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Doctoral Thesis

Orbifolded Partition Function

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Abstract

A 3-dimensional supersymmetric gauge theory has attracted great interests and has been explored extensively since M-theory was proposed. The M-theory is the non-perturbative realization of the string theory, and it is believed to be the most fundamental theory. The M-theory consists of two kinds of fundamental objects, called an M2-brane and an M5-brane. The M2-branes are described by the gauge theory in the low energy limit. There also exist another expression to describe the M2-branes in the low energy limit, which is a supergravity. These different theories are believed to be equivalent to each other. In general, a correspondence between outwardly different but physically equivalent theories is called a duality. The duality between the two theories that illustrate the M2-branes is called a gauge/gravity correspondence, or an AdS/CFT. The benefit of the duality is that the strong coupling region of one of the theories corresponds to the weak coupling region of the other theory. Although perturbative methods are not applicable to explore the strong coupling region the duality enables us to study such a region through the dual theory. Therefore, the duality is an important tool.

Since the duality is so non-trivial it is difficult to show it. One way to check the duality is to compare partition functions calculated from the theories. The coincidence of the partition functions is an evidence of the duality. The problem is that we have to treat the non-perturbative effects on the one side of the duality. Though it is usually difficult to include the non-perturbative effects into a partition function a localization method enables us to exactly calculate a partition function for a supersymmetric gauge theory. We assume a classical gravity with a weak coupling for the gravity side, and calculate the partition function for the gauge theory, exactly. It is shown that the partition functions of the dual pair indeed coincide.

In this thesis we basically consider general 3-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories rather than the specific model of M2-branes, and discuss a formula to derive their exact partition functions so that we can apply the formula to check dualities among the gauge theories and AdS/CFT.

When we calculate a partition function we have to choose the background space on which the theory is defined. If we select a curved space for the background space the geometric parameters are translated into parameters of the partition function. The more parameters sophisticate the partition function as an index for checking dualities. Our goal is to calculate the "correct" partition function on the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$. The "correct" means that the partition functions calculated for a dual pair coincide. A gauge theory on the orbifold has degenerate vacua due to the Aharonov-Bohm effect. These vacua are specified by so called holonomies and the partition function is expressed by the sum of each contribution. The known formula of the partition function on the orbifold gives different values for a pair of dual theories, which means that the formula is incorrect. Numerical results indicate that extra phase factors in the holonomy sum are needed to match the partition functions. Using the fact that there is no ambiguity in the partition function of a non-gauge theory we can determine the phase factors for a gauge theory that is dual to a non-gauge theory. We propose a formula of the extra phase factors.

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Contents

Summary of symbols

$\mathbb{R}$	:	the set of all real numbers.
$\mathbb{Z}$	:	the set of all integers.
$\mathbb{Z}_n$	:	the set whose elements are $\{0, 1, 2, \cdots, n-1\}$.
$\mathcal{M}$	:	a manifold (naively say, a smooth curved space).
$\boldsymbol{S}^n$	:	n-dimensional sphere : $\sum_{i=0}^{n}(x_i)^2 = R^2$.
$\boldsymbol{T}^n$	:	n-dimensional torus : $\boldsymbol{T}^n = \otimes_{i=1}^{n}\boldsymbol{S}^1$.

Chapter 1

Introduction

The goal of this thesis is to derive the correct formula of a partition function on an orbifold (we call this by an orbifolded partition function) of 3-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. Especially, we point out that the known formula for the orbifolded partition function does not work for checking dualities, and some modifications are needed. In this introduction we explain why we consider 3-dimensional supersymmetric gauge theories and why the partition function is important to study.

1.1 String theory and M-theory

The standard model and general relativity have been extensively tested in experiments, and are thought as the most plausible and established theories that describe our world. Nonetheless, there are still lots of problems and unsatisfactory points.

One big problem is that a quantum theory of the gravity is not included in them. General relativity is a powerful tool to describe dynamics of objects, especially for large scale, like stars, galaxies, and even our universe itself. However, as it is a classical theory it cannot explain what happens inside black holes, nor what happens at the extremely early time of the universe, where the quantum effects of the gravity become important. We need the quantum theory of the gravity.

Although we believe our world is described by a simple and unified theory, the standard model is so complicated. It consists of a number of varieties of particles and forces with lots of parameters. Therefore, we somehow want to unify them into a simple theory. We dare to say that one of such attempts is to use supersymmetry. Supersymmetry is symmetry that connects bosonic and fermionic fields (particles) and ties them up. Namely, it associates gauge fields and fermions, and/or fermions and scalar fields. We call one of paired fermion and boson a superpartner against the other. A parameter of the transformation of the supersymmetry is a spinor ϵ due to the difference of spin statistics of fermions and bosons:

$$\delta\Phi = \epsilon\Psi \tag{1.1}$$

where δ is the transformation, and Φ and Ψ are a field and its superpartner, respectively. Furthermore, one can define the number of supersymmetries $\mathcal{N}$. It is the number of parameters of the transformation. The collection of the fermions and the bosons that are connected by the transformation is called a multiplet. The examples of $\mathcal{N}=2$ multiplets are shown in Fig. 1.1), which are extensively used in this thesis. We, unfortunately, cannot say that it really unifies particles and gauge

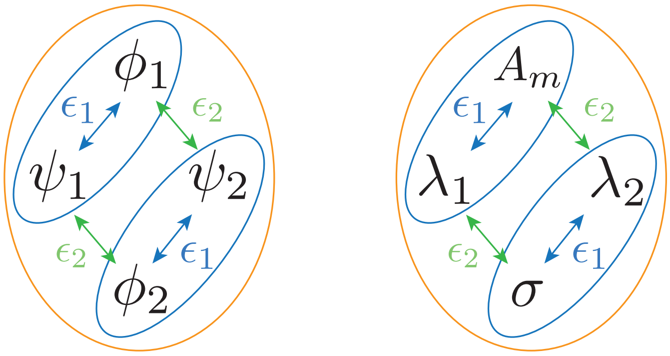

Figure 1.1: $\mathcal{N}=2$ multiplets. The left one consists of two pairs of real scalars $\phi_{1,2}$ and Majorana fermions $\psi_{1,2}$, and it is called a chiral multiplet. The right one consists of a gauge field A_m and Majorana fermions λ_1, λ_2 and a real scalar σ, and it is called a gauge multiplet. The blue circles designate $\mathcal{N}=1$ multiplets.

fields in our world and makes the standard model simple since the particles in our world are not connected to each other by the supersymmetry. Nonetheless, it is a natural unification of matters and gauge fields.

Superstring theory is one of theories that includes quantum gravity. Moreover, it is extremely simple; it has only two parameters. It is defined in 10 dimensions and it consists of spatially one-dimensional extended objects, called fundamental superstrings or simply strings (sometimes denoted by F1). Quantization of the string gives the discrete spectrum of a variety of multi-spin fields including scalars, fermions, gauge fields, and spin-2 objects, one of which is nothing but the graviton. Therefore, the string theory is a quantum theory of the gravity. Furthermore, there are solitonic objects called D-branes on which the string can end [1]. There are various dimensional D-branes from point-like ones in the spacetime up to ones filling the 10-dimensional space. It is customary to denote them by Dp-branes, where p specify the spacial dimension of D-branes; e.g. D2-branes are 3-dimensional objects and D(-1)-branes are 0-dimensional, point-like objects in the spacetime. One of important features of the branes is to give us gauge theories in the low energy limit (IR limit). The number of D-branes N corresponds to the rank of the unitary gauge group $U(N)$. The scalar fields parameterize the position of the branes. Therefore, various supersymmetric gauge theories can be engineered from the branes in the IR limit.

Though the superstring theory is a fascinating theory there is a crucial gap between the superstring theory and the real world; the real world is 4-dimensional, on the other hand the superstring is defined in 10 dimensions. One way to resolve this gap is to compactify the spacetime. Let us image a paper (2-dimensional object) and role it up to a cylinder. If the radius R is small enough it is regarded as a 1-dimensional object, effectively (see Fig. 1.2). The same manipulation can be done for the spacetime, and it is called $\boldsymbol{S}^1$ compactification, where $\boldsymbol{S}^n$ is an n-dimensional sphere (e.g. $\boldsymbol{S}^1$ is a circle, $\boldsymbol{S}^2$ is a sphere etc.). One can repeat the compactification until the dimension reduces to 4, and the standard model maybe realized in the IR

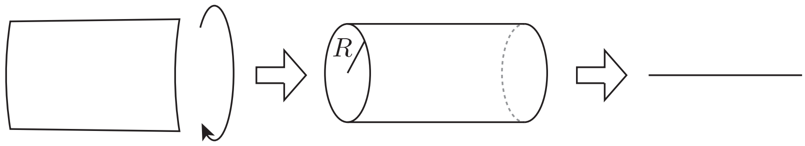

Figure 1.2: Compactification

limit of the string theory.

Perturbative analysis on the superstring theory shows that the superstring theory consists of 5 varieties. These are called type I, type IIA, type IIB, hetero $SO(32)$, and hetero $E_8 \times E_8$ superstring theory. Although we had tried to unify all the theories we arrived at the fact that there are actually five theories. In 1995 E. Witten proposed that these varieties can be unified by M-theory. Inversely, those varieties are realized as a certain limit of this M-theory [2].

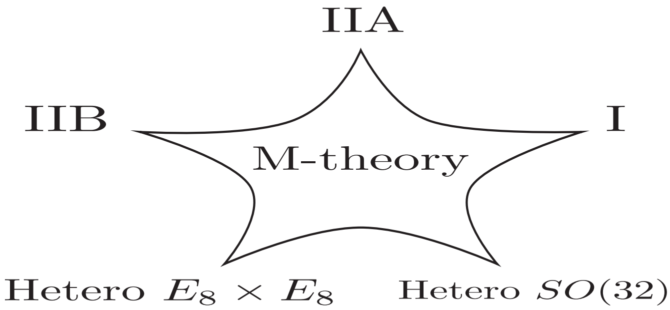

Figure 1.3: M-theory and 5 varieties of superstring theories as a limit

Since then, the M-theory has been being extensively investigated. However, no one knows even its definition that provides the way of quantization and calculating S-matrix. We only know that

- it is defined in 11 dimensions,
- it consists of two fundamental objects, called M2-branes and M5-branes,
- it reduces to 11-dimensional supergravity (SUGRA) in the IR limit,
- and it reduces to type IIA superstring theory when it compactified on a circle $\boldsymbol{S}^1$.

Note that 11 is the highest dimension in which the supersymmetry is realized, and the number of parameters in the M-theory is only one, which is the tension of M2-branes. These are the reasons why we believe that the M-theory is the most fundamental and simple theory.

1.2 M2-branes

The first step of looking into the M-theory is probably to study the fundamental objects. As M2-branes are simpler and easier to investigate than M5-branes we focus on M2-branes. An M2-brane is a spatially 2-dimensional (spacetimely 3d) object. So far what we know about M2-branes is that :

- An M2-brane becomes a D2-brane or a fundamental string when the space is compactified on $\boldsymbol{S}^1$. When an M2-brane wraps the compactified direction $\boldsymbol{S}^1$ it becomes a fundamental string (see Table 1.1 and Figure 1.4). On the other hand, when an M2-brane does not wrap the $\boldsymbol{S}^1$ it becomes a D2-brane (see Table 1.1 and Figure 1.5).

Table 1.1: The directions that are filled with M2-branes. 0 is the time direction and 1 to 9 are the space directions of the string theory. The 11th direction that newly appears in M-theory is customary written as 11, which is compactified so as to derive the string theory. The M2 becomes a fundamental string and the M2' becomes a D2-brane.

	0	1	2	3	4	5	6	7	8	9	11
M2	○	○									○
M2'	○	○	○								

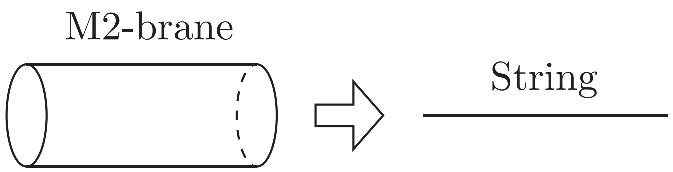

Figure 1.4: An M2-brane wrapping around $\boldsymbol{S}^1$ reduces to the fundamental string.

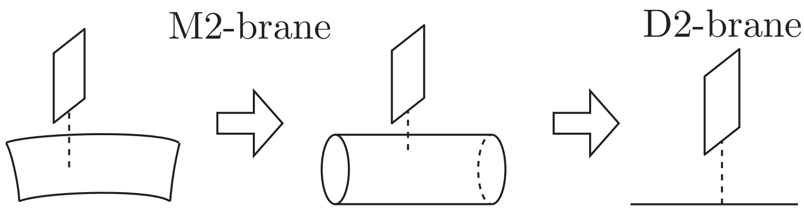

Figure 1.5: An M2-brane that does not wrap $\boldsymbol{S}^1$ becomes a D2-brane.

- M2-branes are effectively described by the solution of the 11-dimensional SUGRA in the IR limit.
- Three years ago, the effective field theory of M2-branes was found in [3]. It is a 3-dimensional $\mathcal{N} = 6$ supersymmetric gauge theory, which describe the dynamical degrees of freedom of N coincident M2-branes in the IR limit. This effective theory is called an ABJM model.

As a $\mathcal{N} = 6$ multiplet can be decomposed into $\mathcal{N} = 2$ multiples, a $\mathcal{N} = 6$ supersymmetric theory is a subgroup of a $\mathcal{N} = 2$ supersymmetric theory. In this thesis we study 3-dimensional $\mathcal{N} = 2$ supersymmetric theories. As the ABJM model can be described by $\mathcal{N} = 2$ multiplets the ABJM model is, of course, in our range.

1.3 Duality

We stated that the M-theory leads to superstring theories in a certain limit. Inversely say, the M-theory is a generalization of superstring theories, and actually, it is a non-perturbative realization of superstring theories. Hence, we need some non-perturbative tool to explore the M-theory. Duality is a significant and powerful tool to analyze the non-perturbative aspects of a theory. A duality connects outwardly different but physically equivalent theories. Physically equivalent means that observables like a spectrum in the dual theories are the same. As global symmetries are a kind of observables those of dual theories should coincide. On the other hand, gauge symmetries are not the observables and do not have to be the same for dual theories. In order to check dualities quantitatively, following observables are useful. As the information of a spectrum is translated into a partition function it can be used to check dualities. A correlation function is related to the physical degrees of freedom of the operators, and hence, we can use it to measure the equivalence of theories. Let us look into a few examples of a duality.

Mirror symmetry

Mirror symmetry connects gauge theories of different gauge groups with different matter contents. As we will use the simplest example of the mirror symmetry let us see it here briefly.

- The first theory is $\mathcal{N}=2$ supersymmetric quantum electrodynamics (SQED) with a fundamental and an anti-fundamental chiral multiplet $q, \widetilde{q}$.
- The other theory is a XYZ model, which consists of three chiral multiplets with a potential. Though the three multiples are equivalent we put different names $Q, \widetilde{Q}, S$ on them.

These theories have common global symmetries summarized in Table 1.2 From Ta-

Table 1.2: Global symmetries of the $\mathcal{N}=2$ SQED and the XYZ model, named $U(1)_V$ and $U(1)_A$. The numbers are the charges of each operator under the symmetries. m and $\widetilde{m}$ are the monopole operators appear in SQED.

	q	$\widetilde{q}$	m	$\widetilde{m}$	Q	$\widetilde{Q}$	S
$U(1)_V$	0	0	1	-1	1	-1	0
$U(1)_A$	1	1	0	0	-1	-1	2

ble 1.2 one can notice that $q\widetilde{q}$ and S have the same charges. Actually, they are the corresponding operators to each other under the duality. If we calculate the correlation functions of those operators (and other corresponding operators) they coincide, and we can convince ourselves that those theories are dual. One can also compare the partition functions of the theories to check the duality.

AdS/CFT correspondence

AdS/CFT is a duality between a gravity theory on Anti-de Sitter (AdS) space and a conformal field theory (CFT) [4]. As we often consider a gauge theory for the CFT it is also called a gauge/gravity correspondence. They are completely different formalism but their physical degrees of freedom and the global symmetries are the same. In order to see the benefit of this duality we firstly summarize the parameters of the gauge theory and the gravity theory. The parameters of our interest in a gauge theory are the rank of the gauge group N and the coupling constant g_{YM}. On the other hand, those of a gravity theory are the length of the string l_s and the Newton constant G_N in terms of the string coupling and the string length. The gravity theory with the small Newton constant and small string length reduces to the general relativity. The outstanding feature of the AdS/CFT is that the strong coupling region of the gauge theory where the perturbative method cannot be applicable corresponds to a gravity theory with small string length. Inversely, the weak coupling region of the gauge theory corresponds to the general relativity with some stringy effects ($l_s \neq 0$). The quantum effect of the gravity corresponds to the inverse of the rank of the gauge symmetry. Hence, the large number of N corresponds to the classical gravity. We assume that the gravity side is described by the general relativity and we consider the large N limit and the strong coupling region on the gauge theory side.

The effective theories of the M2-branes are the examples of the AdS/CFT. Namely, the ABJM model and the classical solution to 11d SUGRA are related by the AdS/CFT. The classical solution for the M2-branes is AdS_4 with a certain 7-dimensional manifold $\mathcal{M}_7$ (A manifold is a smooth and generally curved space). When the manifold $\mathcal{M}_7$ is 7-dimensional sphere $\boldsymbol{S}^7$ it corresponds to the ABJM model. This relation gives the information about the gauge theory from gravity

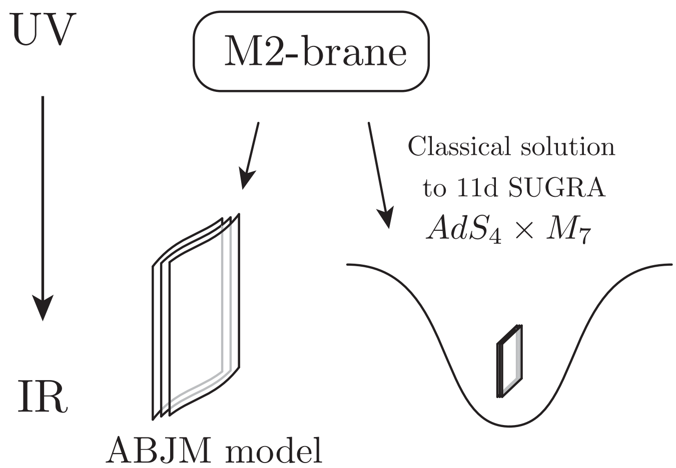

Figure 1.6: A pair of low energy effective theory of M2-brane: one is the gauge field and the other is classical background geometry distorted by the M2-brane.

side. For example, the isometry of the $\mathcal{M}_7$ corresponds to the global symmetry of the dual field theory. The correspondence of the global symmetry of the mirror dual is quite obvious. However, in the ABJM model the manifest global symmetry (symmetry of the Lagrangian) is $SO(6)$ though the isometry of $\boldsymbol{S}^7$ is $SO(8)$. The $SO(8)$

symmetry on the gauge theory side is realized non-perturbatively. Another example is the free energy of the M2-branes, which can be calculated from the gravity side and it tells us that the free energy is proportional to $N^{3/2}$ [5]. This strange exponent is the clue to seek the effective world volume theory of the N coincident M2-branes. Even though this calculation was performed in 1996 the calculation from the gauge theory side was done recently in [6]. We will explicitly see this calculation in Chapter 3.

1.4 Partition function

A partition function Z is usually defined by the state sum :

$$Z = \mathrm{tr}\left(e^{-\beta H}\right) = \sum_n \langle n|e^{-\beta H}|n\rangle = \sum_n e^{-\beta E_n}, \tag{1.2}$$

where H is the Hamiltonian of a theory, $|n\rangle$ is the energy eigenstate vector, $\beta = \frac{1}{k_B T}$ is the inverse temperature. The Boltzmann factor $e^{-\beta H}$ is reminiscent of the time evolution operator e^{-itH}. If one identifies the imaginary time it with the inverse temperature β the Boltzmann factor and the time evolution operator are equivalent. In this sense $\langle n|e^{-\beta H}|n\rangle$ is interpreted as the transition amplitude of a periodic evolution with the period $-i\beta$. Indeed, the inverse temperature is defined as the length of the compactified and Wick rotated time direction in a field theory.

Let us see another expression for the partition function, which is called a path integral. We firstly consider quantum mechanics and extend the result to a field theory case. We start with the state sum in terms of the eigenstates of the position $|q\rangle$:

$$Z = \int dq \langle q|e^{-\beta H}|q\rangle. \tag{1.3}$$

We can insert the complete sets and the partition function becomes

$$Z = \int dq dq_1 \cdots dq_{N-1} \langle q|e^{-\Delta\beta H}|q_{N-1}\rangle\langle q_{N-1}|\cdots|q_1\rangle\langle q_1|e^{-\Delta\beta H}|q\rangle, \tag{1.4}$$

where $\Delta\beta = \beta/N$. If we take the limit $N \to \infty$ it becomes

$$Z = \int \mathcal{D}q e^{-\int d\beta H} = \int \mathcal{D}q e^{-S[q]}, \tag{1.5}$$

where $\mathcal{D}q = \prod_{k=1}^{\infty} dq_k$, $S[q]$ is the action (functional of the path $[q]$). Since the Hamiltonian is equivalent to the Wick rotated (Euclideanized) action: $\int d\beta H = S_E[q]$, we rewrote it by the action and omitted the subscript E above. So far we have considered quantum mechanics. In a field theory q is replaced by a field Φ and the integration in the action is performed over the manifold $\mathcal{M}$ on which a theory is defined;

$$Z = \int_{\mathcal{M}} \mathcal{D}\Phi e^{-S[\Phi]}, \tag{1.6}$$

where note that customarily $\mathcal{M}$ is written as above though it is not the range of the field Φ but the space of the integration inside the action S.

The manifold $\mathcal{M}$ is, for example in quantum mechanics, the circle $\boldsymbol{S}^1$ with radius $R = \beta/2\pi$. In the case of photon system one usually considers the periodic boundary conditions on the walls facing each other of the box in which the photons exist. This box is called 3-dimensional torus $\boldsymbol{T}^3 = \boldsymbol{S}^1 \times \boldsymbol{S}^1 \times \boldsymbol{S}^1$. In addition, we have to consider the time direction $\boldsymbol{S}^1$, and finally the manifold $\mathcal{M}$ is 4-dimensional torus $\boldsymbol{T}^4 = \left(\boldsymbol{S}^1\right)^4$.

As we only consider 3-dimensional theories, $\mathcal{M}$ is a Euclidean 3-dimensional manifold in the remaining part of this thesis. Moreover, in order to avoid the IR divergence associated with infinite volume the manifold is limited to be a compact one.

Note that the partition function of the Minkowski space is positive definite because of the unitarity of the Hamiltonian. However, that of the Euclidean space is generally complex. We are only interested in the absolute value of the partition function, which describe the effective physical degrees of freedom of a theory, and hence, we usually ignore the phase factor. Nonetheless, the phase factor becomes very important when we consider an orbifold, which will be discussed in Section 1.8.

We also use a free energy, whose definition is a bit different from a usual one. The definition of our free energy is as follows.

$$F = -\log |Z| \tag{1.7}$$

1.5 Localization

Since the dualities are so non-trivial the check of the dualities themselves is important. One way to check the dualities is, as discussed, to calculate the partition functions for a pair of dual theories and see if they match or not. In case of the AdS/CFT correspondence we assume that the gravity side is described by the general relativity and we treat the non-perturbative effects on the gauge theory side. In this case we need an exact formula to derive the partition function for the gauge theory.

Localization method enables us to calculate the partition function exactly. If a theory has a fermionic symmetry (e.g. supersymmetry) one can deform the theory without changing the partition function as

$$Z = \int \mathcal{D}\Phi e^{-S_0[\Phi] - t\delta V[\Phi]} \tag{1.8}$$

where Φ symbolically denotes all the fields in the theory, δ is the fermionic transformation and it is assumed to be nilpotent: $\delta^2 = 0$, the action is invariant under the transformation: $\delta S_0 = 0$, and V is a certain functional of the fields. It is easily shown that the partition function does not depend on t;

$$\frac{\partial Z}{\partial t} = -\int \mathcal{D}\Phi \left(\delta V\right) e^{-S_0 - t\delta V} = -\int \mathcal{D}\Phi \delta \left(V e^{-S_0 - t\delta V}\right) = 0. \tag{1.9}$$

We can now take the limit $t \to \infty$ without any change, and the action is dominated by the deformation term $t\delta V$ rather than the original action S_0. Therefore, the path integral is dominated by the saddle points of δV rather than those of S_0. Furthermore, we consider fluctuations around these saddle points:

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\Phi', \tag{1.10}$$

where Φ_0 is a saddle point and Φ' is the fluctuation. Only the quadratic terms of Φ' in $t\delta V[\Phi]$ survive and higher terms will vanish after taking the limit $t \to \infty$. As we will see we can perform the path integral of Φ' explicitly and the partition function becomes

$$Z = \sum_{\Phi_0} e^{-S_0(\Phi_0)} Z_{1-\text{loop}}(\Phi_0) \tag{1.11}$$

where $Z_{1-\text{loop}}$ is the result of the path integral of the fluctuations, and $S_0(\Phi_0) = S_0[\Phi]|_{\Phi=\Phi_0}$ is called a classical contribution and will be denoted by S_{cl}. Note that the saddle points may not be isolated but they may have flat directions. In such a case we have to integrate them along the flat directions. Finally, we stress again that this partition function is exact and can be used to check the dualities.

1.6 Background geometries

The more parameters sophisticate the partition function as an index for checking dualities. Hence, we want to introduce more parameters into the partition function as many as possible. One such a way is to deform the background geometry. We have several simple choices for 3-dimensional manifold $\mathcal{M}$;

- $\mathcal{M} = \boldsymbol{S}^3 \quad \longrightarrow \quad Z$: a number
  This is the simplest 3-dimensional compact space. One may wander that the radius of the $\boldsymbol{S}^3$ would be the parameter of the partition function. As we will see this in Chapter 3 the radius is absorbed by the redefinition of the field as well as the mass parameters.

- $\mathcal{M} = \boldsymbol{S}^2 \times \boldsymbol{S}^1 \quad \longrightarrow \quad Z$: a function of $\beta = R_1/R_2$
  where R_1 and R_2 are the radii of $\boldsymbol{S}^1$ and $\boldsymbol{S}^2$, respectively. This maybe more familiar one than $\boldsymbol{S}^3$ case because the $\boldsymbol{S}^1$ direction can be identified with the time direction. In this sense this background manifold gives a usual partition function. Furthermore, if we introduce an appropriate chemical potentials the partition function with this manifold has special name *superconformal index*. The superconformal index can be written by using trace like in (1.2) because there is $\boldsymbol{S}^1$, and the state $|n\rangle$ can be expressed by a spherical harmonics. In the same as $\boldsymbol{S}^3$ case one of the two radii can be absorbed and only the ratio becomes the parameter.

- $\mathcal{M} = \boldsymbol{S}_b^3 \quad \longrightarrow \quad Z$: a function of b
  $\boldsymbol{S}_b^3$ is a one-parameter deformation of the three-sphere $\boldsymbol{S}^3$, which is called squashed three-sphere. b is the deformation (squashing) parameter. The geometrical parameter b is translated into that of the partition function.

- $\mathcal{M} = \boldsymbol{S}^3/\mathbb{Z}_n \quad \longrightarrow \quad Z$: a function of n and h
  This quotient geometry is called an orbifold. n is the order of the orbifolding (number of dividing) and h is a holonomy, which will be explained later. The partition function on the orbifold is the main topic of this thesis.

For each background, the partition function is exactly calculated using the localization method. Historically, the partition function on $\boldsymbol{S}^4$ is firstly calculated by Pestun in [7]. Kapustin et al. applied the method to $\mathcal{N} \geq 3$ supersymmetric theory and calculate the partition function on $\boldsymbol{S}^3$ [8]. It is generalized to $\mathcal{N} = 2$ case [9, 10]. The partition function has been used to study non-perturbative aspects of three-dimensional field theories, such as dualities among three-dimensional field theories [11, 12, 13, 14, 15, 16, 17] and relation to M-theory via the AdS/CFT correspondence [6, 18, 19, 20, 21, 22, 23, 24]. The superconformal index is computed exactly in [25, 26], and specialized to the large N case [27, 28, 29, 30]. The superconformal index is also used to point out the relation between different dimensions. The squashed partition function can be derived from 4-dimensional superconformal index [31, 32, 33].

1.7 Squashed $\boldsymbol{S}^3$

Although we express the squashed three-sphere simply by $\boldsymbol{S}_b^3$ there are actually two kinds of squashed three-spheres. One is called a biaxially squashed sphere in [34] and the other is called an ellipsoid. Though they are both one-parameter deformation of $\boldsymbol{S}^3$ their isometries are different. The isometry of the $\boldsymbol{S}^3$ is $SO(4) \sim SU(2)_L \times SU(2)_R$. That of the biaxially squashed three-sphere is $SU(2)_L \times U(1)_r$; the deformation breaks one of two $SU(2)$'s of $\boldsymbol{S}^3$. On the other hand, the ellipsoid has only $U(1) \times U(1)$ isometry. Since the supersymmetry is connected to the isometry and since the supersymmetry is used for the localization method, the difference of the isometry affects the partition function on them. The non-trivial partition function on the ellipsoid was derived in [35], which is written in terms of a special function called a double sine function. The authors of [35] also calculated the partition function on the biaxially squashed three-sphere, and found that the result is trivial; the partition function is the same as that on the round three-sphere $\boldsymbol{S}^3$. On the other hand, the partition function on the biaxially squashed three-sphere derived from the reduction of the 4d superconformal index is actually the same as that on ellipsoid [31, 32, 33]. This contradiction was solved in [36]. The origin of the contradiction was the confusion about the supersymmetry that is used for the localization. Though the $SU(2)_L$ singlet supersymmetry leads to the trivial result the $SU(2)_L$ doublet supersymmetry gives the non-trivial result. In this thesis we only look into the biaxially squashed three-sphere and simply call it a squashed sphere.

1.8 Orbifold

An orbifolded manifold or simply an orbifold is realized by an identification of two or more different points in the original manifold. For example, $\boldsymbol{S}^2/\mathbb{Z}_2$ is an orbifolded $\boldsymbol{S}^2$ that the facing two points are identified (see Fig. 1.7), where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$

and $\mathbb{Z}$ is a set of all integers. $\mathbb{Z}_n$ denotes the n-point identification. The non-trivial

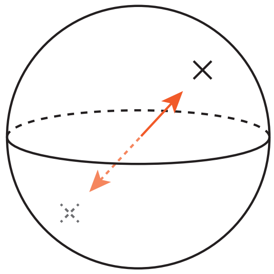

Figure 1.7: $\boldsymbol{S}^2/\mathbb{Z}_2$ geometry. Every pair of points faced each other on the original manifold $\boldsymbol{S}^2$ are identified. For example, if we choose the North hemisphere as the fundamental region the South hemisphere is a copy of the North hemisphere and vice versa.

feature of the orbifolding is to create a cycle on the orbifolded manifold. In the $\boldsymbol{S}^2/\mathbb{Z}_2$ case, for example, a curve on the $\boldsymbol{S}^2/\mathbb{Z}_2$ that joints two identified points is a cycle. Note that the circle along the cycle cannot shrink continuously into a point. However, the union of two such curves are equivalent to a great circle, which can shrink (see Fig. 1.8). A cycle on a manifold is characterized by the fundamental

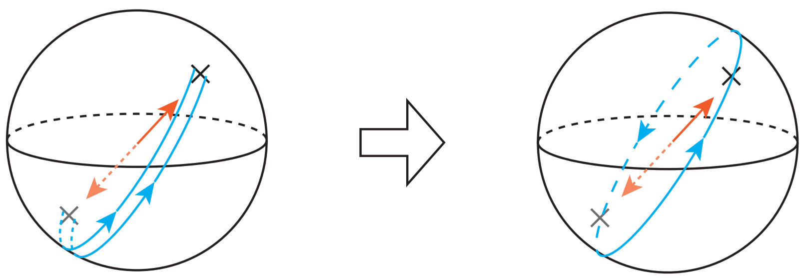

Figure 1.8: Oriented curves on the $\boldsymbol{S}^2/\mathbb{Z}_2$ geometry. Since the crosses are identified one of curves in the left picture is a non-trivial cycle of $\boldsymbol{S}^2/\mathbb{Z}_2$ and cannot shrink into a point. However, the union of the two curves are equivalent to a great circle as in the right picture and it can now shrink into a point.

group $\pi_1(\mathcal{M})$. The circles on $\boldsymbol{S}^1$ never be able to shrink and such a cycle is denoted as $\pi_1(\boldsymbol{S}^1) = \mathbb{Z}$; the integer illustrate the number of the wrapping along the cycle. In $\boldsymbol{S}^2/\mathbb{Z}_2$ case the circle wrapping twice the cycle can shrink, and it is denoted as $\pi_1\left(\boldsymbol{S}^2/\mathbb{Z}_2\right) = \mathbb{Z}_2$; $\mathbb{Z}_2 = \{0, 1\}$ means the number of the wrapping. The fundamental group of $\boldsymbol{S}^3/\mathbb{Z}_n$ is $\pi_1\left(\boldsymbol{S}^3/\mathbb{Z}_n\right) = \mathbb{Z}_n$. The existence of these non-trivial cycle affects the structure of the vacua of a theory through the Aharonov-Bohm (AB) effect.

AB effect and Holonomy

The AB effect tells us that even if the field strength is zero the gauge potential (the gauge field A_μ) affects charged fields. We assume $U(1)$ gauge group here. The typical example is a charged particle traveling around a solenoid. The solenoid creates magnetic flux Φ that is confined to its inside. Namely, the field strength is trivial outside the solenoid. Nevertheless, the wave function of the particle is affected by the non-trivial gauge potential:

$$\oint_C A_\theta d\theta = \int_M \boldsymbol{B} \cdot d\boldsymbol{s} = \Phi, \tag{1.12}$$

$$\therefore \qquad A_\theta = \frac{\Phi}{2\pi r}, \tag{1.13}$$

where C is the path of the particle, which is a circle around the solenoid, θ parametrizes the circle and the period is $2\pi r$, r is the radius of the circle, and M is a disk whose boundary is the circle C. The AB effect is characterized by a holonomy defined as follows.

$$h = \oint_C A_\mu dx^\mu \mod 2\pi. \tag{1.14}$$

The effect on the phase of the wave function is expressed by e^{ih}, which is called Wilson line. In the example above the holonomy is nothing but the flux $h = \Phi$.

Next, let us consider a cylinder $\boldsymbol{S}^1 \times \mathbb{R}$ parametrized by (θ, z). Note that there is no space inside the cylinder. We assume the field strength is trivial everywhere. However, a constant gauge potential A_θ can exist. Since the space is periodic $\theta \sim \theta + 2\pi$ a wave function ψ of a particle living on it should satisfy the boundary condition:

$$\psi(\theta + 2\pi) = \psi(\theta). \tag{1.15}$$

The AB effect due to the constant gauge potential is given by

$$e^{i \oint_C A_\theta d\theta} = e^{2\pi i A_\theta}. \tag{1.16}$$

Now let us consider a gauge transformation:

$$\begin{aligned} A_\mu \to A'_\mu &= A_\mu + \partial_\mu \alpha(x), \\ \psi \to \psi' &= e^{i\alpha(x)}\psi. \end{aligned} \tag{1.17}$$

Using the gauge transformation we can choose a gauge so that the gauge potential vanish:

$$\begin{aligned} A'_\mu &= A_\theta + \partial_\theta \alpha = 0, \\ \therefore \quad \alpha(\theta) &= -\theta A_\theta. \end{aligned} \tag{1.18}$$

In this gauge the AB effect is expressed by the boundary condition as follows.

$$\begin{aligned} \psi(\theta + 2\pi) &= e^{i(\alpha(\theta) - \alpha(\theta + 2\pi))}\psi(\theta) = e^{2\pi i A_\theta + 2\pi i k}\psi(\theta) \quad (k \in \mathbb{Z}) \\ &= e^{2\pi i A_\theta}\psi(\theta) = e^{ih}\psi(\theta), \end{aligned} \tag{1.19}$$

where note that A_θ and h are ones before the gauge transformation, and k appears because the gauge transformation parameter $\alpha(\theta)$ is not single-valued:

$$\alpha(\theta) \sim \alpha(\theta) + 2\pi k \quad (k \in \mathbb{Z}). \tag{1.20}$$

Although this multivalent does not affect in this case it does affect in the next example.

Finally, we consider the AB effect on the orbifold. In this case due to the non-trivial fundamental group $\pi_1(\boldsymbol{S}^3/\mathbb{Z}_n) = \mathbb{Z}_n$, the circle wrapping the cycle n times can shrink into a point. The Stokes theorem tells us that the corresponding holonomy should be trivial:

$$h_n = \oint_{nC} A_\mu dx^\mu = n \times 2\pi A_\theta = 0 \mod 2\pi \tag{1.21}$$

$$\therefore \qquad A_\theta = \frac{k}{n} \quad (k \in \mathbb{Z}), \tag{1.22}$$

where the period of the cycle C is chosen to be 2π, and the subscript n of h_n means that the holonomy is of the path wrapping n times. This result gives discrete holonomies:

$$h = 2\pi\frac{k}{n} \qquad (k \in \mathbb{Z}_n). \tag{1.23}$$

If we rescale the definition of the holonomy as

$$h = \frac{n}{2\pi}\oint A_\mu dx^\mu \mod n, \tag{1.24}$$

the holonomy for $\boldsymbol{S}^3/\mathbb{Z}_n$ takes the value $h = 0, 1, \cdots, n-1 \in \mathbb{Z}_n$. These values specify the vacua of the theory.

The partition function on the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$ is, therefore, expressed by the sum of each contribution:

$$Z_{\text{total}} = Z_{h=0} + Z_{h=1} + \cdots + Z_{h=n-1}. \tag{1.25}$$

Each contribution is calculated by the formula derived in [37, 38]. We call this a (naive) orbifolded partition function.

1.9 Phase problem

We usually focus only on the absolute value of the partition function and the phase is disregarded. This is, however, not allowed when we compute the partition functions of different sectors which are summed up. Even when we are interested only in the absolute value of the total partition function, we need to care about the relative phase of each contribution. The important fact is that the formula (1.25) derived in [37, 38] does not give the same values for a dual pair. This maybe the signal that the duality is not really a duality. We rather assume that the duality is correct and the formula needs some modifications. Let us see an concrete example: the simplest mirror symmetry discussed in Section 1.3. In the $n = 3$ case where there are three sectors specified by the holonomy $h = 0, 1, 2$. The numerical results for the naive orbifolded partition function are summarized in Table 1.3. We can see that the total numerical values are different from each other. However, notice that if one introduce

Table 1.3: Numerical results for the naive orbifolded partition function of the $\mathcal{N}=2$ SQED and the XYZ model show that their total values do not coincide.

Holonomy h	SQED	XYZ model
0	$+0.394$	
1	-0.298	
2	-0.125	
Total	-0.031	0.817

Table 1.4: Numerical results for the modified orbifolded partition function of the SQED and the XYZ model show that their total values coincide.

Holonomy h	Phase factor	SQED	XYZ model
0	$+$	$+0.394$	
1	$-$	-0.298	
2	$-$	-0.125	
Total		0.817	0.817

minus signs to appropriate sectors the results coincide as in Table 1.4. In this case and all the other cases we have checked the phase factors are actually plus or minus signs for this duality. We calculated the partition function numerically up to $n = 10$ and we are succeeded in finding out the correct signs and their patterns, which are discussed in detail in Chapter 5. Hereafter, the correct means that the partition function coincides for a dual pair, and we call the correct partition function on the orbifold a orbifolded partition function.

Our proposal is that the orbifolded partition function is not (1.25) but is expressed as follows.

$$Z_{\text{total}} = e^{i\theta_0} Z_{h=0} + e^{i\theta_1} Z_{h=1} + \cdots + e^{i\theta_{n-1}} Z_{h=n-1}, \tag{1.26}$$

where the $e^{i\theta_h}$ is the appropriate phase.

We will consider the mirror symmetry and another duality proposed by Jafferis and Yin [15] to derive the correct phases. The reason why we use these dualities is because they are known to have dual field theories without vector multiplet. When there is no gauge field there is no holonomy, and hence, no holonomy sum. The orbifolded partition function is calculated without the phase problem on the non-gauge theory side, and we can focus on the problem only on the other side.

Since the proposal for the phase factors is derived from a few specific dualities we have to check whether our formula gives correct values for other dual pairs. In this thesis we will explore the duality between the ABJM model and $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) with a fundamental and an adjoint hypermultiplets. Our proposal passes this test, fortunately.

Finally, we also study the large N behavior of the orbifolded partition function for the ABJM model so as to check the AdS/CFT. Our result shows good coincidence with that on the gravity side.

The thesis is organized as follows. We firstly review the 3-dimensional $\mathcal{N} = 2$ supersymmetric theories in Chapter 2. Their derivation is summarized in Appendix B. We also review briefly the ABJM model. In chapter 3, we extend the SUSY transformation to conformally flat space and calculate the $\boldsymbol{S}^3$ partition function using the localization method. In Chapter 4 we extend the formula of the partition function to the squashed three-sphere $\boldsymbol{S}_b^3$ case. In Chapter 5 we firstly review the formula of the orbifolded partition function derived in [37, 38]. Then, we derive the formula to fix the phase factor so that the numerical values coincide for the dual theories. Chapter 7 is devoted to the conclusions and discussions. Useful formulae and the basic facts are summarized in Appendices.

Note that we adopt the natural units through this thesis. Namely, we set $c = 1$ and $\hbar = 1$, and the length has a mass dimension -1. We also set the electric charge to unity: $e = 1$.

We use subscripts for coordinates like μ in different meanings for each chapter. For example, $\mu, \nu, \cdots$ used as 4d indices in chapter 2, though they are used as global coordinate of 3d in chapter 3. We always give the explanation when the conventions are changed.

Chapter 2

3d supersymmetric gauge theory

We review 3-dimensional (3d) supersymmetric gauge theories with great attention to $\mathcal{N} = 2$ supersymmetry (SUSY). Though we briefly explained what is the supersymmetry, what $\mathcal{N}$ means, and what are the multiples in Introduction, we will restart from defining and explaining them. Our final goal of this chapter is to list the $\mathcal{N} = 2$ supersymmetric actions and the SUSY transformations of fields, and finally, write down the action of the ABJM model. If one is familiar with the 4d SUSY one may prefer to derive the 3D SUSY by dimensional reduction, which is summarized in Section B.2. (See also Appendix B and nice reviews [39, 40].)

In this chapter we use $m, n, p, \cdots$ as vector indices and they run from 0 to 2, and $\alpha, \beta, \gamma, \cdots$ as spinor indices and they take the value 1 or 2.

2.1 Supersymmetry in 3d

Firstly, let us discuss the 3d spinor since the spinor in 3d is quite different from that of 4d. In 3d the Lorentz group is $SO(1,2)$ and the spin group is $SL(2,\mathbb{R}) \sim Sp(2,\mathbb{R})$. Hence, a Majorana spinor can be defined in 3d, which is a real two-component spinor. Our spinor conventions are summarized in Appendix A.

Before the supersymmetry was found there had been Coleman-Mandula theorem, which tells us that only symmetries that the S-matrix can have are the direct products of the Poincare group and a group of internal symmetries. In terms of the algebra it is the direct sum of the Poincare algebra and that of internal symmetries. Therefore, it is concluded that the symmetry like the supersymmetry that is related to the symmetry of spacetime cannot be the symmetry of S-matrix.

However, Haag, Sohnius, and Lopuszanski extended the algebra to a graded algebra and show that the supersymmetry algebra is the unique graded algebra of the symmetry of the S-matrix. A graded algebra is an algebra with anti-commutators. Let us start with admitting the simplest super-Poincare algebra:

$$[M_{mn}, M_{pq}] = \eta_{mp}M_{nq} - \eta_{np}M_{mq} - \eta_{mq}M_{nq} + \eta M_{mp} \tag{2.1}$$

$$[M_{mn}, P_p] = \eta_{mp}P_n - \eta_{np}P_m \tag{2.2}$$

$$[M_{mn}, Q_\alpha] = \frac{1}{2}\left(\gamma_{mn}Q\right)_\alpha \tag{2.3}$$

$$\{Q_\alpha, Q_\beta\} = 2\left(\gamma^m\right)_{\alpha\beta} P_m \tag{2.4}$$

where P_m are the momentum operators, M_{mn} are the Lorentz rotation generators, and Q_α are the generators of the supersymmetry (often called supercharges) and they are the Majorana spinor. Note that we use anti-Hermite representation.

We can actually extend the supersymmetry by putting another index i to the supercharge Q: Q^i, where i runs from 1 to $\mathcal{N}$. (2.4) is slightly modified as

$$\{Q^i_\alpha, Q^j_\beta\} = 2\delta^{ij} \left(\gamma^m\right)_{\alpha\beta} P_m. \tag{2.5}$$

There is a global symmetry that rotates the Q^i. Since the Q^i are Majorana, and hence, real the symmetry is $SO(\mathcal{N})$:

$$\left[R^{ij}, R^{kl}\right] = \delta^{ik} R^{jl} - \delta^{jk} R^{il} - \delta^{il} R^{jk} + \delta^{jl} R^{ik}, \tag{2.6}$$
$$\left[R^{ij}, Q^k\right] = \delta^{ik} Q^j - \delta^{jk} Q^i. \tag{2.7}$$

This is called R-symmetry.

Now we focus on $\mathcal{N} = 2$ case. We combine the $\mathcal{N} = 2$ SUSY generators as

$$Q_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha + iQ^2_\alpha\right), \quad \overline{Q}_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - iQ^2_\alpha\right). \tag{2.8}$$

Then, the $\mathcal{N} = 2$ super Poincare algebra is written as follows.

$$\begin{aligned}
[M_{mn}, M_{pq}] &= \eta_{mp} M_{nq} - \eta_{np} M_{mq} - \eta_{mq} M_{nq} + \eta M_{mp} \\
[M_{mn}, P_p] &= \eta_{mp} P_n - \eta_{np} P_m \\
[M_{mn}, Q_\alpha] &= \frac{1}{2}\left(\gamma_{mn} Q\right)_\alpha \\
[M_{mn}, \overline{Q}_\alpha] &= \frac{1}{2}\left(\gamma_{mn} \overline{Q}\right)_\alpha \\
\{Q_\alpha, \overline{Q}_\beta\} &= 2\left(\gamma^m\right)_{\alpha\beta} P_m
\end{aligned} \tag{2.9}$$

Now let us consider differential operators that satisfy (2.9). Such operators are given as follows.

$$\begin{aligned}
\hat{P}_m &= \partial_m, \\
\hat{Q}_\alpha &= \partial_\alpha + \left(\gamma^m \overline{\theta}\right)_a \partial_m, \\
\hat{\overline{Q}}_\alpha &= \partial_\alpha + \left(\gamma^m \theta\right)_a \partial_m, \\
\hat{M}_{mn} &= x_m \partial_m - x_n \partial_m + \frac{1}{2}\theta \gamma_{mn} \partial + \frac{1}{2}\overline{\theta} \gamma_{mn} \overline{\partial}.
\end{aligned} \tag{2.10}$$

One can actually confirm that these operators satisfy the algebra with opposite sign of those of (2.9). It is not a mistake but a consequence of using anti-Hermite representation.

Let us define following differential operators.

$$\begin{aligned}
D_\alpha &= \partial_\alpha - \left(\gamma^m \overline{\theta}\right)_\alpha \partial_m \\
\overline{D}_\alpha &= \overline{\partial}_\alpha - \left(\gamma^m \theta\right)_\alpha \partial_m
\end{aligned} \tag{2.11}$$

These are called covariant derivatives and satisfy the anti-commutation relations:

$$\{D_\alpha, \overline{D}_\beta\} = 2(\gamma^m)_{\alpha\beta}\partial_m, \quad \{D_\alpha, D_\beta\} = \{\overline{D}_\alpha, \overline{D}_\beta\} = 0. \tag{2.12}$$

The supercharges $Q, \overline{Q}$ anti-commute with the covariant derivatives. If you are familiar with 4d SUSY note that in 3d these operators can be contracted each other;

$$D\overline{D} = D^\alpha \overline{D}_\alpha = \overline{D}^\alpha D_\alpha = \overline{D}D. \tag{2.13}$$

$Q, \overline{Q}$ and $\theta, \overline{\theta}$ are also contracted as well, where θ and $\overline{\theta}$ are constant spinors and they are used as fermionic coordinates. Following notation is understood,

$$\theta^2 = \theta\theta = \theta^\alpha\theta_\alpha, \quad D^2 = DD = D^\alpha D_\alpha \quad \text{etc.} \tag{2.14}$$

The integral with regard to Grassmann number is defined as follows.

$$\int d^2\theta(\theta^2) = 1, \qquad \int d^4\theta(\theta^2\overline{\theta}^2) = 1. \tag{2.15}$$

Now we are ready to define a superfield, which is the function of the superspace coordinates $x, \theta, \overline{\theta}$.

2.2 $\mathcal{N} = 2$ Multiplets

2.2.1 Chiral multiplets

Let us first explore the building blocks of $\mathcal{N} = 2$ SUSY theory. We define a chiral multiplet Φ as a superfield satisfying

$$\overline{D}_\alpha \Phi = 0. \tag{2.16}$$

Under this condition the multiplet can be expanded as

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F - (\theta\gamma^m\overline{\theta})\partial_m\phi - \frac{1}{\sqrt{2}}\theta^2(\overline{\theta}\gamma^m\partial_m\psi) + \frac{1}{4}\theta^2\overline{\theta}^2\partial^m\partial_m\phi. \tag{2.17}$$

Note that since $\theta, \overline{\theta}$ are Grassmann two-component Majorana spinors products of more than three of these spinors vanish automatically. The kinetic term of this multiplet is given as follows.

$$\int d^4\theta\overline{\Phi}\Phi = -\partial_m\overline{\phi}\partial^m\phi + \overline{\psi}\partial\!\!\!/\psi + \overline{F}F \tag{2.18}$$

where we omit the total derivatives. Note that $\overline{F}F$ term has no derivatives, which means that the equation of motion gives a constraint condition. Such a field is called auxiliary field.

We can consider not only single superfield but also many superfields Φ^i. Products of superfields are also superfield and its highest components are SUSY invariant. In component fields it is written by

$$\int d^2\theta W(\Phi) = \frac{\partial W(\phi)}{\partial\phi^i}F^i - \frac{1}{2}\frac{\partial^2 W(\phi)}{\partial\phi^i\partial\phi^j}\psi^i\psi^j, \tag{2.19}$$

where $W(\Phi)$ is a polynomial of the superfields, and it is called superpotential.

2.2.2 Vector multiplets

We consider the "gauge transformation" of the chiral multiplets, namely,

$$\Phi' = e^{i\Lambda}\Phi, \qquad \overline{\Phi}' = \overline{\Phi}e^{-i\overline{\Lambda}}, \tag{2.20}$$

where Λ is a supercoordinate dependent chiral superparameter. When we think Yang-Mills theory the Λ is an element of Lie algebra, and the superfield Φ is a representation of the gauge group G. In order to make the chiral multiple action invariant under the gauge transformation we introduce the "gauge field" V.

$$\int d^4\theta \overline{\Phi} e^{2V} \Phi \tag{2.21}$$

where V is of course not a field but a superfield, and we call V a vector multiplet. Since the usual gauge field is real V should also satisfy a reality condition $V = V^*$. Under this condition the vector multiplet can be expand as

$$\begin{aligned} V = {} & C + \theta\chi + \overline{\theta}\overline{\chi} + \frac{1}{2}\theta^2 (M + iN) + \frac{1}{2}\overline{\theta}^2 (M - iN) \\ & + \theta\overline{\theta}\sigma + i(\theta\gamma^m\overline{\theta})A_m - \theta^2\overline{\theta}\left(\overline{\lambda} + \frac{1}{2}\gamma^m\partial_m\chi\right) - \overline{\theta}^2\theta\left(\lambda + \frac{1}{2}\gamma^m\partial_m\overline{\chi}\right) \\ & - \frac{1}{2}\theta^2\overline{\theta}^2\left(D - \frac{1}{2}\partial^m\partial_m C\right). \end{aligned} \tag{2.22}$$

In order to make the action (2.21) invariant under the gauge transformation 2.20 V should transform as

$$e^{2V'} = e^{i\overline{\Lambda}}e^{2V}e^{-i\Lambda} = \exp\left[2\left(V + \frac{i}{2}(\overline{\Lambda} - \Lambda) + \frac{i}{2}[\overline{\Lambda}, V] - \frac{i}{2}[V, \Lambda] + \cdots\right)\right]. \tag{2.23}$$

Using this gauge freedom we can fix some fields in the vector multiplet. When the group is Abelian all the commutators in (2.23) vanish and the gauge transformation is simply expressed by

$$V \to V' = V + \frac{i}{2}\left(\overline{\Lambda} - \Lambda\right). \tag{2.24}$$

We focus on the Abelian case for a while. We use linear terms of Λ and $\overline{\Lambda}$, and erase the first three terms of (2.22). The linear terms are

$$\begin{aligned} \frac{i}{2}(\overline{\Lambda} - \Lambda) = {} & \frac{i}{2}(\overline{\phi} - \phi) + \frac{i}{\sqrt{2}}\overline{\theta}\overline{\psi} - \frac{i}{\sqrt{2}}\theta\psi + \frac{i}{2}\overline{\theta}^2\overline{F} - \frac{i}{2}\theta^2 F \\ & + \frac{i}{2}\left(\theta\gamma^m\overline{\theta}\right)\partial_m(\overline{\phi} + \phi) - \frac{i}{2\sqrt{2}}\overline{\theta}^2(\theta\partial\!\!\!/\overline{\psi}) + \frac{i}{2\sqrt{2}}\theta^2(\overline{\theta}\partial\!\!\!/\psi) \\ & + \frac{i}{8}\theta^2\overline{\theta}^2\partial^m\partial_m(\overline{\phi} - \phi), \end{aligned} \tag{2.25}$$

where although we used the same notation as the chiral multiplet, the components are not fields but gauge transformation parameters. As a result each component of the vector multiplet transforms as

$$
\begin{aligned}
C &\to C + \frac{i}{2}(\overline{\phi} - \phi) \\
\chi &\to \chi - \frac{i}{\sqrt{2}}\psi \\
M + iN &\to M + iN - iF \\
\sigma &\to \sigma \\
A_m &\to A_m + \frac{1}{2}\partial_m(\overline{\phi} + \phi) \\
\lambda &\to \lambda \\
D &\to D.
\end{aligned} \tag{2.26}
$$

Therefore, we can set $C = \chi = M = N = 0$, and $(\phi + \overline{\phi})/2$ becomes an usual gauge transformation parameter. This gauge is called Wess-Zumino gauge, and V is written as

$$V_{\text{WZ}} = \theta\overline{\theta}\sigma + i(\theta\gamma^m\overline{\theta})A_m - \theta^2\overline{\theta\lambda} - \overline{\theta}^2\theta\lambda - \frac{1}{2}\theta^2\overline{\theta}^2 D. \tag{2.27}$$

From now on we use V in Wess-Zumino gauge and omit the index WZ. The nice feature of this gauge is that the cubic term of V vanish, the exponential is just a quadratic polynomial;

$$e^{2V} \overset{\text{WZ}}{=} 1 + 2V + 2V^2. \tag{2.28}$$

In the non-Abelian case the commutators in (2.23) become non-trivial. Still, there exist the linear terms and we can set some terms. Finally, V can be expanded as in (2.27). The important difference compared to the Abelian case is that σ, λ and D do transform under the gauge transformation due to the commutators.

We can now write down the components of the gauge invariant action of the chiral multiplet;

$$
\begin{aligned}
\int d^4\theta\overline{\Phi}e^{2V}\Phi = &-D_m\overline{\phi}D^m\phi + \overline{\psi}\not{D}\psi + \overline{F}F \\
&+ \sqrt{2}\overline{\phi}(\lambda\psi) + \sqrt{2}(\overline{\lambda\psi})\phi - \overline{\phi}D\phi - \overline{\phi}\sigma\sigma\phi + (\overline{\psi}\sigma\psi)
\end{aligned} \tag{2.29}
$$

where we omitted the total derivative terms, and the covariant derivatives are defined as

$$D_m\phi = \partial_m\phi - iA_m\phi, \qquad D_m\overline{\phi} = \partial_m\overline{\phi} + i\overline{\phi}A_m. \tag{2.30}$$

In order to derive the kinetic term for the vector multiplet we define following quantity;

$$W_\alpha = \frac{1}{8}\overline{D}^\beta\overline{D}_\beta e^{-2V}D_\alpha e^{2V} \overset{\text{WZ}}{=} \frac{1}{4}\overline{D}^\beta\overline{D}_\beta\left(D_\alpha V + [D_\alpha V, V]\right). \tag{2.31}$$

This is gauge covariant and transforms as

$$W'_\alpha = \frac{1}{8}\overline{D}^\beta\overline{D}_\beta\left(e^{i\Lambda}e^{-2V}e^{-i\overline{\Lambda}}D_\alpha\left(e^{i\overline{\Lambda}}e^{2V}e^{-i\Lambda}\right)\right) = e^{i\Lambda}W_\alpha e^{-i\Lambda}, \tag{2.32}$$

where note that the gauge transformation of e^{-2V} is

$$\text{not} \quad e^{-2V} \to e^{-i\overline{\Lambda}} e^{-2V} e^{i\Lambda}, \qquad \text{but} \quad e^{-2V} \to e^{i\Lambda} e^{-2V} e^{-i\overline{\Lambda}}. \tag{2.33}$$

To get the correct transformation law one has to pay attention to the R.H.S of (2.23). In the Wess-Zumino gauge W_α is written as

$$W_\alpha = \lambda_\alpha + \theta_\alpha D + \frac{i}{2}(\gamma^{mn}\theta)_\alpha F_{mn} - (\gamma^m\theta)_\alpha D_m\sigma + (\not{D}\overline{\lambda})_\alpha\theta^2 - [\sigma, \overline{\lambda}_\alpha]\theta^2 + \cdots, \tag{2.34}$$

where $\cdots$ represent terms including $\overline{\theta}$. The reason why the $\overline{\theta}$ terms appear is because we adopted V such that $D_\alpha V \neq 0$ in (2.27). In order to erase those terms we need to shift D appropriately. Here we just ignore such terms and using above expression the kinetic term of the vector multiplet is

$$\frac{1}{2}\int d^2\theta \mathrm{tr} W^\alpha W_\alpha = \mathrm{tr}\left(-\frac{1}{4}F_{mn}F^{mn} + (\overline{\lambda}\not{D}\lambda) - \frac{1}{2}D_m\sigma D^m\sigma + \frac{1}{2}D^2 - (\lambda[\sigma,\lambda])\right). \tag{2.35}$$

When the gauge group is Abelian one can introduce Fayet-Iliopoulos (FI) term into the action.

$$-2\zeta \int d^4\theta \; V = \zeta D \tag{2.36}$$

On the other hand, this term cannot show up in a non-Abelian sector, because the component D is not gauge invariant as discussed before.

2.2.3 $\mathcal{N} = 2$ SUSY transformation law

We are now able to extract the $\mathcal{N} = 2$ SUSY transformation law of component fields from the multiplets. Let us first look into the vector multiplet (the reason will be clear soon).

$$\begin{aligned}\delta V = &\, i(\epsilon\gamma^m\overline{\theta})A_m + (\epsilon\overline{\theta})\sigma - \overline{\theta}^2(\epsilon\lambda) - 2(\epsilon\theta)(\overline{\theta}\overline{\lambda}) - (\epsilon\theta)\overline{\theta}^2 D \\ &+ \frac{i}{2}\overline{\theta}^2(\epsilon\gamma^m\gamma^n\theta)\partial_m A_n - \frac{1}{2}\overline{\theta}^2(\epsilon\gamma^m\theta)\partial_m\sigma + \frac{1}{2}\theta^2\overline{\theta}^2(\epsilon\not{\partial}\overline{\lambda}) \\ &+ \text{terms including } \overline{\epsilon}\end{aligned} \tag{2.37}$$

As you may notice the first three terms break the WZ gauge. Since we started from the WZ gauge we have to restore the WZ gauge by the gauge transformation. The required parameter Λ is given by the following replacement;

$$\phi \to 0, \quad \sqrt{2}\psi \to -2i\left(\sigma + i\gamma^m A_m\right)\overline{\epsilon}, \quad F \to 2i(\overline{\epsilon}\overline{\lambda}). \tag{2.38}$$

Then, the Λ becomes

$$\Lambda = -2i(\theta\overline{\epsilon})\sigma + 2(\theta\gamma^m\overline{\epsilon})A_m + 2i\theta^2(\overline{\epsilon}\overline{\lambda}) + i\theta^2(\overline{\theta}\gamma^m\overline{\epsilon})\partial_m\sigma - \theta^2(\overline{\theta}\gamma^m\gamma^n\overline{\epsilon})\partial_m A_n. \tag{2.39}$$

After the gauge transformation with the parameter derived above we have

$$\begin{aligned}
\delta V &\overset{\text{WZ}}{=} -2(\epsilon\theta)(\overline{\theta\lambda}) - (\epsilon\theta)\overline{\theta}^2 D \\
&\quad - \frac{i}{2}\overline{\theta}^2(\theta\gamma^{mn}\epsilon)F_{mn} + \overline{\theta}^2(\theta\gamma^m\epsilon)D_m\sigma + \frac{1}{2}\theta^2\overline{\theta}^2\left(\epsilon \not{D}\overline{\lambda} - (\epsilon[\sigma,\overline{\lambda}])\right) \\
&\quad + \text{terms including } \overline{\epsilon} \\
&= \theta\overline{\theta}\delta\sigma + i(\theta\gamma^m\overline{\theta})\delta A_m - \theta^2\overline{\theta}\delta\overline{\lambda} - \overline{\theta}^2\theta\delta\lambda - \frac{1}{2}\theta^2\overline{\theta}^2\delta D, \qquad (2.40)
\end{aligned}$$

where $D_m\overline{\lambda} = \partial_m\overline{\lambda} - i[A_m, \overline{\lambda}]$. We can now extract the transformation law for the components.

$$\begin{aligned}
\delta A_m &= i\epsilon\gamma_m\overline{\lambda} - i\overline{\epsilon}\gamma_m\lambda \\
\delta\sigma &= \epsilon\overline{\lambda} + \overline{\epsilon}\lambda \\
\delta\lambda &= \frac{i}{2}\gamma^{mn}\epsilon F_{mn} - \gamma^m\epsilon D_m\sigma + D\epsilon \\
\delta D &= -\epsilon\not{D}\overline{\lambda} - \overline{\epsilon}\not{D}\lambda + \epsilon[\sigma,\overline{\lambda}] + \overline{\epsilon}[\sigma,\lambda] \qquad (2.41)
\end{aligned}$$

Next, we derive the transformation law for the chiral multiplet.

$$\begin{aligned}
\delta\Phi &= \sqrt{2}\epsilon\psi + 2\theta\epsilon F - 2(\theta\gamma^m\overline{\epsilon})\partial_m\phi - \sqrt{2}\theta^2(\overline{\epsilon}\not{\partial}\psi) \\
&\quad + \text{terms including } \overline{\theta}. \qquad (2.42)
\end{aligned}$$

In the case of neutral chiral multiplet we can extract the transformation law from this result. On the other hand, in the charged chiral multiplet case we have to restore the WZ gauge of the vector multiplet, and hence, need to do the gauge transformation with the parameter (2.39). The variation is

$$\begin{aligned}
i\Lambda\Phi &= 2(\theta\overline{\epsilon})\sigma\phi - \sqrt{2}\theta^2(\overline{\epsilon}\sigma\psi) + 2i\theta^2(\overline{\epsilon}\not{A}\psi) - 2\theta^2(\overline{\epsilon}\overline{\lambda})\phi \\
&\quad + \text{terms including } \overline{\theta}. \qquad (2.43)
\end{aligned}$$

The transformation of the charged chiral multiplet is

$$\begin{aligned}
\delta\Phi &\overset{\text{WZ}}{=} \sqrt{2}(\epsilon\psi) + 2(\theta\epsilon)F + 2(\theta\overline{\epsilon})\sigma\phi - 2(\theta\gamma^m\overline{\epsilon})D_m\phi \\
&\quad - \sqrt{2}\theta^2(\overline{\epsilon}\not{D}\psi) - \sqrt{2}\theta^2(\overline{\epsilon}\sigma\psi) - 2\theta^2(\overline{\epsilon}\overline{\lambda})\phi \\
&\quad + \text{terms including } \overline{\theta}. \qquad (2.44)
\end{aligned}$$

Therefore, the transformation law for the component of the chiral multiplet is

$$\begin{aligned}
\delta\phi &= \sqrt{2}\epsilon\psi \\
\delta\psi &= \sqrt{2}\epsilon F + \sqrt{2}\overline{\epsilon}\sigma\phi - \sqrt{2}\gamma^m\overline{\epsilon}D_m\phi \qquad (2.45) \\
\delta F &= -\sqrt{2}(\overline{\epsilon}\not{D}\psi) - \sqrt{2}(\overline{\epsilon}\sigma\psi) - 2(\overline{\epsilon}\overline{\lambda})\phi.
\end{aligned}$$

2.2.4 $\mathcal{N} = 2$ supersymmetric Chern-Simons term

Here we discuss the $\mathcal{N} = 2$ supersymmetric version of Chern-Simons term. There are a couple of ways to derive it. One is to start with the pure Chern-Simons term (B.26), and modify it so as to make it invariant under the transformation (see Subsection B.2.1). The supersymmetric Chern-Simons term is given by

$$S_{\text{CS}} = \frac{k}{2\pi} \int d^3x \ \text{tr} \left[\epsilon^{mnp} \frac{1}{2} \left(A_m \partial_n A_p - \frac{2i}{3} A_m A_n A_p \right) + \bar{\lambda}\lambda - \sigma D \right]. \tag{2.46}$$

One can check that this action is invariant under the SUSY transformation (2.41).

2.3 ABJM model

So far we discussed a general $\mathcal{N} = 2$ supersymmetric gauge theory. In this section we treat the specific model, which describe the low energy effective action of flat M2-branes as in Fig. 2.1, and it is called the ABJM model [3].

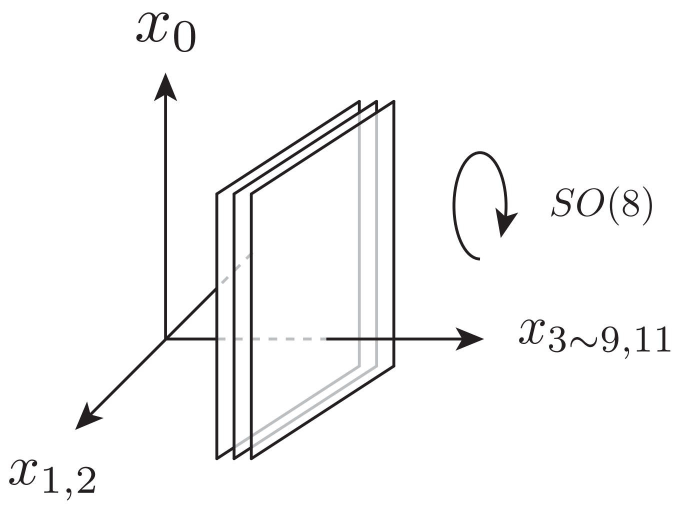

Figure 2.1: Global symmetry of M2-brane

An M2-brane is, as the name express, a 3d spacetime object in 11-dimensional spacetime. The position of the branes is expressed by 8 scalar fields; these scalars express the directions normal to the M2-branes, and this is why 8 scalars are needed. The M2-branes are invariant under rotations of 8 transverse directions. Therefore, the theory should have a global $SO(8)$ symmetry for the scalar fields. Since the theory is supersymmetric the 8 superpartners are also required corresponding to the scalars.

As we consider the IR limit the theory should be conformal. The conformal symmetry mixes up with the supersymmetry and the $SO(8)$ symmetry and becomes the superconformal symmetry $OSp(8|4)$.

Since there are many M2-branes and they are degenerate, we need another index denoting the branes. If the analog to D2-brane in the string theory can be applied the number of the branes corresponds to the rank of the gauge group $U(N)$. However, in the 3d case the Yang-Mills term is dimensionful and it could not be conformal. So the only choice is the Chern-Simons term. One may suppose we are almost reaching the low energy effective theory of M2-branes but we still have a big problem. In order to realize bigger supersymmetry the theory should have appropriate interaction terms. AdS/CFT tells us that the free energy of M2-branes is proportional to $N^{3/2}$; this strange exponent is the effect of the interaction. Therefore, we ought to find the appropriate interaction terms. This had not been achieved for a long time.

Three years ago, a low energy effective theory of M2-branes was found by Aharony, Bergman, Jafferis, and Maldacena [3]. The key point was that they gave up the manifest $\mathcal{N} = 8$ SUSY but kept only the $\mathcal{N} = 6$ SUSY manifestly (manifest means the symmetry of the Lagrangian). It is known that the expected $\mathcal{N} = 8$ SUSY is realized for $k = 1, 2$ non-perturbatively [3]. We do not look into the details of the SUSY enhancement but simply give the action and the SUSY transformation. The ABJM model is $U(N)_k \times U(N)_{-k}$ Chern-Simons matter theory whose matter contents and the representations of the symmetries are summarized in Table 2.1. The Lagrangian

Table 2.1: Gauge symmetries and global symmetries of the ABJM model. $SU(4) \sim SO(6)$ is the R-symmetry.

	A	$\widehat{A}$	Y^A	ψ_A
$U(N)_k$	adj	1	N	N
$U(N)_{-k}$	1	adj	$\overline{N}$	$\overline{N}$
$SU(4)_R$	1	1	$\mathbf{4}$	$\overline{\mathbf{4}}$
$U(1)_B$	0	0	1	1

of the ABJM model is as follows.

$$S_{\text{CS}} = \frac{k}{4\pi}\int d^3x\ \epsilon^{mnp}\ \text{tr}\left[\left(A_m\partial_n A_p - \frac{2i}{3}A_m A_n A_p\right) - \left(\hat{A}_m\partial_n\hat{A}_p - \frac{2i}{3}\hat{A}_m\hat{A}_n\hat{A}_p\right)\right] \tag{2.47}$$

$$S_{\text{kin}} = \frac{1}{2\pi}\int d^3x\ \text{tr}\left[-D^m Y^A D_m\overline{Y}_A + \overline{\psi}^A\gamma^m D_m\psi_A\right] \tag{2.48}$$

$$\begin{aligned} S_{\text{pot}} = \frac{1}{6\pi k^2}\int d^3x\ \text{tr}\big[&Y^A\overline{Y}_A Y^B\overline{Y}_B Y^C\overline{Y}_C + \overline{Y}_A Y^A\overline{Y}_B Y^B\overline{Y}_C Y^C \\ &+4Y^A\overline{Y}_B Y^C\overline{Y}_A Y^B\overline{Y}_C - 6Y^A\overline{Y}_B Y^B\overline{Y}_A Y^C\overline{Y}_C\big] \end{aligned} \tag{2.49}$$

$$\begin{aligned} S_{\text{Yukawa}} = -\frac{1}{2\pi k}\int d^3x\ \text{tr}\Big[&\overline{Y}_A Y^A\overline{\psi}^B\psi_B - Y^A\overline{Y}_A\psi_B\overline{\psi}^B + 2Y^A\overline{Y}_B\psi_A\overline{\psi}^B - 2\overline{Y}_A Y^B\overline{\psi}^A\psi_B \\ &+\epsilon^{ABCD}\overline{Y}_A\psi_B\overline{Y}_C\psi_D - \epsilon^{ABCD}Y^A\overline{\psi}^B Y^C\overline{\psi}^D\Big] \end{aligned} \tag{2.50}$$

where Y^A are complex scalar fields; A runs from 1 to 4, ψ^A are Dirac fermions. We already used the equations of motion for the auxiliary fields and erased them. Note that the covariant derivative is

$$D_\mu Y^A = \partial_\mu Y^A - iA_\mu Y^A + iY^A\hat{A}_\mu. \tag{2.51}$$

Note that Y^A belongs to a fundamental representation of the first gauge group $U(N)_k$ and an anti-fundamental representation of the second gauge group $U(N)_{-k}$, and this representation is called a bi-fundamental. From the Lagrangian we can see that it has $SU(4)$ global symmetry. The matter fields belong to the **6** representation of $SU(4)$, and hence, the theory has $\mathcal{N}=6$ SUSY. The $\mathcal{N}=6$ SUSY transformation is given as follows.

$$\begin{aligned}
\delta Y^A &= \sqrt{2}\xi^{AB}\psi_B \\
\delta\psi_A &= -\sqrt{2}\gamma^m\xi_{AB}D_mY^B + \frac{2\sqrt{2}\pi}{k}\xi_{AB}\left(Y^C\overline{Y}_CY^B - Y^B\overline{Y}_CY^C\right) - \frac{4\sqrt{2}\pi}{k}\xi_{bc}\left(Y^B\overline{Y}_AY^C\right) \\
\delta A_m &= -\frac{2i\sqrt{2}\pi}{k}\left[\xi_{AB}\gamma_m\left(Y^A\overline{\psi}^B\right) + \xi^{AB}\gamma_m\left(\psi_A\overline{Y}_B\right)\right] \\
\delta\widetilde{A}_m &= \frac{2i\sqrt{2}\pi}{k}\left[\xi^{AB}\gamma_m\left(\overline{Y}_A\psi_B\right) + \xi_{AB}\gamma_m\left(\overline{\psi}^AY^B\right)\right]
\end{aligned} \tag{2.52}$$

Where ξ_{AB} is the SUSY parameter and it belongs to **6** representation of $SU(4)$ as we mentioned, and satisfying following relations:

$$\xi_{AB} = -\xi_{BA}, \qquad (\xi_{AB})^* = -\frac{1}{2}\epsilon^{ABCD}\xi_{CD}. \tag{2.53}$$

So far we write down the Lagrangian in terms of component fields. We can write down the same Lagrangian by the $\mathcal{N}=2$ multiplet.

$$\begin{aligned}
S_{\text{CS}} &= \sum_{I=1,2}\frac{k_I}{4\pi}\int d^3x\ \epsilon^{mnp}\ \text{tr}\left(DV_I\overline{D}V_I + \cdots\right) \\
&= \sum_{I=1,2}\frac{k_I}{4\pi}\int d^3x\ \epsilon^{mnp}\ \text{tr}\left[-\sigma_ID_I + \left(A^I_m\partial_nA^I_p - \frac{2i}{3}A^I_mA^I_nA^I_p\right) + (\lambda\overline{\lambda})\right]
\end{aligned} \tag{2.54}$$

$$S_{\text{matter}} = \frac{1}{2\pi}\int d^3xd^4\theta\ \text{tr}\left[A_i^\dagger e^{2V_1}A^ie^{-2V_2} + B^ie^{-2V_1}B_i^\dagger e^{2V_2}\right] \tag{2.55}$$

$$S_{\text{pot}} = \frac{1}{2\pi k}\int d^3xd^2\theta\ \epsilon_{ij}\epsilon_{kl}\text{tr}\left[A^iB^kA^jB^l\right] + c.c. \tag{2.56}$$

where $A_m^{I=1} = A_m$, $A_m^{I=2} = \widehat{A}_m$, $k_{I=1} = k$, and $k_{I=2} = -k$. At the first line we expressed the Chern-Simons terms in terms of vector multiplets. $\cdots$ express the higher terms and they give only a tertiary term of A_m (see Subsection B.2.1 for the details). We can see that the Lagrangian has some global symmetries listed in Table 2.2. These symmetries play an important role in the latter chapters. If we integrate out all the auxiliary field in the multiplets we obtain the Lagrangian in terms of the component fields (2.52).

Table 2.2: Global symmetries of ABJM model

	A_1	A_2	B_1	B_2
$U(1)_A$	$+1$	-1	0	0
$U(1)_B$	0	0	$+1$	-1
$U(1)_T$	$+1$	$+1$	-1	-1

Chapter 3

Partition function on $\boldsymbol{S}^3$

In the previous chapter we discussed $\mathcal{N}=2$ supersymmetric gauge theories: their construction and their transformation laws in 3d Minkowski space. In this chapter we calculate the partition function of those theories using the localization on 3d Euclidean space, rather than on the Minkowski space. The partition function is defined by

$$Z = \int_{\mathcal{M}} \mathcal{D}\Phi e^{-S[\Phi]}. \tag{3.1}$$

where $\mathcal{M}$ is a manifold on which the theory is defined. The reason why we use Euclidean space is purely because of the technical reason; if the space is not compact the volume integration diverges. For simplicity, we consider the most symmetric compact space in 3d as the first example, which is three-sphere $\boldsymbol{S}^3$. The Euclideanization makes no change on the Lagrangians nor the SUSY transformation laws as we have not used any explicit representation for the metric nor the γ-matrices. When the explicit expression for the γ-matrices are needed we use $\gamma^{\widehat{m}} = \sigma_m$, where σ_m are the Pauli's matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.2}$$

Note that in a 3d Euclidean space a vector index runs from 1 to 3 rather than 0 to 2 for making sure that we treat Euclidean space. These Euclidean notations are used through the remaining part of this thesis.

3.1 SUSY on conformally flat space

We extend the SUSY transformation law we derived in the previous chapter to the one on a conformally flat space. Conformally flat means that the geometry is deformed from the flat space by Weyl transformation, which is defined below. Since $\boldsymbol{S}^3$ is conformally flat (see Section C.5) we can use the extended SUSY transformation law for our purpose.

Firstly, we remind ourselves of the $\mathcal{N}=2$ SUSY transformation law of vector multiplet.

$$\begin{aligned}\delta_0 A_\mu &= i\epsilon\gamma_\mu\overline{\lambda} - i\overline{\epsilon}\gamma_\mu\lambda\\ \delta_0\sigma &= \epsilon\overline{\lambda} + \overline{\epsilon}\lambda\\ \delta_0\lambda &= \frac{i}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - \gamma^\mu\epsilon D_\mu\sigma + D\epsilon\\ \delta_0\overline{\lambda} &= -\frac{i}{2}\gamma^{\mu\nu}\overline{\epsilon}F_{\mu\nu} - \gamma^\mu\overline{\epsilon}D_\mu\sigma + D\overline{\epsilon}\\ \delta_0 D &= -\epsilon\gamma^\mu D_\mu\overline{\lambda} - \overline{\epsilon}\gamma^\mu D_\mu\lambda + \epsilon[\sigma,\overline{\lambda}] + \overline{\epsilon}[\sigma,\lambda].\end{aligned} \tag{3.3}$$

where the subscripts 0 mean the transformation is applied for the flat space. Note that the indices $\mu, \nu, \cdots$ are not for 4d indices but 3d world space indices and run from 1 to 3.

Weyl transformation is defined as a local scaling of the vielbein (or metric):

$$\begin{aligned} e_\mu^m &\to e_\mu^{'m} = e^{-\alpha(x)}e_\mu^m,\\ g_{\mu\nu} &\to g'_{\mu\nu} = e^{-2\alpha(x)}g_{\mu\nu}.\end{aligned} \tag{3.4}$$

Under this transformation a field Φ with Weyl weight n transforms as follows.

$$\Phi' = e^{n\alpha}\Phi. \tag{3.5}$$

Weyl weights of the components of a vector multiplet and the SUSY transformation parameters are given in Table 3.1. Note that the weight of A_μ is zero, on the other

Table 3.1: Weyl weights of fields of vector multiplet

Fields	ϵ	$\overline{\epsilon}$	A_μ	σ	λ	$\overline{\lambda}$	D	γ_μ
Weyl weights	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	$\frac{3}{2}$	$\frac{3}{2}$	2	-1

hand, the weight of $A_m = e_m^\mu A_\mu$ is one. Furthermore, the position of the index is important; e.g.

$$\gamma^\mu \to \gamma'^\mu = g'^{\mu\nu}\gamma'_\nu = e^{2\alpha}g^{\mu\nu}e^{-\alpha}\gamma_\nu = e^\alpha g^{\mu\nu}\gamma_\nu. \tag{3.6}$$

The assignment of the weights in the table shows that the weights of both side of the transformations (3.3) are the same. Hence, the transformation is invariant under the Weyl transformation if the parameter $\alpha(x)$ is independent of the coordinate x.

Let us consider the case that the parameter depends on the coordinates. We denote fields and the vielbein on the flat space $\mathcal{M}$ without prime e.g. Φ, e_μ^m, and those on conformally flat space $\mathcal{M}'$ with prime e.g. Φ', $e_\mu'^m$. We define the SUSY transformation of the fields Φ' with conformal weight n on $\mathcal{M}'$ as

$$\delta\Phi' = e^{n\alpha}\delta_0\Phi, \tag{3.7}$$

where $\Phi = e^{-n\alpha}\Phi'$ is the pull-back of the fields to the flat space $\mathcal{M}$. In the case that the SUSY transformation of the field Φ includes no derivative terms the α dependence of the RHS of (3.7) does not remain. However, if the transformation includes derivative terms, extra terms containing $\partial\alpha$ come up. In order to keep the transformation covariant one usually introduces a covariant derivative with corresponding gauge field. Here, we show that the; derivative of the SUSY parameter $D_\mu\epsilon$ can be used as the gauge field.

Using the behavior of the spin connection (C.12) under the Weyl transformation:

$$\omega_{\mu mn}(e') = \omega_{\mu mn}(e) - \left(e_{\mu m}e_n^\lambda - e_{\mu n}e_m^\lambda\right)\partial_\lambda\alpha, \tag{3.8}$$

we can see that the Weyl transformation of the $D_\mu\epsilon$ becomes

$$D'_\mu\epsilon' = e^{-\alpha/2}\left(D_\mu\epsilon - \frac{1}{2}\gamma_\mu\gamma^\lambda\epsilon\partial_\lambda\alpha\right). \tag{3.9}$$

This transformation looks similar to that of gauge transformation of a gauge field in the sense that it is shifted by $\partial\alpha$. Using this fact we can construct the SUSY transformation that transforms covariantly under the Weyl transformation. For instance, we replace the terms with the derivatives as following.

$$\begin{aligned}(D_\mu\sigma)\gamma^\mu\epsilon &\to (D_\mu\sigma)\gamma^\mu\epsilon + \frac{2}{3}\sigma\gamma^\mu D_\mu\epsilon,\\ (\bar{\epsilon}\gamma^\mu D_\mu\lambda) &\to (\bar{\epsilon}\gamma^\mu D_\mu\lambda) + \frac{1}{3}\left(D_\mu\bar{\epsilon}\gamma^\mu\lambda\right).\end{aligned} \tag{3.10}$$

One can check that these terms covariantly transform;

$$\begin{aligned}(D_\mu\sigma')\gamma'^\mu\epsilon' + \frac{2}{3}\sigma'\gamma'^\mu D'_\mu\epsilon' &= e^{\frac{3}{2}\alpha}\left[(D_\mu\sigma)\gamma^\mu\epsilon + \frac{2}{3}\sigma\gamma^\mu D_\mu\epsilon\right],\\ \left(\bar{\epsilon}'\gamma'^\mu D'_\mu\lambda'\right) + \frac{1}{3}\left(D'_\mu\bar{\epsilon}'\gamma'^\mu\lambda'\right) &= e^{2\alpha}\left[(\bar{\epsilon}\gamma^\mu D_\mu\lambda) + \frac{1}{3}\left(D_\mu\bar{\epsilon}\gamma^\mu\lambda\right)\right].\end{aligned} \tag{3.11}$$

Note that newly added terms vanish when the space is flat, and hence, do not cause any problem. The SUSY transformation law with this replacement is

$$\begin{aligned}\delta A_\mu &= i\epsilon\gamma_\mu\bar{\lambda} - i\bar{\epsilon}\gamma_\mu\lambda\\ \delta\sigma &= \epsilon\bar{\lambda} + \bar{\epsilon}\lambda\\ \delta\lambda &= \frac{i}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - \gamma^\mu\epsilon D_\mu\sigma + D\epsilon - \frac{2}{3}\sigma\gamma^\mu D_\mu\epsilon\\ \delta\bar{\lambda} &= -\frac{i}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} - \gamma^\mu\bar{\epsilon}D_\mu\sigma + D\bar{\epsilon} - \frac{2}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon}\\ \delta D &= -\epsilon\gamma^\mu D_\mu\bar{\lambda} - \bar{\epsilon}\gamma^\mu D_\mu\lambda + \epsilon[\sigma,\bar{\lambda}] + \bar{\epsilon}[\sigma,\lambda] + \frac{1}{3}\bar{\lambda}\gamma^\mu D_\mu\epsilon + \frac{1}{3}\lambda\gamma^\mu D_\mu\bar{\epsilon},\end{aligned} \tag{3.12}$$

where we omit the primes. This transformation is applicable for any conformally flat space. Note that ϵ is now not a constant. Rather, it should satisfy the following equation:

$$D_\mu\epsilon = \gamma_\mu\kappa, \tag{3.13}$$

where κ is an arbitrary spinor. This condition is called a Killing spinor equation, and it has the same form after the Weyl transformation with an appropriate redefinition of κ:

$$D'_\mu\epsilon' = \gamma'_\mu\kappa'. \tag{3.14}$$

Similarly, we can construct the SUSY transformation on conformally flat space of a chiral multiplet. Let us consider a chiral multiplet with conformal weight Δ. Namely, its components have weights as in Table 3.2. The transformation for the

Table 3.2: Weyl weights of the fields of chiral multiplet

Fields	ϕ	ψ	F
Weyl weights	Δ	$\Delta+\frac{1}{2}$	$\Delta+1$

flat space is

$$\begin{aligned}
\delta_0\phi &= \sqrt{2}\epsilon\psi \\
\delta_0\psi &= -\sqrt{2}\gamma^m\bar{\epsilon}D_m\phi + \sqrt{2}\bar{\epsilon}\sigma\phi + \sqrt{2}\epsilon F \\
\delta_0 F &= -\sqrt{2}\bar{\epsilon}\gamma^m D_m\psi + \sqrt{2}\bar{\epsilon}\sigma\psi - 2\bar{\epsilon}\bar{\lambda}\phi.
\end{aligned} \tag{3.15}$$

The terms including derivatives should be replaced as

$$\begin{aligned}
\left(D_\mu\phi\right)\gamma^\mu\bar{\epsilon} &\to \left(D_\mu\phi\right)\gamma^\mu\bar{\epsilon} + \frac{2}{3}\Delta\phi\gamma^\mu D_\mu\bar{\epsilon}, \\
\left(\bar{\epsilon}\gamma^\mu D_\mu\psi\right) &\to \left(\bar{\epsilon}\gamma^\mu D_\mu\psi\right) + \frac{2}{3}\left(\Delta-\frac{1}{2}\right)\left(D_\mu\bar{\epsilon}\gamma^\mu\psi\right).
\end{aligned} \tag{3.16}$$

Then, the transformation for a conformally flat space is derived as follows.

$$\begin{aligned}
\delta\phi &= \sqrt{2}\epsilon\psi \\
\delta\psi &= -\sqrt{2}\gamma^m\bar{\epsilon}D_m\phi + \sqrt{2}\bar{\epsilon}\sigma\phi + \sqrt{2}\epsilon F - \frac{2\sqrt{2}}{3}\Delta\phi\gamma^\mu D_\mu\bar{\epsilon} \\
\delta F &= -\sqrt{2}\bar{\epsilon}\gamma^m D_m\psi + \sqrt{2}\bar{\epsilon}\sigma\psi - 2\bar{\epsilon}\bar{\lambda}\phi - \frac{2\sqrt{2}}{3}\left(\Delta-\frac{1}{2}\right)\left(D_\mu\bar{\epsilon}\gamma^\mu\psi\right).
\end{aligned} \tag{3.17}$$

3.2 Calculation of $\boldsymbol{S}^3$ partition function

3.2.1 Vector multiplet

In order to use the localization technique we need supersymmetry on $\boldsymbol{S}^3$. On $\boldsymbol{S}^3$ with radius r we use a Killing spinor satisfying

$$D_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon. \tag{3.18}$$

There are two linearly independent solutions to this equation, and we denote them as ϵ_1 and ϵ_2. The corresponding SUSY transformations are δ_1 and δ_2. In this section we derive the deformation term δV used in the localization (**??**) and calculate the partition function exactly.

The transformation laws of vector multiplet on $\boldsymbol{S}^3$ are given as follows.

$$\begin{aligned}
\delta A_\mu &= i\epsilon\gamma_\mu\overline{\lambda} \\
\delta\sigma &= \epsilon\overline{\lambda} \\
\delta\lambda &= \frac{i}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - \gamma^\mu\epsilon D_\mu\sigma + D\epsilon - \frac{i}{r}\sigma\epsilon \\
\delta\overline{\lambda} &= 0 \\
\delta D &= -\epsilon\gamma^\mu D_\mu\overline{\lambda} + \epsilon[\sigma,\overline{\lambda}] + \frac{i}{2r}\overline{\lambda}\epsilon
\end{aligned} \tag{3.19}$$

Using these we consider $\delta_2\mathcal{V} = \delta_2\delta_1\mathcal{T}$ with $\mathcal{T} = -\frac{1}{2}\lambda\lambda$. After some calculation we have

$$\begin{aligned}
\delta_2\delta_1\mathcal{T}/\epsilon_2\epsilon_1 = &\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\sigma D^\mu\sigma - i\gamma^{\mu\nu\rho}F_{\mu\nu}D_\rho\sigma + \left(iD + \frac{1}{r}\sigma\right)^2 \\
&- 2\overline{\lambda}\not{D}\lambda + 2[\sigma,\overline{\lambda}]\lambda - \frac{i}{r}\overline{\lambda}\lambda.
\end{aligned} \tag{3.20}$$

Hence, the deformation term is

$$\begin{aligned}
t\delta V = t\int d^3x\sqrt{g}\Bigg(&\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\sigma D^\mu\sigma - i\gamma^{\mu\nu\rho}F_{\mu\nu}D_\rho\sigma + \left(iD + \frac{1}{r}\sigma\right)^2 \\
&-2\overline{\lambda}\not{D}\lambda + 2[\sigma,\overline{\lambda}]\lambda - \frac{i}{r}\overline{\lambda}\lambda\Bigg).
\end{aligned} \tag{3.21}$$

Since the auxiliary field D is pure imaginary the boson part is positive definite. Localization method tells us that the only following saddle points contribute to the partition function;

$$F_{\mu\nu} = 0, \quad D_\mu\sigma = 0, \quad iD + \frac{1}{r}\sigma = 0. \tag{3.22}$$

The general solution to these conditions is

$$A_\mu = 0, \quad \sigma = \sigma_0, \quad D = \frac{i}{r}\sigma_0, \tag{3.23}$$

where we took the Lorentz gauge $\partial_\mu A^\mu = 0$. The path integral split into the integration of σ_0 that parametrizes the saddle points and that of fluctuations around them. Though σ_0 takes value in the Lie algebra of the gauge group, we can diagonalize it by the gauge transformation: $\tilde{\sigma}_0 = U^{-1}\sigma_0 U$. Then, the integration can be written by that of the Cartan part.

$$\int d\sigma_0 = \int d\widetilde{\sigma}_0 \left(\prod_{\alpha\in G}\alpha\left(\widetilde{r\sigma}_0\right)\right), \tag{3.24}$$

where Vandermonde determinant appear as Jacobian, and α is a root of the gauge group and it is defined as follows.

$$[\sigma, A_\mu] = \alpha(\sigma)A_\mu \tag{3.25}$$

The product is over all the roots of the gauge group G. For example, for $U(N)$,

$$\prod_{\alpha\in G} \alpha(\sigma_0) = \prod_{i,j=1,i\neq j}^{\dim G} (\sigma_{0i} - \sigma_{0j}) . \tag{3.26}$$

where and from now on we omit the tilde on $\widetilde{\sigma}_0$, and σ_0 takes value in the Cartan part. In order to integrate the fluctuations out we expand all the fields as

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\Phi', \tag{3.27}$$

where the classical part Φ_0 is nonzero only for σ_0 (and D). Taking the weak coupling limit $t\to\infty$ all the higher order terms vanish and only the quadratic terms of Φ' survive.. Finally, the action becomes

$$\begin{aligned} S = \int d^3x\sqrt{g}\Big\{ & (-A'_\nu D_\mu D^\mu A'^\nu) + \frac{2}{r^2}A'_\mu A'^\mu + A'_\mu[\sigma_0,[\sigma_0,A'^\mu]] - \sigma' D_\mu D^\mu \sigma' \\ & -2\left(\overline{\lambda}'\gamma^\mu D_\mu \lambda'\right) - 2\left(\overline{\lambda}'[\sigma_0,\lambda']\right) - \frac{i}{r}\left(\overline{\lambda}'\lambda\right)\Big\} . \end{aligned} \tag{3.28}$$

Note that the gauge field in the covariant derivative is replaced by its classical value $A_\mu = 0$. What we only need to do is to expand each field by harmonic functions on $\boldsymbol{S}^3$ and perform the path integral. The harmonic functions on $\boldsymbol{S}^3$ are summarized in chapter E.

The differential operator acting on the scalar field σ' is the Laplacian, and its eigenvalues are given by

$$-r^2 D_\mu D^\mu = l\,(l+2) \qquad l = 0,1,2,\cdots, \quad \text{degeneracy } (l+1)^2. \tag{3.29}$$

Nonetheless, the integration only gives the constant, and hence, we can ignore it.

A divergenceless vector field is expanded by harmonics in $(l, l+1)$ and $(l+1, l)$ representations of the isometry $SO(4)\sim SU(2)\times SU(2)$ of $\boldsymbol{S}^3$.

The eigenvalues of the Laplacian of each representation are

$$-r^2 D_\mu D^\mu = (l+2)^2 - 2, \quad l = 0,1,2,\cdots, \quad \text{degeneracy } (l+1)(l+3). \tag{3.30}$$

Hence, the eigenvalue of the differential operator

$$D_{\text{vec}} = -D_\mu D^\mu + \frac{2}{r^2} + \alpha(r\sigma_0)^2 \tag{3.31}$$

is given as follows.

$$r^2 D_{\text{vec}} = (l+2)^2 + \alpha(r\sigma_0)^2 = (l+2+i\alpha(r\sigma_0))\,(l+2-i\alpha(r\sigma_0))\,, \quad l = 0,1,2,\cdots . \tag{3.32}$$

Using the eigenvalues of the Dirac operator

$$r\gamma^\mu D_\mu = \pm i\left(l+\frac{3}{2}\right), \quad l = 0,1,2,\cdots, \quad \text{degeneracy } (l+1)(l+2), \tag{3.33}$$

the differential operator for the fermion

$$\frac{r}{2}D_{\text{fermi}} = -r\gamma^\mu D_\mu - \alpha(r\sigma_0) - \frac{i}{2}, \tag{3.34}$$

gives the eigenvalues

$$i\left(l+1+i\alpha(r\sigma_0)\right) \text{ and } -i\left(l+2-i\alpha(r\sigma_0)\right), \quad l = 0,1,2,\cdots. \tag{3.35}$$

Summarizing the results we obtained the 1-loop partition function as follows.

$$\begin{aligned} Z &= \frac{\prod_{l=0}^\infty (l+1+i\alpha(r\sigma_0))^{(l+1)(l+2)} \prod_{l=0}^\infty (l+2-i\alpha(r\sigma_0))^{(l+1)(l+2)}}{\prod_{l=0}^\infty (l+2+i\alpha(r\sigma_0))^{(l+1)(l+3)} \prod_{l=0}^\infty (l+2-i\alpha(r\sigma_0))^{(l+1)(l+3)}} \\ &= \frac{\prod_{l=0}^\infty (l+1+i\alpha(r\sigma_0))^{(l+2)}}{\prod_{l=0}^\infty (l+2-i\alpha(r\sigma_0))^{(l+1)}}. \end{aligned} \tag{3.36}$$

Furthermore, using the fact that we always take the product of all the roots (positive and negative α appear as a pair) we can write the partition function as

$$Z = \prod_{l=1}^\infty (l+i\alpha(r\sigma_0)) \prod_{l=1}^\infty (l-i\alpha(r\sigma_0)) \tag{3.37}$$

Ignoring the divergent constant factor we have

$$Z = \prod_{l=1}^\infty \left(1+\frac{\alpha(r\sigma_0)^2}{l^2}\right) = \frac{\sinh(\pi\alpha(r\sigma_0))}{\pi\alpha(r\sigma_0)}. \tag{3.38}$$

Note that the term in the denominator cancel with the Vandermonde determinant (3.24).

So far we have not taken the contribution of original action into account. Indeed, almost all terms vanish at the saddle points. However, if the original action has Chern-Simons term or Fayet Iliopoulos term, they have classical contributions. From (B.40) we can derive the Euclideanized Chern-Simons term :

$$S_{\text{ESCS}} = -\frac{k}{4\pi}\int d^3x \ \text{tr}\left[\epsilon^{mnp}\frac{1}{2}\left(A_m\partial_n A_p - \frac{2i}{3}A_m A_n A_p\right) + \overline{\lambda}\lambda - \sigma D\right]. \tag{3.39}$$

Substituting (3.27) and taking $t \to \infty$ limit the action becomes

$$S_{\text{ESCS}} = \frac{k}{4\pi}\int d^3x \ \text{tr}\left[\frac{i}{r}\sigma_0^2\right] = i\pi k \ \text{tr}\,(r\sigma_0)^2. \tag{3.40}$$

where we used the $\boldsymbol{S}^3$ volume $2\pi^2 r^3$.

Similarly, for the Fayet Iliopoulos term we have

$$S_{\text{EFI}} = \zeta\int d^3x \ D = \zeta\int d^3x \ \frac{i}{r}\sigma_0^{U(1)} = 4i\pi^2 r^2 \zeta\sigma_0^{U(1)}. \tag{3.41}$$

To sum up we obtained the partition function as

$$Z = \int d\sigma_0 e^{-i\pi k \text{tr}(r\sigma_0)^2 - 2\pi i\zeta r\sigma_{0U(1)}} \prod_{\alpha\in G} \sinh\left(\pi\alpha(r\sigma_0)\right), \tag{3.42}$$

where we rescaled $\zeta \to \xi/(2\pi r)$ so that the expression becomes simple.

3.2.2 Chiral multiplet

What we do here is almost parallel with that of vector multiplet. The SUSY transformation laws with ϵ parameter are

$$\begin{aligned}
&\delta\phi = \sqrt{2}\epsilon\psi, \\
&\delta\psi = \sqrt{2}\epsilon F, \\
&\delta F = 0, \\
&\delta\overline{\phi} = 0, \\
&\delta\overline{\psi} = -\sqrt{2}\gamma^m\epsilon D_m\overline{\phi} + \sqrt{2}\epsilon\overline{\phi}\sigma - \frac{i\sqrt{2}}{r}\Delta\overline{\phi}\epsilon, \\
&\delta\overline{F} = -\sqrt{2}\epsilon\gamma^m D_m\overline{\psi} + \sqrt{2}\epsilon\overline{\psi}\sigma - 2\overline{\phi}\epsilon\lambda + \frac{i\sqrt{2}}{r}\left(\Delta - \frac{1}{2}\right)(\epsilon\overline{\psi}) .
\end{aligned} \tag{3.43}$$

Using these we can derive the following action

$$\begin{aligned}
\mathcal{T} &= -\frac{1}{2}\overline{F}\phi, \\
S &= t\int d^3x\sqrt{g}\delta_2\delta_1\mathcal{T}/\epsilon_2\epsilon_1 \\
&= t\int d^3x\sqrt{g}\Big[-D_\mu D^\mu\overline{\phi}\phi + (D_\mu\overline{\psi}\gamma^\mu\psi) - \overline{F}F + \frac{i(2\Delta-1)}{2r}\overline{\psi}\psi - \frac{\Delta(\Delta-2)}{r^2}\overline{\phi}\phi \\
&\quad -\sqrt{2}(\overline{\psi}\overline{\lambda}) - \sqrt{2}\overline{\phi}(\lambda\psi) - \overline{\psi}\sigma\psi - \frac{i(2\Delta-1)}{r}\overline{\phi}\sigma\phi + \overline{\phi}\sigma\sigma\phi + \overline{\phi}D\phi\Big] .
\end{aligned} \tag{3.44}$$

The boson part is again positive definite and all the fields localize at zero in this case. Then, we rescale all the fields as $\Phi \to \Phi/\sqrt{t}$ and take the $t \to \infty$ limit:

$$\begin{aligned}
S = \int d^3x\sqrt{g}\Big[&\overline{\phi}\left(-D_\mu D^\mu - \frac{\Delta(\Delta-2)}{r^2} - \frac{2i(\Delta-1)}{r}\sigma_0 + \sigma_0^2\right)\phi \\
&+\overline{\psi}\left(-\gamma^\mu D_\mu + \frac{i(2\Delta-1)}{2r} - \sigma_0\right)\psi - \overline{F}F\Big] .
\end{aligned} \tag{3.45}$$

As we did for vector multiplet we can perform the path integral by harmonic function expansion. It is useful to denote the eigenvalue of σ_0 corresponding to the representation of the chiral multiplet by $\rho(\sigma_0)$. Namely,

$$\begin{aligned}
\sigma_0\Phi = \rho(\sigma_0)\Phi &\quad \text{for a fundamental field } \Phi , \\
[\sigma_0, \Phi] = \rho(\sigma_0)\Phi &\quad \text{for an adjoint field } \Phi .
\end{aligned} \tag{3.46}$$

The integration for the auxiliary field F becomes a constant and we can ignore it. The differential operator for the scalar field ϕ can be written as

$$r^2 D_{\text{bos}} = -r^2 D_\mu D^\mu + 1 - (\Delta - 1 + i\rho(r\sigma_0))^2 . \tag{3.47}$$

The eigenvalues of the operator are derived using (3.29):

$$r^2 D_{\text{bos}} = (l+1)^2 - (\Delta - 1 + i\rho(r\sigma_0))^2 = (l + \Delta + i\rho(r\sigma_0))\,(l + 2 - \Delta - i\rho(r\sigma_0)) . \tag{3.48}$$

The operator for the fermion ψ is

$$rD_{\text{fermi}} = -r\gamma^\mu D_\mu + \frac{i(2\Delta - 1)}{2} - \rho(r\sigma_0). \tag{3.49}$$

The eigenvalues for the Dirac operator (3.33) belongs to two series:

$$\begin{aligned} rD_{\text{fermi}} &= i\left(l + 1 + \Delta + i\rho(r\sigma_0)\right) \\ rD_{\text{fermi}} &= -i\left(l + 2 - \Delta - i\rho(r\sigma_0)\right). \end{aligned} \tag{3.50}$$

Finally, the 1-loop partition function for chiral multiplet becomes following form.

$$\begin{aligned} Z_{1-\text{loop}} &= \prod_{\rho\in\mathcal{R}} \frac{\prod_{l=0}^{\infty}\left(l+1+\Delta+i\rho(r\sigma_0)\right)^{(l+1)(l+2)} \prod_{l=0}^{\infty}\left(l+2-\Delta-i\rho(r\sigma_0)\right)^{(l+1)(l+2)}}{\prod_{l=0}^{\infty}\left(l+\Delta+i\rho(r\sigma_0)\right)^{(l+1)^2} \prod_{l=0}^{\infty}\left(l+2-\Delta-i\rho(r\sigma_0)\right)^{(l+1)^2}} \\ &= \prod_{\rho\in\mathcal{R}} \prod_{l=1}^{\infty} \left(\frac{l+1-\Delta-i\rho(r\sigma_0)}{l-1+\Delta+i\rho(r\sigma_0)}\right)^l. \end{aligned} \tag{3.51}$$

In order to make the expression simple we set

$$z = i(1-\Delta) + \rho(r\sigma_0) \tag{3.52}$$

and define following function

$$l(z) = \log \prod_{k=1}^{\infty} \left(\frac{k - iz}{k + iz}\right)^k. \tag{3.53}$$

A similar function is used in [9], though the definition is bit different. We call this function l-function. The derivative of this function is expressed by an elementary function;

$$\frac{dl(z)}{dz} = i\pi z \coth(\pi z), \tag{3.54}$$

where we used the zeta function regularization (see F.1).

The final formula for the partition function is given as follows.

$$Z = \int d\sigma e^{-i\pi k \operatorname{tr}(\sigma^2) - 2\pi i \zeta \sigma_{U(1)}} \prod_{\alpha\in G} \sinh\left(\pi\alpha(\sigma)\right) \prod_{\rho\in\mathcal{R}} e^{l(i(1-\Delta)+\rho(\sigma))}, \tag{3.55}$$

where we renamed as $r\sigma_0 \to \sigma$. The constant factor is again ignored.

3.3 Partition function of ABJM model

Having derived the formula of the $\boldsymbol{S}^3$ partition function let us use it for the ABJM model. The required information to use the formula is

- the gauge group G
- the representations and the Weyl weights

- the levels of the Chern-Simons terms and the coefficients ζ of Fayet-Iliopoulos terms

Note that we especially do not need the explicit form of the interaction terms. As we saw in the derivation of the formula interaction terms do not contribute to the partition function. Nonetheless, in some case the interaction terms fix the Weyl weight of chiral multiplets.

In the ABJM model we have

- Two vector multiplets with $U(N)$ gauge group with Chern-Simons level k and $-k$. Their saddle points are parametrized by the Cartan part

$$\sigma_1 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \widetilde{\lambda}_1 & & \\ & \ddots & \\ & & \widetilde{\lambda}_N \end{pmatrix}. \tag{3.56}$$

  The trace in the Chern-Simons term is

$$k\mathrm{tr}(\sigma_1^2) - k\mathrm{tr}(\sigma_2^2) = k\sum_{i=1}^{N}\left(\lambda_i^2 - \widetilde{\lambda}_i^2\right). \tag{3.57}$$

  The roots of $U(N)$ are given by

$$\alpha_{ij}(\sigma_1) = \lambda_i - \lambda_j, \quad \alpha_{ij}(\sigma_1) = \widetilde{\lambda}_i - \widetilde{\lambda}_j, \tag{3.58}$$

  and the product $\prod_{\alpha\in G}$ becomes $\prod_{i,j=1,\ i\neq j}^{N}$

- Four chiral multiples with the Weyl weight $\frac{1}{2}$ belonging to a bi-fundamental representation. The weight vector for the bi-fundamental is expressed as

$$\rho_{ij}(\sigma) = \lambda_i - \widetilde{\lambda}_j. \tag{3.59}$$

Substituting these to the formula (3.55) We have

$$Z = \frac{1}{(N!)^2}\int d^N\lambda d^N\widetilde{\lambda}\ e^{-i\pi k\sum_{i=1}^{N}(\lambda_i^2-\widetilde{\lambda}_i^2)}\frac{\prod_{1\leq i<j\leq N}\left[\sinh^2\pi(\lambda_i-\lambda_j)\sinh^2\pi(\widetilde{\lambda}_i-\widetilde{\lambda}_j)\right]}{\prod_{i,j=1}^{N}\cosh^2\pi(\lambda_i-\widetilde{\lambda}_j)}. \tag{3.60}$$

There are several ways to calculate this integral. Analytic continuation of Lens space matrix model is used in [41]. In the large N limit the integral is exactly evaluated in [6], Fermi gas approach is applied in [20, 21], and numerical method is first explored in [42]. Here we use the method explored in [18], which is easy to apply and applicable not only to the ABJM model but also many other theories.

According to [18] we do not need to evaluate the integral in (3.60) to obtain the leading order of a large N expansion. The integral is actually evaluated by the stationary point of σ for the leading order. Let us see the free energy $F = -\log Z$:

$$F(\lambda, \widetilde{\lambda}) = i\pi k \sum_i (\lambda_i^2 - \widetilde{\lambda}_i^2) - 2\sum_{i<j} \log\sinh \pi(\lambda_i - \lambda_j) - 2\sum_{i<j} \log\sinh \pi(\widetilde{\lambda}_i - \widetilde{\lambda}_j) + 2\sum_{i,j} \log\cosh \pi(\lambda_i - \widetilde{\lambda}_j). \quad (3.61)$$

What we want is the stationary point of the free energy. It is given by the following simultaneous equations.

$$-\frac{1}{2\pi}\frac{\partial F}{\partial \lambda_i} = -ik\lambda_i + \sum_{j\neq i} \coth \pi(\lambda_i - \lambda_j) - \sum_j \tanh \pi(\lambda_i - \widetilde{\lambda}_j) = 0,$$
$$-\frac{1}{2\pi}\frac{\partial F}{\partial \widetilde{\lambda}_i} = ik\widetilde{\lambda}_i + \sum_{j\neq i} \coth \pi(\widetilde{\lambda}_i - \widetilde{\lambda}_j) + \sum_j \tanh \pi(\lambda_j - \widetilde{\lambda}_i) = 0. \quad (3.62)$$

These are numerically solved for finite N; the eigenvalue distribution is as in Fig. 3.1 and Fig. 3.2. What we can learn from the distribution is

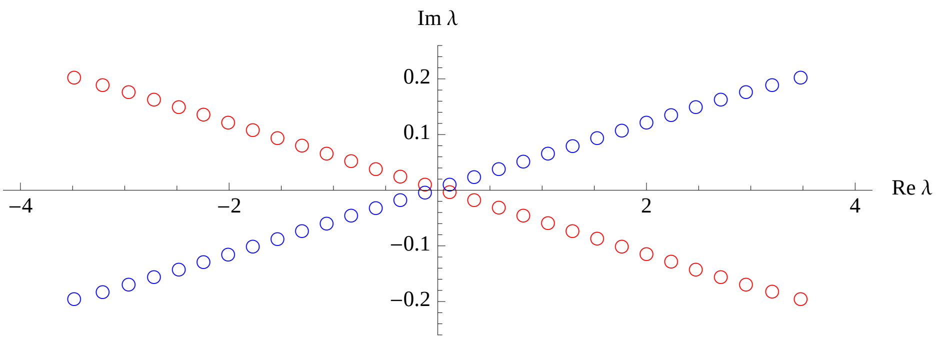

Figure 3.1: The eigenvalue distribution of the ABJM model. The blue and red circles are λ_i and $\widetilde{\lambda}_i$, respectively. N is set to be 30.

- λ_i and $\widetilde{\lambda}_i$ are complex
- $\widetilde{\lambda}_i = \lambda_i^*$.
- In $N \to \infty$ limit the imaginary part is

$$-\frac{1}{4} < \mathrm{Im}\,\lambda_i < \frac{1}{4}. \quad (3.63)$$

On the other hand, the real part is

$$\mathrm{Re}\,\lambda_{\max} \sim \mathcal{O}(N^\alpha), \quad (3.64)$$

where $\mathrm{Re}\,\lambda_{\max}$ is the maximum of $\mathrm{Re}\,\lambda_i$, and α is a certain constant $0 < \alpha < 1$.

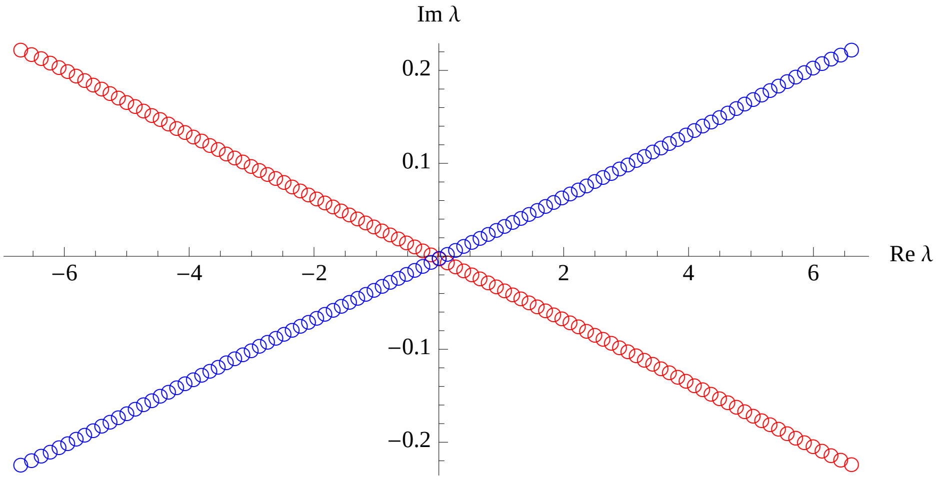

Figure 3.2: The eigenvalue distribution of the ABJM model with $N = 100$.

Therefore, we assume following form for λ and $\widetilde{\lambda}$.

$$\lambda_j = N^{\alpha} x_j + i y_j, \qquad \widetilde{\lambda}_j = N^{\alpha} x_j - i y_j. \tag{3.65}$$

We also introduce the eigenvalue density $\rho(x)$:

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(x - \lambda_i\right). \tag{3.66}$$

By definition it is normalized as follows.

$$\int \rho(x) dx = \frac{1}{N} \sum_{i=1}^{N} 1 = 1 \tag{3.67}$$

In what follows, we replace the sum by the integration with the density:

$$\sum_i \to N \int dx \rho(x). \tag{3.68}$$

Using these we rewrite the free energy. Firstly, the Chern-Simons term becomes

$$\begin{aligned} i\pi k \sum_i \left(\lambda_i^2 - \widetilde{\lambda}_i^2\right) &= i\pi k N \int dx \; \rho(x) \left((N^{\alpha} x + iy)^2 - (N^{\alpha} x - iy)^2\right) \\ &= -4\pi k N^{1+\alpha} \int dx \rho\left(x\right) xy. \end{aligned} \tag{3.69}$$

The other terms in (3.61) are expressed with the following function:

$$f(z) = \log\cosh(\pi z) \tag{3.70}$$

as follows.

$$-2 \sum_{i<j} f\left(\lambda_i - \lambda_j + \frac{i}{2}\right) - 2 \sum_{i<j} f\left(\widetilde{\lambda}_i - \widetilde{\lambda}_j + \frac{i}{2}\right) + 2 \sum_{i,j} f\left(\lambda_i - \widetilde{\lambda}_j\right). \tag{3.71}$$

The real part of z is proportional to N^α and the function $f(z)$ rapidly approaches to the form

$$f_0(z) = \text{sign}\,(\text{Re}\,z)\,(\pi z - \log 2)\,. \tag{3.72}$$

Let us consider terms in (3.71) with $f(z)$ are replaced by $f_0(z)$:

$$\begin{aligned}
&-2\sum_{x_i>x_j} f_0\left(\lambda_i - \lambda_j + \frac{i}{2}\right) - 2\sum_{x_i>x_j} f_0\left(\widetilde{\lambda}_i - \widetilde{\lambda}_j + \frac{i}{2}\right) \\
&+2\sum_{x_i>x_j} f_0\left(\lambda_i - \widetilde{\lambda}_j\right) + 2\sum_{x_i\le x_j} f_0\left(\lambda_i - \widetilde{\lambda}_j\right) \\
&= 2\pi \sum_{x_i>x_j}\left[-\left(\lambda_i - \lambda_j + \frac{i}{2}\right) - \left(\widetilde{\lambda}_i - \widetilde{\lambda}_j + \frac{i}{2}\right) + \left(\lambda_i - \widetilde{\lambda}_j\right) + \left(\widetilde{\lambda}_i - \lambda_j\right)\right].
\end{aligned} \tag{3.73}$$

The terms including λ_i or $\widetilde{\lambda}_i$ cancel out and the constant terms will be ignored because they are negligible in the large N limit; hence, we do not have to consider f_0. Still, we have the remaining part $f(z) - f_0(z)$, which has non-zero value near the $\text{Re}\,z = 0$. Especially, in the large N limit the remaining part proportional to delta function;

$$f(z) - f_0(z) = g(y)\delta(\hat{x}), \tag{3.74}$$

where $\hat{x} = N^\alpha x$. In order to derive the expression for the $g(y)$ we integrate (3.74) over $-\infty < \hat{x} < \infty$.

$$\begin{aligned}
g(y) &= \int d\hat{x} f(z) - f_0(z) = \int_{-\infty}^{\infty} d\hat{x} \log\left(1 + e^{2\pi z\,\text{sign}(\hat{x})}\right) \\
&= \int_{-\infty}^{0} d\hat{x}\log\left(1 + e^{2\pi z}\right) + \int_0^\infty d\hat{x}\log\left(1 - e^{-2\pi z}\right) \\
&= -\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^n \frac{e^{2\pi i n y} + e^{-2\pi i n y}}{n^2}
\end{aligned} \tag{3.75}$$

The infinite sum can be done, and the expression for $-1/2 \le y \le 1/2$ is

$$g(y) = \frac{\pi}{12} - \pi y^2. \tag{3.76}$$

Note that $g(y)$ has a period 1.

Using these results let us derive the free energy. As we discussed f_0 can be ignored. This means that we can replace $f(z)$ by $g(y)\delta(\hat{x}) = N^{-\alpha} g(y)\delta(x)$. This replacement leads to

$$\begin{aligned}
-\sum_{i\neq j}\log\sinh\pi\left(\lambda_i - \lambda_j\right) &= -N^2 \int dx dx' \rho(x)\rho(x') f\left(\lambda - \lambda' + \frac{i}{2}\right) \\
&= -N^{2-\alpha}\int dx \rho(x)^2 g\left(\frac{1}{2}\right)
\end{aligned} \tag{3.77}$$

for the term including $\lambda_i - \lambda_j$ in (3.61), and the similar term as (3.77) for the term including $\widetilde{\lambda}_i - \widetilde{\lambda}_j$. For the terms including $\lambda_i - \widetilde{\lambda}_j$ become

$$\sum_{i,j} \log\cosh\pi\left(\lambda_i - \widetilde{\lambda}_j\right) = 2N^2 \int dx dx' \rho(x)\rho(x') f\left(\lambda - \widetilde{\lambda}\right)$$
$$= 2N^{2-\alpha} \int dx \rho(x)^2 g\left(2y\right). \tag{3.78}$$

From those results the free energy becomes

$$F = \int dx \left(-4\pi N^{1+\alpha} k \rho(x) x y + 2N^{2-\alpha}\left(g(2y) - g\left(\frac{1}{2}\right)\right)\rho(x)^2\right) \tag{3.79}$$

The dependence on N in the free energy is $N^{1+\alpha}$ and $N^{2-\alpha}$. In order for the free energy to have a stationary point these two terms should balance in large N limit. Therefore, $\alpha = \frac{1}{2}$, and hence, $F \propto N^{3/2}$.

We assume the numerical result $|y| < 1/4$ is correct and use the explicit form for $g(y)$ to rewrite the free energy:

$$F = \pi N^{3/2} \int dx \left(-4k\rho(x) x y + \left(\frac{1}{2} - 8y^2\right)\rho(x)^2\right). \tag{3.80}$$

What we need to do now is to fix the $\rho(x)$ and $y(x)$ so that the free energy has the minimum. As $\rho(x)$ satisfies the normalization condition (3.67) we introduce the Lagrange multiplier μ;

$$F = \pi N^{3/2} \int dx \left(-4k\rho(x) x y + \left(\frac{1}{2} - 8y^2\right)\rho^2 - \mu\rho\right) + \mu. \tag{3.81}$$

Stationary condition for $\rho(x)$ is

$$-4kxy + (1 - 16y^2)\rho - \mu = 0. \tag{3.82}$$

That for $y(x)$ is

$$-4k\rho(x)x - 16y\rho(x)^2 = 0. \tag{3.83}$$

These equations give

$$y(x) = -\frac{kx}{4\mu}, \qquad \rho(x) = \mu. \tag{3.84}$$

Note that the eigenvalue density does not depend on x. This means that the range of the integration is limited, and from the normalization condition the range should be

$$|x| \leq \frac{1}{2\mu}. \tag{3.85}$$

Substituting all these results back to the free energy (3.80) we have

$$F = \pi N^{3/2} \int_{-1/(2\mu)}^{1/(2\mu)} \left(\frac{\mu^2}{2} + \frac{k^2 x^2}{2}\right) dx = \pi N^{3/2}\left(\frac{\mu}{2} + \frac{k^2}{24\mu^3}\right). \tag{3.86}$$

This should be the minimum of a function of μ, and this condition gives

$$\mu^2 = \frac{k}{2}. \tag{3.87}$$

Lastly, we reach the final form of the free energy:

$$F = \frac{\pi\sqrt{2k}}{3} N^{3/2}. \tag{3.88}$$

This is exactly the same as the free energy of M2-branes calculated from the gravity side [6]. Therefore, this is a non-trivial evidence of the AdS/CFT correspondence. In the following chapters, we repeat similar calculations for the squashed three-sphere and the orbifold. Although there are technical difficulties and new problems in each case the strategy to check the AdS/CFT using the partition functions is the same.

Chapter 4

Squashed partition function

The goal of this chapter is to derive the formula for the partition function similar to (3.55) but on the squashed three-sphere.

The squashing parameter was first introduced in [35]. They consider two kinds of squashed $\mathbf{S}^3$. The first one is the squashed sphere with the metric

$$ds^2 = r^2 \left[(\mu^1)^2 + (\mu^2)^2 + \frac{1}{v^2}(\mu^3)^2 \right]. \tag{4.1}$$

μ^a $(a = 1, 2, 3)$ are the left-invariant differentials summarized in Section C.2. We define symmetries $SU(2)_L$ and $SU(2)_R$ as left and right $SU(2)$ actions, respectively.

$$g \to g_L g g_R, \quad g_L \in SU(2)_L, \quad g_R \in SU(2)_R, \tag{4.2}$$

where g is an element of $SU(2)$. The parameter v in the metric (4.1) is the squashing parameter. For later convenience we also define u by

$$v^2 = 1 + u^2. \tag{4.3}$$

The round sphere corresponds to $v = 1$ and $u = 0$. The differentials μ^a are invariant under $SU(2)_L$, while they are transformed as a triplet under $SU(2)_R$. Therefore, when $v \neq 1$, the metric (4.1) breaks $SU(2)_R$ to its $U(1)$ subgroup, which is denoted by $U(1)_r$.

$\mathcal{N} = 2$ superconformal theories on round $\mathbf{S}^3$ have eight supersymmetries, and the squashing breaks some of them. [35] shows that it is possible to recover 1/4 of them (two supersymmetries) by turning on a Wilson line for the R-symmetry. It is important that the recovered supersymmetries are $SU(2)_L$ singlets. They computed the $\mathbf{S}^3$ partition function for such theories with the expectation that they may obtain a result depending on the squashing parameter in a non-trivial way. The result was rather disappointing. It was turned out that the partition function is identical to that on the round sphere up to some variable changes.

Having obtained this result, the authors of [35] moved on to study another model in which both $SU(2)_L$ and $SU(2)_R$ are broken. This squashed sphere is often called "ellipsoid." They again turn on an R-symmetry Wilson line to recover 1/4 supersymmetry, and compute the partition function. This time they obtained the 1-loop partition function

$$Z^{1-\text{loop}} = \frac{\prod_{\alpha\in\Delta} s_b\left(\frac{\alpha(\lambda)-i}{v}\right)}{\prod_I s_b\left(\frac{\rho_I(\lambda)-i(1-\Delta_I)}{v}\right)}. \tag{4.4}$$

with the parameter b depending on the squashing parameter of the ellipsoid in a certain way. The numerator is the contribution of vector multiplets, and α runs over all roots of the gauge algebra. The denominator contains the contribution of chiral multiplets. I labels chiral multiplets, and ρ_I and Δ_I are the weight vectors of the representation and the Weyl weight, respectively, of a chiral multiplet I. $s_b(z)$ is the double sine function defined by

$$s_b(z) = \prod_{p,q=0}^{\infty} \frac{b\left(q+\frac{1}{2}\right) + b^{-1}\left(p+\frac{1}{2}\right) - iz}{b\left(p+\frac{1}{2}\right) + b^{-1}\left(q+\frac{1}{2}\right) + iz}. \tag{4.5}$$

The features of the double sine function $s_b(z)$ is summarized in chapter F.

To understand the independence of the partition function on the squashing parameter of the $SU(2)_L \times U(1)_r$ symmetric squashing in [35], let us consider which modes of fields contribute to the partition function. Let us focus on a chiral multiplet. Its contribution to the 1-loop partition function is given by

$$Z^{1-\text{loop}} = \frac{\text{Det}\,\mathcal{D}_F}{\text{Det}\,\mathcal{D}_B}, \tag{4.6}$$

where $\mathcal{D}_B$ and $\mathcal{D}_F$ are certain differential operators appearing in the scalar and fermion actions. Their determinants are the products of eigenvalues of the differential operators. A complex scalar field on $\mathbf{S}^3$ can be expanded by scalar spherical harmonics, which belong to the $SU(2)_L \times SU(2)_R$ representation

$$\left(\bigoplus_{j=0}^{\infty}(j,j)_B\right) \oplus \left(\bigoplus_{j=0}^{\infty}(j,j)_B\right). \tag{4.7}$$

We use subscripts 'B' and 'F' to indicate the statistics of modes. Roughly speaking, the two summations correspond to particles and anti-particles. Similarly, a spinor field is expanded as

$$\left(\bigoplus_{j=0}^{\infty}(j+1/2,j)_F\right) \oplus \left(\bigoplus_{j=0}^{\infty}(j,j+1/2)_F\right). \tag{4.8}$$

Because of supersymmetry, the majority of these modes are paired between bosons and fermions, and their contribution to the partition function (4.6) cancel each other. If there exists an $SU(2)_L$ singlet supercharge, which is actually the case in the $SU(2)_L \times U(1)_r$ symmetric squashing in [35], the cancellation occurs between modes with the same $SU(2)_L$ quantum numbers:

$$(j,j)_B \leftrightarrow (j,j-1/2)_F, \quad \text{or} \quad (j,j)_B \leftrightarrow (j,j+1/2)_F. \tag{4.9}$$

In the first pair in (4.9) the number of bosonic modes in (j,j) is larger than that of the fermionic modes in $(j,j-1/2)$. After the cancellation, only the bosonic modes with the highest or lowest $SU(2)_R$ weight survive and contribute to the 1-loop partition function (4.6). Similarly, in the second pair in (4.9), only the fermionic modes with the highest or lowest $SU(2)_R$ weight contribute to the partition function (4.6). Thus, even if the $SU(2)_R$ symmetry is broken and the degeneracy in each $SU(2)_R$ multiplet is lost, it does not affect the structure of the partition function. This is also the case for vector multiplets.

From the arguments above, we notice that if we can realize squashing without $SU(2)_L$ singlet supercharges we may obtain the partition function depending on the squashing parameter in a non-trivial way even if the $\mathbf{S}^3$ is $SU(2)_L \times U(1)_r$-symmetric. To study such theories is a main purpose of this chapter. One way to construct such theories is to compactify 4d theories by $\mathbf{S}^1$. Let us consider a 4d $\mathcal{N} = 1$ superconformal theory on $\mathbf{S}^3 \times \mathbf{R}$. The isometry of this background is $SU(2)_L \times SU(2)_R \times \mathbf{R}$. The theory has eight supersymmetries, and it is possible to compactify $\mathbf{R}$ to $\mathbf{S}^1$ with preserving four supersymmetries belonging to $SU(2)_L$ doublets[33]. Through this compactification, we can relate the $\mathbf{S}^3$ partition function to the 4d superconformal index[31, 32, 33]. It is pointed out in [32] that if we turn on the $SU(2)_R$ Wilson line, we can reproduce 1-loop partition function (4.4) with $b \neq 1$ from 4d superconformal index. The 3d theory obtained by such a compactification is a theory in squashed $\mathbf{S}^3$ with $SU(2)_L \times U(1)_r$ isometry, and is different from the theories studied in [35]. We give the supersymmetry transformation laws and Lagrangians on the squashed sphere, and compute the partition function.

This chapter is organized as follows. In Section 4.1, we give the supersymmetry transformation laws and supersymmetric Lagrangians without derivations. In Section 4.2 we compute the 1-loop partition function and obtain (4.4) with the parameter

$$b = \frac{1 + iu}{v}. \tag{4.10}$$

In Section 4.3, we explain how we can derive the transformation laws and Lagrangians given in Section 4.1 by the dimensional reduction from 4d theory. In Section 4.4 we study the free energy of large N quiver gauge theories which are expected to have M-theory duals.

Before ending the introduction, we summarize our conventions and notations. We use the $SU(2)_L$-invariant local frame on the squashed sphere with the vielbein

$$e^{\widehat{1}} = r\mu^1, \quad e^{\widehat{2}} = r\mu^2, \quad e^{\widehat{3}} = \frac{r}{v}\mu^3. \tag{4.11}$$

We use Roman characters $k, l, m, n, \ldots, = 1, 2, 3$ for 3d tangent indices, and hatted characters $\widehat{k}, \widehat{l}, \widehat{m}, \widehat{n}, \ldots, = \widehat{1}, \widehat{2}, \widehat{3}$ for local indices.

4.1 $\mathcal{N} = 2$ supersymmetry on the squashed sphere

4.1.1 Transformation laws

$\mathcal{N} = 2$ superconformal theories on round $\mathbf{S}^3$ have eight supercharges. If we turn on real mass parameters, half of the supersymmetries are broken, and we call the unbroken part $\mathcal{N} = 2$ supersymmetry. It is possible to squash the $\mathbf{S}^3$ in such a way that the $\mathcal{N}=2$ supersymmetry is preserved. Killing spinors ϵ and $\bar{\epsilon}$ for the four supersymmetries satisfy the Killing equations

$$D_m\epsilon = -\frac{i}{2vr}\gamma_m\epsilon + \frac{u}{vr}f^n\gamma_{mn}\epsilon,$$
$$D_m\bar{\epsilon} = -\frac{i}{2vr}\gamma_m\bar{\epsilon} - \frac{u}{vr}f^n\gamma_{mn}\bar{\epsilon}, \tag{4.12}$$

where we define the vector field

$$f^m = e^m_{\hat{3}}. \tag{4.13}$$

This vector field generates $U(1)_r$ isometry. Each of the differential equations in (4.12) has two linearly independent solutions which form an $SU(2)_L$ doublet. An explicit form of the solutions are

$$\epsilon = e^{-\theta T_3}g^{-1}\epsilon_0, \quad \bar{\epsilon} = e^{\theta T_3}g^{-1}\bar{\epsilon}_0, \tag{4.14}$$

where ϵ_0 and $\bar{\epsilon}_0$ are arbitrary constant spinors, ,θ is the angle defined by $e^{i\theta} = (1+iu)/v$ and g is the element of $SU(2)$ (see Section C.3).

Supersymmetry transformation laws for component fields of vector multiplets are

$$\begin{aligned}
\delta A_m &= i(\epsilon\gamma_m\bar{\lambda}) - i(\bar{\epsilon}\gamma_m\lambda) + uf_m(\epsilon\bar{\lambda}) + uf_m(\bar{\epsilon}\lambda),\\
\delta\sigma &= v(\epsilon\bar{\lambda}) + v(\bar{\epsilon}\lambda),\\
\delta\lambda &= -\mathcal{F}^{(+)}_{\hat{m}}\gamma_{\hat{m}}\epsilon + D\epsilon,\\
\delta\bar{\lambda} &= \mathcal{F}^{(-)}_{\hat{m}}\gamma_{\hat{m}}\bar{\epsilon} + D\bar{\epsilon},\\
\delta D &= -(\epsilon\gamma^m D_m\bar{\lambda}) + \frac{i}{2vr}(\epsilon\bar{\lambda}) + \frac{1}{v}(\epsilon(1-iu\not{f})[\sigma,\bar{\lambda}])\\
&\quad - (\bar{\epsilon}\gamma^m D_m\lambda) + \frac{i}{2vr}(\bar{\epsilon}\lambda) - \frac{1}{v}(\bar{\epsilon}(1+iu\not{f})[\sigma,\lambda]),
\end{aligned} \tag{4.15}$$

where $\not{f} = f^m\gamma_m$ and $\mathcal{F}^{(\pm)}_{\hat{m}}$ are defined by

$$\mathcal{F}^{(\pm)}_{\hat{m}} = \frac{1}{2}\epsilon_{\hat{m}\hat{p}\hat{q}}F_{\hat{p}\hat{q}} + \frac{u}{v}f^{\hat{p}}\epsilon_{\hat{m}\hat{p}\hat{n}}D_{\hat{n}}\sigma \pm \frac{1}{v}D_{\hat{m}}\sigma. \tag{4.16}$$

Transformation laws for component fields in a chiral multiplet with Weyl weight Δ

are

$$
\begin{aligned}
\delta\phi &= \sqrt{2}(\epsilon\psi), \\
\delta\phi^\dagger &= \sqrt{2}(\overline{\epsilon}\overline{\psi}), \\
\delta\psi &= -\sqrt{2}\gamma^m\overline{\epsilon}D_m\phi + \frac{\sqrt{2}}{v}(1-iu\not{f})\overline{\epsilon}\sigma\phi + \sqrt{2}\epsilon F + \frac{\sqrt{2}\Delta i}{vr}(1-iu\not{f})\overline{\epsilon}\phi, \\
\delta\overline{\psi} &= -\sqrt{2}\gamma^m\epsilon D_m\phi^\dagger + \frac{\sqrt{2}}{v}(1+iu\not{f})\epsilon\phi^\dagger\sigma + \sqrt{2}\overline{\epsilon}F^\dagger + \frac{\sqrt{2}\Delta i}{vr}(1+iu\not{f})\epsilon\phi^\dagger, \\
\delta F &= -\sqrt{2}D_m(\overline{\epsilon}\gamma^m\psi) - \frac{\sqrt{2}(\Delta-2)i}{vr}(\overline{\epsilon}(1+iu\not{f})\psi) \\
&\qquad - \frac{\sqrt{2}}{v}(\overline{\epsilon}(1+iu\not{f})\sigma\psi) - 2(\overline{\epsilon}\overline{\lambda})\phi, \\
\delta F^\dagger &= -\sqrt{2}D_m(\epsilon\gamma^m\overline{\psi}) - \frac{\sqrt{2}(\Delta-2)i}{vr}(\epsilon(1-iu\not{f})\overline{\psi}) \\
&\qquad - \frac{\sqrt{2}}{v}(\epsilon(1-iu\not{f})\overline{\psi})\sigma - 2\phi^\dagger(\epsilon\lambda).
\end{aligned} \tag{4.17}
$$

The commutation relation of the two transformations $\delta(\epsilon,\overline{\epsilon})$ and $\delta(\epsilon',\overline{\epsilon}')$ is

$$
[\delta(\epsilon,\overline{\epsilon}),\delta(\epsilon',\overline{\epsilon}')] = 2\mathcal{L}_{l'} + 2\alpha\left(-i\sigma + \frac{R}{r}\right). \tag{4.18}
$$

R is the R charge, and σ should be understood as the gauge transformation with parameter σ. l' and α are bilinear of the transformation parameters

$$
l'^m = (\overline{\epsilon}\gamma^m\epsilon') + (\epsilon\gamma^m\overline{\epsilon}'), \quad \alpha = \frac{i}{v}\overline{\epsilon}(1+iu\not{f})\epsilon' - \frac{i}{v}\epsilon(1-iu\not{f})\overline{\epsilon}', \tag{4.19}
$$

and $\mathcal{L}_v$ is the Lie derivative associated with a vector field v. It is easily shown by the Killing equations (4.12) that l'^m is a Killing vector and α is a constant on the squashed sphere. l'^m can be divided into a $SU(2)_L$ part l^m and $U(1)_r$ part proportional to f^m:

$$
l'^m = l^m - \frac{u}{v}\alpha f^m. \tag{4.20}
$$

The right hand side in (4.18) contains generators of $SU(2)_L$, $U(1)_r$, and $U(1)_R$. $U(1)_r$ does not rotate the supercharges, and thus is the center of the algebra. Therefore, the supersymmetry algebra on the squashed sphere is $SU(2|1)\ltimes U(1)_r$, a central extension of $SU(2|1)$. If we regard the 3d theory as an $\mathbf{S}^1$ compactification of a 4d theory, α can be regarded as the parameter of a shift along the 4-th direction. If we substitute (4.20) into (4.18), we have $U(1)_r$ transformation with α in the coefficient. This implies the existence of non-vanishing graviphoton background field. From the 4d perspective, a graviphoton field is, roughly speaking, identified with the non-diagonal components g_{m4} of the metric. When the background graviphoton field is non-vanishing, the compactified direction x^4 is tilted, and shift along x^4 generates a shift in 3d proportional to the graviphoton potential field when it is projected onto 3d. (4.20) implies that the graviphoton field in our background is given by

$$
V^m = \frac{u}{v}f^m. \tag{4.21}
$$

We will see in Section 4.3 that the graviphoton field (4.21) is indeed arises in the compactification.

4.1.2 Actions

The supersymmetric kinetic Lagrangian for vector multiplet is

$$\mathcal{L}_{\rm YM}=\mathcal{L}_{\mathcal{A}}+\mathcal{L}_\lambda-\frac{1}{2}{\rm tr}D^2, \tag{4.22}$$

where $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{L}_\lambda$ are bosonic and fermionic terms given by

$$\begin{aligned}\mathcal{L}_{\mathcal{A}}&=\frac{1}{2}{\rm tr}(\mathcal{F}^{(-)}_{\widehat{m}}\mathcal{F}^{(-)}_{\widehat{m}}),\\ \mathcal{L}_\lambda&={\rm tr}\left[-\overline{\lambda}\gamma^mD_m\lambda+\frac{i}{2vr}\overline{\lambda}\lambda-\frac{1}{v}\overline{\lambda}(1+iu\not{f})[\sigma,\lambda]\right].\end{aligned} \tag{4.23}$$

tr is a positive definite gauge invariant inner product of the gauge algebra.

The supersymmetric kinetic Lagrangian for chiral multiplet with Weyl weight Δ is

$$\mathcal{L}_{\rm chiral}=\mathcal{L}_\phi+\mathcal{L}_\psi-F^\dagger F, \tag{4.24}$$

where $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ are given by

$$\begin{aligned}\mathcal{L}_\phi&=-\phi^\dagger D_mD^m\phi+\phi^\dagger\sigma\sigma\phi+\phi^\dagger D\phi-\frac{\Delta^2-2\Delta}{r^2}\phi^\dagger\phi+\frac{2i(\Delta-1)}{r}\phi^\dagger\sigma\phi\\ &\quad+\frac{u}{v}f^m\left[-i\phi^\dagger\sigma D_m\phi-i\phi^\dagger D_m(\sigma\phi)+\frac{2(\Delta-1)}{r}\phi^\dagger D_m\phi\right],\\ \mathcal{L}_\psi&=-(\overline{\psi}\gamma^mD_m\psi)+\frac{i}{2vr}(\overline{\psi}\psi)-\left(\overline{\psi}\frac{i(\Delta-ir\sigma)}{vr}(1+iu\not{f})\psi\right)\\ &\quad-\sqrt{2}\phi^\dagger(\lambda\psi)-\sqrt{2}(\overline{\psi}\overline{\lambda})\phi.\end{aligned} \tag{4.25}$$

Let $\overline{\epsilon}_1$ and $\overline{\epsilon}_2$ be two independent solutions of the second equation in (4.12). The kinetic Lagrangians (4.22) and (4.24) can be obtained from

$$(\overline{\epsilon}_1\overline{\epsilon}_2)\mathcal{L}_{\rm YM}=-\frac{1}{4}\delta(\overline{\epsilon}_1)\delta(\overline{\epsilon}_2){\rm tr}(\overline{\lambda}\overline{\lambda}),\quad(\overline{\epsilon}_1\overline{\epsilon}_2)\mathcal{L}_{\rm chiral}=-\frac{1}{2}\delta(\overline{\epsilon}_1)\delta(\overline{\epsilon}_2)(\phi^\dagger F). \tag{4.26}$$

Because $\delta(\overline{\epsilon}_1)$ and $\delta(\overline{\epsilon}_2)$ commute with each other the right hand side of these equations contains the parameters $\overline{\epsilon}_1$ and $\overline{\epsilon}_2$ only through the scalar product $(\overline{\epsilon}_1\overline{\epsilon}_2)$, and these equations consistently define the Lagrangians $\mathcal{L}_{\rm YM}$ and $\mathcal{L}_{\rm chiral}$. These Lagrangians do not depend on the choice of two independent Killing spinors $\overline{\epsilon}_1$ and $\overline{\epsilon}_2$, and they are exact with respect to $\delta(\overline{\epsilon})$ for any $\overline{\epsilon}$ satisfying (4.12).

The supersymmetric completion of the Chern-Simons term and the FI term are

$$\begin{aligned}\mathcal{L}_{\rm CS}&={\rm tr}_{\rm CS}\Bigg[\frac{i}{2}\epsilon^{mnp}\left(A_m\partial_nA_p-\frac{2i}{3}A_mA_nA_p\right)\\ &\qquad\qquad+(\overline{\lambda}\lambda)-\frac{1}{v}D\sigma+\frac{i}{vr}\sigma^2-\frac{iu}{2v}\sigma\epsilon^{mnp}f_mF_{np}\Bigg],\\ \mathcal{L}_{\rm FI}&=-{\rm tr}_{\rm FI}\left[D-\frac{2i}{r}\sigma+\frac{2ui}{vr}f^mA_m\right],\end{aligned} \tag{4.27}$$

where ${\rm tr}_{\rm CS}$ is a gauge invariant inner product of Lie algebra, which does not have to be positive definite, and ${\rm tr}_{\rm FI}$ is a gauge invariant linear map from the gauge algebra to $\mathbf{R}$.

4.2 Partition function

In this section we compute the partition function of a theory on the squashed $\mathbf{S}^3$. Because of the $\delta(\bar{\epsilon})$-exactness of the kinetic Lagrangians $\mathcal{L}_{\rm YM}$ and $\mathcal{L}_{\rm chiral}$, we can send the coefficients of these Lagrangians to infinity without changing the partition function. The theory becomes free in this limit, and we can perform the path integral to obtain the expression (1.11) of the partition function.

4.2.1 Mode expansion on squashed $\mathbf{S}^3$

Let $\Phi(g)$ be a spin s field on the squashed sphere. We expand it by the spin basis $|s, s_z\rangle$ ($s_z = -s, -s+1, \ldots, s$)

$$\Phi(g) = \sum_{s_z=-s}^{s} \Phi_{s_z}(g)|s, s_z\rangle. \tag{4.28}$$

Because we are using the $SU(2)_L$-invariant frame, $|s, s_z\rangle$ are transformed as the $(0, s)$ representation of $SU(2)_L \times SU(2)_R$. $\Phi_{s_z}(g)$ for each s_z is a scalar function on $\mathbf{S}^3$, and can be expanded by the scalar spherical harmonics $Y^j_{m',m}(g)$ as

$$\Phi_{s_z}(g) = \sum_{j,m',m} \Phi^j_{s_z,m',m} Y^j_{m',m}(g). \tag{4.29}$$

The harmonics $Y^j_{m',m}$ belong to the (j, j) representation of $SU(2)_L \times SU(2)_R$. j is the common azimuthal quantum number for both $SU(2)_L$ and $SU(2)_R$, and m' and m are magnetic quantum numbers for $SU(2)_L$ and $SU(2)_R$, respectively. They take values

$$\begin{aligned} j &= 0, \frac{1}{2}, 1, \ldots, \\ m &= -j, -j+1, \ldots, j-1, j, \\ m' &= -j, -j+1, \ldots, j-1, j. \end{aligned} \tag{4.30}$$

In the following, we use the ket notation for the harmonics $Y^j_{m',m}(g)$

$$|j, m', m\rangle = Y^j_{m',m}(g). \tag{4.31}$$

The expansion of the field $\Phi(g)$ is expressed as

$$\Phi(g) = \sum_{j,m',m,s_z} \Phi^j_{s_z,m',m}|j, m', m\rangle \otimes |s, s_z\rangle. \tag{4.32}$$

The covariant derivative on round $\mathbf{S}^3$ with the left-invariant frame acts on the field $\Phi(g)$ as

$$D^{(0)} = \mu^a(2L_a + S_a), \tag{4.33}$$

where L_a and S_a are $SU(2)$ generators. L_a are the $SU(2)_R$ orbital angular momenta acting on the $SU(2)_R$ index m of $|j, m', m\rangle$, and S_a are the spin operators acting on $|s, s_z\rangle$. These operators are normalized so as to satisfy the commutation relation (C.25).

The covariant derivative on the squashed sphere is obtained from $D^{(0)}$ by replacing the spin connection on the round sphere, $\omega_{(0)}^{\widehat{m}\widehat{n}} = \epsilon_{\widehat{m}\widehat{n}\widehat{p}}\mu^p$, by $\omega^{\widehat{m}\widehat{n}}$, the spin connection on the squashed sphere. $\omega^{\widehat{m}\widehat{n}}$ and $\omega_{(0)}^{\widehat{m}\widehat{n}}$ are related by

$$\begin{aligned}
\omega^{\widehat{1}\widehat{2}} &= \left(2 - \frac{1}{v^2}\right)\mu^3 = \omega_{(0)}^{\widehat{1}\widehat{2}} + \left(1 - \frac{1}{v^2}\right)\mu^3, \\
\omega^{\widehat{2}\widehat{3}} &= \frac{1}{v}\mu^1 = \omega_{(0)}^{\widehat{2}\widehat{3}} + \left(\frac{1}{v} - 1\right)\mu^1, \\
\omega^{\widehat{3}\widehat{1}} &= \frac{1}{v}\mu^2 = \omega_{(0)}^{\widehat{3}\widehat{1}} + \left(\frac{1}{v} - 1\right)\mu^2.
\end{aligned} \tag{4.34}$$

Combining (4.33) and (4.34), we obtain the following algebraic expression for the covariant derivative on the squashed sphere.

$$D = \mu^1\left(2L_1 + \frac{1}{v}S_1\right) + \mu^2\left(2L_2 + \frac{1}{v}S_2\right) + \mu^3\left[2L_3 + \left(2 - \frac{1}{v^2}\right)S_3\right]. \tag{4.35}$$

The non-vanishing components of the spin j representation matrices for generators L_a are

$$\begin{aligned}
\langle j, m', m|L_3|j, m', m\rangle &= im, \\
\langle j, m', m + \frac{1}{2}|L_{1+i2}|j, m', m - \frac{1}{2}\rangle &= i\sqrt{(j + \frac{1}{2})^2 - m^2}, \\
\langle j, m', m - \frac{1}{2}|L_{1-i2}|j, m', m + \frac{1}{2}\rangle &= i\sqrt{(j + \frac{1}{2})^2 - m^2},
\end{aligned} \tag{4.36}$$

where $L_{1\pm i2} \equiv L_1 \pm iL_2$. We also introduce $SU(2)_L$ generators L'_a. The non-vanishing components of L'_3 are

$$\langle j, m', m|L'_3|j, m', m\rangle = im'. \tag{4.37}$$

In the following subsections we compute the determinant of certain differential operators appearing in the Lagrangians. Because the squashed background preserves $SU(2)_L$ and $U(1)_r$, the differential operators commute with operators $L'_aL'_a$, L'_3, and $L_3 + S_3$. Therefore, we can compute the determinant in each eigenspace defined by

$$L'_aL'_a = -j(j+1), \quad L'_3 = im', \quad L_3 + S_3 = im. \tag{4.38}$$

Because L_a and L'_a act on scalar spherical harmonics, $L'_aL'_a = L_aL_a$ holds. This restriction generically defines $2s + 1$ dimensional vector space spanned by

$$\{|j, m', m - s_z\rangle \otimes |s, s_z\rangle\}_{s_z=-s}^{s}, \tag{4.39}$$

and the differential operator reduces to a $(2s+1) \times (2s+1)$ matrix on this subspace. If m is close to $\pm j$ and some $m - s_z$ are out of the allowed range in (4.30), special treatment is needed.

4.2.2 Bosons in vector multiplets

Because of the $\delta(\bar{\epsilon})$-exactness of $\mathcal{L}_{\rm YM}$, we can add $\mathcal{L}_{\rm YM}$ to the Lagrangian of the theory with an arbitrary coefficient without changing the partition function. In the limit in which the coefficient goes to infinity, the path integral for vector multiplet reduces to the Gaussian integral around the saddle points. Let us start with the bosonic part. Saddle points are given by $\mathcal{F}_{\widehat{m}} = D = 0$. This is the case iff

$$A_m = D = 0, \quad \sigma = \sigma_0, \tag{4.40}$$

up to gauge transformations. σ_0 is a constant expectation value of σ, and we assume that it is diagonalized by gauge transformations. At saddle points, the classical values of the Chern-Simons term and FI term in (4.27) are

$$\begin{aligned} S^{\rm cl}_{\rm CS}(\sigma_0) &= \int d^3x\sqrt{g}\mathcal{L}^{\rm cl}_{\rm CS}(\sigma_0) = \frac{2\pi^2 i r^2}{v^2}{\rm tr}_{\rm CS}(\sigma_0^2), \\ S^{\rm cl}_{\rm FI}(\sigma_0) &= \int d^3x\sqrt{g}\mathcal{L}^{\rm cl}_{\rm FI}(\sigma_0) = \frac{4\pi^2 i r^2}{v}{\rm tr}_{\rm FI}(\sigma_0). \end{aligned} \tag{4.41}$$

We define the fluctuation part of the scalar field

$$\varphi = \sigma - \sigma_0. \tag{4.42}$$

The path integral of the auxiliary field D gives constant, and we ignore its contribution.

All component fields in the vector multiplet belong to the adjoint representation of the gauge group G, and have $\dim G$ components. In the following, we focus on one component in each field that satisfies $[\sigma_0, \Phi] = \alpha(\sigma_0)\Phi$. To obtain the final expression, we need to take the product over all weights α in the adjoint representation.

To fix the gauge we introduce the gauge fixing function

$$f = D_{\widehat{m}} A_{\widehat{m}}, \tag{4.43}$$

and add the gauge fixing term

$$\mathcal{L}_{\rm GF} = \frac{1}{2}{\rm tr} f^2, \tag{4.44}$$

to the Lagrangian. We still have residual gauge symmetry with constant transformation parameters. This residual symmetry is fixed by requiring the constant mode of the scalar field σ_0 to be diagonal. The Jacobian factor associated with this gauge fixing is the Vandermonde determinant

$$\prod_{\alpha\in\Delta} \alpha(\sigma_0). \tag{4.45}$$

We should include this factor to the result of the path integral below.

Let us define four-component field $\mathcal{A} = (A_{\widehat{1}}, A_{\widehat{2}}, A_{\widehat{3}}, \varphi)^T$. In the following we ignore higher order terms with respect to the fluctuation fields. The quadratic part of $\mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\rm GF}$ with respect to $\mathcal{A}$ is

$$\mathcal{L}_{\mathcal{A}} + \mathcal{L}_{\rm GF} = \frac{1}{2r^2}(\mathcal{D}_{\mathcal{A}}\mathcal{A})^T(\mathcal{D}_{\mathcal{A}}\mathcal{A}), \tag{4.46}$$

where the differential operator $\mathcal{D}_{\mathcal{A}}$ is defined by

$$\mathcal{D}_{\mathcal{A}} \begin{pmatrix} A_{\widehat{3}} \\ A_{\widehat{1}+i\widehat{2}} \\ A_{\widehat{1}-i\widehat{2}} \\ \varphi \end{pmatrix} = r \begin{pmatrix} \mathcal{F}^{(-)}_{\widehat{3}} \\ \mathcal{F}^{(-)}_{\widehat{1}+i\widehat{2}} \\ \mathcal{F}^{(-)}_{\widehat{1}-i\widehat{2}} \\ f \end{pmatrix}. \tag{4.47}$$

By using (4.35) with spin 1 representation matrix $(S_a)_{\widehat{b}\widehat{c}} = \epsilon_{abc}$, we can rewrite the definition of $\mathcal{F}^{(-)}_{\widehat{m}}$ in (4.16) in the algebraic form

$$\begin{aligned} r\mathcal{F}^{(-)}_{\widehat{3}} &= \frac{2 - ir\alpha(\sigma_0)}{v} A_{\widehat{3}} - iL_{1-i2}A_{\widehat{1}+i\widehat{2}} + iL_{1+i2}A_{\widehat{1}-i\widehat{2}} - \frac{1}{v}2vL_3\varphi, \\ r\mathcal{F}^{(-)}_{\widehat{1}+i\widehat{2}} &= -2iL_{1+i2}A_{\widehat{3}} + \left[2v(1+iL_3) - \frac{1-iu}{v}ir\alpha(\sigma_0)\right] A_{\widehat{1}+i\widehat{2}} - \frac{1-iu}{v}2L_{1+i2}\varphi, \\ r\mathcal{F}^{(-)}_{\widehat{1}-i\widehat{2}} &= 2iL_{1-i2}A_{\widehat{3}} + \left[2v(1-iL_3) - \frac{1+iu}{v}ir\alpha(\sigma_0)\right] A_{\widehat{1}-i\widehat{2}} - \frac{1+iu}{v}2L_{1-i2}\varphi. \end{aligned} \tag{4.48}$$

We also rewrite the gauge fixing function (4.43) as

$$rf = 2vL_3A_{\widehat{3}} + L_{1-i2}A_{\widehat{1}+i\widehat{2}} + L_{1+i2}A_{\widehat{1}-i\widehat{2}}. \tag{4.49}$$

The algebraic form of $\mathcal{D}_{\mathcal{A}}$ is

$$\mathcal{D}_{\mathcal{A}} = \begin{pmatrix} \frac{2-ir\alpha(\sigma_0)}{v} & -iL_{1-i2} & iL_{1+i2} & -2L_3 \\ -2iL_{1+i2} & 2v(1+iL_3) - \frac{1-iu}{v}ir\alpha(\sigma_0) & 0 & -\frac{1-iu}{v}2L_{1+i2} \\ 2iL_{1-i2} & 0 & 2v(1-iL_3) - \frac{1+iu}{v}ir\alpha(\sigma_0) & -\frac{1+iu}{v}2L_{1-i2} \\ 2vL_3 & L_{1-i2} & L_{1+i2} & 0 \end{pmatrix}. \tag{4.50}$$

By restriction to the subspace defined by (4.38), the operator (4.50) becomes 4×4 matrix with each component being a complex number. Its determinant is

$$\det \mathcal{D}_{\mathcal{A}} = \frac{4[j(j+1)+u^2m^2]}{v}(2j+2imu+ir\alpha(\sigma_0))(2j+2-2imu-ir\alpha(\sigma_0)). \tag{4.51}$$

(We use "det" for the determinant of the matrix defined in the subspace (4.38), and "Det" for the functional determinant of differential operators.) We need to divide this by the Jacobian factor associated with the gauge fixing. The algebraic form of the gauge transformation of $\mathcal{A}$ is

$$\delta\mathcal{A} = \begin{pmatrix} \delta A_{\widehat{a}} = D_{\widehat{a}}\lambda \\ \delta\varphi = i[\lambda, \sigma_0] \end{pmatrix} = \frac{1}{r} \begin{pmatrix} 2vL_3 \\ 2L_{1-i2} \\ 2L_{1+i2} \\ -ir\alpha(\sigma_0) \end{pmatrix} \lambda. \tag{4.52}$$

Substituting this into (4.49), we obtain the Jacobian

$$r\frac{\delta f}{\delta\lambda} = -4[j(j+1)+u^2m^2]. \tag{4.53}$$

Therefore, the path integral of physical modes in the restricted vector space with the quantum numbers in (4.38) gives[1]

$$\det{}' \mathcal{D}_{\mathcal{A}} = \frac{\det \mathcal{D}_{\mathcal{A}}}{r\delta f/\delta\lambda} = (2j + 2imu + ir\alpha(\sigma_0))(2j + 2 - 2imu - ir\alpha(\sigma_0)) \qquad (4.54)$$

The two factors in (4.54) correspond to the first two irreducible representation in the decomposition

$$(j, j) \otimes (0, 1) = (j, j-1) \oplus (j, j+1) \oplus (j, j). \qquad (4.55)$$

The last representation corresponds to the gauge degrees of freedom. By taking the product over quantum numbers j, m, and m', we obtain

$$\mathrm{Det}' \mathcal{D}_{\mathcal{A}} = \prod_j \prod_{|m|\le j-1} (2j + 2imu + ir\alpha(\sigma_0))^{2j+1} \prod_{|m|\le j+1} (2j + 2 - 2imu - ir\alpha(\sigma_0))^{2j+1}. \qquad (4.56)$$

Because the four supersymmetries are $SU(2)_R$ singlets, the cancellation between bosons and fermions occurs among the modes with the same $SU(2)_R$ quantum numbers. For this reason, we shift the quantum number j so that the $SU(2)_R$ spins become j. Namely, in the first factor in (4.56) we replace j by $j+1$, and in the second factor by $j-1$. Correspondingly, the first two representations in (4.55) become

$$(j + 1, j) \oplus (j - 1, j) \qquad (4.57)$$

After this shift we obtain

$$\mathrm{Det}' \mathcal{D}_{\mathcal{A}} = \prod_j \prod_{|m|\le j} (2j + 2 + 2imu + ir\alpha(\sigma_0))^{2j+3} (2j - 2imu - ir\alpha(\sigma_0))^{2j-1}. \qquad (4.58)$$

Up to now, we have not specified the region of the spin j. The product with respect to j should be taken over the region for which the spins in (4.57) are non-negative. This means that for the first factor in (4.58) we take $j = 0, 1/2, \ldots$ and for the second factor $j = 1, 3/2, \ldots$. By taking account of this, we obtain

$$\mathrm{Det}' \mathcal{D}_{\mathcal{A}} = (-ir\alpha(\sigma_0)) \prod_{j=0}^{\infty} \prod_{|m|\le j} (2j+2+2imu+ir\alpha(\sigma_0))^{2j+3} (2j-2imu-ir\alpha(\sigma_0))^{2j-1}. \qquad (4.59)$$

The factor $-ir\alpha(\sigma_0)$ is inserted to remove the unwanted contribution of the second factor with $j = 0$.

4.2.3 Fermions in vector multiplets

The action for the fermion field λ at the saddle point (4.40) is

$$\mathcal{L}_\lambda = \frac{1}{r} \overline{\lambda} \mathcal{D}_\lambda \lambda, \qquad (4.60)$$

[1] We ignore constant factor $(-1/v)$.

where the differential operator $\mathcal{D}_\lambda$ is given by

$$\begin{aligned}
\mathcal{D}_\lambda &= -r\gamma^m D_m + \frac{i}{2v} - \frac{1}{v}(1+iu\gamma_{\hat{3}})r\alpha(\sigma_0) \\
&= -2\gamma_{\hat{1}}L_1 - 2\gamma_{\hat{2}}L_2 - 2v\gamma_{\hat{3}}L_3 - iv - \frac{1}{v}(1+iu\gamma_{\hat{3}})r\alpha(\sigma_0) \\
&= \begin{pmatrix} -2vL_3 - iv - \frac{1+iu}{v}r\alpha(\sigma_0) & -2L_{1-i2} \\ -2L_{1+i2} & 2vL_3 - iv - \frac{1-iu}{v}r\alpha(\sigma_0) \end{pmatrix}. \qquad (4.61)
\end{aligned}$$

In the subspace with the quantum numbers (4.38), this becomes 2×2 matrix with the determinant

$$\det \mathcal{D}_\lambda = (2j+1+ir\alpha(\sigma_0)+2imu)(2j+1-ir\alpha(\sigma_0)-2imu). \qquad (4.62)$$

The first and the second factor correspond to the two irreducible representations in

$$(j,j)\otimes(0,\frac{1}{2}) = (j,j-\frac{1}{2})\oplus(j,j+\frac{1}{2}). \qquad (4.63)$$

By taking the product over all possible quantum numbers and ignoring a constant factor, we obtain

$$\mathrm{Det}\,\mathcal{D}_\lambda = \prod_j \prod_{|m|\le j-1/2} (2j+1+ir\alpha(\sigma_0)+2imu)^{2j+1} \prod_{|m|\le j+1/2} (2j+1-ir\alpha(\sigma_0)-2imu)^{2j+1}. \qquad (4.64)$$

Let us shift j by $\pm 1/2$ so that the $SU(2)_R$ spin of the two representations become the same

$$(j+\frac{1}{2},j)\oplus(j-\frac{1}{2},j). \qquad (4.65)$$

After the shift, the determinant becomes

$$\mathrm{Det}\,\mathcal{D}_\lambda = \prod_{j=0}^{\infty}\prod_{|m|\le j}(2j+2+ir\alpha(\sigma_0)+2imu)^{2j+2}(2j-ir\alpha(\sigma_0)-2imu)^{2j}. \qquad (4.66)$$

Combining (4.59), (4.66), and the Vandermonde determinant (4.45), we obtain

$$\begin{aligned}
Z^{\text{1-loop}}_{\text{vector}}(\sigma_0) &= \prod_{\alpha\in\Delta}\frac{\mathrm{Det}\,\mathcal{D}_\lambda}{\mathrm{Det}'\,\mathcal{D}_\mathcal{A}}\prod_{\alpha\in\Delta}\alpha(\sigma_0) \\
&= \prod_{\alpha\in\Delta}\prod_j\prod_{|m|\le j}\frac{2j-ir\alpha(\sigma_0)-2imu}{2j+2+ir\alpha(\sigma_0)+2imu}. \qquad (4.67)
\end{aligned}$$

If we set

$$j=\frac{p+q}{2}, \quad m=\frac{p-q}{2}, \qquad (4.68)$$

we obtain

$$\begin{aligned}
Z^{\text{1-loop}}_{\text{vector}}(\sigma_0) &= \prod_{\alpha\in\Delta}\prod_{p,q=0}^{\infty}\frac{(1-iu)p+(1+iu)q+1-i(r\alpha(\sigma_0)-i)}{(1+iu)p+(1-iu)q+1+i(r\alpha(\sigma_0)-i)} \\
&= \prod_{\alpha\in\Delta} s_b\left(\frac{r\alpha(\sigma_0)-i}{v}\right). \qquad (4.69)
\end{aligned}$$

This is the same as the numerator in (4.4) with b in (4.10).

4.2.4 Bosons in chiral multiplets

We can reduce the path integral with respect to chiral multiplets to Gaussian integrals by sending the coefficient of $\mathcal{L}_{\text{chiral}}$ to infinity.

Let us compute the contribution of bosonic fields in a chiral multiplet with Weyl weight Δ belonging to a gauge representation $\mathcal{R}$. The path integral of the auxiliary field F gives constant, and we can neglect it.

Let us assume that ϕ is eigenmode of σ_0 and $\sigma_0\phi = \rho(\sigma_0)\phi$, where ρ is a weight in the representation $\mathcal{R}$. At the saddle point (4.40), the scalar Lagrangian is

$$\mathcal{L}_\phi = \frac{1}{r^2}\phi^\dagger \mathcal{D}_\phi \phi, \tag{4.70}$$

where the differential operator $\mathcal{D}_\phi$ is given by

$$\mathcal{D}_\phi = -r^2 D_m D^m - (ir\rho(\sigma_0) - \Delta + 2)(ir\rho(\sigma_0) - \Delta) - \frac{2u}{v}(ir\rho(\sigma_0) - \Delta + 1) r D_{\hat{3}}. \tag{4.71}$$

We expand the scalar field with $\mathbf{S}^3$ spherical harmonics $|j, m', m\rangle$. These harmonics are eigenfunctions of the Laplacian $D_m D^m$ and $D_{\hat{3}}$.

$$\begin{aligned} r^2 D_m D^m |j, m', m\rangle &= (-4j(j+1) - 4u^2 m^2)|j, m', m\rangle, \\ r D_{\hat{3}} |j, m', m\rangle &= 2ivm|j, m', m\rangle. \end{aligned} \tag{4.72}$$

The eigenvalue of the differential operator $\mathcal{D}_\phi$ in the subspace defined by (4.38) is

$$\mathcal{D}_\phi = (2j + ir\rho(\sigma_0) - \Delta + 2 + 2ium)(2j - ir\rho(\sigma_0) + \Delta - 2ium). \tag{4.73}$$

By taking the product over all possible quantum numbers, we obtain the determinant of the differential operator

$$\operatorname{Det}\mathcal{D}_\phi = \prod_{j=0}^{\infty}\prod_{|m|\le j}(2j + ir\rho(\sigma_0) - \Delta + 2 + 2ium)^{2j+1}(2j - ir\rho(\sigma_0) + \Delta - 2ium)^{2j+1}. \tag{4.74}$$

4.2.5 Fermions in chiral multiplets

The linearized action of fermion fields ψ and $\overline{\psi}$ at the saddle point (4.40) is

$$\mathcal{L}_\psi = \frac{1}{r}(\overline{\psi}\mathcal{D}_\psi \psi), \tag{4.75}$$

where the differential operator $\mathcal{D}_\psi$ is given by

$$\mathcal{D}_\psi = -r\gamma^m D_m + \frac{i}{2v} - \frac{i(\Delta - ir\rho(\sigma_0))}{v}(1 + iu\gamma^{\hat{3}}). \tag{4.76}$$

By using (4.35), we can rewrite this operator in the algebraic form

$$\begin{aligned} \mathcal{D}_\psi &= -\gamma_{\hat{1}+i\hat{2}} L_{1-i2} - \gamma_{\hat{1}-i\hat{2}} L_{1+i2} - 2v\gamma_{\hat{3}} L_3 - iv - \frac{i(\Delta - ir\sigma_0)}{v}(1 + iu\gamma_{\hat{3}}) \\ &= \begin{pmatrix} -2vL_3 - iv - \frac{i(\Delta - ir\rho(\sigma_0))}{v}(1+iu) & -2L_{1-i2} \\ -2L_{1+i2} & 2vL_3 - iv - \frac{i(\Delta - ir\rho(\sigma_0))}{v}(1-iu) \end{pmatrix}. \end{aligned} \tag{4.77}$$

In the vector space with quantum numbers (4.38), this becomes 2×2 matrix with the determinant

$$\det \mathcal{D}_\psi = (2j + 1 + \Delta - ir\rho(\sigma_0) - 2imu)(2j + 1 - \Delta + ir\rho(\sigma_0) + 2imu). \quad (4.78)$$

The two factors correspond to the two representations in the irreducible decomposition

$$(j, j) \otimes (0, \frac{1}{2}) = (j, j + \frac{1}{2}) \oplus (j, j - \frac{1}{2}). \quad (4.79)$$

The first and the second factor in (4.78) correspond to the first and the second irreducible representations in (4.79). By taking the product over all possible quantum numbers, we obtain

$$\mathrm{Det}\, \mathcal{D}_\psi = \prod_{j=0}^{\infty} \prod_{|m|\leq j} (2j+\Delta-ir\rho(\sigma_0)-2ium)^{2j}(2j-\Delta+ir\rho(\sigma_0)+2+2ium)^{2j+2}, \quad (4.80)$$

where we shifted the quantum number j so that $SU(2)_R$ spins become j.

Combining (4.74) and (4.80) we obtain

$$\begin{aligned} Z_{\text{chiral}}^{1-\text{loop}} &= \prod_{\rho\in\mathcal{R}} \frac{\mathrm{Det}\, \mathcal{D}_\psi}{\mathrm{Det}\, \mathcal{D}_\phi} \\ &= \prod_{\rho\in\mathcal{R}} \prod_{j=0,1/2,\ldots} \prod_{|m|\leq j} \frac{2j - \Delta + 2 + ir\rho(\sigma_0) + 2ium}{2j + \Delta - ir\rho(\sigma_0) - 2ium}. \end{aligned} \quad (4.81)$$

After the variable change (4.68) we obtain

$$\begin{aligned} Z_{\text{chiral}}^{1-\text{loop}} &= \prod_{\rho\in\mathcal{R}} \prod_{p,q=0}^{\infty} \frac{(1 + iu)p + (1 - iu)q + 1 + i(r\rho(\sigma_0) + i\Delta - i)}{(1 - iu)p + (1 + iu)q + 1 - i(r\rho(\sigma_0) + i\Delta - i)} \\ &= 1\Big/ \prod_{\rho\in\mathcal{R}} s_b\left(\frac{r\rho(\sigma_0) - i(1 - \Delta)}{v}\right). \end{aligned} \quad (4.82)$$

This is the contribution of one chiral multiplet belonging to $\mathcal{R}$ with Weyl weight Δ. By multiplying the contributions of all chiral multiplets we obtain the denominator in (4.4) with b in (4.10).

4.3 4d to 3d

4.3.1 4d theory

As we mentioned in Introduction, the 3d theory we investigated can be derived from a 4d theory by dimensional reduction. In this section, we summarize the derivation of the action and the transformation laws.

We first summarize the 4d conventions and notation. We use Greek characters $\kappa, \lambda, \mu, \nu, \ldots, = 1, 2, 3, 4$ for 4d tangent indices, and hatted ones $\widehat{\kappa}, \widehat{\lambda}, \widehat{\mu}, \widehat{\nu}, \ldots, = \widehat{1}, \widehat{2}, \widehat{3}, \widehat{4}$ for 4d local indices. We use the Dirac matrices

$$\gamma_{\widehat{m}} = \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix}, \quad \gamma_{\widehat{4}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (4.83)$$

We call upper half of a Dirac spinor left components and lower half right components. We use unbarred and barred spinors for left-handed and right-handed spinors.

We start from a 4d theory defined in the background $\mathbf{S}^3\times\mathbf{R}$, where $\mathbf{S}^3$ is a round sphere with radius r. We use $g_{(0)}\in SU(2)$ and $x^4\in\mathbf{R}$ to parametrize $\mathbf{S}^3$ and $\mathbf{R}$, respectively. The metric is

$$ds^2=r^2\left[(\mu^1_{(0)})^2+(\mu^2_{(0)})^2+(\mu^3_{(0)})^2\right]+(dx^4)^2, \tag{4.84}$$

where $\mu^a_{(0)}$ is the left-invariant 1-form defined by

$$2\mu^a_{(0)}T_a=g^{-1}_{(0)}dg_{(0)}. \tag{4.85}$$

For later convenience, we define vector fields h and t_a $(a=1,2,3)$ by

$$h=\left(\frac{\partial}{\partial x^4}\right)_{g_{(0)}},\quad t_a g_{(0)}=2g_{(0)}T_a. \tag{4.86}$$

h is the translation along $\mathbf{R}$, and t_a are the dual basis to μ^a. By definition $(t_a,\mu^b)=\delta^b_a$.

This manifold admits four left-handed Killing spinors ϵ_i and four right-handed Killing spinors $\overline{\epsilon}_i$ $(i=1,2,3,4)$. They have the quantum numbers shown in Table 4.1, and satisfy the Killing equations

Table 4.1: Quantum numbers of eight Killing spinors in $\mathbf{S}^3\times\mathbf{R}$

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	$\overline{\epsilon}_1$	$\overline{\epsilon}_2$	$\overline{\epsilon}_3$	$\overline{\epsilon}_4$
R	1	1	1	1	-1	-1	-1	-1
T^L_3	$-\frac{i}{2}$	$\frac{i}{2}$	0	0	$\frac{i}{2}$	$-\frac{i}{2}$	0	0
T^R_3	0	0	$-\frac{i}{2}$	$\frac{i}{2}$	0	0	$\frac{i}{2}$	$-\frac{i}{2}$
$D=-r\partial_4$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$$\begin{aligned}D_\mu\epsilon_{1/2}&=-\frac{1}{2r}\gamma_\mu\not{h}\epsilon_{1/2},\quad D_\mu\epsilon_{3/4}=+\frac{1}{2r}\gamma_\mu\not{h}\epsilon_{3/4},\\ D_\mu\overline{\epsilon}_{1/2}&=+\frac{1}{2r}\gamma_\mu\not{h}\overline{\epsilon}_{1/2},\quad D_\mu\overline{\epsilon}_{3/4}=-\frac{1}{2r}\gamma_\mu\not{h}\overline{\epsilon}_{3/4}.\end{aligned} \tag{4.87}$$

where $\not{h}=h^\mu\gamma_\mu$.

Because the background (4.84) is conformally flat, we can easily obtain the supersymmetry transformation laws from those in the flat spacetime by Weyl transformation. The transformation laws for vector multiplets are

$$\begin{aligned}\delta A_\mu&=i(\epsilon\gamma_\mu\overline{\lambda})-i(\overline{\epsilon}\gamma_\mu\lambda),\\ \delta\lambda&=\frac{i}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu}+D\epsilon,\\ \delta\overline{\lambda}&=-\frac{i}{2}\gamma^{\mu\nu}\overline{\epsilon}F_{\mu\nu}+D\overline{\epsilon},\\ \delta D&=-(\epsilon\gamma^\mu D_\mu\overline{\lambda})-(\overline{\epsilon}\gamma^\mu D_\mu\lambda).\end{aligned} \tag{4.88}$$

Transformation laws for chiral multiplets are

$$\begin{aligned}
\delta\phi &= \sqrt{2}(\epsilon\psi), \\
\delta\phi^\dagger &= \sqrt{2}(\overline{\epsilon}\overline{\psi}), \\
\delta\psi &= -\sqrt{2}\gamma^\mu\overline{\epsilon}D_\mu\phi + \sqrt{2}\epsilon F - \frac{\Delta}{\sqrt{2}}\gamma^\mu D_\mu\overline{\epsilon}\phi, \\
\delta\overline{\psi} &= -\sqrt{2}\gamma^\mu\epsilon D_\mu\phi^\dagger + \sqrt{2}\overline{\epsilon}F^\dagger - \frac{\Delta}{\sqrt{2}}\gamma^\mu D_\mu\epsilon\phi^\dagger, \\
\delta F &= -\sqrt{2}(\overline{\epsilon}\gamma^\mu D_\mu\psi) - 2(\overline{\epsilon}\overline{\lambda})\phi - \frac{\Delta-1}{\sqrt{2}}D_\mu\overline{\epsilon}\gamma^\mu\psi, \\
\delta F^\dagger &= -\sqrt{2}(\epsilon\gamma^\mu D_\mu\overline{\psi}) - 2\phi^\dagger(\epsilon\lambda) - \frac{\Delta-1}{\sqrt{2}}D_\mu\epsilon\gamma^\mu\overline{\psi}.
\end{aligned} \tag{4.89}$$

The kinetic Lagrangians for vector and chiral multiplets can be obtained in the same way as in 3d

$$(\overline{\epsilon}_1\overline{\epsilon}_2)\mathcal{L}^{(4d)}_{\text{YM}} = -\frac{1}{4}\delta(\overline{\epsilon}_1)\delta(\overline{\epsilon}_2)\text{tr}(\overline{\lambda}\overline{\lambda}), \quad (\overline{\epsilon}_1\overline{\epsilon}_2)\mathcal{L}^{(4d)}_{\text{chiral}} = -\frac{1}{2}\delta(\overline{\epsilon}_1)\delta(\overline{\epsilon}_2)(\phi^\dagger F). \tag{4.90}$$

The explicit form of these kinetic Lagrangians is

$$\mathcal{L}^{(4d)}_{\text{YM}} = \mathcal{L}^{(4d)}_{\mathcal{A}} + \mathcal{L}^{(4d)}_{\lambda} - \frac{1}{2}\text{tr}D^2, \quad \mathcal{L}^{(4d)}_{\text{chiral}} = \mathcal{L}^{(4d)}_{\phi} + \mathcal{L}^{(4d)}_{\psi} - F^\dagger F, \tag{4.91}$$

where

$$\begin{aligned}
\mathcal{L}^{(4d)}_{\mathcal{A}} &= \text{tr}\frac{1}{2}\mathcal{F}^{(-)}_{\widehat{m}}\mathcal{F}^{(-)}_{\widehat{m}}, \\
\mathcal{L}^{(4d)}_{\lambda} &= -\text{tr}(\overline{\lambda}\gamma^\mu D_\mu\lambda), \\
\mathcal{L}^{(4d)}_{\phi} &= -\phi^\dagger D_\mu D^\mu\phi + \phi^\dagger D\phi - \frac{\Delta^2-2\Delta}{r^2}\phi^\dagger\phi - \frac{2(\Delta-1)}{r}h^\mu\phi^\dagger D_\mu\phi, \\
\mathcal{L}^{(4d)}_{\psi} &= -(\overline{\psi}\gamma^\mu D_\mu\psi) - \frac{\Delta-1}{r}h^\mu(\overline{\psi}\gamma_\mu\psi) - \sqrt{2}\phi^\dagger(\lambda\psi) - \sqrt{2}(\overline{\psi}\overline{\lambda})\phi.
\end{aligned} \tag{4.92}$$

$\mathcal{F}^{(\pm)}_{\widehat{m}}$ are defined by

$$\mathcal{F}^{(\pm)}_{\widehat{m}} = \frac{1}{2}\epsilon_{\widehat{m}\widehat{p}\widehat{q}}F_{\widehat{p}\widehat{q}} \pm F_{\widehat{m}\widehat{4}}. \tag{4.93}$$

4.3.2 Killing spinors and twisted compactification

To obtain 3d theory, we need to compactify the **R** direction. This is realized by imposing the condition

$$\mathcal{O}\Phi = \Phi, \tag{4.94}$$

on all fields Φ in the theory, where $\mathcal{O}$ is an operator containing shift along x^4 and additional twists. To keep some of supersymmetries unbroken, we should choose $\mathcal{O}$ which keep the corresponding Killing spinors invariant. Our choice is

$$\mathcal{O} = q^{D-\frac{1}{2}R_0-2uT^R_3}, \quad q = e^{-\beta}, \tag{4.95}$$

where $D = -r\partial_4$ is the x^4-translation, and β is the period of the $\mathbf{S}^1$ compactification divided by the $\mathbf{S}^3$ radius r. R_0 is an R-symmetry. This is not the R-symmetry in the superconformal algebra, but one that does not rotate the dynamical scalar components of chiral multiplets.

$$R_0(\phi) = R_0(\phi^\dagger) = 0, \quad R_0(\overline{\epsilon}) = R_0(\psi) = -1, \quad R_0(\lambda) = +1. \tag{4.96}$$

This twist preserves four supersymmetries out of eight corresponding to ϵ_1, ϵ_2, $\overline{\epsilon}_1$, and $\overline{\epsilon}_2$. Note that when $u \neq 0$ this compactification breaks $SU(2)_R$ to $U(1)_r$.

The constraint (4.94) with the operator $\mathcal{O}$ in (4.95) implies the following identification of the points

$$(g_{(0)} e^{\frac{2u}{r}\beta T_3^R}, x^4 + \beta) \sim (g_{(0)}, x^4). \tag{4.97}$$

(Figure 4.1.) We take the small radius limit $\beta \to 0$, and get rid of all Kaluza-Klein

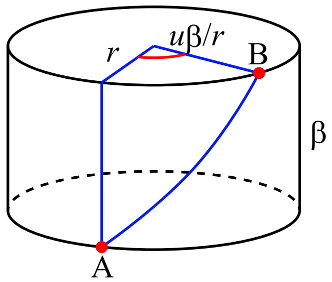

Figure 4.1: Twisted compactification of $\mathbf{S}^3 \times \mathbf{R}$ is shown. Points A and B are identified.

modes except the lowest one for each field to obtain 3d theory. This reduction is realized by imposing the constraint

$$\left(D - 2uT_3^R - \frac{1}{2}R_0\right)\Phi = 0 \tag{4.98}$$

on all fields. By using the vector fields in (4.86), we can rewrite this as the differential equation

$$\left(-\mathcal{L}_{rh+ut_3} - \frac{1}{2}R_0\right)\Phi = 0. \tag{4.99}$$

The constraint (4.99) determines the x^4 dependence of fields from their values on the $x^4 = 0$ slice.

It is convenient to perform the coordinate transformation

$$(g, x^4) = (g_{(0)} e^{-\varphi(x^4) T_3^R}, x^4), \quad \varphi(x^4) = \frac{2u}{r}x^4. \tag{4.100}$$

In the new coordinate system, the identification (4.97) is simplified as

$$(g, x^4 + \beta) \sim (g, x^4). \tag{4.101}$$

The metric in the new coordinate system is

$$ds^2 = r^2\left[(\mu^1)^2 + (\mu^2)^2 + \frac{1}{v^2}(\mu^3)^2\right] + v^2(dx^4 + V)^2, \tag{4.102}$$

where μ^a are defined in (C.28), and the 1-form V is

$$V = \frac{ru}{v^2}\mu^3. \tag{4.103}$$

After dimensional reduction, we obtain squashed sphere (4.1) with the background graviphoton field V. Note that this V is the same as (4.21).

We use the 4d vielbein

$$E^{\widehat{1}} = r\mu^1, \quad E^{\widehat{2}} = r\mu^2, \quad E^{\widehat{3}} = \frac{r}{v}\mu^3, \quad E^{\widehat{4}} = v(dx^4 + V). \tag{4.104}$$

In general, the components of the 4d spin connection Ω of the 4d manifold with the metric

$$ds^2 = E^{\widehat{\mu}}E^{\widehat{\mu}} = e^{\widehat{m}}e^{\widehat{m}} + v^2(dx^4 + V)^2, \tag{4.105}$$

are

$$\Omega_{\widehat{m}-\widehat{n}\widehat{p}} = \omega_{\widehat{m}-\widehat{n}\widehat{p}}, \quad \Omega_{\widehat{4}-\widehat{m}\widehat{n}} = -\frac{v}{2}(dV)_{\widehat{m}\widehat{n}}, \quad \Omega_{\widehat{m}-\widehat{4}\widehat{n}} = -\frac{v}{2}(dV)_{\widehat{m}\widehat{n}}, \quad \Omega_{\widehat{4}-\widehat{4}\widehat{m}} = 0, \tag{4.106}$$

where ω is the spin connection of the 3d manifold with the metric $ds^2 = e^{\widehat{m}}e^{\widehat{m}}$. By using the explicit form of the graviphoton field V, we obtain

$$\Omega_{\widehat{4}-\widehat{1}\widehat{2}} = \Omega_{\widehat{1}-\widehat{4}\widehat{2}} = -\Omega_{\widehat{2}-\widehat{4}\widehat{1}} = -\frac{u}{vr}. \tag{4.107}$$

The components of the vielbein are

$$\begin{pmatrix} E_m{}^{\widehat{n}} & E_m{}^{\widehat{4}} \\ E_4{}^{\widehat{n}} & E_4{}^{\widehat{4}} \end{pmatrix} = \begin{pmatrix} e_m{}^{\widehat{n}} & ue_m^{\widehat{3}} \\ 0 & v \end{pmatrix}, \quad \begin{pmatrix} E_{\widehat{m}}{}^n & E_{\widehat{m}}{}^4 \\ E_{\widehat{4}}{}^n & E_{\widehat{4}}{}^4 \end{pmatrix} = \begin{pmatrix} e_{\widehat{m}}{}^n & -\frac{u}{v}\delta^{\widehat{3}}_{\widehat{m}} \\ 0 & \frac{1}{v} \end{pmatrix}. \tag{4.108}$$

In the new coordinate system, the vector field appearing in the constraint (4.99) is

$$rh + ut_3 = r\frac{\partial}{\partial x^4}. \tag{4.109}$$

By using this and the spin connection in (4.107), the constraint (4.99) is simplified as

$$\left(-r\frac{\partial}{\partial x^4} - \frac{R_0}{2}\right)\Phi = 0. \tag{4.110}$$

4.3.3 Dimensional reduction

We define 3d fields as the restriction of the corresponding 4d fields on the slice $x^4 = 0$. For a 4d left-handed (right-handed) spinor field, we take two components of the left-handed (right-handed) part of the 4d field as the 3d field. For example, for the left-handed spinor field $\lambda^{(4d)}$ we define the corresponding 3d field by

$$\lambda^{(4d)}|_{x^4=0} = \begin{pmatrix} \lambda^{(3d)} \\ 0 \end{pmatrix}. \tag{4.111}$$

A 4d gauge field $A^{(4d)} = A^{(4d)}_\mu dx^\mu$ is decomposed into 3d gauge field $A^{(3d)} = A^{(3d)}_m dx^m$ and 3d adjoint scalar field σ by

$$A^{(4d)}_\mu|_{x^4=0} dx^\mu = A^{(3d)} + \sigma dx^4. \tag{4.112}$$

To obtain 3d Lagrangians and transformation laws, we need to rewrite the 4d covariant derivatives in terms of 3d ones. By using the explicit form of the vielbein and spin connection, we obtain

$$\begin{aligned}\frac{1}{2}E^\mu_{\widehat{m}}\Omega_{\mu-\widehat{\kappa}\widehat{\lambda}}S_{\kappa\lambda} &= \frac{1}{2}e^n_{\widehat{m}}\omega_{n-\widehat{k}\widehat{l}}S_{kl} - \frac{u}{vr}\epsilon_{\widehat{m}\widehat{n}\widehat{3}}S_{4n},\\ \frac{1}{2}E^\mu_{\widehat{4}}\Omega_{\mu-\widehat{\kappa}\widehat{\lambda}}S_{\kappa\lambda} &= -\frac{u}{vr}S_{12},\end{aligned} \tag{4.113}$$

where $S_{\mu\nu}$ are 4d spin operators. With these relations and the constraint (4.110), we can easily obtain

$$\begin{aligned}D^{(4d)}_{\widehat{m}} &= D^{(3d)}_{\widehat{m}} - \frac{u}{vr}\epsilon_{\widehat{m}\widehat{3}\widehat{n}}S_{n4} + \frac{u}{vr}\delta_{\widehat{m}\widehat{3}}\left(\frac{R_0}{2} + ir\sigma\right),\\ D^{(4d)}_{\widehat{4}} &= -\frac{1}{vr}\left(\frac{R_0}{2} + ir\sigma\right) - \frac{u}{vr}S_{12}.\end{aligned} \tag{4.114}$$

By using these relations, it is straightforward to obtain 3d supersymmetry transformation laws and 3d Lagrangians from 4d ones. (The Chern-Simons term cannot be obtained from 4d Lagrangian, and we need to construct it by, for example, Noether procedure.) We will not explain them in detail. We only demonstrate the derivation of the 3d Killing equations (4.12). Our compactification preserves the Killing spinors ϵ_1, ϵ_2, $\overline{\epsilon}_1$, and $\overline{\epsilon}_2$. They satisfy the 4d Killing equations

$$D_{\widehat{\mu}}\epsilon = -\frac{1}{2r}\gamma_{\widehat{\mu}}\not{h}\epsilon, \quad D_{\widehat{\mu}}\overline{\epsilon} = \frac{1}{2r}\gamma_{\widehat{\mu}}\not{h}\overline{\epsilon}. \tag{4.115}$$

For $\widehat{\mu} = \widehat{m}$, the left hand side of these equations are rewritten by (4.114) as

$$\begin{aligned}D^{(4d)}_{\widehat{m}}\epsilon = D^{(3d)}_{\widehat{m}}\epsilon - \frac{u}{2vr}\epsilon_{\widehat{m}\widehat{3}\widehat{n}}\gamma_{\widehat{n}\widehat{4}}\epsilon + \frac{u}{2vr}\delta_{\widehat{m}\widehat{3}}\epsilon = D^{(3d)}_{\widehat{m}}\epsilon + \frac{u}{2vr}\gamma_{\widehat{3}}\gamma_{\widehat{m}}\epsilon,\\ D^{(4d)}_{\widehat{m}}\overline{\epsilon} = D^{(3d)}_{\widehat{m}}\overline{\epsilon} - \frac{u}{2vr}\epsilon_{\widehat{m}\widehat{3}\widehat{n}}\gamma_{\widehat{n}\widehat{4}}\overline{\epsilon} - \frac{u}{2vr}\delta_{\widehat{m}\widehat{3}}\overline{\epsilon} = D^{(3d)}_{\widehat{m}}\overline{\epsilon} - \frac{u}{2vr}\gamma_{\widehat{3}}\gamma_{\widehat{m}}\overline{\epsilon}.\end{aligned} \tag{4.116}$$

The right hand side of the equations in (4.115) are rewritten as

$$\begin{aligned}-\frac{1}{2r}\gamma_{\widehat{m}}\not{h}\epsilon &= -\frac{1}{2vr}\gamma_{\widehat{m}}(\gamma_{\widehat{4}} - u\gamma_{\widehat{3}})\epsilon = -\frac{i}{2vr}\gamma_{\widehat{m}}\epsilon + \frac{u}{2vr}\gamma_{\widehat{m}}\gamma_{\widehat{3}}\epsilon,\\ \frac{1}{2r}\gamma_{\widehat{m}}\not{h}\overline{\epsilon} &= \frac{1}{2vr}\gamma_{\widehat{m}}(\gamma_{\widehat{4}} - u\gamma_{\widehat{3}})\overline{\epsilon} = -\frac{i}{2vr}\gamma_{\widehat{m}}\overline{\epsilon} - \frac{u}{2vr}\gamma_{\widehat{m}}\gamma_{\widehat{3}}\overline{\epsilon}.\end{aligned} \tag{4.117}$$

where we used the fact that the vector field h has the components

$$h^{\widehat{\mu}} = \left(0, 0, -\frac{u}{v}, \frac{1}{v}\right). \tag{4.118}$$

Combining (4.116) and (4.117), we obtain the 3d Killing equations (4.12).

4.4 Large N limit

In this section we investigate the free energy $F = -\log Z$ of large N gauge theories which are expected to have M-theory dual. We consider a quiver gauge theory with gauge group

$$G = \prod_{a=1}^{n_G} U(N)_a. \tag{4.119}$$

In this case the traces in (4.27) are expressed as linear combinations of the traces for $U(N)_a$ gauge groups,

$$\mathrm{tr}_{\mathrm{CS}} = \sum_{a=1}^{n_G} \frac{k_a}{2\pi} \mathrm{tr}_a, \quad \mathrm{tr}_{\mathrm{FI}} = \sum_{a=1}^{n_G} \frac{\zeta_a}{vr} \mathrm{tr}_a, \tag{4.120}$$

where tr_a is the trace over the $U(N)_a$ fundamental representation. The coefficients k_a and ζ_a are Chern-Simons levels and FI parameters, respectively. The Chern-Simons parameters k_a must be integers. The normalization of the FI parameters ζ_a is chosen for later convenience.

It is pointed out in [18] that in order to obtain the leading term of the free energy in the $1/N$ expansion, we do not have to perform the integral over σ_0 in (1.11). We only need to determine the minimum value of the integrand of (1.11). Namely, we obtain the free energy by minimizing

$$F(\sigma_0) = S^{\mathrm{cl}}(\sigma_0) - \log Z^{1-\mathrm{loop}}(\sigma_0). \tag{4.121}$$

It is convenient to decompose this into three parts: the classical action $F_1 = S^{\mathrm{cl}}$, the 1-loop contribution of vector and bi-fundamental chiral multiplets F_2, and the 1-loop contribution of fundamental and anti-fundamental chiral multiplets F_3.

From (4.41), the classical action F_1 is

$$F_1 = S^{\mathrm{cl}}_{\mathrm{CS}} + S^{\mathrm{cl}}_{\mathrm{FI}} = \sum_{a=1}^{n_G} \sum_{i=1}^{N} \left(\frac{\pi i}{v^2} k_a \lambda_{a,i}^2 + \frac{4\pi^2 i}{v^2} \zeta_a \lambda_{a,i} \right), \tag{4.122}$$

where $\lambda_{a,i}$ are diagonal components of the expectation value of the $U(N)_a$ adjoint scalar field rescaled by r

$$r(\sigma_0)_a = \mathrm{diag}\{\lambda_{a,j}\}. \tag{4.123}$$

F_2 is the 1-loop contribution of vector multiplets and bi-fundamental chiral multiplets. It is given by

$$\begin{aligned} F_2 = &- \sum_{a=1}^{n_G} \sum_{j \neq k} f_b \left(\frac{1}{v} (\lambda_{a,j} - \lambda_{a,k} - i) \right) \\ &+ \sum_{I \in \mathrm{bi-fund}} \sum_{j,k} f_b \left(\frac{1}{v} \left(\lambda_{h(I),j} - \lambda_{t(I),k} - i (1 - \Delta_I) \right) \right), \end{aligned} \tag{4.124}$$

where $f_b(z) = \log s_b(z)$ and b is the parameter related to the squashing parameter by (4.10). The first line and the second line are contribution of vector and bi-fundamental chiral multiplets, respectively. I runs over all bi-fundamental chiral multiplets. We use $h(I)$ and $t(I)$ to represent the $U(N)$ factors at the head and the tail of the arrow corresponding to the chiral multiplet I in the quiver diagram. Namely, a chiral multiplet I belongs to the bi-fundamental representation $(N_{h(I)}, \overline{N}_{t(I)})$. Adjoint chiral multiplets are treated as bi-fundamental chiral multiplets with $h(I) = t(I)$, and their contribution is also included in F_2.

The contribution of fundamental and anti-fundamental chiral multiplets is denoted by F_3, and given by

$$F_3 = \sum_{I \in \text{fund}} \sum_j f_b \left(\frac{1}{v} \left(\lambda_{h(I),j} - i\,(1 - \Delta_I) \right) \right) + \sum_{I \in \text{anti-fund}} \sum_j f_b \left(\frac{1}{v} \left(-\lambda_{h(I),j} - i\,(1 - \Delta_I) \right) \right), \tag{4.125}$$

where $I \in$fund and $I \in$anti-fund mean that the index I runs over fundamental and anti-fundamental chiral multiplets, respectively.

In [18] the minimum points are determined numerically in some models, and the eigenvalue distribution is found to behave in the large N limit as

$$\lambda_{a,j} = N^\alpha x_j + i y_{a,j}, \tag{4.126}$$

where x_j and $y_{a,j}$ are real numbers, and α is a certain constant in the region $0 < \alpha < 1$. Note that x_j are common for all $U(N)_a$ factors. In [43], the analysis is extended to a large class of quiver gauge theories, and it is shown that we can consistently determine the free energy proportional to $N^{3/2}$ based on the ansatz (4.126) if the theory satisfies the following conditions.

(A) The theory is non-chiral. This means that the number of bi-fundamental chiral multiplets transforming in $(N, \overline{N})$ of the gauge group $U(N)_a \times U(N)_b$ is the same as that in $(\overline{N}, N)$.

(B) The Weyl weights of chiral multiplets satisfy

$$\sum_{I \in a} (1 - \Delta_I) - 2 = 0, \quad \forall a, \tag{4.127}$$

where Δ_I is the Weyl weight of the bi-fundamental field I. The sum is taken over all bi-fundamental fields coupled by $U(N)_a$. A $U(N)_a$ adjoint chiral multiplet should be included twice. Fundamental and anti-fundamental fields should not be included.

(C) The total number of fundamental fields and anti-fundamental fields should be the same. Note that this condition is not imposed for each $U(N)_a$ factor. The numbers of fundamental and anti-fundamental fields for each $U(N)_a$ factor may be different. Only the total numbers matter.

(D) Chern-Simons levels sum up to zero:

$$\sum_{a=1}^{n_G} k_a = 0. \tag{4.128}$$

In [43], it is shown that the free energy of theories satisfying these condition defined on round $\mathbf{S}^3$ is proportional to $N^{3/2}$. We generalize it to theories in the squashed $\mathbf{S}^3$. We follow the prescription proposed in [43].

The first step to determine the free energy in the large N limit is to rewrite the summations in (4.122), (4.124), and (4.125) by integrals. We define the density function $\rho(x)$ by

$$\rho(x) = \frac{1}{N}\sum_{j=1}^{N}\delta(x - x_j). \tag{4.129}$$

By definition, ρ satisfies the normalization condition

$$\int_{x_{\min}}^{x_{\max}} \rho(x)dx = 1. \tag{4.130}$$

In the large N limit, we can treat ρ as a continuous function of x. Similarly, we replace $y_{a,i}$ by functions $y_a(x)$. The classical action contribution F_1 is rewritten in the continuous form as

$$F_1 = N\sum_{a=1}^{n_G}\int_{x_{\min}}^{x_{\max}} dx\rho\left(\frac{\pi i}{v^2}k_a\lambda_a^2 + \frac{4\pi^2 i}{v^2}\zeta_a\lambda_a\right). \tag{4.131}$$

We substitute the continuous form of (4.126)

$$\lambda_a(x) = N^\alpha x + iy_a(x) \tag{4.132}$$

into (4.131). Thanks to the condition (D), $N^{1+2\alpha}$ terms cancel, and the leading terms are proportional to $N^{1+\alpha}$. If we ignore sub-leading terms, we obtain

$$F_1 = \frac{\pi N^{1+\alpha}}{v^2}\sum_{a=1}^{n_G}\int_{x_{\min}}^{x_{\max}} dx\rho x(-2k_a y_a + 4\pi i\zeta_a). \tag{4.133}$$

F_2 in (4.124) is rewritten as

$$\begin{aligned} F_2 = &- N^2\int_{x_{\min}}^{x_{\max}} dx\int_{x_{\min}}^{x_{\max}} dx'\rho\rho'\sum_a f_b\left(\frac{1}{v}(\lambda_a - \lambda'_a - i)\right) \\ &+ N^2\int_{x_{\min}}^{x_{\max}} dx\int_{x_{\min}}^{x_{\max}} dx'\rho\rho'\sum_{I\in\text{adj}} f_b\left(\frac{1}{v}(\lambda_{h(I)} - \lambda'_{t(I)} - i(1-\Delta_I)\right), \end{aligned} \tag{4.134}$$

where $\rho' \equiv \rho(x')$ and $\lambda'_a \equiv \lambda_a(x')$. The key idea to rewrite these double integrals to tractable form is that if $x \neq x'$ we can replace the function f_b by its asymptotic form

$$f_b^{\text{asym}}(z) = i\pi\left(\frac{z^2}{2} + \frac{b^2 + b^{-2}}{24}\right)\text{sign}(x), \tag{4.135}$$

because the real part of eigenvalues λ scales as N^α in the large N limit. We call this "long range potential." The contribution from $x = x'$ should be taken separately as a "short range potential" proportional to $\delta(x - x')$. In the large N limit, we can replace the function $f_b(z)$ by the sum of long range and short range potentials

$$f_b(x + iy) \to f_b^{\text{asym}}(x + iy) + \delta(x)g_b(y), \tag{4.136}$$

where the function $g_b(y)$ is given by

$$g_b(y) = \frac{\pi}{3}y^3 - \frac{\pi}{12}(b^2 + b^{-2})y. \quad (4.137)$$

See Appendix F.3 for a derivation of (4.135) and (4.137).

Let us consider the contribution of the long-range potential in (4.134). $f_b^{\text{asym}}(z)$ is a quadratic function of z, and after substitution of (4.132), (4.134) contains terms of order $N^{2+2\alpha}$, $N^{2+\alpha}$, and N^2. To obtain the free energy of order $N^{3/2}$, all these terms should cancel. This is indeed the case. We can easily show that the contribution of long range potential in F_2 cancel due to the conditions (A) and (B). As a result, only the short range potential contributes to F_2. Because the short range potential contains δ-function, we can perform one of integrals. After the x' integral, F_2 is given by

$$F_2 = vN^{2-\alpha}\int_{x_{\min}}^{x_{\max}} dx\rho^2 \left[\sum_{I\in\text{bi-fund}} g_b\left(\frac{1}{v}(y_I - (1-\Delta_I))\right) - \sum_{a=1}^{n_G} g_b\left(-\frac{1}{v}\right)\right], \quad (4.138)$$

where we defined

$$y_I = y_{h(I)} - y_{t(I)}. \quad (4.139)$$

By using (4.127), we can rewrite the second term in the brackets in (4.138) as the summation over bi-fundamental chiral multiplets

$$\begin{aligned} F_2 &= vN^{2-\alpha}\int_{x_{\min}}^{x_{\max}} dx\rho^2 \sum_{I\in\text{bi-fund}} \left[g_b\left(\frac{1}{v}(y_I - (1-\Delta_I))\right) - (1-\Delta_I)g_b\left(-\frac{1}{v}\right)\right] \\ &= \frac{\pi N^{2-\alpha}}{v^2}\int_{x_{\min}}^{x_{\max}} dx\rho^2 \sum_{I\in\text{bi-fund}} \frac{1}{3}(y_I+\Delta_I)(y_I-1+\Delta_I)(y_I-2+\Delta_I). \end{aligned} \quad (4.140)$$

To obtain the second line we used $\sum_I y_I = 0$ following from the condition (A).

The continuous form of the contribution of fundamental and anti-fundamental fields, (4.125), is

$$\begin{aligned} F_3 =& N\int_{x_{\min}}^{x_{\max}} dx\rho \sum_{I\in\text{fund}} f_b\left(\frac{\lambda_{h(I)} - i(1-\Delta_I)}{v}\right) \\ &+ N\int_{x_{\min}}^{x_{\max}} dx\rho \sum_{I\in\text{anti-fund}} f_b\left(\frac{-\lambda_{t(I)} - i(1-\Delta_I)}{v}\right). \end{aligned} \quad (4.141)$$

Order $N^{1+2\alpha}$ terms in the long range potential contribution cancel by the condition (C), and the leading non-vanishing terms in F_3 are of order $N^{1+\alpha}$. The contribution of the short range potential is of order $N^{1-\alpha}$, and we can neglect them. The leading terms in F_3 are

$$\begin{aligned} F_3 =& \frac{\pi N^{1+\alpha}}{v^2}\int_{x_{\min}}^{x_{\max}} dx\rho \sum_{I\in\text{fund}} |x|(1-\Delta_I - y_{h(I)}) \\ &+ \frac{\pi N^{1+\alpha}}{v^2}\int_{x_{\min}}^{x_{\max}} dx\rho \sum_{I\in\text{anti-fund}} |x|(1-\Delta_I + y_{t(I)}). \end{aligned} \quad (4.142)$$

Now we have succeeded in writing all the contributions to the free energy as one-dimensional integral. F_1 and F_3 are proportional to $N^{1+\alpha}$, and F_2 is proportional to $N^{2-\alpha}$. To obtain minimum point, these should balance, and this require $\alpha = 1/2$. In this case, the free energy is proportional to $N^{3/2}$, as is expected from the analysis on the gravity side of AdS/CFT.

Let us focus on the dependence on the squashing parameter v. We find that in all terms of order $N^{3/2}$ the v dependence is factored out as the factor $1/v^2$. (For the contribution of FI terms, this is the case when we adopt the normalization of FI parameters in (4.120).) Therefore, the free energy obtained by minimizing the x-integral is always $1/v^2$ times as that for round $\mathbf{S}^3$:

$$F_{\text{squashed}} = \frac{1}{v^2} F_{\text{round}}. \tag{4.143}$$

This fact guarantees that the R charge at the IR fixed point obtained by extremizing Z does not depend on the squashing parameter.

We investigated $\mathcal{N} = 2$ supersymmetric theories on squashed sphere with $SU(2)_L \times U(1)_r$ isometry. The theories have four supercharges,which are transformed by $SU(2)_L$ isometry as a pair of doublets. We constructed supersymmetry transformation laws and Lagrangians by using $\mathbf{S}^1$ compactification of 4d theory. Although the metric of the squashed sphere is the same as that of the $SU(2)_L \times U(1)_r$ symmetric squashing in [35], the supersymmetry group is different. We computed the partition function by using localization, and showed that it depends on the squashing parameter in a non-trivial way.

We also computed the free energy of large N quiver gauge theories on the squashed $\mathbf{S}^3$. We considered a class of quiver gauge theories studied in [43], whose partition function on round $\mathbf{S}^3$ scales as $N^{3/2}$. We confirmed that the free energy on squashed $\mathbf{S}^3$ is proportional to $N^{3/2}$ as well, and the v dependence is factored out as the additional factor $1/v^2$ regardless of the detailed structure of the theory. It would be interesting problem to look for holographic dual of the gauge theories on the squashed sphere, and confirm that the same result is reproduced by the analysis on the gravity side.

Summary

We derive the squashed partition function in this chapter; we summarize the results here:

$$\begin{aligned}
Z &= \int [d\sigma] e^{-S^{\text{cl}}(\sigma)} Z^{1-\text{loop}}(\sigma), \\
Z^{1-\text{loop}}(\sigma) &= \frac{\prod_{\alpha\in\Delta} s_b\left(\alpha(\lambda) - \frac{i}{v}\right)}{\prod_I s_b\left(\rho_I(\lambda) - \frac{i(1-\Delta_I)}{v}\right)}, \\
S^{\text{cl}}(\sigma) &= i\pi k \text{tr}(\sigma^2) + 2\pi i \zeta \text{tr}(\sigma), \\
[d\sigma] &= \frac{1}{|W|} \prod_{a=1}^{\text{rank}G} d\sigma_a
\end{aligned} \tag{4.144}$$

where we omitted the subscript 0 and rescaled the fields so that the expression becomes simple. So far we never mind the normalization constant, however, it is really important to check the duality numerically. We determined the measure (4.144) so that the squashed partition functions calculated for a pair of dual theories coincide numerically.

Chapter 5

Orbifolded partition function

In this chapter, we investigate the partition function of three-dimensional $\mathcal{N} = 2$ supersymmetric field theories on the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$. First of all, as an orbifolding usually breaks the supersymmetry we have to care about which direction is orbifolded. We choose the $\boldsymbol{S}^1$ fiber direction of the squashed $\boldsymbol{S}^3$ for the orbifolding, whose direction corresponds to $U(1)_r$ of $SU(2)_L \times U(1)_r$. Since the supersymmetry we constructed in the previous chapter is doublet of the $SU(2)_L$ and $U(1)_r$ singlet the supersymmetry is not affected by this orbifolding.

Due to the non-trivial homotopy of the orbifold, $\pi_1(\boldsymbol{S}^3/\mathbb{Z}_n) = \mathbb{Z}_n$ (see Section C.4), a gauge theory defined in it has degenerate vacua specified by the holonomy h associated with the gauge symmetry. Their contributions are summed up to obtain the total partition function:

$$Z_{\text{total}} = Z_{h=0} + Z_{h=1} + \cdots + Z_{h=n-1}. \tag{5.1}$$

In general, the partition function of a Euclidean theory is complex. We usually focus only on its absolute value and the phase is disregarded. This is, however, not allowed when we compute the partition functions of different sectors which are *summed up*. Even when we are interested only in the absolute value of the total partition function, we need to care about the relative phase of each contribution.

The purpose of this chapter is to determine appropriate phase factors $e^{-\theta_h}$ in the holonomy sum in some gauge theories and look for a general rule for these phases.

$$Z_{\text{total}} = e^{i\theta_0} Z_{h=0} + e^{i\theta_1} Z_{h=1} + \cdots + e^{i\theta_{n-1}} Z_{h=n-1} \tag{5.2}$$

We consider two gauge theories which are known to have dual field theories without vector multiplet. On one side of the dualities, in the non-gauge theories, we can compute the absolute value of the partition function up to overall constant factor independent of parameters. By comparing the partition functions of gauge and non-gauge theories in each dual pair, we infer the relative phases in the holonomy sum in the gauge theories.

5.1 The $\boldsymbol{S}^3/\mathbb{Z}_n$ partition function

The basic parts of the $\boldsymbol{S}^3/\mathbb{Z}_n$ partition function is easily derived from the squashed partition function by dropping off the modes that do not satisfy the boundary condition specified by the holonomy. We start from the explanation of the holonomy and give the results. Note that the results in the following section are already known in the literatures [37, 38].

5.1.1 $\mathbb{Z}_n$ orbifolding

We consider the left-invariant orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$ with $\mathbb{Z}_n \subset U(1)_r \subset SU(2)_R$. The partition function on the orbifold is obtained in [37] for theories without matter fields in a general Lens space $L(p,q)$ without squashing. It is extended to theories with chiral multiplets in background with nontrivial squashing parameter in [38]. Our orbifold corresponds to $L(n,-1)$.

Because supercharges are $U(1)_r$ neutral, the orbifolding by $\mathbb{Z}_n \subset U(1)_r$ does not break any supersymmetry, and we can define $\mathcal{N}=2$ supersymmetric theories on the orbifold. A gauge theory in this orbifold has degenerate vacua specified by the holonomy

$$h = \frac{n}{2\pi}\oint_C A, \tag{5.3}$$

where C is a non-trivial cycle corresponding to the fundamental group $\pi_1(\boldsymbol{S}^3/\mathbb{Z}_n) = \mathbb{Z}_n$ in the orbifold (see C.4). The consistency to $nC=0$ requires $e^{2\pi i h}=1$. (Note that we define h with the factor n in (5.3).) The holonomy can be turned on for both global and gauge symmetries. The holonomy for gauge symmetries should be summed up in the path integral because it consists of a dynamical gauge field. The partition function is given by

$$Z(h_{\rm global}) = \sum_{h_{\rm local}}\int [d\lambda] e^{-S_0(\lambda,h)} Z^{1-{\rm loop}}(\lambda,h), \tag{5.4}$$

where $S_0(\lambda,h)$ and $Z^{1-{\rm loop}}(\lambda,h)$ are the classical action and the one-loop determinant. The summation is taken over the holonomy associated with gauge symmetry, which is denoted by $h_{\rm local}$ in (5.4). The holonomy for global symmetry $h_{\rm global}$ is not summed, and the partition function Z depends on $h_{\rm global}$.

The integration measure $[d\lambda]$ is defined by

$$[d\lambda] = \frac{1}{|W|}\prod_{a=1}^{{\rm rank}G}\frac{d\lambda_a}{n}. \tag{5.5}$$

We introduce the factor $1/n$ for each integration variable for later convenience.

One may think that the classical action for $\boldsymbol{S}^3/\mathbb{Z}_n$ is obtained by dividing that for $\boldsymbol{S}^3$ by n. This naive expectation is not correct. The classical action $S_0(\lambda,h)$ consists of two parts;

$$S_0^{\boldsymbol{S}^3/\mathbb{Z}_n}(\lambda,h) = \frac{1}{n}S_0^{\boldsymbol{S}^3}(\lambda) - i\Phi(h). \tag{5.6}$$

One is $1/n$ of the classical action for $\boldsymbol{S}^3$, and has the same origin as the $\boldsymbol{S}^3$ case. The other part comes from the Chern-Simons term. Due to the non-trivial topology of $\boldsymbol{S}^3/\mathbb{Z}_n$, the Chern-Simons term gives non-vanishing contribution even for a flat gauge connection [37, 44, 45] (see Section D.2);

$$\Phi = \frac{\pi k}{n}{\rm tr}_{\rm fund}(h^2). \tag{5.7}$$

This phase plays an important role in dualities in $\boldsymbol{S}^3/\mathbb{Z}_n$. The factor $e^{i\Phi}$ may be ill-defined depending on the coefficient. If nk is odd, the holonomies $h = \text{diag}(\cdots, h_i, \cdots)$ and $h = \text{diag}(\cdots, h_i + n, \cdots)$, which are identified in $\mathbb{Z}_n$, give different phases. We will meet such an ambiguity in the example in §5.2.2, and there we will give an additional rule to fix the ambiguity.

The one-loop partition function for the orbifold can be obtained by projecting out the factors in (4.5) which originate from $\mathbb{Z}_n$-variant modes. Let φ be a field with a weight vector ρ. On $\boldsymbol{S}^3$ it is Fourier expanded as

$$\varphi(\psi) = \sum_{m \in \boldsymbol{Z}/2} \varphi_m e^{im\psi}, \tag{5.8}$$

where $0 \leq \psi < 4\pi$ is the coordinate along the Hopf fiber of $\boldsymbol{S}^3$, and m is the $SU(2)_R$ magnetic quantum number. After $\mathbb{Z}_n$ orbifolding, the field must satisfy the boundary condition

$$\varphi\left(\psi + \frac{4\pi}{n}\right) = e^{2\pi i \frac{\rho(h)}{n}} \varphi(\psi), \tag{5.9}$$

and only modes φ_m with the index m satisfying

$$2m = p - q = \rho \cdot h \mod n \tag{5.10}$$

survive after the orbifold projection. We define $s_{b,h}(z)$ as the function obtained from (4.5) by restricting the product over (p, q) by (5.10). This restricted product is realized by substituting

$$p = np' + [k + h]_n, \quad q = nq' + k, \tag{5.11}$$

to (4.5), and perform the product with respect to non-negative integers p' and q', and $k = 0, 1, \ldots, n-1$. $[h]_n$ represents the remainder when h is divided by n. It is convenient to introduce notation $\langle \cdots \rangle_n$ defined by

$$\langle h \rangle_n = \frac{1}{n}\left([h]_n + \frac{1}{2}\right) - \frac{1}{2}. \tag{5.12}$$

This satisfies the relations

$$\langle h + an \rangle_n = \langle h \rangle_n \quad (a \in \mathbb{Z}), \quad \langle -1 - h \rangle_n = -\langle h \rangle_n. \tag{5.13}$$

We rewrite the numerator in (4.5) as

$$\begin{aligned} & b\left(p + \frac{1}{2}\right) + b^{-1}\left(q + \frac{1}{2}\right) - iz \\ &= n\left[b\left(p' + \langle k + h \rangle_n + \frac{1}{2}\right) + b^{-1}\left(q' + \langle k \rangle_n + \frac{1}{2}\right) - i\frac{z}{n}\right]. \end{aligned} \tag{5.14}$$

The denominator in (4.5) is also rewritten in a similar way, and we obtain

$$\begin{aligned} s_{b,h}(z) &= \prod_{k=0}^{n-1} \prod_{p',q'=0}^{\infty} \frac{b(q' + \frac{1}{2}) + b^{-1}(p' + \frac{1}{2}) + b\langle k \rangle_n + b^{-1}\langle k + h \rangle_n - i\frac{z}{n}}{b(p' + \frac{1}{2}) + b^{-1}(q' + \frac{1}{2}) + b\langle k + h \rangle_n + b^{-1}\langle k \rangle_n + i\frac{z}{n}} \\ &= \prod_{k=0}^{n-1} \prod_{p',q'=0}^{\infty} \frac{b(q' + \frac{1}{2}) + b^{-1}(p' + \frac{1}{2}) + b\langle k \rangle_n + b^{-1}\langle k + h \rangle_n - i\frac{z}{n}}{b(p' + \frac{1}{2}) + b^{-1}(q' + \frac{1}{2}) - b\langle k \rangle_n - b^{-1}\langle k + h \rangle_n + i\frac{z}{n}}. \end{aligned} \tag{5.15}$$

In the second line we replaced k in the denominator by $-1-k-h$ and used the second relation in (5.13). In the final expression the product with respect to p' and q' has the same form as that in the definition (4.5) of the double sine function $s_b(z)$, and we obtain

$$s_{b,h}(z) = \prod_{k=0}^{n-1} s_b\left(\frac{z}{n} + ib\langle k\rangle_n + ib^{-1}\langle k+h\rangle_n\right). \tag{5.16}$$

By definition, the product of $s_{b,h}(z)$ over all h reproduces the original double sine function;

$$s_b(z) = \prod_{h=0}^{n-1} s_{b,h}(z). \tag{5.17}$$

The function $s_{b,h}(z)$ satisfies the following formulae, which are analogs of (F.17) and (F.18).

- Self-duality and reflection property

$$s_{b,h}(z) = s_{b^{-1},-h}(z) = \frac{1}{s_{b,-h}(-z)} = \frac{1}{s_{b^{-1},h}(-z)}. \tag{5.18}$$

- Functional equations

$$\begin{aligned}\frac{s_{b,h+1}(z+\frac{ib}{2})}{s_{b,h}(z-\frac{ib}{2})} &= \frac{1}{2\cosh\left(\frac{\pi bz}{n}+\pi i\langle h\rangle\right)},\\ \frac{s_{b,h-1}(z+\frac{ib^{-1}}{2})}{s_{b,h}(z-\frac{ib^{-1}}{2})} &= \frac{1}{2\cosh\left(\frac{\pi b^{-1}z}{n}+\pi i\langle -h\rangle\right)},\\ \frac{s_{b,h}(z+\frac{i}{v})}{s_{b,h}(z-\frac{i}{v})} &= \frac{1}{[2\sinh\left(\frac{\pi bz+\pi ih}{n}\right)][2\sinh\left(\frac{\pi b^{-1}z-\pi ih}{n}\right)]}.\end{aligned} \tag{5.19}$$

The one-loop determinant for $\boldsymbol{S}^3/\boldsymbol{Z}_n$ is obtained simply by replacing $s_b(z)$ in (4.4) by $s_{b,h}(z)$.

$$Z^{1-\mathrm{loop}}(\lambda,h) = \frac{\prod_{\alpha\in\Delta} s_{b,\alpha(h)}\left(\alpha(\lambda)-\frac{i}{v}\right)}{\prod_I s_{b,\rho_I(h)}\left(\rho_I(\lambda)-\frac{i(1-\Delta_I)}{v}\right)}. \tag{5.20}$$

The 1-loop determinant of vector multiplets can be rewritten in terms of elementary functions.

$$\begin{aligned}Z^{1-\mathrm{loop}}_{\mathrm{vector}}(\lambda,h) &= \prod_{\alpha\in\Delta} s_{b,\alpha(h)}\left(\alpha(\lambda)-\frac{i}{v}\right)\\ &= \prod_{\alpha\in\Delta_+}\left[2\sinh\frac{\pi}{n}\left(b\alpha(\lambda)+i\alpha(h)\right)\right]\left[2\sinh\frac{\pi}{n}\left(b^{-1}\alpha(\lambda)-i\alpha(h)\right)\right].\end{aligned} \tag{5.21}$$

When $b=1$, this agree with the partition function in the lens space $L(n,-1)$ given in [37].

5.2 Dualities in $\boldsymbol{S}^3/\mathbb{Z}_n$

A gauge theory in $\boldsymbol{S}^3/\mathbb{Z}_n$ has degenerate vacua labeled by holonomies associated with the gauge symmetry. The contributions of these vacua should be summed up to obtain the total partition function. In this section we consider two dual pairs and confirm that the partition functions of theories dual to each other agree if appropriate phase factors are inserted in the holonomy sum.

5.2.1 $\mathcal{N}=2$ SQED and XYZ model

We first consider the mirror symmetry between an $\mathcal{N}=2$ SQED and the XYZ model [46]. On one side of the duality, we consider $\mathcal{N}=2$ SQED with two chiral multiplets q and $\widetilde{q}$ with $U(1)$ charge $+1$ and -1, respectively. We assume that q and $\widetilde{q}$ have the Weyl weight Δ. The mirror theory, the XYZ model, consists of three chiral multiplets Q, $\widetilde{Q}$ and S interacting through the superpotential

$$W=\widetilde{Q}SQ. \tag{5.22}$$

Note that these three fields are symmetric, though we describe the S looks special. By the operator relation $S=\widetilde{q}q$ and the marginality of the superpotential (5.22), we can determine the Weyl weights of the fields in this theory as

$$\Delta_S=2\Delta,\quad \Delta_Q=\Delta_{\widetilde{Q}}=1-\Delta. \tag{5.23}$$

Although the correct value of Δ at the infra-red fixed point is $\Delta=1/3$, the equality of partition functions holds regardless of Δ [14], and we leave Δ unfixed.

The global symmetry which is the same for these two theories is $U(1)_V\times U(1)_A$. The charge assignments are shown in Table 5.1. We introduce real mass parameters

Table 5.1: The charge assignment of the global symmetry $U(1)_V\times U(1)_A$ of SQED and XYZ model which are mirror to each other. m and $\widetilde{m}$ are the monopole and anti-monopole operators.

	q	$\widetilde{q}$	m	$\widetilde{m}$	Q	$\widetilde{Q}$	S
$U(1)_V$	0	0	1	-1	1	-1	0
$U(1)_A$	1	1	0	0	-1	-1	2

ζ and μ for $U(1)_V$ and $U(1)_A$, respectively. $U(1)_A$ symmetry in SQED is the topological $U(1)$ symmetry acting on monopole operators, and the corresponding mass parameter ζ is the Fayet-Iliopoulos parameter. The $\boldsymbol{S}^3$ partition functions of these theories are

$$\begin{aligned} Z^{\text{SQED}}&=\int\frac{e^{-2\pi i\zeta\lambda}}{s_b(\lambda+\mu-\frac{i(1-\Delta)}{v})s_b(-\lambda+\mu-\frac{i(1-\Delta)}{v})}d\lambda,\\ Z^{\text{XYZ}}&=\frac{1}{s_b(\zeta-\mu-\frac{i\Delta}{v})s_b(-\zeta-\mu-\frac{i\Delta}{v})s_b(2\mu-\frac{i(1-2\Delta)}{v})}. \end{aligned} \tag{5.24}$$

Because two theories are mirror to each other, the partition functions should agree. This agreement is confirmed by using the pentagon relation of the double sine function [47][1]:

$$\int \frac{s_b(x+r)}{s_b(x+s)} e^{-2\pi i t x} dx = e^{\pi i t(r+s)} \frac{s_b(t-\frac{r}{2}+\frac{s}{2}+\frac{i}{v})}{s_b(t+\frac{r}{2}-\frac{s}{2}-\frac{i}{v})} s_b(r-s-\frac{i}{v}). \tag{5.25}$$

By substituting

$$x=\lambda, \quad r=-\mu+\frac{i(1-\Delta)}{v}, \quad s=\mu-\frac{i(1-\Delta)}{v}, \quad t=\zeta, \tag{5.26}$$

to the pentagon relation (5.25), we obtain $Z^{\mathrm{XYZ}} = Z^{\mathrm{SQED}}$. Note that for the agreement of the two partition functions, the integration measure should be chosen as in (4.144).

Let us generalize this to the theories on the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$. On the SQED side, we need to sum up the contribution of n saddle points specified by the holonomy h of the $U(1)$ gauge symmetry. We can also introduce holonomies h_V and h_A for $U(1)_V$ and $U(1)_A$ global symmetries as non-dynamical background gauge potentials. Because $U(1)_V$ current in SQED is the field strength of the dynamical gauge field A, the $U(1)_V$ holonomy is realized by the Chern-Simons term

$$S = \frac{i}{2\pi} \int V dA, \tag{5.27}$$

where V is the non-dynamical $U(1)_V$ background gauge field. In the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$, this term gives rise to the non-trivial phase factor

$$\Phi = 2\pi \frac{h_V h}{n}. \tag{5.28}$$

Taking account of this phase factor, the partition function for each holonomy is

$$Z^{\mathrm{SQED}}(h,h_V,h_A) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i\zeta\lambda/n} e^{2\pi i h_V h/n}}{s_{b,h_A+h}(\mu+\lambda-\frac{i(1-\Delta)}{v}) s_{b,h_A-h}(\mu-\lambda-\frac{i(1-\Delta)}{v})} \frac{d\lambda}{n}. \tag{5.29}$$

On the other hand, the partition function of the XYZ model is

$$Z^{\mathrm{XYZ}}(h_V,h_A) = \frac{1}{s_{b,-h_A+h_V}(-\mu+\zeta-\frac{i\Delta}{v}) s_{b,-h_A-h_V}(-\mu-\zeta-\frac{i\Delta}{v}) s_{b,2h_A}(2\mu-\frac{i(1-2\Delta)}{v})}. \tag{5.30}$$

Naive expectation is that these are related by

$$Z^{\mathrm{XYZ}}(h_V,h_A) = \sum_{h=0}^{n-1} Z^{\mathrm{SQED}}(h,h_V,h_A). \tag{5.31}$$

[1]The pentagon relation usually refers to the operator equation $\varphi_b(\hat{P})\varphi_b(\hat{X}) = \varphi_b(\hat{X})\varphi_b(\hat{X}+\hat{P})\varphi_b(\hat{P})$, where $\hat{X}$ and $\hat{P}$ are operators satisfying $[\hat{P},\hat{X}] = 1/2\pi i$. $\varphi_b(z)$ is the quantum dilogarithm related to the double sine function by $s_b(z) = \exp[-\pi i(\frac{z^2}{2}+\frac{b^2+b^{-2}}{24})]\varphi_b(z)$. This operator equation is equivalent to (5.25) [48, 49], which we refer to as the pentagon relation.

This is actually the case only when $h_A = 0$. We confirmed this relation numerically up to $n = 10$. Again, the choice of the integration measure (5.5) is essential for the equality in (5.31).

The relation (5.31), however, does not hold if we turn on the holonomy h_A for $U(1)_A$ symmetry. Instead, we found that the relation

$$Z^{\mathrm{XYZ}}(h_V, h_A) = \sum_h \sigma(h, h_V, h_A) Z^{\mathrm{SQED}}(h, h_V, h_A) \tag{5.32}$$

hold if we choose an appropriate sign function $\sigma(h, h_V, h_A) = \pm 1$. The analysis for $h_A = 0$ implies

$$\sigma(h, h_V, 0) = 1. \tag{5.33}$$

We can determine $\sigma(h, h_V, h_A)$ for general h_A by the numerical analysis. For $n = 2, 3, 4$, we obtained

$$\sigma_1^{(2)} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \sigma_1^{(3)} = \sigma_2^{(3)} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix},$$
$$\sigma_1^{(4)} = \sigma_3^{(4)} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \quad \sigma_2^{(4)} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \tag{5.34}$$

where we express the function in the matrix form

$$(\sigma_{h_A}^{(n)})_{h, h_V} = \sigma(h, h_V, h_A). \tag{5.35}$$

We determined the signs up to $n = 10$, and found the general form

$$\sigma(h, h_V, h_A) = (-1)^{f(h_A) + g(h_A, h) + g(h_A, h_V)}, \tag{5.36}$$

where

$$f(h) = \min(|h + n\mathbf{Z}|), \quad g(h, h') = \min(f(h), f(h')). \tag{5.37}$$

5.2.2 $SU(2)$ gauge theory and a chiral multiplet

As the second example, we consider the duality proposed by Jafferis and Yin in [15]. The theory on one side is $SU(2)$ Chern-Simons theory with level $k = 1$ coupling to one adjoint chiral multiplet Φ. It is dual to the theory consisting of a single chiral multiplet X. These theories have global symmetry $U(1)_A$ rotating Φ and X with charges 1 and 2, respectively.

Let us first compute the $\boldsymbol{S}^3$ partition function of the $SU(2)$ theory. We parameterize the $SU(2)$ Cartan algebra by

$$\lambda = x T_3, \quad T_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{5.38}$$

and we adopt the integration measure

$$[d\lambda] = \frac{dx}{2\sqrt{2}}, \tag{5.39}$$

where the factor 1/2 comes from the order of the Weyl group of $SU(2)$, and $1/\sqrt{2}$ from the normalization of the $SU(2)$ generators $\mathrm{tr}T_aT_b = (1/2)\delta_{ab}$. The classical value of the Chern-Simons term with level $k = 1$ is

$$S_0 = \pi i \mathrm{tr}(\lambda^2) = \frac{\pi i}{2}x^2. \tag{5.40}$$

The partition function of the $SU(2)$ theory is

$$Z^{SU(2)} = \int \frac{e^{-\frac{\pi}{2}ix^2} s_b(x - \frac{i}{v}) s_b(-x - \frac{i}{v})}{s_b(x - \frac{i(1-\Delta)}{v}) s_b(-\frac{i(1-\Delta)}{v}) s_b(-x - \frac{i(1-\Delta)}{v})} \frac{dx}{2\sqrt{2}}, \tag{5.41}$$

where we denote the Weyl weight of the adjoint chiral multiplet by Δ. If we turn on the real mass parameter μ for $U(1)_A$ the weight Δ is replaced by $\Delta - iv\mu$.

The dual theory contains a single chiral multiplet X. This corresponds to the gauge invariant operator $\mathrm{tr}\Phi^2$ in the $SU(2)$ gauge theory, and has Weyl weight 2Δ. The $\boldsymbol{S}^3$ partition function is

$$Z^X = \frac{1}{s_b(-\frac{i(1-2\Delta)}{v})}. \tag{5.42}$$

We can easily check numerically that these two partition functions coincide up to a phase factor.

$$Z^{SU(2)} = e^{i\phi} Z^X, \quad \phi = -\pi\left(\frac{1}{4} + \frac{2\Delta + \Delta^2}{2v^2}\right). \tag{5.43}$$

This relation is confirmed in [15] for the round sphere. The coincidence of the absolute value is due to our choice of the integration measure. In [15] different measure is used and extra numerical factor arises. We do not argue about this point, and focus only on the phases. For the round sphere, the phase factor

$$e^{i\phi} = \exp\left[\pi i\left(\frac{1}{4} - \frac{(1+\Delta)^2}{2}\right)\right] = \int e^{\pi i t^2 - \sqrt{2}\pi i(1+\Delta)t} dt, \tag{5.44}$$

is interpreted in [15] as the contribution of a decoupled topological sector. For squashed $\boldsymbol{S}^3$, there seems no such a simple explanation for this factor.

We would like to extend this relation to the orbifolds. In the introduction of holonomy, we should note that the gauge group is, precisely speaking, not $SU(2)$ but $SU(2)/\mathbb{Z}_2 = SO(3)$. The allowed holonomies are

$$\exp\left(2\pi i \frac{h}{n} T_3\right), \quad h = 0, \ldots, n-1. \tag{5.45}$$

(If the gauge group were $SU(2)$, 2π in the exponent in (5.45) should be replaced by 4π.) For the flat connection specified by the holonomy h, the classical Chern-Simons action gives the phase factor

$$e^{i\Phi} = e^{\frac{\pi i}{2n}h^2}. \tag{5.46}$$

This is not well defined as a map from $\mathbb{Z}_n$ to $\mathbb{C}$. This gives different phases for h and $h + n$, which are identical in $\mathbb{Z}_n$. We will fix this ambiguity later by an additional rule.

The orbifold partition function of the $SU(2)$ theory is obtained from the $\boldsymbol{S}^3$ partition function (5.41) by

- replacing each $s_b(z)$ by $s_{b,h}(z)$ with an appropriate holonomy,
- replacing the measure dx by dx/n,
- replacing the classical action S_0 in (5.40) by S_0/n,
- and introducing the phase factor $e^{\pi i h^2/2n}$.

We obtain

$$Z^{SU(2)}(h,h_A)=\int\frac{e^{\pi i\frac{h^2}{2n}}e^{-\frac{\pi}{2n}ix^2}s_{b,h}(x-\frac{i}{v})s_{b,-h}(-x-\frac{i}{v})}{s_{b,h_A+h}(x-\frac{i(1-\Delta)}{v})s_{b,h_A}(-\frac{i(1-\Delta)}{v})s_{b,h_A-h}(-x-\frac{i(1-\Delta)}{v})}\frac{dx}{2\sqrt{2}n}. \tag{5.47}$$

The partition function of the chiral multiplet X is

$$Z^X(h_A)=\frac{1}{s_{b,2h_A}(-\frac{i(1-2\Delta)}{v})}. \tag{5.48}$$

We consider two cases with even n and odd n separately. Let us first consider the case with odd n. In this case, (5.46) defines double-valued map from $\mathbb{Z}_n$ to $\mathbb{C}$. For h and $h+n$, which are identified in $\mathbb{Z}_n$, the phase factor takes different values whose phases always differ by $\pi/2$. We denote these two phase factors by $(e^{\frac{\pi i}{2n}h^2})_\pm$. The subscript $\pm$ is chosen so that the two phases satisfy $(e^{\frac{\pi i}{2n}h^2})_+=i(e^{\frac{\pi i}{2n}h^2})_-$. Corresponding to these two choices of the phase factor, we define two partition functions $Z^{SU(2)}_\pm(h,h_A)$.

We take the ansatz

$$\sum_{h=0}^{n-1}\sigma(h,h_A)e^{\mp\frac{\pi i}{4}}Z^{SU(2)}_\pm(h,h_A)=e^{i\phi}Z^X(h_A), \tag{5.49}$$

between the partition functions of the dual theories. $\sigma(h,h_A)$ is an unknown phase function depending on the $SU(2)$ holonomy h and $U(1)_A$ holonomy h_A, and $e^{i\phi}$ is a phase factor independent of holonomies. The double signs on the left hand side are in the same order. The factor $e^{\mp\frac{\pi i}{4}}$ is inserted to cancel the difference of $Z^{SU(2)}_+$ and $Z^{SU(2)}_-$. Although we can choose one of signs as a convention and absorb this factor by $\sigma(h,h_A)$ or $e^{i\phi}$, we separate this factor for later convenience. We carried out the numerical analysis up to $n=29$, and we found

$$\begin{aligned}\sigma(h,h_A)&=(-1)^{g(h_A,h)}\exp\left[i\pi\frac{f(h_A)(f(h_A)+n)}{2n}\right],\\ \phi(h_A)&=-\pi\frac{\Delta^2+2\Delta}{2nv^2},\end{aligned} \tag{5.50}$$

make the equation (5.49) hold, where f and g are the functions defined in (5.37).

Let us turn to the case with even n. In this case we divide n possible holonomies to the $n/2$ satisfying

$$h-\frac{n}{2}\in 2\mathbb{Z}_n, \tag{5.51}$$

and the others. The phase factor (5.46) is well-defined for holonomies satisfying (5.51), while (5.46) has the sign ambiguity for the other holonomies. With the numerical analysis up to $n = 30$, we found that $Z^X(h_A)$ can be given as a linear combination of only $Z^{SU(2)}(h, h_A)$ with h satisfying (5.51),

$$\sum_{h-n/2\in 2\mathbb{Z}_n} \sigma(h, h_A)\sqrt{2}Z^{SU(2)}(h, h_A) = e^{i\phi}Z^X(h_A), \tag{5.52}$$

where $\sigma(h, h_A)$ and $\phi(h_A)$ are functions defined in (5.50). Comparing this to (5.49), we notice that the phase factor $e^{\pm\frac{\pi i}{4}}$ is replaced by $\sqrt{2} = e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}$. Although this factor depends on the choice of the integration measure and this may not have physical significance, it may be interesting to discuss what this factor implies under the assumption that our choice of the integration measure is an appropriate one. One possible interpretation is as follows. In the theory of the chiral multiplet X, the $U(1)_A$ holonomy appear only through $2h_A$. When n is even, there are two holonomies which gives the same $2h_A$. Let h_A be one of them, and $h'_A = h_A + n/2$ the other. It is natural to sum up the contribution of these two holonomies on the $SU(2)$ theory side. If we introduce different phase factors $e^{+\frac{\pi i}{4}}$ and $e^{-\frac{\pi i}{4}}$ for h_A and h'_A in this summation, we obtain the following relation similar to (5.49).

$$\begin{aligned} e^{i\phi}Z^X(h_A) = e^{i\phi}Z^X(h'_A) = &\sum_{h-n/2\in 2\mathbb{Z}_n} \sigma(h, h_A)e^{\frac{\pi i}{4}}Z^{SU(2)}(h, h_A) \\ &+ \sum_{h-n/2\in 2\mathbb{Z}_n} \sigma(h, h'_A)e^{-\frac{\pi i}{4}}Z^{SU(2)}(h, h'_A). \end{aligned} \tag{5.53}$$

5.2.3 $S^3/\mathbb{Z}_{2k+1}$

In the previous subsections, we found that we need non-trivial phase factors to match the partitions functions of dual theories in two examples. For odd n, in fact, we can express these phase factors in a unified way. Let us define σ_h by

$$\sigma_h = (-1)^{[h]_n([h]_n-(-1)^{(n-1)/2})/2}. \tag{5.54}$$

When n is odd, this takes values ±1 depending on $h \in \mathbb{Z}_n$. We can represent $(-1)^{f(h)}$ and $(-1)^{g(h,h')}$ with this function by

$$(-)^{f(h)} = \sigma_{2h}, \quad (-)^{g(h,h')} = \sigma_{h+h'}\sigma_{h-h'}. \tag{5.55}$$

Therefore, the sign function (5.36) in the first example can be given as the product of five σ_h;

$$\sigma(h, h_V, h_A) = \sigma_{h-h_A}\sigma_{h+h_A}\sigma_{h_V+h_A}\sigma_{h_V-h_A}\sigma_{2h_A}. \tag{5.56}$$

The indices of five σ_h coincide up to sign with the holonomy indices of the functions $s_{b,h}(z)$ appearing in the mirror relation (5.32). Because $\sigma_h = \sigma_{-h}$ and the sign of the index of σ_h does not matter, the phases can be absorbed into the definition of the function $s_{b,h}(z)$. Namely, if we define $\widehat{Z}^{\text{SQED}}$ and $\widehat{Z}^{\text{XYZ}}$ from Z^{SQED} and Z^{XYZ}, respectively, by replacing $s_{b,h}(z)$ in these partition functions by $\widehat{s}_{b,h}(z)$ defined by

$$\widehat{s}_{b,h}(z) = \sigma_h s_{b,h}(z), \tag{5.57}$$

the relation

$$\widehat{Z}^{\rm XYZ}(h_V, h_A) = \sum_{h=0}^{n-1} \widehat{Z}^{\rm SQED}(h, h_V, h_A) \tag{5.58}$$

holds without the extra sign factors. This is actually the case in the second example. Because the phase function can be written as

$$\sigma(h, h_A) = \sigma_{h_A+h}\sigma_{h_A-h} \exp\left[i\pi \frac{f(h_A)(f(h_A)+n)}{2n}\right], \tag{5.59}$$

$\widehat{Z}^{SU(2)}$ and $\widehat{Z}^X$ defined with $\widehat{s}_{b,h}(z)$ satisfy the relation

$$\sum_{h=0}^{n-1} e^{\mp\frac{\pi i}{4}} \widehat{Z}_\pm^{SU(2)}(h, h_A) = \omega(h_A) e^{i\phi} \widehat{Z}^X(h_A), \tag{5.60}$$

where $\omega(h_A)$ is a certain factor depending only on h_A.

In the two examples, we found that if we replace $s_{b,h}(z)$ by $\widehat{s}_{b,h}(z)$ the duality relations hold without introducing non-trivial relative phase factors in the holonomy sums. This is simple enough for us to expect that this rule is universal. It would be interesting to check whether this rule really holds for other examples of dual pairs.

Summary

We investigated relative phases in the holonomy sum, which is necessary to obtain the partition functions of gauge theories in $\boldsymbol{S}^3/\mathbb{Z}_n$. We used a few dualities between gauge theories and non-gauge theories to determine the phases.

We first considered mirror symmetry between $\mathcal{N} = 2$ SQED with one flavor and the XYZ model containing three chiral multiplets. We showed that with the appropriate choice of the phases in the holonomy sum the partition functions of these theories coincides. Furthermore, we found that when n is odd, the phase factor is absorbed by the redefinition of the single function $s_{b,h}(z)$, the orbifold extension of the double sine function. We also considered the duality between $SU(2)$ gauge theory and a chiral multiplet proposed by Jafferis and Yin. We could again find phase factors which makes the duality relation hold. When n is odd the phases are absorbed by redefining the function $s_{b,h}(z)$ in the same way as in the first example.

When n is odd, in all these examples, the phase factors can be absorbed by the definition of the function $s_{b,h}(z)$. This fact strongly suggests that the modified function $\widehat{s}_{b,h}$ in (5.57) always gives a "correct" partition function in some sense.

Lastly, we mention the remaining problems. So far we do not have a general formula to derive the correct phases when n is even. In order to have the correct partition functions for every theory the general formula is desired.

When nk is odd the Chern-Simons term is actually ill-defined:

$$e^{\frac{i\pi k}{n}h^2} \longrightarrow e^{\frac{i\pi k}{n}(h+n)^2} = e^{\frac{i\pi k}{n}h^2 + 2\pi i k h + i\pi n k}. \tag{5.61}$$

Though the shift $h \to h+n$ should not make any change, it actually does give a different phase. As we saw in Section 5.2.2 if we choose h or $h+n$ appropriately it works for checking the duality. We have not had any rule which sector h or $h+n$ we should choose. We will face this problem in next chapter again, and give a heuristic rule to select them.

Chapter 6

Orbifolded partition function for the AdS/CFT

When we assume the gravity side is described by the general relativity we need to evaluate the partition function in the large N limit as we did in the previous chapters. In this chapter we firstly review the result of [50], where the orbifolded partition function in the large N limit is evaluated, and the coincidence of the partition functions for the ABJM model and the gravity dual is confirmed. In the case of the orbifolded partition function the large N means the large number of holonomies, which leads to diverging number of terms. Naively, we seem not to deal with these terms. However, in the large N limit only limited terms contribute to the leading order of the $1/N$ expansion of the partition function and we can evaluate it.

Having checked the coincidence of the partition functions in the large N limit our next step maybe to explore next leading order of $1/N$ expansion. It should contain the information of the quantum effect of the gravity. If we use the orbifolded partition function to study the next leading order we have to treat all the contributions, therefore, the phase factor plays a significant role there. The Monte Carlo method, which is first applied to evaluate the ABJM partition function in [42] enable us to calculate the orbifolded partition function up to some number N. For the large number of N it takes an unrealistic time so the practical maximum value is about $N = 20$ or so. Still, the numerical result gives us rich information about the finite N effects.

In order to use the Monte Carlo method we have to prepare an equivalent expression for the partition function of the ABJM model. This is because a numerical analysis is not compatible with the violently oscillating integration; the partition function with the Chern-Simons term is the typical example. It is known that the $\boldsymbol{S}^3$ partition function of the ABJM model can be rewritten as a certain integration without the Chern-Simons term [11]. This replacement is equivalent to consider the partition function of the mirror theory of the ABJM model with unit Chern-Simons level $k = 1$. (the rewriting itself is valid for any k but the mirror symmetry is only valid for $k = 1$). Since we have no formula to rewrite the orbifolded partition function so far let us consider the orbifolded partition function of the mirror theory. However, note that our formula to fix the phase factor is not confirmed for this mirror symmetry, we firstly have to check whether the orbifolded partition functions coincide or not. We perform this consistency check numerically, and the result shows that they coincide in high accuracy for some cases (see Subsection 6.3.2). This is one of our non-trivial result that our proposal for the formula to fix the phase factor is valid for the duality between the ABJM model and its mirror theory.

Finally, we show the free energy of the ABJM model with N up to 9. The result shows a characteristic behavior of the free energy, which is not seen in the $\boldsymbol{S}^3$ case. This is our second result of this chapter.

6.1 Phase factors for the ABJM model and its mirror

Since the formula to fix the phase factors we have derived is only applicable for the specific dualities for even n we need an extended version of the formula, which should at least be applicable to the ABJM model and its mirror theory. The important fact discussed in the previous chapter is that generally the sign factor cannot be absorbed into a single double sine function for general n. Therefore, the extended formula should be related to a pair of double sine functions. Our proposals to fix the relative phases for arbitrary n is that when there is a pair of chiral multiplets with the holonomy $h+h'$ and $h-h'$ we introduce the factor $(-1)^{g(h,h')}$:

$$s_{b,h+h'}\left(z+z'-i\frac{(1-\Delta)}{v}\right)s_{b,h-h'}\left(z-z'-i\frac{(1-\Delta)}{v}\right)\to(-1)^{g(h,h')}. \tag{6.1}$$

We also introduce the factor $(-1)^{f(h)}$ when there is a chiral multiplet with the holonomy $-2h$:

$$s_{b,-2h}\left(-2z-i\frac{(1-\Delta)}{v}\right)\to(-1)^{f(h)}. \tag{6.2}$$

Since these rules are just proposals they should be tested numerically, and we will do the test for the ABJM model and its mirror theory, later.

6.2 Analytical approach to large N limit

Let us evaluate the orbifolded partition function of the ABJM model in the large N limit. The gauge groups, Chern-Simons levels, the matter contents and the global symmetry for the ABJM model are already discussed in Section (2.3). Using those information the general formula (5.4) gives us the orbifolded partition function for the ABJM model as follows.

$$Z^{\mathrm{ABJM}}(h_A,h_B,h_T)=\sum_{h,\widetilde{h}}\sigma(h,\widetilde{h},h_A,h_B,h_T)Z_{\mathrm{ABJM}}(h,\widetilde{h},h_A,h_B,h_T). \tag{6.3}$$

where h and $\widetilde{h}$ are the holonomies for the gauge symmetry $U(N)^2$ and each of them has N components. $Z_{\text{ABJM}}(h,\widetilde{h},h_A,h_B,h_T)$ is given as

$$Z_{\text{ABJM}}(h,\widetilde{h},h_A,h_B,h_T)=\frac{1}{(n^N N!)^2}\int d^N\lambda d^N\widetilde{\lambda}e^{-\frac{i\pi k}{n}\sum_i\left(\lambda_i^2-\widetilde{\lambda}_i^2\right)+\frac{i\pi k}{n}\sum_i\left(h_i^2-\widetilde{h}_i^2\right)}$$
$$\frac{\prod_{i<j}\begin{pmatrix}2\sinh\frac{\pi}{n}\left(b\left(\lambda_i-\lambda_j\right)+i\left(h_i-h_j\right)\right)2\sinh\frac{\pi}{n}\left(b^{-1}\left(\lambda_i-\lambda_j\right)-i\left(h_i-h_j\right)\right)\\ \times 2\sinh\frac{\pi}{n}\left(b\left(\widetilde{\lambda}_i-\widetilde{\lambda}_j\right)+i\left(\widetilde{h}_i-\widetilde{h}_j\right)\right)2\sinh\frac{\pi}{n}\left(b^{-1}\left(\widetilde{\lambda}_i-\widetilde{\lambda}_j\right)-i\left(\widetilde{h}_i-\widetilde{h}_j\right)\right)\end{pmatrix}}{\prod_{i,j}\begin{pmatrix}s_{b,h_A+h_T+h_i-\widetilde{h}_j}(m_A+m_T+\lambda_i-\widetilde{\lambda}_j-\frac{i}{2v})\\ \times s_{b,-h_A+h_T+h_i-\widetilde{h}_j}(-m_A+m_T+\lambda_i-\widetilde{\lambda}_j-\frac{i}{2v})\\ \times s_{b,h_B-h_T-h_i+\widetilde{h}_j}(m_B-m_T-\lambda_i+\widetilde{\lambda}_j-\frac{i}{2v})\\ \times s_{b,-h_B-h_T-h_i+\widetilde{h}_j}(-m_B-m_T-\lambda_i+\widetilde{\lambda}_j-\frac{i}{2v})\end{pmatrix}}. \tag{6.4}$$

Using our proposal (6.1) and (6.2) the sign factor is given as follows.

$$\sigma(h,\widetilde{h},h_A,h_B,h_T)=(-1)^{\sum_{i,j}\left(g(h_A,h_T+h_i-\widetilde{h}_j)+g(h_B,-h_T-h_i+\widetilde{h}_j)\right)} \tag{6.5}$$

In order to derive the leading order of the free energy $F=-\log Z$ in large N limit we only need the information of the stationary point of the integrand; we do not need the integration in the partition function. Furthermore, we assume the leading contribution comes from the one with a certain holonomy $h=\overline{h}$ and $\widetilde{h}=\overline{\widetilde{h}}$. Even if the leading contributions are degenerate for different $\overline{h}$ the effect of degeneracy is only logarithmic;

$$F\sim-\log\left(kZ_{\overline{h},\overline{\widetilde{h}}}\right)=-\log\left(Z_{\overline{h},\overline{\widetilde{h}}}\right)-\log k \tag{6.6}$$

where k is the degree of the degeneracy. Therefore, we can ignore the degeneracy; the information we need is what is the value of $\overline{h}$ and $\overline{\widetilde{h}}$.

For simplicity, we set $b=1$ and $m_A=m_B=m_T=0$ from now on. The stationary point of the free energy satisfies following equations.

$$\frac{\partial F}{\partial\lambda_i}=-\frac{2i\pi k}{n}\lambda_i+\sum_{j\neq i}\frac{\pi}{n}\left(\coth\frac{\pi}{n}\left(\lambda_i-\lambda_j+i(h_i-h_j)\right)+\coth\frac{\pi}{n}\left(\lambda_i-\lambda_j-i(h_i-h_j)\right)\right)$$
$$-\sum_j\sum_{k=0}^{n-1}\sum_{I=1}^{4}(-1)^{\delta_{I,3}+\delta_{I,4}}\ i\pi z_{ij-I}\coth\left(\pi z_{ij-I}\right) \tag{6.7}$$

where

$$z_{ij-1}=\frac{1}{n}\left(\frac{1}{n}\left(\lambda_i-\widetilde{\lambda}_j-\frac{i}{2}\right)+i\langle k\rangle_n+i\langle k+h_A+h_T+h_i-\widetilde{h}_j\rangle_n\right)$$
$$z_{ij-2}=\frac{1}{n}\left(\frac{1}{n}\left(\lambda_i-\widetilde{\lambda}_j-\frac{i}{2}\right)+i\langle k\rangle_n+i\langle k-h_A+h_T+h_i-\widetilde{h}_j\rangle_n\right)$$
$$z_{ij-3}=\frac{1}{n}\left(\frac{1}{n}\left(-\lambda_i+\widetilde{\lambda}_j-\frac{i}{2}\right)+i\langle k\rangle_n+i\langle k+h_B-h_T-h_i+\widetilde{h}_j\rangle_n\right)$$
$$z_{ij-4}=\frac{1}{n}\left(\frac{1}{n}\left(-\lambda_i+\widetilde{\lambda}_j-\frac{i}{2}\right)+i\langle k\rangle_n+i\langle k-h_B-h_T-h_i+\widetilde{h}_j\rangle_n\right). \tag{6.8}$$

We used following formula to derive the expression above.

$$\begin{aligned}\partial_{\lambda_i} \log s_{1,h}\left(\lambda_i - \widetilde{\lambda}_j - \frac{i}{2}\right) &= \partial_{\lambda_i} \log \prod_{k=0}^{n-1} s_1\left(\frac{1}{n}\left(\lambda_i - \widetilde{\lambda}_j - \frac{i}{2}\right) + i\langle k\rangle_n + i\langle k+h\rangle_n\right) \\ &= \sum_{k=0}^{n-1} \frac{i\pi}{n}\left(\frac{1}{n}\left(\lambda_i - \widetilde{\lambda}_j - \frac{i}{2}\right) + i\langle k\rangle_n + i\langle k+h\rangle_n\right) \\ &\qquad \times \coth \pi\left(\frac{1}{n}\left(\lambda_i - \widetilde{\lambda}_j - \frac{i}{2}\right) + i\langle k\rangle_n + i\langle k+h\rangle_n\right).\end{aligned} \tag{6.9}$$

The derivation of this formula is given in F.1. $\frac{\partial F}{\partial \widetilde{\lambda}_i}$ are similar:

$$\begin{aligned}\frac{\partial F}{\partial \widetilde{\lambda}_i} &= \frac{2i\pi k}{n}\widetilde{\lambda}_i + \sum_{j\neq i} \frac{\pi}{n}\left(\coth \frac{\pi}{n}\left(\widetilde{\lambda}_i - \widetilde{\lambda}_j + i(\widetilde{h}_i - \widetilde{h}_j)\right) + \coth \frac{\pi}{n}\left(\widetilde{\lambda}_i - \widetilde{\lambda}_j - i(\widetilde{h}_i - \widetilde{h}_j)\right)\right) \\ &\quad + \sum_j \sum_{k=0}^{n-1} \sum_{I=1}^{4} (-1)^{\delta_{I,3}+\delta_{I,4}}\ i\pi z_{ji-I} \coth\left(\pi z_{ji-I}\right).\end{aligned} \tag{6.10}$$

Numerical results tell us that $\overline{h} = \overline{\widetilde{h}} = \mathrm{diag}(c, c, \cdots, c)$ for a certain integer $0 \leq c \leq n-1$ give us the leading contribution. The stationary point of the integrand of the leading contribution is given by the configurations of λ_i and $\widetilde{\lambda}_i$ shown in Figure 6.1 and 6.2. Therefore, at least for $(h_A, h_B, h_T) = (0, 0, 0)$ and $(1, 1, 1)$ case the

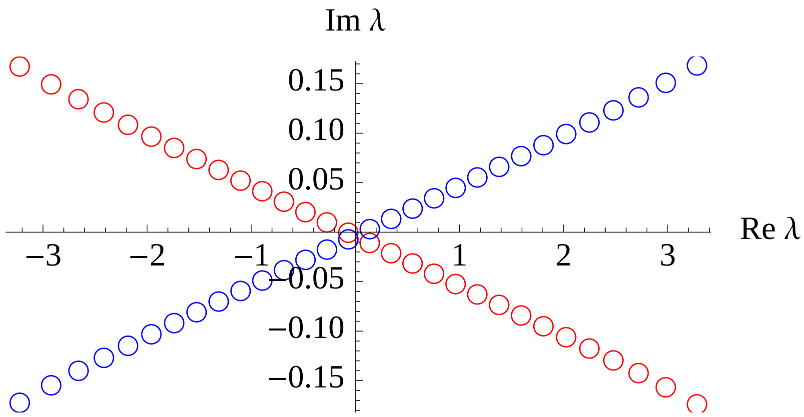

Figure 6.1: Numerical result for $(h_A, h_B, h_T) = (0, 0, 0)$ and $(1, 1, 1)$

same ansatz (4.126) should work for the orbifolded partition function as well, though the calculation is not so easy. We here just show the result of the $v = 1$ case [50]:

$$F = \frac{\pi\sqrt{2k}}{3n} N^{3/2}. \tag{6.11}$$

The difference compared to the round three-sphere case (3.88) is just $1/n$ factor. This is again exactly the same result as that of gravity side.

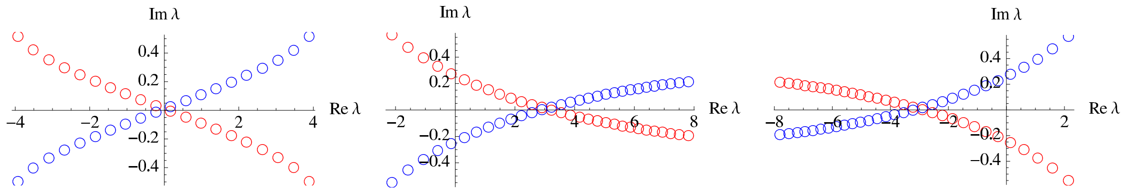

Figure 6.2: Numerical results for $(h_A, h_B, h_T) = (0,0,1)$ and $(1,1,0)$, $(0,1,0)$ and $(1,0,1)$, and $(1,0,0)$ and $(0,1,1)$ from the left to right.

6.3 Numerical approach to finite N

In this section we use Monte Carlo method to approach the finite N effects, which is explored in [42]. Basic idea of this method is to evaluate the integrals in the partition function numerically. The way of the evaluation is to use a Hybrid Monte Carlo simulation method polished in the lattice theory. The benefit of this method is that one can evaluate the partition function for N up to 20 or so. This number 20 is big enough to study the finite N effects. By the way, recently analytical approach to finite N effect is dramatically developed [6, 19, 20, 21, 22, 23, 24], though these methods seem not to be applicable to our orbifold background case.

In order to use the Monte Carlo method we have to prepare the mirror theory as discussed. On the round three-sphere we can analytically show the equivalence of the partition function of those dual theories [11]. However, on the squashed or the orbifolded three-sphere it is difficult to show the equivalence analytically for finite N. Hence firstly, we need to check the duality in some cases.

6.3.1 Mirror theory of the ABJM model

The ABJM model with unit Chern-Simons level is known to be mirror to the $\mathcal{N}=4$ $U(N)$ gauge theory with a fundamental hypermultiplet $(q,\widetilde{q})$ and an adjoint hypermultiplet (Φ_1,Φ_2) [11, 51]. In terms of $\mathcal{N}=2$ language this theory has the superpotential

$$W = \mathrm{tr}\Phi_3\left(q\widetilde{q} + [\Phi_1,\Phi_2]\right), \tag{6.12}$$

where Φ_3 is the chiral multiplet which form the $\mathcal{N}=4$ vector multiplet together with the $\mathcal{N}=2$ vector multiplet. The Weyl weights of the chiral multiplets are

$$\Delta_q = \Delta_{\widetilde{q}} = \Delta_{\Phi_1} = \Delta_{\Phi_2} = \frac{1}{2}, \quad \Delta_{\Phi_3} = 1. \tag{6.13}$$

These theories are expected to have $SO(8)$ global symmetry, which are not manifest in the Lagrangians. We only focus on the Cartan subalgebra

$$U(1)_A \times U(1)_B \times U(1)_T \times U(1)_R \subset SO(8), \tag{6.14}$$

where $U(1)_R$ is the R-symmetry acting on the $\mathcal{N}=2$ supercharges. The charge assignments in the two theories are listed in Table 2.2 and 6.1.

$U(1)_B$ in the mirror theory is a topological $U(1)$ symmetry coupling to monopole operators, and the corresponding mass parameter is the Fayet-Iliopoulos parameter.

Table 6.1: Charge assignments in the mirror theory of the ABJM model. m and $\widetilde{m}$ are monopole and anti-monopole fields.

	Φ_1	Φ_2	Φ_3	m	$\widetilde{m}$	q	$\widetilde{q}$
$U(1)_A$	$+1$	-1	0	0	0	0	0
$U(1)_B$	0	0	0	$+1$	-1	0	0
$U(1)_T$	$+1$	$+1$	-2	0	0	$+1$	$+1$

The partition function of the mirror theory is

$$Z^{\text{mirror}}(h_A, h_B, h_T) = \sum_h \sigma(h, h_A, h_B, h_T) Z_{\text{mirror}}(h, h_A, h_B, h_T). \tag{6.15}$$

$$Z_{\text{mirror}}(h, h_A, h_B, h_T) = \frac{1}{n^N N!} \int d^N \lambda e^{-\frac{2\pi i m_B}{n} \sum_i \lambda_i + \frac{2\pi i}{n} \sum_i h_i h_B}$$
$$\frac{\prod_{i<j} 2\sinh \frac{\pi}{n}\left(b\left(\lambda_i - \lambda_j\right) + i\left(h_i - h_j\right)\right) 2\sinh \frac{\pi}{n}\left(b^{-1}\left(\lambda_i - \lambda_j\right) - i\left(h_i - h_j\right)\right)}{\prod_{i,j} \begin{pmatrix} s_{b, h_A + h_T + h_i - h_j}(m_A + m_T + \lambda_i - \lambda_j - \frac{i}{2v}) \\ s_{b, -h_A + h_T + h_i - h_j}(-m_A + m_T + \lambda_i - \lambda_j - \frac{i}{2v}) \\ s_{b, -2h_T + h_i - h_j}(-2m_T + \lambda_i - \lambda_j - \frac{i}{2v}) \end{pmatrix} \prod_i s_{b, h_T + h_i}(m_T + \lambda_i - \frac{i}{2v}) s_{b, h_T - h_i}(m_T - \lambda_i - \frac{i}{2v})} \tag{6.16}$$

where we introduce mass parameters (m_A, m_B, m_T) and holonomies (h_A, h_B, h_T) corresponding to the flavor symmetry $U(1)_A \times U(1)_B \times U(1)_T$. h stands for h_i's, which are the holonomies for the gauge symmetry. The phase factor is derived using (6.1) and (6.2) as follows,

$$\sigma(h, h_A, h_B, h_T) = (-1)^{\sum_{i,j} g(h_A, h_T + h_i - h_j) + N f(h_T) + \sum_{i<j} g(-2h_T, h_i - h_j) + \sum_i g(h_T, h_i)}. \tag{6.17}$$

It is difficult to show the equivalence of the partition functions (6.15) and (6.4) in general N. However, in Abelian case we can show the equivalence through other dualities. As a preparation we consider another mirror dual theories: $\mathcal{N} = 4$ SQED and hypermultiplet.

It is known that the $\mathcal{N} = 4$ SQED with one flavor is mirror to a hypermultiplet [52]. This mirror pair is obtained from the $\mathcal{N} = 2$ mirror pair in §5.2.1 by adding a chiral multiplet $\widetilde{S}$ on the both sides of the duality. On the SQED side, the new chiral multiplet $\widetilde{S}$ couples to the system through the superpotential $W = \widetilde{q}\widetilde{S}q$. This corresponds to the mass term $W = \widetilde{S}S$ on the other side of the duality, and we can integrate out S and $\widetilde{S}$ to obtain the system with a hypermultiplet $(Q, \widetilde{Q})$. The global symmetry of the resulting mirror pair is $U(1)_V \times U(1)_A$ with the charge assignment summarized in Table 6.2. We again introduce the mass parameters ζ and μ for $U(1)_V$ and $U(1)_A$, respectively. We denote the Weyl weights of q and $\widetilde{q}$ by Δ. Then

Table 6.2: Global symmetries for $\mathcal{N}=4$ SQED and the hypermultiplets. m and $\widetilde{m}$ are again (anti-)monopole operators.

	q	$\widetilde{q}$	$\widetilde{S}$	m	$\widetilde{m}$	Q	$\widetilde{Q}$
$U(1)_V$	0	0	0	1	-1	1	-1
$U(1)_A$	1	1	-2	0	0	-1	-1

the Weyl weight of $\widetilde{S}$ is $1-2\Delta$. The introduction of $\widetilde{S}$ changes the partition functions by the factor

$$\frac{1}{s_{b,-2h_A}(-2\mu+\frac{i(1-2\Delta)}{v})}=s_{b,2h_A}(2\mu-\frac{i(1-2\Delta)}{v}). \tag{6.18}$$

The partition function of two theories are given by

$$\begin{aligned}
&Z^{\mathcal{N}=4}(\zeta,\mu;h,h_V,h_A)\\
&=\int_{-\infty}^{\infty}\frac{e^{-2\pi i\zeta\lambda/n}e^{2\pi i h_V h/n}}{\begin{array}{l}s_{b,h_A+h}(\mu+\lambda-\frac{i(1-\Delta)}{v})\\ \times s_{b,h_A-h}(\mu-\lambda-\frac{i(1-\Delta)}{v})\\ \times s_{b,-2h_A}(-2\mu-\frac{i(2\Delta-1)}{v})\end{array}}\frac{d\lambda}{n},
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
&Z^{\text{hyper}}(\zeta,\mu;h_V,h_A)\\
&=\frac{1}{s_{b,-h_A+h_V}(-\mu+\zeta-\frac{i\Delta}{v})s_{b,-h_A-h_V}(-\mu-\zeta-\frac{i\Delta}{v})}.
\end{aligned} \tag{6.20}$$

Because the factor (6.18) does not depend on h, it is rather trivial that the partition functions match if we use the same sign function (5.36) as in the $\mathcal{N}=2$ case. Namely, the following relation holds.

$$Z^{\text{hyper}}(\zeta,\mu;h_V,h_A)=\sum_{h=0}^{n-1}\sigma(h,h_V,h_A)Z^{\mathcal{N}=4}(\zeta,\mu;h,h_V,h_A). \tag{6.21}$$

Now we are ready to show the equivalence of the ABJM partition function and that of the mirror theory in Abelian case. The partition function of the mirror theory in Abelian gauge group becomes following relatively simple from.

$$Z^{\text{mirror}}(h_A,h_B,h_T)=\sum_h\sigma(h,h_B,h_T)Z_{\text{mirror}}(h,h_A,h_B,h_T) \tag{6.22}$$

with the contribution of each holonomy sector

$$\begin{aligned}
&Z_{\text{mirror}}(h,h_A,h_B,h_T)\\
&=\frac{1}{s_{b,h_A+h_T}\left(m_A+m_T-\frac{i}{2v}\right)s_{b,-h_A+h_T}\left(-m_A+m_T-\frac{i}{2v}\right)s_{b,-2h_T}(-2m_T)}\\
&\quad\times\int\frac{dx}{n}\frac{e^{2\pi i\frac{hh_B}{n}}e^{-2\pi i\frac{m_Bx}{n}}}{s_{b,h+h_T}(x+m_T-\frac{i}{2v})s_{b,-h+h_T}(-x+m_T-\frac{i}{2v})}\\
&=Z^{\text{hyper}}(m_A,-m_T;h_A,-h_T)|_{\Delta=\frac{1}{2}}Z^{\mathcal{N}=4}(m_B,m_T;h,h_B,h_T)|_{\Delta=\frac{1}{2}}.
\end{aligned} \tag{6.23}$$

With the relation (6.21), we can rewrite the partition function (6.22) as the product of two Z^{hyper};

$$Z^{\text{mirror}}(h_A,h_B,h_T)=Z^{\text{hyper}}(m_A,-m_T;h_A,-h_T)|_{\Delta=\frac{1}{2}}Z^{\text{hyper}}(m_B,m_T;h_B,h_T)|_{\Delta=\frac{1}{2}}. \tag{6.24}$$

On the ABJM side, we need to sum up n^2 contributions parameterized by a pair of holonomies (h_1,h_2) for the gauge group $U(1)_1\times U(1)_2$. The partition function of the sector specified by (h_1,h_2) is

$$\begin{aligned}&Z^{\text{ABJM}}(h_1,h_2,h_A,h_B,h_T)\\&=e^{i\Phi(h_1,h_2)}\int\frac{d\lambda}{n}\frac{d\widetilde{\lambda}}{n}\frac{\exp\left[-\frac{\pi i}{n}(\lambda^2-\widetilde{\lambda}^2)\right]}{\begin{array}{l}s_{b,h_A+h_T+h_1-h_2}(m_A+m_T+\lambda-\widetilde{\lambda}-\frac{i}{2v})\\ \times s_{b,-h_A+h_T+h_1-h_2}(-m_A+m_T+\lambda-\widetilde{\lambda}-\frac{i}{2v})\\ \times s_{b,h_B-h_T-h_1+h_2}(m_B-m_T+\widetilde{\lambda}-\lambda-\frac{i}{2v})\\ \times s_{b,-h_B-h_T-h_1+h_2}(-m_B-m_T+\widetilde{\lambda}-\lambda-\frac{i}{2v})\end{array}}.\end{aligned} \tag{6.25}$$

A question is if it is possible to choose an appropriate phases in the holonomy sum. The answer is rather simple. We do not need any non-trivial phases in this sum. Let us confirm this by summing up (6.25) over holonomies h_1 and h_2. If we define $h_{12}\equiv h_1-h_2$ and replace h_1 by $h_{12}+h_2$, h_2 appears only in the phase factor

$$\Phi=\frac{\pi}{n}(h_1^2-h_2^2)=\frac{\pi}{n}(h_{12}^2+2h_2h_{12}). \tag{6.26}$$

The summation with respect to h_2 gives non-vanishing result only when $h_{12}=0$, and we obtain

$$\begin{aligned}Z^{\text{ABJM}}(h_A,h_B,h_T)&=\sum_{h_1,h_2=0}^{n-1}Z^{\text{ABJM}}(h_1,h_2,h_A,h_B,h_T)\\&=\sum_{h=0}^{n-1}Z^{\text{ABJM}}(h,h,h_A,h_B,h_T)=nZ^{\text{ABJM}}(0,0,h_A,h_B,h_T)\\&=n\int\frac{d\lambda}{n}\frac{d\widetilde{\lambda}}{n}\frac{\exp\left[-\frac{\pi i}{n}(\lambda^2-\widetilde{\lambda}^2)\right]}{\begin{array}{l}s_{b,h_A+h_T}(m_A+m_T+\lambda-\widetilde{\lambda}-\frac{i}{2v})\\ \times s_{b,-h_A+h_T}(-m_A+m_T+\lambda-\widetilde{\lambda}-\frac{i}{2v})\\ \times s_{b,h_B-h_T}(m_B-m_T+\widetilde{\lambda}-\lambda-\frac{i}{2v})\\ \times s_{b,-h_B-h_T}(-m_B-m_T+\widetilde{\lambda}-\lambda-\frac{i}{2v})\end{array}}.\end{aligned} \tag{6.27}$$

We can easily perform the integral and have

$$\begin{aligned}&Z^{\text{ABJM}}(h_A,h_B,h_T)\\&=\frac{1}{\begin{array}{l}s_{b,h_A+h_T}(m_A+m_T-\frac{i}{2v})s_{b,-h_A+h_T}(-m_A+m_T-\frac{i}{2v})\\ \times s_{b,h_B-h_T}(m_B-m_T-\frac{i}{2v})s_{b,-h_B-h_T}(-m_B-m_T-\frac{i}{2v})\end{array}}\\&=Z^{\text{hyper}}(m_A,-m_T;h_A,-h_T)|_{\Delta=\frac{1}{2}}Z^{\text{hyper}}(m_B,m_T;h_B,h_T)|_{\Delta=\frac{1}{2}}.\end{aligned} \tag{6.28}$$

This result precisely agrees with the partition function of the mirror theory (6.24).

6.3.2 Numerical check of the mirror symmetry on $S^3/\mathbb{Z}_n$

Let us move on the non-Abelian case. We set mass parameters to zero for simplicity from now on. Though we have given the rules for the phase factors (6.1) and (6.2) they are just proposals so we need to check the equivalence of the orbifolded partition functions of the ABJM model and the mirror theory. Here, we check the equivalence for $N = 2$, $n = 2, 3$ cases numerically.

For the ABJM model it is technically difficult to evaluate the integral of the partition function because of the Chern-Simons factor, which oscillates violently. Therefore, we rotate the phases of $\lambda, \widetilde{\lambda}$ in Chern-Simons factor in (6.4) so that the integration converge:

$$-\frac{i\pi k}{n}\lambda_i^2 \to -\frac{i\pi k}{n}(e^{-i\pi\alpha}\lambda_i)^2 \quad \frac{i\pi k}{n}\widetilde{\lambda}_i^2 \to \frac{i\pi k}{n}(e^{i\pi\alpha}\widetilde{\lambda}_i)^2, \tag{6.29}$$

and try to find the asymptotic value of the α dependent partition function $Z(\alpha)$ at $\alpha = 0$ by plotting it as a function of α. We adopt an extrapolated value of a quadratic function fitted from the five points of $Z(\alpha)$ at $\alpha = 0.001, 0.002, 0.003, 0.004, 0.005$. On the other hand, the mirror side is easily calculated numerically by Mathematica etc.

$n = 2$ **case**

Since the ABJM model and the Mirror dual have three global symmetries we can introduce the holonomies for them. The holonomies take the integer $h \in \mathbb{Z}_n$, and hence, we have n^3 different combinations of the holonomies and the values for the partition functions. As the integrations of the partition functions take long time we firstly compare the integrand of the partition function. Fortunately, many of them have the same function form. For example let us consider $(h_A, h_B, h_T) = (1, 1, 1)$ in $n = 2$ case. The integrand of the partition function (6.25) becomes

$$e^{i\Phi(h_1,h_2)}\frac{\exp\left[-\frac{\pi i}{n}(\lambda^2 - \widetilde{\lambda}^2)\right]}{\begin{array}{l}s_{b,h_1-h_2}(\lambda - \widetilde{\lambda} - \frac{i}{2v})s_{b,h_1-h_2}(\lambda - \widetilde{\lambda} - \frac{i}{2v})\\ \times s_{b,-h_1+h_2}(\widetilde{\lambda} - \lambda - \frac{i}{2v})s_{b,-h_1+h_2}(\widetilde{\lambda} - \lambda - \frac{i}{2v})\end{array}}, \tag{6.30}$$

and this is exactly the same as that of $(h_A, h_B, h_T) = (0, 0, 0)$. Similarly the integrand of the partition function of $(h_A, h_B, h_T) = (0, 0, 1)$ is equivalent to that of $(h_A, h_B, h_T) = (1, 1, 0)$, the one of $(h_A, h_B, h_T) = (1, 0, 0)$. is the same as $(h_A, h_B, h_T) = (0, 1, 1)$. Therefore, the integrands of the partition functions of the ABJM model are divided into 3. One can also those of the mirror theory are divided into 4 (in the mirror case the equivalences are checked numerically). Therefore, we only need to perform the integration for these 7 out 16. The final results are actually categorized into three for $n = 2$ case. 2 out of 4 different integrand in the mirror side give the same value after the integration. We distinguish the difference of the integrands in the mirror theory side by putting different numbers of sector. The results for $v = 1$ are listed in Table 6.3. Quantitatively, some of them are identified in reasonable accuracy but other coincidences are not so accuracy. These mismatches mainly come from the technical difficulties in the ABJM model side. One may say

Table 6.3: The results for $n = 2$, $v = 1$ case. They are categorized by the values of results. The sector distinguish the difference of the integrand.

Category	Sector	(h_A, h_B, h_T)	Mirror theory	ABJM model
1		$(0,0,0)$ $(1,1,1)$	1.11906	1.13290
2		$(0,0,1)$ $(1,1,0)$	−0.10861	−0.10861
3	3-1	$(0,1,0)$ $(1,0,1)$	0.176777	0.176577
	3-2	$(1,0,0)$ $(0,1,1)$	0.176777	

that those coincidence between the ABJM model and its mirror might be accident, but we believe that this is the consequence of the mirror symmetry. In order to convince ourselves we show $v = 1.2$ case in Table 6.4. This also indicate that the

Table 6.4: The results for $n = 2, v = 1.2$.

Category	Sector	(h_A, h_B, h_T)	Mirror theory	ABJM model
1		$(0,0,0)$ $(1,1,1)$	3.28943	3.26610
2		$(0,0,1)$ $(1,1,0)$	−0.257084	−0.257269
3	3-1	$(0,1,0)$ $(1,0,1)$	0.439046	0.438456
	3-2	$(1,0,0)$ $(0,1,1)$	0.439046	

partition functions for the ABJM model and the mirror theory are the same.

$n = 3$ case

In this case and, generally say, in odd n case ($k = 1$) we need special treatment for the holonomies from the Chern-Simons term because of their sign ambiguity for h and $h + n$;

$$e^{\frac{i\pi}{n}h^2}, \qquad e^{\frac{i\pi}{n}(h+n)^2} = -e^{\frac{i\pi}{n}h^2}. \tag{6.31}$$

We thought the Chern-Simons terms for h and $h + n$ would be identified in $\mathbb{Z}_n$. Actually, they are not identified as in (6.31); they are the same only up to the sign. Therefore, we need to choose "correct signs" somehow for each holonomy, and check its validity by numerical analysis.

For the $U(2)$ gauge group case we checked all the possibility of the sign ambiguity for each holonomy and found that if we choose the plus for even h and the minus for odd h the partition functions of the ABJM model and the mirror theory coincide in some accuracy (the other combination of the signs and even/odd also works). Namely, the correct Chern-Simons term for the holonomy is

$$(-1)^h e^{\frac{i\pi}{n}h^2}. \tag{6.32}$$

The other choices of the sign give a quite big gap for the most accurate Category (Category 2 in the Table 6.5).

Let us see the numerical results for $n = 3$ and $v = 1$ case, which is summarized in Table 6.5. The number of sectors increases compare to $n = 2$ case and the

Table 6.5: The results for $n = 3$, $v = 1$ case. The results falls into 4 categories and 11 sectors for the mirror theory.

Category	Sector	(h_A, h_B, h_T)	Mirror theory	ABJM model
1		$(0,0,0)$	7.48359	7.42241
2	2-1	$(0,0,1)$ $(0,0,2)$	-0.0596831	-0.0596810
	2-2	$(1,1,0)$ $(1,2,0)$ $(2,1,0)$ $(2,2,0)$	-0.0596831	
3	3-1	$(0,1,0)$ $(0,2,0)$	0.57735	0.576233
	3-2	$(1,0,0)$ $(2,0,0)$	0.57735	
	3-3	$(1,1,1)$ $(1,2,2)$ $(2,1,1)$ $(2,2,2)$	0.57735	
	3-4	$(1,1,2)$ $(1,2,1)$ $(2,1,2)$ $(2,2,1)$	0.57735	
4	4-1	$(0,1,1)$ $(0,2,2)$	0.00000	-0.00010
	4-2	$(0,1,1)$ $(0,2,2)$	0.00000	
	4-3	$(0,1,2)$ $(0,2,1)$	0.00000	
	4-4	$(1,0,1)$ $(1,0,2)$ $(2,0,1)$ $(2,0,2)$	0.00000	

coincidence of the partition functions in different Sectors and Categories improves the reliability of the mirror symmetry on $\boldsymbol{S}^3/\mathbb{Z}_n$.

From these results we conclude that the ABJM partition functions on $\boldsymbol{S}^3/\mathbb{Z}_n$ can be calculated from that of the mirror theory.

We show the values of the free energy of the mirror theory up to $N = 9$ for $n = 2$ in Fig. 6.3. Even at this level we can see that the extrapolated values of the free energy at $1/N = 0$ give almost the same result of as (6.11), which is about 0.74.

The interesting features of this result are that :

- The large N values for each category are almost the same.
- The ways of approaches to the large N value are completely different, depending on the Category.

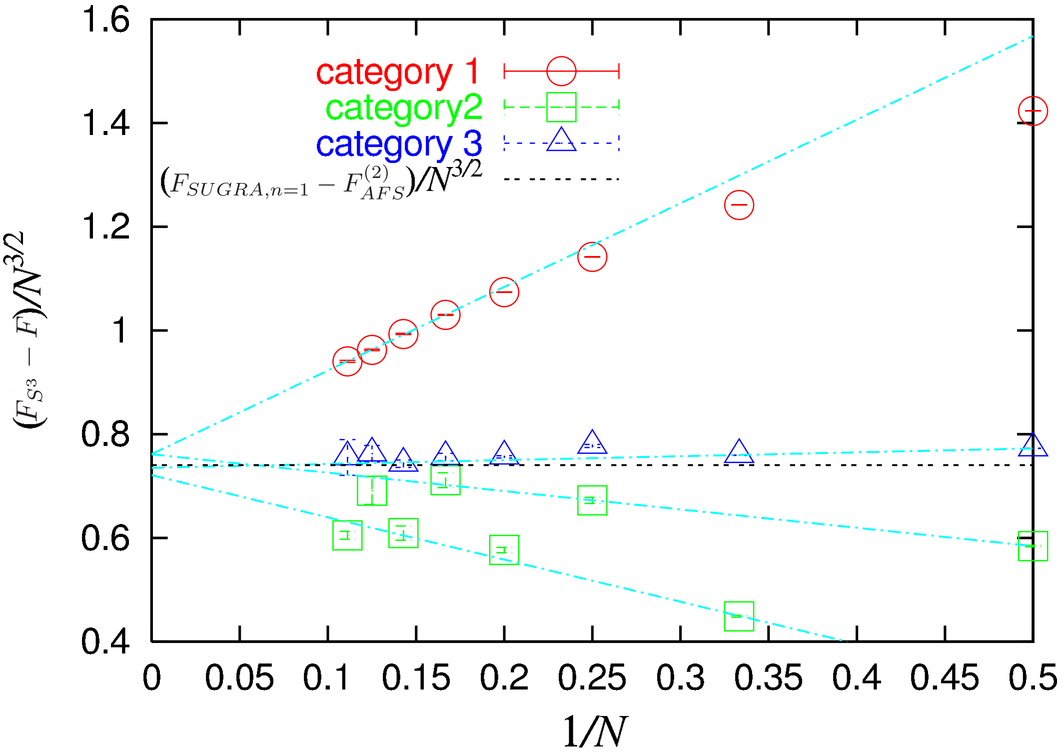

Figure 6.3: Numerical results for the free energy of the ABJM model on $\boldsymbol{S}^3/\mathbb{Z}_n$. The different categories correspond to the different values of h_A, h_B, h_T (see Table 6.4). $F^{(2)}_{AFS}$ is the free energy for the ABJM model with $n = 2$ in the large N limit. We plot the difference from that on $\boldsymbol{S}^3$.

At the moment we are not sure what the difference of the way of the approaches to the large N value means. The difference of the holonomies for global symmetry should correspond to a difference of some parameters on the gravity side. Since the holonomies are related to the global symmetries the corresponding parameters should be related to the isometry of the background geometry $\boldsymbol{S}^7$ of $AdS_4/\mathbb{Z}_n \times \boldsymbol{S}^7$. We have to calculate the orbifolded partition function for a different order of the orbifold n as well as to look into the gravity side to point out what is the origin of the gap of the way of the approach.

Chapter 7

Conclusions and Discussions

We started from the basics of the 3d supersymmetries and calculated a partition function of $\mathcal{N} = 2$ supersymmetric gauge theories on various backgrounds using the localization method. We also evaluate the large N value of free energies of the theories. Finally, we calculated the free energies of the ABJM model in the large N limit and checked the AdS/CFT correspondence between the low energy effective theories of the M2-branes throught the coincidences of the free energies.

In chapter 2 $\mathcal{N} = 2$ chiral and vector supermultiplets are defined and the transformation laws are derived from the supermultiplets. We also reviewed Lagrangians of the ABJM model in terms of $\mathcal{N} = 2$ supermultiplets as well as the component fields, which show the manifest $\mathcal{N} = 6$ supersymmetry.

In chapter 3 we extended the SUSY transformation laws on the flat space to those on conformally flat spaces, which include the three-sphere $\boldsymbol{S}^3$. Then, using the localization method we calculated the exact partition function for $\mathcal{N} = 2$ supersymmetric gauge theories on the round three-sphere $\boldsymbol{S}^3$. We evaluated the partition function for large N limit in order to compare the result with that of gravity dual; they coincide each other.

In chapter 4 we investigated $\mathcal{N} = 2$ supersymmetric theories on squashed sphere with $SU(2)_L \times U(1)_r$ isometry. The theories have four supercharges, which are transformed by $SU(2)_L$ isometry as a pair of doublets. We constructed supersymmetry transformation laws and Lagrangians by using $\mathbf{S}^1$ compactification of 4d theory. Although the metric of the squashed sphere is the same as that of the $SU(2)_L \times U(1)_r$ symmetric squashing in [35], the supersymmetry group is different. We computed the partition function by using localization, and showed that it depends on the squashing parameter in a non-trivial way.

We also computed the free energy of large N quiver gauge theories on the squashed three-sphere $\mathbf{S}^3_b$. We considered a class of quiver gauge theories studied in [43], whose partition function on round $\mathbf{S}^3$ scales as $N^{3/2}$. We confirmed that the free energy on squashed $\mathbf{S}^3$ is proportional to $N^{3/2}$ as well, and the v dependence is factored out as the additional factor $1/v^2$ regardless of the detailed structure of the theory.

We investigated relative phases in the holonomy sum, which is necessary to obtain the partition functions of gauge theories in $\boldsymbol{S}^3/\mathbb{Z}_n$ in chapter 5. We used dualities between gauge theories and non-gauge theories to determine the phases.

We first considered mirror symmetry between $\mathcal{N} = 2$ SQED with one flavor and the XYZ model containing three chiral multiplets. We showed that with the appropriate choice of the phases in the holonomy sum the partition functions of these theories coincide. Furthermore, we found that when n is odd, the phase factor is absorbed by the redefinition of the single function $s_{b,h}(z)$, the orbifold extension of the double sine function. We also considered the duality between a certain $SU(2)$ gauge theory and a chiral multiplet proposed by Jafferis and Yin. We could again find phase factors which makes the duality relation hold. When n is odd the phases are absorbed by redefining the function $s_{b,h}(z)$ in the same way as in the first example. This fact strongly suggests that the modified function $\widehat{s}_{b,h}$ in (5.57) always gives a "correct" partition function in some sense. However, there still two remaining problems.

- We need the formulae to fix the phase factor for even n case.
- Though we could fix the ambiguity of the holonomy for the Chern-Simons term in some cases as in (6.32) we need to check whether the formula is applicable to other dualties.

We need further exploration on those subjects as well as to seek the origin or the reason of these sign ambiguities.

In chapter 6 we tried to evaluate the orbifolded partition function for the ABJM model with finite N effects. Since we believe that the finite N effects include the information of the quantum effects of the gravity research on this direction is important and exciting.

The large N limit of the partition function of the ABJM model on the orbifold is considered in [50]. Though the partition function consists of the contributions of different sectors specified by the holonomies, it is found that the partition function in the large N limit is dominated by the specific contribution with a certain holonomy configuration. However, when one consider the next leading order of $1/N$ the other contributions become important, and hence, the phase factor plays a significant role there.

Appendix A

Spinor convention

We summarize the 3-dimensional spinor conventions. We use mostly + metric.

$$\eta_{mn} = \mathrm{diag}(-1, +1, +1) \tag{A.1}$$

We use m, n, p, q as 3-dimensional space indices and they run as $m = 0, 1, 2$. We define the gamma matrices as follows.

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}, \qquad \gamma^{mnp} = \mathbb{1}_2 \epsilon^{mnp} \tag{A.2}$$

where γ^{mnp} is the anti-symmetrized gamma matrix;

$$\gamma^{mnp} = \frac{1}{3!} \left[\gamma^m \gamma^n \gamma^p + \gamma^n \gamma^p \gamma^m + \gamma^p \gamma^m \gamma^n - \gamma^n \gamma^m \gamma^p - \gamma^p \gamma^n \gamma^m - \gamma^m \gamma^p \gamma^n \right]. \tag{A.3}$$

Similarly,

$$\gamma^{mn} = \frac{1}{2!} \left[\gamma^m \gamma^n - \gamma^n \gamma^m \right]. \tag{A.4}$$

Lorentz group in 3-dimension is SO(1, 2) and its spinor representation is 2-dimensional representation and it can be chosen to be real;

$$\mathrm{SO}(1, 2) \sim \mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R}). \tag{A.5}$$

For a while, we use following expressions for the γ-matrices.

$$\gamma^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.6}$$

We choose a standard spinor index position for the gamma matrices and spinors as follows.

$$(\gamma^m)_\alpha{}^\beta \qquad \psi_\alpha, \tag{A.7}$$

where α, β, (γ, δ will be appeared later) are used for spinor indices and run as $\alpha = 1, 2$. Note that in 3-dimension we do not have dotted spinor indices as in 4-dimension since there is no chirality in 3d. Spinor indices are raised and lowered by an anti-symmetric tensor ϵ as follows.

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta \qquad \psi_\alpha = \psi^\beta \epsilon_{\beta\alpha} \tag{A.8}$$

where ϵ is defined as

$$\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta}, \quad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon^{12} = 1. \tag{A.9}$$

Hence,

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = -\delta_\alpha^\gamma. \tag{A.10}$$

We use North-West and South-East convention. Namely, we omit the spinor indices when left-upper and right-lower indices are contracted. The order is important and when it is reversed the minus sign is needed;

$$\psi\lambda = \psi^\alpha\lambda_\alpha = -\psi_\alpha\lambda^\alpha = \lambda^\alpha\psi_\alpha = \lambda\psi, \tag{A.11}$$

where note that ψ and λ are Grassmann odd spinors and when they are exchanged they give minus sign as like the third equal above.

From those definition we can derive following formulae for the γ-matrices.

$$\epsilon^{\alpha\gamma}(\gamma^m)_\gamma{}^\beta = (\gamma^m)^{\alpha\beta} = (\gamma^m)^{\beta\alpha} \tag{A.12}$$

$$(\gamma^{m_1}\gamma^{m_2}\cdots\gamma^{m_k})^{\alpha\beta} = (-1)^{k-1}(\gamma^{m_k}\cdots\gamma^{m_2}\gamma^{m_1})^{\beta\alpha} \tag{A.13}$$

$$(\psi\gamma^{m_1}\gamma^{m_2}\cdots\gamma^{m_k}\lambda) = (-1)^k(\lambda\gamma^{m_k}\cdots\gamma^{m_2}\gamma^{m_1}\psi) \tag{A.14}$$

When the γ-matrices are put between the same spinor θ we have following expressions;

$$\theta\gamma_m\theta = 0, \quad \theta\gamma_m\gamma_n\theta = \theta^2\eta_{mn}, \quad \theta\gamma_m\gamma_n\gamma_p\theta = \theta^2\epsilon_{mnp}. \tag{A.15}$$

We define the complex conjugate * for Grassmann number A and B.

$$(AB)^* = A^*B^* \tag{A.16}$$

This leads the complex conjugate for spinors as follows.

$$(\psi\lambda)^* = \psi^*\lambda^* \tag{A.17}$$

We denote a derivative of the Grassmann number as follows.

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \tag{A.18}$$

In order to see the power of this conventions, let us compare it to a vector (matrix) like one. We stop omitting the spinor indices for a while and when the spinors appear without spinor index we think it as a vector. In this vector like notation (A.11) is rewritten as

$$\psi^\alpha\lambda_\alpha = \epsilon^{\alpha\beta}\psi_\beta\lambda_\alpha = (\epsilon\psi)^T\lambda = \psi^T\epsilon^T\lambda. \tag{A.19}$$

where

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \epsilon = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{A.20}$$

In 3-dimension the Majorana spinor can be defined. We define a charge conjugate spinor as follows.

$$\psi_C = C\overline{\psi}^T \tag{A.21}$$

where $\overline{\psi}$ is the Dirac conjugate of ψ and written as $\overline{\psi} = \psi^\dagger \gamma^0$. C is a charge conjugation matrix and it satisfies

$$C^T = -C, \tag{A.22}$$
$$C^\dagger C = \mathbf{1}_2, \tag{A.23}$$
$$C^\dagger \gamma^m C = -(\gamma^m)^T. \tag{A.24}$$

We choose the charge conjugate matrix as

$$C = \gamma^0. \tag{A.25}$$

The Majorana condition show that

$$\psi = \psi_C = \gamma^0(\psi^\dagger\gamma^0)^T = \psi^*. \tag{A.26}$$

Therefore, the Majorana condition for this case is just a reality condition.

When the spinor is real the Dirac conjugate of ψ becomes

$$\overline{\psi} = \psi^T\gamma^0 = \psi^T\epsilon^T = \psi^\alpha. \tag{A.27}$$

Therefore, we do not need Dirac conjugate of ψ in our notation. For example, Lagrangian for a spinor can be written as

$$\mathcal{L} = \psi^\alpha \left((\gamma^m)_\alpha{}^\beta \partial_m - m\delta_\alpha^\beta \right) \psi_\beta \tag{A.28}$$
$$= \psi \left(\gamma^m \partial_m - m \right) \psi, \tag{A.29}$$

where we omit the spinor indices in the last line. We never use overlines as the Dirac conjugate nor epsilons as the charge conjugate matrix in the body of this thesis.

A.1 Fierz transformation

Fierz transformation is to expand the product of two spinors by the complete set of the γ-matrix. The results in 3d can be described as follows.

$$(A\psi)(\chi B) = -\frac{1}{2}(\chi\psi)(AB) - \frac{1}{2}(\chi\gamma^m\psi)(A\gamma_m B), \tag{A.30}$$

where ψ, χ are the spinors we expanded and A, B are arbitrary spinors. Typically useful expression is

$$(A\theta)(\theta B) = -\frac{1}{2}\theta^2(AB). \tag{A.31}$$

This is because of the anti-symmetry of $\theta\gamma^m\theta = -\theta\gamma^m\theta = 0$.

The rest of this section is devoted how to derive the formula. If you are not interested in you can skip this section now.

In 3d the complete set of the γ-matrix is

$$\mathbb{1}_2, \quad \gamma^m. \tag{A.32}$$

Note that in odd-dimension we cannot define Weyl spinors, and hence, we do not have an analogy of γ_5 in 4d. Also, γ^{mn} and γ^{mnp} are not independent in 3d;

$$\gamma^{mn} = \epsilon^{mnp}\gamma_p, \qquad \gamma^{mnp} = \mathbb{1}_2\epsilon^{mnp}. \tag{A.33}$$

Thus we expand the product of two spinors ψ and χ by those γ-matrices;

$$\psi_\alpha\chi^\beta = a(\mathbb{1}_2)_\alpha^{\ \beta} + b_m(\gamma^m)_\alpha^{\ \beta}, \tag{A.34}$$

where a and b^m are a certain scalar and vector, which we derive now. The trace of (A.34) is

$$\begin{aligned} \mathrm{tr}\left(\psi_\alpha\chi^\beta\right) &= \psi_\alpha\chi^\alpha = -\chi^\alpha\psi_\alpha = -\chi\psi \\ &= a\ \mathrm{tr}\left(\mathbb{1}_2\right) = 2a. \end{aligned} \tag{A.35}$$

After applying the γ^n from the right to (A.34) its trace is

$$\begin{aligned} \mathrm{tr}\left(\psi_\alpha\chi^\beta(\gamma^n)_\beta^{\ \delta}\right) &= \psi_\alpha\chi^\beta(\gamma^n)_\beta^{\ \alpha} = -\chi^\beta(\gamma^n)_\beta^{\ \alpha}\psi_\alpha = -(\chi\gamma^n\psi) \\ &= b_m\mathrm{tr}\left(\gamma^m\gamma^n\right) = b_m\mathrm{tr}\left(\gamma^{mn} + \mathbb{1}_2\eta^{mn}\right) = 2b^n. \end{aligned} \tag{A.36}$$

Hence,

$$a = -\frac{1}{2}\chi\psi, \qquad b_m = -\frac{1}{2}(\chi\gamma_m\psi), \tag{A.37}$$

and we have (A.30). Of course one can do the similar calculation in any dimension. The general formula for the Fierz transformation depends on whether the dimension is even or odd. In even case

$$(A\psi)(\chi B) = -\frac{1}{2^{D/2}}\sum_{k=0}^{D}\frac{(-1)^{k(k-1)/2}}{k!}\left(\chi\gamma^{\mu_1\cdots\mu_k}\psi\right)\left(A\gamma_{\mu_1\cdots\mu_k}B\right). \tag{A.38}$$

In odd case one just need to replace $2^{D/2}$ by $2^{[D/2]}$ in front and the upper limit of the sum D by $[D/2]$, where [ ] means the integer part of the number inside.

Appendix B

Construction of $\mathcal{N} = 2$ supersymmetric theories

We construct 3-dimensional supersymmetric gauge theories through two different approaches. We firstly construct $\mathcal{N} = 1$ supersymmetric theories and derive the SUSY transformation in a brute force way as an exercise. Then, we move onto the $\mathcal{N} = 2$ case. One way is to use the super Poincare algebra to drive the transformation laws. One may notice that this approach is reminiscent of Wess and Bagger's great book [53]. Especially, 3d $\mathcal{N} = 2$ case is almost the same as 4d $\mathcal{N} = 1$ SUSY. So one may suppose whether there are any relation between them. Indeed, we can derive the 3d $\mathcal{N} = 2$ SUSY Lagrangian and the transformation from that of 4d using dimensional reduction, and this is our final approach.

B.1 Brute force approach to $\mathcal{N} = 1$ supersymmetric theories

In this section we construct the supersymmetric transformation laws for fields. This section is based on the nice review of 3d SUSY [39]. SUSY connects bosons and fermions, and naively say, the transformation changes the bosons to fermions, and vice versa. The mass dimensions of fields are good indicators to determine the form of the transformation. In 3d scalars have mass dimension 1/2 and fermions have mass dimension 1. This means that the transformation is controlled by a certain constant ϵ which has mass dimension 1/2. Now the transformation δ is naively expressed as follows.

$$\delta b(x) \sim \epsilon f(x), \qquad \delta f(x) \sim \epsilon \partial b(x) \tag{B.1}$$

where ∂ is a space-time derivative, which is introduced to fill the mass dimension gap. Notice that from the statistics the ϵ have to be a fermionic constant, and it is find that actually ϵ is a spinor. Let us consider the commutator of the transformation.

$$\delta_1(\delta_2 b) \sim \delta_1(\epsilon_2 f) \sim \epsilon_2(\epsilon_1 \partial b) \sim \epsilon_2 \epsilon_1 \partial b \tag{B.2}$$

$$\delta_1(\delta_2 f) \sim \delta_1(\epsilon_2 \partial b) \sim \epsilon_2 \partial(\epsilon_1 f) \sim \epsilon_2 \epsilon_1 \partial f \tag{B.3}$$

The commutator (equations above are not really ones of commutator though) of SUSY transformation leads to translation. When the commutator leads to symmetries of the theory we say that the transformation closes.

Although discussion above looks clear and good, the naive transformation law discussed above generally does not close. We now have to seriously consider what is the real meaning of SUSY. It means that degrees of freedom (D.O.F.) of bosons and fermions in a supersymmetric theory are the same. However, the D.O.F. depends on whether the fields are on shell or off shell. We listed the D.O.F. of fields in Table B.1. From the Table B.1 when we consider scalars and its superpartners the D.O.F.

Table B.1: Degrees of freedom of fields counted in real degree

Fields	On shell	Off shell
Real scalar ϕ	1	1
Majorana fermion ψ	1	2
Real auxiliary field F	0	1
Gauge field A_m	1	2
Gaugino λ	1	2

of those match in on shell but does not match in off shell. Therefore, in order to close the SUSY transformation we need either equations of motion for the fermions or auxiliary field F. F has one D.O.F. in off shell but when the E.O.M. is considered F should vanish or give regularizations to other fields. Typically,

$$\mathcal{L}_F \sim FF \tag{B.4}$$

gives E.O.M. $F = 0$. Notice that F has mass dimension 3/2.

Note that D.O.F. of gauge fields and its superpartners in 3d match in both on shell and off shell, and hence, auxiliary field is not needed. From now on we consider off shell SUSY and let us derive the off shell SUSY transformation laws.

B.1.1 Wess-Zumino model

We begin with a real scalar field ϕ and its superpartner ψ, which is the Majorana spinor, with auxiliary fields. This is called Wess-Zumino model, which is the simplest supersymmetric theory. The action can be written as follows.

$$S_{\mathrm{WZ}} = \int dx^3 \left[-\frac{1}{2}\partial^m\phi\partial_m\phi + \frac{1}{2}\psi\not\partial\psi + \frac{a}{2}FF \right] \tag{B.5}$$

where a is an undetermined constant. Slash means a contraction with a γ-matrix (e.g. $\not\partial = \gamma^m\partial_m$) through this thesis. From the analysis on the mass dimension we are motivated to consider following transformation laws.

$$\delta\phi = \epsilon\psi \tag{B.6}$$
$$\delta\psi = -b\not\partial\phi\epsilon + c\epsilon F \tag{B.7}$$
$$\delta F = -\epsilon\not\partial\psi \tag{B.8}$$

where we rescaled ϵ and F so that the coefficients become simple, b and c are to be determined. After the SUSY transformation of the action (B.5) you will find that $a = c$ and $b = 1$ make the variation vanish $\delta S = 0$, meaning the theory is supersymmetric.

Next, we discuss some interaction terms of those fields. Firstly, the mass term in general set up is

$$S_m = \int d^3x \; m \left(F\phi - \frac{d}{2}\psi\psi \right) \tag{B.9}$$

where m is mass parameter that has mass dimension 1, and d is a certain constant. This is supersymmetric when $bd = 1$ and $cd = 1$. Therefore, kinetic term + mass term is supersymmetric provided that $a = b = c = d = 1$.

The cubic term can be

$$S_g = \int d^3x \; g \left(F\phi^2 - e\psi\psi\phi \right), \tag{B.10}$$

where g is a coupling constant that has mass dimension $1/2$, and e is again an unfixed constant. This is supersymmetric when $be = 1$ and $ce = 1$. These conditions with that of kinetic term give $a = b = c = e = 1$.

Finally, we check the commutators of the transformations. The commutator of the transformation for the scalar is

$$[\delta_2, \delta_1]\phi = 2b \left(\epsilon_2 \gamma^m \epsilon_1 \right) \partial_m \phi. \tag{B.11}$$

This is exactly the translation as discussed, and the parameter is given by $\xi^m = 2b\epsilon_2\gamma^m\epsilon_1$.

For the fermion

$$[\delta_2, \delta_1]\psi = 2b \left(\epsilon_2 \gamma^m \epsilon_1 \right) \partial_m \phi. \tag{B.12}$$

where we used the Fierz transformation A.1, and we set $b = c$ otherwise the commutator does not become translation.

For the auxiliary field

$$[\delta_2, \delta_1]F = 2c \left(\epsilon_2 \gamma^m \epsilon_1 \right) \partial_m \phi. \tag{B.13}$$

When $b = c$ all the commutators become translations with the same parameter.

Let us summarize the results. The consistent and beautiful choice of those constants is $a = b = c = d = e = 1$. Then ,the SUSY transformations are

$$\begin{aligned} \delta\phi &= \epsilon\psi, \\ \delta\psi &= -\not{\partial}\phi\epsilon + \epsilon F, \\ \delta F &= -\epsilon\not{\partial}\psi, \end{aligned} \tag{B.14}$$

and the commutator of those become the same translation. The general action is

$$S = S_K + S_m + S_g + (\text{other interaction terms}). \tag{B.15}$$

Other interaction terms can be derived in a similar way order by order. We call this set of fields ϕ, ψ and F as a scalar multiplet. There is actually such a formalism which tie up those fields in one superfield, though we will not discuss it here.

B.1.2 Supersymmetric Yang-Mills theory

Let us consider a non-Abelian gauge field $A_m = A_m^a T^a$. We use a as the Lie algebra index and T^a as generators of the Lie algebra. T^a satisfies the commutation relation $[T^a, T^b] = if^{abc}T^c$. We use the normalization such that $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. The corresponding superpartner is $\lambda = \lambda^a T^a$, and no auxiliary field as discussed. The action is

$$S_{\mathrm{YM}} = \frac{1}{g^2}\int d^3x \ \mathrm{tr}\left[-\frac{1}{2}F_{mn}F^{mn} + \lambda \not{D}\lambda\right] \tag{B.16}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m - i[A_m, A_n]$, the field strength and $D_m = \partial_m - i[A_m, *]$, the covariant derivative. The gauge transformations are

$$\delta_g A_m = D_m \Lambda, \tag{B.17}$$

$$\delta_g \lambda = -i[\lambda, \Lambda], \tag{B.18}$$

which leads to the invariance of the action (B.16). Note that the gauge coupling g has the mass dimension 1/2, so we can construct the SUSY transformation laws using it to fill the mass gap. The most general form of the transformation with Lorentz covariance is

$$\delta A_m = a\epsilon\gamma_m\lambda, \quad \delta\lambda = bF_{mn}\gamma^{mn}\epsilon + c\partial^m A_m\epsilon + dg^2\not{A}\epsilon, \tag{B.19}$$

where a, b, c, and d are some constants. From the invariance of the action (B.16) under this transformation laws it is shown that $c = d = 0$ and $a - 2b = 0$. Therefore,

$$\delta A_m = a\epsilon\gamma_m\lambda, \quad \delta\lambda = \frac{a}{2}F_{mn}\gamma^{mn}\epsilon. \tag{B.20}$$

Let us see the SUSY commutation of the gauge field.

$$[\delta_2, \delta_1]A_m = -2a^2 F_{mn}\epsilon_2\gamma^n\epsilon_1 \tag{B.21}$$

This expression is further deconstructed into the covariant derivative and the gauge transformation.

$$[\delta_2, \delta_1]A_m = 2a^2\epsilon_2\gamma^n\epsilon_1\partial_n A_m - 2a^2 D_m A_n\epsilon_2\gamma^n\epsilon_1 \tag{B.22}$$

The first term in the right-hand side is the translation with parameter, $2a^2\epsilon_2\gamma^n\epsilon_1$, and the second term is the gauge transformation with parameter, $\Lambda = -2a^2 A_n\epsilon_2\gamma^n\epsilon_1$. The SUSY commutation of gaugino gives covariant translation.

$$[\delta_2, \delta_1]\lambda = 2a^2\epsilon_2\gamma^p\epsilon_1 D_p\lambda \tag{B.23}$$

$$= 2a^2\epsilon_2\gamma^p\epsilon_1\left(\partial_p\lambda - i[A_p, \lambda]\right) \tag{B.24}$$

Again, these are the translation and the gauge transformation with the same parameter as the gauge field.

In order to have the same translation as those of Wess-Zumino model a^2 should be set to 1. Since the sign of a can be absorbed into the definition of the gaugino λ we set $a = 1$. The final form of the gauge transformations is

$$\delta A_m = \epsilon\gamma_m\lambda,$$

$$\delta\lambda = \frac{1}{2}F_{mn}\gamma^{mn}\epsilon. \tag{B.25}$$

B.1.3 Chern-Simons term

In three dimension we can consider a following action without any dimensionful constant.

$$S_{\rm CS} = \frac{\kappa}{2} \int d^3x \; \epsilon^{mnp} {\rm tr} \left(A_m \partial_n A_p - \frac{2i}{3} A_m A_n A_p \right) \tag{B.26}$$

where κ is a dimensionless constant. This action is called pure Chern-Simons action and has trivial field equation;

$$F_{mn} = 0. \tag{B.27}$$

Therefore, the Chern-Simons action does not have any physical D.O.F, and it is called a topological action. The action is invariant under the infinitesimal gauge transformation (B.17). However, the invariance of a finite gauge transformation (or sometimes called large gauge transformation)

$$A_m \to A'_m = U^{-1} A_m U + i U^{-1} \partial_m U, \quad U = e^{i\Lambda}, \tag{B.28}$$

requires a restriction on κ:

$$\kappa = \frac{k}{2\pi} \qquad (k \in \mathbb{Z}). \tag{B.29}$$

See Section D.1 for the derivation of this constraint.

Let us consider the fermionic part of the Chern-Simons action. As we discussed the Chern-Simons action has no D.O.F., hence, the corresponding fermionic action should not have any D.O.F. In addition the dimensional analysis tells us that the mass dimension should be 3. Summarizing all we conclude that the supersymmetric Chern-Simons action is given as follows.

$$S_{\rm CS} = \frac{k}{4\pi} \int d^3x \; {\rm tr} \left[\epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + \lambda\lambda \right] \tag{B.30}$$

This is invariant under the same supersymmetric transformation as that of Super-Yang-Mills (B.25).

B.2 Dimensional reduction method

It is well known that 3d $\mathcal{N} = 2$ SUSY can be derived from 4d $\mathcal{N} = 1$ SUSY by dimensional reduction. We will see how to get the transformation law and the action from 4d in this subsection. Here, we denote Greek letters $\mu, \nu, \cdots$ as 4d spacetime indices (e.g. $\mu = 0, 1, 2, 3$) and $m, n, \cdots$ as 3d spacetime indices ($m = 0, 1, 2$). We distinguish 4d gamma-matrix and that of 3d by Γ^μ and γ^m, respectively, where the 4d Γ and 3d γ are related as follows.

$$\Gamma^m = \begin{pmatrix} 0 & \gamma^m \\ \gamma^m & 0 \end{pmatrix}, \qquad \Gamma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{B.31}$$

Firstly, we list the action and the transformation law in 4d. The action is

$$S = \int d^4x \text{ tr} \left[-D_\mu \overline{\phi} D^\mu \phi + \overline{\psi} \not{D} \psi + \overline{F} F - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + \overline{\lambda} \not{D} \lambda + D^2 \right]. \tag{B.32}$$

And the transformation is

$$\begin{aligned}
\delta\phi &= \sqrt{2}\epsilon\psi \\
\delta\psi &= -\sqrt{2}\Gamma^\mu \overline{\epsilon} D_\mu \phi + \sqrt{2}\epsilon F \\
\delta F &= -\sqrt{2}\overline{\epsilon}\Gamma^\mu D_\mu \psi - 2\overline{\epsilon}\overline{\lambda}\phi \\
\delta\overline{\phi} &= \sqrt{2}\overline{\epsilon}\overline{\psi} \\
\delta\overline{\psi} &= -\sqrt{2}\Gamma^\mu \epsilon D_\mu \overline{\phi} + \sqrt{2}\overline{\epsilon}\overline{F} \\
\delta\overline{F} &= -\sqrt{2}\epsilon\Gamma^\mu D_\mu \overline{\psi} - 2\epsilon\lambda\overline{\phi}
\end{aligned}$$

$$\begin{aligned}
\delta A_\mu &= i\epsilon\Gamma_\mu\overline{\lambda} - i\overline{\epsilon}\Gamma_\mu\lambda \\
\delta\lambda &= \frac{i}{2}\Gamma^{\mu\nu}\epsilon F_{\mu\nu} + D\epsilon \\
\delta\overline{\lambda} &= -\frac{i}{2}\Gamma^{\mu\nu}\overline{\epsilon}F_{mn} + D\overline{\epsilon} \\
\delta D &= -\epsilon\Gamma^\mu D_\mu\overline{\lambda} - \overline{\epsilon}\Gamma^\mu D_\mu\lambda,
\end{aligned} \tag{B.33}$$

where the 4d spinors are Majorana spinors.

The dimensional reduction, or often called compactification, is naively illustrated in Fig. B.1. We roll up a flatspace to a cylinder and introduce a boundary condition

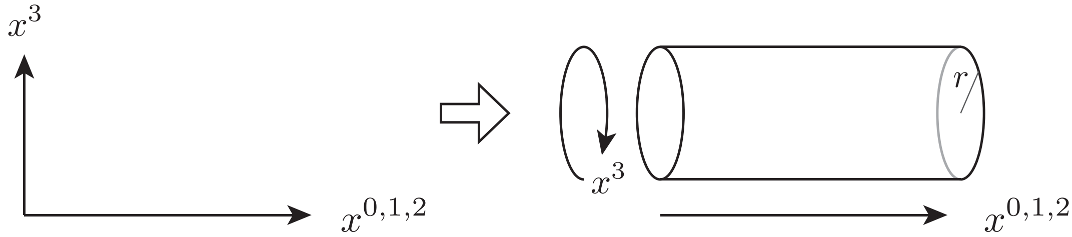

Figure B.1: Dimensional reduction

to fields on it. The fields are expanded as

$$\Phi^{(4d)}(x^m, x^3) = \sum_k e^{ikx^3/r}\Phi_k(x^m), \tag{B.34}$$

where r is the radius of the cylinder, $\Phi^{(4d)}$ is any field in a theory, and Φ_k is the coefficients of the Fourier expansion. In case of the scalar field ϕ the field equation becomes

$$\begin{aligned}
\left(\partial^\mu\partial_\mu - m^2\right)\phi &= \left(\partial^m\partial_m + \partial_3^2 - m^2\right)\phi \\
&= \left(\partial^m\partial_m - \left(\frac{k}{r}\right)^2 - m^2\right)\phi \quad (k \in \mathbb{Z}).
\end{aligned} \tag{B.35}$$

This is an infinite tower of the massive scalar field; its effective mass is $m_{\text{eff}}^2 = m^2 + k^2/r^2$. When the radius is extremely small the fields $k \neq 0$ becomes very massive and decouple from the theory. In other word, it is possible to ignore the dependence on the x^3 direction, or equivalently we can set $\partial_3 = 0$. Other than that, we need to decompose the 4d Majorana spinor to 3d complex spinor due to the difference of the representation;

$$\lambda^{(4d)} \to \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \qquad \overline{\lambda}^{(4d)} \to \begin{pmatrix} 0 \\ \overline{\lambda} \end{pmatrix}. \tag{B.36}$$

Also, we need to decompose the gauge field into two part;

$$A_\mu \to (A_m, \sigma)\,. \tag{B.37}$$

The fourth direction (x^3 direction) is no longer the component of the vector, albeit the adjoint scalar.

Along these strategy we can decompose the action as well as the transformation.

$$\begin{aligned} -D_\mu\overline{\phi}D^\mu\phi &\to -D_m\overline{\phi}D^m\phi - \overline{\phi}\sigma\sigma\phi \\ \overline{\psi}\not{D}\psi &\to \overline{\psi}\not{D}\psi + \overline{\psi}\sigma\psi \\ \overline{F}F &\to \overline{F}F \\ -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} &\to -\frac{1}{2}F^{mn}F_{mn} - D_m\sigma D^m\sigma \\ \overline{\lambda}\not{D}\lambda &\to \overline{\lambda}\not{D}\lambda + \overline{\lambda}\sigma\lambda \\ D^2 &\to D^2 \end{aligned} \tag{B.38}$$

The transformations become

$$\begin{aligned} \delta\phi &= \sqrt{2}\epsilon\psi \\ \delta\psi &= -\sqrt{2}\gamma^m\overline{\epsilon}D_m\phi + \sqrt{2}\overline{\epsilon}\sigma\phi + \sqrt{2}\epsilon F \\ \delta F &= -\sqrt{2}\overline{\epsilon}\gamma^m D_m\psi + \sqrt{2}\overline{\epsilon}\sigma\psi - 2\overline{\epsilon}\overline{\lambda}\phi \\ \delta\overline{\phi} &= \sqrt{2}\overline{\epsilon}\overline{\psi} \\ \delta\overline{\psi} &= -\sqrt{2}\gamma^m\epsilon D_m\overline{\phi} + \sqrt{2}\epsilon\sigma\overline{\phi} + \sqrt{2}\overline{\epsilon}\overline{F} \\ \delta\overline{F} &= -\sqrt{2}\epsilon\gamma^m D_m\overline{\psi} + \sqrt{2}\epsilon\sigma\overline{\psi} - 2\epsilon\lambda\overline{\phi} \end{aligned}$$

$$\begin{aligned} \delta A_m &= i\epsilon\gamma_m\overline{\lambda} - i\overline{\epsilon}\gamma_m\lambda \\ \delta\sigma &= \epsilon\overline{\lambda} + \overline{\epsilon}\lambda \\ \delta\lambda &= \frac{i}{2}\gamma^{mn}\epsilon F_{mn} - \gamma^m\epsilon D_m\sigma + D\epsilon \\ \delta\overline{\lambda} &= -\frac{i}{2}\gamma^{mn}\overline{\epsilon}F_{mn} - \gamma^m\overline{\epsilon}D_m\sigma + D\overline{\epsilon} \\ \delta D &= -\epsilon\gamma^m D_m\overline{\lambda} - \overline{\epsilon}\gamma^m D_m\lambda + \epsilon[\sigma, \overline{\lambda}] + \overline{\epsilon}[\sigma, \lambda]. \end{aligned} \tag{B.39}$$

These are exactly the same as ones we derived in different ways.

B.2.1 $\mathcal{N} = 2$ supersymmetric Chern-Simons term

Here we discuss the $\mathcal{N} = 2$ supersymmetric version of Chern-Simons term. There are a couple of ways to derive it. One is to start with the pure Chern-Simons term (B.26), and modify it so as to make it invariant under the transformation. The other way is to use the vector multiplet and covariant derivative to construct a SUSY invariant Lagrangian. Point is that the Chern-Simons term does not exist in 4d. Hence, we cannot use the dimensional reduction method. This fact gives a clue to derive the superfield expression for the Chern-Simons term; the expression should be the combination that cannot be constructed in 4d.

Firstly, we attack the first way, which is a brute-force approach. Here as we cannot show our effort we just show the result. The supersymmetric Chern-Simons term is given by

$$S_{\text{SCS}} = \frac{k}{2\pi} \int d^3x \ \text{tr} \left[\epsilon^{mnp} \frac{1}{2} \left(A_m \partial_n A_p - \frac{2i}{3} A_m A_n A_p \right) + \overline{\lambda}\lambda - \sigma D \right]. \tag{B.40}$$

Next, we use the superfield and the covariant derivatives to construct (B.40). This formalism was given in [54]. Notice that in 4d the covariant derivatives D_α and $\overline{D}_{\dot{\alpha}}$ cannot contract each other due to the chirality. However, in 3d there is no chirality, and hence, we can contract them. Let us consider the Abelian case first.

$$\begin{aligned} S_{\text{CS}} &= \frac{k}{4\pi} \int d^3x d^4\theta \ \left(\overline{D}^\alpha V \right) \left(D_\alpha V \right) \\ &= \frac{k}{2\pi} \int d^3x \ \left(\frac{1}{2} \epsilon^{mnp} A_m \partial_n A_p + \overline{\lambda}\lambda - \sigma D \right) \end{aligned} \tag{B.41}$$

The SUSY invariance is obvious because we used the superfields. The gauge invariance is easily checked by using the partial integration and the fact that

$$\begin{aligned} \overline{D}D &= D\overline{D} \\ D_\alpha \overline{\Lambda} &= \overline{D}_\beta \Lambda = 0. \end{aligned} \tag{B.42}$$

In the non-Abelian case, though the derivation is non-trivial, the action is given as follows.

$$\begin{aligned} S_{\text{CS}} &= -\frac{k}{2\pi} \int d^3x d^4\theta \int_0^1 dt \ \text{tr} \left[V \overline{D}^\alpha \left(e^{-2tV} D_\alpha e^{2tV} \right) \right] \\ &= \frac{k}{2\pi} \int d^3x d^4\theta \int_0^1 dt \ \text{tr} \left[\overline{D}V D V - \frac{2}{3} \left(V D V \overline{D} V - V \overline{D} V D V \right) \right] \\ &= \frac{k}{2\pi} \int d^3x \ \text{tr} \left[\frac{1}{2} \epsilon^{mnp} \left(A_m \partial_n A_p - \frac{2i}{3} A_m A_n A_p \right) + \overline{\lambda}\lambda - \sigma D \right] \end{aligned} \tag{B.43}$$

Though in order to show the gauge invariance we need some calculation (see Section D.3) one can show that this is gauge invariant.

Appendix C

S^3 geometry

C.1 Curved geometry

Since we treat curved geometry we have to prepare some tools. On the curved space a spin is defined locally. Namely, the spin is define on the tangent space rather than the curved space. We need to introduce so called vielbein e_μ^m (typically, it is called dreibein in 3d) as a map from curved space to the tangent space;

$$e_m^{\ \mu}(x)e_n^{\ \nu}(x)g_{\mu\nu}(x) = \eta_{mn} \tag{C.1}$$

where $\eta_{mn} = \text{diag}(-,+,+)$. Here we use $m, n, \cdots$ as local Lorentz indices and $\mu, \nu, \cdots$ as world indices. The vielbein satisfies

$$e_\mu^{\ m}e_m^{\ \nu} = \delta_\mu^\nu, \qquad e_m^{\ \mu}e_\mu^{\ n} = \delta_m^n. \tag{C.2}$$

$\epsilon_m^{\ \mu}$ and $\epsilon_\mu^{\ m}$ are an inverse matrix each other. Using the inverse matrix we can express the metric by vielbein

$$g_{\mu\nu} = e_\mu^{\ m}e_\nu^{\ n}\eta_{mn}. \tag{C.3}$$

An index of a gauge field can be changed by the vielbein;

$$A_m = e_m^{\ \mu}A_\mu, \qquad A_\mu = e_\mu^{\ m}A_m. \tag{C.4}$$

$e_\mu^{\ m}$ transform under the local Lorentz transformation as

$$\delta_L e_\mu^{\ m} = -\lambda^m_{\ n}e_\mu^{\ n} \tag{C.5}$$

where $\lambda_{mn} = -\lambda_{nm}$ is the parameter of the infinitesimal Lorentz transformation. In order to make the covariant derivative we need a gauge field. We call the gauge field to the local Lorentz by spin connection, and it is written as

$$\omega_\mu^{\ m}{}_n(x). \tag{C.6}$$

The spin connection is anti-symmetric for the local Lorentz index

$$\omega_\mu^{\ mn} = -\omega_\mu^{\ nm}. \tag{C.7}$$

The spin connection transforms under the local Lorentz as

$$\delta_L \omega_\mu{}^{mn} = D_\mu \lambda^{mn} \equiv \partial_\mu \lambda^{mn} + \omega_\mu{}^m{}_p \lambda^{pn} + \omega_\mu{}^n{}_p \lambda^{mp}. \tag{C.8}$$

Using the spin connection the covariant derivative is defined e.g. for a vector V^m as

$$D_\mu V^m = \partial_\mu V^m + \omega_\mu{}^m{}_n V^n, \tag{C.9}$$

and for a spinor:

$$D_\mu = \left(\partial_\mu + \frac{1}{4} \omega_\mu{}^{mn} \gamma_{mn} \right) \psi. \tag{C.10}$$

The covariant derivative for a scalar is usual derivative $D_\mu \phi = \partial_\mu \phi$.

The vielbein and the spin connection is not isolated each other. We can actually write down the spin connection in terms of the vielbein using so called torsionless condition:

$$D_\mu e_\nu{}^m - D_\nu e_\mu{}^m = \partial_\mu e_\nu{}^m + \omega_\mu{}^m{}_n e_\nu{}^n - \partial_\nu e_\mu{}^m - \omega_\nu{}^m{}_n e_\mu{}^n = 0. \tag{C.11}$$

Then, we have

$$\begin{aligned} \omega_{\mu mn} &= \frac{1}{2} \left(e_m{}^\nu \Omega_{\mu\nu n} - e_n{}^\nu \Omega_{\mu\nu m} - e_m{}^\rho e_n{}^\sigma e_\mu{}^p \Omega_{\rho\sigma p} \right) \\ \Omega_{\mu\nu m} &= \partial_\mu e_{\nu m} - \partial_\nu e_{\mu m}. \end{aligned} \tag{C.12}$$

From the spin connection we can construct the Riemann tensor as follows.

$$R_{\mu\nu}{}^m{}_n = \partial_\mu \omega_\nu{}^m{}_n - \partial_\nu \omega_\mu{}^m{}_n + \omega_\mu{}^m{}_p \omega_\nu{}^p{}_n - \omega_\nu{}^m{}_p \omega_\mu{}^p{}_n. \tag{C.13}$$

C.2 Differential form

So far we have seen the explicit form of the vielbein, spin connection and Riemann tensor. There is actually simpler way to express them; it is a differential form. In general, for the rank n anti-symmetric tensor we define n-form as follows.

$$A_n = \frac{1}{n!} A_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}, \tag{C.14}$$

where dx^μ is the basis and $\wedge$ is a wedge product (exterior product). The wedge product is anti-symmetric:

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu. \tag{C.15}$$

$A_{\mu_1 \cdots \mu_n}$ is called the component of the form A_n. When it is obvious we omit the subscript n and simply denote A_n as A. For example, for a gauge field A_μ we write it in the differential form as

$$A = A_\mu dx^\mu. \tag{C.16}$$

Similarly the field strength is expressed as

$$
\begin{aligned}
F &= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \\
&= d \wedge A = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = \frac{1}{2} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) dx^\mu \wedge dx^\nu ,
\end{aligned} \tag{C.17}
$$

where $d = \partial_\mu dx^\mu$ is an exterior derivative. We also abbreviate the wedge product e.g. $d \wedge A \to dA$.

Using the differential form the vielbein is

$$
e^m = e_\mu{}^m dx^\mu , \tag{C.18}
$$

and the torsionless condition becomes

$$
\begin{aligned}
De^m &= D_\mu e_\nu{}^m dx^\mu dx^\nu \\
&= \left(\delta^m_n d + \omega^m{}_n \right) e^n = \left(\delta^m_n \partial_\mu + \omega_\mu{}^m{}_n \right) e_\nu{}^n dx^\mu dx^\nu = 0
\end{aligned} \tag{C.19}
$$

Moreover, when we think e^m as vector and $\omega^m{}_n$ as matrix, we can also omit the local Lorentz indices as well;

$$
De = (d + \omega) e . \tag{C.20}
$$

Hence, the torsionless condition can be written by

$$
De = 0 . \tag{C.21}
$$

In this manner the Riemann tensor can be written by

$$
R = D^2 = d\omega + \omega^2 , \tag{C.22}
$$

where we used the fact that $d^2 = 0$. Note that ω is a matrix, and hence, $\omega^2 \neq 0$.

C.3 $\boldsymbol{S}^3$ geometry

$\boldsymbol{S}^3$ is usually described by Cartesian coordinate as

$$
x^2 + y^2 + z^2 + w^2 = r^2 . \tag{C.23}
$$

Here we illustrate the $\boldsymbol{S}^3$ as a group manifold $\boldsymbol{S}^3 \sim SU(2)$ with Euler angles (θ, ϕ, ψ). Their periods are $(\pi, 2\pi, 4\pi)$. $SU(2)$ element g can be expressed as follows.

$$
g = e^{\phi T_3} e^{\theta T_2} e^{\psi T_3} \tag{C.24}
$$

where $T^a = i\sigma^a/2$ is an anti-Hermite representation of $SU(2)$ algebra

$$
[T_a, T_b] = -\epsilon_{abc} T_c . \tag{C.25}
$$

The coordinate is expressed as

$$
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = g \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{C.26}
$$

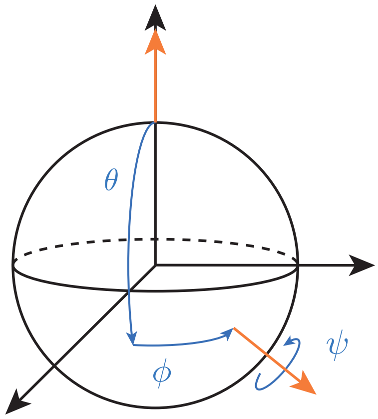

Figure C.1: Arrow with rotation angle on $\boldsymbol{S}^2$. Equivalently, $U(1)$ fiber over $\boldsymbol{S}^2$, which is called Hopf fibration.

with a complex coordinate z_1, z_2;

$$|z_1|^2 + |z_2|^2 = \begin{pmatrix} z_1^* & z_2^* \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} r & 0 \end{pmatrix} g^\dagger g \begin{pmatrix} r \\ 0 \end{pmatrix} = r^2. \tag{C.27}$$

Graphically this can be show as $U(1)$ fiber over $\boldsymbol{S}^2$; see Fig. C.1.

Let us define so called left-invariant 1-form μ^a through the following equation.

$$2\mu^a T_a = g^{-1} dg \tag{C.28}$$

Using the Euler formula

$$\begin{aligned} e^{\theta T_a} &= \mathbf{1} + \theta T_a + \frac{1}{2!} (\theta T_a)^2 + \cdots \\ &= \mathbf{1} \cos\frac{\theta}{2} + 2T_a \sin\frac{\theta}{2} \end{aligned} \tag{C.29}$$

the left-invariant 1-form is explicitly given as follows.

$$\mu^1 = \frac{1}{2} \left(-\sin\psi d\theta + \cos\psi \sin\theta d\phi \right) \tag{C.30}$$

$$\mu^2 = \frac{1}{2} \left(\cos\psi d\theta + \sin\psi \sin\theta d\phi \right) \tag{C.31}$$

$$\mu^3 = \frac{1}{2} \left(d\psi + \cos\theta d\phi \right) \tag{C.32}$$

The metric of $\boldsymbol{S}^3$ is

$$\begin{aligned} ds^2 &= r^2 \left((\mu^1)^2 + (\mu^2)^2 + (\mu^3)^2 \right) \\ &= \frac{r^2}{4} \left[d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2 \right]. \end{aligned} \tag{C.33}$$

This means that the vielbein is

$$e^a = r\mu^a. \tag{C.34}$$

The volume is also calculated using the 1-form.

$$\int r^3 \ \mu^1 \wedge \mu^2 \wedge \mu^3 = -\int \frac{r^2}{8} \sin\theta \ d\theta \wedge d\phi \wedge d\psi = -2\pi^2 r^3 \tag{C.35}$$

$$\mathrm{Vol} = \int r^3 \ \left|\mu^1 \wedge \mu^2 \wedge \mu^3\right| = 2\pi^2 r^3 \tag{C.36}$$

Let us calculate the spin connection of $\boldsymbol{S}^3$. One can calculate the spin connection from the torsionless condition (C.21). We take a derivative of (C.28):

$$\begin{aligned}
& d\left(g^{-1}dg = 2\mu^a T_a\right) \\
\Rightarrow \quad & -g^{-1}dgg^{-1}dg = 2d\mu^a T_a \quad \Rightarrow \quad -4\mu^a \wedge \mu^b \frac{1}{2}\left[T_a, T_b\right] = 2d\mu^c T_c \\
\Rightarrow \quad & \mu^a \wedge \mu^b \epsilon^{abc} T_c = d\mu^c T_c \\
\therefore \quad & d\mu^a = \epsilon^{abc}\mu^b \wedge \mu^c \quad \Rightarrow \quad d\mu^a + \epsilon^{abc}\mu^c \wedge \mu^b = 0
\end{aligned} \tag{C.37}$$

This is nothing but the torsionless condition (C.21):

$$\begin{aligned}
De^a = 0 \quad \Rightarrow \quad D\mu^a = 0 \\
d\mu^a + \omega^{ab}\mu^b = 0
\end{aligned} \tag{C.38}$$

Therefore, the spin connection is

$$\omega^{ab} = \epsilon^{abc}\mu^c. \tag{C.39}$$

C.4 Fundamental group of $\boldsymbol{S}^3$ and $\boldsymbol{S}^3/\mathbb{Z}_n$

The fundamental group of $\boldsymbol{S}^3$ is trivial: $\pi_1\left(\boldsymbol{S}^3\right) = 1$. , the orbifold $\boldsymbol{S}^3/\mathbb{Z}_n$ has non-trivial fundamental group: $\pi_1\left(\boldsymbol{S}^3/\mathbb{Z}_n\right) = \mathbb{Z}_n$. This is very significant feature of the orbifold because this is the origin of the holonomy. Let us see this explicitly. Since the Euler angle is not adequate for global coordinate of $\boldsymbol{S}^3$; it has singular points at the north and south pole. We consider two patches of $\boldsymbol{S}^3$ as in Fig. C.2. However,

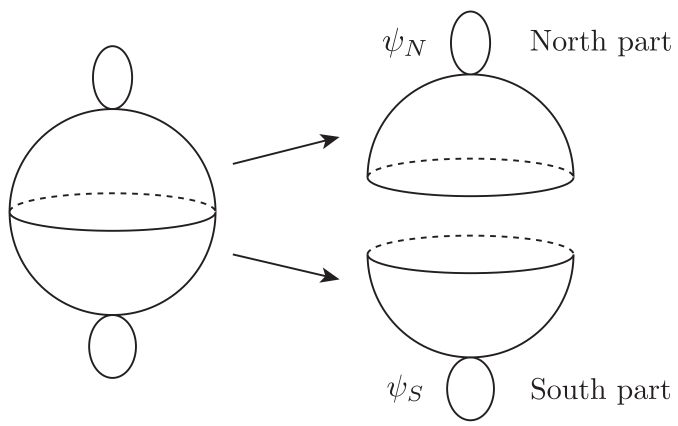

Figure C.2: Separated $\boldsymbol{S}^3$; the circle put on $\boldsymbol{S}^2$ parametrize the fiber direction.

ϕ and ψ themselves are already ill-defined for $\boldsymbol{S}^2$. We get rid of the singularity by tuning the third coordinate ψ patch by patch. Let us see the region around the North pole $\theta \sim \epsilon \quad (\epsilon << 1)$.

$$T_2(\phi) = e^{i\sigma_2 \epsilon/2} \simeq \mathbb{1} + i\epsilon\sigma_2/2 = \begin{pmatrix} 1 & \epsilon/2 \\ \epsilon/2 & 1 \end{pmatrix} \tag{C.40}$$

$$T_3(\phi) = e^{i\sigma_3 \phi/2} = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \tag{C.41}$$

$$\therefore \quad T_3(\phi)T_2(\epsilon)T_3(\psi) = \begin{pmatrix} e^{i(\phi+\psi)/2} & \frac{\epsilon}{2}e^{i(\phi-\psi)/2} \\ -\frac{\epsilon}{2}e^{i(-\phi+\psi)/2} & e^{-i(\phi+\psi)/2} \end{pmatrix} \tag{C.42}$$

Taking $\epsilon \to 0$ and utilize $(1\ 0)^T$ from the right we have

$$z_1 = e^{i(\phi+\psi)/2} \tag{C.43}$$

As there should not be ϕ dependence at the point we set $\psi_N = \phi + \psi$. At the South pole $\theta = \pi - \epsilon$ we repeat the same thing and get $\psi_S = -\phi + \psi$. Therefore, we need the junction condition when change the patch:

$$\psi_N = \psi_S + 2\phi. \tag{C.44}$$

Let us consider the circle around the North pole

$$\begin{aligned} \theta &= R << 1 \\ \phi &= 2\pi t \quad (0 \leq t < 1) \\ \psi &= \text{const} = 0, \end{aligned} \tag{C.45}$$

and move it to the South pole (see Fig C.3). Then, after the change of the patch

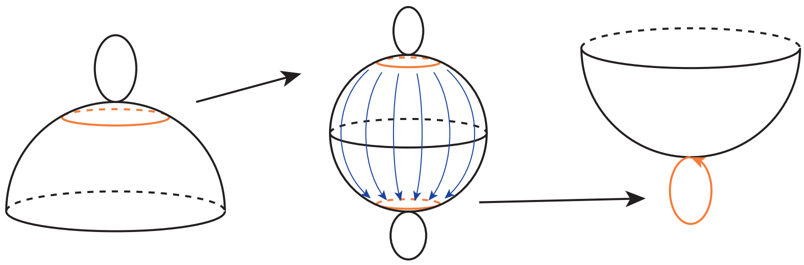

Figure C.3: Circle created from the point can lap the $\boldsymbol{S}^1$.

the circle wind the $\boldsymbol{S}^1$ once.

$$\begin{aligned} \theta &= \pi \\ \phi &= 2\pi t \\ \psi_S &= \psi_N - 2\phi = -4\pi t \end{aligned} \tag{C.46}$$

Of course one can repeat the same thing to wind the circle any times. Therefore, inversely, a circle winding many times can shrink into a point, which is expressed as $\pi_1(\boldsymbol{S}^3) = 1$.

$S^3/\mathbb{Z}_n$ case is similar but gives a non-trivial result. The orbifolding affect the $U(1)$ fiber part and it reduces the period from 4π to $4\pi/n$. Then, the junction condition becomes

$$\psi_N = \psi_S + 2n\phi \tag{C.47}$$

Thus, a circle winding S^1 n times can shrink into a point. Inversely, a circle winding less than n times can not shrink into a point, which is expressed as $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$.

C.5 Weyl transformation and S^3

Let us firstly derive the S^1 from $\mathbb{R}^1$. The S^1 is parametrized by $r\theta$ with $\theta \sim \theta + 2\pi$ from the north pole to clockwise direction. The $\mathbb{R}^1$ is parametrized by x; see Fig. C.4. From the geometric calculation we have the following relation between two;

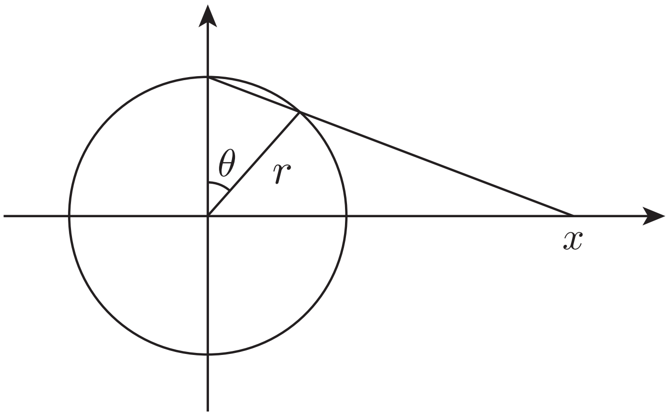

Figure C.4: Circle with radius r and its stereographic projection from the north pole

$$\tan\frac{\theta}{2} = \frac{r}{x}. \tag{C.48}$$

Note that $\theta = 0$ corresponds to $x = \infty$; there we have to use another patch to parametrize the point e.g. use the stereographic projection from the south pole.

From (C.48) we can also have the relation of the lengths.

$$r d\theta = -\frac{2r^2}{r^2 + x^2} dx. \tag{C.49}$$

The square of those gives the metric

$$ds^2 = r^2 d\theta^2 = \frac{4r^4}{(r^2 + x^2)^2} dx^2 = g(x) dx^2. \tag{C.50}$$

This means that S^1 is conformally flat. Similarly, we can create S^3 from $\mathbb{R}^3$ by Weyl transformation. In the case we only need to replace x^2 by $x^m x_m$ in $g(x)$;

$$\eta_{mn} \to g_{mn} = \frac{4r^4}{(r^2 + x^2)^2} \eta_{mn}. \tag{C.51}$$

Appendix D

Chern-Simons term

D.1 Chern-Simons level quantization

Let us see the quantization of Chern-Simons level explicitly. We use differential form here; we summarized it in Section C.2. The finite gauge transformation of the Chern-Simons action (B.26) with ignoring the surface term becomes

$$\delta_g S_{\mathrm{CS}} = \frac{\kappa}{6} \int d^3x \ \mathrm{tr} \left(U^{-1} dU \right)^3 \tag{D.1}$$

where we omitted the wedge products. Since non-Abelian gauge groups always include $SU(2)$ subgroup let us consider the $SU(2)$ gauge group. Due to the non-trivial fundamental group $\pi_3 \left(SU(2) \right) = \mathbb{Z}$ (D.1) is not zero. As one may notice the integrand of (D.1) is the left-invariant 1-form. We can explicitly write down the left-invariant 1-form by the generators of $SU(2)$. Then, the integration gives the volume of $\boldsymbol{S}^3$:

$$\begin{aligned} \delta_g S_{\mathrm{CS}} &= \frac{\kappa}{6} \int d^3x \ \mathrm{tr} \left(U^{-1} dU \right)^3 = \frac{4\kappa}{3} \int d^3x \ \mathrm{tr} \left(T^a T^b T^c \right) \mu_a \mu_b \mu_c \\ &= -2\kappa \int \ \mu^1 \wedge \mu^2 \wedge \mu^3 \\ &= 4\pi^2 \kappa. \end{aligned} \tag{D.2}$$

Though we used anti-Hermite representation one can do the same thing in a Hermite one and reach the same result. The result means that the action is not invariant under the large gauge transformation (B.28). However, the partition function is invariant;

$$Z' = \int \mathcal{D}\Phi e^{iS_{\mathrm{CS}} + i\delta_g S_{\mathrm{CS}}} = \int \mathcal{D}\Phi e^{iS_{\mathrm{CS}} + 2\pi i (2\pi\kappa)} \tag{D.3}$$

if one impose the constraint

$$2\pi\kappa = k \in \mathbb{Z}. \tag{D.4}$$

We call this integer k Chern-Simons level. The final form of the action is then,

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int d^3x \ \epsilon^{\mu\nu\rho} \mathrm{tr} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right). \tag{D.5}$$

D.2 Holonomy for Chern-Simons term

We firstly reconsider the quantization of the Chern-Simons term in a different method. Then, we discuss the holonomy for the Chern-Simons term. In this section we extensively use the differential form, which is summarized in Section C.2.

We consider the two gauge fields A and B on S^3 and their cross term in Chern-Simons term:

$$S = \kappa \int AdB \tag{D.6}$$

The gauge transformations of these gauge fields are given as follows.

$$\delta A = d\lambda_A \tag{D.7}$$
$$\delta B = d\lambda_B \tag{D.8}$$

We describe the $\boldsymbol{S}^3$ as in Appendix C The coordinate is the Euler angle (θ, ϕ, ψ). As the coordinate is not well-defined we separate it into two patches and introduce local coordinates for them; we distinguish them by putting subscripts N and S for upper-half and lower-half.

The junction condition for the fields are given as follows.

$$\begin{aligned} A_N + d\alpha &= A_S \\ B_N + d\beta &= B_S \end{aligned} \tag{D.9}$$

These conditions should not depend on the gauge choice. We denote the fields after the gauge transformation with prime (e.g. $A' = A + d\lambda_A$), and the conditions become

$$A'_N + d\alpha' = A'_S \tag{D.10}$$
$$\rightarrow \quad A_N + d\lambda_{A_N} + d\alpha' = A_S + d\lambda_{A_S}. \tag{D.11}$$

In order to make these be the same as (D.9) we have to set

$$\alpha' = \alpha - \lambda_{A_N} + \lambda_{A_S}. \tag{D.12}$$

We fix the coefficient κ of the Chern-Simons action so that the action is gauge invariant. The gauge transformation of the action (D.6) is

$$\delta S = \kappa \left(\int_N d\lambda_{A_N} dB_N + \int_S d\lambda_{A_S} dB_S \right). \tag{D.13}$$

We use the Stokes theorem but note that λ has period 2π and multi-valued. However, its derivative $d\lambda$ is well-defined. Thus, the Stokes theorem should be used so that those derivatives remain:

$$\begin{aligned} \delta S &= -\kappa \left(\int_N d(d\lambda_{A_N} B_N) + \int_S d(d\lambda_{A_S} B_S) \right) = -\kappa \left(\int_{\partial N} d\lambda_{A_N} B_N + \int_{\partial S} d\lambda_{A_S} B_S \right) \\ &= -\kappa \left(\int_{\partial N} d\lambda_{A_N} B_N - \int_{\partial N} d\lambda_{A_S} (B_N + d\beta) \right) \\ &= \kappa \left(\int_{\partial N} (d\lambda_{A_S} - d\lambda_{A_N}) B_N + \int_{\partial N} d\lambda_{A_S} d\beta \right), \end{aligned} \tag{D.14}$$

We used the fact the direction of the boundary ∂N and ∂S are opposite each other between the first and second line. Generally, the remaining terms have non-zero values, and hence, we introduce a term that erase them. The possible candidates are A, B, α, β. We use the derivative of α, and its gauge transformation

$$\delta(d\alpha) = -d\lambda_{A_N} + d\lambda_{A_S}. \tag{D.15}$$

As this is the same as the first term of (D.14), We add the following term to the action.

$$S_1 = -\kappa \int_{\partial N} d\alpha B_N \tag{D.16}$$

The gauge transformation of the modified action is

$$\delta(S + S_1) = \kappa \int_{\partial N} d\lambda_{A_S} d\beta - \kappa \int_{\partial N} d\alpha d\lambda_{B_N} \tag{D.17}$$

We explicitly integrate the action on the torus. The path of the integration is taken as in Fig. D.1.

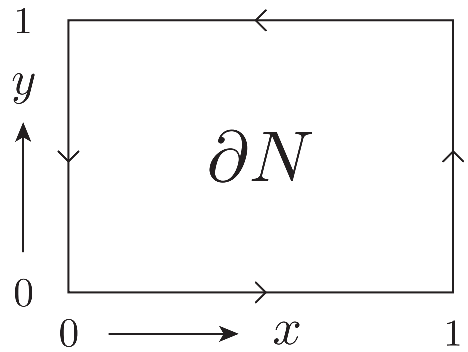

Figure D.1: Integration path along the boundary of torus T^2

$$\begin{aligned}
\int_{\partial N} d\lambda_{A_S} d\beta &= \oint \lambda_{A_S}(x, y) d\beta \\
&= \int \lambda_{A_S}(x, 0)\partial_x \beta dx + \int \lambda_{A_S}(1, y)\partial_y \beta dy - \int \lambda_{A_S}(x, 1)\partial_x \beta dx - \int \lambda_{A_S}(0, y)\partial_y \beta dy \\
&= \int [\lambda_{A_S}(x, 0) - \lambda_{A_S}(x, 1)]\partial_x \beta dx + \int [\lambda_{A_S}(1, y) - \lambda_{A_S}(0, y)]\partial_y \beta dy \\
&= -\Delta_y \lambda_{A_S} \int \partial_x \beta dx + \Delta_x \lambda_{A_S} \int \partial_y \beta dy \\
&= -\Delta_y \lambda_{A_S} \Delta_x \beta + \Delta_x \lambda_{A_S} \Delta_y \beta \\
&= (2\pi)^2 k \qquad (m \in \mathbb{Z})
\end{aligned} \tag{D.18}$$

Therefore, if we set $\kappa = k/2\pi$ $(k \in \mathbb{Z})$ the action is well-defined.

Now we are ready to consider the orbifold case. Due to the non-trivial fundamental group $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$ the vacua is specified by holonomies h_A, h_B. The holonomies are related to the gauge fields as follows.

$$\frac{2\pi h_A}{n} = \oint A, \qquad \frac{2\pi h_B}{n} = \oint B \tag{D.19}$$

We explicitly introduce the coordinates $\theta,\ \phi,\ \psi_{N,S}$. As discussed in Section C.4, the junction condition is

$$\psi_S = \psi_N + 2n\phi_N \tag{D.20}$$

From (D.19) the gauge fields are written in terms of the coordinate:

$$A_N = \frac{h_A}{2n} d\psi_N, \qquad A_S = \frac{h_A}{2n} d\psi_S, \tag{D.21}$$

where note that the period of ψ's are 4π. The junction for the fields (D.9) tells us that

$$\begin{aligned} \frac{h_A}{2n}(d\psi_S - d\psi_N) &= d\alpha \\ \rightarrow \quad h_A d\phi_N &= d\alpha. \end{aligned} \tag{D.22}$$

The same for B. Substitute this to the action (D.16) with $\kappa = k/2\pi$ we have

$$\begin{aligned} S &= -\frac{k}{2\pi}\int_{\partial N} d\alpha B_N = -\frac{k}{2\pi}\int h_A d\phi_N \frac{h_B}{2n} d\psi_N = -\frac{k}{4\pi n} h_A h_B (8\pi^2) \\ &= -\frac{2\pi k}{n} h_A h_B. \end{aligned} \tag{D.23}$$

This is the distinctive feature for the Chern-Simons term on the orbifold. Note that we considered the cross term of A and B, hence, it is multiplied by 2. For the Chern-Simons term with single gauge field leads to $\pi k h_A^2/n$.

D.3 Gauge invariance of the Chern-Simons term in superspace

We introduce useful expression

$$g = e^{2tV}, \tag{D.24}$$

and rewrite the action;

$$S_{\text{CS}} = -\frac{k}{8\pi}\int d^3x d^4\theta \int_0^1 dt \underbrace{\text{tr}\left[\left(g^{-1}\partial_t g\right)\overline{D}\left(g^{-1}Dg\right)\right]}_{I}, \tag{D.25}$$

where we denote I as the integrand. This is topological and is written by the total derivative. In order to see this we introduce a symbol;

$$[d] = g^{-1}dg. \tag{D.26}$$

From the definition the following formula is easily derived.

$$\delta[d] = [\delta d] - [\delta][d] \tag{D.27}$$

The gauge transformation is then given as follows.

$$\begin{aligned}\delta I &= \delta \ \mathrm{tr}\left([\partial_t]\overline{D}[D]\right) \\ &= \mathrm{tr}\left([\delta\partial_t]\overline{D}[D]\right) - \mathrm{tr}\left([\delta][\partial_t]\overline{D}[D]\right) + \mathrm{tr}\left([\partial_t]\overline{D}[\delta D]\right) - \mathrm{tr}\left([\partial_t]\overline{D}[\delta][D]\right)\end{aligned} \tag{D.28}$$

After the partial integration of x, θ, t these terms become zero. Note that the total derivative of x and θ is set to zero as usual. However, we still have to pay attention to the surface term of t. It is

$$\delta S_{\mathrm{CS}} = -\frac{k}{8\pi}\int d^3x d^4\theta \int_0^1 dt \ \partial_t \mathrm{tr}\left([\delta]\overline{D}[D]\right) = -\frac{k}{8\pi}\int d^3x d^4\theta \ \ \mathrm{tr}\left([\delta]\overline{D}[D]\right)\Big|_{t=1} . \tag{D.29}$$

The explicit infinitesimal gauge transformation for $[\delta]$:

$$[\delta]\big|_{t=1} \, g^{-1}\delta g\Big|_{t=1} = e^{-2V}e^{i\overline{\Lambda}}e^{2V}e^{-i\Lambda} - 1 = ie^{-2V}\overline{\Lambda}e^{2V} - i\Lambda \tag{D.30}$$

shows that (D.29) is zero, and therefore, the action is invariant under the gauge transformation.

Appendix E

Spherical harmonic function

Our goal in this chapter is to construct the formula to derive the eigenvalues of the Laplacian on a symmetric space $\mathcal{M}$. This chapter is base on [55].

E.1 Frame bundle

We describe the spin R field ψ on a d dimensional manifold $\mathcal{M}$ using the bra-ket representation $|\psi\rangle$:

$$|\psi\rangle = \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, i\rangle \psi_i(\boldsymbol{x}) \tag{E.1}$$

where $\boldsymbol{x}$ is a point on the manifold $\mathcal{M}$, i is the index of spin basis of $SO(d)$, and $\psi_i(\boldsymbol{x})$ is the wave function. The basis $|\boldsymbol{x}, i\rangle$ is defined by the following tensor product.

$$|\boldsymbol{x}\rangle \otimes |i\rangle \tag{E.2}$$

The basis for the spin is usually defined on each point $\boldsymbol{x}$, hence it should be written $|i\rangle_{\boldsymbol{x}} \in V_{\boldsymbol{x}}$. But they are not isolated and we can actually define the map from non-trivial basis $|i\rangle_{\mathbf{0}}$ to the basis on every points $\boldsymbol{x}$ as $|i\rangle_{\boldsymbol{x}}$.

$$f(\boldsymbol{x}) \;:\; |i\rangle_{\mathbf{0}} \;\longmapsto\; |i\rangle_{\boldsymbol{x}} \tag{E.3}$$

We can now construct the spin basis at any point from $|i\rangle_{\mathbf{0}}$ and $f(x)$. $f(x)$ is the section of $SO(d)$ fiber bundle over $\mathcal{M}$. This fiber bundle is called frame bundle $F\mathcal{M}$.

Benefit of the frame bundle is that we can set frame without a specific spin.

According to this change we rewrite the basis as follows.

$$|\boldsymbol{x}, f(\boldsymbol{x})\rangle \otimes |i\rangle_{\mathbf{0}} \tag{E.4}$$

Using this the field ψ is rewritten as follows.

$$|\psi\rangle = \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f(\boldsymbol{x})\rangle \otimes |i\rangle_0 \psi_i(\boldsymbol{x}) \tag{E.5}$$

the section f always acts on the element of spin space V_x, henceforth we identify the action leads to the same result. Namely,

$$|\boldsymbol{x}, f(\boldsymbol{x})h\rangle \otimes |i\rangle_0 \sim |\boldsymbol{x}, f(\boldsymbol{x})\rangle \otimes h|i\rangle_0, \quad h \in H = SO(d). \tag{E.6}$$

The tensor product with the identification above is written as $\otimes_H$. Using this product we can derive the relation between the wave functions with different section $f' = fh$;

$$\begin{aligned}
|\psi\rangle &= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f'(\boldsymbol{x})\rangle \otimes_H |i\rangle_0 \psi'_i(\boldsymbol{x}) \\
&= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f(\boldsymbol{x})h(\boldsymbol{x})\rangle \otimes_H |i\rangle_0 \psi'_i(\boldsymbol{x}) \\
&= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f(\boldsymbol{x})\rangle \otimes_H h(\boldsymbol{x})|i\rangle_0 \psi'_i(\boldsymbol{x}) \\
&= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i,j=1}^{\dim R} |\boldsymbol{x}, f(\boldsymbol{x})\rangle \otimes_H |j\rangle_0 h(\boldsymbol{x})_{ji} \psi'_i(\boldsymbol{x})
\end{aligned} \tag{E.7}$$

where we defined

$$h(\boldsymbol{x})_{ij} = {}_0\langle i|h(\boldsymbol{x})|j\rangle_0 \tag{E.8}$$

, which is the representation matrix of representation R of $h(\boldsymbol{x})$. Comparison between (E.5) and (E.8) give us following relation:

$$\psi_i(\boldsymbol{x}) = \sum_{j=1}^{\dim R} h(\boldsymbol{x})_{ij} \psi'_j(\boldsymbol{x}). \tag{E.9}$$

This is the local rotation of the spin R field.

In order to relate the field $|\psi\rangle$ and its component (or wave function) $\psi_i(\boldsymbol{x})$ the section $f(\boldsymbol{x})$ of frame bundle has to be chosen. As the section is locally chosen when we consider the translation the section also changes. To keep the parallelism of the spin we need to cancel the change of the section caused by the translation.

Let us consider the explicit translation that bring a particle located at a point $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\epsilon}$ to $\boldsymbol{x}$ (see Fig. E.1). The translation determine the unitary transformation U_ϵ that change $V_{\boldsymbol{x}'}$ to $V_{\boldsymbol{x}}$;

$$U_\epsilon \;:\; V_{\boldsymbol{x}'} \longmapsto V_{\boldsymbol{x}}. \tag{E.10}$$

We denote this translation P, and its action to the basis is

$$P|\boldsymbol{x}', f(\boldsymbol{x}')\rangle = |\boldsymbol{x}, U_\epsilon f(\boldsymbol{x}')\rangle, \tag{E.11}$$

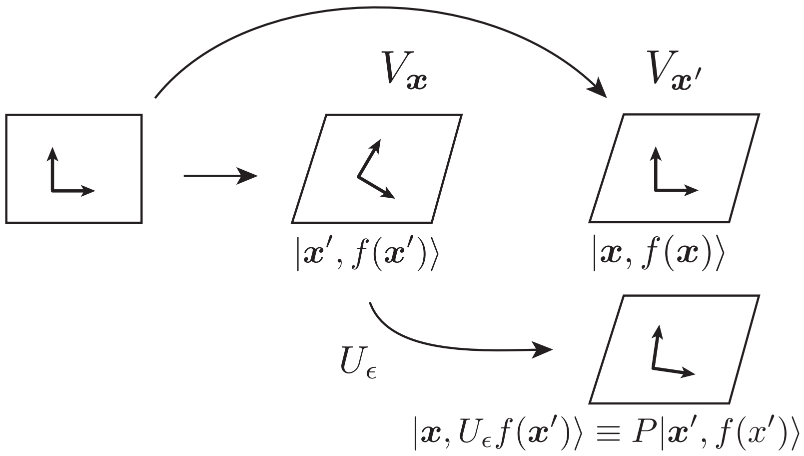

Figure E.1: Translation, the change of section and pullback of the basis

which changes the frame. Since the new section $U_\epsilon f(\boldsymbol{x}')$ is different from the original section $f(\boldsymbol{x})$ we have to modify it to compare the two. The translated field $P|\psi\rangle$ is

$$\begin{aligned} P|\psi\rangle &= P\int_{\mathcal{M}} d\boldsymbol{x}' \sum_{i=1}^{\dim R} |\boldsymbol{x}', f(\boldsymbol{x}')\rangle \otimes_H |i\rangle_0 \psi_i(\boldsymbol{x}') \\ &= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, U_\epsilon f(\boldsymbol{x}')\rangle \otimes_H |i\rangle_0 \psi_i(\boldsymbol{x}') \\ &= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f(\boldsymbol{x})\Omega_{\boldsymbol{\epsilon}}\rangle \otimes_H |i\rangle_0 \psi_i(\boldsymbol{x}') \\ &= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f(\boldsymbol{x})\rangle \otimes_H \Omega_{\boldsymbol{\epsilon}}|i\rangle_0 \psi_i(\boldsymbol{x}') \\ &= \int_{\mathcal{M}} d\boldsymbol{x} \sum_{i,j=1}^{\dim R} |\boldsymbol{x}, f(\boldsymbol{x})\rangle \otimes_H |i\rangle_0 \Omega_{\boldsymbol{\epsilon},ij} \psi_j(\boldsymbol{x}') \end{aligned} \tag{E.12}$$

where we defined

$$\Omega_{\boldsymbol{\epsilon}} = f^{-1}(\boldsymbol{x}) U_{\boldsymbol{\epsilon}} f(\boldsymbol{x}') \ : \ V_{\boldsymbol{0}} \to V_{\boldsymbol{x}'} \to V_{\boldsymbol{x}} \to V_{\boldsymbol{0}}, \tag{E.13}$$

and its representation matrix $\Omega_{\boldsymbol{\epsilon},ij}$. As $\Omega_{\boldsymbol{\epsilon}}$ is isomorphic mapping on $V_{\boldsymbol{0}}$ it is the element of $SO(d)$. Hence, we can compare the two spinors healthily with the wave functions $\psi_i(\boldsymbol{x})$ and

$$\psi_i^P(\boldsymbol{x}) = \sum_{j=1}^{\dim R} \Omega_{\boldsymbol{\epsilon},ij} \psi_j(\boldsymbol{x}'). \tag{E.14}$$

Using the infinitesimal translation parameter ϵ^μ we can denote $\Omega_{\boldsymbol{\epsilon}} = 1 + \epsilon^\mu \omega_\mu$ with the element of Lie algebra $so(d)$, ω_μ, Then, the covariant derivative is defined as follows.

$$\epsilon^\mu D_\mu \psi_i(\boldsymbol{x}) = \psi_i^P(\boldsymbol{x}) - \psi_i(\boldsymbol{x})$$
$$= \epsilon^\mu \left[\partial_\mu \psi_i(\boldsymbol{x}) + \sum_{j=1}^{\dim R} \omega_{\mu,ij} \psi_j(\boldsymbol{x}) \right], \tag{E.15}$$

where ω_μ is the spin connection.

This complexity comes from choosing the section. Hence, let us define the field as a function on the frame bundle $F\mathcal{M}$ without the choice of the section. Namely, we define the map $f : V_{\mathbf{0}} \to V_{\boldsymbol{x}}$ that is independent of the point $\boldsymbol{x}$ as an isolated coordinate. In this notation the basis is

$$|\boldsymbol{x}, f\rangle. \tag{E.16}$$

This pair gives a point of the frame bundle $F\mathcal{M}$, and hence, it is the basis of the function on the frame bundle. You can suppose that the pair $(\boldsymbol{x}, f)$ put the spin space V_0 with the direction specified by f on the manifold $\mathcal{M}$. Using this basis we can rewrite the general state as follows.

$$|\psi\rangle = \int_f d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f\rangle \otimes_H |i\rangle_0 \Phi_i(\boldsymbol{x}, f) \tag{E.17}$$

The difference compared to (E.5) is that the wave function $\Psi_i(\boldsymbol{x}, f)$ is defined over the whole frame bundle, and the integration is performed over the section f. In order to make the integrand be independent of the choice of the section, the integrand should not depend on the section. Especially, the state is unchanged under the replacement $f \to fh$ (local rotation) the integrand should satisfy following relation:

$$\sum_{i=1}^{\dim R} |\boldsymbol{x}, f\rangle \otimes_H |i\rangle_0 \Phi_i(\boldsymbol{x}, f) = \sum_{i=1}^{\dim R} |\boldsymbol{x}, fh\rangle \otimes_H |i\rangle_0 \Phi_i(\boldsymbol{x}, fh)$$
$$= \sum_{i=1}^{\dim R} |\boldsymbol{x}, f\rangle \otimes_H h|i\rangle_0 \Phi_i(\boldsymbol{x}, fh)$$
$$= \sum_{i,j=1}^{\dim R} |\boldsymbol{x}, f\rangle \otimes_H |i\rangle_0 h_{ij} \Phi_j(\boldsymbol{x}, fh) \tag{E.18}$$

Namely, the wave function should satisfy the relation

$$\Phi_i(\boldsymbol{x}, f) = \sum_{j=1}^{\dim R} h_{ij} \Phi_j(\boldsymbol{x}, fh). \tag{E.19}$$

If the wave function satisfy this relation the state vector $|\psi\rangle$ does not depend on the choice of the section.

The translation of the state in this notation is written as follows.

$$P_\epsilon |\psi\rangle = \int_f d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, U_\epsilon f\rangle \otimes_H |i\rangle_0 \Phi_i(\boldsymbol{x}', f)$$
$$= \int_f d\boldsymbol{x} \sum_{i=1}^{\dim R} |\boldsymbol{x}, f\rangle \otimes_H |i\rangle_0 \Phi_i(\boldsymbol{x}', U_\epsilon^{-1} f) \tag{E.20}$$

The translated wave function is simply illustrated by

$$\Psi_i^P(\boldsymbol{x}, f) = \Psi_i(\boldsymbol{x}', U_\epsilon^{-1} f). \tag{E.21}$$

We do not have to consider the spin connection and so on. What we have to do is that specify the wave function satisfying the condition (E.19), and the translation U_ϵ. This is easily done in the case of symmetric space as we will see coming sections.

E.2 Frame bundle of G/H

Let us consider a group G transitively act on a manifold $\mathcal{M}$. This means that the manifold $\mathcal{M}$ is a homogeneous space. We choose the origin O of the manifold. We call a subgroup that do not move the origin by H:

$$H = \{g \in G | gO = O\}. \tag{E.22}$$

We consider the element $g\ inG$ moves the origin to a point x, namely, we set $x = gO$. Then, gH also moves the origin to the point x. Hence, we have the manifold $\mathcal{M}$ by the identification $g \sim gh$:

$$\mathcal{M} = G/H. \tag{E.23}$$

This means that G is the H-bundle over $\mathcal{M}$. We define a projection π that project G to $\mathcal{M}$.

We describe the Lie algebra of G and H by $\mathcal{G}$ and $\mathcal{H}$, respectively. Also, we define the orthogonal complement $\mathcal{K}$. These satisfy the following commutation relations:

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{K}] \subset \mathcal{H}. \tag{E.24}$$

The first two relations are just the definition of subgroup. The last relation is quite non-trivial. We regard $\mathcal{K}$ as the translation, $\mathcal{H}$ as local rotation, and as we will see the last relation means the torsionless condition.

In previous section we saw that the field on the manifold $\mathcal{M}$ can be denoted by that on the frame bundle $F\mathcal{M}$. A point on the frame bundle $F\mathcal{M}$ gives both a point of the manifold $\mathcal{M}$ and the local orthogonal set. In the case here $\mathcal{M} = G/H$ this is realized by choosing the element of G. Namely, if we denote the basis on the origin by $|i\rangle_0$ and the basis moved by g from the origin by $|i\rangle_{\boldsymbol{x}}$, we can define the homeomorphism of G and $F\mathcal{M}$. As like this we can unify the coordinates $\boldsymbol{x}$ of the manifold and f fixing the section into $g \in G$. We reexpress the configuration of the spin R field on $\mathcal{M}$ (E.17) by

$$|\Psi\rangle = \int dg |g\rangle \otimes_H |i\rangle_0 \Psi_i(g), \tag{E.25}$$

where the product $\otimes_H$ is defined by the identification:

$$|gh\rangle \otimes_H |i\rangle_0 = |g\rangle \otimes_H h|i\rangle_0. \tag{E.26}$$

The condition (E.19) becomes

$$\Psi_i(g) = \sum_{j=1}^{\dim R} h_{ij} \Psi_j(gh). \tag{E.27}$$

In case of $\mathcal{M} = G/H$ we can have the complete set of the wave function. We denote the representation $\widetilde{R}$ of G including the branch point into the spin representation R. We express the representation matrix for $\widetilde{R}$ by $\rho^{\widetilde{R}}_{AB}$, and the space that the matrix acts by $V_{\widetilde{R}}$. The basis indices of the space $V_{\widetilde{R}}$ are written by $A, B = 1, \cdots, \dim \widetilde{R}$. $\widetilde{R}$ branching into R means that there exist the subspace $V_R \in V_{\widetilde{R}}$ that transform as the representation R. The basis indices of this subspace are written by $i, j = 1, \cdots, \dim R$.

In our case this means that there exist the representation $R \in \widetilde{R}$ that is invariant under the H action. We define the function for every representation $\widetilde{R}$ of G that branches into R:

$$Y^{\widetilde{R}}_{i,A}(g) = \rho^{\widetilde{R}}(g^{-1})_{iA}. \tag{E.28}$$

This function satisfy the condition (E.27), and it makes the complete system. Namely, the function $\Psi_i(g)$ satisfy the condition is expressed by the linear combination of (E.28):

$$\Psi_i(g) = \sum_{\widetilde{R}} \sum_{A=1}^{\dim \widetilde{R}} \rho^{\widetilde{R}}(g^{-1})_{iA} \Psi^{\widetilde{R}}_A, \tag{E.29}$$

where $\Psi^{\widetilde{R}}_A$ are constant coefficient. If $\widetilde{R}$ includes more than two branches the sum over the $\widetilde{R}$ is taken over each subgroup R. It is easily checked that (E.29) satisfy (E.27).

E.3 Isometry and Lie derivative

From the relation G and $F\mathcal{M}$ we can define an isometry for $\mathcal{M}$ by the right action:

$$\mathrm{iso}(g')|g\rangle = |g'g\rangle. \tag{E.30}$$

The isometry action on (E.25) gives the transformation of the wave function:

$$\begin{aligned} \mathrm{iso}|\Psi\rangle &= \int dg \sum_{i,\widetilde{R},A} |g'g\rangle \otimes_H |i\rangle_0 \rho^{\widetilde{R}}(g^{-1})_{iA} \Psi^{\widetilde{R}}_A \\ &= \int dg \sum_{i,\widetilde{R},A} |g\rangle \otimes_H |i\rangle_0 \rho^{\widetilde{R}}(g^{-1}g')_{iA} \Psi^{\widetilde{R}}_A \\ &= \int dg \sum_{i,\widetilde{R},A,B} |g'g\rangle \otimes_H |i\rangle_0 \rho^{\widetilde{R}}(g^{-1})_{iA} \rho^{\widetilde{R}}(g')_{AB} \Psi^{\widetilde{R}}_B. \end{aligned} \tag{E.31}$$

The isometry is expressed as a rotation of the constant vector $\Psi_A^{\widetilde{R}}$:

$$\Psi_A^{\widetilde{R}} \longrightarrow \Psi_A'^{\widetilde{R}} = \sum_B \rho^{\widetilde{R}}(g')_{AB} \Psi_B^{\widetilde{R}}. \tag{E.32}$$

Lie derivative corresponding to the generator X of the isometry is defined by an infinitesimal transformation of a field under the isometry:

$$\mathcal{L}_X = \mathrm{iso}(1+X) - 1. \tag{E.33}$$

In terms of (E.31)

$$\mathcal{L}_X \Psi_i(g) = \Psi_i\left((1+X)^{-1} g\right) - \Psi_i(g) = \Psi_i(-Xg). \tag{E.34}$$

Though action of the isometry on g is the left one, we rewrite it as the right action as follows.

$$(1+X)g = gg^{-1}(1+X)\,g = g\,(1 + v^m K_m + \omega^a H_a) \tag{E.35}$$

In this notation v^m is the Killing vector for X, and $\omega^a H_a$ is the rotation attached to the isometry.

E.4 Covariant derivative

We fix the section f, and define a orthogonal coordinate ξ^m near an arbitrary point $\boldsymbol{x}$ $(m = 1, \cdots, \dim \mathcal{M})$. This is give by the projection from ξ^m to the neighbor of the point $\boldsymbol{x}$.

$$\boldsymbol{x}(\xi) = \pi\left(g(1 + \xi^m K_m)\right) \tag{E.36}$$

We abbreviate the sum symbol from now on. The metric around the origin is given by

$$ds^2 = \xi^m \xi^m. \tag{E.37}$$

The normalization of the K_m includes the information of the size of the manifold $\mathcal{M}$.

When we determine the orthogonal set the translation of from ξ^m to $\xi^m - \epsilon^m$ is given as follows.

$$g \longrightarrow g\,(1 - \epsilon^m K_m)\,e. \tag{E.38}$$

We denote this translation as P_ϵ:

$$P_\epsilon |g\rangle = |g\,(1 - \epsilon^m K_m)\rangle. \tag{E.39}$$

P_ϵ acting on (E.25) gives

$$\begin{aligned} P_\epsilon |\Psi\rangle &= \int dg |g\,(1 - \epsilon^m K_m)\rangle \otimes_H |i\rangle_0 \Psi_i(g) \\ &= \int dg |g\rangle \otimes_H |i\rangle_0 \Psi_i(g\,(1 + \epsilon^m K_m)). \end{aligned} \tag{E.40}$$

Using this P_ϵ we can define the covariant derivative as follows.

$$\begin{aligned}\epsilon^m D_m|\Psi\rangle &= P_\epsilon|\Psi\rangle - |\Psi\rangle \\ &= \int dg|g\rangle \otimes_H |i\rangle_0 \left[\Psi_i(g\,(1+\epsilon^m K_m)) - \Psi_i(g)\right].\end{aligned} \tag{E.41}$$

For the wave function ψ_i it is given as follows.

$$D_m\Psi_i = \Psi_i(gK_m) \tag{E.42}$$

The right hand side is defined by

$$\epsilon^m\Psi_i(gK_m) = \Psi_i(g\,(1+\epsilon^m K_m)) - \Psi_i(g) \tag{E.43}$$

Using the expansion of the wave function (E.29):

$$\Psi_i^{\widetilde{R}}(g) = \rho^{\widetilde{R}}(g^{-1})_{iA}\Psi_A^{\widetilde{R}}. \tag{E.44}$$

In this notation (E.43) is written as follows.

$$\begin{aligned}\Psi_i(g\,(1+\epsilon^m K_m)) - \Psi_i(g) &= \left[\rho^{\widetilde{R}}((1-\epsilon^m K_m)\,g^{-1})_{iA} - \rho^{\widetilde{R}}(g^{-1})_{iA}\right]\Psi_A^{\widetilde{R}} \\ &= \rho^{\widetilde{R}}((-\epsilon^m K_m))_{iA}\rho^{\widetilde{R}}(g^{-1})_{AB}\Psi_B^{\widetilde{R}} \\ &= \rho^{\widetilde{R}}((-\epsilon^m K_m))_{iA}\Psi_A^{\widetilde{R}}(g),\end{aligned} \tag{E.45}$$

where we defined the wave function extended to the $\widetilde{R}$ by $\Psi_A^{\widetilde{R}}(g)$:

$$\Psi_A^{\widetilde{R}}(g) = \rho^{\widetilde{R}}(g^{-1})_{AB}\Psi_B^{\widetilde{R}}. \tag{E.46}$$

Using these the covariant derivative is expressed as follows.

$$D_m\Psi_i^{\widetilde{R}}(g) = \Psi_i^{\widetilde{R}}(gK_m) = -\rho^{\widetilde{R}}(K_m)_{iA}\Psi_A^{\widetilde{R}}(g). \tag{E.47}$$

Note that the covariant derivative satisfy the condition (E.27):

$$D_m\Psi(gh) = \Psi_i(ghK_m) = \Psi\left(g(hK_mh^{-1})h\right) = h^{-1}_{mn}h^{-1}_{ij}\Psi_j(gK_n) = h^{-1}_{mn}h^{-1}_{ij}D_n\Psi_j(g), \tag{E.48}$$

where note that the index of the covariant derivative m also transform as the spin index. The state vector for the covariant derivative can be defined as follows.

$$|D\Psi\rangle = \int dg|g\rangle \otimes_H (|m\rangle_0 \otimes |i\rangle_0)\, D_m\Psi_i(g) \tag{E.49}$$

The identifying product $\otimes_H$ is defined as follows.

$$|gh\rangle \otimes_H (|m\rangle_0 \otimes |i\rangle_0) \sim |g\rangle \otimes_H (h|m\rangle_0 \otimes h|i\rangle_0) \tag{E.50}$$

To take the derivative twice we just repeat the (E.47) twice:

$$D_mD_n\Psi_i^{\widetilde{R}}(g) = D_m\Psi_i^{\widetilde{R}}(gK_n) = \Psi_i^{\widetilde{R}}(gK_mK_n). \tag{E.51}$$

Then, the Laplacian is given as follows.

$$\Delta\Psi_i^{\widetilde{R}}(g) = \Psi_i^{\widetilde{R}}(gK_mK_m) = \left[C_2(\widetilde{R}) - C_2(R)\right]\Psi_i(g), \tag{E.52}$$

where $C_2(\widetilde{R})$ is the Casimir operator that created from K_AK_B contracted by the G invariant metric g^{AB}, and $C_2(R)$ is that of limited region inside the representation R:

$$C_2(\widetilde{R}) = -g^{AB}T_AT_B, \qquad C_2(R) = -g^{ij}H_iH_j. \tag{E.53}$$

Since we use anti-Hermite generators we defined the Casimir operators to be positive. The eigenvalue of the Laplacian is completely determined algebraically.

The commutator of the covariant derivative

$$[D_m, D_n]\Psi_i^{\widetilde{R}}(g) = \Psi_i^{\widetilde{R}}(g[K_m, K_n]) = \rho^R\left(-[K_m, K_n]\right)_{ij}\Psi_j^{\widetilde{R}}(g) \tag{E.54}$$

gives the Riemann tensor:

$$R_{mn} = [D_m, D_n] = -[K_m, K_n]. \tag{E.55}$$

E.5 Spin connection and vielbein

We can extract the information of the spin connection and the vielbein by comparing to the covariant derivative, which is given by

$$e^m D_m\Psi_i = d\Psi_i + \omega_{ij}\Psi_j. \tag{E.56}$$

We rewrite the derivative in RHS:

$$d\Psi_i(g) = \Psi_i(dg), \tag{E.57}$$

where we regard the RHS expanded as in (E.29) and d acts on the $\rho\ \widetilde{R}(g^{-1})$. The spin connection is rewritten as $\omega\Psi(g) = \Psi(-g\omega)$. Using these expressions and the algebraic form of the covariant gives

$$e^m\Psi_i(gK_m) = \Psi_i(dg) + \Psi_i(-g\omega). \tag{E.58}$$

This should be satisfied for any field, and hence,

$$\mu = e + \omega, \qquad \mu = g^{-1}dg, \tag{E.59}$$

where $e = e^mK_m$. μ, by the definition, satisfy following equation.

$$d\mu + \mu^2 = 0 \tag{E.60}$$

Substitute the expression for the μ and divide the result into $\mathcal{H}$ component and $\mathcal{K}$ component, and we have

$$\begin{aligned} de + \omega e + e\omega &= 0 \\ d\omega + \omega^2 + e^2 &= 0. \end{aligned} \tag{E.61}$$

The first equation illustrates the torsionless condition. The last one gives the Riemann tensor.

E.6 $\boldsymbol{S}^3$ case

Any dimensional sphere is given as a group manifold $SO(n+1)/SO(n)$ We especially look into the $\boldsymbol{S}^3$ case. Since $SO(4) \sim SU(2)_L \times SU(2)_R$ we construct the $\boldsymbol{S}^3$ using $SU(2)$ group.

We use the anti-Hermite $SU(2)$ generator

$$[T_a, T_b] = -\epsilon_{abc} T_c. \tag{E.62}$$

Spinor and vector representation of the spin operator is given as follows.

$$T_a^{(1/2)} = \frac{i}{2}\sigma_a, \qquad \left(T_a^{(1)}\right)_{bc} = \epsilon_{abc}. \tag{E.63}$$

Rasing and lowering operators are defined by

$$T_\pm = T_1 \pm i T_2. \tag{E.64}$$

The matrix element of the general spin representation is as follows.

$$\begin{aligned} \langle m|T_3|m\rangle &= im \\ \langle m+1|T_+|m\rangle &= i\sqrt{(j-m)(j+m+1)} \\ \langle m-1|T_-|m\rangle &= i\sqrt{(j+m)(j-m+1)} \end{aligned} \tag{E.65}$$

Using the Dirac matrix generator for $SO(N)$ against spinor representation is chosen as

$$T_{MN} = \frac{1}{2}\gamma_{MN} \tag{E.66}$$

For vector representation we choose the generator so that the action to the basis $|M\rangle$ becomes

$$|1\rangle = T_{12}|2\rangle. \tag{E.67}$$

When we define the $\boldsymbol{S}^3 = SO(4)/SO(3)$ we use T_{12}, T_{23}, T_{31} for the subgroup $H = SO(3)$. These generator satisfy the same commutation relation as that of (T_1, T_2, T_3). Therefore,

$$T_{12} = T_3^L + T_3^R. \tag{E.68}$$

The remaining generators like T_{m4} is chosen so that

$$T_{34} = \frac{1}{2}\gamma^{34} = \frac{1}{2}\gamma^5\gamma^{12} = \frac{1}{2}\left(P_L\gamma^{12} - P_R\gamma^{12}\right). \tag{E.69}$$

Hence,

$$T_{34} = T_3^L - T_3^R. \tag{E.70}$$

We denote the element of $SO(4)$ by (g_l, g_r) as a pair of $SU(2)$. H is its diagonal subgroup. The general field is expressed as follows.

$$|\Psi\rangle = |g_l, g_r\rangle|i\rangle\Psi_i(g_l, g_r). \tag{E.71}$$

Spin and local rotation act on the field as

$$\text{spin}(h)|\Psi\rangle = |g_l, g_r\rangle h|i\rangle\Psi_i(g_l, g_r).\text{local}(h)|\Psi\rangle \quad = |g_l h, g_r h\rangle|i\rangle\Psi_i(g_l, g_r). \tag{E.72}$$

Therefore, following relation holds.

$$|g_l, g_r\rangle h|i\rangle\Psi_i(g_l, g_r) = |g_l h, g_r h\rangle|i\rangle\Psi_i(g_l, g_r). \tag{E.73}$$

Or, we can abbreviate the wave function and simply consider that there is following gauge symmetry:

$$|g_l, g_r\rangle h|i\rangle \sim |g_l h.g_r h\rangle|i\rangle \tag{E.74}$$

The infinitesimal transformation is expressed as

$$|g_l T_m, g_r\rangle|i\rangle + |g_l, g_r T_m\rangle|i\rangle \sim |g_l, g_r\rangle T_m|i\rangle \tag{E.75}$$

Scalar function on $\boldsymbol{S}^3$ has trivial spin basis $|0\rangle$, and this leads to

$$|g_l h, g_r h\rangle|0\rangle \sim |g_l, g_r\rangle|0\rangle \tag{E.76}$$

The general form of the scalar field satisfying the relation above is

$$|\psi\rangle = |g_l, g_r\rangle\rho^j(g_l g_r^{-1})_{AB}\Psi_{AB}, \tag{E.77}$$

where ρ^j is representation matrix of spin j, and this function transform under the isometry as (j, j)). Hence, we use the following basis instead of $|g_l, g_r\rangle$ to treat the scalar function.

$$||g_l g_r^{-1}\rangle\rangle = \{|g_l, g_r\rangle\} \tag{E.78}$$

where the RHS is equivalence class define by the (E.76). Since the scalar field does not need to introduce the frame, the equivalence class is enough.

$$|\psi\rangle = ||g\rangle\rangle\rho^j(g)_{AB}\Psi_{AB}, \tag{E.79}$$

General spin field can be define by the basis $||g\rangle\rangle$, we need to introduce the gauge fixing condition so as to determine the frame. For example, the left-invariant frame is given as follows.

$$||g\rangle\rangle_{LI} = |g, e\rangle. \tag{E.80}$$

Lie derivative and the covariant derivative generally do not respect this gauge. Hence, we need the gauge transformation regaining the gauge.

For example, using the left-invariant frame the Lie derivative and the covariant derivative is

$$\begin{aligned}
\mathcal{L}_{(T_m,0)}||g\rangle\rangle_{LI}|i\rangle &= |T_m g, e\rangle|i\rangle = ||T_m g\rangle\rangle_{LI}|i\rangle \\
\mathcal{L}_{(0,T_m)}||g\rangle\rangle_{LI}|i\rangle &= |g_l, T_m\rangle|i\rangle \sim ||gT_m\rangle\rangle_{LI}|i\rangle + ||g\rangle\rangle_{LI}T_m|i\rangle \\
rD_m||g\rangle\rangle_{LI}|i\rangle &= -|gT_m, e\rangle_{LI}|i\rangle + |g, T_m\rangle_{LI}|i\rangle \sim -2||gT_m\rangle\rangle_{LI}|i\rangle + ||g\rangle\rangle_{LI}T_m|i\rangle.
\end{aligned} \tag{E.81}$$

Generator $\mathcal{K}$ is define as

$$K_m = \frac{1}{r}\left(T^l_m - T^r_m\right). \tag{E.82}$$

$||g\rangle\rangle_{LI}$ is regarded as scalar function. Then, we regard this part as orbit part and $|i\rangle$ part as spin part, and isometry action on the $||g\rangle\rangle_{LI}$ is identified with L^l_m and L^r_m, and action on $|i\rangle$ is identified with spin operator S_m . Hence, the total orbital angular momentum and covariant derivative is given as follows.

$$\begin{aligned}
J^l_m &\overset{LI}{=} L^l_m \\
J^r_m &\overset{LI}{=} L^r_m + S_m \\
rD_m &\overset{LI}{=} 2L^r_m + S_m
\end{aligned} \tag{E.83}$$

For the right invariant frame we have

$$\begin{aligned}
J^l_m &\overset{RI}{=} L^l_m + S_m \\
J^r_m &\overset{RI}{=} L^r_m \\
rD_m &\overset{RI}{=} -2L^r_m - S_m
\end{aligned} \tag{E.84}$$

Using these we can rewrite the Laplacian:

$$\begin{aligned}
r^2 D_m D_m &= (2L^r_m + S_m)^2 \\
&= 2(L^r_m)^2 + 2(L^r_m + S_m)^2 - S^2_m
\end{aligned} \tag{E.85}$$

Furthermore, using $(L^r_m)^2 = (L^l_m)^2$ we have

$$r^2 D_m D_m = 2(J^l_m)^2 + 2(J^r_m)^2 - S^2_m. \tag{E.86}$$

Appendix F

Double sine function

F.1 Jafferis' l-function

We introduce how to derive the derivative of l-function using the zeta-function regularization. The definition of the l-function is

$$e^{l(z)} = \prod_{n=1}^{\infty} \left(\frac{n - iz}{n + iz} \right)^n \tag{F.1}$$

This is from the 1-loop determinant of the chiral multiplet. Taking log and derivative:

$$\begin{aligned} \partial_z l(iz) &= \sum_{n=1}^{\infty} n \partial_z \log \left(\frac{n+z}{n-z} \right) = \sum_{n=1}^{\infty} \left(\frac{n}{n+z} + \frac{n}{n-z} \right) \\ &= \sum_{n=1}^{\infty} \left(1 - \frac{z}{n+z} + 1 + \frac{z}{n-z} \right) = \sum_{n=1}^{\infty} \left(2 + \frac{2z^2}{n^2 - z^2} \right) \\ &= 2\zeta(0) - \sum_{n=1}^{\infty} \left(\frac{2z^2}{z^2 - n^2} \right) = -1 - \sum_{n=1}^{\infty} \left(\frac{2z^2}{z^2 - n^2} \right) \end{aligned} \tag{F.2}$$

where we used the zeta-function regularization $\zeta(0) = -1/2$ in the last equality. Then, the partial fraction expansion of the trigonometric function

$$\pi \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \tag{F.3}$$

tells us that (F.2) becomes

$$\partial_z l(iz) = -\pi z \cot \pi z. \tag{F.4}$$

Integration of this function (maybe by Mathematica) gives the explicit and regularized function;

$$l(z) = -z \log(1 - e^{2\pi i z}) + \frac{i}{2} \left(\pi z^2 + \frac{1}{\pi} (\mathrm{Li}_2(e^{2\pi i z})) \right) - \frac{i\pi}{12} \tag{F.5}$$

We show the relation between the l-function and the double sine function here.

$$s_1(x) = \prod_{p,q=0}^{\infty} \frac{p+q+1-ix}{p+q+1+ix} = \prod_{n=1}^{\infty} \left(\frac{n-ix}{n+ix}\right)^n \tag{F.6}$$

$$\therefore \qquad s_1(x) = e^{l(-ix)} \tag{F.7}$$

Actually, $l_r(z) = l(-iz)$ is more useful to be used for numerical calculation. In this definition some formulae become simpler.

$$l(-z) = -l(z) \tag{F.8}$$

$$e^{l(z-\frac{i}{2})-l(z+\frac{i}{2}))} = 2\cosh(\pi z) \tag{F.9}$$

$$e^{l(z-i)-l(z+i))} = 4\sinh^2(\pi z) \tag{F.10}$$

These formulae can be derived from the double sine function, or directly from the infinite products (F.1).

For a numerical calculation the integral form of the polylog function is useful:

$$\mathrm{Li}_{1+s}(x) = \frac{1}{\Gamma(s+1)} \int_0^{\infty} dk \frac{k^s}{e^k/z-1}. \tag{F.11}$$

F.2 Double sine function

Double sine function looks very complicated, though it is a simple building block of the squashed partition function. It has several definition. The infinite product form is directly connected to the partition function and actually it is a regularized form. However, in order to explore asymptotic form the integration form is the best, and when one calculate the squashed partition function the expression using the q-Pochhammer symbol is a nice because Mathematica knows them.

$$s_b(x) = \prod_{p,q=0}^{\infty} \frac{bp+b^{-1}q+\frac{b+b^{-1}}{2}-ix}{bp+b^{-1}q+\frac{b+b^{-1}}{2}+ix} \tag{F.12}$$

$$= \exp\left[-i\pi\left(\frac{x^2}{2}+\frac{(b^2+b^{-2})}{24}\right)+\int_{\mathbb{R}+i0}\frac{dt}{4t}\frac{e^{-2itx}}{\sinh bt \sinh b^{-1}t}\right] \tag{F.13}$$

$$= \exp\left[-i\pi\left(\frac{x^2}{2}+\frac{2\eta^2-1}{12}\right)\right]\frac{(-qe^{2\pi bx};q^2)_\infty}{(-\widetilde{q}e^{2\pi b^{-1}x};\widetilde{q}^2)_\infty} \qquad \text{for Im } b^2>0 \tag{F.14}$$

where

$$\eta = \frac{b+b^{-1}}{2} = \frac{1}{v}\,, \qquad q = e^{i\pi b^2}\,, \qquad \widetilde{q} = e^{i\pi b^{-2}}, \tag{F.15}$$

$$(x;q)_\infty = \prod_{k=9}^{\infty}(1-q^k x) \qquad \text{called q-deformed Pochhammer function.} \tag{F.16}$$

In order to show the duality explicitly following relations are important.

- Self-duality and reflection property

$$s_b(z) = s_{b^{-1}}(z) = \frac{1}{s_b(-z)}. \tag{F.17}$$

- Functional equations

$$\frac{s_b(z+\frac{ib}{2})}{s_b(z-\frac{ib}{2})} = \frac{1}{2\cosh(\pi bz)},$$
$$\frac{s_b(z+\frac{ib^{-1}}{2})}{s_b(z-\frac{ib^{-1}}{2})} = \frac{1}{2\cosh(\pi b^{-1}z)},$$
$$\frac{s_b(z+\frac{i}{v})}{s_b(z-\frac{i}{v})} = \frac{1}{[2\sinh(\pi bz)][2\sinh(\pi b^{-1}z)]}. \tag{F.18}$$

Relation to other functions like double gamma function is nicely summarized in [56].

F.3 Separation of long-range and short-range potentials

In this appendix we determine the explicit form of the long-range and the short-range potentials.

Let x and y be the real and imaginary parts of z. Namely,

$$z = x + iy. \tag{F.19}$$

In the region $|y| < 1/v$, the function $f_b(z)$ is given by[57, 58]

$$\begin{aligned} f_b(z) \equiv \log s_b(z) &= i\pi\left(\frac{z^2}{2}+\frac{b^2+b^{-2}}{24}\right) + \int_{C_-} F(z,t)dt \\ &= -i\pi\left(\frac{z^2}{2}+\frac{b^2+b^{-2}}{24}\right) + \int_{C_+} F(z,t)dt, \end{aligned} \tag{F.20}$$

where the function $F(z,t)$ is

$$\begin{aligned} F(z,t) &= \frac{e^{-2itz}}{4t\sinh bt \sinh \frac{t}{b}} \\ &= \frac{1}{4t^3} - \frac{iz}{2t^2} - \frac{1}{t}\left(\frac{z^2}{2}+\frac{b^2+b^{-2}}{24}\right) + \frac{i}{3}z^3 + \frac{iz}{12}(b^2+b^{-2}) + \mathcal{O}(t). \end{aligned} \tag{F.21}$$

The function $F(z,t)$ has poles at $t = n\pi ib$ and $t = n\pi ib^{-1}$ ($n \in \mathbf{Z}$). $C_\pm$ are the contours shown in Fig F.1. The first and the second expressions in (F.20) are useful for $x > 0$ and $x < 0$, respectively, because when $x \to +\infty$ the integral in (F.20) along C_- vanishes, and when $x \to +\infty$ the integral along C_+ vanishes. From this fact, we obtain the asymptotic form

$$f_b^{\text{asym}}(z) = i\pi\left(\frac{z^2}{2}+\frac{b^2+b^{-2}}{24}\right)\text{sign}(x). \tag{F.22}$$

The difference of $f_b(z)$ from the asymptotic form is

$$f_b(z) - f_b^{\text{asym}}(z) = \int_C F(z,t)dt = \int_C F(iy,t)e^{-2itx}dt, \tag{F.23}$$

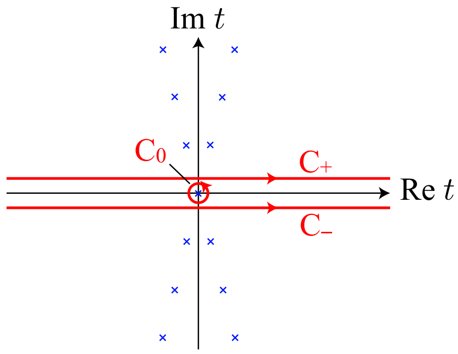

Figure F.1: Integration contours $C_\pm$ and C_0 on the t-plane are shown. The crosses are poles of function $F(z,t)$.

where $C = C_-$ for $x > 0$ and $C = C_+$ for $x < 0$. Because this almost vanishes when $|x|$ is large, we can approximately express this difference by using $\delta(x)$ as

$$f_b(z) - f_b^{\text{asym}}(z) \sim \delta(x) g_b(y) \tag{F.24}$$

We can determine the function $g_b(y)$ by integrating the right hand side over x.

$$\begin{aligned} g_b(y) &= \int_{-\infty}^{\infty} (f_b(z) - f_b^{\text{asym}}(z)) dx \\ &= \int_0^{\infty} \left(\int_{C_-} F(iy,t) e^{-2itx} dt \right) dx + \int_{-\infty}^{0} \left(\int_{C_+} F(iy,t) e^{-2itx} dt \right) dx. \end{aligned} \tag{F.25}$$

Thanks to small imaginary part of t along the contours $C_\pm$, these x integrals converge, and we obtain

$$\begin{aligned} g_b(y) &= \frac{1}{2i} \int_{C_-} \frac{F(iy,t)}{t} dt - \frac{1}{2i} \int_{C_+} \frac{F(iy,t)}{t} dt \\ &= \frac{1}{2i} \oint_{C_0} \frac{F(iy,t)}{t} dt \\ &= \frac{\pi}{3} y^3 - \frac{\pi}{12} (b^2 + b^{-2}) y. \end{aligned} \tag{F.26}$$

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