y u_k],\end{aligned}$$

for all x and $e : |e|_1 = 1$. Then all limit point of $\{\phi_b^k\}_b$ is an element of translation covariant closed form.

Proof. From the definition, $\{\phi_b^k\}_b$ is translation covariant. Hence we have only to prove that $\{\phi_b^k\}_b$ is bounded uniformly in k and all limit point is an element of the closed form. But the proof of these facts is parallel to that of Lemma 9.2. Hence we only state the strategy. We prove that $\{\phi_b^k\}_b$ is bounded uniformly in k, by using Lemma 4.3. Hence we can take (a subsequence and take) a weak limit of $\{\phi_b^k\}_b$. We have only to prove that all limit point is an element of the closed form. Since

$$\psi_b^k := \frac{1}{|\Lambda_k|}\sum_{y\in\mathbf{Z}^d} \mathcal{L}_b \tau_y u_k$$

is an element of the closed form, we consider the difference $\phi_b^k - \psi_b^k$. We prove that all limit point of $\phi_b^k - \psi_b^k$ is an element of the closed form, furthermore we can prove that the representation of all limit point of $\{\phi_b^k - \psi_b^k\}_b$ is a element of $\mathcal{G}_0$ which is defined in Lemma 9.2 below. □

Given a closed form $\{\Phi_b\}$ for each finite box Λ_n, we can "integrate " Φ_b and get unique G^n such that

$$\begin{aligned}\mathcal{L}_b G^n(\eta) &= \Phi_b\\ E_{\Lambda_n,y,E}[G^n] &= 0\end{aligned}$$

if $b \in \Lambda_n$, and for all y, E.

Lemma 9.2 *We have $\mathcal{G}_c \subset \overline{\mathcal{G}_0 + \mathcal{G}_E}$ where $\mathcal{G}_0$ is the linear space generated by*

$$
\begin{aligned}
&1_{\{\eta_0>0\}}1_{\{\eta_e=0\}},\\
&\eta_0 1_{\{\eta_e=0\}} + 1_{\{\eta_0\geq 2\}}1_{\{1\leq\eta_e\leq k-1\}},
\end{aligned}
$$

$\mathcal{G}_E = \{\mathcal{L}_{0,e}\sum_y \tau_y g : g$ is a local function$\}$, and $\overline{\mathcal{G}_0 + \mathcal{G}_E}$ is the closure of $\mathcal{G}_0 + \mathcal{G}_E$ relative to standard L^2 norm of P.

Proof. Since $\mathcal{L}_b$ and conditional expectation $\bar{E}[\cdot|\mathcal{F}_{\Lambda_n}]$ commute if $b \in \Lambda_n$, $\{\bar{E}[\Phi_b|\mathcal{F}_{\Lambda_{3n}}]\}_{b\in\Lambda_{3n}}$ is also closed on Λ_{3n}. We can "integrate" and construct G^{3n} such that

$$
\begin{aligned}
&\mathcal{L}_b G^{3n}(\eta) = \bar{E}[\Phi_b|\mathcal{F}_{\Lambda_{3n}}],\\
&E_{\Lambda_{3n},y,E}[G^{3n}] = 0,
\end{aligned}
$$

for all y, E. We define h^n and Ψ_b^n by

$$
\begin{aligned}
h^n &:= \bar{E}[G^{3n}|\mathcal{F}_{\Lambda_n}],\\
\Psi_b^n &:= \frac{1}{(N)^d}\mathcal{L}_b\sum_{x\in\mathbf{Z}^d}\tau_x h^n.
\end{aligned}
$$

Since Ψ_b^n are shift covariant, we only consider $\Psi_{(0,e)}^n$. Since h^n is $\mathcal{F}_{\Lambda_n}$ measurable function, we decompose Ψ^n as interior part Ω_1^n and boundary part Ω_2^n as

$$
\begin{aligned}
\Omega_1^n &:= \frac{1}{(2n+1)^d}\mathcal{L}_{0,e}\sum_{y,y+e\in\Lambda_n}\tau_y h^n,\\
\Omega_2^n &:= \frac{1}{(2n+1)^d}\mathcal{L}_{0,e}\sum_{y\in\Lambda_n,y+e\notin\Lambda_n}\tau_y h^n + \frac{1}{(2n+1)^d}\mathcal{L}_{0,e}\sum_{y\notin\Lambda_n,y+e\in\Lambda_n}\tau_y h^n.
\end{aligned}
$$

First we show the next lemma:

Lemma 9.3 *It holds that*

$$\bar{E}[|\Phi_{0,e} - \Omega_1^n|^2] \to 0$$

as $n \to \infty$.

Proof. Since $\mathcal{L}_b$ and conditional expectation $\bar{E}[\cdot|\mathcal{F}_{\Lambda_n}]$ commute if b is in Λ_n, we have

$$\Phi_{0,e} - \Omega_1^n = \frac{1}{(2n+1)^d} \sum_{y,y+e\in\Lambda_n} [\Phi_{0,e} - \bar{E}[\Phi_{0,e}|\mathcal{F}_{\Lambda_{y,n}}]] + \frac{1}{2n+1}\Phi_{0,e}.$$

Since Φ_b is L^2 bounded, the second term of the right hand side of the last equality is negligible. Hence we have to show that

$$\lim_{n\to\infty} \bar{E}[|\Phi_{0,e} - \bar{E}[\Phi_{0,e}|\mathcal{F}_{\Lambda_{y,n}}]|^2] = 0 \tag{55}$$

for any y. It is standard to see that $\bar{E}[\Phi_{0,e}|\mathcal{F}_{\Lambda_{y,n}}]$ converges weakly to $\Phi_{0,e}$. But by Jensen's inequality, we have $\bar{E}[\bar{E}[\Phi_{0,e}|\mathcal{F}_{\Lambda_{y,n}}]^2] \leq \bar{E}[\Phi_{0,e}^2]$, hence we have (55). □

Next step we show that the boundary term is bounded.

Lemma 9.4 *There exists a constant C such that*

$$\sup_n \bar{E}[|\Omega_2^n|^2] \leq C.$$

Proof. Using Lemma 4.3, we have to estimate two terms in (22). By spectral gap inequality,

$$V(G^{3n}) \leq C(3n)^2 \sum_{b\in\Lambda_{3n}} \bar{E}[\Phi_b] \leq C' n^{d+2}.$$

By simple computation

$$\bar{E}[(1 + c_{\mathrm{ge}}(\eta_y))(\mathcal{L}_b G^{3n})^2] \leq C\bar{E}[\Phi_b] \leq C'.$$

These two inequality and (22) shows that there exists a constant C such that

$$\bar{E}[|\Omega_2^n|^2] \leq C.$$

□

Since the boundary has $2d$ surfaces, we can decompose the boundary term Ω_2^n into $2d$ parts corresponding to each surfaces. Since the boundary term Ω_2^n is uniformly bounded, then we can take a weak limit of each parts of the boundary term, and we write $b_{e_i}^{\pm}$ for each of them respectively.

Lemma 9.5 *For $1 \le i \le d$ the limit point of $b^{\pm}_{e_i}(\eta)$ depends only on η_0 and η_{e_i}.*

Proof. We only show that $b^-_{e_i}$ depends only on η_0 and η_{e_i}, and in this proof we omit e_i. By the construction, b^{n-} is measurable with respect to the σ-algebra generated by $\{\eta_{e_i}, \eta_x : x \in \mathbf{Z}^d, x_i \le 0\}$. The weak limit b^- inherits this property.

Next step we will show that if the bond $b = (z, y)$ satisfies

$$|z - y|_1 = 1, z_i \le 0 \text{ and } z, y \ne 0, \tag{56}$$

then $\mathcal{L}_{z,y} b^- = 0$.

For each fixed n, let

$$\partial\Lambda_n = \partial\Lambda_{e_i,-,n} := \{x \in \Lambda_n : x_i = -n\}.$$

Then for each fixed bond $b = (z, y)$ satisfying the condition (56), there exists $n_0 = n_0(z)$ such that for any $n \ge n_0$ we have

$$\begin{aligned}
&\#\{x \in \Lambda_n : z \in \Lambda_{x,n}, y \notin \Lambda_{x,n} \text{ or } z \notin \Lambda_{x,n}, y \in \Lambda_{x,n}\} \\
&\le \begin{cases} 0 & d = 1, \\ 2 & d = 2, \\ 2(2N+1)^{d-2} & d \ge 3, \end{cases} \\
&\#\{x \in \Lambda_n : z \in \Lambda_{x,n}, y \in \Lambda_{x,n}\} \le (2n+1)^{d-1}, \\
&\#\{x \notin \Lambda_n : z \in \Lambda_{x,n}, y \notin \Lambda_{x,n}\} \le (2n+1)^{d-1}.
\end{aligned} \tag{57}$$

By simple computation we have

$$\mathcal{L}_b \tau_x h^n = \mathcal{L}_b \tau_x \bar{E}[G^{3n} | \mathcal{F}_n] = \begin{cases} \tau_x \bar{E}[\mathcal{L}_{\tau_{-x} b} G^{3n} | \mathcal{F}_n] & \text{if both } z, y \in \Lambda_{n,x} \\ 0 & \text{if both } z, y \notin \Lambda_{n,x}. \end{cases} \tag{58}$$

On using (57), (58), Lemma 4.3, and the Schwarz inequality, there exists a constant C such that

$$\bar{E}[(\mathcal{L}_b b^{n-})^2] \le \frac{C}{n^2}$$

Hence we conclude that $\mathcal{L}_{z,y} b^- = 0$. This implies that b^- is the exchangeable function for $\{\eta_x : x_i < 0\}$. Using the Hewwitt-Savage 0-1 law, we conclude the proof of the lemma. □

We will make up equations for the boundary term. Without risk of confusion we omit e_i and simply write 1 for e_i, 2 for $2e_i$, and -1 for $-e_i$. By our construction $b_n^- = \mathcal{L}_{0,1} \sum_{x\in\partial\Lambda} \tau_x h^n$. We write H_n for $\sum_{x\in\partial\Lambda} \tau_x h^n$, then for any n H_n depends only on $\{\eta_x : x \le 0\}$, i.e. H_n does not depend on η_1. By simple computation, we have

$$\begin{aligned}
\mathcal{L}_{2,1} b_n^-(\eta) &= \mathcal{L}_{2,1}(\mathcal{L}_{0,1} H_n)(\eta) \\
&= -\{1_{\{\eta_0=1\}}1_{\{\eta_1=0\}}1_{\{1\le\eta_2\le k-1\}} + 1_{\{\eta_0\ge 2\}}1_{\{\eta_1=0\}}1_{\{\eta_2=k\}} \\
&\quad +1_{\{\eta_0\ge 2\}}1_{\{\eta_1=k-1\}}1_{\{\eta_2\ge 2\}}\} b_n^-(\eta) \\
&\quad -1_{\{\eta_0\ge 2\}}1_{\{\eta_1=0\}}1_{\{1\le\eta_2\le k-1\}}\mathcal{L}_{2,1} b_n^-(\eta).
\end{aligned}$$

Since b^- is the weak limit of b_n^-, this equality holds for b^-.

Now we turn to the $\mathcal{L}_{0,-1}$ part. Denote the configuration by $(\eta) = (\eta', \eta_{-1}, \eta_0)$ where $\eta' = \eta_{\mathbf{Z}\setminus\{0,1\}}$.

$$\begin{aligned}
\mathcal{L}_{0,-1} b_n^-(\eta) &= \mathcal{L}_{0,-1}(\mathcal{L}_{0,1} H_n)(\eta) \\
&= -\{1_{\{\eta_{-1}=0\}}1_{\{\eta_0>0\}}1_{\{\eta_1=0\}} + 1_{\{\eta_{-1}=0\}}1_{\{\eta_0\ge 2\}}1_{\{1\le\eta_1\le k-1\}}\} b^-(\eta) \\
&\quad +1_{\{1\le\eta_{-1}\le k-1\}}1_{\{\eta_0\ge 2\}}1_{\{\eta_1=0\}} \\
&\quad \times\{(H_n(\eta', \eta_{-1}+1, 0) - H_n(\eta', \eta_{-1}+1, \eta_0 - 1)) \\
&\quad -(H_n(\eta', \eta_{-1}, 0) - H_n(\eta', \eta_{-1}, \eta_0))\} \\
&\quad +1_{\{1\le\eta_{-1}\le k-1\}}1_{\{\eta_0\ge 2\}}1_{\{1\le\eta_1\le k-1\}} \\
&\quad \times\{(H_n(\eta', \eta_{-1}+1, \eta_0-2) - H_n(\eta', \eta_{-1}+1, \eta_0 - 1)) \\
&\quad -(H_n(\eta', \eta_{-1}, \eta_0-1) - H_n(\eta', \eta_{-1}, \eta_0))\}.
\end{aligned}$$

For the second and third term,

$$\begin{aligned}
H_n(\eta', \eta_{-1}+1, \eta_0-1) - H_n(\eta', \eta_{-1}, \eta_0) &= \mathcal{L}_{0,-1} H_n(\eta', \eta_{-1}, \eta_0) \\
&= \frac{1}{2n+1}\Phi_b^n(\eta', \eta_{-1}, \eta_0), \\
H_n(\eta', \eta_{-1}+1, \eta_0-2) - H_n(\eta', \eta_{-1}, \eta_0-1) &= \frac{1}{2n+1}\Phi_b^n(\eta', \eta_{-1}, \eta_0-1), \\
H_n(\eta', \eta_{-1}+1, \eta_0-1) - H_n(\eta', \eta_{-1}, \eta_0) &= \frac{1}{2n+1}\Phi_b^n(\eta', \eta_{-1}, \eta_0),
\end{aligned}$$

where $\Phi_b^n = \bar{E}[\Phi_b|\mathcal{F}_{\Lambda_n}]$ and $b = (0,1)$. Since Φ_b is uniformly bounded in L^2, each term vanishes as $n \to \infty$. We must estimate the next;

$$H_n(\eta', \eta_{-1}+1, 0) - H_n(\eta', \eta_{-1}, 0)$$

$$
\begin{aligned}
&= \Phi_b^n(\eta', 0, \eta_{-1}+1) - \Phi_b^n(\eta', 0, \eta_{-1}) \\
&\quad + H_n(\eta', 0, \eta_{-1}+1) - H_n(\eta', 0, \eta_{-1})
\end{aligned}
$$

and the first two terms vanish as $n \to \infty$ since Φ_b is uniformly bounded in L^2. We can rewrite the last two terms as

$$
H_n(\eta', 0, \eta_{-1}+1) - H_n(\eta', 0, \eta_{-1}+1) = \mathcal{L}_{0,1} H_n(\eta', 0, \eta_{-1})
$$

if $1 \le \eta_1 \le k-1$. On the other hands if we consider the configuration $\omega^l = \omega^l(\eta)$ defined by

$$
\left(\omega^l(\eta)\right)_z := \begin{cases} \eta_{-1} & \text{if } z = 0 \\ l & \text{if } z = 1 \\ \eta_z & \text{otherwise,} \end{cases} \tag{59}
$$

then

$$
H_n(\eta', 0, \eta_{-1}+1) - H_n(\eta', 0, \eta_{-1}+1) = \mathcal{L}_{0,1} H_n(\omega^l)
$$

for all l, since H_n does not depend on η_1. This term tends to $b^-(\omega^l)$ as $n \to \infty$ for $1 \le l \le k-1$. We conclude that

$$
\begin{aligned}
&\mathcal{L}_{0,-1} b^-(\eta) \\
&\quad = -\{1_{\{\eta_{-1}=0\}} 1_{\{\eta_0>0\}} 1_{\{\eta_1=0\}} + 1_{\{\eta_{-1}=0\}} 1_{\{\eta_0\ge 2\}} 1_{\{1\le \eta_1 \le k-1\}}\} b^-(\eta) \\
&\qquad + 1_{\{1\le \eta_{-1}\le k-1\}} 1_{\{\eta_0 \ge 2\}} 1_{\{\eta_1=0\}} b^-(\omega^l)
\end{aligned}
$$

for all $1 \le l \le k-1$, where $\omega^l = \omega^l(\eta)$ defined by (59). We get the equations for the limit of the boundary term b^-;

$$
\begin{aligned}
&\mathcal{L}_{2,1} b^-(\eta) \\
&\quad = -\{1_{\{\eta_0=1\}} 1_{\{\eta_1=0\}} 1_{\{1\le\eta_2\le k-1\}} + 1_{\{\eta_0\ge 2\}} 1_{\{\eta_1=0\}} 1_{\{\eta_2=k\}} \\
&\qquad + 1_{\{\eta_0\ge 2\}} 1_{\{\eta_1=k-1\}} 1_{\{\eta_2\ge 2\}}\} b^-(\eta) \\
&\qquad - 1_{\{\eta_0\ge 2\}} 1_{\{\eta_1=0\}} 1_{\{1\le\eta_2\le k-1\}} \mathcal{L}_{2,1} b^-(\eta). \\
&\mathcal{L}_{0,-1} b^-(\eta) \\
&\quad = -\{1_{\{\eta_{-1}=0\}} 1_{\{\eta_0>0\}} 1_{\{\eta_1=0\}} + 1_{\{\eta_{-1}=0\}} 1_{\{\eta_0\ge 2\}} 1_{\{1\le \eta_1 \le k-1\}}\} b^-(\eta) \\
&\qquad + 1_{\{1\le \eta_{-1}\le k-1\}} 1_{\{\eta_0 \ge 2\}} 1_{\{\eta_1=0\}} b^-(\omega^l).
\end{aligned}
$$

for all $1 \le l \le k-1$, where $\omega^l = \omega^l(\eta)$ defined by (59). We can solve this system of equations and find that a family of solutions is the linear combination of

$$
\begin{aligned}
&1_{\{\eta_0>0\}} 1_{\{\eta_1=0\}}, \\
&\eta_0 1_{\{\eta_1=0\}} + 1_{\{\eta_0\ge 2\}} 1_{\{1\le\eta_1\le k-1\}}.
\end{aligned}
$$

10 Proof for well-known facts

In this section we give elementary proofs for well-known facts.

For the proof of Lemma 3.5 we have used following lemma

Lemma 10.1 *Let $\mathcal{L}$ be the generator of Markov process X_t with reversible invariant measure ν. Let*

$$u(x,t) := E_x\Big[\exp\Big(\int_0^t V(X(s))ds\Big)\Big],$$

where E_x denotes the expectation with respect to the process starting at x. Then

$$\frac{1}{T}\log\int u(x,T)d\nu(x) \leq \sup\operatorname{spec}\{V-(-\mathcal{L})\}.$$

Proof. By the Feynman-Kac formula, u is a solution of the equation $\frac{\partial u}{\partial t} = \mathcal{L}u + Vu$ with the initial condition $u(x,0)\equiv 1$. Multiplying the equation by u and integrating by parts we have

$$\frac{\partial}{\partial t}\frac{1}{2}\int u^2 d\nu = \int\{Vu^2 - u(-\mathcal{L})u\}d\nu \leq \sup\operatorname{spec}\{V-(-\mathcal{L})\}\int u^2 d\nu.$$

Since $u(x,0)\equiv 1$, we have

$$\int u(x,T)^2 d\nu \leq \exp[2T\sup\operatorname{spec}\{V-(-\mathcal{L})\}].$$

This inequality is the same as

$$\frac{1}{T}\log\int u(x,T)d\nu \leq \sup\operatorname{spec}\{V-(-\mathcal{L})\}.$$

□

Proof of Lemma 3.5. First we consider the equilibrium process, namely $\mu_N = \nu_N$. Using Lemma 10.1 and Chebyshev's inequality, we have

$$\begin{aligned}
&\frac{1}{\gamma N^d}\sup\operatorname{spec}[\gamma N^d V - (-\mathcal{L}_N)] \\
&\geq \frac{1}{T}\frac{1}{\gamma N^d}\log E^N_{\nu_N}\Big[\exp\Big\{\int_0^T \gamma N^d V(X(s))ds\Big\}\Big] \\
&\geq \exp[\delta\gamma N^d]\Big]\frac{1}{T}\Big(\frac{1}{\gamma N^d}\log P^N_{\nu_N}\Big[\int_0^T \gamma N^d V(X(s))ds > \delta\Big] + \delta\Big).
\end{aligned}$$

Since $\delta > 0$ and $\gamma > 0$ are arbitrary, we have (15) for equilibrium process.

From equilibrium process to nonequilibrium process, we use entropy inequality or the Hölder inequality. First we claim that

$$\frac{dP_{\mu_N}}{dP_{\nu_N}} = \frac{d\mu_N}{d\nu_N}.$$

If we assume entropy bound (11), we use a special case of entropy inequality, that is

$$P^N_{\mu_N}(A) \leq \frac{\log 2 + H(\mu_N|\nu_N)}{\log(1 + P^N_{\nu_N}(A)^{-1})}$$

where $A = A_N = \{X_N : \int_0^T V(X_N(s))ds > \delta\}$. Hence we have (13). If we assume (14), using the Hölder inequality we have

$$P^N_{\mu_N}(A) \leq (||\frac{d\mu_N}{d\nu_N}||_{p,\nu_N})(P^N_{\nu_N}(A))^{1/q},$$

where $q = p/(1-p)$. Hence we have (15). □

Proof of Lemma 3.7. Let φ be the first eigenvector of $\mathcal{L} + f$ that is φ is the function satisfies

$$\begin{aligned}
&\sup \operatorname{spec}\Big(\mathcal{L} + f\Big) = E[\varphi\Big(\mathcal{L} + f\Big)\varphi], \\
&E[\varphi^2] = 1.
\end{aligned}$$

Let $\bar{\varphi} = \varphi - E[\varphi]$. Since $\mathcal{L}E[\varphi] = 0$, we have

$$E[\varphi\mathcal{L}\varphi] = E[\bar{\varphi}\mathcal{L}\bar{\varphi}].$$

Since $E[f] = 0$, we have

$$\sup \operatorname{spec}\Big(\mathcal{L} + f\Big) = E[\bar{\varphi}\mathcal{L}\bar{\varphi}] + 2E[\varphi]E[f\bar{\varphi}] + E[f\bar{\varphi}^2].$$

Using the Schwarz inequality, we have

$$\begin{aligned}
&E[\bar{\varphi}\mathcal{L}\bar{\varphi}] + 2E[\varphi]E[f\bar{\varphi}] \\
&\leq \ -E[\bar{\varphi}(-\mathcal{L})\bar{\varphi}] + 2|E[\varphi]|\sqrt{E[f(-\mathcal{L})^{-1}f]E[\bar{\varphi}(-\mathcal{L})\bar{\varphi}]} \\
&\leq \ \Big(E[\varphi]\Big)^2 E[f(-\mathcal{L})^{-1}f] \\
&\leq \ E[f(-\mathcal{L})^{-1}f].
\end{aligned}$$

On the other hands, since $E[\bar{\varphi}]=0$, using the spectral gap for $-\mathcal{L}$, we have

$$E[f\bar{\varphi}^2] \le ||f||_\infty E[\bar{\varphi}^2] \le ||f||_\infty \frac{1}{\delta} E[\bar{\varphi}(-\mathcal{L})\bar{\varphi}] = ||f||_\infty \frac{1}{\delta} E[\varphi(-\mathcal{L})\varphi].$$

Since $E[f]=0$, $\sup \operatorname{spec}\Big(\mathcal{L}+f\Big) \ge 0$, then we have

$$\begin{aligned}
&E[\varphi(-\mathcal{L})\varphi] \\
&\quad \le \; E[f\varphi^2] \le E[f(\varphi^2 - E[\varphi]^2)] \\
&\quad \le \; E[f\varphi^2 - E[\varphi]^2] \\
&\quad = \; E[f(\varphi + E[\varphi])(\varphi - E[\varphi])] \\
&\quad \le \; ||f||_\infty \sqrt{E[(\varphi + E[\varphi])^2] E[(\varphi - E[\varphi])^2]} \\
&\quad = \; 2||f||_\infty \sqrt{E[\bar{\varphi}^2]} \\
&\quad \le \; 2||f||_\infty \sqrt{\frac{E[\varphi(-\mathcal{L})\varphi]}{\delta}}.
\end{aligned}$$

This inequality shows that

$$E[\varphi(-\mathcal{L})\varphi] \le 4\frac{||f||_\infty^2}{\delta}.$$

We conclude that

$$\begin{aligned}
&\sup \operatorname{spec}\Big(\mathcal{L}+f\Big) \\
&\quad = \; E[\bar{\varphi}\mathcal{L}\bar{\varphi}] + 2E[\varphi]E[f\bar{\varphi}] + E[f\bar{\varphi}^2] \\
&\quad \le \; E[f(-\mathcal{L})^{-1}f] + 4\frac{||f||_\infty^3}{\delta^2}.
\end{aligned}$$

□

Proof of Proposition 3.9. Denote by M_ε the standard mollifier defined by means of the smooth function

$$M(p,\rho) := \begin{cases} c\exp\Big[\dfrac{1}{p^2+\rho^2-1}\Big] & \text{if } p^2+\rho^2<1, \\ 0 & \text{if } p^2+\rho^2 \ge 1, \end{cases}$$

where the constant c is chosen so that

$$\int M(p,\rho)\,dp\,d\rho = 1.$$

Let us define

$$M_\varepsilon(p,\rho) := \frac{1}{\varepsilon^2} M(\frac{p}{\varepsilon}, \frac{\rho}{\varepsilon}).$$

Let us define $M_\varepsilon * f$ by

$$M_\varepsilon * f(p,\rho) := \int M_\varepsilon(p-p', \rho-\rho') f(p',\rho') dp' d\rho'.$$

Suppose that ε is small enough. For each n we can find the function $g_n = \{g^q_{n,e}(p,\rho,\eta)\}$ satisfying following two properties.
For each p, ρ, each component is $\mathcal{F}_{\Lambda_n}$ measurable function.
For each p, ρ,

$$\begin{aligned} & V\Big(\sum_e \{\alpha^E_e \phi^E_e(g^q_{n,e}(p,\rho,\cdot)) + \alpha^P_e \phi^P_e(g^q_{n,e}(p,\rho,\cdot))\}; p, \rho\Big) \\ & \leq V\Big(\sum_e \{\alpha^E_e \phi^E_e(f) + \alpha^P_e \phi^P_e(f)\}; p, \rho\Big) \end{aligned}$$

for any $f \in \mathcal{F}_{\Lambda_n}$. Let us define

$$\begin{aligned} f_n(p,\rho) &:= \begin{cases} V\Big(\sum_e \{\alpha^E_e \phi^E_e(g^q_{n,e}(p,\rho,\cdot)) + \alpha^P_e \phi^P_e(g^q_{n,e}(p,\rho,\cdot))\}; p, \rho\Big) \\ \qquad\qquad \text{if } 0 \leq p \leq 1, p \leq \rho \leq kp, \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \\ F_n(p,\rho) &:= M_\varepsilon * f_n(p,\rho) \end{aligned}$$

for each n. From the definition f_n is bounded function for each n, hence F_n is well defined. From the definition of g_n,$f_n(p,\rho)$ is not increasing and tends to 0 as n tends to ∞ for each p, ρ. Clearly $F_n(p,\rho)$ inherits these properties. Applying Dini's theorem, $F_n(p,\rho)$ converges to 0 uniformly, i.e. for any $\delta > 0$ there exists n_0 such that for any $n \geq n_0$

$$\sup_{p,\rho} F_n(p,\rho) \leq \frac{\delta}{2}. \tag{60}$$

Given a $\delta > 0$, pick n such that (60) holds and fix it. Then define

$$\begin{aligned} s_\varepsilon(p,\rho) \quad &:= \sup_{p',\rho' : (p-p')^2 + (\rho-\rho')^2 \leq \varepsilon} \\ & \Big| V\Big(\sum_e \{\alpha^E_e \phi^E_e(g^q_{n,e}(p,\rho,\cdot)) + \alpha^P_e \phi^P_e(g^q_{n,e}(p,\rho,\cdot))\}; p, \rho\Big) \\ & - V\Big(\sum_e \{\alpha^E_e \phi^E_e(g^q_{n,e}(p,\rho,\cdot)) + \alpha^P_e \phi^P_e(g^q_{n,e}(p,\rho,\cdot))\}; p', \rho'\Big)\Big|. \end{aligned}$$

Since for each g, V is a continuous function of p, ρ, s_ε is not increasing and tends to 0 as ε tends to 0 for each p, ρ. If we consider $M_\varepsilon * s_\varepsilon(p,\rho)$, then applying Dini's theorem again $M_\varepsilon * s_\varepsilon(p,\rho)$ converges to 0 uniformly, i.e. there exists $\varepsilon > 0$ such that

$$M_\varepsilon * s_\varepsilon(p) \le \frac{\delta}{2}.$$

Since V is a nonnegative quadratic form, we conclude that

$$\begin{aligned}
&V\Big(\sum_e \{\alpha_e^E \phi_e^E((M_\varepsilon * g_{n,e}^q)(p,\rho)(\eta)) + \alpha_e^P \phi_e^P((M_\varepsilon * g_{n,e}^q)(p,\rho)(\eta))\}; p, \rho\Big) \\
&\le \int M_\varepsilon(p-p', \rho-\rho') V\Big(\sum_e \{\alpha_e^E \phi_e^E(g_{n,e}^q(p',\rho',\cdot)) \\
&\quad + \alpha_e^P \phi_e^P(g_{n,e}^q(p',\rho',\cdot))\}; p,\rho\Big) dp' d\rho' \\
&\le F_n(p,\rho) + M_\varepsilon s_\varepsilon(p,\rho) \le \delta
\end{aligned}$$

for all p, ρ. □

Proof of Theorem 5.1. First, we will construct strictly decreasing sequences $\{t_n\}$ and $\{d_n\}$, satisfying certain conditions. Let us define $d_{-1} := T$, and

$$I(t) := \int_0^T \Phi\Big(\frac{|\phi(t) - \phi(s)|}{g(|t-s|)}\Big) ds.$$

Since $\int_0^T I(t)dt \le B$, there exists $t_0 \in (0, d_{-1})$ such that

$$I(t_0) \le \frac{B}{T}.$$

For t_{n-1}, we take d_{n-1} holding

$$g(d_{n-1}) = \frac{1}{2} g(t_{n-1}).$$

Since g is strictly increasing function, we have $d_{n-1} < t_{n-1}$. For the d_{n-1} there exists $t_n \in (0, d_{n-1})$ such that

$$I(t_n) \le \frac{2B}{d_{n-1}}, \tag{61}$$

$$\Phi\Big(\frac{|\phi(t_n) - \phi(t_{n-1})|}{g(|t_n - t_{n-1}|)}\Big) \le \frac{2I(t_{n-1})}{d_{n-1}}, \tag{62}$$

since if we could not choose such t_n, we would have either

$$\int_0^T I(t)dt > B,$$
$$I(t_{n-1}) > \int_0^{t_{n-1}} \Phi\Big(\frac{|\phi(t)-\phi(s)|}{g(|t-s|)}\Big)ds.$$

From our definition of t_n and g, we conclude that $\{t_n\}$ is a strictly decreasing sequence and $\lim t_n = 0$. We also have

$$p(t_{n-1}-t_n) \le 4(g(d_{n-1}) - g(d_n)) \tag{63}$$

Using (62),(63),(61), in this order, we have

$$|\phi(t_{n-1}) - \phi(t_n)| \le 4\int_{d_n}^{d_{n-1}} \Phi^{-1}\Big(\frac{4B}{u^2}\Big)g(du).$$

Taking summation over n, we have

$$|\phi(t_0) - \phi(0)| \le 4\int_0^T \Phi^{-1}\Big(\frac{4B}{u^2}\Big)g(du).$$

Using the same construction for $\phi(T-t)$, we have

$$|\phi(T) - \phi(t_0)| \le 4\int_0^T \Phi^{-1}\Big(\frac{4B}{u^2}\Big)g(du).$$

Hence we have

$$|\phi(T) - \phi(0)| \le 8\int_0^T \Phi^{-1}\Big(\frac{4B}{u^2}\Big)g(du). \tag{64}$$

This inequality holds for any ϕ satisfying (30), and any g, Φ. Hence let us define

$$\bar{\phi}(u) := \phi(s + \frac{t-s}{T}u)$$
$$\bar{g}(u) := g(\frac{t-s}{T}u)$$

on $[0,T]$. We have

$$\int_0^T\int_0^T \Phi(\frac{|\bar{\phi}(u)-\bar{\phi}(v)|}{\bar{g}(|u-v|)})dudv \le (\frac{T}{t-s})^2 B =: \bar{B}.$$

From (64) we have

$$|\bar{\phi}(T) - \bar{\phi}(0)| \leq 8 \int_0^T \Phi^{-1}(\frac{4\bar{B}}{u^2})\bar{g}(du),$$

that is

$$|\phi(t) - \phi(s)| \leq 8 \int_0^{|t-s|} \Phi^{-1}(\frac{4B}{u^2})g(du).$$

□

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Spectral gap for zerorange-exclusion process

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1 Introduction

In this paper we introduce the "zerorange-exclusion process", and obtain an estimate of the spectral gap for it. The process is a kind of lattice gas. It has two conserved quantities, the number of particles and the total energy. It will be proved that the spectral gap, for the process confined to a cube in $\mathbf{Z}^d$ with width n, is bounded below by Cn^{-2}, where C is independent of n but depends on the particle and energy densities. This estimate is motivated by the study of the hydrodynamic limit of the process; it is sufficient for obtaining the characterization of the space of germs of closed forms [2].

Let Λ_N be a d-dimensional cube with width N, centered at origin. and $\eta = (\eta_x)_{x\in\Lambda_N}$ denote the configuration of a lattice gas with energy where for each x, $\eta_x \in \mathbf{Z}_+$, and $\xi_x := 1_{\{\eta_x\neq 0\}}$. $\xi_x = 0$ means that a site x is vacant, and $\xi_x = 1$ means that there exists particle at site x with energy η_x.

We consider the two types of jump; firstly particles jump to vacant sites and secondly the energy on an occupied site is transfered to one of its neighboring occupied sites according to the zerorange law.

Let $\eta^{(x,y)}$ and $\eta^{x\to y}$ be the configurations related with two types of jump defined by

$$
\begin{aligned}
(\eta^{(x,y)})_z &= \begin{cases} \eta_y, & \text{if } z = x, \\ \eta_x, & \text{if } z = y, \\ \eta_z, & \text{otherwise,} \end{cases} \\
(\eta^{x\to y})_z &= \begin{cases} \eta_x - 1, & \text{if } z = x, \\ \eta_y + 1, & \text{if } z = y, \\ \eta_z, & \text{otherwise,} \end{cases}
\end{aligned}
$$

Then define the operators $\pi^{(x,y)}$ and $\pi^{x\to y}$ by

$$
\begin{aligned}
\pi^{(x,y)} f(\eta) &:= f(\eta^{(x,y)}) - f(\eta), \\
\pi^{x\to y} f(\eta) &:= f(\eta^{x\to y}) - f(\eta).
\end{aligned}
$$

Let $c_{\text{ex}}(r), c_{\text{ze}}(r)$ be functions of $r \in \mathbf{Z}_+$ such that

$$
\begin{aligned}
& c_{\text{ex}}(0) = 0, \\
& c_{\text{ex}}(l) > 0, \ \text{for any } l \geq 1, \\
& c_{\text{ze}}(0) = c_{\text{ze}}(1) = 0, \\
& c_{\text{ze}}(l) > 0, \ \text{for any } l \geq 2.
\end{aligned}
$$

We assume technical conditions; there exists constant a_1 such that

$$|c_{\rm ze}(k) - c_{\rm ze}(k+1)| \le a_1 \tag{65}$$

and there exists k_0 and $a_2 > 0$ such that for any $k > l + k_0$

$$c_{\rm ze}(k) - c_{\rm ze}(l) \ge a_2 \tag{66}$$

and there exists $a_3 > 0$ such that

$$c_{\rm ex}(k) \ge a_3 c_{\rm ze}(k).$$

For each directed bond $b = (x, y)$, let L_b be the operator defined by

$$L_b f(\eta) := c_{\rm ex}(\eta_x)(1-\xi_y)\pi^{(x,y)} f(\eta) + c_{\rm ze}(\eta_x)\xi_y \pi^{x\to y} f(\eta).$$

Then L_N is defined by

$$L_N := \sum_{b=(x,y)\in\Lambda_N, |x-y|=1} L_b.$$

We call the Markov process generated by L_N the zerorange-exclusion process.

Consider the product measure whose marginals are given by

$$\bar{P}_{p,\rho}(\eta_x = l) := \begin{cases} 1-p & \text{if } l = 0, \\ p\dfrac{1}{Z_{\alpha(p,\rho)}} & \text{if } l = 1, \\ p\dfrac{1}{Z_{\alpha(p,\rho)}} \dfrac{(\alpha(p,\rho))^{l-1}}{c_{\rm ze}(2)c_{\rm ze}(3)\cdots c_{\rm ze}(l)} & \text{if } l \ge 2, \end{cases}$$

for all x. Here Z is the normalizing constant and $\alpha(p, \rho)$ is a positive constant depending on p and ρ and determined by the relation

$$\bar{E}_{p,\rho}[\eta_x] = \rho.$$

The canonical measure on Λ_n of the particle number y and the total energy E is defined by

$$P_{n,y,E}[\ \cdot\ ] = \bar{P}_{p,\rho}[\ \cdot\ \mid \sum_{x\in\Lambda_n} \xi_x = y \sum_{x\in\Lambda_n} \eta_x = E].$$

From the definition, we have reversibility such that

$$\begin{aligned} & c_{\rm ze}(\eta_x)\xi_w P_{n,y,E}(\{\eta\}) = c_{\rm ze}((\eta^{x\to w})_w)(\xi^{x\to w})_x P_{n,y,E}(\{\eta^{x\to w}\}) \\ & P_{n,y,E}(\{\eta\}) = P_{n,y,E}(\{\eta^{(x,y)}\}), \end{aligned} \tag{67}$$

for any $n, y, E \in \mathbf{Z}_+$, for any $x, w \in \Lambda_n$ and for any configuration η on Λ_n.

Theorem 1.1 *There exists a constant C such that for any positive integer n, y, E, satisfying $y \le |\Lambda_n|$ and $E \ge y$,*

$$V_{n,y,E}(f) \le Cn^2 \left(\frac{|\Lambda_n|}{y}\right)^2 \frac{E}{y} D_{n,y,E}(f),$$

where V denotes the variance and D denotes the Dirichlet form defined by

$$D_{n,y,E}(f) := -E_{n,y,E}[f L_n f].$$

2 Outline of Proof

In this section, we fix n, y, E, and simply write $E[\cdot]$ for $E_{n,y,E}[\cdot]$. For $n \in \mathbf{N}$ let $\mathcal{F}_{\xi,n}$ denote the

$$\sigma\text{-algebra generated by } \{\xi_x : x \in \Lambda_n\}.$$

Then

$$V_{n,y,E}(f) \le 2E\Big[(f - E[f|\mathcal{F}_{\xi,n}])^2\Big] + 2E\Big[(E[f|\mathcal{F}_{\xi,n}] - E[f])^2\Big]. \tag{68}$$

To estimate the first term, we deduce the following lemma from results of [1].

Lemma 2.1 *There exists a constant C such that*

$$E\Big[(f - E[f|\mathcal{F}_{\xi,n}])^2|\mathcal{F}_{\xi,n}\Big] \le C\frac{1}{y}E\Big[\sum_{x,y\in\Lambda} c_{\mathrm{ze}}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2|\mathcal{F}_{\xi,n}\Big].$$

Lemma 2.2 *There exists a constant C such that*

$$E\Big[\sum_{x,y\in\Lambda_n} c_{\mathrm{ze}}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2\Big] \le Cn^2|\Lambda_n|\frac{|\Lambda_n|}{y}\frac{E}{y}D(f).$$

We shall give proofs of Lemmas 2.1 and 2.2 in Section 4 and 3 respectively. By successively applying Lemma 2.1 and 2.2, the first term on the right hand side of (68) is bounded by

$$\begin{aligned} &C\frac{|\Lambda_n|}{y}\frac{1}{|\Lambda_n|}E\Big[\sum_{x,y\in\Lambda_n} c_{\mathrm{ze}}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2\Big] \\ &\le C'\left(\frac{|\Lambda_n|}{y}\right)^2\frac{E}{y}n^2D(f). \end{aligned}$$

The second term on the right hand side of (68) is easy to dispose of. Since $E[f|\mathcal{F}_{\xi,n}]$ depends only on ξ, we can use a spectral gap estimate for the simple exclusion process to see that

$$E\Big[(E[f|\mathcal{F}_{\xi,n}] - E[f])^2\Big] \le Cn^2 E\Big[\sum_{x,y\in\Lambda_n,|x-y|=1} (\pi^{(x,y)}E[f|\mathcal{F}_{\xi,n}])^2\Big].$$

Since if $x, y \in \Lambda_n$ then the operator $\pi^{(x,y)}$ and the conditional expectation $E[\cdot|\mathcal{F}_{\xi,n}]$ commute, on using Jensen's inequality, the last term is at most

$$Cn^2 D(f).$$

Hence we conclude that

$$V(f) \le C\left(\frac{|\Lambda_n|}{y}\right)^2 \frac{E}{y} n^2 D(f).$$

3 Proof of Lemma 2.2

For any $x \in \mathbf{Z}^d$ put

$$|x| := \sum_{i=1}^{d} |x_i|.$$

Let $\gamma(x,y)$ denote the canonical path from x to y defined by

$$\gamma(x,y) := \{z(i) : 0 \le i \le |x-y|\}, \tag{69}$$

where $z(i)$ is defined by

$$\begin{aligned}
&z(i) = (z(i)_1, \dots, z(i)_d) \text{ such that} \\
&z(i)_j = x_j \text{ for } i \le \sum_{k=1}^{j-1} |x_k - y_k|, \\
&z(i)_j = y_j \text{ for } i \ge \sum_{k=1}^{j} |x_k - y_k|, \\
&z(i)_j = x_j + (i - \sum_{k=1}^{j-1} |x_k - y_k|)\frac{y_i - x_i}{|y_i - x_i|} \\
&\qquad \text{for } \sum_{k=1}^{j-1} |x_k - y_k| < i < \sum_{k=1}^{j} |x_k - y_k|,
\end{aligned}$$

namely $\gamma(x,y)$ denotes the nearest neighbor path that goes from x to y, moving successively as far as it has to in each of the coordinate directions, following the natural order for the different coordinate directions.

Lemma 3.1 *For any $x, y \in \Lambda_n$, let $\gamma(x,y)$ be a canonical path from x to y. Then we have*

$$
\begin{aligned}
&E[c_{\text{ze}}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2] \\
&\leq C|x-y| \sum_{z,w\in\gamma(x,y),|z-w|=1} E[c_{\text{ze}}(\eta_z)\xi_w(\pi^{z\to w}f(\eta))^2 \\
&+\{c_{\text{ex}}(\eta_z)+c_{\text{ex}}(\eta_y)\}(1-\xi_w)(\pi^{(z,w)}f(\eta))^2].
\end{aligned}
$$

In particular

$$
\begin{aligned}
&E[c_{\text{ze}}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2 1_{\{1\leq\eta_x\leq 2E/y\}}] \\
&\leq C\frac{2E}{y}n \sum_{z,w\in\gamma(x,y),|z-w|=1} E[c_{\text{ze}}(\eta_z)\xi_w(\pi^{z\to w}f(\eta))^2 \\
&+c_{\text{ex}}(\eta_z)(1-\xi_w)(\pi^{(z,w)}f(\eta))^2].
\end{aligned}
$$

Proof. Let us define the configurations $T^{x,y}\eta$ and $S^{x,y}\eta$ by

$$
\begin{aligned}
T^{x,y}\eta &:= \begin{cases} \eta^{(x,y)} & \text{if } \xi_x = 1, \xi_y = 0, \\ \eta^{x\to y} & \text{if } \eta_x \geq 2, \xi_y = 1, \\ \eta & \text{otherwise,} \end{cases} \\
S^{x,y}\eta &:= \begin{cases} \eta^{(x,y)} & \text{if } \xi_x = 1, \xi_y = 0, \\ \eta & \text{otherwise.} \end{cases}
\end{aligned}
$$

Let $z(i) := z_{x,y}(i)$ $(0 \leq i \leq |x-y|)$ be defined by (69), namely $z(i)$ is the i-th point from x on the canonical path $\gamma(x,y)$. If $\eta_x \geq 2$, we can rewrite $\eta^{x\to y}$ as

$$
\begin{aligned}
\eta^{x\to y} &= S^{z(1),z(0)} \circ S^{z(2),z(1)} \circ \cdots \circ S^{z(|x-y|-1),z(|x-y|-2)} \circ T^{z(|x-y|-1),z(|x-y|)} \\
&\quad \circ T^{z(|x-y|-1),z(|x-y|-2)} \circ \cdots \circ T^{z(0),z(1)}\eta.
\end{aligned}
$$

Let us define $T_i\eta$ by $T_0\eta := \eta$,

$$T_i\eta = T^{z(i-1),z(i)}T_{i-1}\eta,$$

for $1 \leq i \leq |x-y|$, and

$$T_i\eta = S^{z(2|x-y|-i),z(2|x-y|-i-1)}T_{i-1}\eta,$$

for $|x-y|+1 \le i \le 2|x-y|-1$. Using reversibility (67), we can check that

$$c_{\rm ze}(\eta_x)\xi_y P(\{\eta\}) = c_{\rm ze}((T_i\eta)_{z(i)})(T_i\xi)_y P(\{T_i\eta\})$$

for $0 \le i \le |x-y|$, and

$$c_{\rm ze}(\eta_x)\xi_y P(\{\eta\}) = c_{\rm ze}((T_i\eta)_y)(T_i\xi)_{z(2|x-y|-i)} P(\{T_i\eta\})$$

for $|x-y|+1 \le i \le 2|x-y|-1$. Using these notations and equalities, we have

$$\begin{aligned}
&E[c_{\rm ze}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2] \\
&\le E[c_{\rm ze}(\eta_x)\xi_y(\sum_{i=1}^{2|x-y|-1}(f(T_i\eta)-f(T_{i-1}\eta)))^2] \\
&\le 2|x-y|\sum_{i=1}^{2|x-y|-1}E[c_{\rm ze}(\eta_x)\xi_y(f(T_i\eta)-f(T_{i-1}\eta))^2] \\
&\le 2|x-y|\Big\{\sum_{i=1}^{|x-y|}\sum_{\eta}P(\{T_i\eta\})[c_{\rm ze}((T_i\eta)_{z(i)})(T_i\xi)_y(f(T_i\eta)-f(T_{i-1}\eta))^2] \\
&\quad+\sum_{i=|x-y|+1}^{2|x-y|-1}\sum_{\eta}P(\{T_i\eta\})[c_{\rm ze}((T_i\eta)_y)(T_i\xi)_{z(2|x-y|-i)}(f(T_i\eta)-f(T_{i-1}\eta))^2]\Big\}.
\end{aligned}$$

From the condition imposed on the jump rates, the last term is at most

$$\begin{aligned}
&2|x-y|\Big\{\sum_{i=1}^{|x-y|}E[c_{\rm ze}(\eta_{z(i)})\xi_{z(i+1)}(f(\eta^{z(i)\to z(i+1)})-f(\eta))^2] \\
&+\sum_{i=1}^{|x-y|}E[c_{\rm ex}(\eta_{z(i)})(1-\xi_{z(i+1)})(f(\eta^{(z(i),z(i+1))})-f(\eta))^2] \\
&+\sum_{i=|x-y|+1}^{2|x-y|-1}E[c_{\rm ex}(\eta_y)(1-\xi_{z(i+1)})(f(\eta^{(z(i),z(i+1))})-f(\eta))^2]\Big\}.
\end{aligned}$$

From the definition of the $z(i)$, we conclude the proof. □

Proof of Lemma 2.2. From our setting that $\sum_x \xi_x = y$ and $\sum_x \eta_x = E$, we have

$$\frac{y}{2} \le \sum_x 1_{\{1\le\eta_x\le 2E/y\}} \le y.$$

For any $w \in \Lambda_n$ we have

$$
\begin{aligned}
&E[c_{\text{ze}}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2] \\
&\leq 2E[c_{\text{ze}}(\eta_x)\xi_w(\pi^{x\to w}f(\eta))^2] + 2E[c_{\text{ze}}(\eta_x)\xi_w(\pi^{y\to w}f(\eta))^2],
\end{aligned}
$$

by reversibility (67). Using these results, we have

$$
\begin{aligned}
&E\Big[\sum_{x,y\in\Lambda_n} c_{\text{ze}}(\eta_x)\xi_y(\pi^{x\to y}f(\eta))^2\Big] \\
&\leq \frac{4}{y}\sum_{x,y,w\in\Lambda_n}\Big(E\Big[c_{\text{ze}}(\eta_x)1_{\{1\leq\eta_w\leq 2E/y\}}(\pi^{x\to w}f(\eta))^2\Big] \\
&+E\Big[c_{\text{ze}}(\eta_y)1_{\{1\leq\eta_w\leq 2E/y\}}(\pi^{y\to w}f(\eta))^2\Big]\Big).
\end{aligned}
$$

By symmetry, we only consider the first term. On using Lemma 3.1, the first term is not greater than

$$
\begin{aligned}
&C|\Lambda_n|\frac{4}{y}\frac{2E}{y}n\sum_{x,w\in\Lambda_n}\sum_{u,v\in\gamma(x,w),|u-v|=1}E[c_{\text{ze}}(\eta_u)\xi_v(\pi^{u\to v}f(\eta))^2 \\
&+c_{\text{ex}}(\eta_u)(1-\xi_v)(\pi^{(u,v)}f(\eta))^2].
\end{aligned}
$$

By simple computation, we have

$$
\sum_{x,w\in\Lambda_n}\sum_{u,v\in\gamma(x,w),|u-v|=1}f(u,v) \leq 2n|\Lambda_n|\sum_{u,v\in\Lambda_n|u-v|=1}f(u,v)
$$

for any positive function $f(u,v)$. Hence we conclude the proof. □

4 Proof of Lemma 2.1

Let us consider a meanfield type zerorange process on $\mathbf{N}^{S_n}$ with jump rate c_{ze} where $S_n := \{1, 2, \ldots, n\}$. The generator of the meanfield type zerorange process is defined by

$$
\bar{L}_n := \frac{1}{n}\sum_{x,y\in S_n} c_{\text{ze}}(\eta_x)\pi^{x\to y}f(\eta).
$$

The grand canonical measure is a product measure whose marginal distribution is given by

$$\bar{P}_\rho(\eta_x = l) := \begin{cases} \dfrac{1}{Z_{\alpha(\rho)}} & \text{if } l = 1, \\ \dfrac{1}{Z_{\alpha(\rho)}} \dfrac{(\alpha(\rho))^{l-1}}{c_{\text{ze}}(2)c_{\text{ze}}(3)\cdots c_{\text{ze}}(l)} & \text{if } l \geq 2, \end{cases}$$

where Z is the normalizing constant and $\alpha(\rho)$ is determined by the relation

$$\bar{E}_\rho[\eta_x] = \rho.$$

The canonical measures are defined by

$$P_{n,E}[\ \cdot\ ] = \bar{P}_\rho[\ \cdot\ \mid \sum_{x \in S_n} \eta_x = E].$$

The Dirichlet form is defined by

$$\bar{D}_{n,E}(f) := -E_{n,E}[f \bar{L}_n f].$$

Lemma 2.1 clearly follows from the following proposition.

Proposition 4.1 *Given the conditions (65) and (66), there exists a constant C such that*

$$E_{n,E}[(f - E_{n,E}[f])^2] \leq C\bar{D}_{n,E}(f).$$

To prove Proposition 4.1, we quote results of [1].

Lemma 4.2 *(Lemma 2.1 of [1]) There exists a constant C such that for any $x \in S_n$ for any function depending only on η_x, we have*

$$E_{n,E}[(f - E_{n,E}[f])^2] \leq C\bar{D}_{n,E}(f)$$

for all $n \geq 2$.

The following two lemmas are immediately deduced from Lemma 3.2 and Proposition 3.1 of [1], respectively.

Lemma 4.3 *Assume that there exists a constant W_{n-1} such that*

$$E_{n-1,E}[(f - E_{n-1,E}[f])^2] \leq W_{n-1}\bar{D}_{n-1,E}(f)$$

for all E. Then there exists a constant C which does not depend on n such that

$$\bar{D}_{n,E}(E_{n,E}[f|\eta_x]) \leq CW_{n-1}\bar{D}_{n,E}(f)$$

for all $n \geq 2$, $E \in \mathbf{N}$, and $x \in S_n$.

Lemma 4.4 *For every $\varepsilon > 0$, there exists n_0 and a constant $C(\varepsilon)$, such that*

$$\bar{D}_{n,E}(E_{n,E}[f|\eta_x]) \leq \frac{C(\varepsilon)}{n}\bar{D}_{n,E}(f) + \frac{\varepsilon}{n}E_{n,E}[(f - E_{n,E}[f])^2]$$

for all $n \geq n_0, E \in \mathbf{N}$, and $x \in S_n$.

Proof of Proposition 4.1. Our strategy of proof may be as follows. For any n, assume that there exists a constant W_{n-1} such that

$$E_{n-1,E}[(f - E_{n-1,E}[f])^2] \leq W_{n-1}\bar{D}_{n-1,E}(f) \tag{70}$$

for all E. We shall deduce from (70) that there exists a constant W_n depending only on n and W_{n-1} such that

$$E_{n,E}[(f - E_{n,E}[f])^2] \leq W_n\bar{D}_{n,E}(f)$$

for all E. From the relation between W_{n-1} and W_n, we can take a sequence $\{W_n\}$ inductively which satisfies

$$E_{n,E}[(f - E_{n,E}[f])^2] \leq W_n\bar{D}_{n,E}(f)$$

for all n, E, and $W_n \leq C_0$ for some constant C_0 uniformly in n.

By Lemma 4.2, there exists a constant C_1 such that

$$E_{2,E}[(f - E_{2,E}[f])^2] \leq C_1\bar{D}_{2,E}(f) \tag{71}$$

for all $E \in \mathbf{N}$. By simple computation, we have

$$\begin{aligned} &E_{n,E}[(f - E_{n,E}[f])^2] \\ &\leq \frac{1}{n}\sum_{x \in S_n}\{E_{n,E}[(f - E_{n,E}[f|\eta_x])^2] + E_{n,E}[(E_{n,E}[f|\eta_x] - E_{n,E}[f])^2]\}. \end{aligned} \tag{72}$$

By (70), the first term on the right hand side of (72) is not greater than

$$\frac{n-2}{n-1}W_{n-1}\bar{D}_{n,E}(f).$$

On using Lemmas 4.2 and 4.3, there exists C_2 which does not depend on n, E such that the second term on the right hand side of (72) is not greater than

$$C_2W_{n-1}\bar{D}_{n,E}(f)$$

for all, $n > 2$. Then we have

$$E_{n,E}[(f - E_{n,E}[f])^2] \leq (1 + C_2)\, W_{n-1}\bar{D}_{n,E}(f) \tag{73}$$

for all $n > 2$. On the other hands, we take $\varepsilon = 1/10$ in Lemma 4.4 then there exists a constant C_3 and n_0 such that the second term on the right hand side of (72) is not greater than

$$\frac{C_3}{n}\bar{D}_{n,E}(f) + \frac{1}{10n}E_{n,E}[(f - E_{n,E}[f])^2]$$

for all $n \geq n_0$. Then we have

$$E_{n,E}[(f - E_{n,E}[f])^2] \leq \left\{ \left(1 - \frac{1}{2(n-1)}\right) W_{n-1} + \frac{C_3}{n} \right\} \bar{D}_{n,E}(f) \tag{74}$$

for all $n \geq n_0$. From (71)(73)(74), if we take

$$W_n := \begin{cases} C_1 & \text{if } n = 2, \\ (1 + C_2)W_{n-1} & \text{if } 3 \leq n \leq n_0 - 1, \\ \left(1 - \dfrac{1}{2n-1}\right) W_{n-1} + \dfrac{C_3}{n} & \text{if } n \geq n_0, \end{cases} \tag{75}$$

inductively, then we have

$$E_{n,E}[(f - E_{n,E}[f])^2] \leq W_n\bar{D}_{n,E}(f)$$

for all $n \geq 2, E \in \mathbf{N}$. To complete the proof of Proposition 4.1, we only have to show that W_n defined by (75) satisfies that there exists a constant C_0 such that $W_n \leq C_0$ for all $n \geq 2$. From the definition

$$W_n = C_1(1 + C_2)^{n-2} \leq C_1(1 + C_2)^{n_0 - 1}$$

for $2 \leq n \leq n_0 - 1$. If $n \geq n_0$, $C \geq 2C_3$ and $W_{n-1} \leq C$, then $W_n \leq C$. If we take

$$C_0 = \max\{2C_3, C_1(1 + C_2)^{n_0 - 1}\}$$

then we have $W_n \leq C_0$ for all $n \geq 2$. □

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The gradient condition for one-dimensional symmetric exclusion processes

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1 Introduction

Given a Gibbs measure on the one dimensional lattice $\mathbf{Z}$ with translation-invariant potential of finite range, we construct an exchange rate for one-dimensional lattice gas which satisfy both the detailed balance condition relative to the Gibbs measure and the gradient condition. For the construction, we use an infinite system of linear equations indexed by finite sets which is given in [3]. Since this system of equations has plenty of freedom, it has many solutions, most of which do not possess properties necessary for constructing the desired exchange rate. Our strategy is to find a suitable condition such that the system with it added becomes uniquely solvable and the unique solution satisfies the required properties.

Based on an exchange rate which satisfies both the detailed balance condition and the gradient condition, we can prove the hydrodynamic limit for every one-dimensional lattice gas reversible under the Gibbs measure that is not necessarily of gradient type, in a way parallel to [1] and [4] with the help of the result of [2] on the spectral gap.

Let $\eta = (\eta_x; x \in \mathbf{Z})$, $\eta_x = 0$ or 1, denotes an element of $\{0,1\}^{\mathbf{Z}}$, the state space of one-dimensional lattice gas. The site x is interpreted as vacant if $\eta_x = 0$ and occupied if $\eta_x = 1$. The potential $\{J_A\}_{A\subset\mathbf{Z}}$ is supposed to have a finite range : there exists a constant p such that

$$J_A = 0 \text{ whenever } \mathrm{diam} A > p, \tag{76}$$

and to be translation-invariant :

$$J_A = J_{A+a} \text{ for all } A \subset \mathbf{Z} \text{ and } a \in \mathbf{Z}. \tag{77}$$

We define a Hamiltonian $H_\Lambda(\eta) = H^J_\Lambda(\eta)$ by

$$H_\Lambda(\eta) := \sum_{A\subset\mathbf{Z}, A\cap\Lambda\neq\emptyset} J_A \eta^A \quad , \quad \eta^A := \prod_{x\in A} \eta_x,$$

and a shift operator τ_x by

$$(\tau_x\eta)_z = \eta_{x+z} \text{ for all } x, z \in \mathbf{Z},$$
$$\tau_x A = A + x \text{ for all } A \subset \mathbf{Z} \text{ and } x \in \mathbf{Z}.$$

Our main result is stated as follows.

Theorem 1.1 *There exists an exchange rate $c(x, x+1, \eta)$ which satisfies the following conditions:*

1. *Locality: $c(x, x+1, \eta)$ depends only on $\{\eta_z; |z-x| \leq r\}$ for some $r > 0$.*
2. *Translation invariance: $c(x, x+1, \eta) = c(0, 1, \tau_x \eta)$, for all $x \in \mathbf{Z}$.*
3. *Positivity and exclusion : $c(x, x+1, \eta) > 0$ if $\eta_x \neq \eta_{x+1}$, and $c(x, x+1, \eta) = 0$ if $\eta_x = \eta_{x+1}$.*
4. *Detailed balance condition:*

$$\begin{aligned} & c(x, x+1, \eta) \exp[-H_{\{0,1\}}(\eta)] \\ = \ & c(x, x+1, \eta^{x,x+1}) \exp[-H_{\{0,1\}}(\eta^{x,x+1})], \end{aligned} \tag{78}$$

   where

$$(\eta^{x,y})_z = \begin{cases} \eta_y & \text{if } z = x, \\ \eta_x & \text{if } z = y, \\ \eta_z & \text{otherwise} \end{cases}$$

5. *Gradient condition: there exists a local function $h(\eta)$ such that*

$$c(x, x+1, \eta)(\eta_x - \eta_{x+1}) = h(\tau_x \eta) - h(\tau_{x+1} \eta). \tag{79}$$

Remark 1.2 *The function $c(x, x+1, \eta)$ that we shall construct depends only on $\{\eta_z; z \in \{x-p, x-p+1, \ldots, x+p, x+p+1\}\}$; Hence the function $h(\eta)$ of (79) depends only on $\{\eta_z; z \in \{-p, -p+1, \ldots, p-1, p\}\}$.*

2 A system of linear equations for $\{a(A)\}$

In this section we describe a system of linear algebraic equations given in [3]. By the condition 2 we have only to consider the case $x = 0$. We define $\Delta H(\eta)$ by

$$\Delta H(\eta) = \sum_A{}^* (J_{A \cup \{0\}} - J_{A \cup \{1\}}) \eta^A. \tag{80}$$

Here $\sum^*{}_A$ stands for summation over all finite subsets $A \subset \mathbf{Z}$ which contain neither 0 nor 1. Note that $\Delta H(\eta)$ does not depend on (η_0, η_1). By the conditions 3 and 4 of Theorem 1.1 $c(0, 1, \eta)$ must be given in the form,

$$c(0, 1, \eta) = \eta_0(1 - \eta_1) g(\eta) + (1 - \eta_0) \eta_1 g(\eta) e^{-\Delta H(\eta)} \tag{81}$$

where g is a positive local function which dose not depend on η_0, η_1. Let us write $A \subset\subset \mathbf{Z}$ if A is a finite subset of $\mathbf{Z}$ and if we expand the function h appearing in the gradient condition in the form

$$h(\eta) = \sum_{A \subset\subset \mathbf{Z}} a(A)\eta^A. \tag{82}$$

Then

$$h(\tau_1\eta) - h(\eta) = \sum_{A \subset\subset \mathbf{Z}} \{a(\tau_{-1}A) - a(A)\}\eta^A. \tag{83}$$

We rewrite (81) as

$$c(0,1,\eta)(\eta_1 - \eta_0) = -\eta_0 g(\eta) + \eta_1 g(\eta) e^{-\Delta H(\eta)} + \eta_0\eta_1 g(\eta)(1 - e^{-\Delta H(\eta)}).$$

Equating the right side of this with that of (83) and comparing the coefficient of $1, \eta_0, \eta_1$ and $\eta_0\eta_1$ we deduce the following system of equations

$$\begin{aligned} 0 &= \sum_A{}^* \{a(\tau_{-1}A) - a(A)\}\eta^A & (84)\\ -g(\eta) &= \sum_A{}^* \{a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\})\}\eta^A & (85)\\ g(\eta)e^{-\Delta H(\eta)} &= \sum_A{}^* \{a(\tau_{-1}(A \cup \{1\})) - a(A \cup \{1\})\}\eta^A & (86)\\ g(\eta)(1 - e^{-\Delta H(\eta)}) &= \sum_A{}^* \{a(\tau_{-1}(A \cup \{0,1\})) - a(A \cup \{0,1\})\}\eta^A. & (87) \end{aligned}$$

The equation (84) holds if and only if

$$a(\tau_{-1}A) = a(A), \text{ for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}. \tag{88}$$

Since the sum on the left sides of the equations (85),(86),(87) vanishes, they imply that

$$\begin{aligned} &a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\}) + a(\tau_{-1}(A \cup \{1\})) - a(A \cup \{1\}) \\ &+ a(\tau_{-1}(A \cup \{0,1\})) - a(A \cup \{0,1\})\} = 0, \\ &\qquad\qquad \text{for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}. \end{aligned} \tag{89}$$

Since $e^{\Delta H(\eta)}$ dose not depend on η_0, η_1 we can expand it in the form

$$e^{\Delta H(\eta)} = \sum_B{}^* d(B)\eta^B. \tag{90}$$

From the equations (85),(86) and (90) it then follows that

$$
\begin{aligned}
&a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\}) \\
&+ \sum_{B \cup C = A} \{a(\tau_{-1}(C \cup \{1\})) - a(C \cup \{1\})\} d(B) = 0, \qquad (91) \\
&\qquad\qquad\qquad\qquad \text{for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}.
\end{aligned}
$$

Conversely, if a collection $\{a(A)\}_{A \subset\subset \mathbf{Z}}$ solves (88),(89) and (91), and we define $g(\eta)$ by (85), then we have the equations (84) through (87), so that the exchange rate given by (81) satisfies both the gradient condition and the detailed balance condition, provided that the function $h(\eta)$ given by (82) is local and the function $g(\eta)$ given by (85) satisfies $g(\eta) > 0$.

3 Notations and some results

For $A \subset \mathbf{Z} \setminus \{0,1\}$ we define $\Delta H(A)$ by

$$\Delta H(A) = \sum_{B \subset\subset A} (J_{B \cup \{0\}} - J_{B \cup \{1\}}).$$

It immediately follows that

$$\Delta H(\eta) = \Delta H(S(\eta) \setminus \{0,1\}),$$

where $S(\eta)$ is the support of η, i.e., $S(\eta) = \{x \in \mathbf{Z} : \eta_x = 1\}$.

We consider the sets

$$\Lambda = \{-p, -p+1, \ldots\} \text{ and } \Gamma = \{-p, -p+1, \ldots, p, p+1\}.$$

In view of (76) it holds that

$$
\begin{aligned}
\Delta H(A) &= \Delta H(A \cap \Gamma) \qquad (92) \\
\Delta H(\emptyset) &= 0 \qquad (93)
\end{aligned}
$$

Remark 3.1 *In the sequel we shall make use of some elementary formulas on the summation over subsets of a finite set, which are recalled here.*
(i) By the binomial expansion of $(1-1)^{\#A}$ *we have* $\sum_{B \subset A} (-1)^{\#(A \setminus B)} = 0$ *if* $A \neq \emptyset$.
(ii) Let Ω *be a finite set. If* f *and* g *are functions of subsets of* Ω*, then by using (i) it is easy to check that the following two conditions are equivalent.*

1. $f(A) = \sum_{B \subset A} g(B)$ *for all* $A \subset \Omega$.

2. $g(A) = \sum_{B \subset A} (-1)^{\#(A \setminus B)} f(B)$ *for all* $A \subset \Omega$.

Lemma 3.2 *The coefficients* $d(B)$ *defined by (90) satisfies that*

$$d(B) = 0 \text{ for all } B \subset \mathbf{Z} \setminus \{0,1\} \text{ such that } B \cap \Gamma^c \neq \emptyset.$$

Proof. Decompose B into $C = B \cap \Gamma$ and $D = B \setminus \Gamma$. Then by Remark 3.1 and (92)

$$\begin{aligned} d(B) &= \sum_{E \subset B} (-1)^{\#(B \setminus E)} e^{\Delta H(E)} \\ &= \sum_{G \subset D} (-1)^{\#(D \setminus G)} \sum_{F \subset C} (-1)^{\#(C \setminus F)} e^{\Delta H(F)} \end{aligned}$$

but the first factor of the last line equals zero according to Remark 3.1 (i).
□

Lemma 3.3 *If a collection* $\{a(A)\}_{A \subset\subset \mathbf{Z}}$ *satisfies the following two conditions*
i) $a(A) = 0$ *for all* A *such that* $A \cap \Lambda^c \neq \emptyset$, *and*
ii) $\{a(A)\}_{A \subset\subset \Lambda}$ *solves the equations (88),(89),(91) for all* $A \subset\subset \Lambda \setminus \{0,1\}$, *then the collection* $\{a(A)\}_{A \subset\subset \mathbf{Z}}$ *solves the equations (88),(89),(91) for all* $A \subset\subset \mathbf{Z} \setminus \{0,1\}$.

Proof. We have only to check (91) for A such that $A \cap \Lambda^c \neq \emptyset$. The left side of (91) is written as

$$\begin{aligned} & a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\}) \\ & \quad + \sum_{D \subset A, D \cap \Lambda^c \neq \emptyset, D \cup E = A} \{a(\tau_{-1}(E \cup \{1\})) - a(E \cup \{1\})\} d(D) \\ & \quad + \sum_{D \subset (A \cap \Lambda), D \cup E = A} \{a(\tau_{-1}(E \cup \{1\})) - a(E \cup \{1\})\} d(D), \end{aligned}$$

of which the second term vanishes, since if $D \subset A$ and $D \cap \Lambda^c \neq \emptyset$, then $d(D) = 0$ by Lemma 3.2. By condition i), $a(A \cup \{0\}) = a(\tau_{-1}(A \cup \{0\})) = 0$, and if $E \subset A$ and $E \cap \Lambda^c \neq \emptyset$ then $a(E \cup \{1\}) = a(\tau_{-1}(E \cup \{1\})) = 0$, so that the other two terms also vanish. □

Put

$$\tilde{a}(A) = a(A) - a(\tau_{-1}A). \tag{94}$$

Then it is easy to check that the conditions i) and ii) of Lemma 3.3 hold if and only if the following system of equations holds

$$a(A) = 0 \text{ whenever } A \cap \Lambda^c \neq \emptyset, \tag{95}$$

$$\tilde{a}(A) = 0 \text{ for all } A \subset\subset \Lambda \setminus \{0,1\}, \tag{96}$$

$$\sum_{D \subset A} \{\tilde{a}(D \cup \{0\}) + \tilde{a}(D \cup \{1\}) + \tilde{a}(D \cup \{0,1\})\} = 0 \tag{97}$$

$$\text{for all } A \subset\subset \Lambda \setminus \{0,1\},$$

$$\sum_{D \subset A} \tilde{a}(D \cup \{0\}) + \sum_{D \subset A} \tilde{a}(D \cup \{1\}) e^{\Delta H(A)} = 0 \tag{98}$$

$$\text{for all } A \subset\subset \Lambda \setminus \{0,1\}.$$

Recall the remark given at the end of section 2 , where we state the conditions for $h(\eta)$ and $g(\eta)$, that is, $h(\eta)$ is local and $g(\eta) > 0$. These are written as

$$\sum_{D \subset A} \tilde{a}(D \cup \{0\}) > 0 \text{ for all } A \subset\subset \Lambda \setminus \{0,1\} \tag{99}$$

and

$$a(A) = 0 \text{ whenever } A \cap C^c \neq \emptyset \tag{100}$$

where C is some finite set.

Given a set function $\{b(A)\}_{A \subset\subset \Lambda \setminus \{0,1\}}$, we introduce an additional system of equations

$$b(A) = \sum_{D \subset A} \tilde{a}(D \cup \{0\}), \tag{101}$$

so that we will get a unique solution of (94)-(98) and (101). If $b(A) > 0$ for all $A \subset\subset \Lambda \setminus \{0,1\}$, then the unique solution satisfies (99). Thus our problem of constructing h is solved if we can find $b(A) > 0$ so that the corresponding solution $a(A)$ also satisfies (100). Our proof of Theorem 1.1 in the next section consists of proving (100) for a suitably chosen $\{b(A)\}$.

4 Constructing an exchange rate

Definition 4.1 *We define a mapping $\tilde{\tau}$ from all finite subsets of $\mathbf{Z} \setminus \{0,1\}$ into themselves by*

$$\tilde{\tau}A = \begin{cases} \tau_1 A & -1 \notin A, \\ (\tau_2 A \setminus \{1\}) \cup \{2\} & -2 \notin A, -1 \in A, \\ \vdots & \\ (\tau_k A \setminus \{1\}) \cup \{k\} & -k \notin A, \{-1,-2,\ldots,-(k-1)\} \subset A, \\ \vdots & \end{cases}$$

and then $\tilde{\tau}^n, n \geq 2$, inductively by $\tilde{\tau}^n A = \tilde{\tau}(\tilde{\tau}^{n-1}A)$. We define $\tilde{\tau}^{-1}$ in the same way but with the position $-k$ replaced by $k+1$ and at the same time the shift τ_k by τ_{-k}, and define $\tilde{\tau}^n$ for $n < -1$ by iteration. Clearly $\tilde{\tau}^{-1}$ is the inverse of $\tilde{\tau}$.

For $A \subset\subset \mathbf{Z} \setminus \{0,1\}$ we define $b(A)$ by

$$b(A) = \prod_{n=1}^{\infty} e^{-\Delta H(\tilde{\tau}^n A)}. \tag{102}$$

From (92) and (93) it follows that

$$b(A) = \prod_{n=1}^{|\min A|+p} e^{-\Delta H(\tilde{\tau}^n A)} \quad \text{for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}.$$

Lemma 4.2 $\{b(A)\}$ *has the following properties*

1. *Locality:*
$$b(A) = b(A \cap \Gamma) \textit{ for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}. \tag{103}$$

2. *Relation between $b(A)$ and $b(\tilde{\tau}A)$:*
$$b(A) = b(\tilde{\tau}A)e^{-\Delta H(\tilde{\tau}A)} \textit{ for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}. \tag{104}$$

3. *Positivity:*
$$b(A) > 0 \textit{ for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}. \tag{105}$$

The property 1 will follow from the next lemma.

Lemma 4.3 *It holds that*

$$\sum_{k=-\infty}^{\infty} \Delta H(\tilde{\tau}^k A) = 0 \textit{ for all } A \subset\subset \mathbf{Z} \setminus \{0,1\}. \tag{106}$$

Proof. Let $A \subset\subset \mathbf{Z} \setminus \{0,1\}$ and k be the number of connected components of A. We may suppose $\min A = 2$ since there can always be found n such that $\min \tilde{\tau}^n A = 2$. We write

$$\tilde{A}_1 = A = A_1 \cup A_2 \cup \ldots \cup A_k$$

where A_i are connected components arranged in order from the left to the write; $A_i = \{a_i, a_i+1, \ldots, b_i\}$, $2 = a_1 \leq b_1 < a_2 - 1 < b_2 < \ldots < a_k - 1 < b_k$. We define $\tilde{A}_l$ $(1 \leq l \leq 2k)$ by

$$\tilde{A}_{2i-1} = \tilde{\tau}^{f(i)} \tilde{A}_1, \quad \tilde{A}_{2i} = \tilde{\tau}^{f(i)-1} \tilde{A}_1,$$

where

$$f(i) = \begin{cases} 0 & i = 1, \\ -\sum_{j=2}^{i} (a_j - b_{j-1} - 1) & 2 \leq i \leq k. \end{cases}$$

Notice that the mapping $\tilde{\tau}$ conserves the number of connected components as well as the number of elements. The function f is chosen so that the left end of the i-th component of $\tilde{A}_{2i-1}$ is 2 and the right end of the i-th component of $\tilde{A}_{2i}$ is -1. Now, if $\tilde{\tau}^k A = \tau_1 \tilde{\tau}^{k-1} A$ (i.e., $-1 \notin \tilde{\tau}^{k-1} A$), then $\sum_{D \subset \tilde{\tau}^k A} J_{D \cup \{1\}} = \sum_{D \subset \tilde{\tau}^{k-1} A} J_{D \cup \{0\}}$. So

$$-\sum_{l=-\infty}^{\infty} \Delta H(\tilde{\tau}^l A) = \sum_{i=1}^{k} \left(\sum_{D \subset \tilde{A}_{2i-1}} J_{D \cup \{1\}} - \sum_{D \subset \tilde{A}_{2i}} J_{D \cup \{0\}} \right). \tag{107}$$

We must show that the right side of (107) vanishes. To this end we construct the one-to-one mapping from $\cup_{i=1}^k \mathcal{P}(\tilde{A}_{2i-1})$ into $\cup_{i=1}^k \mathcal{P}(\tilde{A}_{2i})$, where $\mathcal{P}(A)$ denotes a power set of A. To define the mapping, first we decompose $\tilde{A}_i$ into connected components $A_{i,j}$ for $1 \leq j \leq k$:

$$\tilde{A}_i = A_{i,1} \cup A_{i,2} \cup \ldots \cup A_{i,k}.$$

Now consider a subset $D \subset \tilde{A}_{2i-1}$. Put $D_j = D \cap A_{2i-1,j}$ if $j \neq i$, and $D_i = (D \cap A_{2i-1,i}) \cup \{1\}$, and define k_{D_j} by

$$k_{D_j} = \begin{cases} \min\{k : k > 0, \tau_k D_j \cap A^c_{2i-1,j} \neq \emptyset\} - 1 & D_j \neq \emptyset, \\ \infty & D_j = \emptyset. \end{cases}$$

Because $k_{D_i} < \infty$, there exists $1 \leq p \leq k$ and positive integers $j_1, j_2, \dots j_p$ which satisfy $k_{D_{j_1}} = k_{D_{j_2}} = \dots = k_{D_{j_p}} < k_{D_m}$ for $m \neq j_q$ $(q = 1, 2, \dots, p)$. We order the number $1, 2, \dots, k$ by $i, i+1, \dots, k, 1, 2, \dots, i-1$ and let j_q be the first member of $\{j_1, \dots, j_p\}$ in this ordering. Then we can find $E \subset \tilde{A}_{2j_q}$ such that

$$\tau_{-l}(E \cup \{0\}) = D \cup \{1\}$$

where $l = b_{j_q} - k_{D_{j_q}}$. We can determine the inverse mapping in the same manner but with the position k replaced with $-k+1$ and the ordering is reversed. It would be clear that by means of this one-to-one correspondence the sum on the right side of (107) vanishes by cancellation. □

Proof of Lemma 4.2. If $A = B \cup C$ where $B = A \cap (\Lambda \setminus \Gamma)$ and $C = A \setminus B$ then

$$-\Delta H(\tilde{\tau} A) = -\Delta H(\tilde{\tau} C),$$

because $(\tilde{\tau} A) \cap \Gamma$ does not depend on the part B of A. Hence

$$b(A) = b(C).$$

On the other hand, by Lemma 4.3,

$$b(A) = \prod_{k=0}^{\infty} e^{\Delta H(\tilde{\tau}^{-k} A)},$$

which shows $b(C) = b(C \cap \Gamma)$ by the same reasoning as above. Thus $b(A) = b(C \cap \Gamma) = b(A \cap \Gamma)$. The property 1 has been verified. The properties 2 and 3 are trivial by definition. □

Lemma 4.4 *Let $b(A)$ be given by (102). Then the unique solution of (94)-(98) and (101) satisfies the condition (100).*

Proof. Clearly it suffices to prove
(i) if $0 \notin A$ then $a(A) = 0$,
and
(ii) if $\mathrm{diam} A > \mathrm{diam}\Gamma$ then $a(A) = 0$.
The proof of (i) is carried out by double induction on $\#A$ and $\max A$. Given a set A such that $0 \notin A$, we will assume that $a(B) = 0$ if either $\#B < \#A, 0 \notin B$ or $\#B = \#A, \max B < \max A, 0 \notin B$. The equation (96) and the assumption imply that

$$a(A) = \tilde{a}(A) + a(\tau_{-1} A) = 0 \text{ for all } A \text{ such that } 0, 1 \notin A. \tag{108}$$

The equations (96) and (101) imply that

$$\begin{aligned}\sum_{D\subset A} a(D\cup\{0\}) &= \sum_{D\subset A}\tilde a(D\cup\{0\}) + \sum_{D\subset\tau_{-1}A} a(D\cup\{-1\}) \\ &= b(A\cap\Lambda) + \sum_{D\subset\tau_{-1}A\cup\{-1\}} a(D) - \sum_{D\subset\tau_{-1}A} a(D), \quad (109) \\ &\qquad \text{for all } A \text{ such that } 0,1\notin A\end{aligned}$$

The equations (96)-(98) and (101) imply that

$$\begin{aligned}\sum_{D\subset A}\tilde a(D\cup\{1\}) &= -b(A\cap\Lambda)e^{-\Delta H(A\cap\Lambda\setminus\{1\})} \\ \sum_{D\subset A}\tilde a(D\cup\{0,1\}) &= b(A\cap\Lambda)(e^{-\Delta H(A\cap\Lambda)}-1) \\ &\qquad \text{for all } A \text{ such that } 0,1\notin A\end{aligned}$$

Hence for all A such that $0\notin A$ and $1\in A$ we have

$$\begin{aligned}&\sum_{D\subset A} a(D) \\ &= \sum_{D\subset A\setminus\{1\}}\{a(D\cup\{1\})+a(D)\} \\ &= \sum_{D\subset A\setminus\{1\}}\{\tilde a(D\cup\{1\})+\tilde a(D)\} + \sum_{D\subset\tau_{-1}A\setminus\{0\}}[a(D\cup\{0\})+a(D)] \\ &= -b(A\cap\Lambda\setminus\{1\})e^{-\Delta H(A\cap\Lambda\setminus\{1\})} + \sum_{D\subset\tau_{-1}A\setminus\{0\}}[a(D\cup\{0\})+a(D)], \quad (110)\end{aligned}$$

and

$$\begin{aligned}&\sum_{D\subset A} a(D\cup\{0\}) \\ &= \sum_{D\subset A\setminus\{1\}}\{a(D\cup\{0,1\})+a(D\cup\{0\})\} \\ &= \sum_{D\subset A\setminus\{1\}}\{\tilde a(D\cup\{0,1\})+\tilde a(D\cup\{0\})\} \\ &\quad + \sum_{D\subset\tau_{-1}A\setminus\{0\}}[a(D\cup\{-1,0\})+a(D\cup\{-1\})] \\ &= b(A\cap\Lambda\setminus\{1\})e^{-\Delta H(A\cap\Lambda\setminus\{1\})} - \sum_{D\subset\tau_{-1}A\setminus\{0\}}[a(D\cup\{0\})+a(D)] \\ &\quad + \sum_{D\subset\tau_{-1}A\cup\{-1\}\setminus\{0\}}[a(D\cup\{0\})+a(D)] \quad (111)\end{aligned}$$

The two equations (110) and (111) imply that

$$\sum_{D\subset A}[a(D)+a(D\cup\{0\})]=\sum_{D\subset\tau_{-1}A\cup\{-1\}\setminus\{0\}}[a(D\cup\{0\})+a(D)] \qquad (112)$$

for all A such that $0\notin A$, $1\in A$. By (96) and (108), we have only to consider $A\subset\subset\Lambda$ such that $0\notin A$ and $1\in A$. By the assumption of induction $a(A)=\sum_{D\subset A}a(D)$ for $0\notin A$. We may suppose that $A\subset\subset\Lambda$ is a union of $\{1,2,\ldots,k\}(k\geq 1)$ and $\tilde{A}$ for which $\tilde{A}\cap\{1,2,\ldots,k,k+1\}=\emptyset$. Then we have by (110)

$$\begin{aligned} a(A) &= \sum_{D\subset\{1,2,\ldots,k\}\cup\tilde{A}} a(D) \\ &= -b((\{2,3,\ldots,k\}\cup\tilde{A})\cap\Lambda)e^{-\Delta H((\{2,3,\ldots,k\}\cup\tilde{A})\cap\Lambda} \\ &\quad+\sum_{D\subset\{1,2,\ldots,k-1\}\cup\tau_{-1}\tilde{A}}[a(D\cup\{0\})+a(D)] \end{aligned}$$

By (112) the last sum equals

$$\sum_{D\subset\{1,2,\ldots,k-2\}\cup\{-1\}\cup\tau_{-2}\tilde{A}}[a(D\cup\{0\})+a(D)],$$

and repeating the same procedure we arrive at

$$\begin{aligned} &= \sum_{D\subset\{-1,-2,\ldots,-(k-1)\}\cup\tau_{-k}\tilde{A}}[a(D\cup\{0\})+a(D)] \\ &= \sum_{D\subset\{-1,-2,\ldots,-(k-1)\}\cup\tau_{-k}\tilde{A}} a(D\cup\{0\}) \end{aligned}$$

Therefore by (109)

$$\begin{aligned} a(A) &= -b((\{2,3,\ldots,k\}\cup\tilde{A})\cap\Lambda)e^{-\Delta H((\{2,3,\ldots,k\}\cup\tilde{A})\cap\Lambda)} \\ &\quad+b((\{-1,-2,\ldots,-(k-1)\}\cup\tilde{A}_{-k}\cap\Lambda)), \end{aligned}$$

which vanishes in view of (103) and (104) since $\tilde{\tau}(\{-1,-2,\ldots,-(k-1)\}\cup\tau_{-k}\tilde{A})=\{1,2,\ldots,k\}\cup\tilde{A}$. Claim (i) has been verified.
For the proof of (ii) it suffices, by virtue of the first claim (i), to prove

$$\tilde{a}(A)=0 \text{ whenever } \operatorname{diam}(A)>\operatorname{diam}\Gamma. \qquad (113)$$

If $0,1 \notin A$ then (113) is trivial. Suppose $0,1 \notin B$ and consider the cases $A = B \cup \{0\}$, $B \cup \{1\}$ or $B \cup \{0,1\}$.
We decompose B into $C = B \cap \Gamma$ and $D = B \setminus \Gamma$ and apply Remark 3.1 (ii), the defining relation (101), (103) and Remark 3.1 (i) in turn to see

$$\begin{aligned}
\tilde{a}(B \cup \{0\}) &= \sum_{F \subset B} (-1)^{\#(B \setminus F)} b(F) \\
&= \left(\sum_{F \subset C} (-1)^{\#(C \setminus F)} b(F) \right) \left(\sum_{G \subset D} (-1)^{\#(D \setminus G)} \right) \\
&= 0
\end{aligned}$$

(notice that $D \neq \emptyset$ since $\mathrm{diam} A > \mathrm{diam}\Gamma$). We can show $\tilde{a}(B \cup \{1\}) = \tilde{a}(B \cup \{0,1\}) = 0$ in the same way. □

For the proof of Remark 1.2 we prove that the exchange rate constructed by the solution of (94)-(98) , (101) and (102) depends only on $\{\eta_z; z \in \{-p, -p+1, \ldots, p, p+1\}\}$, and the function $h(\eta)$ in (79) depends only on $\{\eta_z; z \in \{-p, -p+1, \ldots, p-1, p\}\}$. To this end we first notice that the exchange rate $c(0,1,\eta)$ is rewritten as

$$\begin{aligned}
c(0,1,\eta) &= \eta_0(1-\eta_1)g(\eta) + (1-\eta_0)\eta_1 g(\eta) e^{-\Delta H(\eta)} \\
&= \eta_0(1-\eta_1) b(S(\eta) \setminus \{0,1\}) \\
&\quad + (1-\eta_0)\eta_1 b(S(\eta) \setminus \{0,1\}) e^{-\Delta H(S(\eta) \setminus \{0,1\})},
\end{aligned}$$

where $S(\eta)$ is the support of η, i.e. $S(\eta) = \{x \in \mathbf{Z} : \eta_x = 1\}$. Since $b(A) = b(A \cap \Gamma)$ and $e^{-\Delta H(A)} = e^{-\Delta H(A \cap \Gamma)}$, $c(0,1,\eta)$ depends only on $\{\eta_z; z \in \Gamma\} = \{\eta_z; z \in \{-p, -p+1, \ldots, p, p+1\}\}$; hence so does $\tau_1 h(\eta) - h(\eta) = c(0,1,\eta)(\eta_1 - \eta_0)$. It would be obvious that the function $h(\eta)$ depends only on $\{\eta_z; z \in \{-p, -p+1, \ldots, p-1, p\}\}$.

5 Biased exchange rate

In this section, we consider the driven lattice gas on a discrete torus $\mathbf{T}_N = \mathbf{Z}/N\mathbf{Z}$. Assume $c(x, x+1, \eta)$ satisfies the condition 1–4 of the Theorem 1.1. Let L^N be the generator defined by

$$L^N f(\eta) = \sum_{x \in \mathbf{T}_N} c(x, x+1, \eta)(f(\eta^{x,x+1}) - f(\eta)),$$

and X^N be the Markov process whose generator is L^N. Let $c_p(x,x+1,\eta)$ be biased exchange rate if

$$c_p(x,x+1,\eta) = p\eta_x c(x,x+1,\eta) + (1-p)\eta_{x+1}c(x,x+1,\eta)$$

for $0 \le p \le 1$, and L_p^N be the generator defined by

$$L_p^N f(\eta) = \sum_{x\in\mathbf{T}_N} c_p(x,x+1,\eta)(f(\eta^{x,x+1}) - f(\eta)),$$

and X_p^N be the Markov process whose generator is L_p^N.

Proposition 5.1 *For each sufficiently large N, the class of invariant measures for X_p^N coincides with that of X^N if and only if $c(x,x+1,\eta)$ satisfies the condition 5 of Theorem 1.1*

Proof. Assume μ_N is an invariant measure of X^N, which is a Gibbs measure on $\mathbf{T}_N$ with Hamiltonian. By a simple computation,

$$\int L_p^N f(\eta)\mu_N(d\eta) = \int \sum_{x\in\mathbf{T}_N} (1-2p)(\eta_x - \eta_{x+1})c(x,x+1,\eta)f(\eta)\mu_N(d\eta) \quad (114)$$

The sufficiency follows from the condition 5 of Theorem 1.1, because the right side of (114) is equal to

$$\int \sum_{x\in\mathbf{T}_N} (1-2p)(\tau_x h(\eta) - \tau_{x+1}h(\eta))f(\eta)\mu_N(d\eta) = 0.$$

The proof of necessity is immediate from (114) and the following lemma.

Lemma 5.2 *Let $F(\eta) = \sum_{A\subset\mathbf{T}_N} f(A)\eta^A$ be a local function and satisfies*

$$\sum_{x\in\mathbf{T}_N} \tau_x F(\eta) = 0 \quad (115)$$

for all η. Then there exists a local function $g(\eta)$ such that

$$F(\eta) = g(\eta) - \tau_1 g(\eta)$$

Proof. First, we show that the coefficient $f(A)$ satisfies that

$$\sum_{x\in\mathbf{T}_N} f(\tau_x A) = 0 \quad (116)$$

for all $A \subset \mathbf{T}_N$ by induction on the cardinality of A. Let $A = \emptyset$, considering $\eta : \eta_x = 0$ for all $x \in \mathbf{T}_N$, then we have

$$0 = \sum_{x \in \mathbf{T}_N} f(\tau_x \emptyset).$$

Assume that if $B \subset A$ but $B \neq A$, then $\sum_{x \in \mathbf{T}_N} f(\tau_x B) = 0$. Considering $\eta : \eta_x = 1$ for all $x \in A$ and $\eta_x = 0$ for all $x \notin A$, then we have

$$0 = \sum_{x \in \mathbf{T}_N} f(\tau_x A) + \sum_{B \subset A, B \neq A} \sum_{x \in \mathbf{T}_N} f(\tau_x B).$$

We partition the power set $\mathcal{P}(\mathbf{T}_N)$ into the equivalence classes of congruence. Denote by $\mathcal{T}$ the set of representations, i.e., $\mathcal{T}$ is a family of sets which satisfies that
(i) If $A \in \mathcal{T}$, then $\tau_x A \notin \mathcal{T}$ for $x \neq 0$
(ii) $\{\tau_x A\}_{x \in \mathbf{T}_N, A \in \mathcal{T}} = \mathcal{P}(\mathbf{T}_N) \setminus \emptyset$.
By means of $\mathcal{T}$, the function F may be written as

$$F(\eta) = \sum_{A \in \mathcal{T}} \sum_{x \in \mathbf{T}} f(\tau_x A) \eta^{\tau_x A}.$$

Because $F(\eta)$ is a local function, for each $A \in \mathcal{T}$ $f(\tau_x A)$ vanishes except for a finite number of x. We can choose n and $\{x_i\}_{i=1}^n$ such that $x_i < x_{i+1}$ and $f(\tau_x A) = 0$ if $x \neq x_i$ for all i. Now we decompose

$$\begin{aligned} \sum_{x \in \mathbf{T}} f(\tau_x A) \eta^{\tau_x A} &= \sum_{i=1}^{n} f(\tau_{x_i}) \eta^{\tau_{x_i} A} \\ &= f(\tau_{x_1} A)(\eta^{\tau_{x_1} A} - \eta^{\tau_{x_2} A}) \\ &\quad + (f(\tau_{x_1} A) + f(\tau_{x_2} A)) \eta^{\tau_{x_2} A} + \sum_{i=3}^{n} f(\tau_{x_i}) \eta^{\tau_{x_i} A} \end{aligned}$$

repeating the same procedure we arrive at

$$\sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} f(\tau_{x_k} A) \right) (\eta^{\tau_{x_i} A} - \eta^{\tau_{x_{i+1}} A}) + \sum_{k=1}^{n} f(\tau_{x_k} A) \eta^{\tau_{x_n} A} \tag{117}$$

The second sum is equal to zero according to the equality (116). It is easy to see that the first sum is of the form $g_A(\eta) - \tau_1 g_A(\eta)$. □

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