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著者(和文)	西場正浩
Author(English)	Masahiro Nishiba
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Pricing exotic options and American options : an asymptotic expansion approach and symbolic computing

Masahiro Nishiba
Tokyo Institute of Technology
2-12-1 Ookayama Meguro-ku
Tokyo 152-8552 Japan
E-mail: nishiba.m@gmail.com

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Abstract

This thesis introduces approximation formulas and a method of calculation for the prices of financial derivatives. To approximate these values, the author applies an asymptotic expansion method to multidimensional diffusion processes and develops a package for a symbolic computing software. With the application of the asymptotic expansion method, the prices of some financial derivatives can be approximated explicitly. The package automatically derives the approximation of the law of multidimensional diffusion processes.

The author applies the asymptotic expansion method to the problems of pricing European swaptions in the SABR/LIBOR market model, credit valuation adjustments of interest rate swaps in the SABR swap rate model, call options on the maximum of two assets in the constant elasticity of variance (CEV) model, average strike options in the Black-Scholes model and American options in the Heston model. The asymptotic expansion method is versatile and can be applied to other models.

To calculate the prices of American options, the author introduces a method that is based on the combination of the asymptotic expansion method and backward induction.

Numerical examples show practical effectiveness of the proposed method.

Keywords American option; Asymptotic expansion; Average strike option; Black-Scholes model; CEV model; Call option on the maximum of two assets; CVA; Heston model; Monte Carlo; quasi-Monte Carlo; SABR model; simulation;

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Chapter 1

Introduction

The price of a financial derivative is equal to the expectation of discounted value of its payoff under the risk neutral measure. This means that the price is given by the calculation of $\mathbb{E}\left[F\left(X_T\right)\right]$. Here X_T is the value at time T of the D-dimensional diffusion process X_t that describes its underlying asset prices, and F is the payoff function which determines the financial derivative. Furthermore, in the case that the payoff depends not only on the asset prices at time T but also on the asset prices at times T_1,..., T_{N-1} and T_N, the calculation of the price that is given by $\mathbb{E}\left[F\left(X_{T_1},\ldots,X_{T_N}\right)\right]$ becomes more difficult. Therefore, it is highly important to find fast and reliable methods for the numerical evaluation of $\mathbb{E}\left[F\left(X_T\right)\right]$ and $\mathbb{E}\left[F\left(X_{T_1},\ldots,X_{T_N}\right)\right]$. A number of studies on this subject has been conducted (see [4]).

In the case that the law of X_T is known explicitly, $\mathbb{E}\left[F\left(X_T\right)\right]$ can be calculated, e.g., by using Gaussian quadrature [17]. This condition is not necessarily satisfied in many practical problems. There are three approaches to these problems: the partial differential equation approach, simulation, and the asymptotic expansion method. The partial differential equation approach works only when the dimension is relatively small. One of the major methods of simulation is Monte Carlo simulation. A distinct advantage of Monte Carlo simulation is that its convergence rate does not depend on the number of state variables. The last approach is the asymptotic expansion method. This method is based on Watanabe theory [29] and has been applied to problems in mathematical finance [7, 8, 11, 12, 21, 22, 23, 26, 27]. An advantage of this method is that it is possible to obtain the analytic approximation formulas for the prices of financial derivatives. This advantage enables rapid computation of the prices. In this research, we focus on this method.

We derive higher order approximations and the approximations of the laws of complicated diffusion models by using the asymptotic expansion method. However these calculations can be difficult, because the asymptotic expansion method requires a huge amount of symbolic manipulation. To conquer this difficulty, we develop a package for a symbolic computing software, Maxima [20]. This package enables us to automatically derive the approximations of the laws of diffusion processes by the asymptotic expansion method.

To show the usefulness of the asymptotic expansion method, we consider three practical problems in finance. The first problem is one to which we can naively apply the asymptotic expansion method. That means that the calculation of $\mathbb{E}\left[F_1\left(F_2\left(X_T\right)\right)\right]$ where X_T is D-dimensional diffusion process, $F_1 : \mathbb{R} \to \mathbb{R}$ has polynomial growth, and $F_2 : \mathbb{R}^D \to \mathbb{R}$ is a smooth function whose all derivatives have polynomial growth. As numerical examples, we consider interest rate swaptions in the SABR/LIBOR market model and the credit valuation adjustment of interest rate swaps in the SABR swap rate model.

The second one is the calculation of $\mathbb{E}\left[F\left(X_T\right)\right]$, where F has polynomial growth. We extend the asymptotic expansion method [24] and call this extended method the multidimensional asymptotic expansion method (MAE). As numerical examples, we consider "the call option on the maximum of two assets" [5] in the constant elasticity of variance (CEV) model and the average strike option in the Black-Scholes model. These numerical results show that MAE is practically accurate enough. The computation time is less than 10^{-1} seconds.

The third problem is the calculation of $\mathbb{E}\left[F\left(X_{T_1}, \dots, X_{T_N}\right)\right]$. In this research, we focus on the calculation of the prices of the American options. There are many methods for the calculation of the prices of the American options, for example, the lattice method, the partial differential equation approach and simulation(see [3, 4]). We consider the calculation of the prices of the American options with simulation. Rogers [19] introduces the dual representation approach and calculates the upper bounds on several American options by using Monte Carlo simulation. Longstaff and Schwartz [10] introduce the least squares Monte Carlo simulation to evaluate the prices of American options. They approximate the conditional continuation values of the option by using the least squares method. Broadie and Glasserman [1] introduce the stochastic mesh method. They assume that the law of X is known explicitly, which is not necessarily satisfied in many practical problems.

In this research, we introduce a new method to calculate the prices of the American options by using the combination of MAE and backwards induction. We call this new method the multidimensional asymptotic expansion mesh method (MAEM). We also introduce MAEM with simulation (MAEMS). We apply MAEM and MAEMS to the American put options in the Heston model. This numerical experiment shows that the method is relatively accurate compared with least squares Monte Carlo simulations and other methods [2, 6, 16, 28, 30].

The chapter 2 describes the asymptotic expansion method and introduces MAE. The chapter 3 describes the package for a symbolic computing software. The chapter 4 discusses the applications of the asymptotic expansion method. The chapter 5 discusses the results of this research and concludes with issues in the future.

Chapter 2

The asymptotic expansion method

In this chapter, we briefly describe the asymptotic expansion method proposed in [24]. We then extend the asymptotic expansion method and introduce the multidimensional asymptotic expansion method.

We consider the approximation of the value of $\mathbb{E}\left[F\left(X\left(T\right)\right)\right]$ where $F : \mathbb{R}^D \to \mathbb{R}$ and D is an integer greater than one. X is a D-dimensional diffusion process that satisfies some conditions we describe in the section 2.1.

In [24] it is supposed that there exist a function $F_1 : \mathbb{R} \to \mathbb{R}$ that has polynomial growth and a smooth function $F_2 : \mathbb{R}^D \to \mathbb{R}$ whose all derivatives have polynomial growth such that $\mathbb{E}\left[F\left(X\left(T\right)\right)\right] = \mathbb{E}\left[F_1\left(F_2\left(X\left(T\right)\right)\right)\right]$.

In this thesis, we suppose that $F : \mathbb{R}^D \to \mathbb{R}$ has polynomial growth. Our setting includes the setting in [24].

There are following five steps to approximate $\mathbb{E}\left[F\left(X\left(T\right)\right)\right]$ by using the method proposed in [24].

1. We calculate the asymptotic expansion of X^ϵ that is the perturbed process of X by $\epsilon \in (0, 1]$.

2. We calculate the characteristic function of the normalized process of X^ϵ by using the Taylor expansion around $\epsilon = 0$.

3. We calculate the conditional expectations in the Taylor expansion of the characteristic function.

4. We calculate the law of X^ϵ.

5. We approximate $\mathbb{E}\left[F\left(X\left(T\right)\right)\right]$ by using the law of $X\left(T\right)$.

In the third step, we need to calculate the conditional expectations given the values of Ito integrals. To calculate these conditional expectations we need to generalize Proposition 1 in [24] to higher dimensions. The generalized statement, Theorem 2.1, is given in the section 2.3.

A scheme to calculate the conditional expectations is proposed in [25]. The difference between Theorem 2.1 and the scheme in [25] is discussed in the section 2.3.

2.1 The asymptotic expansion of X^ϵ

First, we calculate the asymptotic expansion of a perturbed process X^ϵ following the method proposed in [24]. We begin with an introduction of basic notions. Let $\left(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0 \leq t \leq T}\right)$ be a complete probability space that satisfies the usual hypotheses, that is to say, $\mathcal{F}$ is P-complete, $\mathcal{F}_0$ contains all P-null sets, $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is right-continuous where $T \in (0, \infty)$ denotes some fixed horizon of economy. Moreover, let $W(t) = (W_1(t), \ldots, W_d(t))$ be a d-dimensional standard Brownian motion. We consider a D-dimensional diffusion process $X(t) = (X_1(t), \ldots, X_D(t))$ that is the solution to the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}X(t) &= \alpha(X(t))\, \mathrm{d}t + \sigma(X(t))\, \mathrm{d}W(t), \\ X(0) &= x_0 \end{aligned} \tag{2.1}$$

where $x_0 \in \mathbb{R}^D$, $\alpha : \mathbb{R}^D \to \mathbb{R}^D$ and $\sigma : \mathbb{R}^D \to \mathbb{R}^D \times \mathbb{R}^d$. We suppose that

$$\begin{aligned} &\alpha \in C_b^\infty\left(\mathbb{R}^D; \mathbb{R}^D\right), \\ &\sigma \in C_b^\infty\left(\mathbb{R}^D; \mathbb{R}^D \times \mathbb{R}^d\right) \end{aligned} \tag{2.2}$$

where for $Y \in \left\{\mathbb{R}^D, \mathbb{R}^D \times \mathbb{R}^d\right\}$, $C_b^\infty\left(\mathbb{R}^D; Y\right)$ denotes the set of Y-valued infinitely differentiable functions defined on $\mathbb{R}^D$ whose derivatives are all bounded.

We consider the perturbed process X^ϵ of the process X by putting a small number $\epsilon \in (0, 1]$ following [24], that is to say, we define X^ϵ by the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}X^\epsilon(t) &= \alpha(X^\epsilon(t))\, \mathrm{d}t + \epsilon\sigma(X^\epsilon(t))\, \mathrm{d}W(t), \\ X^\epsilon(0) &= x_0 \end{aligned}$$

Then, when $\epsilon \searrow 0$, $X^\epsilon(T)$ has its asymptotic expansion as follows:

$$X^\epsilon(T) = \sum_{i=0}^{\infty} \epsilon^i X_{(i)}(T) / i! \tag{2.3}$$

in $L^p(\Omega)$ for any $p \geq 1$ where $X_{(i)}(T) = \left(X_{(i),1}(T), \ldots, X_{(i),D}(T)\right)$ is a D-dimensional diffusion process that is given by

$$X_{(i)}(T) = \left.\frac{\mathrm{d}^i X^\epsilon(T)}{\mathrm{d}\epsilon^i}\right|_{\epsilon=0} \tag{2.4}$$

for $i \in \mathbb{N}$. Here $i!$ means the factorial of i for $i \in \mathbb{N}$. Coefficients $X_{(i)}(T)$ are obtained by the Taylor expansion and represented based on multiple Wiener-Ito integrals. See [24] for the details of this expansion method.

2.2 Calculation of the characteristic function of the normalized process of X^ϵ

We calculate the characteristic function of $X^\epsilon(T)$ in the equation (2.3). We denote by G^ϵ the normalized process of $X^\epsilon(T)$ that is given by

$$G^\epsilon = \left(X^\epsilon(T) - X_{(0)}(T)\right) / \epsilon.$$

The following is a useful fact that can be easily shown. $X_{(0)}$ is a deterministic function that is defined by the following ordinary differential equation:

$$\begin{aligned} &X_{(0)}(t) = \alpha\left(X_{(0)}(t)\right)\,\mathrm{d}t, \\ &X_{(0)}(0) = x_0. \end{aligned}$$

Next, $X_{(1)}$ is defined by the following stochastic differential equation:

$$\begin{aligned} &X_{(1)}(t) = \alpha_x\left(X_{(0)}(t)\right) X_{(1)}(t)\,\mathrm{d}t + \sigma\left(X_{(0)}(t)\right)\,\mathrm{d}W(t), \\ &X_{(1)}(0) = 0, \end{aligned}$$

where $\alpha_x = \left(\frac{\partial \alpha}{\partial x_1}, \ldots, \frac{\partial \alpha}{\partial x_D}\right)$. Let Σ be a matrix with size $D \times D$. For i, $j \in \{1, \ldots, D\}$, the element at ith row and jth column is defined by

$$\Sigma_{i,j} = \mathbb{E}\left[X_{(1),i}(T)\, X_{(1),j}(T)\right].$$

We assume that $\det(\Sigma) > 0$. Here $\det(\Sigma)$ means the determinant of Σ. This assumption means that the distribution of $X_{(1)}(T)$ does not degenerate and $X_{(1)}(T)$ is a normal random variable with the D-dimensional mean vector 0 and the covariance matrix Σ.

Let φ_{G^ϵ} be the characteristic function of G^ϵ. For $\xi \in \mathbb{R}^D$, $\varphi_{G^\epsilon} : \mathbb{R}^D \to \mathbb{R}$ is defined by

$$\varphi_{G^\epsilon}(\xi) = \mathbb{E}\left[\exp\left(\sqrt{-1}\,\langle \xi, G^\epsilon \rangle\right)\right]$$

where $\langle \xi, G^{\epsilon} \rangle$ means the inner product of ξ and G^{ϵ}. Applying the Taylor expansion around $\epsilon = 0$ to $\varphi_{G^{\epsilon}}(\xi)$, we have

$$\begin{aligned}
\varphi_{G^{\epsilon}}(\xi) &= \sum_{i=0}^{\infty} \left. \frac{\mathrm{d}^i \varphi_{G^{\epsilon}}(\xi)}{\mathrm{d}\epsilon^i} \right|_{\epsilon=0} \frac{\epsilon^i}{i!} \\
&= \mathbb{E}\left[\exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \frac{\epsilon}{2}\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(2)} \right\rangle \exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \frac{\epsilon^2}{8}\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(2)} \right\rangle^2 \exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \frac{\epsilon^2}{6}\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(3)} \right\rangle \exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \cdots \\
&= \mathbb{E}\left[\exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \frac{\epsilon}{2}\mathbb{E}\left[\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(2)} \right\rangle \middle| X_{(1)}\right] \exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \frac{\epsilon^2}{8}\mathbb{E}\left[\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(2)} \right\rangle^2 \middle| X_{(1)}\right] \exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \frac{\epsilon^2}{6}\mathbb{E}\left[\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(3)} \right\rangle \middle| X_{(1)}\right] \exp\left(\sqrt{-1}\left\langle \xi, X_{(1)} \right\rangle\right)\right] \\
&+ \cdots .
\end{aligned} \tag{2.5}$$

Here, $X_{(i)}(T)$ is abbreviated as $X_{(i)}$ for $i = 1, 2, 3$. To obtain the Taylor expansion of $\varphi_{G^{\epsilon}}(\xi)$ explicitly, we need to calculate conditional expectations, e.g., $\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(2)} \right\rangle \middle| X_{(1)}\right]$, $\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(2)} \right\rangle \middle| X_{(1)}\right]$, $\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(2)} \right\rangle^2 \middle| X_{(1)}\right]$ and $\mathbb{E}\left[\left\langle \sqrt{-1}\,\xi, X_{(3)} \right\rangle \middle| X_{(1)}\right]$ in the equation (2.5). Theorem 2.1 in the next section is useful for the calculations of these conditional expectations.

2.3 Calculation of conditional expectations

We calculate the conditional expectations in the equation (2.5). The following theorem, which plays an essential role in the calculations, enables us to obtain the law of $X(T)$ in multidimensional cases. Because we can explicitly obtain the Taylor expansion of $\varphi_{G^{\epsilon}}(\xi)$, we can calculate the law of X^{ϵ} by using Lemma 2.1 in the section 2.4.

Theorem 2.1. *Let* $\mathbf{T} = [0, T]$, $f \in L^2(\mathbf{T}^n)$ *for* $n \geq 1$, $q_j \in L(\mathbf{T})$ *for* $1 \leq j \leq m$ *and* $(\mathbf{t})$ *denote* $(t_1, t_2, \ldots, t_n)$. *Suppose that* $\{W_i\}_{i=1,\ldots,n}$ *is an*

n-dimensional Brownian motion and $\{Z_i\}_{i=1,\ldots,m}$ is an m-dimensional Brownian motion with $\mathrm{d}\left\langle W_i, W_j\right\rangle = \rho^W_{i,j}\,\mathrm{d}t$, $\mathrm{d}\left\langle Z_i, Z_j\right\rangle = \rho^Z_{i,j}\,\mathrm{d}t$ and $\mathrm{d}\left\langle W_i, Z_j\right\rangle = \rho_{i,j}\,\mathrm{d}t$. Define a matrix Σ_{22} with size $m\times m$ as follows:

$$\Sigma_{22} = \left\{\int_0^T q_i\left(t\right)\, q_j\left(t\right)\, \mathrm{d}t\right\}_{i,j=1,\ldots,m}$$

and suppose that $\det\left(\Sigma_{22}\right) > 0$. Then the following equality:

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f\left(\mathbf{t}\right)\,\mathrm{d}W_n\left(t_n\right)\cdots\,\mathrm{d}W_2\left(t_2\right)\,\mathrm{d}W_1\left(t_1\right)\Bigg| \\
&\qquad\left(\int_0^T q_1\left(t\right)\,\mathrm{d}Z_1\left(t\right),\ldots\int_0^T q_m\left(t\right)\,\mathrm{d}Z_m\left(t\right)\right) = \left(x_1,\ldots,x_m\right)\Bigg] \\
&= \int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f\left(\mathbf{t}\right) H_n\left(\mathbf{t}\right)\,\mathrm{d}t_n\cdots\,\mathrm{d}t_2\,\mathrm{d}t_1
\end{aligned}$$

holds, where

$$\begin{aligned}
\Sigma_{12}\left(\mathbf{t}\right) &= \left\{\rho_{i,j}\, q_j\left(t_i\right)\right\}_{\substack{i=1,\ldots,n\\ j=1,\ldots,m}}, \\
\Sigma_{21}\left(\mathbf{t}\right) &= {}^t\Sigma_{12}\left(\mathbf{t}\right), \\
\xi &= \left(\xi_1,\ldots,\xi_n\right), \\
H_n\left(\mathbf{t}\right) &= \left.\frac{\mathrm{d}^n\,\exp\left(x\,\Sigma_{22}^{-1}\,\Sigma_{21}\left(\mathbf{t}\right)\ {}^t\xi - \frac{1}{2}\,\xi\,\Sigma_{12}\left(\mathbf{t}\right)\,\Sigma_{22}^{-1}\,\Sigma_{21}\left(\mathbf{t}\right)\ {}^t\xi\right)}{\mathrm{d}\,\xi_1\cdots\,\mathrm{d}\,\xi_n}\right|_{\xi=0}.
\end{aligned}$$

Here ${}^tA$ means the transpose of matrix A.

Proof. Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$ defined by

$$\mathcal{G} = \left\{\left(\int_0^T q_1\left(t\right)\,\mathrm{d}Z_1\left(t\right),\ldots,\int_0^T q_m\left(t\right)\,\mathrm{d}Z_m\left(t\right)\right) = \left(x_1,\ldots,x_m\right)\right\}.$$

Here $\mathcal{G}$ depends on $\left(x_1,\ldots,x_m\right)$.

For $\mathbf{t}\in\mathbf{T}^n$, we define an $\mathbb{R}$-valued function f_n by

$$f_n\left(\mathbf{t}\right) = \mathbf{1}_{\{t_n<\cdots<t_1\}}\left(\mathbf{t}\right) f\left(\mathbf{t}\right).$$

By the definition of f_n, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\int_0^T\cdots\int_0^T f_n\left(\mathbf{t}\right)\,\mathrm{d}W_n\left(t_n\right)\cdots\,\mathrm{d}W_2\left(t_1\right)\,\mathrm{d}W_1\left(t_1\right)\middle|\mathcal{G}\right] \\
&= \mathbb{E}\left[\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f\left(\mathbf{t}\right)\,\mathrm{d}W_n\left(t_n\right)\cdots\,\mathrm{d}W_2\left(t_1\right)\,\mathrm{d}W_1\left(t_1\right)\middle|\mathcal{G}\right].
\end{aligned} \tag{2.6}$$

As $f_n \in L^2(\mathbf{T}^n)$, there exists a sequence $\left\{f_n^{(l)}\right\}_{l\in\mathbb{N}}$ such that $f_n^{(l)} \nearrow f_n$ and $f_n^{(l)}$ is represented in the following form:

$$f_n^{(l)} = \sum_{I_1,\ldots,I_n=1}^{k} c_{I_1\cdots I_n} \mathbf{1}_{A_{I_1}\times\cdots\times A_{I_n}}(\mathbf{t})$$

where $k \in \mathbb{N}$, $A_1, \ldots, A_k \subset \mathbf{T}$ are pairwise-disjoint sets and the coefficients $c_{I_1\cdots I_n} \in \mathbb{R}$ are zero if any two of the indices $I_1, \ldots, I_n$ are equal.

By the monotone convergence theorem, we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\, \mathrm{d}W_n(t_n) \cdots \mathrm{d}W_1(t_1) \middle| \mathcal{G}\right] \\ &\to \mathbb{E}\left[\int_0^T \cdots \int_0^T f_n(\mathbf{t})\, \mathrm{d}W_n(t_n) \cdots \mathrm{d}W_1(t_1) \middle| \mathcal{G}\right] \quad \text{(a.s.)}. \end{aligned}$$

We also have

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\, \mathrm{d}W_n(t_n) \cdots \mathrm{d}W_1(t_1) \middle| \mathcal{G}\right] \\ &= \sum_{I_1,\ldots,I_n=1}^{k} c_{I_1\cdots I_n} \mathbb{E}\left[\int_0^T \mathbf{1}_{A_{I_n}}(t)\, \mathrm{d}W_n(t) \cdots \int_0^T \mathbf{1}_{A_{I_1}}(t)\, \mathrm{d}W_1(t) \middle| \mathcal{G}\right] \qquad (2.7) \\ &= \sum_{I_1,\ldots,I_n=1}^{k} c_{I_1\cdots I_n} \widetilde{H}_n \end{aligned}$$

where

$$\widetilde{H}_n = \left. \frac{\mathrm{d}^n \exp\left(x\, \Sigma_{22}^{-1}\, {}^t\widetilde{\Sigma}_{12}\, {}^t\xi - \frac{1}{2}\, \xi\, \widetilde{\Sigma}_{12}\, \Sigma_{22}^{-1}\, {}^t\widetilde{\Sigma}_{12}\, {}^t\xi\right)}{\mathrm{d}\,\xi_1 \cdots \mathrm{d}\,\xi_n} \right|_{\xi=0}$$

and

$$\widetilde{\Sigma}_{12} = \left\{\rho_{i,j} \int_0^T \mathbf{1}_{A_{I_i}}(t)\, q_j(t)\, \mathrm{d}t\right\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}.$$

The second equality in the equation (2.7) is ensured by Corollary A.1 in Appendix. Since we can rewrite $\widetilde{H}_n$ as follows:

$$\widetilde{H}_n = \int_0^T \cdots \int_0^T \mathbf{1}_{A_{I_1}\times\cdots\times A_{I_n}}(\mathbf{t})\, H_n(\mathbf{t})\, \mathrm{d}t_n \cdots \mathrm{d}t_1,$$

we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\,\mathrm{d}W_n(t_n)\cdots \mathrm{d}W_1(t_1)\,\middle|\,\mathcal{G}\right] \\
&= \int_0^T \cdots \int_0^T \sum_{I_1,\ldots,I_n=1}^{k} c_{I_1\cdots I_n}\mathbf{1}_{A_{I_1}\times\cdots\times A_{I_n}}(\mathbf{t})\,H_n(\mathbf{t})\,\mathrm{d}t_n\cdots \mathrm{d}t_1 \qquad (2.8) \\
&= \int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\,H_n(\mathbf{t})\,\mathrm{d}t_n\cdots \mathrm{d}t_1.
\end{aligned}$$

By the equations (2.6), (2.7) and (2.8), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(\mathbf{t})\,\mathrm{d}W_n(t_n)\cdots \mathrm{d}W_2(t_2)\,\mathrm{d}W_1(t_1)\,\middle|\,\mathcal{G}\right] \\
&= \lim_{l\to\infty}\int_0^T \cdots \int_0^T f_n^{(l)}(\mathbf{t})\,H_n(\mathbf{t})\,\mathrm{d}t_n\cdots \mathrm{d}t_1 \\
&= \int_0^T \cdots \int_0^T f_n(\mathbf{t})\,H_n(\mathbf{t})\,\mathrm{d}t_n\cdots \mathrm{d}t_1 \\
&= \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(\mathbf{t})\,H_n(\mathbf{t})\,\mathrm{d}t_n\cdots \mathrm{d}t_2\,\mathrm{d}t_1.
\end{aligned}$$

Thus the statement is proved. □

As mentioned in the beginning of this chapter, a scheme to calculate the conditional expectations given the value of a random variable with Gaussian distribution is proposed in [25]. In the scheme, the conditional expectations are represented by the expressions that include not conditional expectations but unconditional expectations. These unconditional expectations are calculated by solving some ordinary differential equations. We think that the scheme is more complex than Theorem 2.1, and that Theorem 2.1 is easy to implement.

2.4 Derivation of the law of X^ϵ

We calculate the law of X^ϵ that we obtain in the previous section. We denote by f_{G^ϵ} the probability density function of G^ϵ. By a property of the characteristic function, f_{G^ϵ} is given by

$$f_{G^\epsilon}(g) = \left(\frac{1}{2\pi}\right)^D \int_{\mathbb{R}^D} \exp\left(\sqrt{-1}\,\langle \xi, g\rangle\right)\varphi_{G^\epsilon}(\xi)\,\mathrm{d}\xi$$

for $g \in \mathbb{R}^D$. By using Theorem 2.1 in the previous section, we calculate the conditional expectations in the equation (2.5) in the section 2.2. Those conditional expectations are represented as polynomial functions of $x = (x_1, \ldots, x_D)$ and $\xi = (\xi_1, \ldots, \xi_D)$. Thus we can obtain the closed form of φ_{G^ϵ}. Now we can calculate f_{G^ϵ} explicitly by using the following lemma.

Lemma 2.1. *Let a D-dimensional random variable $X = (X_1, \ldots, X_D)$ have normal density function $f(x; 0, \Sigma)$ with a mean vector 0 and a covariance matrix Σ. Let h and z be polynomial functions from $\mathbb{R}^D$ to $\mathbb{R}$. Then*

$$\begin{aligned}\left(\frac{1}{2\,\pi}\right)^D \int_{\mathbb{R}^D} z\left(-\sqrt{-1}\,\xi\right)\,\exp\left(-\sqrt{-1}\,\langle \xi, x\rangle\right)\,\mathbb{E}\left[h(X)\exp\left(\sqrt{-1}\,\langle \xi, X\rangle\right)\right]\,\mathrm{d}\xi \\ = z\left(\frac{\partial}{\partial x}\right)\,\left(h(x)\,f(x;0,\Sigma)\right)\end{aligned} \tag{2.9}$$

holds.

Proof. By the property of the characteristic function,

$$\left(\frac{1}{2\pi}\right)^D \int_{\mathbb{R}^D} e^{\left(-\sqrt{-1}\langle\xi,x\rangle\right)}\mathbb{E}\left[e^{\left(\sqrt{-1}\langle\xi,X\rangle\right)}\right]\,\mathrm{d}\xi = f(x;0,\Sigma)$$

holds. By integration by parts, we have

$$\begin{aligned}&\left(\frac{1}{2\pi}\right)^D \int_{\mathbb{R}^D} e^{\left(-\sqrt{-1}\langle\xi,x\rangle\right)}\mathbb{E}\left[X_i e^{\left(\sqrt{-1}\langle\xi,X\rangle\right)}\right]\,\mathrm{d}\xi \\ =&-\sqrt{-1}\left(\frac{1}{2\pi}\right)^D \int_{\mathbb{R}^{D-1}} \left[e^{\left(-\sqrt{-1}\langle\xi,x\rangle\right)}\mathbb{E}\left[e^{\left(\sqrt{-1}\langle\xi,X\rangle\right)}\right]\right]_{\xi_i=-\infty}^{\xi_i=\infty} \prod_{j=i,i\neq j}^{D} \mathrm{d}\xi_j \\ &+\left(\frac{1}{2\pi}\right)^D \int_{\mathbb{R}^D} x_i e^{\left(-\sqrt{-1}\langle\xi,x\rangle\right)}\mathbb{E}\left[e^{\left(\sqrt{-1}\langle\xi,X\rangle\right)}\right]\,\mathrm{d}\xi \\ =&x_i f(x;0,\Sigma)\end{aligned}$$

for $i \in \{1, \ldots, D\}$. Repeating this manipulation, we have

$$\left(\frac{1}{2\pi}\right)^D \int_{\mathbb{R}^D} e^{\left(-\sqrt{-1}\langle\xi,x\rangle\right)}\mathbb{E}\left[h(X)\,e^{\left(\sqrt{-1}\langle\xi,X\rangle\right)}\right]\,\mathrm{d}\xi = h(x)\,f(x;0,\Sigma)\,.$$

Differentiating this equation, we obtain the equation (2.9). □

2.5 Approximation of $\mathbb{E}\left[F\left(X\left(T\right)\right)\right]$

Finally, we approximate $\mathbb{E}\left[F\left(X\left(T\right)\right)\right]$ by using an approximation of the law of $X\left(T\right)$. By the definitions of X^{ϵ} and X, $X^{\epsilon} = X$ almost surely when $\epsilon = 1$. So we denote the nth order approximation of X, $\varphi_{G^{\epsilon}}$ and $f_{G^{\epsilon}}$ by $X^{(n)}$, $\varphi_{G^{\epsilon}}^{(n)}$ and $f_{G^{\epsilon}}^{(n)}$, respectively. For $n \in \mathbb{N}$ these are defined by

$$X^{(n)}\left(T\right) = \sum_{i=0}^{n} \frac{1}{i!} X_{(i)}\left(T\right),$$

$$\varphi_{G^{\epsilon}}^{(n)}\left(\xi\right) = \sum_{i=0}^{n} \frac{1}{i!} \left.\frac{\mathrm{d}^{i}\varphi_{G^{\epsilon}}\left(\xi\right)}{\mathrm{d}\epsilon^{i}}\right|_{\epsilon=0}$$

for $\xi \in \mathbb{R}^{D}$ and

$$f_{G^{\epsilon}}^{(n)}\left(g\right) = \left(\frac{1}{2\pi}\right)^{D} \int_{\mathbb{R}^{D}} \exp\left(\sqrt{-1}\left\langle \xi, g\right\rangle\right) \varphi_{G^{\epsilon}}^{(n)}\left(\xi\right) \mathrm{d}\xi \tag{2.10}$$

for $g \in \mathbb{R}^{D}$. By the equations (2.5), (2.9) and the fact that $X_{(1)}$ is a normal random variable, the right-hand side of the equation (2.10) is integrable. So $f_{G^{\epsilon}}^{(n)}$ can be defined by the equation (2.10).

Now the nth order approximation of the law of $X\left(T\right)$ is given by

$$f^{(n)}\left(x\right) = f_{G^{\epsilon}}^{(n)}\left(x - X_{(0)}\left(T\right)\right)$$

for $x \in \mathbb{R}^{D}$. The nth order approximation of $\mathbb{E}\left[F\left(X\left(T\right)\right)\right]$ is given by

$$V^{(n)} = \int_{\mathbb{R}^{D}} F\left(x\right) f^{(n)}\left(x\right) \mathrm{d}x.$$

Here we suppose that F has polynomial growth.

As described at the beginning of this chapter, we have extended the asymptotic expansion method. Now we can approximate the value of $\mathbb{E}\left[F\left(X\left(T\right)\right)\right]$. We call this method the multidimensional asymptotic expansion method.

2.6 Practical remark

We suppose that σ and α satisfy the condition (2.2). This condition is not necessarily satisfied in many models in finance. But when we apply this method to models that do not satisfy this condition, we still can obtain good approximations of the laws of them. Theoretical support of these results is an issue in the future.

Chapter 3

The asymptotic expansion method with Maxima

3.1 Introduction

The difficulty of asymptotic expansion method

As mentioned in the chapter 2, the asymptotic expansion method requires a huge amount of symbolic manipulation that includes the application of Ito's lemma, Fubini's theorem, and Theorem 2.1. As Theorem 2.1 shows, we have to calculate multiple integrals to obtain the approximations explicitly. So to apply the method to complicated models and to derive higher order approximations are difficult in practice without symbolic computing. Moreover, to use the approximations derived with the method in practice, we have to implement the approximations. The approximations usually have long expressions, which causes difficulty to implement them.

A Maxima package

To conquer the difficulties in the application of the asymptotic expansion method and in the implementation of the long expressions, we develop a package written by LISP programming language for a symbolic computing software, Maxima [20]. We call this package SymAE. SymAE enables us to automatically calculate the approximation of the law of multidimensional diffusion processes by using the asymptotic expansion method. That means we can apply the asymptotic expansion method to complicated models and derive higher order approximation. SymAE also enables us to automatically generate C++ code of the long expressions of the approximations. So the input of SymAE is the definition of diffusion processes, and its output is C++ source files.

To obtain more accurate approximations, we have to apply the asymptotic expansion method to diffusion models with different ways. Sometimes we obtain more accurate approximations by numeraire change or by applying the method to $C\left(X_T\right)$ instead of X_T where $C : \mathbb{R}^D \to \mathbb{R}^D$ is an injective function. However, we do not know what function C or what numeraire are better before we actually derive the approximation. That is why we have to apply the asymptotic expansion method to the models several times to obtain good approximations. However, SymAE enables us to automattically calculate the approximation by the asymptotic expansion method and make it easy to find a good function C and numeraire.

The package does not require the knowledge of LISP for users, and its interface is Maxima. The grammar of Maxima is easy to understand. This package requires only ten and several lines to apply the asymptotic expansion method and generate C++ file.

3.2 The functional design of SymAE

We describe the key features of SymAE. The functions in SymAE fall into three groups A, B, and C. First, group A consists of functions that are related to Ito calculus, for example, Ito's formula and Fubini's theorem. Second, group B consists of functions that are related to conditional expectations. Finally, group C consists of functions that are related to the asymptotic expansion method. These three groups are not independent. Group B depends on group A. Group C depends on group A and B.

Group A: functions related to Ito calculus

Originally, Maxima does not support the Ito calculus. SymAE provides Ito calculus, e.g., Ito's integral, Ito formula and Fubini's theorem. To provide Ito's formula by SymAE, we need a function to find products of Ito's integrals in expressions.

SymAE also provides Fubini's theorem. More precisely, the package provides the exchange of Ito integral and Riemann-Stieltjes integral. For this function, the package needs a function that finds multiple integrals that consist of Ito integral and Riemann-Stieltjes integral.

SymAE has a function that solves stochastic differential equations. In many cases the stochastic differential equations that occur when using the asymptotic expansion method can be solved. So, the package supports some formulas for solving stochastic differential equations.

Group B: functions related to conditional expectations

SymAE provides a function that calculate some conditional expectations by using Theorem 2.1. For this function, the package needs a function that calculate function $H_n(\mathbf{t})$ in the equation 2.1 and the functions generated in Group A.

Group C: functions related to the asymptotic expansion method

For the asymptotic expansion method, SymAE provides a function that derives the ith term in the equation (2.3) and a function that derives the ith term in the equation (2.5). In these functions, the functions generated in Group A and B are required.

3.3 The implementation of functions

In this section, we describe the implementation of the main functions in SymAE.

3.3.1 Group A

To use the asymptotic expansion method, we must handle Riemann-Stieltjes integrals and Ito integrals. They are represented in SymAE as follows:

- `int_(f(s),s,l,u)`
  $= \int_u^l f(s)\,\mathrm{d}s$ where f is an integrand, s is an integrating variable, l is a lower bound and u is an upper bound.

- `ito_(f(s),s,W,u,l)`
  $= \int_l^u f(s)\,\mathrm{d}W(s)$ where f is an integrand, s is an integrating variable, W is a Brownian motion, l is a lower bound and u is an upper bound.

The functions `int_` and `ito_` do not evaluate the expressions. To evaluate the expressions, we have to call the following function.

- `cal_int_(expr)`
  replaces the `int_` in the expression `expr` with `integrate` which is a default function of Maxima.

Example:

```
(%i1) cal_int_(int_(f(x),x,l,u));
(%o1) integrate(f(x),x,l,u)
```

We briefly describe some main functions related to Ito calculus.

- `apply_ito_formula_(expr)`
  applies the Ito's lemma to the expression `expr`.
  Example:

```
(%i1)ito_(f1(s),s,w[1],0,t)*ito_(f2(s),s,w[2],0,t)
(%o1)ito_(f1(t_0),t_0,w[1],0,t)*ito_(f2(t_0),t_0,w[2],0,t)

(%i2)apply_ito_formula_(expr)
(%o2)ito_(ito_(f1(t_0),t_0,w[1],0,t_1)*f2(t_1),t_1,w[2],0,t)
     +ito_(ito_(f2(t_0),t_0,w[2],0,t_1)*f1(t_1),t_1,w[1],0,t)
     +corr_(w[2],w[1])*int_(f1(t_0)*f2(t_0),t_0,0,t)
```

- `apply_fubini_(expr)`
  applies the Fubini's theorem to the expression `expr`.
  Example:

```
(%i1)expr:int_(ito_(f0(s),s,w,0,u),u,0,t);
(%o1)int_(ito_(f0(t_0),t_0,w,0,t_1),t_1,0,t)

(%i2)apply_fubini_(expr);
(%o2)int_(1,t_0,0,t)*ito_(f0(t_0),t_0,w,0,t)
     -ito_(int_(1,t_0,0,t_1)*f0(t_1),t_1,w,0,t)
```

- `sexpand_(expr)`
  applies the Fubini's theorem and the Ito's lemma to the expression `expr` as much as possible.

The function `sexpand_` is time consuming. For example, when we apply this function to $\left(\int_0^T f_1(t)\,\mathrm{d}W_1(t)\right)^3 \left(\int_0^T f_2(t)\,\mathrm{d}W_2(t)\right)^3$, the function `apply_ito_formula_` is called 364 times and the function `apply_fubini_` is called 400 times. In this case, the function takes about 122 seconds and 2728 megabytes.

To derive the approximation of the law of a diffusion process, we have to apply the function `sexpand_` to the products of multidimensional Ito integrals. For example, `sexpand_` is called more than 1000 times to derive the 5th order approximation of the law of the SABR model. To reduce the computational cost of `sexpand_`, we implement the result of the application of the function `sexpand_` to some products of multidimensional Ito integrals as the formulae in Maxima. For example, Table 3.1 shows the amount of computation time required to apply sexpand_ to products of the following expressions:

$$F_2\left(T\right) = \int_0^T f_1(s) \left(\int_0^s f_2(t) \, \mathrm{d}W_2(t) \right) \mathrm{d}W_1(s),$$

$$F_3\left(T\right) = \int_0^T f_1(s) \left(\int_0^s f_2(t) \left(\int_0^t f_3(u) \, \mathrm{d}W_3(u) \right) \mathrm{d}W_2(t) \right) \mathrm{d}W_1(s),$$

$$F_4\left(T\right) = \int_0^T f_1(s) \left(\int_0^s f_2(t) \left(\int_0^t f_3(u) \left(\int_0^u f_4(v) \, \mathrm{d}W_4(v) \right) \mathrm{d}W_3(u) \right) \mathrm{d}W_2(t) \right) \mathrm{d}W_1(s),$$

and

$$H_2\left(T\right) = \int_0^T h_1(s) \left(\int_0^s h_2(t) \, \mathrm{d}W_2(t) \right) \mathrm{d}W_1(s).$$

Table 3.1 shows that the amount of computation time of sexpand_ is improved if the result of the applications of sexpand_ are implemented as formulae.

Table 3.1: The amount of computation time of sexpand_.

	without formulae	with formulae
$F_3(T)F_4(T)$	6.940 (s)	0.499 (s)
$F_2(T)^2F_3(T)$	67.483 (s)	2.249 (s)
$F_2(T)^2H_2(T)^3$	15617.693 (s)	18.530 (s)

3.3.2 Group B

We briefly describe some main functions related to conditional expectations. The expectations of expressions that include Ito integrals are calculated with the function expect_(expr). The function expect_(expr) applies sexpand_ to an expression expr and substitutes 0 to the expectations of Ito integrals .

- expect_(expr)
  calculates the expectation of the expression expr.
  Example:

```
(%i1)expect_((ito_(f(s),s,w,0,t)));
(%o1) 0

(%i2)expect_((ito_(f(s),s,w,0,t))^2);
(%o2)int_(f(t_[1])^2,t_[1],0,t)
```

The conditional expectations that are given in the form of the equation 2.1 are calculated with the function multi_cexpect_.

- multi_cexpect_(expr,C,c)
  calculates the conditional expectation of the expression expr given C=c.
  Example:

```
(%i1)define(F0(t),ito_(f0(s),s,w,l,u))$
(%i2)define(F1(t),ito_(f1(s),s,w,l,u))$
(%i3)define(F2(t),ito_(f2(s),s,w,l,u))$
(%i4)multi_cexpect_(F0(t),[F1(t),F2(t)],[x1,x2]);
(%o4) -(int_(f0(t_0)*f1(t_0),t_0,l,u)
      *int_(f1(t_0)*f2(t_0),t_0,0,u)*x2
      -int_(f1(t_0)^2,t_0,0,u)*int_(f0(t_0)*f2(t_0),t_0,l,u)*x2
      -int_(f0(t_0)*f1(t_0),t_0,l,u)*int_(f2(t_0)^2,t_0,0,u)*x1
      +int_(f0(t_0)*f2(t_0),t_0,l,u)
      *int_(f1(t_0)*f2(t_0),t_0,0,u)*x1)
      /(int_(f1(t_0)^2,t_0,0,u)*int_(f2(t_0)^2,t_0,0,u)
      -int_(f1(t_0)*f2(t_0),t_0,0,u)^2)
```

The function multi_cexpect_ is applied to conditional expectations in the step 3 that is described in the beginning of the chapter 2. This function sometimes generates long equations. For example, Table 3.2 shows the number of terms in equations that are applied the function multi_cexpect_ to. Here G_1, G_2, and G_3 are given by

$$G_1(T) = \int_0^T g_1(s)\, \mathrm{d}W_1(s),$$

$$G_2(T) = \int_0^T g_2(s)\, \mathrm{d}W_2(s),$$

$$G_3(T) = \int_0^T g_3(s)\, \mathrm{d}W_3(s).$$

Table 3.2 shows that conditional expectations have a huge number of terms when they are applied the function multi_cexpect_ to. So, to apply Theorem 2.1 to the conditional expectations can be difficult when we derive the higher order approximations without symbolic computing.

3.3.3 Group C

We briefly describe two main functions related to the asymptotic expansion method. The function nth_term_asym_ and nth_joint_pdf_ derive the nth terms that are given in the form of the equation (2.4) and (2.10), respectively.

Table 3.2: The number of terms in equations that are applied multi_cexpect_ to.

	the number of terms
$\mathbb{E}\left[F_2(T)\vert G_1(t)=x_1, G_2(T)=x_2\right]$	20
$\mathbb{E}\left[F_3(T)\vert G_1(t)=x_1, G_2(T)=x_2\right]$	78
$\mathbb{E}\left[F_2(T)\vert G_1(t)=x_1, G_2(T)=x_2, G_3(T)=x_3\right]$	291
$\mathbb{E}\left[F_3(T)\vert G_1(t)=x_1, G_2(T)=x_2, G_3(T)=x_3\right]$	3597

- `nth_term_asym_(expr(t),S,[x1,..., xl],epsilon,n,t,&rest S1,S2,...)` derives the nth term of the asymptotic expansion of the expression `expr(t)` with respect to `epsilon`.
  `expr(t)` is the definition of `S` and `[x1,..., xl]` is the argument of `S`, and `S1,S2,...` are stochastic process which `S` depends on.
  Examples:

```
(%i1)define(def_S(T), s0+int_(r*S(s),s,0,T)
       +epsilon*ito_(sigma*S(s,epsilon),s,w[1],0,T));
(%o1)def_S(T):=ito_(S(t_[2],epsilon),t_[2],w[1],0,T)*epsilon*sigma
       +s0+int_(S(t_[1]),t_[1],0,T)*r

(%i2)nth_term_asym_(def_S(t),S,[t],epsilon,0,t)
(%o2)S[0](t):=s0*e^(r*t)

(%i3)nth_term_asym_(def_S(t),S,[t],epsilon,1,t);
(%o3)S[1](t):=ito_(e^(t_[5]*r),t_[5],w[1],0,t)*s0*sigma
```

- `nth_joint_pdf_(pdfn,[S1,S2,...],[x1,x2,...],[s1,s2,...],n,epsilon)` calculates the `n`th term of an approximation of the joint probability density function `pdfn` of `[S1,S2,...]` with respect to `epsilon`. `[x1,x2,...]` is the arguments of `[S1,S2,...]` and `[s1,s2,...]` is the arguments of `pdfn`.

One of the major goals of SymAE is the implementation of the function `nth_joint_pdf_`. This function generates the equation (2.5) and apply Lemma 2.1 to the equation (2.5). We can straightforwardly implement this function. However, this function depends on functions generated in Group A and B, especially on `sexpand_` and `multi_cexpect_`. As described in the subsection 3.3.1, the function `sexpand_` is time consuming. So to reduce the computational cost of `sexpand_` is important for `nth_joint_pdf_`. Furthermore, this function can sometimes generate long expressions, because the equation (2.5) includes conditional expectations. The computation of the conditional expectations requires a huge amount of symbolic manipulation as described in the subsection 3.3.2.

3.4 Procedure of programming

We provide an example of deriving the approximation of a stochastic differential equation by using SymAE. The following is the code for approximating the law of the SABR model by using the asymptotic expansion method.

First, we load SymAE in Maxima.

```
load("AsymptoticExpansionMaxima/v2.0/index.lisp");
```

The SABR model is given by

$$x\left(t\right)=x_0+\int_0^t \epsilon y(s)x(s)^{\beta}\,\mathrm{d}w_1(s),$$

$$y\left(t\right)=y_0+\int_0^t \epsilon\alpha y(s)\,\mathrm{d}w_2(s),$$

and represented as follows:

```
define(def_x(T),
    x0
    + ito_(epsilon*sigma*y(s,epsilon)*x(s,epsilon)^beta,
     s,w[1],0,T));
define(def_y(T),
    y0 + ito_(epsilon*alpha*y(s),s,w[2],0,T));
```

The correlation of the Brownian motion is given by

$$\mathrm{d}\left\langle w_1,w_2\right\rangle_t=\rho\,\mathrm{d}t$$

and represented as follows:

```
correlation:true;
set_correlation_(rho,[w[1],w[2]]);
```

The model parameters x_0, y_0, β and α are declared as follows:

```
set_model_param_(x0,y0,beta,alpha);
```

The 0th, 1st, 2nd, and 3rd terms in the equation (2.3) are calculated as follows:

```
nth_term_asym_(def_y(t),y,[t],epsilon,0,t);
nth_term_asym_(def_y(t),y,[t],epsilon,1,t);
nth_term_asym_(def_y(t),y,[t],epsilon,2,t);
nth_term_asym_(def_y(t),y,[t],epsilon,3,t);
nth_term_asym_(def_x(t),x,[t],epsilon,0,t,y);
nth_term_asym_(def_x(t),x,[t],epsilon,1,t,y);
nth_term_asym_(def_x(t),x,[t],epsilon,2,t,y);
nth_term_asym_(def_x(t),x,[t],epsilon,3,t,y);
```

The 0th, 1st, and 2nd terms in the equation (2.5) are clculated as follows:

```
nth_joint_pdf_(npdf0, [x,y], [t], [a,b], 0, epsilon);
nth_joint_pdf_(npdf1, [x,y], [t], [a,b], 1, epsilon);
nth_joint_pdf_(npdf2, [x,y], [t], [a,b], 2, epsilon);
```

The functions `npdf0`, `npdf1`, and `npdf2` include `diff` and `int`, and these function are calculated as follows:

```
define(npdf0(a,b),cal_pdf_(npdf0(a,b)));
define(npdf1(a,b),cal_pdf_(npdf1(a,b)));
define(npdf2(a,b),cal_pdf_(npdf2(a,b)));
```

The C++ code of the functions `npdf0`, `npdf1`, and `npdf2` are generated as follows:

```
make_c_class_file2_(t,"class_name","file_path",
’npdf0(a,b),’npdf1(a,b),’npdf2(a,b));
```

3.5 Conclusion

As we mentioned in the section 3.1, the asymptotic expansion method has difficulties in the application and implementation. By using SymAE, the higher order approximation of the law of the diffusion model can be easily derived. Moreover, SymAE automatically generate the C++ code of the approximations. As the section 3.4 shows, the programming code on Maxima to derive the C++ code of the asymptotic expansion method is easy to implement for users of SymAE.

The numerical examples of the approximations that are generated by SymAE are shown in the chapter 4.

Chapter 4

Applications of the asymptotic expansion method

4.1 Pricing interest rate swaptions in the SABR/LIBOR market model

4.1.1 The SABR/LIBOR market model

We consider the interest rate swaps in the SABR/LIBOR market model. In the pricing problem of interest rate derivatives, the volatility smile of forward rates is important. The SABR model can capture the volatility smile and the LIBOR market model is a market standard model in practice. So the SABR/LIBOR market model is an important model in pricing interest rate derivative.

Let $F_i(t)$ denote a forward rate with respect to term $[T_i, T_{i+1}]$ at time $t \leq T_i$ and $\sigma_i(t)$ denote a volatility function of forward rate $F_i(t)$. Hence $i = 0, \ldots, n$ and $0 = T_0 < T_i < \cdots < T_n$. For $k \in \{1, \ldots, n\}$, let F_k be numeraire and the SABR/LIBOR market model is given by

$$dF_j\left(t\right) = \sigma_j\left(t\right) F_j\left(t\right)^{\beta_j} \times \begin{cases} -\sum_{i=j+1}^{k} \dfrac{\rho_{i,j}\delta_i\sigma_i\left(t\right) F_i\left(t\right)^{\beta_i}}{1+\delta_i F_i\left(t\right)} dt + dW_j^{k+1}\left(t\right) & \text{if } j < k \\ dW_j^{k+1}\left(t\right) & \text{if } j = k \\ \sum_{i=k+1}^{j} \dfrac{\rho_{i,j}\delta_i\sigma_i\left(t\right) F_i\left(t\right)^{\beta_i}}{1+\delta_i F_i\left(t\right)} dt + dW_j^{k+1}\left(t\right) & \text{if } j > k \end{cases}$$

and

$$\mathrm{d}\sigma_j\left(t\right)=\alpha_j\left(t\right)\sigma_j\left(t\right)\times\begin{cases}-\sum_{i=j+1}^{k}\dfrac{r_{i,j}\delta_i\sigma_i\left(t\right)F_i\left(t\right)^{\beta_i}}{1+\delta_iF_i\left(t\right)}\,\mathrm{d}t+\,\mathrm{d}Z_j^{k+1}\left(t\right) & \text{if } j<k\\ \mathrm{d}Z_j^{k+1}\left(t\right) & \text{if } j=k\\ \sum_{i=k+1}^{j}\dfrac{r_{i,j}\delta_i\sigma_i\left(t\right)F_i\left(t\right)^{\beta_i}}{1+\delta_iF_i\left(t\right)}\,\mathrm{d}t+\,\mathrm{d}Z_j^{k+1}\left(t\right) & \text{if } j>k\end{cases}$$

where $\left\{W_i^{k+1}\right\}$ and $\left\{Z_j^{k+1}\right\}$ are Brownian motions with

$$\mathrm{d}\left\langle W_i^{k+1},W_j^{k+1}\right\rangle_t=\rho_{i,j}\,\mathrm{d}t,$$

$$\mathrm{d}\left\langle W_i^{k+1},Z_j^{k+1}\right\rangle_t=r_{i,j}\,\mathrm{d}t$$

and

$$\mathrm{d}\left\langle Z_i^{k+1},Z_j^{k+1}\right\rangle_t=\eta_{i,j}\,\mathrm{d}t$$

for $i,j=1,\ldots,n$.

We consider interest rate swaptions. The price of the payer swaption $V(t)$ at time t is given as the following expectation.

$$\begin{aligned}V\left(t\right)&=D_\mu\left(t\right)\mathbb{E}^{T_\mu}\left[\left.\left(\sum_{i=\mu}^{M}\delta_iB\left(T_\mu,T_{i+1}\right)\left(F_i\left(T_\mu\right)-K\right)\right)^+\right|\mathcal{F}_t\right]\\&=D_M\left(t\right)\mathbb{E}^{T_{M+1}}\left[\left.\left(\sum_{i=\mu}^{M}\delta_i\left(F_i\left(T_\mu\right)-K\right)\prod_{j=i+1}^{M}\left(1+\delta_jF_j\left(T_\mu\right)\right)\right)^+\right|\mathcal{F}_t\right]\\&=D_M\left(t\right)\mathbb{E}^{T_{M+1}}\left[\left.\left(S\left(T_\mu\right)\right)^+\right|\mathcal{F}_t\right],\end{aligned}$$

where

$$S\left(T_\mu\right)=\sum_{i=\mu}^{M}\delta_i\left(F_i\left(T_\mu\right)-K\right)H_i(T_\mu),$$

$$H_i(T_\mu)=\prod_{j=i+1}^{M}\left(1+\delta_jF_j\left(T_\mu\right)\right).$$

Here $\mathbb{E}^{T_{M+1}}\left[\cdot\right]$ means the expectation with numeraire F_{M+1} and filtration $\mathcal{F}_t$.

We apply the asymptotic expansion method to the forward rate F_j where $j < k$. The stochastic differential equation that is perturbed by parameter $\epsilon \in (0,1]$ is given by

$$\mathrm{d}F_j^{\epsilon}(t) = \sigma_j^{\epsilon}(t)\,F_j^{\epsilon}(t)^{\beta_j}\left(-\epsilon^2 \sum_{i=j+1}^{k} \frac{\rho_{i,j}\delta_i\sigma_i^{\epsilon}(t)\,F_i^{\epsilon}(t)^{\beta_i}}{1+\delta_i F_i^{\epsilon}(t)}\mathrm{d}t + \epsilon\,\mathrm{d}W_j^{k+1}(t)\right)$$

and

$$\mathrm{d}\sigma_j^{\epsilon}(t) = \alpha_j(t)\,\sigma_j^{\epsilon}(t)\left(-\epsilon^2 \sum_{i=j+1}^{k} \frac{r_{i,j}\delta_i\sigma_i^{\epsilon}(t)\,F_i^{\epsilon}(t)^{\beta_i}}{1+\delta_i F_i^{\epsilon}(t)}\mathrm{d}t + \epsilon\,\mathrm{d}Z_j^{k+1}(t)\right).$$

By the asymptotic expansion method, we obtain the nth order approximation of the law of $S(T_\mu)$ that is denoted by $f_S^{(n)}$. Now we have the nth order approximation of the value $V(0)$ that is given by

$$V^{(n)}(0) = D_M(0)\int_0^{\infty} s f_S^{(n)}(s)\,\mathrm{d}s.$$

4.1.2 Numerical results

In the test case, we compute $f_S^{(n)}$ for $n = \{1,2,3\}$, and in these cases the integrals included in $f_S^{(n)}$ can be analytically calculated.

We use the following model parameters

$$\mu = 1,\quad M = 6,\quad \delta_i = 1.0,\quad T_i = i,\quad \beta_i = 0.8,\quad F_i(0) = 0.01,\quad \sigma_i(0) = 0.2.$$

We use values that are calculated with quasi-Monte Carlo simulation with Ninomiya-Victoir method and Romberg extrapolation([15]). In the simulations the number of sample paths is 2,000,000 and the numbers of division are 16 + 32. Table 4.1 shows the numerical results. In this table, AE2, AE3 and QMC indicate the 2nd asymptotic expansion, 3rd order asymptotic expansion and the quasi-Monte Carlo simulation, respectively. The numerical results show the 3rd order expansion is more accurate than 2nd order expansion. This fact is important in practice because we do not know that a higher order expansion is more accurate.

Table 4.1: Prices of swaptions

Fixed Rate	AE2	AE3	QMC	AE2-QMC	AE3-QMC
0.004	0.034708	0.035028	0.035041	-0.000333	-0.000013
0.006	0.024781	0.024911	0.024922	-0.000141	-0.000011
0.008	0.016780	0.016875	0.016932	-0.000152	-0.000057
0.010	0.010988	0.011102	0.011203	-0.000215	-0.000101
0.012	0.007113	0.007207	0.007328	-0.000215	-0.000121
0.014	0.004550	0.004682	0.004790	-0.000240	-0.000108
0.016	0.002778	0.003111	0.003151	-0.000373	-0.000040
0.018	0.001544	0.002105	0.002093	-0.000549	0.000012
0.020	0.000749	0.001356	0.001405	-0.000656	-0.000049

4.2 Application to exotic option pricing problems

4.2.1 Call option on the maximum of two assets in the CEV model

The problem

We consider options that are called "call options on the maximum of two assets"[5] and suppose that two assets S_1 and S_2 satisfy the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}S_1(t) &= r\,S_1(t)\ \mathrm{d}t + \sigma_1\,S_1(t)^{\beta_1}\ \mathrm{d}W_1(t)\,,\\ S_1(0) &= S_1^0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}S_2(t) &= r\,S_2(t)\ \mathrm{d}t + \sigma_2\,S_2(t)^{\beta_2}\ \mathrm{d}W_2(t)\,,\\ S_2(0) &= S_2^0 \end{aligned}$$

where $r \in \mathbb{R}$, σ_i and $S_i^0 \in \mathbb{R}_+$ for $i = 1, 2$. $\mathbb{R}_+$ means $\{x \in \mathbb{R}; x \geq 0\}$. Here, (W_1, W_2) is a 2-dimensional Brownian motion with the instantaneous correlation $\rho \in [-1, 1]$. Then the price of this option is given by

$$V(r, \sigma_1, \sigma_2, K, T) = \mathbb{E}\left[\exp(-r\,T)\left(\max(S_1(T), S_2(T)) - K\right)^+\right]$$

where $K \in \mathbb{R}_+$ and $(x)^+$ means $\max(x, 0)$ for any $x \in \mathbb{R}$.

To simplify the approximation of the law of $(S_1(T), S_2(T))$, we define diffusion processes $Y_1(t)$ and $Y_2(t)$ by

$$Y_i(t) = \exp(-rt)\,S_i(t)\big/S_i(0)$$

for $i = 1, 2$. The diffusion processes $Y_1(t)$ and $Y_2(t)$ satisfy the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}Y_i(t) &= \sigma_i (S_i(0))^{-(1-\beta_i)} \exp(-(1-\beta_i) rt) Y_i(t)^{\beta_i} \, \mathrm{d}W_i(t), \\ Y_i(0) &= 1 \end{aligned}$$

for $i = 1, 2$. We consider the following perturbed processes:

$$\begin{aligned} \mathrm{d}Y_i^{\epsilon}(t) &= \epsilon\sigma_i (S_i(0))^{-(1-\beta_i)} \exp(-(1-\beta_i) rt) Y_i^{\epsilon}(t)^{\beta_i} \, \mathrm{d}W_i(t), \\ Y_i^{\epsilon}(0) &= 1 \end{aligned}$$

for $i = 1, 2$.

By MAE, we obtain the nth order approximation of the low of $(Y_1(T), Y_2(T))$ that is denoted by $f_Y^{(n)}$. Now we have the nth order approximation of the value $V(r, \sigma_1, \sigma_2, K, T)$ that is given by

$$\begin{aligned} &V^{(n)}(r, \sigma_1, \sigma_2, K, T) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\max\left(S_1^0 y_1, S_2^0 y_2\right) - \exp(-rT) K\right)^+ f_Y^{(n)}(y_1, y_2) \, \mathrm{d}y_2 \, \mathrm{d}y_1. \end{aligned} \tag{4.1}$$

Numerical results

We show the accuracy and the computational cost of the approximation in the equation (4.1). We compute $f_Y^{(n)}$ for $n = \{1, 2, 3, 4, 5\}$, and in these cases the integrals included in $f_S^{(n)}$ can be analytically calculated. We use the following parameters:

$$S_i^0 = 1, \quad \beta_i = 0.7, \quad \sigma_i = 0.4, \quad \rho = -0.3, \quad r = 0$$

for $i = 1, 2$.

We consider the values calculated with quasi-Monte Carlo simulation [9, 14] as the true values of the call options on the maximum of two assets. To calculate these values faster than the Euler-Maruyama scheme, we use the higher-order weak approximation scheme [13, 15]. We use Ninomiya-Victoir scheme with Romberg extrapolation [15] as a discretization scheme. We use 8+16 time steps per year as the number of partitions and generate 10^9 paths in each simulation. We consider the values calculated with these simulations as their limit values in the following experiments.

Table 4.2 shows the values that are computed with quasi-Monte Carlo simulation and the approximation $V^{(n)}$. In this table, QMC and MAEn for

$n \in \{1,2,3,4,5\}$ indicate quasi-Monte Carlo simulation and the nth order approximation, respectively. IV indicates the implied volatility and these values are calculated with the Black-Scholes formula for the European call option with underlying asset $\max(S_1, S_2)$, strike K and maturity T.

These numerical results show that the 3rd approximation is accurate within 1 bp. Thus we can say that the 3rd approximation is accurate enough in practice.

Table 4.2: Numerical results of call options on the maximum of two assets in the CEV model.

	QMC	MAE1	MAE2	MAE3	MAE4	MAE5
			$K = 0.8$			
Price	0.46162	0.46407	0.46513	0.46165	0.46162	0.46159
IV	1.02996	1.03797	1.04146	1.03004	1.02994	1.02984
Error(bps)		80.2	115.0	0.8	-0.1	-1.1
			$K = 0.9$			
Price	0.37327	0.37260	0.37640	0.37327	0.37326	0.37324
IV	0.87427	0.87230	0.88346	0.87426	0.87425	0.87419
Error(bps)		-19.6	91.9	-0.1	-0.2	-0.8
			$K = 1.0$			
Price	0.29407	0.28823	0.29673	0.29406	0.29407	0.29405
IV	0.75467	0.73897	0.76183	0.75463	0.75465	0.75460
Error(bps)		-157.0	71.6	-0.4	-0.1	-0.6
			$K = 1.1$			
Price	0.22605	0.21379	0.22842	0.22605	0.22604	0.22603
IV	0.66458	0.63336	0.67064	0.66458	0.66456	0.66453
Error(bps)		-312.1	60.6	0.1	-0.1	-0.5
			$K = 1.2$			
Price	0.16993	0.15148	0.17226	0.16994	0.16993	0.16992
IV	0.59757	0.55131	0.60341	0.59760	0.59756	0.59754
Error(bps)		-462.7	58.4	0.3	-0.1	-0.3

The amount of computation time required to calculate the approximation $V^{(n)}$ is summarized in Table 4.3. This table shows that the approximation can be calculated in less than a millisecond. The CPU used in this experiment is Intel Core i7 by Intel Corporation.

Table 4.3: CPU time required to calculate the approximations.

Method	MAE1	MAE2	MAE3	MAE4	MAE5
CPU time (s)	5.1×10^{-6}	1.9×10^{-5}	7.1×10^{-5}	2.1×10^{-4}	5.3×10^{-4}

Remark

We use the implied volatilities to compare the approximation formula with the benchmarks. This is because the relative error of the prices of the financial derivatives is not suitable to evaluate the error as we can reduce the relative error by adding some cash flows to the payoff of the derivatives.

4.2.2 Average strike option in the BS model

The problem

We consider an average strike option that is a type of the Asian option. The strike of this option is an average of the underlying asset rate over a predetermined period of time.

We consider two diffusion processes S and A whose dynamics are given by

$$\begin{aligned} \mathrm{d}S(t) &= rS(t)\ \mathrm{d}t + \sigma S(t)\ \mathrm{d}W(t), \\ S(0) &= S_0, \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}A(t) &= S(t)\ \mathrm{d}t, \\ A(t) &= 0, \end{aligned}$$

where r, σ and S_0 are in $\mathbb{R}_+$. Then the price of the average strike option is given by

$$V(S_0, r, \sigma, T, \eta) = \mathbb{E}\left[e^{-rT}\left(\eta S(T) - \frac{\eta A(T)}{T}\right)^+\right] \tag{4.2}$$

where η is 1 or -1.

To obtain a good approximation of V in the equation (4.2), we introduce an equivalent measure $\mathbb{Q}^*$ given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^*} = Z(T)$$

where

$$Z(t) = \exp\left(-\frac{r^2}{2\sigma^2}t + \frac{r}{\sigma}W^*(t)\right),$$

for any $t \in [0, T]$ and W^* is a Brownian motion under $\mathbb{Q}^*$. So

$$S(t) = S_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma W^*(t)\right)$$

and

$$A(t) = \int_0^t S(u)\,\mathrm{d}u$$

hold.

Now we have

$$V(S_0, r, \sigma, T, \eta) = \mathbb{E}^*\left[e^{-rT}Z(T)\left(\eta S(T) - \frac{\eta A(T)}{T}\right)^+\right] \tag{4.3}$$

where $\mathbb{E}^*[\cdot]$ is the expectation under $\mathbb{Q}^*$.

We perturb S, A and Z by putting a small number $\epsilon \in (0,1]$ and the dynamics of the perturbed processes S^ϵ, A^ϵ and Z^ϵ are

$$\begin{aligned} \mathrm{d}S^\epsilon(t) &= \epsilon\sigma S^\epsilon(t)\,\mathrm{d}W^*(t), \\ S^\epsilon(0) &= S_0, \end{aligned}$$

$$\begin{aligned} \mathrm{d}A^\epsilon(t) &= S^\epsilon(t)\,\mathrm{d}t, \\ A^\epsilon(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}Z^\epsilon(t) &= \frac{\epsilon r}{\sigma}Z^\epsilon(t)\,\mathrm{d}W^*(t), \\ Z^\epsilon(0) &= 1. \end{aligned}$$

By applying MAE to ϵW^* and A^ϵ, we obtain the nth order approximation of the law of $W^*(T)$ and $A(T)$ that is denoted by $f^{(n)}_{W^*,A}$. We then have an nth order approximation of V given by

$$\begin{aligned} &V^{(n)}(S_0, r, \sigma, T, \eta) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\left(r+\frac{r^2}{2\sigma^2}\right)T+\frac{r}{\sigma}w}\left(\eta S_0 e^{-\frac{\sigma^2}{2}T+\sigma w} - \eta\frac{a}{T}\right)^+ f^{(n)}_{W^*,A}(w,a)\,\mathrm{d}w\,\mathrm{d}a \end{aligned} \tag{4.4}$$

Numerical results

We show the accuracy and the computational cost of the approximation in the equation (4.4). The implied volatilities are calculated by using the Black-Scholes formula for the plain vanilla option with strike $\mathbb{E}\left[\int_0^T S_t \, \mathrm{d}t\right] \Big/ T$ and maturity T. We compute $f^{(n)}_{W^*,A}$ for $n = \{1, 2, 3, 4\}$, and in these cases the integrals included in $f^{(n)}_S$ can be analytically calculated.

We use the following parameters:

$$S_0 = 1, \quad r = 0.05, \quad \sigma = 0.3.$$

We consider the values calculated with quasi-Monte Carlo simulation as the true values of the average strike options. We use Ninomiya-Victoir scheme with Romberg extrapolation [15] as a discretization scheme. We use 32+64 time steps per year as the number of partitions and generate 10^9 paths in each simulation. We consider the values calculated with these simulations as their limit values in the following experiments.

The results are in Table 4.4 and Table 4.5. In these tables, QMC and MAEn for $n \in \{1, 2, 3, 4\}$ indicate quasi-Monte Carlo simulation and the nth order approximation. Error indicates the difference between implied volatility of QMC and that of MAEn. The results show that the 3rd order approximation is accurate within 1 bp in cases $T = 0.25$ and $T = 0.5$, and within 5 bps in cases $T = 1$ and $T = 2$.

The amount of computation time required to calculate the approximation $V^{(n)}$ is summarized in Table 4.6. This table shows that the approximation can be calculated in less than 0.1 seconds. The CPU used in this experiment is Intel Core i7 by Intel Corporation.

4.3 American option pricing

We introduce a new method for pricing American options on multidimensional diffusion models when there is a finite but possibly large number of exercise dates. This new method that is called the multidimensional asymptotic expansion mesh method (MAEM) is designed to solve general optimal stopping problems. This method is similar to the stochastic mesh method [1]. The stochastic mesh method assumes the law of the diffusion model that describes the asset prices is given. The mesh of the state state space is constructed by generating the sample paths of the diffusion model. Our method approximates the law of the diffusion models and deterministically constructs the mesh of the state space.

Table 4.4: Numerical results of average strike put options.

	QMC	MAE1	MAE2	MAE3	MAE4
$T = 0.25$					
Price	0.03177	0.03194	0.03169	0.03177	0.03176
IV	0.20022	0.20111	0.19981	0.20022	0.20021
Error(bps)		8.9	-4.1	-0.1	-0.1
$T = 0.5$					
Price	0.04355	0.04402	0.04332	0.04356	0.04355
IV	0.18256	0.18428	0.18177	0.18260	0.18258
Error(bps)		17.3	-7.9	0.4	0.2
$T = 1$					
Price	0.05914	0.06046	0.05850	0.05922	0.05923
IV	0.17983	0.18322	0.17816	0.18002	0.18006
Error(bps)		33.9	-16.7	1.9	2.3
$T = 2$					
Price	0.07958	0.08317	0.07761	0.07982	0.08042
IV	0.19194	0.19865	0.18826	0.19239	0.19351
Error(bps)		67.0	-36.8	4.5	15.6

Let the exercise dates be $0 = t_0 < t_1 < \cdots < t_N = T$, the payoff function be $h : \{t_0, \ldots, t_N\} \times \mathbb{R}^D \to \mathbb{R}$ and $X(t)$ be a diffusion process on $\mathbb{R}^D$ with a fixed initial state $X(0) = x_0$. We suppose that $X(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}X(t) &= \alpha(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \\ X(0) &= x_0 \end{aligned}$$

where $x_0 \in \mathbb{R}^D$. We suppose that α and σ satisfy the condition (2.2).

The value of this option is given by

$$V = \max_{\tau \in \mathcal{T}} \mathbb{E}\left[h(\tau, X(\tau))\right]$$

where $\mathcal{T}$ is a set of $\{\mathcal{F}_t\}_{0 \le t \le T}$-stopping times that take values in the set $\{t_0, \ldots, t_N\}$. We can calculate V with backwards induction and the value at time t_n in state $x \in \mathbb{R}^D$ is

$$V(t_n, x) = \max\left(h(t_n, x), \mathbb{E}\left[V(t_{n+1}, X(t_{n+1})) \,|X(t_n) = x\right]\right) \tag{4.5}$$

Table 4.5: Numerical results of average strike call options.

	QMC	MAE1	MAE2	MAE3	MAE4
			$T = 0.25$		
Price	0.03807	0.03827	0.03799	0.03807	0.03807
IV	0.20063	0.20166	0.20022	0.20062	0.20062
Error(bps)		10.3	-4.1	-0.1	-0.1
			$T = 0.5$		
Price	0.05626	0.05684	0.05604	0.05627	0.05626
IV	0.18369	0.18579	0.18290	0.18372	0.18371
Error(bps)		21.0	-7.9	0.4	0.2
			$T = 1$		
Price	0.08499	0.08673	0.08435	0.08507	0.08508
IV	0.18307	0.18755	0.18142	0.18326	0.18330
Error(bps)		44.7	-16.6	1.9	2.3
			$T = 2$		
Price	0.13305	0.13834	0.13112	0.13329	0.13388
IV	0.20146	0.21137	0.19785	0.20191	0.20302
Error(bps)		99.1	-36.1	4.5	15.7

Table 4.6: CPU time required to calculate the approximations.

Method	MAE1	MAE2	MAE3	MAE4
CPU time (s)	4.9×10^{-3}	6.2×10^{-3}	8.4×10^{-3}	2.4×10^{-2}

for $i = 0, \ldots, N-1$ and $V(t_N, x) = h(t_N, x)$. We are interested in computing $V \equiv V(0, x_0)$. However the numerical calculation of V is usually difficult because the law of $\{X(t_1), \ldots, X(t_N)\}$ is not necessarily known in many practical problems. As described in the chapter 1, a number of studies on this problem have been conducted(see [4]).

4.3.1 The mesh method

We introduce the mesh method for pricing American options on multidimensional diffusion process. We abbreviate $X(t_n)$, $V(t_n, \cdot)$ and $h(t_n, \cdot)$ as X_n, $V_n(\cdot)$ and $h_n(\cdot)$, respectively. The mesh method begins with the construction of a mesh on $\{t_0, \ldots, t_N\} \times K^M$ where the set K^M is defined by

$$K^M = \prod_{k=1}^{D} [-a_k M, b_k M] \subset \mathbb{R}^D$$

for $M \in \mathbb{N}$, a_k and b_k are positive numbers.

Suppose that X_{n+1} has density $f_n(x, \cdot) : \mathbb{R}^D \to \mathbb{R}$ when $X_n = x \in \mathbb{R}^D$. To introduce the method, we provide some definitions. We define an adapted process $X^{(M)} = \left(X_1^{(M)}, \ldots, X_D^{(M)}\right)$ as follows:

$$X_k^{(M)}(t) = \max\left(\min\left(X(t), -a_k M\right), b_k M\right) \tag{4.6}$$

for $k \in \{1, \ldots, D\}$ and $M \in \mathbb{N}$, and abbreviate $X^{(M)}(t_n)$ as $X_n^{(M)}$. We define a set of functions $\left\{f_n^{(M)} : K^M \times K^M \to \mathbb{R}\right\}_{n=1,\ldots,N}$ by

$$f_n^{(M)}(x, y) = f_n(x, y)$$

for x and $y \in K^M$.

We also define sets of functions $\left\{V_n^{(M)} : K^M \to \mathbb{R}\right\}_{n=0,\ldots,N}$ and $\left\{V_n^{(m,M)} : K^M \to \mathbb{R}\right\}_{\substack{n=0,\ldots,N \\ m\in\mathbb{N}}}$ by

$$V_n^{(M)}(x) = \begin{cases} h_n(x) & \text{if } n = N \\ \max\left(h_n(x), \int_{K^M} V_{n+1}^{(M)}(y) f_n^{(M)}(x, y)\, \mathrm{d}y\right) & \text{if } n < N \end{cases} \tag{4.7}$$

and

$$V_n^{(m,M)}(x) = \begin{cases} h_n(x) & \text{if } n = N \\ \max\left(h_n(x), \sum_{i\in I_m} V_{n+1}^{(m,M)}(y_i) f_n^{(M)}(x, y_i)\Delta^m\right) & \text{if } n < N \end{cases} \tag{4.8}$$

where for $k \in \{1, \ldots, D\}$ and $j \in \{1, \ldots, 2^m\}$, $\Delta_k^m = \frac{M(b_k + a_k)}{2^m}$, $\Delta^m = \prod_{k=1}^{D} \Delta_k^m$, $a_{k,j}^m = -a_k M + \Delta_k^m (j-1)$, $b_{k,j}^m = -a_k M + \Delta_k^m j$, $I_m = \{1, \ldots, 2^m\}^D$ and $C_i^m = \prod_{k=1}^{D} [a_{k,i_k}^m, b_{k,i_k}^m)$ for $i \in I_m$.

Theorem 4.1. *Suppose that for any $n \in \{0, \ldots, N\}$, $h_n(\cdot)$ is continuous and bounded on $\mathbb{R}^D$, and that for any $n \in \{0, \ldots, N\}$ and $M \in \mathbb{N}$, $f_n(\cdot, \cdot)$ is continuous on $\mathbb{R}^D \times \mathbb{R}^D$.*

Then for any $n \in \{0, \ldots, N\}$ and $x \in \mathbb{R}^D$

$$\lim_{M\to\infty} \lim_{m\to\infty} \left| V_n^{(m,M)}(x) - V_n(x) \right| = 0 \tag{4.9}$$

holds.

To prove Theorem 4.1, we introduce the following lemma.

Lemma 4.1. *Suppose that the conditions are the same as in Theorem 4.1. Then for any $n \in \{0, \ldots, N\}$ and $M \in \mathbb{N}$*

$$\lim_{m\to\infty} \sup_{x\in K^M} \left| V_n^{(m,M)}(x) - V_n^{(M)}(x) \right| = 0 \tag{4.10}$$

holds.

Proof of Lemma 4.1. We define $\overline{V}_n^{(m,M)} : K^M \to \mathbb{R}$ and $\underline{V}_n^{(m,M)} : K^M \to \mathbb{R}$ for $m \in \mathbb{N}$ and $n \in \{0, \ldots, N\}$ by

$$\overline{V}_n^{(m,M)}(x) = \begin{cases} h_n(x) & \text{if } n = N \\ \max\left(h_n(x), \displaystyle\sum_{i\in I_m} \sup_{y\in C_i^m} \overline{V}_{n+1}^{(m,M)}(y)\, f_n^{(M)}(x,y)\Delta^m \right) & \text{if } n < N \end{cases} \tag{4.11}$$

and

$$\underline{V}_n^{(m,M)}(x) = \begin{cases} h_n(x) & \text{if } n = N \\ \max\left(h_n(x), \displaystyle\sum_{i\in I_m} \inf_{y\in C_i^m} \underline{V}_{n+1}^{(m,M)}(y)\, f_n^{(M)}(x,y)\Delta^m \right) & \text{if } n < N. \end{cases}$$

First we prove

$$V_n^{(m,M)}(x) \le \overline{V}_n^{(m,M)}(x) \tag{4.12}$$

for any $m \in \mathbb{N}$, $n \in \{0, \ldots, N\}$ and $x \in K^M$ by induction, proceeding backwards from $n = N$ to $n = 0$. When $n = N$, by the definition of $V_n^{(m,M)}$ given by the equation (4.11), we have the equation $\overline{V}_N^{(m,M)}(x) = V_N^{(m,M)}(x)$. Then we have the inequality (4.12) for $n = N$. We assume that the inequality (4.12) holds for $n = n' + 1$. Then

$$\begin{aligned}
V_{n'}^{(m,M)}(x) &= \max\left(h_{n'}(x), \sum_{i\in I_m} V_{n'+1}^{(m,M)}(y_i)\, f_{n'}^{(M)}(x,y_i)\Delta^m\right) \\
&\le \max\left(h_{n'}(x), \sum_{i\in I_m} \overline{V}_{n'+1}^{(m,M)}(y_i)\, f_{n'}^{(M)}(x,y_i)\Delta^m\right) \\
&\le \max\left(h_{n'}(x), \sum_{i\in I_m} \sup_{y\in C_i^m} \overline{V}_{n'+1}^{(m,M)}(y)\, f_{n'}^{(M)}(x,y)\Delta^m\right) \\
&= \overline{V}_{n'}^{(m,M)}(x).
\end{aligned}$$

Thus $V_{n'}^{(m,M)}(x) \le \overline{V}_{n'}^{(m,M)}(x)$. This proves that the inequality (4.12) holds for $n = n'$. Since the inductive step has been proved, it has now been proved by induction that the inequality (4.12) holds for any $n \in \{0,\ldots,N\}$.

Using a method similar to that employed in the proof of the inequality (4.12), we can show that

$$\underline{V}_{n}^{(m,M)}(x) \le V_{n}^{(m,M)}(x)$$

for any $n \in \{0,\ldots,N\}$. Now, we have

$$\underline{V}_{n}^{(m,M)}(x) \le V_{n}^{(m,M)}(x) \le \overline{V}_{n}^{(m,M)}(x). \tag{4.13}$$

Repeating the discussion we also have

$$\underline{V}_{n}^{(m,M)}(x) \le V_{n}^{(M)}(x) \le \overline{V}_{n}^{(m,M)}(x). \tag{4.14}$$

Before we prove the equation (4.10), we define statements $\mathbf{S}_1(n)$, $\mathbf{S}_2(n)$, $\mathbf{S}_3(n)$ and $\mathbf{S}_4(n)$ for any $n \in \{0,\ldots,N\}$ by

$\mathbf{S}_1(n)$: $V_n^{(M)}$ is continuous on K^M,

$\mathbf{S}_2(n)$: $\overline{V}_n^{(m,M)}$ is continuous on K^M for any $m \in \mathbb{N}$,

$\mathbf{S}_3(n)$: $\overline{V}_n^{(m,M)}(x) \ge \overline{V}_n^{(m+1,M)}(x)$ for any $x \in K^M$ and $m \in \mathbb{N}$,

$\mathbf{S}_4(n)$: $\lim_{m\to\infty} \sup_{x\in K^M} \left|\overline{V}_n^{(m,M)}(x) - V_n^{(M)}(x)\right| = 0.$

We prove that $\mathbf{S}_1(n)$, $\mathbf{S}_2(n)$, $\mathbf{S}_3(n)$ and $\mathbf{S}_4(n)$ are true for any $n \in \{0,\dots,N\}$ by induction, proceeding backwards form $n = N$ to $n = 0$.

By the definitions of $V_N^{(M)}$ and $\overline{V}_N^{(m,M)}$, we have

$$V_N^{(M)}(x) = h_N(x) = \overline{V}_N^{(m,M)}(x)$$

for any $x \in K^M$. Since h_N is continuous on K^M then $\mathbf{S}_1(N)$, $\mathbf{S}_2(N)$, $\mathbf{S}_3(N)$ and $\mathbf{S}_4(N)$ are true.

We prove that if $\mathbf{S}_1(n'+1)$, $\mathbf{S}_2(n'+1)$, $\mathbf{S}_3(n'+1)$ and $\mathbf{S}_4(n'+1)$ are true for some $n' \in \{0,\dots,N-1\}$, then $\mathbf{S}_1(n')$, $\mathbf{S}_2(n')$, $\mathbf{S}_3(n')$ and $\mathbf{S}_4(n')$ are true.

h_n and $f_n^{(M)}$ are continuous on K^M and $K^M \times K^M$, respectively. By the assumptions that $\mathbf{S}_1(n'+1)$ and $\mathbf{S}_2(n'+1)$ are true, $V_{n'+1}^{(M)}$ and $\overline{V}_{n'+1}^{(m,M)}$ are continuous on K^M. Then $V_{n'}^{(M)}$ and $\overline{V}_{n'}^{(m,M)}$ are continuous on K^M by the definition of $V_{n'}^{(M)}$ and $\overline{V}_{n'}^{(m,M)}$. That means $\mathbf{S}_1(n')$ and $\mathbf{S}_2(n')$ are true.

By the definition of $\overline{V}_{n'}^{(m,M)}(x)$, we have

$$\begin{aligned}
\overline{V}_{n'}^{(m,M)}(x) &= \max\left(h_{n'}(x), \sum_{i\in I_m} \sup_{y\in C_i^m} \overline{V}_{n'+1}^{(m,M)}(y)\, f_{n'}^{(M)}(x,y)\Delta^m\right) \\
&\geq \max\left(h_{n'}(x), \sum_{i\in I_m} \sup_{y\in C_i^m} \overline{V}_{n'+1}^{(m+1,M)}(y)\, f_{n}^{(M)}(x,y)\Delta^m\right) \\
&\geq \max\left(h_{n'}(x), \sum_{i\in I_{m+1}} \sup_{y\in C_i^{m+1}} \overline{V}_{n'+1}^{(m+1,M)}(y)\, f_{n'}^{(M)}(x,y)\Delta^{m+1}\right) \\
&= \overline{V}_{n'}^{(m+1,M)}(x)
\end{aligned} \tag{4.15}$$

The first inequality in (4.15) is ensured by the statement $\mathbf{S}_3(n'+1)$ and the second inequality in (4.15) is ensured by the fact that $\left\{C_i^{m+1}\right\}_{i\in I_{m+1}}$ is a refinement of $\{C_i^m\}_{i\in I_m}$. Now we have proved that $\mathbf{S}_3(n')$ is true.

Finally we show that $\mathbf{S}_4(n')$ is true. By the inequalities (4.14) and the assumption that $\mathbf{S}_4(n'+1)$ is true, for any $\epsilon > 0$, there exists $m^* \in \mathbb{N}$ such that for any $m > m^*$

$$0 \leq \sup_{x\in K^M}\left(\overline{V}_{n'+1}^{(m,M)}(x) - V_{n'+1}^{(M)}(x)\right) < \epsilon$$

and then we have

$$\sup_{x\in K^M} \overline{V}_{n'+1}^{(m,M)}(x) < \sup_{x\in K^M} V_{n'+1}^{(M)}(x) + \epsilon. \tag{4.16}$$

We also have

$$\lim_{m\to\infty}\sum_{i\in I_m}\sup_{y\in C_i^m} V_{n'+1}^{(M)}(y)\, f_{n'}^{(M)}(x,y)\Delta^m = \int_{K^M} V_{n+1}^{(M)}(y)\, f_n^{(M)}(x,y)\ \mathrm{d}y \tag{4.17}$$

and

$$\lim_{m\to\infty}\sum_{i\in I_m}\sup_{y\in C_i^m} f_{n'}^{(M)}(x,y)\Delta^m = \int_{K^M} f_n^{(M)}(x,y)\ \mathrm{d}y \le 1. \tag{4.18}$$

Hence for any $m > m^*$ and $x \in K^M$ we have

$$\begin{aligned} &\left|\overline{V}_{n'}^{(m,M)}(x) - V_{n'}^{(M)}(x)\right| \\ &\le \sum_{i\in I_m}\sup_{y\in C_i^m} \overline{V}_{n'+1}^{(m,M)}(y)\, f_n^{(M)}(x,y)\Delta^m - \int_{K^M} V_{n'+1}^{(M)}(y)\, f_n^{(M)}(x,y)\ \mathrm{d}y \\ &\le \sum_{i\in I_m}\sup_{y\in C_i^m}\left(V_{n'+1}^{(M)}(y)+\epsilon\right) f_{n'}^{(M)}(x,y)\Delta^m - \int_{K^M} V_{n'+1}^{(M)}(y)\, f_n^{(M)}(x,y)\ \mathrm{d}y. \end{aligned} \tag{4.19}$$

The first inequality in (4.19) is ensured by the fact that $|\max(x_1,x_2) - \max(x_1,x_3)| \le |x_2 - x_3|$ for any x_1, x_2 and $x_3 \in \mathbb{R}$ and the inequalities (4.14). The second inequality in (4.19) is ensured by the inequality (4.16). Hence by (4.17), (4.18) and (4.19) we have

$$\lim_{n'\to\infty}\left|\overline{V}_{n'}^{(m,M)}(x) - V_{n'}^{(M)}(x)\right| \le \epsilon$$

for any $\epsilon > 0$ and $x \in K^M$.

Now by Dini's theorem, $\left\{\overline{V}_{n'}^{(m,M)}(x)\right\}_{m\in\mathbb{N}}$ converges uniformly to $V_{n'}^{(M)}(x)$. Then $\mathbf{S}_4(n')$ is true.

Since the inductive step has been proved, it has now been proved by induction that $\mathbf{S}_1(n)$, $\mathbf{S}_2(n)$, $\mathbf{S}_3(n)$ and $\mathbf{S}_4(n)$ are true for any $n \in \{0,\ldots,N\}$.

Using the method which proved that $\mathbf{S}_4(n)$ is true for any $n \in \{0,\ldots,N\}$, we can show that the following statement:

$$\mathbf{S}_4'(n):\ \lim_{m\to\infty}\sup_{x\in K^M}\left|\underline{V}_n^{(m,M)}(x) - V_n^{(M)}(x)\right| = 0$$

holds for any $n \in \{0,\ldots,N\}$.

Then we have

$$\lim_{m\to\infty}\sup_{x\in K^M}\left|V_n^{(m,M)}(x) - V_n^{(M)}(x)\right| = 0$$

for any $n \in \{0,\ldots,N\}$. □

Now we prove Theorem 4.1.

Proof of Theorem 4.1. By the condition (2.2) and the definition (4.6)

$$\lim_{M\to\infty} X^{(M)}(t) = X(t) \tag{4.20}$$

holds for $t \in [0,T]$.

We define functions $\widetilde{V}_n^{(M)} : \mathbb{R}^D \to \mathbb{R}$ and $\tilde{f}_n^{(M)} : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$ by

$$\widetilde{V}_n^{(M)}(x) = \begin{cases} V_n^{(M)}(x) & \text{if } x \in K^M \\ 0 & \text{if } x \notin K^M \end{cases}$$

for $x \in \mathbb{R}^D$, and

$$\tilde{f}_n^{(M)}(x,y) = \begin{cases} f_n^{(M)}(x,y) & \text{if } (x,y) \in K^M \times K^M \\ 0 & \text{if } (x,y) \notin K^M \times K^M \end{cases}$$

for $(x,y) \in \mathbb{R}^D \times \mathbb{R}^D$, respectively.

We also define statements $\mathbf{S}_1(n)$, $\mathbf{S}_2(n)$ and $\mathbf{S}_3(n)$ by

$\mathbf{S}_1(n)$: $\widetilde{V}_n^{(M)}$ is bounded on $\mathbb{R}^D$,

$\mathbf{S}_2(n)$: V_n is bounded on $\mathbb{R}^D$,

$\mathbf{S}_3(n)$: $\lim_{M\to\infty} \left|\widetilde{V}_n^{(M)}(x) - V_n(x)\right| = 0$ for any $x \in \mathbb{R}^D$.

We prove $\mathbf{S}_1(n)$, $\mathbf{S}_2(n)$ and $\mathbf{S}_3(n)$ are true for $n \in \{0,\ldots,N\}$ by induction, proceeding backwards from $n = N$ to $n = 0$. We note that for any $x \in \mathbb{R}^D$, there exists $M^* \in \mathbb{N}$ such that $x \in K^{M^*}$. By the definitions of $\widetilde{V}_N^{(M)}$ and V_N, and the equation (4.20), $\mathbf{S}_1(N)$, $\mathbf{S}_2(N)$ and $\mathbf{S}_3(N)$ are true.

Next we suppose that $\mathbf{S}_1(n'+1)$, $\mathbf{S}_2(n'+1)$ and $\mathbf{S}_3(n'+1)$ are true for some $n' \in \{0,\ldots,N-1\}$. By the assumptions that $h_{n'}$ is bounded on $\mathbb{R}^D$ and the definitions of $\widetilde{V}_N^{(M)}$ and V_N, $\mathbf{S}_1(n')$ and $\mathbf{S}_2(n')$ are true.

Finally, we prove that $\mathbf{S}_3(n')$ is true. By the definitions of $\widetilde{V}_{n'}^{(M)}$ and $V_{n'}$, we have

$$\begin{aligned}
&\lim_{M\to\infty} \left|\widetilde{V}_{n'}^{(M)}(x) - V_{n'}(x)\right| \\
&\le \lim_{M\to\infty} \left| \int_{\mathbb{R}^D} \widetilde{V}_{n'+1}^{(M)}(y)\, \tilde{f}_{n'}^{(M)}(x,y)\, \mathrm{d}y - \mathbb{E}\left[V_{n'+1}(X_{n'+1}) \middle| X_{n'} = x\right] \right| \\
&\le \lim_{M\to\infty} \int_{\mathbb{R}^D} \left| \widetilde{V}_{n'+1}^{(M)}(y)\, \tilde{f}_{n'}^{(M)}(x,y) - V_{n'+1}(y)\, f_n(x,y) \right| \mathrm{d}y \\
&= \int_{\mathbb{R}^D} \lim_{M\to\infty} \left| \widetilde{V}_{n'+1}^{(M)}(y)\, \tilde{f}_{n'}^{(M)}(x,y) - V_{n'+1}(y)\, f_n(x,y) \right| \mathrm{d}y
\end{aligned} \tag{4.21}$$

The last equality in (4.21) is ensured by Lebesgue's dominated convergence theorem. By the definition of $\tilde{f}_{n'}^{(M)}$, the assumption $\mathbf{S}_3\left(n'+1\right)$ are true, and the equation (4.20), we have

$$\begin{aligned}
&\lim_{M\to\infty}\left|\widetilde{V}_{n'+1}^{(M)}\left(y\right)\tilde{f}_{n'}^{(M)}\left(x,y\right)-V_{n'+1}\left(y\right)f_n\left(x,y\right)\right| \\
\leq &\lim_{M\to\infty}\left|\widetilde{V}_{n'+1}^{(M)}\left(y\right)-V_{n'+1}\left(y\right)\right|\tilde{f}_{n'}^{(M)}\left(x,y\right) \\
&+\lim_{M\to\infty}V_{n'+1}\left(y\right)\left|\tilde{f}_{n'}^{(M)}\left(x,y\right)-f_n\left(x,y\right)\right| \\
=&0
\end{aligned}$$

Thus $\mathbf{S}_1\left(n\right)$, $\mathbf{S}_2\left(n\right)$ and $\mathbf{S}_3\left(n\right)$ are true for $n\in\{0,\ldots,N\}$.

Now we prove the equation (4.9). By Lemma 4.1 and the fact that $\mathbf{S}_3\left(n\right)$ is true, we have

$$\begin{aligned}
&\lim_{M\to\infty}\lim_{m\to\infty}\left|V_n^{(m,M)}\left(x\right)-V_n\left(x\right)\right| \\
&\leq\lim_{M\to\infty}\lim_{m\to\infty}\left(\left|V_n^{(m,M)}\left(x\right)-V_n^{(M)}\left(x\right)\right|+\left|V_n^{(M)}\left(x\right)-\widetilde{V}_n^{(M)}\left(x\right)\right|+\left|\widetilde{V}_n^{(M)}\left(x\right)-V_n\left(x\right)\right|\right) \\
&=0.
\end{aligned}$$

□

By Theorem 4.1, $V_n^{(m,M)}$ converges to V_n when $m\to\infty$ and $M\to\infty$. So we calculate $V_n^{(m,M)}$ instead of V_n in the evaluations of the American options.

4.3.2 The mesh method with simulation

In the method in the previous chapter, the computational effort in the recursive pricing of the equation (4.24) is proportional to $m^{2D}\times N$. Here m means the number of nodes at each time and N means the number of divisions for time. To reduce the computing costs, we introduce the combination of the mesh method and Monte Carlo or quasi-Monte Carlo simulation.

As discussed in the subsection 4.3.1, we can construct a mesh structure that describes the optimal exercise boundary. Let $\tau^{(m,M)}$ denote the $\{\mathcal{F}_t\}_{0\leq t\leq T}$-stopping time that is defined by the mesh:

$$\widehat{V}^{(m,M)}\left(t_n,x\right)=\begin{cases}0 & \text{if } n=N\\ \displaystyle\sum_{i\in I_m}V^{(m,M)}\left(t_{n+1},y_i\right)f_n(x,y_i)\Delta^m & \text{if } n=0,\ldots,N-1\end{cases} \tag{4.22}$$

and

$$\widehat{V}\left(t_n, x\right) = \begin{cases} 0 & \text{if } n = N \\ \mathbb{E}\left[V\left(t_n, X_{n+1}\right)|X_n = x\right] & \text{if } n = 0, \ldots, N-1. \end{cases}$$

Now $\tau^{(m,M)}$ is given by

$$\tau^{(m,M)} = \min\left(t : h\left(t, X_n\right) \geq \widehat{V}^{(m,M)}\left(t, X_n\right), t \in \{t_1, t_2, \ldots, t_N\}\right).$$

We have

$$E\left[h\left(\tau^{(m,M)}, X\left(\tau^{(m,M)}\right)\right)\right] \leq \max_{\tau \in \mathcal{T}} \mathbb{E}\left[h\left(\tau, X\left(\tau\right)\right)\right]$$

for any $m \in \mathbb{N}$.

The convergence of $E\left[h\left(\tau^{(m,M)}, X\left(\tau^{(m,M)}\right)\right)\right]$ is given by the following theorem.

Theorem 4.2. *Suppose that the conditions are the same as in Theorem 4.1 and in addition*

$$P\left(h\left(t, X_t\right) = \widehat{V}\left(t, X_t\right)\right) = 0.$$

Then

$$\lim_{M\to\infty} \lim_{m\to\infty} E\left[h\left(\tau^{(m,M)}, X\left(\tau^{(m,M)}\right)\right)\right] = E\left[h\left(\tau, X\left(\tau\right)\right)\right].$$

Proof. First we will show that $\lim_{M\to\infty} \lim_{m\to\infty} P\left(\tau^{(m,M)} \neq \tau\right) = 0$. There exists $U \in \mathbb{R}$ such that $\left|\widehat{V}^{(m,M)}\left(t_n, x\right) - \widehat{V}\left(t_n, x\right)\right| < U$ for any $x \in K$, because $\widehat{V}\left(t_n, \cdot\right)$ and $\widehat{V}^{(m,M)}\left(t_n, \cdot\right)$ are continuous on K^M by Theorem 4.1. By Lebesgue's dominated convergence theorem,

$$\lim_{M\to\infty} \lim_{m\to\infty} \mathbb{E}\left[\left|\widehat{V}^{(m,M)}\left(t_n, X_n\right) - \widehat{V}\left(t_n, X_n\right)\right|\right] = 0.$$

The above expression implies that for any $\epsilon > 0$

$$\lim_{M\to\infty} \lim_{m\to\infty} P\left(\left|\widehat{V}^{(m,M)}\left(t_n, X_n\right) - \widehat{V}\left(t_n, X_n\right)\right| > \epsilon\right) = 0. \qquad (4.23)$$

Hence $\tau^{(m,M)} \neq \tau$ means that $\tau^{(m,M)}$ stops before τ or $\tau^{(m,M)}$ does not stop at time τ. Thus

$$\begin{aligned}
P\left(\tau^{(m,M)} \neq \tau\right) \leq & \sum_{n=0}^{N-1} P\left(\widehat{V}^{(m,M)}\left(t_n, X_n\right) \leq h\left(n, X_n\right) < \widehat{V}\left(t_n, X_n\right)\right) \\
& + P\left(h\left(\tau, X_\tau\right) < \widehat{V}^{(m,M)}\left(\tau, X_\tau\right)\right) \\
\leq & \sum_{n=0}^{N-1} P\left(\widehat{V}^{(m,M)}\left(t_n, X_n\right) \leq h\left(t_n, X_n\right) < \widehat{V}\left(t_n, X_n\right)\right) \\
& + \sum_{n=0}^{N-1} P\left(\widehat{V}\left(t_n, X_n\right) \leq h\left(t_n, X_n\right) < \widehat{V}^{(m,M)}\left(t_n, X_n\right)\right)
\end{aligned}$$

holds. Under the assumptions in this theorem, there exists $\epsilon > 0$ such that

$$\begin{aligned}
P\left(\tau^{(m,M)} \neq \tau\right) & \leq 2 \sum_{n=0}^{N-1} P\left(\left|\widehat{V}^{(m,M)}\left(t_n, X_n\right) - \widehat{V}\left(t_n, X_n\right)\right| > \epsilon\right) \\
& \to 0
\end{aligned}$$

when $m \to \infty$ and $M \to \infty$. The inequality is ensured by the equation (4.23).

Now we can prove the statement in Theorem 4.2 as follows:

$$\begin{aligned}
0 & \leq \mathbb{E}\left[h\left(\tau, X\left(\tau\right)\right)\right] - \mathbb{E}\left[h\left(\tau^{(m,M)}, X\left(\tau^{(m,M)}\right)\right)\right] \\
& = \mathbb{E}\left[h\left(\tau, X\left(\tau\right)\right) - h\left(\tau^{(m,M)}, X\left(\tau^{(m,M)}\right)\right)\right] \\
& = \mathbb{E}\left[h\left(\tau, X\left(\tau\right)\right) - h\left(\tau^{(m,M)}, X\left(\tau^{(m,M)}\right)\right) ; \tau^{(m,M)} \neq \tau\right] \\
& \leq \mathbb{E}\left[1_{\tau^{(m,M)} \neq \tau} h\left(\tau, X\left(\tau\right)\right)\right] \\
& \leq \mathbb{E}\left[1_{\left\{\tau^{(m,M)} \neq \tau\right\}}\right]^{\frac{1}{2}} \mathbb{E}\left[h\left(\tau, X\left(\tau\right)\right)^2\right]^{\frac{1}{2}} \\
& \leq P\left(\tau^{(m,M)} \neq \tau\right)^{\frac{1}{2}} \mathbb{E}\left[h\left(\tau, X\left(\tau\right)\right)^2\right]^{\frac{1}{2}} \\
& \to 0
\end{aligned}$$

when $m \to \infty$ and $M \to \infty$. The third inequality is ensured by Hölder's inequality. □

4.3.3 Multidimensional asymptotic expansion mesh method

When $f_n\left(x, \cdot\right)$ is known explicitly for any $n \in \{0, 1, \ldots, N\}$ and $x \in \mathbb{R}^D$, we can calculate the value of $V_n^{(m,M)}\left(x\right)$. This condition is not necessarily satisfied in many practical problems. So we use MAE to obtain an approximation of the law of X_{n+1} given $X_n = x$. By using this approximation, we can approximate $V_n^{(m,M)}$ defined by the equation (4.8) in the previous section. The lth order approximation of $V_n^{(m,M)}$ is given by

$$V_n^{(m,M)(l)}(x) = \begin{cases} h_n(x) & \text{if } n = N \\ \max\left(h_n(x), \sum_{i \in I_m} V_{n+1}^{(m,M)(l)}(y_i)\, f_n^{(M)(l)}(x, y_i)\Delta^m \right) & \text{if } n < N \end{cases} \tag{4.24}$$

where $f_n^{(M)(l)}$ is the lth order approximation of $f_n^{(M)}$. We call the method that is the combination of MAE and the mesh method the multidimensional asymptotic expansion mesh method (MAEM). We also call the combination of MAE and the mesh with simulation the multidimensional asymptotic expansion mesh method with simulation (MAEMS).

In this thesis, the combination of the mesh method and MAE is proposed, but the mesh method is available if the law of the diffusion process or its approximation are given.

4.3.4 Practical remark

To reduce the computational cost of the method described in the subsections 4.3.1 and 4.3.2, we provide some approximations.

First we use a stopping time τ'^M instead of τ^M defined by (4.6). The stopping time τ'^M is defined by

$$X_k^{'(M)}(t_n) = \max\left(\min\left(X(t), -a_k(n)M\right), b_k(n)M\right) \tag{4.25}$$

for $k \in \{1, \ldots, D\}$, $M \in \mathbb{N}$, and $a_k(n)$ and $b_k(n) \in \mathbb{R}_+$.

Second we construct the mesh with the number of time steps $N' \in \mathbb{N}$ that satisfies $N' \leq N$ and approximate $\widehat{V}^{(m,M)}$ defined by (4.22) by

$$\widehat{V'}^{(m,M)}(t_n, x) = \frac{\widehat{V}^{(m,M)}(t_{n'+1}, x) - \widehat{V}^{(m,M)}(t_{n'}, x)}{t_{n'+1} - t_{n'}}(t_n - t_{n'}) + \widehat{V}^{(m,M)}(t_{n'}, x) \tag{4.26}$$

for all $n \in \{0, \ldots, N-1\}$ and $x \in \mathbb{R}^D$ where $n' \in \{0, \ldots, N'-1\}$ that satisfies $t_{n'} \leq t_n < t_{n'+1}$.

Third to reduce the error of numerical integrations involved in (4.24), we use

$$V_n^{\prime(m,M)(l)}(x) = \begin{cases} h_n(x) & \text{if } n = N \\ \max\left(h_n(x), \sum_{i \in I_m} V_{n+1}^{\prime(m,M)(l)}(y_i) \dfrac{f_n^{(M)(l)}(x, y_i)\Delta^m}{\sum_{i \in I_m} f_n^{(M)(l)}(x, y_i)\Delta^m}\right) & \text{if } n < N \end{cases} \tag{4.27}$$

instead.

Fourth we use the number of division $m'(n)$ at time t_n for $n \in \{1, \ldots, N\}$ instead of m. $m'(n)$ is given by

$$m'(n) = \left\lfloor \frac{m_N - m_1}{t_N - t_1}(t_n - t_1) + m_1 \right\rfloor \tag{4.28}$$

for m_1 and $m_N \in \mathbb{N}$. Here $\lfloor x \rfloor$ means the largest integer not greater than x for $x \in \mathbb{R}$.

4.3.5 Numerical examples

We calculate the prices of the American put options in the Heston model by using MAEM and MAEMS, and test the practical feasibility of the methods. We use quasi-Monte Carlo method in MAEMS in the following experiments.

The Heston model is given by

$$\mathrm{d}S(t) = rS(t)\,\mathrm{d}t + \sqrt{V_t}S(t)\,\mathrm{d}W^1(t),$$

$$\mathrm{d}V(t) = \kappa(\theta - V(t))\,\mathrm{d}t + \omega\sqrt{V(t)}\,\mathrm{d}W^2(t)$$

where $\mathrm{d}\left\langle W^1, W^2 \right\rangle_t = \rho\,\mathrm{d}t$, $S(0) = S_0$ and $V(0) = V_0$. We then have

$$S(t) = S_0 \exp(rt)\exp(Z(t))$$

where

$$Z(t) = -\int_0^t \frac{V(u)}{2}\,\mathrm{d}u + \int_0^t \sqrt{V(u)}\,\mathrm{d}W^1(u).$$

We consider the following perturbed processes of $V(t)$ and $Z(t)$:

$$\mathrm{d}V^\epsilon(t) = \kappa(\theta - V^\epsilon(t))\,\mathrm{d}t + \epsilon\omega\sqrt{V^\epsilon(t)}\,\mathrm{d}W^2(t),$$

$$V_0^\epsilon = V_0$$

and

$$\mathrm{d}Z^\epsilon(t) = -\frac{V^\epsilon(t)}{2}\,\mathrm{d}t + \epsilon\sqrt{V^\epsilon(t)}\,\mathrm{d}W^1(t),$$

$$Z_0^\epsilon = 0$$

for $\epsilon \in (0,1]$. By applying the multidimensional asymptotic expansion method to V^ϵ and Z^ϵ, we can obtain an nth order approximation of the law of $V(t)$ and $Z(t)$ that is denoted by $f_{V,Z}^{(n)}$. An nth order approximation of the law of $S(t)$ and $V(t)$ can be obtained by

$$f_{S,V}^{(n)}(s,v) = f_{V,Z}^{(n)}(v, \log(s/S_0) - rt)/s.$$

We use the same parameters as in earlier papers that introduce the methods to evaluate the American put options in the Heston model [2, 6, 16, 28, 30]. The relevant parameter values taken in all those papers are:

$$\kappa = 5.0, \quad \theta = 0.16, \quad \omega = 0.9, \quad \rho = 0.1, \quad r = 0.1.$$

We set the strike $K = 10$, the initial volatility $V_0 = (0.25)^2$ and the time to maturity $T = 0.25$ that means three months. In numerical results, the different values of the initial stock price S_0 are considered.

Table 4.7: European put option prices and implied volatilities that are calculated with quasi-Monte Carlo simulation and the exact formula.

		S_0				
		8	9	10	11	12
QMC	Price	1.83879	1.04822	0.50142	0.20820	0.08044
	IV	0.50535	0.42359	0.38901	0.37606	0.37338
Exact	Price	1.83886	1.04834	0.50146	0.20818	0.08042
	IV	0.50545	0.42367	0.38903	0.37604	0.37335
	Error(bps)	-0.9	-0.9	-0.2	0.2	0.4

Table 4.7 gives the European option prices that are calculated with the exact formula and quasi-Monte Carlo simulation with the number of paths 10^8 and the number of time steps 512. The table shows that the prices that are calculated with the quasi-Monte Carlo simulation are accurate enough. We use same paths for the least squares Monte Carlo simulation and MAS.

Table 4.8 gives benchmarks for the American options. In this table, CP, IT, OO, VN, ZEV and LSM indicate [2], [6], [16], [28], [30] and [10], respectively. In the least square Monte Carlo simulation, we use the polynomials of S and V of degrees up to 2, 3, 4, 5 and 6 as base functions. We then use the maximum of the values that are calculated with these base functions as the prices of the American put options. IV indicates the implied volatility that is calculated with the Black-Scholes formula for the European put option.

Table 4.8: Benchmarks.

		S_0				
	Method	8	9	10	11	12
Price	VN	1.9968	1.1076	0.5202	0.2134	0.0815
	ZFV	2.0000	1.1076	0.5202	0.2134	0.0821
	IT	2.0000	1.1074	0.5190	0.2130	0.0818
	OO	2.0000	1.1070	0.5170	0.2120	0.0815
	CP	2.0000	1.1080	0.5316	0.2261	0.0907
	LSM	1.9995	1.1070	0.5192	0.2133	0.0817
IV	VN	0.74068	0.46760	0.40056	0.38004	0.37603
	ZFV	0.74638	0.46760	0.40056	0.38004	0.37672
	IT	0.74638	0.46745	0.39982	0.37973	0.37637
	OO	0.74638	0.46715	0.39858	0.37897	0.37603
	CP	0.74638	0.46790	0.40763	0.38970	0.38706
	LSM	0.74553	0.46712	0.39997	0.37994	0.37623

In MAEM and MAS, we use the equations (4.25), (4.26) and (4.27), and we set the parameters $a_1(n)$, $a_2(n)$, $b_1(n)$, $b_2(n)$, and M to satisfy the following equations:

$$a_1(n)M = b_1(n)M = 5\mathbb{E}\left[S\left(t_n\right)^2\right]^{1/2}$$

and

$$a_2(n)M = b_2(n)M = 5\mathbb{E}\left[V\left(t_n\right)^2\right]^{1/2}$$

for any $n \in \{1, \ldots, N'\}$ where $X_1 = S$ and $X_2 = V$.

We use the values in Table 4.9 for N' and m that indicate the number of time steps and the number of division for V and Z.

Table 4.9: Parameters for MAEM and MAEMS.

Case	(i)	(ii)	(iii)	(iv)
N'	64	64	96	96
m	96	144	96	144

Tables 4.10 and 4.11 give the American put option prices and implied volatilities that are calculated with MAEM and MAEMS, respectively. In these tables, Error indicates the difference between the implied volatilities that are calculated with case (iv) and those with other methods. These

tables show that the values calculated with MAEMS are higher than the values calculated with MAEM except two cases. Table 4.11 shows that the values calculated with MAEMS are higher than the values calculated with LSM except in the case of $S_0 = 8$. So we can say that MAEMS is more accurate than MAEM and LSM in this test case because we use the same paths for MAEMS and LSM, and a good stopping time gives a higher price. Compared with the other methods, MAEMS is accurate enough in this test case.

Table 4.10: American put option prices and implied volatilities that are calculated with "the multidimensional asymptotic expansion mesh method" and their errors.

		S_0				
	Case	8	9	10	11	12
Price	(i)	1.99670	1.10630	0.51894	0.21325	0.08175
	(ii)	1.99670	1.10665	0.51899	0.21295	0.08161
	(iii)	1.99771	1.10584	0.51933	0.21413	0.08220
	(iv)	1.99772	1.10683	0.51932	0.21330	0.08177
IV	(i)	0.74050	0.46662	0.39978	0.37992	0.37632
	(ii)	0.74051	0.46688	0.39981	0.37969	0.37615
	(iii)	0.74229	0.46627	0.40002	0.38059	0.37684
	(iv)	0.74231	0.46702	0.40002	0.37996	0.37634
Error	VN	16.3	-5.8	-5.4	-0.8	3.1
	ZFV	-40.7	-5.8	-5.4	-0.8	-3.8
	IT	-40.7	-4.3	2.0	2.3	-0.3
	OO	-40.7	-1.3	14.3	9.9	3.1
	CP	-40.7	-8.9	-76.1	-97.4	-107.1
	LSM	-32.3	-1.0	0.5	0.1	1.1

The amount of computation time required to construct the mesh is summarized in Table 4.12. The CPU used in this experiment is Intel Core i7 by Intel Corporation. This table shows that this method has the limitation in computation time. To conquer this limitation, we use $m'(n)$ in the equation (4.28).

In the second test, we use parameters in Table 4.13. Table 4.14 gives the prices that are calculated with MAEMS and LSM, and their computation time. We generate 2^{17} paths to calculate the exercising boundary with 64 time steps in LSM. In this table Error means the difference between the

Table 4.11: American put option prices and implied volatilities that are calculated with “the multidimensional asymptotic expansion mesh method with simulation” and their errors.

		S_0				
	Case	8	9	10	11	12
Price	(i)	1.99951	1.10737	0.51979	0.21362	0.08204
	(ii)	1.99951	1.10739	0.51990	0.21374	0.08207
	(iii)	1.99951	1.10745	0.51934	0.21340	0.08199
	(iv)	1.99951	1.10750	0.51992	0.21369	0.08206
IV	(i)	0.74550	0.46743	0.40031	0.38020	0.37665
	(ii)	0.74550	0.46744	0.40037	0.38029	0.37669
	(iii)	0.74550	0.46749	0.40003	0.38003	0.37659
	(iv)	0.74550	0.46752	0.40038	0.38026	0.37668
Error	VN	48.3	-0.8	-1.7	2.2	6.5
	ZFV	-8.7	-0.8	-1.7	2.2	-0.4
	IT	-8.7	0.7	5.7	5.3	3.1
	OO	-8.7	3.8	18.0	12.9	6.5
	CP	-8.7	-3.8	-72.4	-94.4	-103.8
	LSM	-0.3	4.1	4.1	3.2	4.5

Table 4.12: CPU time required to construct the mesh.

Case	(i)	(ii)	(iii)	(iv)
CPU time (s)	292	385	1460	1919

implied volatilities in these test cases and that of the case (iv) in Table 4.11. This table shows that the method with $m'(n)$ calculates the mesh much faster than the method with m.

4.4 Evaluating the CVA of interest rate swaps in the SABR model

4.4.1 The problem

We consider the CVA of interest rate swaps. Let $0 = T_0 < T_1 < \cdots < T_N$ for $N \in \mathbb{N}$ where $T_{n+1} - T_n = \delta$ for $n \in \mathbb{N}$ and $\delta \in \mathbb{R}_+$. Let $P(t, T_i)$ denote the price of the discount bound with maturity T_i at time t for $i \in \{1, \ldots, N\}$.

Table 4.13: Parameters for MAEMS.

Case	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	(xii)	(xiii)
N'	32	32	32	64	64	64	96	96	96
m_1	32	32	32	32	32	32	32	32	32
m_N	96	128	160	96	128	160	96	128	160

Table 4.14: American put option prices and implied volatilities that are calculated with “the multidimensional asymptotic expansion mesh method with simulation” , their errors and the computation time.

Case	Price	IV	Error(bps)	CPU time(s)
(v)	0.51980	0.40031	-0.7	43
(vi)	0.51976	0.40029	-1.0	117
(vii)	0.51978	0.40030	-0.9	263
(viii)	0.51983	0.40033	-0.5	68
(ix)	0.51993	0.40039	0.1	185
(x)	0.51992	0.40038	0.0	415
(xi)	0.51886	0.39973	-6.6	90
(xii)	0.51991	0.40038	0.0	242
(xiii)	0.51996	0.40041	0.3	546
LSM	0.51915	0.39991	-4.7	13

$P(t, T_i)$ satisfies that $P(T_i, T_i) = 1$ and $P(t < T_i)$ for $t > T_i$. The forward LIBOR rate with respect to the term $[T_{i-1}, T_i]$ is

$$F(t, T_{i-1}, T_i) = \frac{1}{\delta}\left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right)$$

for $t \in [0, T_{i-1}]$ and $i = 1, \ldots, N$. The forward swap rate with respect to the term $[T_a, T_b]$ at time $t \leq T_a$ is

$$S_{(a,b)}(t) = \frac{P(t, T_a) - P(t, T_b)}{\delta \sum\limits_{k=a+1}^{b} P(t, T_k)}.$$

Define the positive $\mathbb{Q}^{T_a}$-martingale

$$D_{(a,b)}(t) = \sum_{k=a+1}^{b} \frac{P(t, T_k)}{P(t, T_a)}$$

for $t \in [0, T_a]$. Here $\mathbb{Q}^{T_a}$ means T_a forward measure.

We consider the price of interest rate swaps with nominal 1, strike rate K, first reset date T_a and last reset date T_{b-1}. The price at time t is give by

$$\begin{aligned} V_{(a,b)}(t) &= \delta P(0,T_a)\,\mathbb{E}^{T_a}\left[D_{(a,b)}(T_a)\left(S_{(a,b)}(T_a)-K\right)\middle|\mathcal{F}_t\right] \\ &= \delta P(t,T_a)\,D_{(a,b)}(t)\,\mathbb{E}_{(a,b)}\left[S_{(a,b)}(T_a)-K\middle|\mathcal{F}_t\right] \\ &= \delta P(t,T_a)\,D_{(a,b)}(t)\left(S_{(a,b)}(t)-K\right) \end{aligned}$$

where $\mathbb{E}_{(a,b)}[\cdot]$ means the expectation under the forward swap measure $\mathbb{Q}_{(a,b)}$.

The price of a payer swaption with maturity $T \leq T_a$ and underlying $V_{(a,b)}$ at time t is given by

$$\widehat{V}_{(a,b)}(t,T) = \delta P(t,T)\,D_{(a,b)}(t)\,\mathbb{E}_{(a,b)}\left[\left(S_{(a,b)}(T)-K\right)^+\middle|\mathcal{F}_t\right]$$

The definition of CVA is given by the following.

Definition 4.1. *The CVA of a derivative whose price at time t is $V(t)$ and maturity is T is defined as follows:*

$$\mathrm{CVA} = (1-R)\mathbb{E}^{\mathbb{Q}}\left[\int_0^T B(t)^{-1}\max\left(V(t),0\right)\,\mathrm{dDP}(t)\right]$$

where $R \in [0,1]$ and $\mathrm{DP}(t)$ denote the recovery rate and the default probability of a counterparty, respectively. Here $B(t)$ means the price of zero coupon bond with maturity t.

We suppose that the default probability of a counterparty is given by

$$\mathrm{DP}(t) = 1 - e^{-\int_0^t \lambda\,\mathrm{d}t} = 1 - e^{-\lambda t}$$

where $\lambda \in \mathbb{R}_+$ is a hazard rate. Then by Definition 4.1, the CVA of an interest rate swap is given by

$$\begin{aligned} \mathrm{CVA} =&(1-R)\mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_b} B(t)^{-1}\max\left(V_{(a,b)}(t),0\right)\,\mathrm{dDP}(t)\right] \\ =&(1-R)\int_0^{T_b}\mathbb{E}^{\mathbb{Q}}\left[B(t)^{-1}\max\left(V_{(a,b)}(t),0\right)\right]\lambda e^{-\lambda t}\,\mathrm{d}t \\ =&(1-R)\left\{\int_0^{T_a}\mathbb{E}^{\mathbb{Q}}\left[B(t)^{-1}\max\left(V_{(a,b)}(t),0\right)\right]\lambda e^{-\lambda t}\,\mathrm{d}t\right. \\ &\left.+\sum_{i=a}^{b-1}\int_{T_i}^{T_{i+1}}\mathbb{E}^{\mathbb{Q}}\left[B(t)^{-1}\max\left(V_{(i+1,b)}(t),0\right)\right]\lambda e^{-\lambda t}\,\mathrm{d}t\right\} \\ =&(1-R)\left\{\int_0^{T_a}\widehat{V}_{(a,b)}(0,t)\,\lambda e^{-\lambda t}\,\mathrm{d}t + \sum_{i=a}^{b-1}\int_{T_i}^{T_{i+1}}\widehat{V}_{(i+1,b)}(0,t)\,\lambda e^{-\lambda t}\,\mathrm{d}t\right\} \end{aligned} \tag{4.29}$$

We suppose that a swap rate $S_{(a,b)}$ satisfies the following stochastic differential equation:

$$\begin{aligned}\mathrm{d}S_{(a,b)}(t) &= \left(S_{(a,b)}(t)\right)^{B}\sigma_{(a,b)}(t)\,\mathrm{d}W^{1}_{(a,b)}(t),\\ S_{(a,b)}(0) &= S^{0}_{(a,b)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\sigma_{(a,b)}(t) &= v\sigma_{(a,b)}(t)\,\mathrm{d}W^{2}_{(a,b)}(t),\\ \sigma_{(a,b)}(0) &= \sigma^{0}_{(a,b)}\end{aligned}$$

where (W_1, W_2) is a 2-dimensional standard Brownian motion with the instantaneous correlation $\rho \in [-1,1]$, Here v, $S^{0}_{(a,b)}$ and $\sigma^{0}_{(a,b)} \in \mathbb{R}_{+}$, and $B \in [0,1]$.

By the asymptotic expansion method, we obtain the nth order approximation of the law of $S_{(a,b)}(t)$ that is denoted by $f^{(n)}_{S_{(a,b)}(t)}$. Now we have the nth order approximation of the value $\widehat{V}_{(a,b)}(0,t)$ that is given by

$$\widehat{V}^{(n)}_{(a,b)}(0,t) = \delta P(0,T)\,D_{(a,b)}(0)\int_{K}^{\infty}(s-K)\,f^{(n)}_{S_{(a,b)}(t)}(s)\,\mathrm{d}s. \tag{4.30}$$

By the equations 4.29 and 4.30, the nth order approximation of the CVA is given by

$$\mathrm{CVA}^{(n)} = (1-R)\left\{\int_{0}^{T_a}\widehat{V}^{(n)}_{(a,b)}(0,t)\,\lambda e^{-\lambda t}\,\mathrm{d}t + \sum_{i=a}^{b-1}\int_{T_i}^{T_{i+1}}\widehat{V}^{(n)}_{(i+1,b)}(0,t)\,\lambda e^{-\lambda t}\,\mathrm{d}t\right\}. \tag{4.31}$$

4.4.2 Numerical results

We show the accuracy and the computational cost of the approximation in the equation 4.31. We use the following parameters:

$$B = 0.5, \delta = 0.5, a = 10, b = 20, K \in \{3.5\%, 4\%, 4.5\%\}, R = 0.$$

The other parameters used in this test case are given in Table 4.15. These parameters are calibrated from the market date in Table 4.16 using the method that is proposed in [18].

Table 4.15: Parameters.

first reset date	$S(0)$	$\sigma(0)$	v	ρ
5	0.0477	0.012	0.158	0.000145
5.5	0.0481	0.011	0.149	0.000409
6	0.0486	0.011	0.142	0.000741
6.5	0.049	0.011	0.135	0.000472
7	0.0495	0.011	0.13	0.000113
7.5	0.0498	0.011	0.125	-0.000121
8	0.0502	0.01	0.12	-0.000472
8.5	0.0504	0.01	0.116	-0.000181
9	0.0509	0.01	0.112	0.000401
9.5	0.0509	0.01	0.109	0.000401

Table 4.16: European cap prices (in basis points) on 18 November 2008.

Y(year)-K(%)	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
2.0	25.0	11.0	5.0	2.5	1.5	1.0	0.5	0.0	0.0
3.0	77.0	40.5	21.5	12.0	7.0	4.0	2.5	1.5	1.5
4.0	148.5	86.0	48.5	27.0	16.0	10.0	6.5	4.5	4.0
5.0	230.5	140.5	82.0	47.5	28.5	17.5	11.5	8.0	7.5
6.0	325.5	206.0	125.5	74.5	45.5	29.0	19.0	13.5	12.5
7.0	431.5	283.5	178.0	109.0	68.0	44.5	29.5	21.0	20.5
8.0	545.5	368.5	238.0	149.0	95.0	62.5	42.5	30.0	29.0
9.0	664.0	459.0	304.5	196.5	127.0	85.0	58.5	42.0	40.0
10.0	786.0	554.5	376.5	248.5	164.0	111.0	77.0	56.0	53.0

We consider the values calculated with Monte Carlo simulation as the true values of the CVA. We use Euler-Maruyama scheme as a discretization scheme. The number of partitions and the number of sample paths are 1000 time steps per year and 5×10^6 paths in each simulation. We consider the values calculated with these simulations as their limit values in the following experiments.

Table 4.17: Numerical results of the CVA of interest rate swaps.

fixed rate	MC	AE2	AE3	AE4	AE5
3.5%	0.018513	0.018500	0.018503	0.018503	0.018503
4.0%	0.011602	0.011557	0.011573	0.011572	0.011573
4.5%	0.005348	0.005317	0.005364	0.005363	0.005364

Table 4.18: Difference between MC and Asymptotic expansion

fixed rate	AE2-MC	AE3-MC	AE4-MC	AE5-MC
3.5%	-1.34797E-05	-1.03684E-05	-1.06949E-05	-1.00477E-05
4.0%	-4.41953E-05	-2.86798E-05	-2.98305E-05	-2.84797E-05
4.5%	-3.01402E-05	1.60005E-05	1.51823E-05	1.59793E-05

Table 4.19: CPU time required to calculate the approximation in the case of $K = 3.5\%$

Method	#Partition	#Sample	CPU time(sec)
AE2	-	-	0.000221
AE3	-	-	0.000417
AE4	-	-	0.000587
AE5	-	-	0.001039

The results in Table 4.17 and 4.18 show that the difference between the values calculated with approximation and Monte Carlo simulation are relatively small to nominal value 1. The amount of computational time required to calculate the approximation is summarized in Table 4.19. This table shows that the approximation can be calculated in about a millisecond. The CPU used in this experiment is Intel Core i7 by Intel Corporation.

Chapter 5

Conclusion

We have introduced MAE to derive an approximation of the law of a D-dimensional diffusion process $X(t)$. MAE is a method to combine the asymptotic expansion method and Theorem 2.1 in the section 2.3. Here the process $X(t)$ satisfies the equation (2.1), and we suppose that the condition (2.2) holds. We have also developed the package SymAE to automatically derive the approximation by using MAE. The package enables us to easily use MAE and to derive more higher order approximations and facilitates researches related to the asymptotic expansion method. The implementation of the scheme that is proposed in [25] to compute conditional expectations is an issue in the future.

By using the approximation of the law, we can derive the approximation of the value $\mathbb{E}\left[F\left(X_T\right)\right]$ where F is an $\mathbb{R}$-valued function that are defined on $\mathbb{R}^D$ and has polynomial growth.

As numerical examples, we have provided the numerical experiments: interest rate swaptions in the SABR/LIBOR market model, “call option on the maximum of two assets” in the CEV model, average strike options in the Black-Scholes model and the CVA of interest rate swaps in the SABR model These results show that MAE enables us to calculate the approximations within a second, and these approximations are accurate enough in practice.

We have also introduced MAEM and MAEMS to calculate the prices of the American options by using the combination of the MAE and backwards induction. In the mesh method, the accuracy can be improved if the way to determine the points of the mesh is appropriately modified. To find a better way to determine the points in the equation (4.8) is important, and this topic is reserved for future work. As a numerical example of these method, we have considered the American put options in the Heston model. The numerical results show that MAEMS is more accurate than the least square Monte Carlo method. However, MAEM and MAEMS have the limitation in computation time, and to apply these methods to higher dimensional models is practically difficult. To reduce the computation time is an issue in the future. As mentioned in the section 4.3, MAEM and MAEMS are based on the idea of the stochastic mesh method. We consider the combination of the stochastic mesh method and the asymptotic expansion method as an issue in the future.

As mentioned in the chapter 1, the calculation of the expectation $\mathbb{E}\left[F\left(X_T\right)\right]$ is important in practice. This thesis shows that accurate approximations of $\mathbb{E}\left[F\left(X_T\right)\right]$ can be derived by using the asymptotic expansion method in some practical problems, and the computation time of these approximations is fast enough in practice.

Appendix: Multivariate normal distribution

We describe some definitions and theorems that are related to the multivariate normal distribution.

Definition A.1. *The multivariate normal distribution of an n-dimensional random vector $X = \left(X^{(1)}, \ldots, X^{(n)}\right)$ can be written in the following notation:*

$$X = \left(X^{(1)}, \ldots, X^{(n)}\right) \sim \mathcal{N}\left(\mu, \Sigma\right)$$

with n-dimensional mean vector μ and $n \times n$ covariance matrix Σ. If Σ is non-degenerate then X has density

$$f\left(x|\mu, \Sigma\right) = \frac{1}{\left(2\,\pi\right)^{n/2}\left|\Sigma\right|^{1/2}} \exp\left(-\frac{1}{2}\left(x-\mu\right)\,\Sigma^{-1}\,{}^{t}\left(x-\mu\right)\right).$$

Theorem A.1. *Let $X = \left(X^{(1)}, \ldots, X^{(n)}\right) \sim \mathcal{N}\left(\mu, \Sigma\right)$. If X, μ and Σ are partitioned as follows:*

$$\begin{aligned}
X &= \left(X_1, X_2\right) \textit{ with size } \left(1 \times q, 1 \times \left(n-q\right)\right),\\
\mu &= \left(\mu_1, \mu_2\right) \textit{ with size } \left(1 \times q, 1 \times \left(n-q\right)\right),\\
\Sigma &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \textit{ with size } \begin{pmatrix} \left(q \times q\right) & q \times \left(n-q\right) \\ \left(n-q\right) \times q & \left(n-q\right) \times \left(n-q\right) \end{pmatrix}
\end{aligned}$$

for some counting number $q < n$, then the distributions of X_1 and X_2 are

$$X_1 \sim \mathcal{N}\left(\mu_1, \Sigma_{11}\right),$$

and

$$X_2 \sim \mathcal{N}\left(\mu_2, \Sigma_{22}\right),$$

respectively. Moreover, the conditional distribution of X_1 *given* X_2 *is*

$$X_1|X_2 = x_2 \sim \mathcal{N}\left(\tilde{\mu}, \widetilde{\Sigma}\right)$$

where

$$\begin{aligned}\tilde{\mu} &= \mu_1 + (x_2 - \mu_2)\,\Sigma_{22}^{-1}\,\Sigma_{21},\\ \widetilde{\Sigma} &= \Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}.\end{aligned}$$

Proof. Let ξ be partitioned as $\xi = (\xi_1, \xi_2)$ with size $(1 \times q, 1 \times (n-q))$. Then we have the moment generating function of X as follows:

$$\begin{aligned}m\,(\xi|\mu, \Sigma) &= m\,(\xi_1, \xi_2|\mu, \Sigma)\\ &= \exp\left((\mu_1, \mu_2)\ {}^t(\xi_1, \xi_2) + \frac{1}{2}\,(\xi_1, \xi_2)\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\ {}^t(\xi_1, \xi_2)\right).\end{aligned}$$

If let $\xi_2 = 0$, we have

$$m\,(\xi_1, 0|\mu, \Sigma) = \exp\left(\mu_1\ {}^t\xi_1 + \frac{1}{2}\xi_1\,\Sigma_{11}\ {}^t\xi_1\right).$$

Hence $X_1 \sim \mathcal{N}(\mu_1, \Sigma_{11})$. In the manner now described, we can show $X_2 \sim \mathcal{N}(\mu_2, \Sigma_{22})$.

Next, let f_{x_2} denote the conditional probability density function of X_1 given the value x_2 of X_2 and we have

$$\begin{aligned}f_{x_2}(x_1) &= \frac{f\,(x_1, x_2|\mu, \Sigma)}{f\,(x_2|\mu_2, \Sigma_{22})}\\ &= (2\,\pi)^{-n/2}\,|\Sigma|^{-1/2}\,|\Sigma_{22}|^{1/2}\exp\left(-\frac{Q - Q_2}{2}\right)\end{aligned}$$

where

$$\begin{aligned}Q &= (x-\mu)\,\Sigma^{-1\,t}(x-\mu),\\ Q_2 &= (x_2-\mu_2)\,\Sigma_{22}^{-1\,t}(x_2-\mu_2).\end{aligned}$$

We define C as

$$C = \begin{pmatrix}I_1 & -\Sigma_{12}\,\Sigma_{22}^{-1}\\ 0 & I_2\end{pmatrix}$$

where I_1 and I_2 are unit matrices of size q and $n-q$, respectively. Then Q is

$$\begin{aligned}Q &= (x-\mu)\ {}^tC\left(C\,\Sigma^{-1}\ {}^tC\right)C^{\,t}(x-\mu)\\ &= y_1\widetilde{\Sigma}^{-1\,t}y_1 + y_2\Sigma_{22}^{-1\,t}y_2\\ &= y_1\widetilde{\Sigma}^{-1\,t}y_1 + Q_2\end{aligned}$$

where

$$\begin{aligned}(y_1, y_2) &= (x_1 - \mu_1, x_2 - \mu_2)\ {}^tC \\ &= \big(x_1 - \mu_1 - (x_2 - \mu_2)\,\Sigma_{22}^{-1}\,\Sigma_{21}, x_2 - \mu_2\big) \\ &= (x_1 - \tilde{\mu}, x_2 - \mu_2)\,.\end{aligned}$$

And also we obtain

$$C\Sigma\,{}^tC = \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$$

and

$$|\Sigma| = \left|C\Sigma\,{}^tC\right| = \left|\widetilde{\Sigma}\right| |\Sigma_{22}|\,.$$

Thus

$$f_{x_2}(x_1) = (2\,\pi)^{-n/2} \left|\widetilde{\Sigma}\right|^{-1/2} \exp\left(-\frac{(x_1 - \mu_1)\,\widetilde{\Sigma}^{-1\,t}\,(x_1 - \mu_1)}{2}\right).$$

□

Corollary A.1.

$$\mathbb{E}\left[\prod_{i=1}^{q} X^{(i)}\middle| X_2 = x\right] = \left.\frac{\mathrm{d}^q\, m\left(\xi\middle|\tilde{\mu}, \widetilde{\Sigma}\right)}{\mathrm{d}\,\xi_1 \cdots \mathrm{d}\,\xi_q}\right|_{\xi=0}.$$

Moreover, if $\mathbb{E}\left[X^{(i)}\,X^{(j)}\right] = 0$ *for* $i \neq j$, $1 \leq i, j \leq q$

$$\mathbb{E}\left[\prod_{i=1}^{q} X^{(i)}\middle| X_2 = x\right] = \left.\frac{\mathrm{d}^q\, m\left(\xi|\tilde{\mu}, -\Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}\right)}{\mathrm{d}\,\xi_1 \cdots \mathrm{d}\,\xi_q}\right|_{\xi=0}.$$

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