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Study of the Orthogonal Drawing of Graphs

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Chapter 1

Introduction

1.1 Background

We consider the problem of generating orthogonal drawings of graphs in the plane and space. The problem has obvious applications in the design of 2-D and 3-D VLSI circuits and optoelectronic integrated systems: see for example [1], [8], and [10].

Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of vertices of G. G is called a k-graph if $\Delta(G) \leq k$. The connectivity of a graph is the minimum number of vertices whose removal results in a disconnected graph or a single vertex graph. A graph is said to be k-connected if the connectivity of the graph is at least k.

It is well-known that every graph can be drawn in the space so that its edges intersect only at their ends. Such a drawing of a graph G is called a 3-D drawing of G. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a 2-D drawing of G.

A 2-D orthogonal drawing of a planar graph G is a 2-D drawing of G such that each edge is drawn by a sequence of contiguous horizontal and vertical line segments. A 3-D orthogonal drawing of a graph G is a 3-D drawing of G such that each edge is drawn by a sequence of contiguous axis-parallel line segments. Notice that a graph G has a 2-D[3-D] orthogonal drawing only if $\Delta(G) \leq 4[\Delta(G) \leq 6]$. An orthogonal drawing with no more than b bends per edge is called a b-bend orthogonal drawing.

Biedl and Kant [3], and Liu, Morgana, and Simeone [9] showed that every planar 4-graph has a 2-bend 2-D orthogonal drawing with the only exception of the octahedron, which has a 3-bend 2-D orthogonal drawing. Moreover, Kant [7] showed that every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception of K_4. On the other hand, Garg and Tamassia proved that it is NP-complete to decide if a given planar 4-graph has a 0-bend 2-D orthogonal drawing [6]. Battista, Liotta, and Vargiu showed that the problem can be solved in polynomial time for planar 3-graphs and series-parallel graphs [2].

Eades, Symvonis, and Whitesides [5], and Papakostas and Tollis [11] showed that every 6-graph has a 3-bend 3-D orthogonal drawing. Moreover, Wood showed that every 5-graph has a 2-bend 3-D orthogonal drawing [14]. On the other hand, Eades, Stirk, and Whitesides proved that it is NP-complete to decide if a given 5-graph has a 0-bend 3-D orthogonal drawing [4].

1.2 Thesis Outline

The rest of this thesis is organized as follows.

In Chapter 2, we introduce outerplanar graph and series-parallel graph. We consider the orthogonal drawing of their graphs in this thesis.

In Chapter 3, we discuss orthogoanl drawing for outerplanar graphs. We show in Section 3.1 that an outerplanar 3-graph G has a 0-bend 2-D orthogonal drawing if and only if G contains no triangle as a subgraph. We show in Section 3.2 that a series-parallel 4-graphs G has a 1-bend 2-D orthogonal drawing.

In Chapter 4, we discuss orthogonal drawing for series-parallel graphs. We show in Section 4.1 that an outerplanar 6-graph G has a 0-bend 3-D orthogonal drawing if and only if G contains no triangle as a subgraph. We show in Sectio 4.2 that a series-parallel 6-graphs G has a 2-bend 3-D orthogonal drawing.

Chapter 2

Preliminaries

2.1 Outerplanar Graphs

A 2-D drawing of a planar graph G is regarded as a graph isomorphic to G, and referred to as a plane graph. A plane graph partitions the rest of the plane into connected regions. A face is a closure of such a region. The unbounded region is referred to as the external face. We denote the boundary of a face f of a plane graph Γ by $b(f)$. If Γ is 2-connected then $b(f)$ is a cycle of Γ.

Given a plane graph Γ, we can define another graph Γ^* as follows: corresponding to each face f of Γ there is a vertex f^* of Γ^*, and corresponding to each edge e of Γ there is an edge e^* of Γ^*; two vertices f^* and g^* are joined by the edge e^* in Γ^* if and only if the edge e in Γ lies on the common boundary of faces f and g of Γ. Γ^* is called the (geometric-)dual of Γ.

A graph is said to be outerplanar if it has a 2-D drawing such that every vertex lies on the boundary of the external face. Such a drawing of an outerplanar graph is said to be outerplane. Let Γ be an outerplane graph with the external face f_o, and $\Gamma^* - f_o^*$ be a graph obtained from Γ^* by deleting the vertex f_o^* together with the edges incident to f_o^*. It is easy to see that if Γ is an outerplane graph then $\Gamma^* - f_o^*$ is a forest. In particular, an outerplane graph Γ is 2-connected if and only if $\Gamma^* - f_o^*$ is a tree.

2.2 Series-Parallel Graphs

A *series-parallel graph* is defined recursively as follows:

(1) A graph consisting of two vertices joined by a single edge is a series-parallel graph. The vertices are the terminals.

(2) If G_1 is a series-parallel graph with terminals s_1 and t_1, and G_2 is a series-parallel graph with terminals s_2 and t_2, then a graph G obtained by either of the following operations is also a series-parallel graph:

(i) *Series composition*: identify t_1 with s_2. Vertices s_1 and t_2 are the terminals of G.

(ii) *Parallel composition*: identify s_1 and s_2 into a vertex s, and t_1 and t_2 into a vertex t. Vertices s and t are the terminals of G.

A series-parallel graph G is naturally associated with a binary tree $T(G)$, which is called a *decomposition tree* of G. The nodes of $T(G)$ are of three types, S-nodes, P-nodes, and Q-nodes. $T(G)$ is defined recursively as follows:

(1) If G is a single edge, then $T(G)$ consists of a single Q-node.

(2-i) If G is obtained from series-parallel graphs G_1 and G_2 by the series composition, then the root of $T(G)$ is a S-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.

(2-ii) If G is obtained from series-parallel graphs G_1 and G_2 by the parallel composition, then the root of $T(G)$ is a P-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.

Notice that the leaves of $T(G)$ are the Q-nodes, and an internal node of $T(G)$ is either an S-node or P-node. Notice also that every P-node has at most one Q-node as a child, since G is a simple graph. If G has n vertices then $T(G)$ has $O(n)$ nodes, and $T(G)$ can be constructed in $O(n)$ time [13].

Chapter 3

Orthogonal Drawing of Outerplanar Graphs

3.1 2-D Orthogonal Drawing

An edge of a plane graph Γ which is incident to exactly one vertex of a cycle C and located outside C is called a leg of C. A cycle C of Γ is said to be k-legged if C has exactly k legs.

The planar representation $P(\Gamma)$ of a plane graph Γ is the collection of circular permutations of the edges incident to each vertex. Plane graphs Γ and Γ' are said to be equivalent if $P(\Gamma)$ is isomorphic to $P(\Gamma')$.

The following interesting theorem was proved by Rahman, Naznin, and Nishizeki [12].

Theorem I *A plane* 3*-graph* Γ *has an equivalent* 0*-bend* 2*-D orthogonal drawing if and only if every* k*-legged cycle in* Γ *contains at least* $4-k$ *vertices of degree* 2 *in* Γ *for any* k*,* $0 \leq k \leq 3$*.* ∎

We show in this section the following theorem.

Theorem 3.1 *An outerplanar* 3*-graph* G *has a* 0*-bend* 2*-D orthogonal drawing if and only if* G *contains no triangle as a subgraph.*

Proof: The necessity is obvious. We show the sufficiency. Let G be an outerplanar 3-graph with no triangles, and Γ be an outerplane graph isomorphic to G. We show that Γ satisfies the condition of Theorem I.

Lemma 3.1 *If* Γ *is* 2*-connected then the boundary of the external face* f_o *contains at least* 4 *vertices of degree* 2 *in* Γ*.*

Proof of Lemma 3.1: If Γ is a cycle then the lemma is obvious. Suppose that Γ has more than one cycle. Since Γ is 2-connected, $\Gamma^* - f_o^*$ is a tree. Since Γ contains no triangles, the boundary of a face of Γ corresponding to a leaf of $\Gamma^* - f_o^*$ contains at least 2 vertices of degree 2 in Γ, which also lie on the boundary of f_o. Since a tree has at least 2 leaves, we obtain the lemma. ∎

It is easy to see that every cycle C of Γ is the boundary of the external face of a 2-connected outerplane subgraph of Γ. Thus, by Lemma 3.1, C contains at least 4 vertices of degree 2 in the subgraph. It follows that if C is a k-legged cycle in Γ then C contains at least $4 - k$ vertices of degree 2 in Γ. This completes the proof of the theorem. ∎

It should be noted that there exists an outerplanar 4-graph with no triangles that has no 0-bend 2-D orthogonal drawings. Fig. 3.1 shows such a graph F. F has a pentagon and five squares. If F has a 0-bend 2-D orthogonal drawing then the pentagon and squares are drawn as rectangles. All the squares must lie outside a rectangle R representing the pentagon. This is impossible, however, since there exists a pair of consecutive squares which lie to the same side of R.

3.2 3-D Orthogonal Drawing

We show in this section the following theorem.

Theorem 3.2 *An outerplanar 6-graph G has a 0-bend 3-D orthogonal drawing if and only if G contains no triangle as a subgraph.* ∎

The necessity is obvious. We will show the sufficiency in the rest of the section.

3.2.1 2-Connected Outerplanar Graphs

We first consider the case when G is 2-connected. Let G be a 2-connected outerplanar 6-graph with no triangles, and Γ be an outerplane graph isomorphic to G. Since Γ is 2-connected, $T^* = \Gamma^* - f_o^*$ is a tree. A leaf r^* of T^* is designated as a root, and T^* is considered as a rooted tree. If g^* is a child of f^* in T^*, f is called the parent face of g, and g is called a child face of f in Γ. The unique edge in $b(f) \cap b(g)$ is called the base of g. The base of r is defined as an edge with both ends of degree 2. Let S^* be a tree rooted at r^* consisting of r^* together with a subtree rooted at a child of r^* and an edge connecting r^* and the child. If r^* has no child then S^* is consisting of just r^*. $\Gamma(S^*)$ is a subgraph of Γ induced by the vertices on boundaries of faces of Γ corresponding to the vertices of S^*. If S^* is consisting of just r^* then $\Gamma(S^*)$ is denoted by $\Gamma(r^*)$. It should be noted that $\Gamma(S^*)$ is a 2-connected outerplane graph with no triangles. Let f^* be a vertex of S^*, and $f_c^* \in V(T^*) - V(S^*)$ be a child of f^* in T^*. $S^* + f_c^*$ is a rooted tree obtained from S^* by adding f_c^* and an edge (f^*, f_c^*).

For any face f of Γ, $b(f)$ is a cycle, since Γ is 2-connected. Let $b(f) = \{e_0, e_1, \ldots, e_{k-1}\}$, where e_0 is the base of f, and edges e_i and $e_{i+1(\bmod k)}$ are adjacent. A 0-bend 2-D orthogonal drawing of f is said to be canonical if f is drawn as a rectangle such that the edges $e_2, e_3, \ldots,$ and e_{k-2} are drawn on a side of the rectangle. A drawing of $\Gamma(S^*)$ is said to be canonical if every face is drawn canonically.

Fig. 3.2.2 shows a rooted tree T^* for F shown in Fig. 3.1, where r is a square face, and a 0-bend 3-D orthogonal canonical drawing of F.

Roughly speaking, we will show that if $\Gamma(S^*)$ has a 0-bend 3-D orthogonal canonical drawing then $\Gamma(S^*+f_c^*)$ also has a 0-bend 3-D orthogonal canonical drawing. The following theorem immediately follows by induction.

Theorem 3.3 *A 2-connected outerplanar 6-graph with no triangles has a 0-bend 3-D orthogonal drawing.* ∎

3.2.1.1 Proof of Theorem 3.3

For any $v \in V(\Gamma)$, we define that f_v is a face such that v is on $b(f_v)$ and f_v^* is the nearest vertex to r^* in T^*. We denote by $I_\Gamma(v)$ the set of edges incident with v in Γ.

Let $\Lambda(S^*)$ be a 0-bend 3-D orthogonal canonical drawing of $\Gamma(S^*)$. We assume without loss of generality that each vertex of $\Lambda(S^*)$ is positioned at a grid-point in the three-dimensional space. Let

$$\phi : V(\Gamma(S^*)) \to V(\Lambda(S^*))$$

be an isomorphism between $\Gamma(S^*)$ and $\Lambda(S^*)$. The mapping ϕ is called a layout of $\Gamma(S^*)$. If $\phi(v) = (v_x, v_y, v_z)$, we denote $v_x = \phi_x(v), v_y = \phi_y(v)$, and $v_z = \phi_z(v)$. Notice that $\Lambda(S^*)$ is uniquely determined by ϕ.

Let $\boldsymbol{e}_x = (1,0,0), \boldsymbol{e}_y = (0,1,0), \boldsymbol{e}_z = (0,0,1)$ and define that $\mathcal{D} = \{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z, -\boldsymbol{e}_x, -\boldsymbol{e}_y, -\boldsymbol{e}_z\}$. For any $v \in V(\Gamma(S^*))$, β_v is a one-to-one mapping from $I_\Gamma(v)$ to $\mathcal{D}$. If f is a face with the base $e = (u,v)$, and f' is a child face of f with the base $e' = (u, v')$, let $\{e''\} = b(f') \cap I_\Gamma(u) - \{e'\}$. A mapping β_u is said to be admissible if f' does not exist or $\beta_u(e'')$ is orthogonal with both $\beta_u(e)$ and $\beta_u(e')$. A set

$$\mathcal{B}(S^*) = \{\beta_v \mid v \in V(\Gamma(S^*))\}$$

is called a canonical orientation for $\Lambda(S^*)$ if the following conditions are satisfied:

(B1) For any $v \in V(\Gamma(S^*))$ and $e, e' \in I_\Gamma(v)$, if $e, e' \in b(f)$ for a face $f \neq f_v$ of Γ then $\beta_v(e)$ and $\beta_v(e')$ are orthogonal.

(B2) If $e_0 = (u, v) \in E(\Gamma(S^*))$ is the base of a face f in Γ, $\{e_1\} = b(f) \cap I_\Gamma(u) - \{e_0\}$, and $\{e_{k-1}\} = b(f) \cap I_\Gamma(v) - \{e_0\}$ then $\beta_u(e_1) = \beta_v(e_{k-1})$.

(B3) For any $e = (u, v) \in E(\Gamma(S^*))$, $\phi(u) = \phi(v) + m\beta_v(e)$ and $\phi(v) = \phi(u) + m\beta_u(e)$ for some integer m.

(B4) If $e_0 = (u, v) \in E(\Gamma(S^*))$ is the base of a face f in Γ then β_u or β_v is admissible.

We prove the theorem by induction. The basis of the induction is stated in the following lemma, whose proof is obvious.

Lemma 3.2 *$\Gamma(r^*)$ has a 0-bend 3-D orthogonal drawing with a canonical orientation.* ∎

Let f^* be a vertex of S^* with a child $f^*_c \in V(T^*) - V(S^*)$.

Lemma 3.3 *If $\Gamma(S^*)$ has a 0-bend 3-D orthogonal canonical drawing with a canonical orientation then $\Gamma(S^* + f^*_c)$ also has a 0-bend 3-D orthogonal canonical drawing with a canonical orientation.*

Proof of Lemma 3.3 Let $\Lambda(S^*)$ be a 0-bend 3-D orthogonal canonical drawing of $\Gamma(S^*)$ with a canonical orientation $\mathcal{B}(S^*) = \{\beta_v \mid v \in V(\Gamma(S^*))\}$, and ϕ be the layout of $\Gamma(S^*)$.

Let $b(f_c) = \{e_0 = (v_0, v_{k-1}), e_1 = (v_0, v_1), \ldots, e_{k-1} = (v_{k-2}, v_{k-1})\}$, where e_0 is the base of f_c. We assume without loss of generality that $\beta_{v_0}(e_0) = \boldsymbol{e}_x$ and $\beta_{v_0}(e_1) = \boldsymbol{e}_y$. Now we define a layout ϕ' of $\Gamma(S^* + f^*_c)$. For the vertices $v_1, v_2, \ldots, v_{k-3}$ on $b(f_c)$, we define that

$$\phi'(v_i) = \phi(v_0) + \boldsymbol{e}_y + (i-1)\boldsymbol{e}_x, \quad 1 \leq i \leq k-3.$$

We also define that

$$\begin{aligned}
\phi'(v_0) &= \phi(v_0), \\
\phi'_x(v_{k-1}) &= \max\{\phi_x(v_{k-1}), \phi_x(v_0) + k - 3\}, \\
\phi'_y(v_{k-1}) &= \phi_y(v_{k-1}), \\
\phi'_z(v_{k-1}) &= \phi_z(v_{k-1}), \text{ and} \\
\phi'(v_{k-2}) &= \phi'(v_{k-1}) + \boldsymbol{e}_y.
\end{aligned}$$

Let $l = \phi'_x(v_{k-1}) - \phi_x(v_{k-1})$. For each vertex $v \in V(\Gamma(S^*)) - \{v_0, v_{k-1}\}$, we define that

$$\begin{aligned}
\phi'_x(v) &= \begin{cases} \phi_x(v) & \text{if } \phi_x(v) \leq \phi_x(v_0), \\ \phi_x(v) + l & \text{if } \phi_x(v) > \phi_x(v_0), \end{cases} \\
\phi'_y(v) &= \begin{cases} \phi_y(v) & \text{if } \phi_y(v) \leq \phi_y(v_0), \\ \phi_y(v) + 1 & \text{if } \phi_y(v) > \phi_y(v_0), \end{cases} \\
\phi'_z(v) &= \phi_z(v).
\end{aligned}$$

Since $\mathcal{B}(S^*)$ satisfies (B1), (B2), and (B3), ϕ' is well-defined and induces a 0-bend 3-D orthogonal drawing $\Lambda(S^* + f_c^*)$ of $\Gamma(S^* + f_c^*)$, as easily seen.

It remains to show a canonical orientation $\mathcal{B}(S^* + f_c^*)$ for $\Lambda(S^* + f_c^*)$. Let f_i be a child face of f_c such that $e_i \in b(f_c)$ is the base of f_i, $1 \leq i \leq k-1$, if any. We need a mapping α from the child faces of f_c to $\{0, 1, -1\}$. The mapping α will be used to indicate where each face f_i should be drawn. Each face f_i is drawn on the plane in which f_c is drawn if $\alpha(f_i) = 0$, above the plane if $\alpha(f_i) = 1$, and below the plane if $\alpha(f_i) = -1$. We first consider a mapping

$$\alpha' : \{f_1, f_2, \ldots, f_{k-1}\} \to \{0, 1, -1\},$$

which is an extension of α.

Claim 3.1 *A partial mapping α' on $\{f_1, f_{k-1}\}$ with $|\alpha'(f_1)| + |\alpha'(f_{k-1})| \neq 0$ can be extended to a mapping on $\{f_1, f_2, \ldots, f_{k-1}\}$ satisfying the following conditions:*

(A1′) $\alpha'(f_i) \neq \alpha'(f_{i+1})$ *for* $i = 1, k-2$.

(A2′) $|\alpha'(f_i) - \alpha'(f_{i+1})| = 1$ *for* $2 \leq i \leq k-3$.

Proof of Claim 3.1 It suffices to consider the following cases by symmetry.

Case 1 $\alpha'(f_1) \leq 0$ and $\alpha'(f_{k-1}) = -1$: We define that

$$\alpha'(f_i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Case 2 $\alpha'(f_1) = 1$ and $\alpha'(f_{k-1}) = -1$: We define that

$$\alpha'(f_i) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that α' defined above is a desired mapping. □

$\mathcal{B}(S^* + f_c^*)$ will be defined as an extension of $\mathcal{B}(S^*)$, that is, $\mathcal{B}(S^*) \subseteq \mathcal{B}(S^* + f_c^*)$. It suffices to define β_{v_i} $(1 \le i \le k-2)$ for the vertices $v_1, v_2, \ldots, v_{k-2}$ on $b(f_c)$. Let $\{e_1'\} = I_\Gamma(v_0) \cap b(f_1) - b(f_c)$ and $\{e_{k-1}'\} = I_\Gamma(v_{k-1}) \cap b(f_{k-1}) - b(f_c)$, if any. We define a partial mapping α' on $\{f_1, f_{k-1}\}$ as follows. If f_1 is a child face of f_c,

$$\alpha'(f_1) = \begin{cases} 1 & \text{if } \beta_{v_0}(e_1') = \beta_{v_0}(e_0) \times \beta_{v_0}(e_1), \\ -1 & \text{if } \beta_{v_0}(e_1') = -\beta_{v_0}(e_0) \times \beta_{v_0}(e_1), \\ 0 & \quad \text{otherwise,} \end{cases}$$

where $\times$ denotes the exterior product of verctors. If f_{k-1} is a child face of f_c,

$$\alpha'(f_{k-1}) = \begin{cases} 1 & \text{if } \beta_{v_{k-1}}(e_{k-1}') = \beta_{v_{k-1}}(e_0) \times \beta_{v_{k-1}}(e_{k-1}), \\ -1 & \text{if } \beta_{v_{k-1}}(e_{k-1}') = -\beta_{v_{k-1}}(e_0) \times \beta_{v_{k-1}}(e_{k-1}), \\ 0 & \quad \text{otherwise.} \end{cases}$$

If f_i, $i \in \{1, k-1\}$, is not a child face, we define that $\alpha'(f_i) = 1$. Since $\mathcal{B}(S^*)$ satisfies (B4), we have $|\alpha'(f_1)| + |\alpha'(f_{k-1})| \neq 0$. So let α' be a mapping on $\{f_1, f_2, \ldots, f_{k-1}\}$ satisfying the conditions of Claim 3.1. Let α be a restriction of α' to the child faces of f_c. Since α' satisfies the conditions (A1$'$) and (A2$'$), α satisfies the following conditions:

(A1) $\alpha(f_1) \neq \alpha(f_2)$ if f_2 is a child face of f_c, and $\alpha(f_{k-2}) \neq \alpha(f_{k-1})$ if f_{k-2} is a child face of f_c.

(A2) $|\alpha(f_i) - \alpha(f_{i+1})| = 1$ if f_i and f_{i+1} are child faces of f_c, $2 \le i \le k-3$.

For each vertex v_i $(1 \le i \le k-2)$, we label the edges in $I_\Gamma(v_i)$ as follows. Let $e_{v_i}^{(1)} = e_i$. If $e \in I_\Gamma(v_i)$ is the base of a child face $f_i^{(2)}$ of face f_i, let $e_{v_i}^{(2)} = e$. In general, if $e \in I_\Gamma(v_i)$ is the base of a child face $f_i^{(j+1)}$ of face $f_i^{(j)}$ $(2 \le j \le 4)$, let $e_{v_i}^{(j+1)} = e$, if any. If $f_i^{(j)}$ has no such child face and $\{e\} = b(f_i^{(j)}) \cap I_\Gamma(v_i) - \{e_{v_i}^{(j)}\}$, then we defined that $e_{v_i}^{(j+1)} = e$. Let $e_{v_i}^{(6)} = e_{i+1}$. If $e \in I_\Gamma(v_i)$ is the base of a child face $f_{i+1}^{(5)}$ of face f_{i+1}, let $e_{v_i}^{(5)} = e$. In general, if $e \in I_\Gamma(v_i)$ is the base of a child face $f_{i+1}^{(j-1)}$ of $f_{i+1}^{(j)}$ $(3 \le j \le 5)$, let $e_{v_i}^{(j-1)} = e$, if any. If $f_{i+1}^{(j)}$ has no such child face and $\{e\} = b(f_{i+1}^{(j)}) \cap I_\Gamma(v_i) - \{e_{v_i}^{(j)}\}$, then we defined that $e_{v_i}^{(j-1)} = e$.

We first define β_{v_i} for $e^{(1)}_{v_i}$, $e^{(2)}_{v_i}$, $e^{(5)}_{v_i}$, and $e^{(6)}_{v_i}$, if any:

$$\begin{aligned}\beta_{v_i}(e^{(1)}_{v_i}) &= \begin{cases} -\boldsymbol{e}_y & \text{if } i = 1, \\ -\boldsymbol{e}_x & \text{if } 2 \leq i \leq k-2, \end{cases} \\ \beta_{v_i}(e^{(6)}_{v_i}) &= \begin{cases} \boldsymbol{e}_x & \text{if } 1 \leq i \leq k-3, \\ -\boldsymbol{e}_y & \text{if } i = k-2, \end{cases}\end{aligned}$$

and

$$\beta_{v_i}(e^{(2)}_{v_i}) = \begin{cases} -\boldsymbol{e}_x & \text{if } \alpha(f_i) = 0 \text{ and } i = 1, \\ \boldsymbol{e}_y & \text{if } \alpha(f_i) = 0 \text{ and } 2 \leq i \leq k-2, \\ \boldsymbol{e}_z & \text{if } \alpha(f_i) = 1, \\ -\boldsymbol{e}_z & \text{if } \alpha(f_i) = -1, \end{cases}$$

$$\beta_{v_i}(e^{(5)}_{v_i}) = \begin{cases} \boldsymbol{e}_y & \text{if } \alpha(f_{i+1}) = 0 \text{ and } 1 \leq i \leq k-3, \\ \boldsymbol{e}_x & \text{if } \alpha(f_{i+1}) = 0 \text{ and } i = k-2, \\ \boldsymbol{e}_z & \text{if } \alpha(f_{i+1}) = 1, \\ -\boldsymbol{e}_z & \text{if } \alpha(f_{i+1}) = -1. \end{cases}$$

We next define β_{v_i} for $e^{(3)}_{v_i}$ and $e^{(4)}_{v_i}$, if any. We define that:

$$\begin{aligned}\beta_{v_1}(e^{(3)}_{v_1}) &= \begin{cases} -\boldsymbol{e}_x & \text{if } \alpha(f_1) = \pm 1, \\ \boldsymbol{e}_z & \text{if } \alpha(f_1) = 0 \text{ and } \alpha(f_2) = -1, \\ -\boldsymbol{e}_z & \text{if } \alpha(f_1) = 0 \text{ and } \alpha(f_2) = 1, \end{cases} \\ \beta_{v_1}(e^{(4)}_{v_1}) &= \begin{cases} \boldsymbol{e}_y & \text{if } \alpha(f_2) = \pm 1, \\ \boldsymbol{e}_z & \text{if } \alpha(f_2) = 0 \text{ and } \alpha(f_1) = -1, \\ -\boldsymbol{e}_z & \text{if } \alpha(f_2) = 0 \text{ and } \alpha(f_1) = 1. \end{cases}\end{aligned}$$

if any. It should be noted that $\alpha(f_1) \neq \alpha(f_2)$, since α satisfies (A1). For $2 \leq i \leq k-3$, we define that:

$$\begin{aligned}\beta_{v_i}(e^{(3)}_{v_i}) &= -\beta_{v_i}(e^{(5)}_{v_i}), \\ \beta_{v_i}(e^{(4)}_{v_i}) &= -\beta_{v_i}(e^{(2)}_{v_i}),\end{aligned}$$

if any. We define that:

$$\begin{aligned}\beta_{v_{k-2}}(e^{(3)}_{v_{k-2}}) &= \begin{cases} \boldsymbol{e}_y & \text{if } \alpha(f_{k-2}) = \pm 1, \\ \boldsymbol{e}_z & \text{if } \alpha(f_{k-2}) = 0 \text{ and } \alpha(f_{k-1}) = -1, \\ -\boldsymbol{e}_z & \text{if } \alpha(f_{k-2}) = 0 \text{ and } \alpha(f_{k-1}) = 1, \end{cases} \\ \beta_{v_{k-2}}(e^{(4)}_{v_{k-2}}) &= \begin{cases} \boldsymbol{e}_x & \text{if } \alpha(f_{k-1}) = \pm 1, \\ \boldsymbol{e}_z & \text{if } \alpha(f_{k-1}) = 0 \text{ and } \alpha(f_{k-2}) = -1, \\ -\boldsymbol{e}_z & \text{if } \alpha(f_{k-1}) = 0 \text{ and } \alpha(f_{k-2}) = 1, \end{cases}\end{aligned}$$

if any. It should be noted that $\alpha(f_{k-2}) \neq \alpha(f_{k-1})$, since α satisfies (A1).

Since α satisfies (A1) and (A2), it is not difficult to verify that $\mathcal{B}(S^* + f_c^*)$ defined so far satisfies the conditions (B1) through (B4), and is a canonical orientation for $\Lambda(S^* + f_c^*)$. (See Fig.3.) ∎

3.2.2 General Outerplanar Graphs

We next consider the general case when G is a connected outerplanar 6-graph with no triangles, and complete the proof of Theorem 2.

For graphs G_1 and G_2, $G_1 \cup G_2$ is a graph defined as $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A subset $\mathcal{S}$ of $\mathcal{D}$ is said to be suitable if vectors in $\mathcal{D} - \mathcal{S}$ can be linearly arranged such that adjacent vectors are orthogonal. Notice that $\mathcal{S}$ is not suitable if and only if $\mathcal{D} - \mathcal{S} = \{\boldsymbol{e}_a, -\boldsymbol{e}_a\}$ for some $a \in \{x, y, z\}$. Let Γ be an outerplane graph isomorhpic to G, Λ be a 0-bend 3-D orthogonal canonical drawing of Γ with a canonical orientation $\mathcal{B} = \{\beta_v | v \in V(\Gamma)\}$, and $\mathcal{F}_\Gamma(v) = \{\beta_v(e) | e \in I_\Gamma(v)\}$. Λ is said to be suitable if $\mathcal{F}_\Gamma(v)$ is suitable for every vertex $v \in V(\Gamma)$.

In order to complete the proof of Theorem 2, it is sufficient to show the following.

Lemma 3.4 *Let Γ_1 be an outerplane graph without triangles, and Λ_1 be a suitable 0-bend 3-D orthogonal canonical drawing of Γ_1 with a canonical orientation $\mathcal{B} = \{\beta_v | v \in V(\Gamma_1)\}$ and a layout ϕ^1. Let Γ_2 be a 2-connected outerplane graph without triangles or a graph consisting of an edge such that $|V(\Gamma_1) \cap V(\Gamma_2)| = 1$ and $\Gamma_1 \cup \Gamma_2$ is a 6-graph. Then, $\Gamma_1 \cup \Gamma_2$ also has a suitable 0-bend 3-D orthogonal canonical drawing.*

Proof of Lemma 3.4 : Let $\{w\} = V(\Gamma_1) \cap V(\Gamma_2)$ and $\mathcal{D} - \mathcal{F}_{\Gamma_1}(w) = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_s\}$ such that $\boldsymbol{a}_i$ and $\boldsymbol{a}_{i+1}$ are orthogonal ($s \leq 5$). We assume without loss of generality that $\boldsymbol{a}_1 = \boldsymbol{e}_x$ and $\boldsymbol{a}_2 = \boldsymbol{e}_y$. We distinguish two cases.

Case 1 Γ_2 is an edge: Let $E(\Gamma_2) = \{(w, w')\}$. Then, we define a layout ϕ' of $\Gamma_1 \cup \Gamma_2$ as follows:

$$\begin{aligned} \phi'(v) &= \begin{cases} \phi^1(v) & \text{if } \phi_x^1(v) \leq \phi_x^1(w), \\ \phi^1(v) + \boldsymbol{e}_x & \text{if } \phi_x^1(v) \geq \phi_x^1(w) + 1, \end{cases} \\ \phi'(w') &= \phi^1(w) + \boldsymbol{e}_x. \end{aligned}$$

If we define $\beta_w((w, w')) = \boldsymbol{e}_x$, it is easy to see that ϕ' induces a suitable 0-bend 3-D orthogonal canonical drawing of $\Gamma_1 \cup \Gamma_2$ with a canonical orientation.

Case 2 Γ_2 is 2-connected: Let r_2^* be a leaf of T_2^* such that w is on $b(r_2)$, and $b(r_2) = \{(w, v_1), (v_1, v_2), \ldots, (v_{k-2}, v_{k-1}), (v_{k-1}, w)\}$. Let e be the base of the unique child face of r_2. We show a layout of $\Gamma_1 \cup \Gamma_2$ for the case of $e = (w, v_1)$. Layouts of $\Gamma_1 \cup \Gamma_2$ for other cases can be obtained by similar ways. We define a layout ϕ' of $\Gamma_1 \cup \Gamma_2(r_2^*)$ as follows: for each $u \in V(\Gamma_1)$,

$$\begin{aligned}
\phi'_x(u) &= \begin{cases} \phi^1_x(u) & \text{if } \phi^1_x(u) \leq \phi^1_x(w), \\ \phi^1_x(u) + k - 3 & \text{if } \phi^1_x(u) \geq \phi^1_x(w) + 1, \end{cases} \\
\phi'_y(u) &= \begin{cases} \phi^1_y(u) & \text{if } \phi^1_y(u) \leq \phi^1_y(w), \\ \phi^1_y(u) + 1 & \text{if } \phi^1_y(u) \geq \phi^1_y(w) + 1, \end{cases} \\
\phi'_z(u) &= \phi^1_z(u),
\end{aligned}$$

and, for each $v_i \in V(\Gamma_2(r_2^*))$,

$$\phi'(v_i) = \begin{cases} \phi^1(w) + \boldsymbol{e}_y + (i-1)\boldsymbol{e}_x & 1 \leq i \leq k-2, \\ \phi^1(w) + (k-3)\boldsymbol{e}_x & i = k-1. \end{cases}$$

If we define $\beta_w(e) = \boldsymbol{e}_y$ and $\beta_w((w, v_{k-1})) = \boldsymbol{e}_x$, it is easy to verify that ϕ' induces a 0-bend 3-D orthogonal canonical drawing $\Lambda_1 \cup \Lambda(r_2^*)$ of $\Gamma_1 \cup \Gamma_2(r_2^*)$ with a canonical orientation.

Since $\mathcal{F}_{\Gamma_1 \cup \Gamma(r_2^*)}(w)$ is suitable for $\Lambda_1 \cup \Lambda(r_2^*)$, we can produce a suitable 0-bend 3-D orthogonal canonical drawing of $\Gamma_1 \cup \Gamma_2$ with a canonical orientation by Lemma 3. ∎

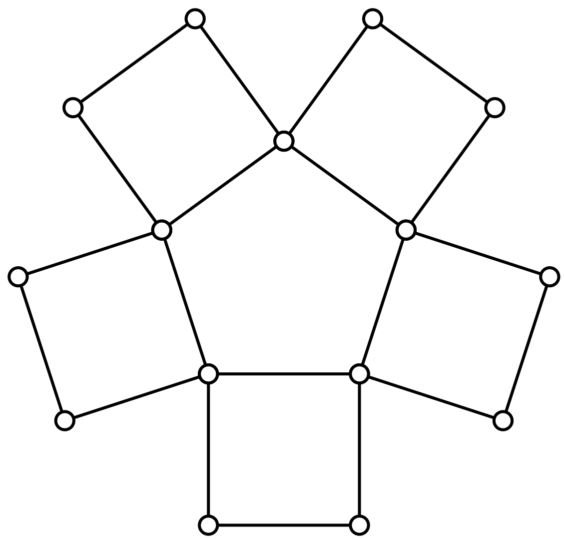

Figure 3.1: An outerplanar 4-graph F.

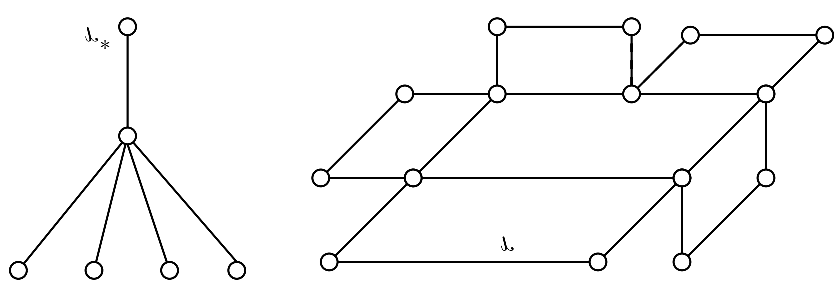

Figure 3.2: T^* for F and a 0-bend 3-D orthogonal canonical drawing of F.

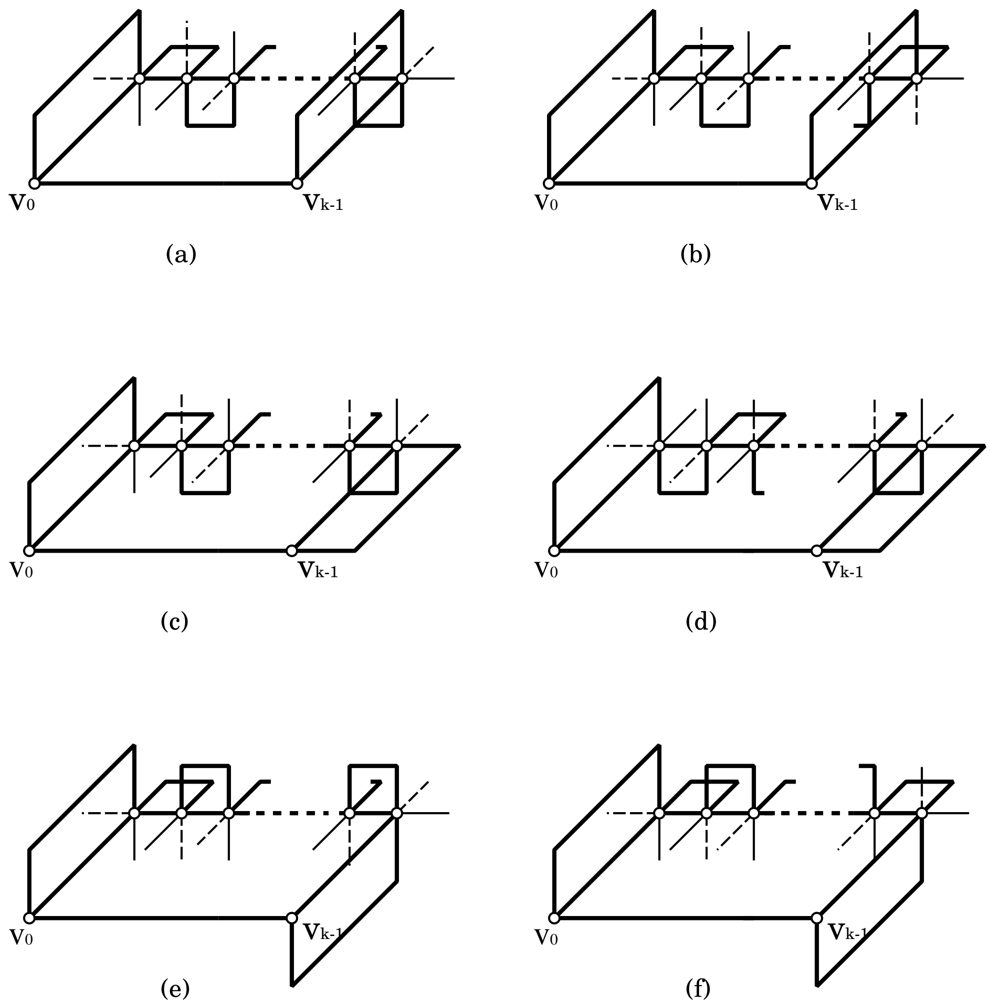

Figure 3.3: All the cases when $\alpha(f_1) = 1$: (a) $\alpha(f_{k-1}) = 1$ and k is odd, (b) $\alpha(f_{k-1}) = 1$ and k is even, (c) $\alpha(f_{k-1}) = 0$ and k is odd, (d) $\alpha(f_{k-1}) = 0$ and k is even, (e) $\alpha(f_{k-1}) = -1$ and k is odd, (f) $\alpha(f_{k-1}) = -1$ and k is even. The face f_c and child faces f_i of f_c $(1 \leq i \leq k-1)$ are shown in boldface, and every $\boldsymbol{e}_{v_i}^{(3)}$ and $\boldsymbol{e}_{v_i}^{(4)}$ are shown in lightface broken and solid lines, respectively.

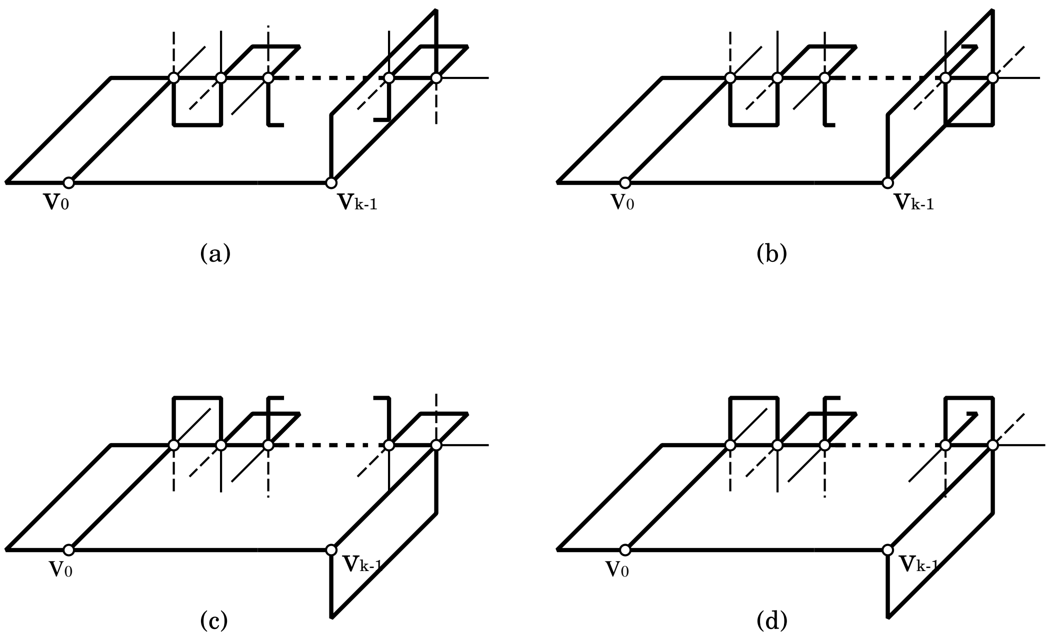

Figure 3.4: All the cases when $\alpha(f_1) = 0$: (a) $\alpha(f_{k-1}) = 1$ and k is odd, (b) $\alpha(f_{k-1}) = 1$ and k is even, (c) $\alpha(f_{k-1}) = -1$ and k is odd, (d) $\alpha(f_{k-1}) = -1$ and k is even. The face f_c and child faces f_i of f_c $(1 \leq i \leq k-1)$ are shown in boldface, and every $\boldsymbol{e}_{v_i}^{(3)}$ and $\boldsymbol{e}_{v_i}^{(4)}$ are shown in lightface broken and solid lines, respectively.

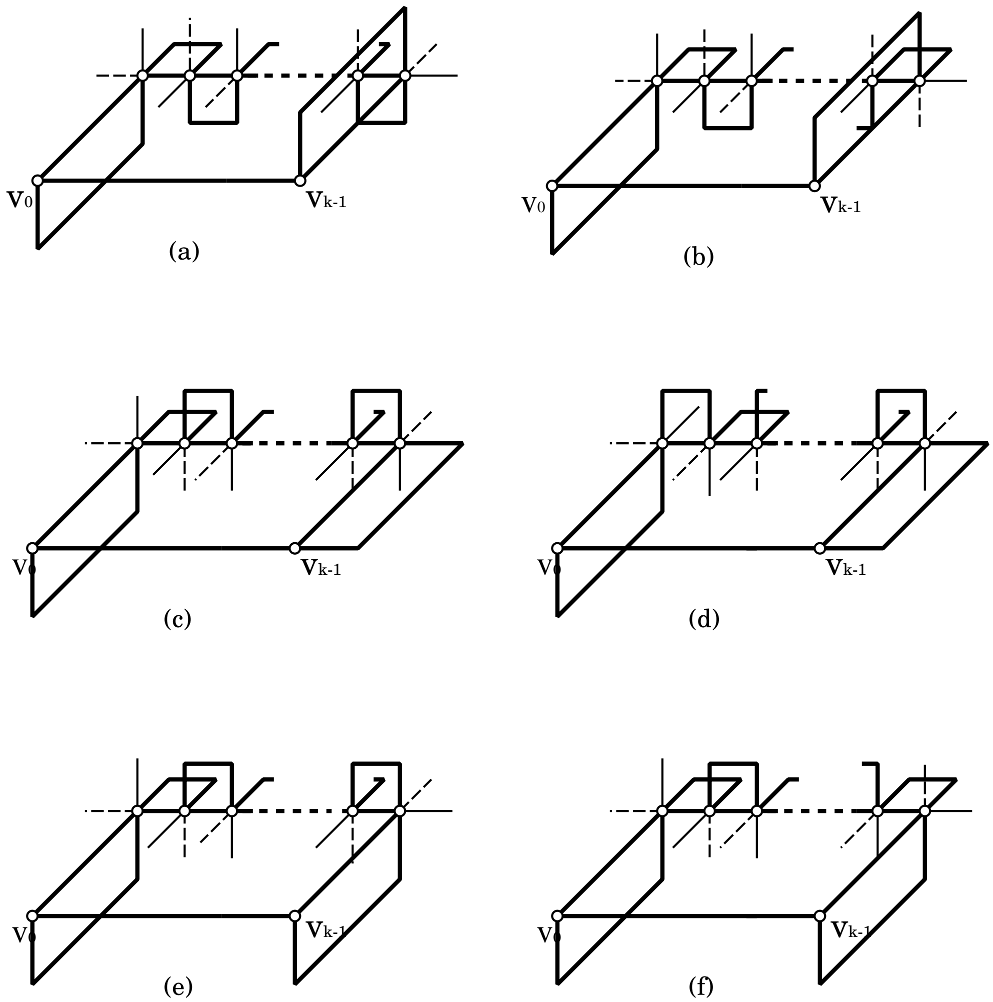

Figure 3.5: All the cases when $\alpha(f_1) = -1$: (a) $\alpha(f_{k-1}) = 1$ and k is odd, (b) $\alpha(f_{k-1}) = 1$ and k is even, (c) $\alpha(f_{k-1}) = 0$ and k is odd, (d) $\alpha(f_{k-1}) = 0$ and k is even, (e) $\alpha(f_{k-1}) = -1$ and k is odd, (f) $\alpha(f_{k-1}) = -1$ and k is even. The face f_c and child faces f_i of f_c $(1 \leq i \leq k-1)$ are shown in boldface, and every $\boldsymbol{e}_{v_i}^{(3)}$ and $\boldsymbol{e}_{v_i}^{(4)}$ are shown in lightface broken and solid lines, respectively.

Chapter 4

Orthogonal Drawing of Series-Parallel Graphs

4.1 2-D Orthogonal Drawing

We show in this section the following theorem.

Theorem 4.1 *Every series-parallel* 4*-graph has a* 1*-bend* 2*-D orthogonal drawing.* ∎

We will show the proof of theorem in the rest of the section.

4.1.1 Proof of Theorem 4.1

Let G be a series-parallel 4-graph with terminals s and t. We generate for G several 1-bend 2-D orthogonal drawings of distinct types depending on $d_G(s)$ and $d_G(t)$. The number of distinct types $\nu(d_G(s), d_G(t))$ is no more than 4 for every pair of $d_G(s)$ and $d_G(t)$. We denote by $\tau(d_G(s), d_G(t), i)$ a type of drawing for G, where $0 \leq i \leq \nu(d_G(s), d_G(t))$. Fig. 4.1 shows the types of 1-bend 2-D orthogonal drawings of G, where terminals are indicated by circles. We denote by $\Gamma_i(G)$ a 1-bend 2-D orthogonal drawing of type $\tau(d_G(s), d_G(t), i)$ for G. We assume without loss of generality that each vertex of $\Gamma(G)$ is positioned at a grid-point in the three-dimensional space. Let

$$\phi : V(G) \rightarrow V(\Gamma(G))$$

be an isomorophism between G and $\Gamma(G)$. The mapping ϕ is called a layout of $\Gamma(G)$. If $\phi(v) = (v_x, v_y, v_z)$, we denote $v_x = \phi_x(v)$, $v_y = \phi_y(v)$, and $v_z = \phi_z(v)$.

The drawings $\Gamma_i(G)$ are generated by Algorithm 1 below.

Algorithm 1 :2D-DRAW($G, \tau(d_G(s), d_G(t), k)$**)**
Input: a series-parallel 4-graph G with terminal s and t, $\tau(d_G(s), d_G(t), k)$
Output: 2-bend 3-D orthogonal drawing $\Gamma(G)$ of type $\tau(d_G(s), d_G(t), k)$
begin
  Compute $T(G)$
  if G consists of a single edge
  then draw $\Gamma(G)$ satisfying type of $\tau(d_G(s), d_G(t), k)$
  else
    if G is the series composition of G_1 and G_2
    then define $\tau(d_{G_1}(s_1), d_{G_1}(t_1), l)$ and $\tau(d_{G_2}(s_2), d_{G_2}(t_2), m)$ depending on $\tau(d_G(s), d_G(t), k)$ shown in Table 1.
    end if
    if G is the parallel composition of G_1 and G_2
    then define $\tau(d_{G_1}(s_1), d_{G_1}(t_1), l)$ and $\tau(d_{G_2}(s_2), d_{G_2}(t_2), m)$ depending on $\tau(d_G(s), d_G(t), k)$ shown in Table 2.
    end if
    $\Gamma(G_1)$ =2D-DRAW($G_1, \tau(d_{G_1}(s_1), d_{G_1}(t_1), l)$)
    $\Gamma(G_2)$ =2D-DRAW($G_2, \tau(d_{G_2}(s_2), d_{G_2}(t_2), m)$)
    if G is the seires composition of G_1 and G_2,
      SER-2D-COM($\Gamma(G_1), \Gamma(G_2)$) (in Section 4.1.1.1)
    end if
    if G is the parallel composition of G_1 and G_2,
      PAR-2D-COM($\Gamma(G_1), \Gamma(G_2)$) (in Section 4.1.1.2)
    end if
  end if
end

4.1.1.1 SER-2D-COM($\Gamma(G_1), \Gamma(G_2)$)

Input: $\Gamma(G_1), \Gamma(G_2)$,

Output: $\Gamma(G)$

Step 1 Generate $\Gamma'(G_1)$ and $\Gamma'(G_2)$ by rotating and $\Gamma(G_1)$ and $\Gamma(G_2)$ such that $\Gamma(G)$, which is drawing type of $\tau(d_G(s), d_G(t), k)$, can be generated by identifying t_1 with s_2.

Step 2 Let ϕ^1 and ϕ^2 be a layout of $\Gamma'(G_1)$ and $\Gamma'(G_2)$, respectively. And let $k = \max\{|\phi^1_x(v)|, |\phi^1_y(v)|, |\phi^2_x(w)|, |\phi^2_y(w)|\}$, $v \in V(\Gamma'(G_1))$ and $w \in V(\Gamma'(G_2))$. It is without loss of generality that $\phi^1(t_1) = \phi^2(t_2) = (0, 0)$, $\phi^1_x(s_1) > 0$, and $\phi^1_y(v) > 0$. Now we output a layout ϕ of $\Gamma(G)$. For the vertex $t \in V(\Gamma(G))$, we output that $\phi(t) = \phi^2(t_2)$. For each vertex $v \in V(\Gamma(G_2)) - \{t_2\}$, output that

$$\phi_x(v) = \begin{cases} \phi_x^2(v) & \text{if } \phi_x^2(v) < \phi_x^2(s_2), \\ \phi_x^2(v) + k & \text{if } \phi_x^2(s_2) \leq \phi_x^2(v), \end{cases}$$

$$\phi_y(v) = \begin{cases} \phi_y^2(v) & \text{if } \phi_y^2(v) < \phi_y^2(s_2), \\ \phi_y^2(v) + k & \text{if } \phi_y^2(s_2) \leq \phi_y^2(v). \end{cases}$$

For each vertex $v \in V(\Gamma(G_1)) - \{t_1\}$, output $\phi(v)$ depending on $\phi_x^2(s_2)$ and $\phi_y^2(s_2)$ as follows:

Case 1 $\phi_x^2(s_2) > 0$ and $\phi_y^2(s_2) > 0$

$$\phi_x(v) = \begin{cases} \phi_x^1(v) + \phi_x(s_1) & \text{if } \phi_x^1(v) \leq 0, \\ \phi_x^1(v) + \phi_x(s_1) + k & \text{if otherwise}, \end{cases}$$

$$\phi_y(v) = \begin{cases} \phi_y^1(v) + \phi_y(s_1) & \text{if } \phi_y^1(v) \leq 0, \\ \phi_y^1(v) + \phi_y(s_1) + k & \text{otherwise}, \end{cases}$$

Case 2 $\phi_x^2(s_2) < 0$ and $\phi_y^2(s_2) > 0$

$$\phi_x(v) = \begin{cases} \phi_x^1(v) + \phi_x(s_1) & \text{if } \phi_x^1(v) \leq 0, \\ \phi_x^1(v) + \phi_x(s_1) + 2k & \text{otherwise}, \end{cases}$$

$$\phi_y(v) = \begin{cases} \phi_y^1(v) + \phi_y(s_1) & \text{if } \phi_y^1(v) \leq 0, \\ \phi_y^1(v) + \phi_y(s_1) + k & \text{otherwise}. \end{cases}$$

4.1.1.2 PAR-2D-COM($\Gamma(G_1), \Gamma(G_2)$)

Input: $\Gamma(G_1), \Gamma(G_2)$,

Output: $\Gamma(G)$

Step 1 Generate $\Gamma'(G_1)$ and $\Gamma'(G_2)$ by rotating and $\Gamma(G_1)$ and $\Gamma(G_2)$ such that $\Gamma(G)$, which is drawing type of $\tau(d_G(s), d_G(t), k)$, can be generated by identifying t_1 with t_2, and s_1 and s_2.

Step 2 Let ϕ^1 and ϕ^2 be a layout of $\Gamma'(G_1)$ and $\Gamma'(G_2)$, respectively. It is without loss of generality that $\phi^1(t_1) = \phi^2(t_2) = (0,0)$, $\phi_x^1(s_1) > 0$, and $\phi_y^1(t_1) > 0$. Now we output a layout ϕ of $\Gamma(G)$. For terminals s, t of G, output that

$$\phi(t) = \phi^1(t_1)$$

$$\phi(s) = (2k, 2k).$$

Let

$$\begin{aligned} l_x^1 &= \phi_x(s) - \phi_x^1(s_1), \\ l_y^1 &= \phi_y(s) - \phi_y^1(s_1), \\ l_x^2 &= \phi_x(s) - \phi_x^2(s_2), \\ l_y^2 &= \phi_y(s) - \phi_x^2(s_2). \end{aligned}$$

For each vertex $v \in V(\Gamma(G_1)) - \{s_1, t_1\}$, output a layout $\phi(v)$ that

$$\begin{aligned} \phi_x(v) &= \begin{cases} \phi_x^1(v) - k & \text{if } \phi_x^1(v) < 0, \\ \phi_x^1(v) & \text{if } 0 \leq \phi_x^1(v) < \phi_x^1(s_1), \\ \phi_x^1(v) + l_x^1 & \text{if } \phi_x^1(s_1) \leq \phi_x^1(v), \end{cases} \\ \phi_y(v) &= \begin{cases} \phi_y^1(v) & \text{if } \phi_y^1(v) \leq 0, \\ \phi_y^1(v) + l_y^1 & \text{if } 0 < \phi_y^1(v) \leq \phi_y^1(s_1), \\ \phi_y^1(v) + l_y^1 + k & \text{if } \phi_y^1(s_1) < \phi_y^1(v). \end{cases} \end{aligned}$$

For each vertex $v \in V(\Gamma(G_2)) - \{s_2, t_2\}$, output a layout $\phi(v)$ that

$$\begin{aligned} \phi_x(v) &= \begin{cases} \phi_x^2(v) & \text{if } \phi_x^2(v) \leq 0, \\ \phi_x^2(v) + l_x^2 & \text{if } 0 < \phi_x^2(v) \leq \phi_x^2(s_2), \\ \phi_x^2(v) + l_x^2 + k & \text{if } \phi_x^2(s_2) < \phi_x^2(v), \end{cases} \\ \phi_y(v) &= \begin{cases} \phi_y^2(v) - k & \text{if } \phi_y^2(v) < 0, \\ \phi_y^2(v) & \text{if } 0 \leq \phi_y^2(v) < \phi_y^2(s_2), \\ \phi_y^2(v) + l_y^2 & \text{if } \phi_y^2(s_2) \leq \phi_y^2(v). \end{cases} \end{aligned}$$

4.1.1.3 Analysis of Algorithm 1

The correctness of the algorithm is guaranteed by the following lemma.

Lemma 4.1 *If G contains more than one edge, then for any $\tau(d_G(s), d_G(t), i)$, $1 \leq i \leq \nu(d_G(s), d_G(t))$, there always exist a drawing $\Gamma_j(G_1)$, $0 \leq j \leq \nu(d_{G_1}(s_1), d_{G_1}(t_1))$, and a drawing $\Gamma_k(G_2)$, $0 \leq k \leq \nu(d_{G_2}(s_2), d_{G_2}(t_2))$, such that we can generate $\Gamma_i(G)$ by combining $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ with the only exception of $\tau(3, 3, 2)$ for G with edge (s, t).* ∎

The proof of the lemma is obvious from the tables 1 and 2 below, which show types of such $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ for each type of $\Gamma_i(G)$, where $\tau(i, j, k)$ is indicated by (i, j, k) in the tables. It is tedious but easy to check the tables. For example, Fig. 4.2 show that $\tau(4, 4, 1)$ is generated by parallel composition of $\tau(1, 1, 1)$ and $\tau(3, 3, 2)$.

4.2 3-D Orthogonal Drawing

We show in this section the following theorem.

Theorem 4.2 *Every series-parallel* 6*-graph has a* 2*-bend* 3*-D orthogonal drawing.* ∎

We will show the proof of theorem in the rest of the section.

4.2.1 Proof of Theorem 4.2

Let G be a series-parallel 6-graph with terminals s and t. We assume without loss of generality that each vertex of $\Gamma(G)$ is positioned at a grid-point in the three-dimensional space. Let

$$\phi : V(G) \to V(\Gamma(G))$$

be an isomorophism between G and $\Gamma(G)$. The mapping ϕ is called a layout of $\Gamma(G)$. If $\phi(v) = (v_x, v_y, v_z)$, we denote $v_x = \phi_x(v)$, $v_y = \phi_y(v)$, and $v_z = \phi_z(v)$. We use a vector $R(G) \in \{+1, -1\}^3$ to represent relative positions of terminals in the space. If $R(G) = (r_x, r_y, r_z)$, we denote $r_x = R_x(G)$, $r_y = R_y(G)$, and $r_z = R_z(G)$. For vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$, define that $\boldsymbol{a} * \boldsymbol{b} = (a_1b_1, a_2b_2, a_3b_3)$. Let $\boldsymbol{e}_x = (1, 0, 0)$, $\boldsymbol{e}_y = (0, 1, 0)$, and $\boldsymbol{e}_z = (0, 0, 1)$. and let $\mathcal{D}^+ = \{X, Y, Z\}$, $\mathcal{D}^- = \{-X, -Y, -Z\}$, $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$, and let $D_G(s)$ and $D_G(t)$ be subsets of $\mathcal{D}$ satisfying the following conditions:

1. $|D_G(s)| = d_G(s)$ and $|D_G(t)| = d_G(t)$.

2. There exist $A \in D_G(s)$ and $B \in D_G(t)$ such that $A \neq -B$.

The conditions above implies that the elements of $D_G(s)$ and $D_G(t)$ can be ordered $A_1, A_2, \ldots, A_{d_G(s)}$ and $B_1, B_2, \ldots B_{d_G(t)}$, respectively, such that $A_i \neq -B_i$ for each i, $1 \leq i \leq \min\{d_G(s), d_G(t)\}$. We denote by $[D_G(s)]$ and $[D_G(t)]$ such sequesces of elements. $D_G(s)$ and $D_G(t)$ are said to be inner-directed if there exist $A \in D_G(s)$ and $B \in D_G(t)$ satisfying the following conditions:

1. $A \in \mathcal{D}^-$ and $B \in \mathcal{D}^+$

2. $A \neq -B$

3. If $D_G(s) - \{A\} \neq \phi$ and $D_G(t) - \{B\} \neq \phi$ then there exist $A' \in D_G(s) - \{A\}$ and $B' \in D_G(t) - \{B\}$ such that $A' \neq -B'$.

A 2-bend 3-D orthogonal drawing $\Gamma(G)$ of G is generated by Algorithm 2 in section 4.1.

as the initially conditions, define $D_G(s)$, $D_G(t)$, and $R(G)$ as follows:

$$\begin{aligned} D_G(s) &= D_G(t) = \mathcal{D}, \quad \text{and} \\ R(G) &= (1,1,1) \end{aligned}$$

4.2.1.1 Algorithm 2: 3D-DRAW($G, D_G(s), D_G(t), R(G)$)

3D-DRAW($G, D_G(s), D_G(t), R(G)$)
Input: a series-parallel 6-graph G with terminal s and t, $D_G(s), D_G(t)$, and $R(G)$
Output: 2-bend 3-D orthogonal drawing $\Gamma(G)$
begin
  Compute $T(G)$
  if G consists of a single edge
    SINGLE-3D-DRAW($G, D_G(s), D_G(t), R(G)$) (in Section 4.2.1.2)
  else
    if G is the series composition of G_1 and G_2
      SER-3D-DECOM($G, G_1, G_2, D_G(s), D_G(t), R(G)$)
      (in Section 4.2.1.3)
    end if
    if G is the parallel composition of G_1 and G_2,
      PAR-3D-DECOM($G, G_1, G_2, D_G(s), D_G(t), R(G)$)
      (in Section 4.2.1.4)
    end if
    $\Gamma(G_1)$ =3D-DRAW($G_1, D_{G_1}(s_1), D_{G_1}(t_1), R(G_1)$)
    $\Gamma(G_2)$ =3D-DRAW($G_2, D_{G_2}(s_2), D_{G_2}(t_2), R(G_2)$)
    if G is the seires composition of G_1 and G_2,
      SER-3D-COM($\Gamma(G_1), \Gamma(G_2)$) (in Section 4.2.1.5)
    end if
    if G is the parallel composition of G_1 and G_2,
      PAR-3D-COM($\Gamma(G_1), \Gamma(G_2)$) (in Section 4.2.1.6)
    end if
  end if
end

4.2.1.2 SINGLE-3D-DRAW($G, D_G(s), D_G(t), R(G)$)

Input: $G, D_G(s), D_G(t), R(G)$

Output: $\Gamma(G)$

Output a layout $\phi(s)$ and $\phi(t)$ of $\Gamma(G)$ that

$$\begin{aligned}\phi(s) &= R(G) * (1,1,1),\\ \phi(t) &= (0,0,0).\end{aligned}$$

Construct an edge route with 2-bend edge route for the single edge $e = (s,t)$ as follows:

$$\phi(t) \rightarrow \boldsymbol{e}_i * R(G) \rightarrow (\boldsymbol{e}_i + \boldsymbol{e}_k) * R(G) \rightarrow \phi(s),$$

where $i \in \{x,y,z\}$ such that $i = |D_G(s)|$, $j = |D_G(t)|$, and $i \neq j \neq k$. Fig. 4.3 shows the edge route for $D_G(s) = -X$, $D_G(t) = Z$, and $R(G) = (1,1,1)$.

4.2.1.3 SER-3D-DECOM($G, G_1, G_2, D_G(s), D_G(t), R(G)$)

Input: $G, G_1, G_2, D_G(s), D_G(t), R(G)$

Output: $D_{G_1}(s_1), D_{G_1}(t_1), D_{G_2}(s_2), D_{G_2}(t_2), R(G_1), R(G_2)$

Step 1 Define that $(X_G, Y_G, Z_G) = (X,Y,Z) * R(G)$, $\mathcal{D}_G^+ = \{X_G, Y_G, Z_G\}$, and $\mathcal{D}_G^- = \{-X_G, -Y_G, -Z_G\}$.

Step 2 If $D_G(s)$ and $D_G(t)$ are inner-directed, then select $A \in D_G(s)$ and $B \in D_G(t)$ such that $A \in \mathcal{D}_G^-$ and $B \in \mathcal{D}_G^+$. Else if $D_G(s) \cap \mathcal{D}_G^- \neq \emptyset$ or $D_G(t) \cap \mathcal{D}_G^+ \neq \emptyset$, select $A \in D_G(s)$ and $B \in D_G(t)$ such that $A \neq -B$ and $A \in \mathcal{D}_G^-$ or $B \in \mathcal{D}_G^+$. Else select $A \in D_G(s)$ and $B \in D_G(t)$ such that $A \neq -B$.

Step 3 Output $D_{G_1}(s_1)$, $D_{G_1}(t_1)$, $D_{G_2}(s_2)$, $D_{G_2}(t_2)$, $R(G_1)$, and $R(G_2)$ depending on A and B as follows:

Case 1 $A \in \mathcal{D}_G^-, B \in \mathcal{D}_G^+$:

Case 1-1 $B \in \{X_G, Z_G\}$: Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{-A\}$. If $d_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-Y_G\}$. If $d_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $\{-Y\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_{G_2}(t_2) = D_G(t)$. Let $R(G_1) = R(G)$ and $R(G_2) = R(G)$.(See Fig. 4.4)

Case 1-2 $B = Y_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{-A\}$. If $d_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that

$|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-X_G\}$. If $d_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $\{-X\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = R(G)$ and $R(G_2) = R(G)$.

Case 2 $A \in \mathcal{D}_G^+, B \in \mathcal{D}_G^-$:

Case 2-1 $A = X_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{A\}$. If $d_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - \{-A\}$. If $d_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $\{-A\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = (-1, +1, +1) * R(G)$ and $R(G_2) = (+1, -1, -1) * R(G)$. (See Fig. 4.5)

Case 2-2 $A = Y_G$: $D_{G_1}(s_1)$,$D_{G_1}(t_1)$, $D_{G_2}(s_2)$, and $D_{G_2}(t_2)$ are same as Case 2-1. Let $R(G_1) = (+1, +1, -1) * R(G)$ and $R(G_2) = (-1, -1, +1) * R(G)$.

Case 2-3 $A = Z_G$: $D_{G_1}(s_1)$,$D_{G_1}(t_1)$, $D_{G_2}(s_2)$, and $D_{G_2}(t_2)$ are same as Case 2-1. Let $R(G_1) = (+1, -1, +1) * R(G)$ and $R(G_2) = (-1, +1, -1) * R(G)$.

Case 3 $A \in \mathcal{D}_G^-, B \in \mathcal{D}_G^-$:

Case 3-1 $A = B = -Z_G$: Let $D_{G_2}(t_2) = D_G(t)$. If $d_{G_2}(s_2) \leq 2$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $S' \subseteq \mathcal{D}_G^- - \{B\}$. If $3 \leq d_{G_2}(s_2) \leq 4$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D}_B^- + \{X_G\}$. If $d_{G_2}(s_2) = 5$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = 5$ and $S' = \mathcal{D} - \{Y_G\}$. Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$, $\{Y_G\} \subseteq S \subseteq \mathcal{D}_G^+$, and $D_{G_2}(s_2) \cap S = \emptyset$. If $d_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - S'$. Let $R(G_1) = (-1, +1, +1) * R(G)$ and $R(G_2) = (+1, -1, -1) * R(G)$.

Case 3-2 $A = B = -Y_G$: Let $D_{G_2}(t_2) = D_G(t)$. If $d_{G_2}(s_2) \leq 2$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\{-X_G\} \subseteq S' \subseteq \mathcal{D}_G^- - \{B\}$. If $3 \leq d_{G_2}(s_2) \leq 4$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D}_G^- + \{X_G\}$. If $d_{G_2}(s_2) = 5$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = 5$ and $S = \mathcal{D} - \{Z_G\}$. Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$, $\{Z_G\} \subseteq S \subseteq$

$\mathcal{D}_G^+$, and $D_{G_2}(s_2) \cap S = \emptyset$. If $d_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - S'$. Let $R(G_1) = (-1, +1, +1) * R(G)$ and $R(G_2) = (+1, -1, -1) * R(G)$. (See Fig. 4.6)

Case 3-3 $A = B = -X_G$: Let $D_{G_2}(t_2) = D_G(t)$. If $d_{G_2}(s_2) \leq 2$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\{-Y_G\} \subseteq S' \subseteq \mathcal{D}_G^- - \{B\}$. If $3 \leq d_{G_2}(s_2) \leq 4$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D}_G^- + \{X_G\}$. If $d_{G_2}(s_2) = 5$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = 5$ and $S' = \mathcal{D} - \{Y_G\}$. Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$, $\{Z_G\} \subseteq S \subseteq \mathcal{D}_G^+$, and $D_{G_2}(s_2) \cap S = \emptyset$. If $d_{G_1}(t_1) \geq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = D_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D} - S'$. Let $R(G_1) = (+1, -1, +1) * R(G)$ and $R(G_2) = (-1, +1, -1) * R(G)$.

Case 3-4 $A \neq B$: Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $S \subseteq \mathcal{D}_G^+ - \{-A\}$. If $d_{G_1}(t_1) \geq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ - \{-A\} \subseteq S \subseteq \mathcal{D} - \{-A\}$. If $d_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $\{-A\} \subseteq S' \subseteq \mathcal{D}_G^- - \{A\} + \{-A\}$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_2}(s_2) \geq 4$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- + \{-A\} \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = R(G)$ and $R(G_2) = (-1, -1, -1) * R(G)$. (See Fig. 4.7)

Case 4 $A \in \mathcal{D}_G^+, B \in \mathcal{D}_G^+$:

Case 4-1 $A = B = Z_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\{Z_G\} \subseteq S \subseteq \mathcal{D}_G^+ - \{A\}$. If $3 \leq d_{G_1}(t_1) \leq 4$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D}_G^+ + \{-Z_G\}$. If $d_{G_1}(t_1) = 5$, let $D_{G_1}(t_1)$ be any set S such that $|S| = 5$ and $S = \mathcal{D} - \{-Y_G\}$. If $d_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $\{-Y_G\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = (-1, -1, +1) * R(G)$ and $R(G_2) = (+1, +1, -1) * R(G)$.

Case 4-2 $A = B = Y_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\{Z_G\} \subseteq S \subseteq \mathcal{D}_G^+ - \{A\}$. If $3 \leq d_{G_1}(t_1) \leq 4$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D}_G^+ + \{-Z_G\}$. If $d_{G_1}(t_1) = 5$, let $D_{G_1}(t_1)$ be any set S such that $|S| = 5$ and

$S = \mathcal{D} - \{-X_G\}$. If $d_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $\{-X_G\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = (-1, -1, +1) * R(G)$ and $R(G_2) = (+1, +1, -1) * R(G)$.

Case 4-3 $A = B = X_G$: Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 2$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\{Z_G\} \subseteq S \subseteq \mathcal{D}_G^+ - \{A\}$. If $3 \leq d_{G_1}(t_1) \leq 4$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ \subseteq S \subseteq \mathcal{D}_G^+ + \{-X_G\}$. If $d_{G_1}(t_1) = 5$, let $D_{G_1}(t_1)$ be any set S such that $|S| = 5$ and $S = \mathcal{D} - \{-Y_G\}$. If $d_{G_2}(s_2) \leq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $\{-Y_G\} \subseteq S' \subseteq \mathcal{D}_G^-$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- \subseteq S' \subseteq \mathcal{D} - S$. Let $D_G(t_2) = D_G(t)$. Let $R(G_1) = (+1, -1, -1) * R(G)$ and $R(G_2) = (-1, +1, +1) * R(G)$. (See Fig. 4.8)

Case 4-4 $A \neq B$: If $d_{G_2}(s_2) \leq 2$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$, $S' \subseteq \mathcal{D}_G^- - \{-B\}$. If $d_{G_2}(s_2) \geq 3$, let $D_{G_2}(s_2)$ be any set S' such that $|S'| = d_{G_2}(s_2)$ and $\mathcal{D}_G^- - \{-B\} + \{B\} \subseteq S' \subseteq \mathcal{D} - \{-B\}$. Let $D_G(t_2) = D_G(t)$. Let $D_{G_1}(s_1) = D_G(s)$. If $d_{G_1}(t_1) \leq 3$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\{-B\} \subseteq S \subseteq \mathcal{D}_G^+ - \{B\} + \{-B\}$, and $D_{G_1}(t_1) \cap S' = \emptyset$. If $d_{G_1}(t_1) \geq 4$, let $D_{G_1}(t_1)$ be any set S such that $|S| = d_{G_1}(t_1)$ and $\mathcal{D}_G^+ + \{-B\} \subseteq S \subseteq \mathcal{D} - S'$. Let $R(G_1) = (-1, -1, -1) * R(G)$ and $R(G_2) = R(G)$. (See Fig. 4.9)

4.2.1.4 PAR-3D-DECOM($G, G_1, G_2, D_G(s), D_G(t), R(G)$)

Input: $G, G_1, G_2, D_G(s), D_G(t), R(G)$

Output: $D_{G_1}(s_1), D_{G_1}(t_1), D_{G_2}(s_2), D_{G_2}(t_2), R(G_1), R(G_2)$

Step 1 Define that $(X_G, Y_G, Z_G) = (X, Y, Z) * R(G)$, $\mathcal{D}_G^+ = \{X_G, Y_G, Z_G\}$, and $\mathcal{D}_G^- = \{-X_G, -Y_G, -Z_G\}$.

Step 2 Construct $[D_G(s)] = (A_1, A_2, \ldots, A_{D_G(s)})$ and $[D_G(t)] = (B_1, B_2, \ldots, B_{D_G(t)})$ such that $A_i \neq -B_i$, $1 \leq i \leq \min\{d_G(s), d_G(t)\}$. If $D_G(s)$ and $D_G(t)$ are inner-directed, we assume without loss of generality that $A_1 \in \mathcal{D}_G^-$ and $B_1 \in \mathcal{D}_G^+$.

Step 3 Output $D_{G_1}(s_1)$, $D_{G_1}(t_1)$, $D_{G_2}(s_2)$, $D_{G_2}(t_2)$, $R(G_1)$, and $R(G_2)$ depending on $d_{G_1}(s_1)$ and $d_{G_1}(t_1)$ as follows:

Case 1 $k_1 = d_{G_1}(s_1) \leq d_{G_1}(t_1)$:

Case 1-1 $e = (s,t) \in G_1$:
$D_{G_1}(s_1) = \{A_1, A_2, \ldots, A_{k_1}\}$,
$D_{G_1}(t_1) = \{B_1, B_2, \ldots, B_{k_1}, B_{k_1+D_{G_2}(t_2)+1}, \ldots, B_{D_G(t)}\}$,
$D_{G_2}(s_2) = \{A_{k_1+1}, A_{k_1+2}, \ldots, A_{D_G(s)}\}$,
$D_{G_2}(t_2) = \{B_{k_1+1}, B_{k_1+2}, \ldots, B_{k_1+D_{G_2}(t_2)}\}$,
$R(G_1) = R(G_2) = R(G)$.

Case 1-2 $e = (s,t) \in G_2$:
$D_{G_1}(s_1) = \{A_2, A_3, \ldots, A_{k_1+1}\}$,
$D_{G_1}(t_1) = \{B_2, B_3, \ldots, B_{k_1+1}, B_{k_1+D_{G_2}(t_2)+1}, \ldots, B_{D_G(t)}\}$,
$D_{G_2}(s_2) = \{A_1, A_{k_1+2}, A_{k_1+3}, \ldots, A_{D_G(s)}\}$,
$D_{G_2}(t_2) = \{B_1, B_{k_1+2}, B_{k_1+3}, \ldots, B_{k_1+D_{G_2}(t_2)}\}$,
$R(G_1) = R(G_2) = R(G)$.

Case 2 $d_{G_1}(s_1) \geq d_{G_1}(t_1) = k_1$:

Case 2-1 $e = (s,t) \in G_1$:
$D_{G_1}(s_1) = \{A_1, A_2, \ldots, A_{k_1}, A_{k_1+D_{G_2}(s_2)+1}, \ldots, A_{D_G(s)}\}$,
$D_{G_1}(t_1) = \{B_1, B_2, \ldots, B_{k_1}\}$,
$D_{G_2}(s_2) = \{A_{k_1+1}, A_{k_1+2}, \ldots, A_{k_1+D_{G_2}(s_2)}\}$,
$D_{G_2}(t_2) = \{B_{k_1+1}, B_{k_1+2}, \ldots, B_{D_G(t)}\}$,
$R(G_1) = R(G_2) = R(G)$.

Case 2-2 $e = (s,t) \in G_2$:
$D_{G_1}(s_1) = \{A_2, A_3, \ldots, A_{k_1+1}, A_{k_1+D_{G_2}(s_2)+1}, \ldots, A_{D_G(s)}\}$,
$D_{G_1}(t_1) = \{B_2, B_3, \ldots, B_{k_1+1}\}$,
$D_{G_2}(s_2) = \{A_1, A_{k_1+2}, A_{k_1+3}, \ldots, A_{k_1+D_{G_2}(s_2)}\}$,
$D_{G_2}(t_2) = \{B_1, B_{k_1+2}, B_{k_1+3}, \ldots, B_{D_G(t)}\}$,
$R(G_1) = R(G_2) = R(G)$.

4.2.1.5 SER-3D-COM($\Gamma(G_1), \Gamma(G_2), R(G_1), R(G_2), R(G)$)

Input: $\Gamma(G_1), \Gamma(G_2), R(G_1), R(G_2), R(G)$

Output: $\Gamma(G)$

Let ϕ^1 and ϕ^2 be a layout of $\Gamma(G_1)$ and $\Gamma(G_2)$, respectivly. Let

$$k = \max\{|\phi_x^1(v)|, |\phi_x^2(w)|, |\phi_y^1(v)|, |\phi_y^2(w)|, |\phi_z^1(v)|, |\phi_z^2(w)|, \},$$

for $\forall v \in V(\Gamma(G_1))$ and $\forall w \in V(\Gamma(G_2))$. And let

$$\begin{aligned}\psi^1(v) &= R(G_1) * \phi^1(v), \quad \text{and} \\ \psi^2(w) &= R(G_2) * \phi^2(w),\end{aligned}$$

where $v \in V(\Gamma(G_1)$ and $w \in V(\Gamma(G_2))$. If $\psi^1(v) = (v_x, v_y, v_z)$ and $\psi^2(w) = (w_x, w_y, w_z)$, we denote that $v_x = \psi^1_x(v)$, $v_y = \psi^1_y(v)$, $v_z = \psi^1_z(v)$, $w_x = \psi^2_x(w)$, $w_y = \psi^2_y(w)$, and $w_z = \psi^2_z(w)$. Now we assume without loss of generality that G obtained by identifing t_1 with s_2. Output that

$$\phi(t) = \phi^2(t_2).$$

For each vertex $v \in V(\Gamma(G)) - \{t\}$, output a layout $\phi_i(v)$ depending on $R_i(G)$, $R_i(G_1)$, and $R_i(G_2)$, $i \in \{x, y, z\}$ as follows:

Case 1 $R_i(G) = R_i(G_1) = R_i(G_2)$

For the vertices $t_1 = s_2$ and $s = s_1$, output that

$$\begin{aligned}\phi_i(t_1) &= \phi_i(s_2) = 2k, \\ \phi_i(s) &= 4k.\end{aligned}$$

Let $l^1_i = \phi_i(s) - \phi^1_i(s_1)$ and $l^2_i = \phi_i(s) - \phi^2_i(s_2)$, $i \in \{x, y, z\}$. For each vertex $v \in V(\Gamma(G_1)) - \{s_1, t_1\}$, output that

$$\phi_i(v) = \begin{cases} \phi^1_i(v) - 2k * R_i(G_1) & \text{if } \psi^1_i(v) < 0, \\ \phi^1_i(v) + \phi_i(t_1) + k * R_i(G_1) & \text{if } 0 < \psi^1_i(v) < \psi^1_i(s_1), \\ \phi_i(t_1) + k * R_i(G_1) + \phi^1_i(v) + l^1_i & \text{if } \psi^1_i(v) > \psi^1_i(s_1), \end{cases}$$

where $i \in \{x, y, z\}$. For each vertex $v \in V(\Gamma(G_2)) - \{s_2, t_2\}$, output that

$$\phi_i(v) = \begin{cases} \phi^2_i(v) & \text{if } \psi^2_i(v) < 0, \\ \phi^2_i(v) + k * R_i(G) & \text{if } 0 < \psi^2_i(v) < \psi^2_i(s_2), \\ \phi_i(s) + l^2_i & \text{if } \psi^2_i(v) > \psi^2_i(s_2), \end{cases}$$

Case 2 $R_i(G) = -R_i(G_1) = R_i(G_2)$: For the vertices $t_1 = s_2$ and $s = s_1$, output that

$$\begin{aligned}\phi_i(s) &= \phi_i(t) + R_i(G), \\ \phi_i(t_1) &= \phi_i(s_2) = \phi_i(s) + 2k * R_i(G).\end{aligned}$$

For each vertex $v \in V(\Gamma(G_1)) - \{s_1, t_1\}$, output that

$$\phi_i(v) = \begin{cases} \phi_i(t_1) + \phi^1_i(v) + k * R_i(G) & \text{if } \psi^1_i(v) < 0, \\ \phi_i(t_1) - k * R_i(G) + \phi^1_i(v) & \text{if } 0 < \psi^1_i(v) < \psi^1_i(s_1), \\ \phi_i(t) - k * R_i(G) + \phi^1_i(v) - l^1_i & \text{if } \psi^1_i(v) > \psi^1_i(s_1), \end{cases}$$

where $i \in \{x, y, z\}$. For each vertex $v \in V(\Gamma(G_2)) - \{s_2, t_2\}$, output that

$$\phi_i(v) = \begin{cases} \phi_i^2(v) & \text{if } \psi_i^2(v) < 0, \\ \phi_i(s) + k * R_i(G) + \phi_i^2(v) & \text{if } 0 < \psi_i^2(v) < \psi_i^2(s_2), \\ \phi_i(s_2) + \phi_i^2(v) - \phi_i^2(s_2) & \text{if } \psi_i^2(v) > \psi_i^2(s_2), \end{cases}$$

where $i \in \{x, y, z\}$.

Case 3 $R_i(G) = R_i(G_1) = -R_i(G_2)$

For the vertices $t_1 = s_2$ and $s = s_1$, output that

$$\begin{aligned} \phi_i(s) &= \phi_i(t) + R_i(G), \\ \phi_i(t_1) &= \phi_i(s_2) = -2k * R_i(G). \end{aligned}$$

For each vertex $v \in V(\Gamma(G_1)) - \{s_1, t_1\}$, output that

$$\phi_x(v) = \begin{cases} \phi_i(t_1) + \phi_i^1(v) - k * R_i(G) & \text{if } \psi_i^1(v) < 0, \\ \phi_i(t_1) + k * R_i(G) + \phi_i^1(v) & \text{if } 0 < \psi_i^1(v) < \psi_i^1(s_1), \\ \phi_i(s) + k * R_i(G) + \phi_i^1(v) - \phi_i^1(s_1) & \text{if } \psi_i^1(v) > \psi_i^1(s_1), \end{cases}$$

where $i \in \{x, y, z\}$. For each vertex $v \in V(\Gamma(G_2)) - \{s_2, t_2\}$, output that

$$\phi_x(v) = \begin{cases} \phi_i(s) + \phi_i^2(v) & \text{if } \psi_i^2(v) < 0, \\ \phi_i(t) - k * R_i(G) + \phi_i^2(v) & \text{if } 0 < \psi_i^2(v) < \psi_i^2(s_2), \\ \phi_i(s_2) + \phi_i^2(v) - \phi_i^2(s_2) & \text{if } \psi_i^2(v) > \psi_i^2(s_2), \end{cases}$$

where $i \in \{x, y, z\}$.

4.2.1.6 PAR-3D-COM($\Gamma(G_1), \Gamma(G_2), R(G_1), R(G_2), R(G)$)

Input: $\Gamma(G_1), \Gamma(G_2), R(G_1), R(G_2), R(G)$

Output: $\Gamma(G)$

Let ϕ^1 and ϕ^2 be a layout of $\Gamma(G_1)$ and $\Gamma(G_2)$, respectivly. Let

$$k = \max\{|\phi_x^1(v)|, |\phi_x^2(w)|, |\phi_y^1(v)|, |\phi_y^2(w)|, |\phi_z^1(v)|, |\phi_z^2(w)|, \},$$

for $\forall v \in V(G_1)$ and $\forall w \in V(G_2)$. And let

$$\begin{aligned} \psi^1(v) &= R(G_1) * \phi^1(v), \quad \text{and} \\ \psi^2(w) &= R(G_2) * \phi^2(w), \end{aligned}$$

where $v \in V(\Gamma(G_1)$ and $w \in V(\Gamma(G_2))$. If $\psi^1(v) = (v_x, v_y, v_z)$ and $\psi^2(w) = (w_x, w_y, w_z)$, we denote that $v_x = \psi^1_x(v)$, $v_y = \psi^1_y(v)$, $v_z = \psi^1_z(v)$, $w_x = \psi^2_x(w)$, $w_y = \psi^2_y(w)$, and $w_z = \psi^2_z(w)$. Output a layout $\phi(s)$ and $\phi(t)$ of $\Gamma(G)$ that

$$\begin{aligned} \phi(t) &= \phi^2(t_2), \\ \phi(s) &= (2k, 2k, 2k) * R(G). \end{aligned}$$

For every $v \in V(G_1) - \{s_1, t_1\}$, output a layout $\phi(v)$ of $\Gamma(G)$ as follows:

$$\phi_i(v) = \begin{cases} \phi^1_i(v) - k * R(G) & \text{if } \psi^1_i(v) < 0, \\ \phi^1_i(v) + k * R(G) & \text{if } 0 < \psi^1_i(v) < \psi^1_i(s_1), \\ \phi^1_i(v) - \phi^1_i(s_1) + k * R_i(G) + \phi_i(s) & \text{if } \psi^1_i(v) > \psi^1_i(s_1), \end{cases}$$

where $i \in \{x, y, z\}$. For every $v \in V(G_2) - \{s_2, t_2\}$, output a layout $\phi(v)$ of $\Gamma(G)$ as follows:

$$\phi_i(v) = \begin{cases} \phi^2_i(v) & \text{if } \psi^2_i(v) < \psi^2_i(s_2). \\ \phi^2_i(v) - \phi^2_i(s_2) + \phi_i(s) & \text{if } \psi^2_i(v) > \psi^2_i(s_2), \end{cases}$$

where $i \in \{x, y, z\}$.

4.2.1.7 Analysis of Algorithm 2

We first show that every edge $e \in E(\Gamma(G))$ have just 2-bend. If G consists of a single edge $e = (s, t)$ and the edge e can be drawn with 2-bend then $D_G(s)$ and $D_G(t)$ must be inner-directed. Thus we show the following Lemma.

Lemma 4.2 *If G consists of an edge $e = (s, t)$, $D_G(s)$ and $D_G(t)$ are inner-directed.*

Proof: Initial conditions $D_G(s)$ and $D_G(t)$ of Algorithm 2 are innner-directed.

Suppose $D_G(s)$ and $D_G(t)$ are inner-directed. If G is the series composition of G_1 and G_2, it is not difficult to see that $D_{G_1}(s_1)$ and $D_{G_1}(t_1)$, and $D_{G_2}(s_2)$ and $D_{G_2}(t_2)$ are inner-directed by Algorithm SER-3D-DECOM. If G is the parallel composition of G_1 and G_2, it is not difficult to see that $D_{G_1}(s_1)$ and $D_{G_1}(t_1)$ or $D_{G_2}(s_2)$ and $D_{G_2}(t_2)$ are inner-directed by Algorithm PER-3D-DECOM. Since G is a simple connected graph, G_1 or G_2 contains the edge $e = (s, t)$.

Suppose $D_G(s)$ and $D_G(t)$ are not inner-directed. If G is the series composition of G_1 and G_2, it is not difficult to see that $D_{G_1}(s_1)$ and $D_{G_1}(t_1)$, and $D_{G_2}(s_2)$ and $D_{G_2}(t_2)$ are inner-directed by Algorithm SER-3D-DECOM. If G is the parallel composition of G_1 and G_2, there is no edge $e = (s, t)$ in G.

Thus if G_1 and G_2 contain $e_1 = (s_1, t_1)$ and $e_2 = (s_2, t_2)$, $D_{G_1}(s_1)$ and $D_{G_1}(t_1)$, and $D_{G_2}(s_2)$ and $D_{G_2}(t_2)$ are inner-directed,respectivly. $\blacksquare$

It remains to show that there are not cross over edges in $\Gamma(G)$. It is not difficult to prove the following Lemmas from Algorithm SER-3D-COM and PER-3D-COM.

Lemma 4.3 *For every vertices $v, w \in V(\Gamma(G))$, a layout $\phi(v), \phi(w)$ is satisfied the following condition:*

$$\begin{aligned} \phi_x(v) &\neq \phi_x(w), \\ \phi_y(v) &\neq \phi_y(w), \\ \phi_z(v) &\neq \phi_z(w). \end{aligned}$$

$\blacksquare$

Lemma 4.4 *For every $v \in \Gamma(G)$, it is satisfied an one of the following two conditions:*

(A) $R_i(G) * \phi_i(s) < R_i(G) * \phi_i(v) < R_i(v) * \phi_i(t), \forall i \in \{x, y, z\}$,

(B) $R_i(G) * \phi_i(s) > R_i(G) * \phi_i(v)$ *or* $R_i(G) * \phi_i(t) < R_i(G) * \phi_i(v)$, $\forall i \in \{x, y, z\}$.

$\blacksquare$

From Lemma 4.4, Algorithm SER-3D-COM, and Algorithm PER-3D-COM, it is not difficult to see the following lemma.

Lemma 4.5 *If G is obtained by a series-compisition or parallel-composition of G_1 and G_2 then it is satisfied one of the following two conditions:*

(A) $|\phi_i(w_1) - \phi_i(v)| > |\phi_i(w_2) - \phi_i(v)|, \forall i \in \{x, y, z\}$,

(B) $|\phi_i(w_1) - \phi_i(v)| < |\phi_i(w_2) - \phi_i(v)|, \forall i \in \{x, y, z\}$.

$\blacksquare$

Lemma 4.6 *There are no cross over edges in $\Gamma(G)$.*

Proof: Suppose the edge $e_1 = (v_1, w_1)$ and $e_2 = (v_2, w_2)$ are crossing and let u be a cross point between e_1 and e_2. Since u is laid on the e_1, two among of $\phi_i(u)$, $i \in \{x, y, z\}$, are same as $\phi_i(v_1)$ or $\phi_i(w_1)$. Since u is laid on the e_2, two among of $\phi_i(u)$, $i \in \{x, y, z\}$, are same as $\phi_i(v_2)$ or $\phi_i(w_2)$. Thus there is an $i \in\in \{x, y, z\}$ such that $\phi_i(j) = \phi_i(k)$ where $j = \{v_1, w_1\}$ and $k = \{v_2, w_2\}$.

From Lemma 4.3, this is contradiction. Thus we consider the case of the edges $e_1 = \{v, w_1\}$ and $e_2 = \{v, w_2\}$. We can prove this case by induction. If G is a single esge then it is clear that $\Gamma(G)$ has no edge crossing. Suppose $\Gamma(G_1)$ and $\Gamma(G_2)$ have no edge crossing. We first assume that $\Gamma(G)$ obtained by series composition of $\Gamma(G_1)$ and $\Gamma(G_2)$. Since $\Gamma(G_1)$ and $\Gamma(G_2)$ have no cross over edges, if $e_1, e_2 \in E(\Gamma(G_1))$ or $e_1, e_2 \in E(\Gamma(G_2))$ then e_1 and e_2 are not crossing in $\Gamma(G)$. Suppose $e_1 \in E(\Gamma(G_1))$ and $e_2 \in E(\Gamma(G_2))$. From Lemma 4.5, every e_1 and e_2 do not cross over. Assume that $\Gamma(G)$ obtained by parallel composition of $\Gamma(G_1)$ and $\Gamma(G_2)$. Since $\Gamma(G_1)$ and $\Gamma(G_2)$ have no cross over edges, if $e_1, e_2 \in E(\Gamma(G_1))$ or $e_1, e_2 \in E(\Gamma(G_2))$ then e_1 and e_2 are not crossing in $\Gamma(G)$. Suppose $e_1 \in E(\Gamma(G_1))$ and $e_2 \in E(\Gamma(G_2))$. From Lemma 4.5, every e_1 and e_2 do not cross over. ∎

From Lemma 4.2 and Lemma 4.6, $\Gamma(G)$ obtained by Algorithm 2 is 2-bend 3-D orthogonal drawing of G.

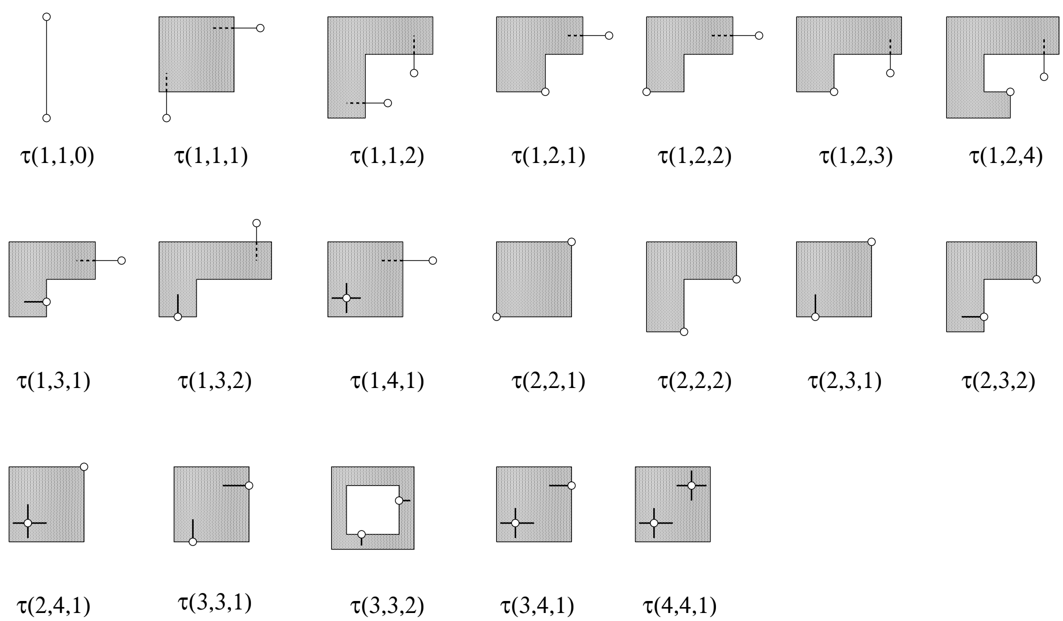

Figure 4.1: Types of 1-bend 2-D orthogonal drawings, where $\tau(i, j, k) = \tau(j, i, k)$.

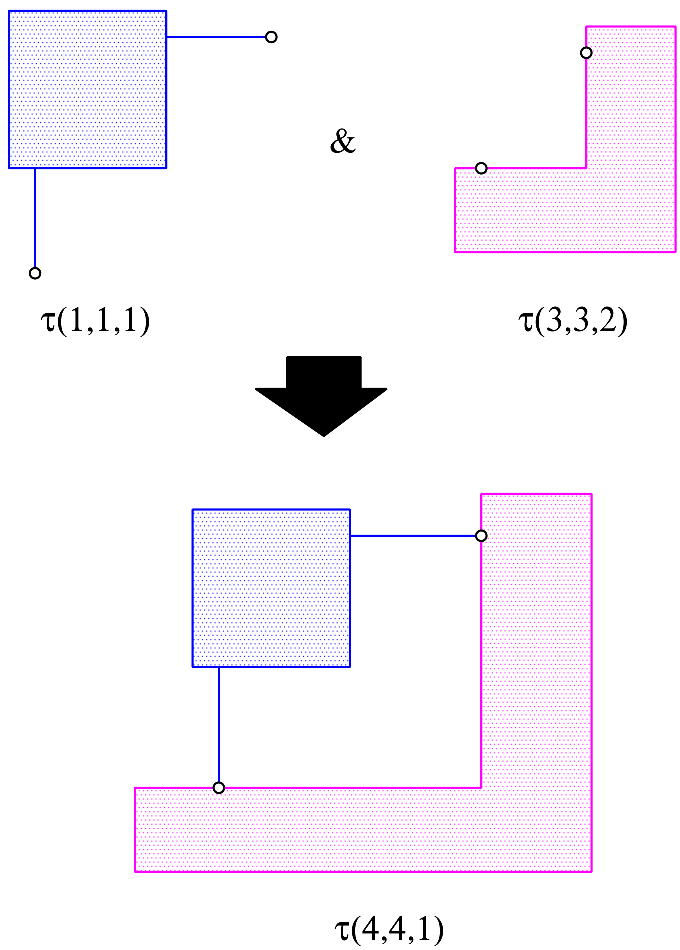

Figure 4.2: $\tau(4,4,1)$ is generated by parallel composition $\tau(1,1,1)$ and $\tau(3,3,2)$.

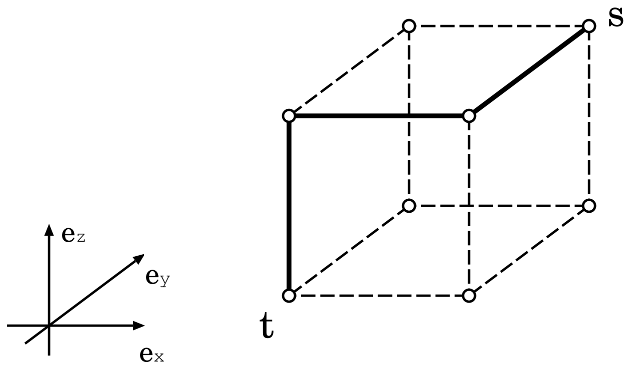

Figure 4.3: An edge route with 2-bend edge route for $D_G(s) = -X$, $D_G(t) = Z$, and $R(G) = (1, 1, 1)$.

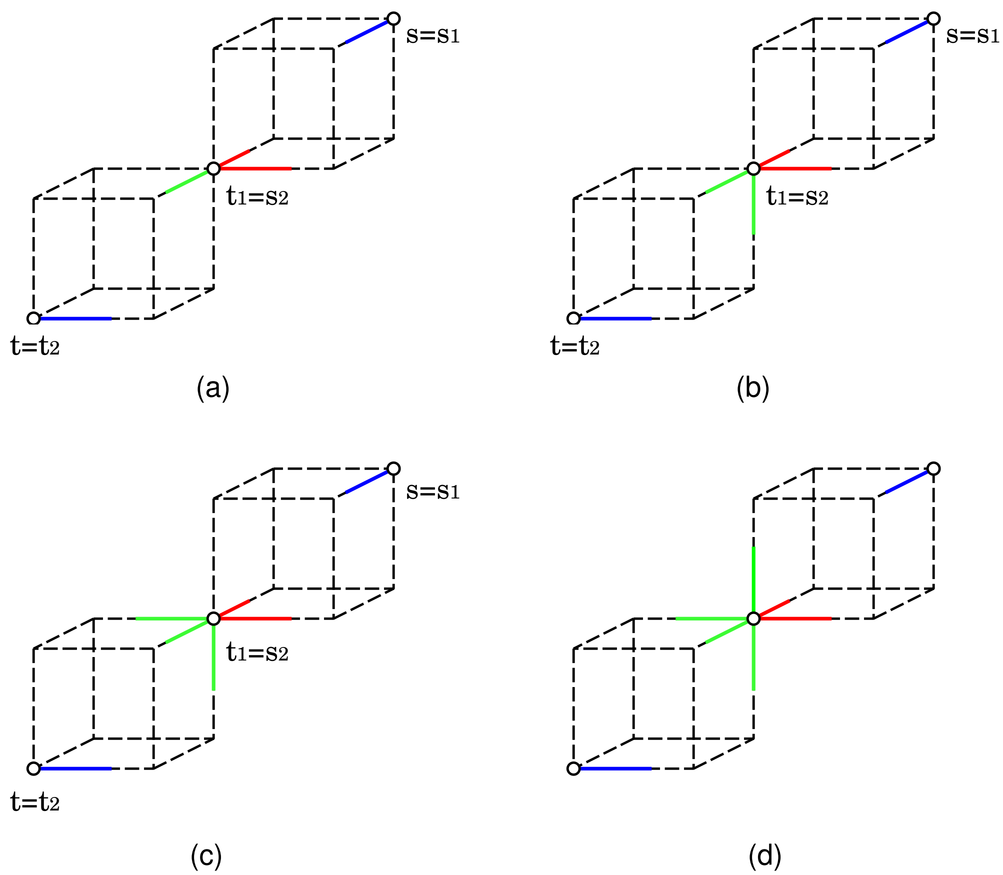

Figure 4.4: All the cases when $A = -Y, B = X$, and $d_{G_1}(t_1) = 2$: (a)$d_{G_2}(s_2) = 1$, (b)$d_{G_2}(s_2) = 2$, (c)$d_{G_2}(s_2) = 3$, and (d)$d_{G_2}(s_2) = 4$. A and B are shown in blue line, the direction including of $D_{G_1}(t_1)$ are shown in red line, and the direction including of $D_{G_2}(s_2)$ are shown in green line.

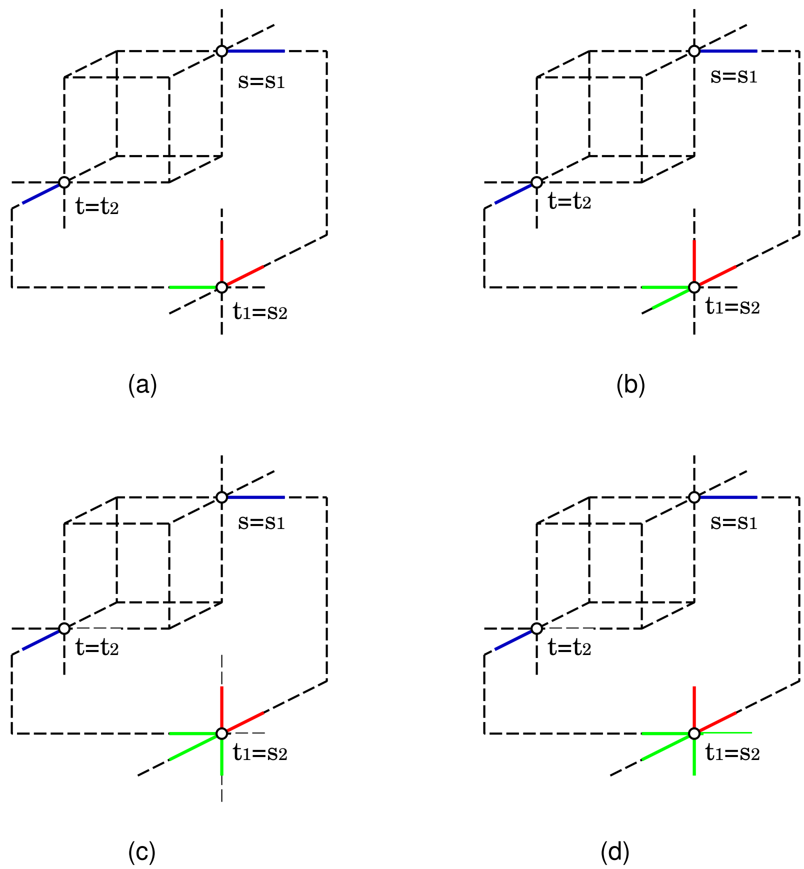

Figure 4.5: All the cases when $A = X, B = -Y$, and $d_{G_1}(t_1) = 2$: (a)$d_{G_2}(s_2) = 1$, (b)$d_{G_2}(s_2) = 2$, (c)$d_{G_2}(s_2) = 3$, and (d)$d_{G_2}(s_2) = 4$. A and B are shown in blue line, the direction including of $D_{G_1}(t_1)$ are shown in red line, and the direction including of $D_{G_2}(s_2)$ are shown in green line.

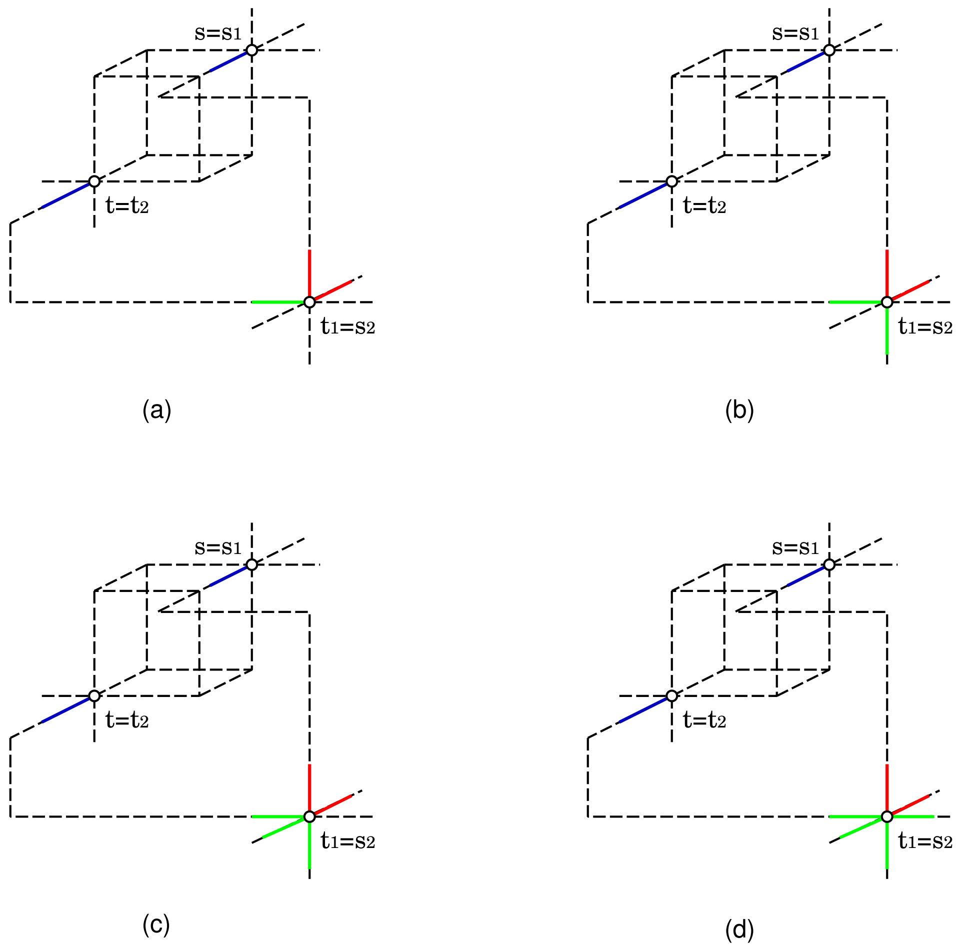

Figure 4.6: All the cases when $A = -Y, B = -Y$, and $d_{G_1}(t_1) = 2$: (a)$d_{G_2}(s_2) = 1$, (b)$d_{G_2}(s_2) = 2$, (c)$d_{G_2}(s_2) = 3$, and (d)$d_{G_2}(s_2) = 4$. A and B are shown in blue line, the direction including of $D_{G_1}(t_1)$ are shown in red line, and the direction including of $D_{G_2}(s_2)$ are shown in green line.

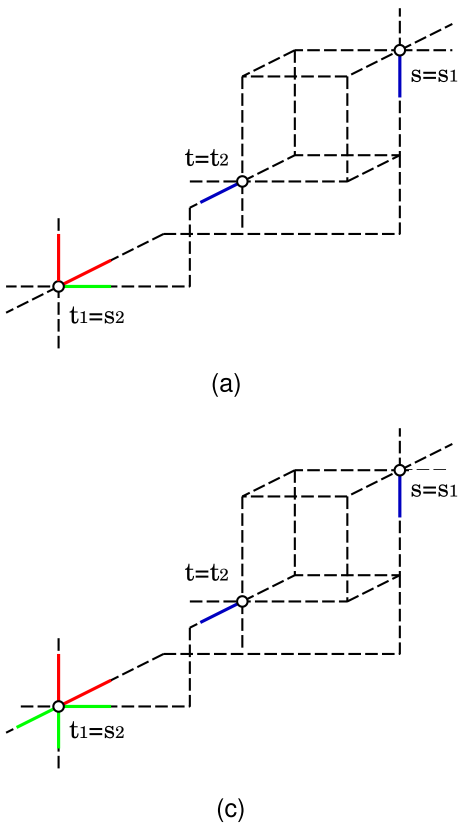

Figure 4.7: All the cases when $A = -Z, B = -Y$, and $d_{G_1}(t_1) = 2$: (a)$d_{G_2}(s_2) = 1$, (b)$d_{G_2}(s_2) = 2$, (c)$d_{G_2}(s_2) = 3$, and (d)$d_{G_2}(s_2) = 4$. A and B are shown in blue line, the direction including of $D_{G_1}(t_1)$ are shown in red line, and the direction including of $D_{G_2}(s_2)$ are shown in green line.

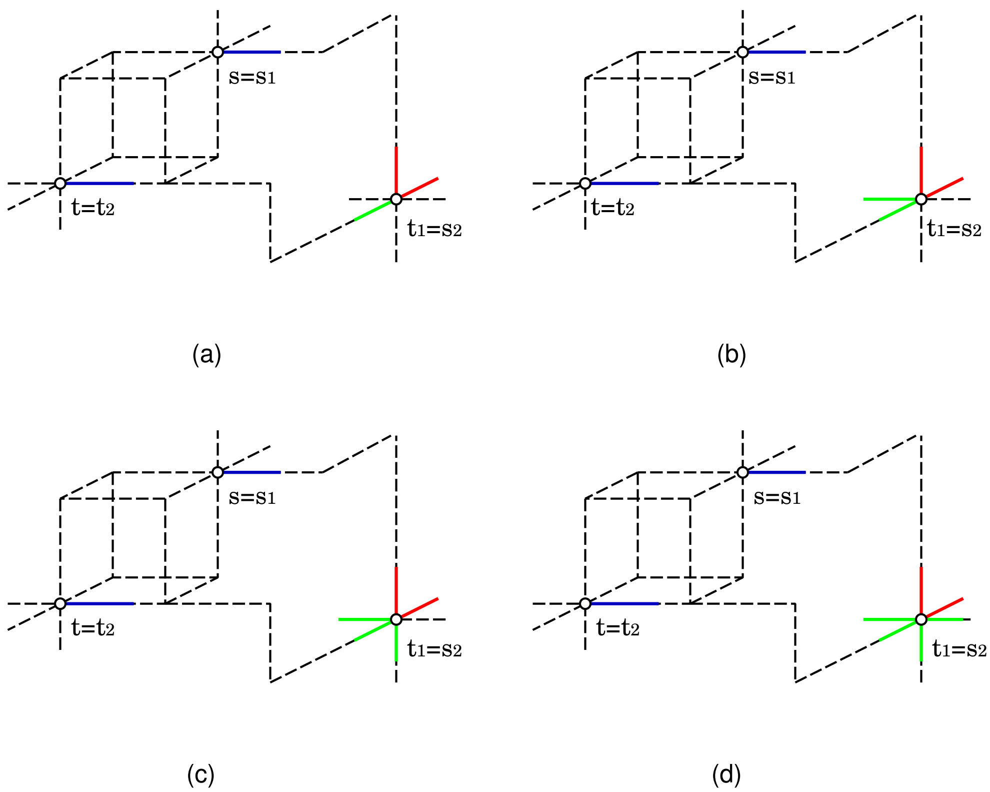

Figure 4.8: All the cases when $A = X, B = X$, and $d_{G_1}(t_1) = 2$: (a)$d_{G_2}(s_2) = 1$, (b)$d_{G_2}(s_2) = 2$, (c)$d_{G_2}(s_2) = 3$, and (d)$d_{G_2}(s_2) = 4$. A and B are shown in blue line, the direction including of $D_{G_1}(t_1)$ are shown in red line, and the direction including of $D_{G_2}(s_2)$ are shown in green line.

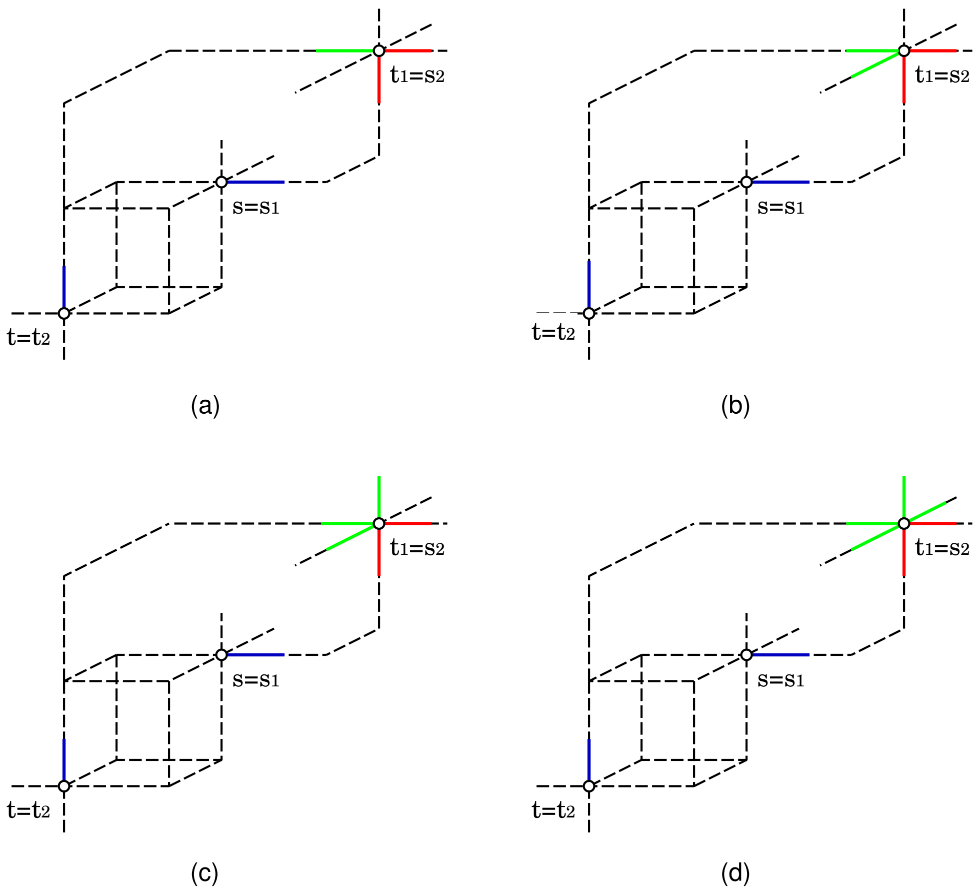

Figure 4.9: All the cases when $A = X, B = Z$, and $d_{G_1}(t_1) = 2$: (a)$d_{G_2}(s_2) = 1$, (b)$d_{G_2}(s_2) = 2$, (c)$d_{G_2}(s_2) = 3$, and (d)$d_{G_2}(s_2) = 4$. A and B are shown in blue line, the direction including of $D_{G_1}(t_1)$ are shown in red line, and the direction including of $D_{G_2}(s_2)$ are shown in green line.

$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$
(1,1,1)	(1,1,1)	(1,1,1)		(1,2,1)	(2,2,2)		(2,2,1)	(1,3,2)
	(1,1,1)	(2,1,2)		(1,3,1)	(1,2,1)		(2,2,1)	(2,3,1)
	(1,1,1)	(3,1,2)	(1,3,1)	(1,1,1)	(1,3,1)		(2,3,1)	(1,3,2)
	(1,2,2)	(1,1,1)		(1,1,1)	(2,3,1)	(2,3,2)	(2,1,1)	(1,3,1)
	(1,2,1)	(2,1,2)		(1,1,1)	(3,3,1)		(2,1,2)	(2,3,2)
	(1,3,2)	(1,1,1)		(1,2,2)	(1,3,1)		(2,1,1)	(3,3,1)
(1,1,2)	(1,1,1)	(1,1,1)		(1,2,1)	(2,3,1)		(2,2,2)	(1,3,2)
	(1,1,1)	(2,1,1)		(1,3,1)	(1,3,2)		(2,2,1)	(2,3,2)
	(1,1,1)	(3,1,1)	(1,3,2)	(1,1,1)	(1,3,1)		(2,3,2)	(1,3,2)
	(1,2,1)	(1,1,1)		(1,1,1)	(2,3,1)	(2,4,1)	(2,1,2)	(1,4,1)
	(1,2,1)	(2,1,1)		(1,1,1)	(3,3,1)		(2,1,2)	(2,4,1)
	(1,3,1)	(1,1,1)		(1,2,2)	(1,3,2)		(2,1,2)	(3,4,1)
(1,2,1)	(1,1,1)	(1,2,1)		(1,2,2)	(2,3,1)		(2,2,1)	(1,4,1)
	(1,1,1)	(2,2,1)		(1,3,2)	(1,3,2)		(2,2,1)	(2,4,1)
	(1,1,1)	(3,2,1)	(1,4,1)	(1,1,1)	(1,4,1)		(2,3,1)	(1,4,1)
	(1,2,1)	(1,2,2)		(1,1,1)	(2,4,1)	(3,3,1)	(3,1,2)	(1,3,2)
	(1,2,1)	(2,2,1)		(1,1,1)	(3,4,1)		(3,1,2)	(2,3,1)
	(1,3,1)	(1,2,2)		(1,2,1)	(1,4,1)		(3,1,2)	(3,3,1)
(1,2,2)	(1,1,1)	(1,2,2)		(1,2,1)	(2,4,1)		(3,2,1)	(1,3,2)
	(1,1,1)	(2,2,1)		(1,3,1)	(1,4,1)		(3,2,1)	(2,3,1)
	(1,1,1)	(3,2,1)	(2,2,1)	(2,1,2)	(1,2,2)		(3,3,1)	(1,3,2)
	(1,2,2)	(1,2,2)		(2,1,2)	(2,2,1)	(3,3,2)	(3,1,1)	(1,3,1)
	(1,2,2)	(2,2,1)		(2,1,2)	(3,2,1)		(3,1,1)	(2,3,1)
	(1,3,2)	(1,2,2)		(2,2,1)	(1,2,2)		(3,1,2)	(3,3,2)
(1,2,3)	(1,1,1)	(1,2,2)		(2,2,1)	(2,2,1)		(3,2,1)	(1,3,1)
	(1,1,1)	(2,2,2)		(2,3,1)	(1,2,2)		(3,2,2)	(2,3,2)
	(1,1,2)	(3,2,1)	(2,2,2)	(2,1,1)	(1,2,1)		(3,3,2)	(1,3,2)
	(1,1,0)	(3,2,2)		(2,1,1)	(2,2,1)	(3,4,1)	(3,1,1)	(1,4,1)
	(1,2,1)	(1,2,2)		(2,1,1)	(3,2,1)		(3,1,1)	(2,4,1)
	(1,2,2)	(2,2,2)		(2,2,1)	(1,2,1)		(3,1,2)	(3,4,1)
	(1,3,1)	(1,2,2)		(2,2,1)	(2,2,2)		(3,2,1)	(1,4,1)
(1,2,4)	(1,1,1)	(1,2,1)		(2,3,1)	(1,2,1)		(3,2,1)	(2,4,1)
	(1,1,1)	(2,2,2)	(2,3,1)	(2,1,2)	(1,3,2)		(3,3,1)	(1,4,1)
	(1,1,1)	(3,2,2)		(2,1,2)	(2,3,1)			
	(1,2,1)	(1,2,1)		(2,1,2)	(3,3,1)			

Table 4.1: Series composition.

$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$	$\Gamma_i(G)$	$\Gamma_j(G_1)$	$\Gamma_k(G_2)$
$(2,2,1)$	$(1,1,1)$	$(1,1,1)$	$(3,3,2)$	$(1,2,2)$	$(2,1,3)$	$(4,4,1)$	$(1,1,1)$	$(3,3,2)$
$(2,2,2)$	$(1,1,1)$	$(1,1,2)$		$(1,1,2)$	$(2,2,2)$		$(1,1,2)$	$(3,3,1)$
$(2,3,1)$	$(1,1,1)$	$(1,2,1)$		$(1,2,1)$	$(2,1,4)$		$(1,2,1)$	$(3,2,2)$
$(2,3,2)$	$(1,1,1)$	$(1,2,4)$	$(3,4,1)$	$(1,1,1)$	$(2,3,2)$		$(1,3,1)$	$(3,1,1)$
$(2,4,1)$	$(1,1,1)$	$(1,3,1)$		$(1,2,1)$	$(2,2,2)$		$(2,2,2)$	$(2,2,2)$
$(3,3,1)$	$(1,1,1)$	$(2,2,2)$		$(1,3,1)$	$(2,1,1)$			

Table 4.2: Parallel composition.

Chapter 5

Conclusion

In this thesis, we consider the problem of generating orthogonal drawing of outerplanar graphs and series-parallel graphs.

In Chapter 3, we prove that every outerplanar 3-graph G has a 0-bend 2-D orhogonal drawing if and only if G contains no triangle as a subgraph. And we prove that every outerplanar 6-graph G has a 0-bend 3-D orhogonal drawing if and only if G contains no triangle as a subgraph. It is interesting to note that a complete bipartite graph $K_{2,3}$, which is a minimal non-outerplanar graph with no triangles, has no 0-bend 2-D or 3-D orthogonal drawing.

In chapter 4, we prove that every series-parallel graph G has a 1-bend 2-D orthogonal drawing. From this results, every outerplanar 4-graph has a 1-bend 2-D orthogonal drawing. And we prove that every series-parallel graph G has 2-bend 3-D orthogonal drawing.

It is an interesting an open problem to decide if every outerplanar 6-graph has a 1-bend 3-D orthogonal drawing, and if every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

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1 Kumiko Nomura, Satoshi Tayu, and Shuichi Ueno. An Orthogonal Drawing for Series-Parallel Graphs. *IPSJ SIG Technical Report*, vol.2004, No.109, pp.25–32, 2004-AL-98, 2004.

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