論文 / 著書情報
Article / Book Information

題目(和文)	非線形写像に対する強エルゴート定理と収束定理
Title(English)	Strong ergodic theorems and convergence theorems for nonlinear mappings
著者(和文)	中條一秀
Author(English)	
出典(和文)	学位:博士(理学), 学位授与機関:東京工業大学, 報告番号:甲第5178号, 授与年月日:2002年3月26日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor (Science), Conferring organization: Tokyo Institute of Technology, Report number:甲第5178号, Conferred date:2002/3/26, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Strong Ergodic Theorems and Convergence Theorems for Nonlinear Mappings

Kazuhide Nakajo

Contents

Chapter 1

Introduction

Nonlinear functional analysis is an area of mathematics which has recently made growth. It is no longer a subsidiary of linear functional analysis. Of course, its applicable domain is far and away wider than that of the linear case because most of natural and social phenomena are nonlinear.

In this thesis, we consider strong ergodic theorems for asymptotically nonexpansive mappings and semigroups and strong convergence theorems for nonexpansive mappings through the hybrid method in the mathematical programming. And we consider convergence theorems by a modified splitting method.

In Chapter 2, 3, we study strong ergodic theorems for asymptotically nonexpansive mappings and semigroups. Let E be a real Banach space. Let $B = \{x \in E \mid \|x\| = 1\}$. Then, E is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for every $x, y \in B$ with $\|x - y\| \geq \epsilon$. E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for $x, y \in B$ with $x \neq y$. It is known that a uniformly convex Banach space is strictly convex. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in B$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

is attained uniformly for $x \in B$. The norm of E is said to be *Fréchet differentiable* if for every $x \in B$, the limit (1.1) is attained uniformly for $y \in B$ and E is said to be *uniformly smooth* if the limit (1.1) is attained uniformly for $x, y \in B$. Let C be a nonempty closed convex subset of E. We denote by $\mathbf{N}$ and $\mathbf{R}^+$, the set of all positive integers and the set of all nonnegative real numbers, respectively. A mapping $T : C \longrightarrow C$ is said to be *asymptotically nonexpansive* [17] if there exists a sequence $\{k_n\}$ of nonnegative real numbers with $\limsup_{n\to\infty} k_n \leq 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for every $x, y \in C$

and $n \in \mathbf{N}$. T is said to be *nonexpansive* if $k_n = 1$ for all $n \in \mathbf{N}$. We denote by $F(T)$ the set of the fixed points of T. A family $\mathcal{S} = \{S(t) \mid 0 \le t < \infty\}$ is said to be an *asymptotically nonexpansive semigroup* on C with Lipschitz constants $\{k(t) \mid 0 \le t < \infty\}$ if

(i) for each $t \in [0, \infty)$, $S(t)$ is a mapping of C into itself and $\|S(t)x - S(t)y\| \le k(t)\|x - y\|$ for each $x, y \in C$;

(ii) $S(t+s)x = S(t)S(s)x$ for each $t, s \in [0, \infty)$ and $x \in C$;

(iii) $S(0)x = x$ for each $x \in C$;

(iv) for each $x \in C$, the mapping $t \longmapsto S(t)x$ is continuous.

(v) $t \longmapsto k(t)$ is continuous mapping of the set of nonnegative real numbers into itself;

(vi) $\limsup_{t \to \infty} k(t) \le 1$.

$\mathcal{S}$ is said to be a *nonexpansive semigroup* on C if $k(t) = 1$ for all $t \in [0, \infty)$. We denote by $F(\mathcal{S})$, the set of common fixed points of $\mathcal{S}$.

Baillon [3] proved the first nonlinear ergodic theorem for nonexpansive mappings with bounded domains : Let C be a nonempty bounded closed convex subset of a real Hilbert space and let T be a nonexpansive mapping of C into itself. Then, for every $x \in C$, the Cesàro means $\frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converge weakly to some $y \in F(T)$. Brézis and Browder [7] proved a nonlinear strong ergodic theorem for nonexpansive mappings of odd-type in a real Hilbert space (see also Reich [32]). Bruck [10] extended Baillon's result [3] to a uniformly convex Banach space whose norm is Fréchet differentiable (see also [20, 33]). Hirano and Takahashi [21] extended Baillon's result [3] to asymptotically nonexpansive mappings in a real Hilbert space. Oka [29] and Tan and Xu [46] extended Bruck's result [10] to asymptotically nonexpansive mappings. On the other hand, The first nonlinear ergodic theorem for nonexpansive mappings with compact domains was proved by Edelstein [15]. Atsushiba and Takahashi [1] generalized Edelstein's result : Let D be a nonempty closed convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping of D into itself such that $T(D) \subset K$ for some compact subset K of D. Let $x \in D$. Then, the Cesàro means $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h} x$ converge strongly to a fixed point of T uniformly in $h \in \mathbf{N} \cup \{0\}$. In this case, if $Qx = \lim_{n \to \infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ for every $x \in D$, then Q is a nonexpansive mapping of D onto $F(T)$ such that $QT^k = T^k Q = Q$ for all $k \in \mathbf{N}$ and $Qx \in \overline{co}\{T^k x \mid k \in \mathbf{N} \cup \{0\}\}$ for all $x \in D$.

In Chapter 2, by using the methods employed in Atsushiba and Takahashi

[1], Bruck [10, 11] and Shioji and Takahashi [40], we extend Atsushiba and Takahashi's theorem to asymptotically nonexpansive mappings.

Theorem 1.1 *Let E be a strictly convex Banach space and let D be a nonempty closed convex subset of E. Let T be an asymptotically nonexpansive mapping of D into itself such that $T(D) \subset K$ for some compact subset K of D and let $x \in D$. Then, the Cesàro means $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$ converge strongly to a fixed point of T uniformly in $h \in \boldsymbol{N} \cup \{0\}$. In this case, if $Qx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ for every $x \in D$, then Q is a nonexpansive mapping of D onto $F(T)$ such that $QT^k = T^kQ = Q$ for all $k \in \boldsymbol{N} \cup \{0\}$ and $Qx \in \overline{co}\{T^k x \mid k \in \boldsymbol{N} \cup \{0\}\}$ for all $x \in D$.*

Next, Baillon and Brézis [5] proved the first nonlinear ergodic theorem for nonexpansive semigroups in a real Hilbert space (see also Baillon [4] and Reich [31]). Hirano and Takahashi [21] extended Baillon and Brézis's theorem to asymptotically nonexpansive semigroups. Hirano and Takahashi's theorem was extended to a uniformly convex Banach space whose norm is Fréchet differentiable by Tan and Xu [47]. On the other hand, Dafermos and Slemrod [13] proved the first nonlinear ergodic theorem for nonexpansive semigroups with compact domains. Atsushiba and Takahashi [2] generalized Dafermos and Slemrod's result : Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{S(t) \mid 0 \leq t < \infty\}$ be a nonexpansive semigroup on C. Let $x \in C$. Then, $\frac{1}{t}\int_0^t S(\tau + h)x d\tau$ converges strongly as $t \to \infty$ to some $y \in F(\mathcal{S})$ uniformly in $h \in \mathbf{R}^+$. In this case, if $Qx = \lim_{t\to\infty} \frac{1}{t}\int_0^t S(\tau)x d\tau$ for every $x \in C$, then Q is a nonexpansive mapping of C onto $F(\mathcal{S})$ such that $QS(t) = S(t)Q = Q$ for every $t \in [0, \infty)$ and $Qx \in \overline{co}\{S(t)x \mid 0 \leq t < \infty\}$ for every $x \in C$.

In Chapter 3, we extend Atsushiba and Takahashi's theorem to asymptotically nonexpansive semigroups by using the methods employed in Atsushiba and Takahashi [1, 2], Bruck [10, 11] and Shioji and Takahashi [39].

Theorem 1.2 *Let C be a nonempty compact convex subset of a strictly convex Banach space E and let $\mathcal{S} = \{S(t) \mid 0 \leq t < \infty\}$ be an asymptotically nonexpansive semigroup on C. Let $x \in C$. Then, $\frac{1}{t}\int_0^t S(\tau + h)x d\tau$ converges strongly as $t \to \infty$ to some $y \in F(\mathcal{S})$ uniformly in $h \in \boldsymbol{R}^+$. In this case, if $Qx = \lim_{t\to\infty} \frac{1}{t}\int_0^t S(\tau)x d\tau$ for every $x \in C$, then Q is a nonexpansive mapping of C onto $F(\mathcal{S})$ such that $QS(t) = S(t)Q = Q$ for every $t \in [0, \infty)$ and $Qx \in \overline{co}\{S(t)x \mid 0 \leq t < \infty\}$ for every $x \in C$.*

In Chapter 4, 5, we study strong convergence theorems for nonexpansive mappings and semigroups by the hybrid method in the mathematical programming and study the splitting method modified by the hybrid method and Mann's type iteration in a real Hilbert space. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We denote by $P_C(\cdot)$, the metric projection onto C. Let T be a nonexpansive mapping of C into itself. It is known that $F(T)$ is closed and convex. Halpern [19] introduced an iteration scheme as follows:

$$x_0 = x \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n \tag{1.2}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, 1]$. Reich [34, 35] proved that $\{x_n\}$ generated by (1.2) converges strongly to some $y \in F(T)$ in the case of a uniformly smooth Banach space when $\alpha_n = n^{-a}$ for some $a \in (0, 1)$ and $F(T)$ is nonempty. Wittmann [49] extended Reich's result in the case of a real Hilbert space H. On the other hand, the hybrid method (the hybrid projection-proximal point method) in the mathematical programming was introduced by Solodov and Svaiter [42, 43, 44]. They get strong convergence by combining proximal point iterations with projection steps. So, in Chapter 4, motivated by the hybrid method, we consider the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n\,, \\ C_n = \{z \in C \mid \|y_n - z\| \le \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \ge 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.3}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. Then, we obtain the following strong convergence theorem.

Theorem 1.3 *If $F(T)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}(x_0)$.*

And by using the lemma in Shimizu and Takahashi [38], we get the strong convergence theorem for nonexpansive semigroups. Here, note that $F(\mathcal{S})$ is closed and convex.

Theorem 1.4 *Let C be a nonempty closed convex subset of H and let $\mathcal{S} = \{S(t) \mid 0 \le t < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. The sequence $\{x_n\}$ is generated by*

$$\begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} S(\tau)x_n d\tau\,, \\ C_n = \{z \in C \mid \|y_n - z\| \le \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \ge 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N} \cup \{0\}$, *where* $\{\alpha_n\}$ *is a sequence in* $[0, a]$ *for some* $a \in [0, 1)$ *and* $\{t_n\}$ *is a positive real divergent sequence. Then, the sequence* $\{x_n\}$ *converges strongly to* $P_{F(S)}(x_0)$.

By this method, we also study the proximal point iteration. A multivalued operator $A : H \longrightarrow 2^H$ is said to be *monotone* if $(x_1 - x_2, y_1 - y_2) \geq 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. A monotone operator A is said to be *maximal* if the graph of A is not properly contained in the graph of any other monotone operator. It is known that a monotone operator A is maximal iff $R(I + rA) = H$ for every $r > 0$, where $R(I + rA) = \cup\{z + rAz \mid z \in H, Az \neq \emptyset\}$. It is also known that $A^{-1}0 = \{x \in H \mid 0 \in Ax\}$ is closed and convex for any maximal monotone A. If A is monotone, then we can define, for each $r > 0$, a nonexpansive mapping $J_r : R(I+rA) \longrightarrow D(A)$ by $J_r = (I+rA)^{-1}$, where $D(A) = \{z \in H \mid Az \neq \emptyset\}$. J_r is called the *resolvent* of A. We know that $F(J_r) = A^{-1}0$ for each $r > 0$, where $F(J_r) = \{z \in D(A) \mid J_r z = z\}$; see [45] for more details. Let $A : H \longrightarrow 2^H$ be a maximal monotone operator. The proximal point iteration generates, for any initial data $x_0 = x \in H$, a sequence $\{x_n\}$ in H by the rule $x_{n+1} = J_{r_n} x_n$ for each $n \in \mathbf{N} \cup \{0\}$, where J_r is the resolvent of A for $r > 0$ and $\{r_n\}$ is a sequence of positive real numbers. The proximal point iteration was first introduced by Martinet [26, 27] and generally studied by Rockafellar [37]. Further, Kamimura and Takahashi [22] and several authors study the proximal point iteration. And we get the following theorem.

Theorem 1.5 *Let* $A : H \longrightarrow 2^H$ *be a maximal monotone operator such that* $A^{-1}0$ *is nonempty. The sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = J_{r_n}(x_n + f_n)\,, \\ C_n = \{z \in H \mid \|y_n - z\| \leq \|x_n + f_n - z\|\}\,, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.4}$$

for every $n \in \mathbf{N} \cup \{0\}$, *where*

$$\{r_n\} \subset (0, \infty), \ \liminf_{n \to \infty} r_n > 0 \ \textit{and} \ \lim_{n \to \infty} \|f_n\| = 0$$

. *Then,* $\{x_n\}$ *converges strongly to* $P_{A^{-1}0}(x_0)$.

Next, we study the splitting method introduced by Passty [30] and Lions and Mercier [23]. Let H be a real Hilbert space and let $A : H \longrightarrow 2^H$ and $B : H \longrightarrow 2^H$ be maximal monotone operators such that $D(B) \subset D(A)$ and

$(A+B)^{-1}0 \neq \emptyset$, where $D(A)$ is the domain of A. Let C be a nonempty closed convex subset of H and let ∂i_C denote the subdifferential of the indicator function of C. Then, the splitting method is one of the methods of finding an element of $(A+B)^{-1}0$ and its iteration is the following:

$$\begin{cases} x_1 = x \in D(A)\,, \\ x_{n+1} = J^B_{\lambda_n}(x_n - \lambda_n w_n) \end{cases}$$

for every $n \in \mathbf{N}$, where $w_n \in Ax_n$, $\{\lambda_n\} \subset (0,\infty)$ and $J^B_{\lambda_n}$ is the resolvent of B, i.e. $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$. If $A = 0$, it is the proximal point iteration [26, 27, 37]. If $B = \partial i_C$, it is the projection method for variational inequalities by Brézis and Sibony [8] (see also Sibony [41]). Further if A is the gradient of a continuously Fréchet differentiable convex functional on H, it is the gradient projection method by Goldstein [18]. Later, the splitting method was extensively analyzed by Gabay [16] and was further studied in Chen and Rockafellar [12], Moudafi and Théra [28], Tseng [48] and references therein. Let $\alpha > 0$. A single valued operator $A : H \longrightarrow H$ is said to be *α-inverse-strongly-monotone* (see Browder and Petryshyn [9], Baillon and Haddad [6], Dunn [14], Liu and Nashed [24]) if $(x - y, Ax - Ay) \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in D(A)$. Let $A : H \longrightarrow H$ be a single valued α-inverse-strongly-monotone operator with $D(A) = H$ and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A + B)^{-1}0 \neq \emptyset$. Then, we know that A and $A + B$ are maximal monotone operators and $F(J^B_\lambda(I - \lambda A)) = (A + B)^{-1}0$ for every $\lambda \in (0,\infty)$ (see e.g. [14, 36, 50]). Then, Gabay [16] proved the following theorem : Let $\alpha > 0$. Let $A : H \longrightarrow H$ be a single valued α-inverse-strongly-monotone operator with $D(A) = H$ and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A + B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$, $x_{n+1} = J^B_\lambda(I - \lambda A)x_n$ for every $n \in \mathbf{N}$, where $0 < \lambda < 2\alpha$. Then, $\{x_n\}$ converges weakly to some $z \in (A + B)^{-1}0$.

In Chapter 5, using an iteration of Mann's type [25], we obtain the following weak convergence theorem which generalizes the result of Gabay.

Theorem 1.6 *Let $\alpha > 0$. Let $A : H \longrightarrow H$ be a single valued α-inverse-strongly-monotone operator with $D(A) = H$ and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in H\,, \\ y_n = J^B_{\lambda_n}(I - \lambda_n A)x_n\,, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n \end{cases}$$

for every $n \in \mathbf{N}$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ with $a < b$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in (0,1)$. Then, $\{x_n\}$ converges weakly to some $z \in (A + B)^{-1}0$.

And we show the following strong convergence theorem by combining the splitting method and the hybrid method.

Theorem 1.7 *Let α, A and B be as in Theorem 1.6. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = J^B_{\lambda_n}(I - \lambda_n A)x_n\,, \\ C_n = \{z \in H \,|\, \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in H \,|\, (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \boldsymbol{N} \cup \{0\}$, where $\{\lambda_n\} \subset [a, 2\alpha]$ for some $a \in (0, 2\alpha)$. Then, $\{x_n\}$ converges strongly to $P_{(A+B)^{-1}0}(x_0)$.

Let A be a mapping of C into H. Then, an element x in C is a solution of the variational inequality if $(y - x, Ax) \geq 0$ for all $y \in C$. We apply this result to the problem of the variational inequalities and get the following.

Corollary 1.8 *Let C be a nonempty closed convex subset of H and let $\alpha > 0$. Let $A : H \longrightarrow H$ be a single valued α-inverse-strongly-monotone operator with $D(A) = H$ and $\{x \in C \,|\, (y - x, Ax) \geq 0\, (\forall y \in C)\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = P_C(x_n - \lambda_n A x_n)\,, \\ C_n = \{z \in C \,|\, \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \,|\, (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \boldsymbol{N} \cup \{0\}$, where $\{\lambda_n\} \subset [a, 2\alpha]$ for some $a \in (0, 2\alpha)$. Then, $\{x_n\}$ converges strongly to the element z_0 in the set $\{x \in C \,|\, (y - x, Ax) \geq 0\, (\forall y \in C)\}$ nearest to x_0.

Acknowledgement. I would like to express my sincere appreciation to Professor Dr. Wataru Takahashi for his valuable advice and constant encouragement during the preparation of this thesis. And I am very thankful to my wife Naruko for her support and encouragement, too.

Bibliography

[1] S. Atsushiba and W. Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domains*, Math. Japonica, 52(2000), 183-195.

[2] S. Atsushiba and W. Takahashi, *Strong convergence theorems for one-parameter nonexpansive semigroups with compact domains*, in Nonlinear Analysis and Its Applications (S.P.Singh and Bruce Watson, Eds.), Marcel Dekker Inc. to appear.

[3] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér A-B, 280(1975), 1511-1514.

[4] J. B. Baillon, *Quelques propriétés de convergence asymptotique pour les semi-groupe de contractions impaires*, C. R. Acad. Sci. Paris Sér A-B, 283(1976), 75-78.

[5] J. B. Baillon and H. Brézis, *Une remarque sur le comportement asymptotique des semigroupes non lineaires*, Houston J. Math., 2(1976), 5-7.

[6] J .B. Baillon and G. Haddad, *Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones*, Israel J. Math., 26(1977), 137-150.

[7] H. Brézis and F. E. Browder, *Nonlinear ergodic theorems*, Bull. Amer. Math. Soc., 82(1976), 959-961.

[8] H. Brézis and M. Sibony, *Méthodes d'approximation et d'itération pour les opérateurs monotones*, Arch. Rational Mech. Anal., 27(1969), 59-82.

[9] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., 20(1967), 197-228.

[10] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., 32(1979), 107-116.

[11] R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math., 38(1981), 304-314.

[12] G. H-G. Chen and R. T. Rockafellar, *Convergence rates in forward-backward splitting*, SIAM J. Optim., 7(1997), 421-444.

[13] C. M. Dafermos and M. Slemrod, *Asymptotic behavior of nonlinear contraction semigroups*, J. Funct. Anal., 13(1973), 97-106.

[14] J. C. Dunn, *Convexity, monotonicity, and gradient processes in Hilbert space*, J. Math. Anal. Appl., 53(1976), 145-158.

[15] M. Edelstein, *On non-expansive mappings of Banach spaces*, Proc. Camb. Phil. Soc., 60(1964), 439-447.

[16] D. Gabay, *Applications of the method of multipliers to variational inequalities*, in Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary-Value Problems (M.Fortin and R.Glowinski, Eds.), Studies in Mathematics and Its Applications, North Holland, Amsterdam, Holland, Vol.15, 299-331, 1983.

[17] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., 35(1972), 171-174.

[18] A. A. Goldstein, *Convex programming in Hilbert space*, Bull. Amer. Math. Soc., 70(1964), 709-710.

[19] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., 73(1967), 957-961.

[20] N. Hirano, *A proof of the mean ergodic theorems for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., 78(1980), 361-365.

[21] N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces*, Kodai Math. J., 2(1979), 11-25.

[22] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory, 106(2000), 226-240.

[23] P. L. Lions and B. Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM J. Numer. Anal., 16(1979), 964-979.

[24] F. Liu and M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal., 6(1998), 313-344.

[25] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., 4(1953), 506-510.

[26] B. Martinet, *Regularisation d'inequations variationnelles par approximations successives*, Rev. Franc. Inform. Rech. Opér., 4(1970), 154-159.

[27] B. Martinet, *Determination approachée d'un point fixe d'une application pseudo-contractante*, C. R. Acad. Sci. Paris, Ser. A, 274(1972), 163-165.

[28] A. Moudafi and M. Théra, *Finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl., 94(1997), 425-448.

[29] H. Oka, *On the nonlinear mean ergodic theorems for asymptotically nonexpansive mappings in Banach spaces*, RIMS (Kyoto Univ.) Kokyuroku, 730(1990), 1-20.

[30] G. B. Passty, *Ergodic convergence to a zero of the sum of monotone operators in Hilbert space*, J. Math. Anal. Appl., 72(1979), 383-390.

[31] S. Reich, *Nonlinear evolution equations and nonlinear ergodic theorems*, Nonlinear Anal., 1(1977), 319-330.

[32] S. Reich, *Almost convergence and nonlinear ergodic theorems*, J. Approx. Theory, 24(1978), 269-272.

[33] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., 67(1979), 274-276.

[34] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., 75(1980), 287-292.

[35] S. Reich, *Some problems and results in fixed point theory*, Contemp. Math., 21(1983), 179-187.

[36] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., 149(1970), 75-88.

[37] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., 14(1976), 877-898.

[38] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl., 211(1997), 71-83.

[39] N. Shioji and W. Takahashi, *Strong convergence theorems for continuous semigroups in Banach spaces*, Math. Japonica, 50(1999), 57-66.

[40] N. Shioji and W. Takahashi, *Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces*, J. Approx. Theory, 97(1999), 53-64.

[41] M. Sibony, *Méthodes itératives pour les équations et inéquations aux dérivées partielles nonlinéares de type monotone*, Calcolo, 7(1970), 65-183.

[42] M. V. Solodov and B. F. Svaiter, *A hybrid projection-proximal point algorithm*, J. Convex Anal., 6(1999), 59-70.

[43] M. V. Solodov and B. F. Svaiter, *A new projection method for variational inequality problems*, SIAM J. Control Optim., 37(1999), 765-776.

[44] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Programming Ser. A, 87(2000), 189-202.

[45] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).

[46] K. K. Tan and H. K. Xu, *The nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., 114(1992), 399-404.

[47] K. K. Tan and H. K. Xu, *An ergodic theorem for nonlinear semigroups of Lipschitzian mappings in Banach spaces*, Nonlinear Anal., 19(1992), 805-813.

[48] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., 38(2000), 431-446.

[49] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., 58(1992), 486-491.

[50] E. Zeidler, *Nonlinear Functional Analysis and its Applications: Nonlinear Monotone Operators, II/B* , Springer-Verlag, New York, 1990.

Chapter 2

A Nonlinear Strong Ergodic Theorem for Asymptotically Nonexpansive Mappings with Compact Domains

2.1 Introduction

Throughout this chapter, a Banach space is real and we denote by $\mathbf{N}$ and R^+, the set of all positive integers and the set of all nonnegative real numbers, respectively. Let C be a nonempty closed convex subset of a Banach space. A mapping $T : C \mapsto C$ is said to be asymptotically nonexpansive [6] if there exists a sequence $\{k_n\}$ of nonnegative real numbers with $\limsup_{n\to\infty} k_n \leq 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for every $x, y \in C$ and $n \in \mathbf{N}$. T is said to be nonexpansive if $k_n = 1$ for all $n \in \mathbf{N}$. The first nonlinear ergodic theorem for nonexpansive mappings with bounded domains was proved by Baillon [2] : Let C be a nonempty bounded closed convex subset of a Hilbert space and let T be a nonexpansive mapping from C into itself. Then, for every $x \in C$, the Cesàro means $\frac{1}{n}\sum_{i=0}^{n-1} T^i x$ converge weakly to some $y \in F(T)$. Bruck [3] extended Baillon's theorem to a uniformly convex Banach space whose norm is Fréchet differentiable. Hirano and Takahashi [7] extended Baillon's theorem to an asymptotically nonexpansive mapping in Hilbert spaces. Oka [8] and Tan and Xu [10] extended Bruck's result to an asymptotically nonexpansive mapping in Banach spaces. On the other hand, Atsushiba and Takahashi [1] obtained the following nonlinear ergodic theorem for nonexpansive mappings with compact domains which generalizes Edelstein's result [5] : Let D be a nonempty closed convex subset of a strictly convex Banach space. Let

T be a nonexpansive mapping from D into itself such that $T(D) \subset K$ for some compact subset K of D and let $x \in D$. Then, $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$ converges strongly to a fixed point of T uniformly in $h \in \mathbf{N} \cup \{0\}$.

In this chapter, we extend Atsushiba and Takahashi's theorem to an asymptotically nonexpansive mapping by using the methods employed in Atsushiba and Takahashi [1], Bruck [3, 4] and Shioji and Takahashi [9].

2.2 Preliminaries and lemmas

We denote by Δ^n the set $\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \,|\, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$ for $n \in \mathbf{N}$. Let E be a Banach space and let $r > 0$. We denote by $D_r(x)$ the open ball in E with center x and radius r. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let C be a subset of E, let T be a mapping from C into itself and let $\varepsilon > 0$. By $F(T)$ and $F_\varepsilon(T)$, we mean the set $\{x \in C \,|\, Tx = x\}$ and $\{x \in C \,|\, \|x - Tx\| \leq \varepsilon\}$, respectively. Let $K > 0$. We denote by $Lip(C, K)$ the set of all mappings from C into itself satifying $\|Tx - Ty\| \leq K\|x - y\|$ for each $x, y \in C$. We denote by Γ the set of all strictly increasing, continuous convex functions γ $:R^+ \longmapsto R^+$ with $\gamma(0) = 0$. A nonempty subset C of a Banach space is said to satisfy the convex approximation property if foy any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that $coM \subset co_m M + D_\varepsilon(0)$ for every subset M of C, where $co\,M$ is the convex hull of M and

$$co_m M = \{\sum_{i=1}^m \lambda_i x_i \,|\, x_i \in M,\ \lambda_i \geq 0\,(i = 1, 2, \ldots, m),\ \sum_{i=1}^m \lambda_i = 1\}.$$

The following lemma was obtained by Bruck [3]. For the proof of the lemmas, see [1].

Lemma 2.1 *Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Then, there exists $\gamma \in \Gamma$ such that for each $K > 0$ and $T \in Lip(C, K)$, there holds*

$$\|T(\lambda x + (1-\lambda)y) - (\lambda Tx + (1-\lambda)Ty)\| \leq K\gamma^{-1}\Big(\|x - y\| - \frac{1}{K}\|Tx - Ty\|\Big)$$

for every $x, y \in C$ and $\lambda \in [0, 1]$.

The following lemma was also obtained by Bruck [4].

Lemma 2.2 *Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Then, for each $p \in N$, there exists $\gamma_p \in \Gamma$ such that for each $K > 0$ and $T \in Lip(C, K)$, there holds*

$$\Big\| T\Big(\sum_{i=1}^p \lambda_i x_i\Big) - \sum_{i=1}^p \lambda_i Tx_i \Big\| \leq K\gamma_p^{-1}\Big(\max_{1\leq i,j\leq p}\Big\{\|x_i - x_j\| - \frac{1}{K}\|Tx_i - Tx_j\|\Big\}\Big)$$

for every $x_1, x_2, \dots, x_p \in C$ *and* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p) \in \Delta^p$.

Following an idea in Atsushiba and Takahashi [1], we can show the following lemma.

Lemma 2.3 *Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be an asymptotically nonexpansive mapping from C into itself. Let $x \in C$ and $n \in \boldsymbol{N}$. Then, for each $\varepsilon > 0$, there exist $l_0 = l_0(n, \varepsilon) \in \boldsymbol{N}$ and $m_0 = m_0(n, \varepsilon) \in \boldsymbol{N}$ such that*

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{l+j+m}x - T^l\Big(\frac{1}{n}\sum_{j=0}^{n-1} T^{j+m}x\Big) \right\| < \varepsilon$$

for all $l \geq l_0$ *and* $m \geq m_0$.

Proof. Let $\{k_n\}$ be Lipschitz constants of T. Without loss of generality, we may assume that $k_l > 0$ for all $l \in \mathbf{N}$. By Lemma 2.2, there exists $\gamma_n \in \Gamma$ such that

$$\begin{aligned} &\left\| T^l\Big(\frac{1}{n}\sum_{j=0}^{n-1} T^{j+m}x\Big) - \frac{1}{n}\sum_{j=0}^{n-1} T^{l+j+m}x \right\| \qquad (2.1)\\ &\leq \; k_l\gamma_n^{-1}\Big(\max_{0\leq i,j\leq n-1}\Big\{\|T^{i+m}x - T^{j+m}x\| - \frac{1}{k_l}\|T^{l+i+m}x - T^{l+j+m}x\|\Big\}\Big) \end{aligned}$$

for all $l \in \mathbf{N}$ and $m \in \mathbf{N} \cup \{0\}$. From $\gamma_n \in \Gamma$ and $\limsup_{n\to\infty} k_n \leq 1$, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $k_l\gamma_n^{-1}(\delta) < \varepsilon$ for all $l \in \mathbf{N}$.

For $0 \leq i, j \leq n-1$, we set $r_{i,j} = \inf_{m\in\mathbf{N}\cup\{0\}} \|T^{m+i}x - T^{m+j}x\|$. There exists $m_1 = m_1(i,j) \in \mathbf{N}\cup\{0\}$ such that $r_{i,j} \leq \|T^{m_1+i}x - T^{m_1+j}x\| < r_{i,j} + \dfrac{\delta}{4}$. From $\limsup_{n\to\infty} k_n \leq 1$, there exists $n_1 \in \mathbf{N}$ such that $k_n < \dfrac{r_{i,j}+\frac{\delta}{2}}{\|T^{m_1+i}x - T^{m_1+j}x\| + \frac{\delta}{4}}$ for all $n \geq n_1$. So, we have

$$\|T^{n+m_1+i}x - T^{n+m_1+j}x\| \leq k_n\|T^{m_1+i}x - T^{m_1+j}x\| < r_{i,j} + \frac{\delta}{2}$$

for all $n \geq n_1$. Put $m_2 = m_2(i,j) = m_1 + n_1$. Then, there holds $r_{i,j} \leq \|T^{m+i}x - T^{m+j}x\| < r_{i,j} + \frac{\delta}{2}$ for all $m \geq m_2$. Similarly, there exists $l_2 = l_2(i,j) \in \mathbf{N}$ such that $r_{i,j} - \dfrac{\delta}{2} \leq \dfrac{1}{k_l}\|T^{l+m+i}x - T^{l+m+j}x\|$ for every $m \in \mathbf{N}$ and $l \geq l_2$. So, we have

$$0 \leq \Big\{\|T^{m+i}x - T^{m+j}x\| - \frac{1}{k_l}\|T^{l+m+i}x - T^{l+m+j}x\|\Big\} \leq \delta$$

for every $m \geq m_2$ and $l \geq l_2$. Let $l_0 = \max\{l_2(i,j) \mid 0 \leq i,\ j \leq n-1\}$ and $m_0 = \max\{m_2(i,j) \mid 0 \leq i,\ j \leq n-1\}$. Then, we get

$$0 \leq \max_{0 \leq i,j \leq n-1} \Big\{ \|T^{m+i}x - T^{m+j}x\| - \frac{1}{k_l}\|T^{l+m+i}x - T^{l+m+j}x\| \Big\} \leq \delta$$

for every $l \geq l_0$ and $m \geq m_0$. So, it follows from (2.1) that

$$\begin{aligned} & \Big\| T^l\Big(\frac{1}{n}\sum_{j=0}^{n-1} T^{j+m}x\Big) - \frac{1}{n}\sum_{j=0}^{n-1} T^{l+j+m}x \Big\| \\ \leq\ & k_l\gamma_n^{-1}\Big(\max_{0 \leq i,j \leq n-1}\Big\{\|T^{m+i}x - T^{m+j}x\| - \frac{1}{k_l}\|T^{l+m+i}x - T^{l+m+j}x\|\Big\}\Big) \\ \leq\ & k_l\gamma_n^{-1}(\delta) < \varepsilon \end{aligned}$$

for every $l \geq l_0$ and $m \geq m_0$. This is the desired result. □

The following lemma was obtained by Atsushiba and Takahashi [1].

Lemma 2.4 *Let C be a nonempty compact subset of a Banach space E . Then, C satisfies the convex approximation property.*

From Lemma 2.2 and Lemma 2.4, we obtain the following lemma.

Lemma 2.5 *Let C be a nonempty compact convex subset of a strictly convex Banach space. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$ holds for every $T \in Lip(C, 1+\delta)$, where $\overline{co}\,A$ is the closure of the convex hull of A.*

Proof. We use an idea in Bruck [3]. Set $R = diam\, C$. Let $\varepsilon > 0$ and $\varepsilon_0 > 0$ with $(3+\varepsilon_0)\varepsilon_0 < \varepsilon$. From Lemma 2.4, there exists $p \in \mathbf{N}$ such that $coM \subset co_pM + D_{\varepsilon_0}(0)$ for all subset M of C. There exists $\gamma_p \in \Gamma$ which satisfies the condition in Lemma 2.2. Since $\gamma_p \in \Gamma$, there exists $\delta > 0$ such that $(1+\delta)\gamma_p^{-1}(2\delta + R\delta) + \delta < \varepsilon_0$. Let $T \in Lip(C, 1+\delta)$. Then, it follows that $co_pF_\delta(T) \subset F_{\varepsilon_0}(T)$. In fact, let $x_i \in F_\delta(T)$ for all $i = 1, 2, \ldots, p$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Delta^p$. Then,

$$\begin{aligned} & \Big\| T\Big(\sum_{i=1}^{p}\lambda_i x_i\Big) - \sum_{i=1}^{p}\lambda_i x_i \Big\| \\ \leq\ & \Big\| T\Big(\sum_{i=1}^{p}\lambda_i x_i\Big) - \sum_{i=1}^{p}\lambda_i Tx_i \Big\| + \delta \\ \leq\ & (1+\delta)\gamma_p^{-1}\Big(\max_{1 \leq i,j \leq p}\Big\{\|x_i - x_j\| - \frac{1}{1+\delta}\|Tx_i - Tx_j\|\Big\}\Big) + \delta \\ \leq\ & (1+\delta)\gamma_p^{-1}\Big(\max_{1 \leq i,j \leq p}\Big\{\|x_i - Tx_i\| + \|x_j - Tx_j\| + \frac{\delta}{1+\delta}\|Tx_i - Tx_j\|\Big\}\Big) + \delta \\ \leq\ & (1+\delta)\gamma_p^{-1}(2\delta + R\delta) + \delta < \varepsilon_0. \end{aligned}$$

Hence, $co_pF_\delta(T) \subset F_{\varepsilon_0}(T)$. So, we obtain $coF_\delta(T) \subset F_{\varepsilon_0}(T)+D_{\varepsilon_0}(0)$. Further it follows that $coF_\delta(T) \subset F_\varepsilon(T)$. In fact, for any $z \in coF_\delta(T)$, there exist $z_1 \in F_{\varepsilon_0}(T)$ and $z_2 \in D_{\varepsilon_0}(0)$ with $z = z_1 + z_2$. And there holds

$$\begin{aligned}\|z - Tz\| &= \|T(z_1 + z_2) - (z_1 + z_2)\| \\ &\leq \|T(z_1 + z_2) - Tz_1\| + \|Tz_1 - z_1\| + \|z_2\| \\ &\leq (1 + \delta)\|z_2\| + \|Tz_1 - z_1\| + \|z_2\| \\ &\leq (1 + \delta)\varepsilon_0 + \varepsilon_0 + \varepsilon_0 = (3 + \delta)\varepsilon_0 \leq (3 + \varepsilon_0)\varepsilon_0 < \varepsilon.\end{aligned}$$

So, we have $z \in F_\varepsilon(T)$. Since $F_\varepsilon(T)$ is closed, we can get $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$. □

Using the method employed in Bruck [3], we obtain the following lemma.

Lemma 2.6 *Let C be a nonempty closed bounded convex subset of a Banach space. Let $\gamma \in \Gamma$, $L \geq 1$ and $T \in Lip(C, L)$ such that*

$$\|T(\lambda x + (1 - \lambda)y) - (\lambda Tx + (1 - \lambda)Ty)\| \leq L\gamma^{-1}\Big(\|x - y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all $x, y \in C$ and $\lambda \in [0, 1]$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $\frac{1}{n}\sum_{i=1}^{n} \|x_{i+1} - Tx_i\| \leq a_n$ and $\frac{1}{n}\sum_{i=1}^{n} \|y_{i+1} - Ty_i\| \leq a_n$ for all $n \in \boldsymbol{N}$, where $\{a_n\}$ is a sequence in R^+. Then, for each $\lambda \in [0, 1]$ and $n \in \boldsymbol{N}$,

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n} \|\lambda x_{i+1} + (1 - \lambda)y_{i+1} - T(\lambda x_i + (1 - \lambda)y_i)\| \\ &\qquad \leq L\gamma^{-1}\Big(\frac{R}{n} + (L - 1)R + 2a_n\Big) + a_n, \quad \text{where} \quad R = diam\, C.\end{aligned}$$

Proof. Let $\lambda \in [0, 1]$ and $n \in \mathbf{N}$. We have

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n} \|\lambda x_{i+1} + (1 - \lambda)y_{i+1} - T(\lambda x_i + (1 - \lambda)y_i)\| \\ &\leq \frac{1}{n}\sum_{i=1}^{n} \|\lambda Tx_i + (1 - \lambda)Ty_i - T(\lambda x_i + (1 - \lambda)y_i)\| \\ &\quad + \lambda\Big(\frac{1}{n}\sum_{i=1}^{n} \|x_{i+1} - Tx_i\|\Big) + (1 - \lambda)\Big(\frac{1}{n}\sum_{i=1}^{n} \|y_{i+1} - Ty_i\|\Big) \\ &\leq \frac{1}{n}\sum_{i=1}^{n} \|\lambda Tx_i + (1 - \lambda)Ty_i - T(\lambda x_i + (1 - \lambda)y_i)\| + a_n. \qquad (2.2)\end{aligned}$$

From the assumption of T, we get

$$\gamma\Big(\frac{1}{L}\|T(\lambda x_i + (1 - \lambda)y_i) - (\lambda Tx_i + (1 - \lambda)Ty_i)\|\Big)$$

$$
\begin{aligned}
&\leq \|x_i - y_i\| - \frac{1}{L}\|Tx_i - Ty_i\| \\
&\leq \|x_i - y_i\| - \|x_{i+1} - y_{i+1}\| \\
&\quad + \|x_{i+1} - Tx_i\| + \|y_{i+1} - Ty_i\| + \frac{L-1}{L}\|Tx_i - Ty_i\| \\
&\leq \|x_i - y_i\| - \|x_{i+1} - y_{i+1}\| \\
&\quad + \|x_{i+1} - Tx_i\| + \|y_{i+1} - Ty_i\| + (L-1)R, \qquad (2.3)
\end{aligned}
$$

for every $i \in \{1, 2, \ldots, n\}$. Using (2.3) and covexity of γ, we obtain

$$
\begin{aligned}
&\gamma\Big(\frac{1}{nL}\sum_{i=1}^{n} \|\lambda Tx_i + (1-\lambda)Ty_i - T(\lambda x_i + (1-\lambda)y_i)\|\Big) \\
&\leq \frac{1}{n}\sum_{i=1}^{n} \gamma\Big(\frac{1}{L}\|\lambda Tx_i + (1-\lambda)Ty_i - T(\lambda x_i + (1-\lambda)y_i)\|\Big) \\
&\leq \frac{1}{n}(\|x_1 - y_1\| - \|x_{n+1} - y_{n+1}\|) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\|x_{i+1} - Tx_i\| + \frac{1}{n}\sum_{i=1}^{n}\|y_{i+1} - Ty_i\| + (L-1)R \\
&\leq \frac{R}{n} + (L-1)R + 2a_n. \qquad (2.4)
\end{aligned}
$$

From (2.2) and (2.4), we obtain

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\|\lambda x_{i+1} + (1-\lambda)y_{i+1} - T(\lambda x_i + (1-\lambda)y_i)\| \\
&\qquad\qquad \leq L\gamma^{-1}\Big(\frac{R}{n} + (L-1)R + 2a_n\Big) + a_n.
\end{aligned}
$$

□

We can show the following lemma from Lemma 2.1, Lemma 2.5 and Lemma 2.6.

Lemma 2.7 *Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \in \boldsymbol{N}$ such that for every $T \in Lip(C, 1+\delta)$ and $\{x_n\}$ in C satisfying $\|x_{n+1} - Tx_n\| \leq \delta$ for all $n \in \boldsymbol{N} \cup \{0\}$, there holds $\frac{1}{n}\sum_{i=0}^{n-1} x_i \in F_\varepsilon(T)$ for every $n \geq N_0$.*

Proof. We use an idea in Bruck [3]. Put $R = diam\, C$ and let $\varepsilon > 0$ and $\varepsilon_0 > 0$ with $(3+\varepsilon_0)\varepsilon_0 < \varepsilon$. By Lemma 2.5, there exists $\eta > 0$ such that for each $T \in Lip(C, 1+\eta)$, $\overline{co}F_\eta(T) \subset F_{\varepsilon_0}(T)$ and $0 < \eta < \dfrac{\varepsilon_0}{2R}$ hold. Choose $p \in \mathbf{N}$ such that $R < \dfrac{p\eta^2}{2}$. There exists $\gamma \in \Gamma$ which satisfies the condition in Lemma 2.1. Define $q : R^+ \longmapsto R^+$ by $q(t) = (1+t)\gamma^{-1}(Rt+2t)+t$ and $q_n : R^+ \longmapsto R^+$ by $q_n(t) = (1+t)\gamma^{-1}\left(\dfrac{R}{n} + Rt + 2t\right) + t$ for all $n \in \mathbf{N}$. Choose $\delta > 0$ so small that $q^{p-1}(\delta) < \dfrac{\eta^2}{2}$ and $\delta < \eta$. Since $\lim_{n\to\infty} q_n(t) = q(t)$, there exists $N_0 \in \mathbf{N}$ with $N_0 \geq p$ such that there hold $q_n^{p-1}(\delta) < \dfrac{\eta^2}{2}$ and $\dfrac{pR}{n} < \varepsilon_0$ for every $n \geq N_0$. Let $T \in Lip(C, 1+\delta)$ and $\{x_n\}$ in C satisfying $\|x_{n+1} - Tx_n\| \leq \delta$ for all $n \in \mathbf{N} \cup \{0\}$. Set $w_i = \dfrac{1}{p}\sum_{j=0}^{p-1} x_{i+j}$. It follows from Lemma 2.6 and induction that

$$\frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+1} - Tw_i\| \leq q_n^{p-1}(\delta) < \frac{\eta^2}{2} \text{ for every } n \geq N_0.$$

So, we have

$$\begin{aligned} &\frac{1}{n}\sum_{i=0}^{n-1}\|w_i - Tw_i\| \\ \leq\ & \frac{1}{n}\sum_{i=0}^{n-1}\|w_i - w_{i+1}\| + \frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+1} - Tw_i\| \leq \frac{\eta^2}{2} + \frac{\eta^2}{2} = \eta^2 \end{aligned} \tag{2.5}$$

for every $n \geq N_0$. For $n \in \mathbf{N}$, put $A(n) = \{i \in \mathbf{N} \mid 0 \leq i \leq n-1 \text{ and } \|w_i - Tw_i\| \geq \eta\}$ and $B(n) = \{0, 1, 2, \ldots, n-1\} \setminus A(n)$. From (2.5), we get

$$\frac{\sharp A(n)}{n} \leq \eta \tag{2.6}$$

for each $n \geq N_0$. From

$$\frac{1}{n}\sum_{i=0}^{n-1} x_i = \frac{1}{n}\sum_{i=0}^{n-1} w_i + \frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i-1} - x_{n+i-1})$$

and

$$\left\| \frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i-1} - x_{n+i-1}) \right\| \leq \frac{R}{np}\frac{(p-1)p}{2} \leq \frac{Rp}{2n} < \frac{\varepsilon_0}{2}$$

for all $n \geq N_0$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} x_i \in \frac{1}{n}\sum_{i=0}^{n-1} w_i + D_{\frac{\varepsilon_0}{2}}(0) \tag{2.7}$$

for all $n \geq N_0$. Let $f \in F(T)$. Then, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} w_i = \Big(\frac{1}{n}\cdot \sharp A(n) f + \frac{1}{n}\sum_{i\in B(n)} w_i\Big) + \frac{1}{n}\sum_{i\in A(n)} (w_i - f). \tag{2.8}$$

It follows from (2.6) and the selection of η that

$$\Big\| \frac{1}{n}\sum_{i\in A(n)} (w_i - f)\Big\| \leq \frac{\sharp A(n)}{n}\cdot R \leq R\cdot\eta < \frac{\varepsilon_0}{2} \tag{2.9}$$

for every $n \geq N_0$. If $i \in B(n)$, then $\|w_i - Tw_i\| < \eta$. This implies that $w_i \in F_\eta(T)$ for each $i \in B(n)$. Hence, $\Big(\frac{1}{n}\cdot \sharp A(n) f + \frac{1}{n}\sum_{i\in B(n)} w_i\Big)$ is a convex combination of elements of $F_\eta(T)$. Therefore, from (2.7), (2.8) and (2.9), we have

$$\frac{1}{n}\sum_{i=0}^{n-1} x_i \in \overline{co}F_\eta(T) + D_{\frac{\varepsilon_0}{2}}(0) + D_{\frac{\varepsilon_0}{2}}(0) \subset F_{\varepsilon_0}(T) + D_{\varepsilon_0}(0)$$

for every $n \geq N_0$. As in the proof of Lemma 2.5, we get $\frac{1}{n}\sum_{i=0}^{n-1} x_i \in F_\varepsilon(T)$ for every $n \geq N_0$. □

Using an idea in Shioji and Takahashi [9], we obtain the following lemma from Lemma 2.5 and Lemma 2.7.

Lemma 2.8 *Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, for each $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \in \mathbf{N}$ such that for every $l \in \mathbf{N}$ and mapping T from C into itself satisfying $T^l \in Lip(C, 1+\delta)$, there holds*

$$\Big\| \frac{1}{m}\sum_{i=0}^{m-1} T^i x - T^l\Big(\frac{1}{m}\sum_{i=0}^{m-1} T^i x\Big)\Big\| \leq \varepsilon$$

for all $m \in \mathbf{N}$ with $m - 1 \geq lN_0$ and $x \in C$.

Proof. Let $\varepsilon > 0$. From Lemma 2.5, there exists $\varepsilon_0 > 0$ such that for each pair $l \in \mathbf{N}$ and mapping T from C into itself with $T^l \in Lip(C, 1+\varepsilon_0)$, $\overline{co}F_{\varepsilon_0}(T^l) \subset F_\varepsilon(T^l)$ holds. From Lemma 2.7, there exist $\eta > 0$ and $N_0 \in \mathbf{N}$ such that for each pair $l \in \mathbf{N}$ and mapping T from C into itself satisfying $T^l \in Lip(C, 1+\eta)$, there holds $\frac{1}{n}\sum_{i=0}^{n-1} T^{il}x \in F_{\varepsilon_0}(T^l)$ for all $n \geq N_0$ and $x \in C$. Put $\delta = \min\{\varepsilon_0, \eta\}$. Let $l \in \mathbf{N}$ and mapping T from C into itself with $T^l \in Lip(C, 1+\delta)$. From the above, there hold $\overline{co}F_{\varepsilon_0}(T^l) \subset F_\varepsilon(T^l)$ and $\frac{1}{n}\sum_{i=0}^{n-1} T^{il}x \in F_{\varepsilon_0}(T^l)$ for all $n \geq N_0$ and $x \in C$. Let $m-1 \geq lN_0$. Choose $n \in \mathbf{N}$ and $s \in \{0,1,2,\ldots,l-1\}$ such that $m-1 = ln+s$. Then $n \geq N_0$. If $s \in \{0,1,2,\ldots,l-2\}$, we obtain

$$\begin{aligned} \frac{1}{m}\sum_{i=0}^{m-1} T^i x &= \frac{n+1}{m}\sum_{j=0}^{s}\Big(\frac{1}{n+1}\sum_{i=0}^{n} T^{il+j}x\Big) + \frac{n}{m}\sum_{j=s+1}^{l-1}\Big(\frac{1}{n}\sum_{i=0}^{n-1} T^{il+j}x\Big) \\ &\in \overline{co}F_{\varepsilon_0}(T^l) \subset F_\varepsilon(T^l) \end{aligned}$$

for all $m \in \mathbf{N}$ with $m-1 \geq lN_0$ and $x \in C$. We similarly get conclusion in case of $s = l-1$. □

The following corollary is a direct consequence of Lemma 2.8.

Corollary 2.9 *Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be an asymptotically nonexpansive mapping from C into itself. Then,*

$$\limsup_{l\to\infty}\limsup_{m\to\infty}\sup_{x\in C}\Big\| \frac{1}{m}\sum_{i=0}^{m-1} T^i x - T^l\Big(\frac{1}{m}\sum_{i=0}^{m-1} T^i x\Big)\Big\| = 0.$$

2.3 Strong ergodic theorem

The following lemma is crucial to prove our nonlinear strong ergodic theorem.

Lemma 2.10 *Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be an asymptotically nonexpansive mapping from C into itself. Let $x \in C$. Then, there exists a sequence $\{i_n\}$ in $\mathbf{N}$ such that for every $z \in F(T)$, $\lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - z\Big\|$ exists.*

Proof. We use the method employed in Atsushiba and Takahashi [1]. From Lemma 2.3, there exist sequences $\{l_n\}$ in $\mathbf{N}$ and $\{i_n\}$ in $\mathbf{N}$ such that

$$\Big\| \frac{1}{n}\sum_{j=0}^{n-1} T^{l+j+i}x - T^l\Big(\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i}x\Big)\Big\| < \frac{1}{n} \tag{2.10}$$

for every $n \in \mathbf{N}$, $l \geq l_n$ and $i \geq i_n$. Let $z \in F(T)$. For every $m, n \in \mathbf{N}$, consider

$$\begin{aligned} I &= \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m+i_n} x - z \right\| \\ &= \left\| \frac{1}{mn} \sum_{j=1}^{n-1} (n-j)(T^{j+i_m+i_n-1}x - T^{j+i_m+i_n+m-1}x) \right. \\ &\quad \left. + \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{n} \sum_{h=0}^{n-1} T^{j+h+i_m+i_n} x - z \right\|, \\ I_1 &= \left\| \frac{1}{mn} \sum_{j=1}^{n-1} (n-j)(T^{j+i_m+i_n-1}x - T^{j+i_m+i_n+m-1}x) \right\|, \\ I_2 &= \left\| \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{n} \sum_{h=0}^{n-1} T^{j+h+i_m+i_n} x - \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} \Big(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \Big) \right\| \end{aligned}$$

and

$$I_3 = \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} \Big(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \Big) - z \right\|.$$

Then, we have $I \leq I_1 + I_2 + I_3$. Fix $n \in \mathbf{N}$ and put $R = diam\, C$. We get $I_1 \leq \frac{1}{mn} \sum_{j=1}^{n-1} (n-j) R \leq \frac{nR}{2m}$ for every $m \in \mathbf{N}$. It follows from (2.10) that

$$\begin{aligned} I_2 &\leq \frac{1}{m} \sum_{j=0}^{m-1} \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{j+h+i_m+i_n} x - T^{j+i_m} \Big(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \Big) \right\| \\ &\leq \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{n} = \frac{1}{n} \end{aligned}$$

for each $m \in \mathbf{N}$ with $i_m \geq l_n$. By $z \in F(T)$, we obtain

$$\begin{aligned} I_3 &\leq \frac{1}{m} \sum_{j=0}^{m-1} \left\| T^{j+i_m} \Big(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \Big) - z \right\| \\ &\leq \frac{1}{m} \sum_{j=0}^{m-1} k_{j+i_m} \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x - z \right\| \end{aligned}$$

for every $m \in \mathbf{N}$, where $\{k_n\}$ is Lipschitz constants of T. Therefore, since $\lim_{m \to \infty} I_1 = 0$ and $\{k_n\}$ is Lipschitz constants of T, we have

$$\limsup_{m \to \infty} \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} x - z \right\|$$

$$
\begin{aligned}
&= \limsup_{m\to\infty} \left\| \frac{1}{m}\sum_{j=0}^{m-1} T^{j+i_m+i_n}x - z \right\| \\
&= \limsup_{m\to\infty} I \le \limsup_{m\to\infty}(I_1+I_2+I_3) \\
&\le \frac{1}{n} + \left\| \frac{1}{n}\sum_{h=0}^{n-1} T^{h+i_n}x - z \right\| \cdot \limsup_{m\to\infty} \frac{1}{m}\sum_{j=0}^{m-1} k_{j+i_m} \\
&\le \frac{1}{n} + \left\| \frac{1}{n}\sum_{h=0}^{n-1} T^{h+i_n}x - z \right\|
\end{aligned}
$$

for every $n \in \mathbf{N}$. So, we get

$$
\limsup_{m\to\infty} \left\| \frac{1}{m}\sum_{j=0}^{m-1} T^{j+i_m}x - z \right\| \le \liminf_{n\to\infty} \left\| \frac{1}{n}\sum_{h=0}^{n-1} T^{h+i_n}x - z \right\|.
$$

Hence, $\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - z \right\|$ exists. □

Remark 2.11 *In Lemma 2.10, take a sequence $\{i'_n\}$ in $\mathbf{N}$ such that $i'_n \ge i_n$ for each $n \in \mathbf{N}$. Then, we can obtain*

$$
\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{j=0}^{n-1} T^{j+i'_n}x - z \right\| = \lim_{n\to\infty} \left\| \frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - z \right\| \quad \textit{for every} \quad z \in F(T).
$$

Theorem 2.12 *Let E be a strictly convex Banach space and let D be a closed convex subset of E. Let T be an asymptotically nonexpansive mapping from D into itself such that $T(D) \subset K$ for some compact subset K of D and let $x \in D$. Then, $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$ converges strongly to a fixed point of T uniformly in $h \in \mathbf{N}\cup\{0\}$. In this case, if $Qx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} T^i x$ for every $x \in D$, then Q is a nonexpansive mapping from D onto $F(T)$ such that $QT^k = T^kQ = Q$ for all $k \in \mathbf{N}\cup\{0\}$ and $Qx \in \overline{co}\{T^k x \mid k \in \mathbf{N}\cup\{0\}\}$ for all $x \in D$.*

Proof. We use the method employed in Atsushiba and Takahashi [1]. $C = \overline{co}\{\{x\} \cup T(D)\}$ is a nonempty compact convex subset of E which is invariant under T and contains $\overline{co}\{T^k x \mid k \in \mathbf{N}\cup\{0\}\}$. From Lemma 2.10, there exists a sequence $\{i_n\}$ in $\mathbf{N}$ such that for each $z \in F(T)$,

$$
\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - z \right\| \tag{2.11}
$$

exists. From Corollary 2.9,

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\left\| \frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x - T^m\Big(\frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x\Big)\right\| = 0. \tag{2.12}$$

Let $\Phi_n = \frac{1}{n}\sum_{j=0}^{n-1} T^{j+i_n}x$ for every $n \in \mathbf{N}$. Then, we first prove that Φ_n converges strongly to a fixed point of T. From the compactness of C, there exists a subsequence $\{\Phi_{n_k}\}$ of $\{\Phi_n\}$ which converges strongly to a point y_0 in C. Then, from (2.12), we have

$$\begin{aligned} 0 &= \limsup_{m\to\infty}\limsup_{n\to\infty}\|\Phi_n - T^m\Phi_n\| \\ &= \limsup_{m\to\infty}\limsup_{k\to\infty}\|\Phi_{n_k} - T^m\Phi_{n_k}\| = \limsup_{m\to\infty}\|y_0 - T^m y_0\| \end{aligned}$$

and hence

$$\begin{aligned} \|y_0 - Ty_0\| &\le \limsup_{m\to\infty}\|y_0 - T^m y_0\| + \limsup_{m\to\infty}\|T^m y_0 - Ty_0\| \\ &\le 0 + k_1\cdot 0 = 0, \end{aligned}$$

where $\{k_n\}$ is Lipschitz constants of T. So, we get $y_0 \in F(T)$. From (2.11), $\lim_{n\to\infty}\|\Phi_n - y_0\| = \lim_{k\to\infty}\|\Phi_{n_k} - y_0\| = 0$. This implies that $\Phi_n \to y_0$. Next we prove that $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+h+i_n}x$ converges strongly to $y_0 \in F(T)$ uniformly in $h \in \mathbf{N}\cup\{0\}$. Take a sequence $\{i'_n\}$ in $\mathbf{N}$ such that $i'_n \ge i_n$ for every $n \in \mathbf{N}$. Then, from Remark 2.11, we have $\frac{1}{n}\sum_{i=0}^{n-1} T^{j+i'_n}x \longrightarrow y_0 \in F(T)$. Since $\{i'_n\}$ is any sequence in $\mathbf{N}$ such that $i'_n \ge i_n$ for every $n \in \mathbf{N}$, it follows that $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+h+i_n}x$ converges strongly to y_0 uniformly in $h \in \mathbf{N}\cup\{0\}$. Let $\varepsilon > 0$. Then, there exists $m \in \mathbf{N}$ such that

$$\left\| \frac{1}{n}\sum_{j=0}^{n-1} T^{j+h+i_n}x - y_0 \right\| < \varepsilon \qquad \text{for every } n \ge m \text{ and } h \in \mathbf{N}\cup\{0\}.$$

So, we have

$$\begin{aligned} &\left\| \frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x - y_0 \right\| \\ &= \left\| \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{m}\sum_{j=0}^{m-1} T^{i+j+h}x + \frac{1}{mn}\sum_{i=1}^{m-1}(m-i)(T^{i+h-1}x - T^{i+h+n-1}x) - y_0 \right\| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{n}\sum_{i=0}^{n-1}\left\| \frac{1}{m}\sum_{j=0}^{m-1} T^{i+j+h}x - y_0 \right\| + \frac{1}{mn}\sum_{i=1}^{m-1}(m-i)\left\| T^{i+h-1}x - T^{i+h+n-1}x \right\| \\
&= \frac{1}{n}\sum_{i=0}^{i_m-1}\left\| \frac{1}{m}\sum_{j=0}^{m-1} T^{i+j+h}x - y_0 \right\| + \frac{1}{n}\sum_{i=0}^{n-i_m-1}\left\| \frac{1}{m}\sum_{j=0}^{m-1} T^{i+j+h+i_m}x - y_0 \right\| \\
&\qquad + \frac{1}{mn}\sum_{i=1}^{m-1}(m-i)\left\| T^{i+h-1}x - T^{i+h+n-1}x \right\| \\
&\le \frac{i_m}{n}R + \frac{n-i_m}{n}\varepsilon + \frac{m}{2n}R
\end{aligned}
$$

for every $n \ge i_m$ and $h \in \mathbf{N} \cup \{0\}$, where $R = diam\, C$. Since $\varepsilon > 0$ is arbitrary, it follows that $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$ converges strongly to y_0 uniformly in $h \in \mathbf{N} \cup \{0\}$. If $Qx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ for every $x \in D$, then Q is a nonexpansive mapping from D onto $F(T)$. In fact, let $\{k_n\}$ be Lipschitz constants of T. Then,

$$
\begin{aligned}
&\left\| \frac{1}{n}\sum_{i=0}^{n-1} T^i x - \frac{1}{n}\sum_{i=0}^{n-1} T^i y \right\| \\
&\qquad \le \frac{1}{n}\sum_{i=0}^{n-1} \|T^i x - T^i y\| \le \|x-y\| \cdot \frac{1}{n}\sum_{i=0}^{n-1} k_i
\end{aligned}
$$

which implies $\|Qx - Qy\| \le \|x - y\|$ for every $x, y \in D$.

Moreover $QT^k = T^kQ = Q$ for every $k \in \mathbf{N} \cup \{0\}$ and $Qx \in \overline{co}\{T^k x \mid k \in \mathbf{N} \cup \{0\}\}$ for every $x \in D$ hold. □

Bibliography

[1] S.Atsushiba and W.Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domains*, Math. Japonica, **52**(2000), 183-195.

[2] J.B.Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér.A-B, **280**(1975), 1511-1514.

[3] R.E.Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., **32**(1979), 107-116.

[4] R.E.Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math., **38**(1981), 304-314.

[5] M.Edelstein, *On non-expansive mappings of Banach spaces*, Proc. Camb. Phil. Soc., **60**(1964), 439-447.

[6] K.Goebel and W.A.Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972), 171-174.

[7] N.Hirano and W.Takahashi, *Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces*, Kodai Math. J., **2**(1979), 11-25.

[8] H.Oka, *On the nonlinear mean ergodic theorems for asymptotically nonexpansive mappings in Banach spaces*, RIMS(Kyoto Univ.) Kokyuroku, **730**(1990), 1-20.

[9] N.Shioji and W.Takahashi, *Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces*, J. Approximation Theory, **97**(1999), 53-64.

[10] K.K.Tan and H.K.Xu, *The nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., **114**(1992), 399-404.

Chapter 3

A Nonlinear Strong Ergodic Theorem for Families of Asymptotically Nonexpansive Mappings with Compact Domains

3.1 Introduction

The first nonlinear ergodic theorem for nonexpansive mappings with bounded domains in a Hilbert space was proved by Baillon [3]. Baillon and Brezis [4] also proved the following nonlinear ergodic theorem for nonexpansive semigroups in a Hilbert space : Let C be a nonempty closed convex subset of a Hilbert space and let $\mathcal{S} = \{S(t) | t \geq 0\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Then, for every $x \in C$, $\frac{1}{t}\int_0^t S(\tau)x d\tau$ converges weakly to some $y \in F(\mathcal{S})$. Hirano and Takahashi [8] extended Baillon and Brezis's theorem to an asymptotically nonexpansive semigroup. Hirano and Takahashi's theorem was extended to a uniformly convex Banach space whose norm is Fréchet differentiable by Tan and Xu [11]. On the other hand, Atsushiba and Takahashi [2] obtained a nonlinear ergodic theorem for nonexpansive semigroups with compact domains in a Banach space which generalizes Dafermos and Slemrod's result [7] : Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) | t \geq 0\}$ be a nonexpansive semigroup on C. Then, for every $x \in C$, $\frac{1}{t}\int_0^t S(\tau + h)x d\tau$ converges strongly to some $y \in F(\mathcal{S})$ uniformly in $h \geq 0$.

In this chapter, we extend Atsushiba and Takahashi's theorem to an asymptotically nonexpansive semigroup by using the methods employed in Atsushiba and Takahashi [1, 2], Bruck [5, 6] and Shioji and Takahashi [10].

3.2 Preliminaries and lemmas

Throughout this chapter, a Banach space is real and we denote by $\mathbf{N}$ and R^+, the set of all positive integers and the set of all nonnegative real numbers, respectively. We denote by Δ^n the set $\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$ for $n \in \mathbf{N}$. Let E be a Banach space and let $r > 0$. We denote by $D_r(x)$ the open ball in E with center x and radius r. For a subset C of E, we denote by $co\, C$ the convex hull of C. E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let C be a subset of E, let T be a mapping from C into itself and let $\varepsilon > 0$. By $F_\varepsilon(T)$, we mean the set $\{x \in C \mid \|x - Tx\| \leq \varepsilon\}$. Let $K > 0$. We denote by $Lip(C, K)$, the set of all mappings from C into itself satisfying $\|Tx - Ty\| \leq K\|x - y\|$ for each $x, y \in C$. We denote by Γ the set of all strictly increasing, continuous convex functions $\gamma{:}R^+ \longmapsto R^+$ with $\gamma(0) = 0$. Let C be a nonempty subset of E. C is said to satisfy the convex approximation property if for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that $coM \subset co_m M + D_\varepsilon(0)$ for every subset M of C, where $co_m M = \{\sum_{i=1}^m \lambda_i x_i \mid x_i \in M, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1\}$.

A family $\mathcal{S} = \{S(t) \mid t \geq 0\}$ is said to be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) \mid t \geq 0\}$ if

(i) for each $t \geq 0$, $S(t)$ is a mapping from C into itself and $\|S(t)x - S(t)y\| \leq k(t)\|x - y\|$ for each $x, y \in C$;

(ii) $S(t+s)x = S(t)S(s)x$ for each $t, s \geq 0$ and $x \in C$;

(iii) $S(0)x = x$ for each $x \in C$;

(iv) for each $x \in C$, the mapping $t \mapsto S(t)x$ is continuous.

(v) $t \mapsto k(t)$ is continuous mapping from the set of nonnegative real numbers into itself;

(vi) $\limsup_{t\to\infty} k(t) \leq 1$.

$\mathcal{S}$ is said to be a nonexpansive semigroup on C if $k(t) = 1$ for all $t \geq 0$. We denote by $F(\mathcal{S})$, the set of common fixed points of $\mathcal{S} = \{S(t) \mid t \geq 0\}$, i.e., $\cap_{t\geq 0}\{x \in C \mid S(t)x = x\}$. The following lemmas was obtained by Bruck [5, 6].

Lemma 3.1 *Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, there exists $\gamma \in \Gamma$ such that for each $K > 0$ and $T \in Lip(C, K)$,*

$$\|T(\lambda x + (1-\lambda)y) - (\lambda Tx + (1-\lambda)Ty)\| \leq K\gamma^{-1}\Big(\|x-y\| - \frac{1}{K}\|Tx - Ty\|\Big)$$

holds for every $x, y \in C$ and $\lambda \in [0,1]$.

Lemma 3.2 *Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, for each $p \in \boldsymbol{N}$, there exists $\gamma_p \in \Gamma$ such that for each $K > 0$ and $T \in Lip(C, K)$,*

$$\Big\| T\Big(\sum_{i=1}^{p} \lambda_i x_i\Big) - \sum_{i=1}^{p} \lambda_i T x_i \Big\| \leq K\gamma_p^{-1}\Big(\max_{1\leq i,j\leq p}\Big\{\|x_i - x_j\| - \frac{1}{K}\|Tx_i - Tx_j\|\Big\}\Big)$$

holds for every $x_1, x_2, \ldots, x_p$ in C and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Delta^p$.

Following ideas in Atsushiba and Takahashi [1, 2], we can show the following lemma.

Lemma 3.3 *Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \mid t \geq 0\}$ be an asymptotically nonexpansive semigroup on C. Let $x \in C$ and $t > 0$. Then, for each $\varepsilon > 0$, there exist $l_0 = l_0(t, \varepsilon) \geq 0$ and $m_0 = m_0(t, \varepsilon) \geq 0$ such that*

$$\Big\| \frac{1}{t}\int_0^t S(l+m+\tau)x\, d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\, d\tau\Big) \Big\| < \varepsilon$$

for every $l \geq l_0$ and $m \geq m_0$.

Proof. Let $x \in C$, $t > 0$ and $\varepsilon > 0$. Let $\{k(t) \mid t \geq 0\}$ be Lipschitz constants of $\mathcal{S}$. Put $\sup\{k(t) \mid t \geq 0\} = M_0$. Since $\{k(t) \mid t \geq 0\}$ is bounded, $M_0 < \infty$ holds. From the assumption of $\mathcal{S}$, we have

$$\begin{aligned}
&\Big\| \frac{1}{t}\int_0^t S(l+m+\tau)x\, d\tau - \frac{1}{n}\sum_{i=1}^{n} S\Big(l+m+\frac{t}{n}i\Big)x \Big\| \\
&\leq \frac{1}{t}\sum_{i=1}^{n}\int_{\frac{i-1}{n}t}^{\frac{i}{n}t} \Big\| S(l+m+\tau)x - S\Big(l+m+\frac{t}{n}i\Big)x \Big\|\, d\tau \\
&\leq \frac{M_0}{t}\sum_{i=1}^{n}\int_{\frac{i-1}{n}t}^{\frac{i}{n}t} \Big\| S(\tau)x - S\Big(\frac{t}{n}i\Big)x \Big\|\, d\tau \\
&\leq \frac{M_0}{t}\sum_{i=1}^{n}\Big\{M_0 \cdot \frac{t}{n}\Big(\sup_{0\leq\tau\leq\frac{t}{n}} \Big\| S(\tau)x - S\Big(\frac{t}{n}\Big)x \Big\|\Big)\Big\} \\
&= M_0^2 \cdot \sup_{0\leq\tau\leq\frac{t}{n}} \Big\| S(\tau)x - S\Big(\frac{t}{n}\Big)x \Big\| \longrightarrow 0,
\end{aligned}$$

as $n \to \infty$, uniformly in $l, m \geq 0$. Similarly, we have

$$\left\| S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\,d\tau\Big) - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\big(m+\frac{t}{n}i\big)x\Big) \right\| \longrightarrow 0,$$

as $n \to \infty$, uniformly in $l, m \geq 0$. So, there exists $N_1 \in \mathbf{N}$ such that

$$\left\| \frac{1}{t}\int_0^t S(l+m+\tau)x\,d\tau - \frac{1}{n}\sum_{i=1}^{n} S\big(l+m+\frac{t}{n}i\big)x \right\| < \frac{\varepsilon}{3}$$

and

$$\left\| S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\,d\tau\Big) - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\big(m+\frac{t}{n}i\big)x\Big) \right\| < \frac{\varepsilon}{3}$$

for every $n \geq N_1$ and $l, m \geq 0$. Hence we get

$$\begin{aligned} &\left\| \frac{1}{t}\int_0^t S(l+m+\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\,d\tau\Big) \right\| \qquad (3.1)\\ \leq\ & \left\| \frac{1}{t}\int_0^t S(l+m+\tau)x\,d\tau - \frac{1}{n}\sum_{i=1}^{n} S\big(l+m+\frac{t}{n}i\big)x \right\| \\ &+ \left\| \frac{1}{n}\sum_{i=1}^{n} S\big(l+m+\frac{t}{n}i\big)x - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\big(m+\frac{t}{n}i\big)x\Big) \right\| \\ &+ \left\| S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\big(m+\frac{t}{n}i\big)x\Big) - S(l)\Big(\frac{1}{t}\int_0^t S(m+\tau)x\,d\tau\Big) \right\| \\ \leq\ & \frac{2}{3}\varepsilon + \left\| \frac{1}{n}\sum_{i=1}^{n} S\big(l+m+\frac{t}{n}i\big)x - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S(m+\frac{t}{n}i)x\Big) \right\| \end{aligned}$$

for every $n \geq N_1$ and $l, m \geq 0$. Fix $n \in \mathbf{N}$ with $n \geq N_1$. Without loss of generality, we assume that $k(l) > 0$ for all $l \in R^+$. From Lemma 3.2, there exists $\gamma_n \in \Gamma$ such that

$$\begin{aligned} &\left\| \frac{1}{n}\sum_{i=1}^{n} S\big(l+m+\frac{t}{n}i\big)x - S(l)\Big(\frac{1}{n}\sum_{i=1}^{n} S\big(m+\frac{t}{n}i\big)x\Big) \right\| \qquad (3.2)\\ &\qquad \leq\ k(l)\gamma_n^{-1}\Big(\max_{1\leq i,j\leq n}\Big\{\Big\| S\big(m+\frac{t}{n}i\big)x - S\big(m+\frac{t}{n}j\big)x \Big\| \\ &\qquad\qquad - \frac{1}{k(l)}\Big\| S\big(l+m+\frac{t}{n}i\big)x - S\big(l+m+\frac{t}{n}j\big)x \Big\|\Big\}\Big) \end{aligned}$$

for every $l, m \geq 0$. From $\gamma_n \in \Gamma$, there exists $\delta > 0$ such that

$$k(l)\gamma_n^{-1}(\delta) < \frac{\varepsilon}{3} \qquad (3.3)$$

for every $l \geq 0$. For $1 \leq i, j \leq n$, we set

$$r_{i,j} = \inf_{m \geq 0} \left\| S\Big(m + \frac{t}{n}i\Big)x - S\Big(m + \frac{t}{n}j\Big)x \right\|.$$

There exists $m_1 \geq 0$ such that

$$\left\| S\Big(m_1 + \frac{t}{n}i\Big)x - S\Big(m_1 + \frac{t}{n}j\Big)x \right\| < r_{i,j} + \frac{\delta}{4}.$$

By $\limsup_{l \to \infty} k(l) \leq 1$, there exists $l_1 > 0$ such that

$$k(l) \leq \frac{r_{i,j} + \frac{\delta}{2}}{\|S(m_1 + \frac{t}{n}i)x - S(m_1 + \frac{t}{n}j)x\| + \frac{\delta}{4}}$$

for every $l \geq l_1$. So, we have

$$\begin{aligned} &\left\| S\Big(l + m_1 + \frac{t}{n}i\Big)x - S\Big(l + m_1 + \frac{t}{n}j\Big)x \right\| \\ &\leq \; k(l) \left\| S\Big(m_1 + \frac{t}{n}i\Big)x - S\Big(m_1 + \frac{t}{n}j\Big)x \right\| \leq r_{i,j} + \frac{\delta}{2} \end{aligned}$$

for every $l \geq l_1$. Put $m_2 = m_2(i, j) = l_1 + m_1$. Then, there holds

$$\left\| S\Big(m + \frac{t}{n}i\Big)x - S\Big(m + \frac{t}{n}j\Big)x \right\| \leq r_{i,j} + \frac{\delta}{2}$$

for every $m \geq m_2$. Similarly, there exists $l_2 = l_2(i, j) \geq 0$ such that

$$r_{i,j} - \frac{\delta}{2} \leq \frac{1}{k(l)} \left\| S\Big(l + m + \frac{t}{n}i\Big)x - S\Big(l + m + \frac{t}{n}j\Big)x \right\|$$

for every $l \geq l_2$ and $m \geq m_2$. Let $l_0 = \max\{l_2(i, j) \,|\, 1 \leq i, j \leq n\}$ and $m_0 = \max\{m_2(i, j) \,|\, 1 \leq i, j \leq n\}$. Then, we have

$$\begin{aligned} 0 \leq \max_{1 \leq i,j \leq n} \Big\{ &\left\| S\Big(m + \frac{t}{n}i\Big)x - S\Big(m + \frac{t}{n}j\Big)x \right\| \\ &- \frac{1}{k(l)} \left\| S\Big(l + m + \frac{t}{n}i\Big)x - S\Big(l + m + \frac{t}{n}j\Big)x \right\| \Big\} \leq \delta \end{aligned} \tag{3.4}$$

for every $l \geq l_0$ and $m \geq m_0$. So, it follows from (3.1), (3.2), (3.3) and (3.4) that

$$\begin{aligned} &\left\| \frac{1}{t} \int_0^t S(l + m + \tau)x \, d\tau - S(l)\Big(\frac{1}{t} \int_0^t S(m + \tau)x \, d\tau\Big) \right\| \\ &\leq \; \frac{2}{3}\varepsilon + k(l)\gamma_n^{-1}(\delta) \leq \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for every $l \geq l_0$ and $m \geq m_0$. □

The following lemma was obtained by Atsushiba and Takahashi [1].

Lemma 3.4 *Let C be a nonempty compact subset of a Banach space. Then, C satisfies the convex approximation property.*

The following lemmas were obtained by Nakajo and Takahashi [9].

Lemma 3.5 *Let C be a nonempty compact convex subset of a strictly convex Banach space. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$ holds for every $T \in Lip(C, 1+\delta)$, where $\overline{co}\,A$ is the closure of the convex hull of A.*

Lemma 3.6 *Let C be a nonempty closed bounded convex subset of a Banach space. Let $\gamma \in \Gamma$, $L \geq 1$ and $T \in Lip(C, L)$ such that*

$$\|T(\lambda x + (1-\lambda)y) - (\lambda Tx + (1-\lambda)Ty)\| \leq L\gamma^{-1}\Big(\|x-y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all $x, y \in C$ and $\lambda \in [0,1]$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $\frac{1}{n}\sum_{i=1}^{n}\|x_{i+1} - Tx_i\| \leq a_n$ and $\frac{1}{n}\sum_{i=1}^{n}\|y_{i+1} - Ty_i\| \leq a_n$ for all $n \in \boldsymbol{N}$, where $\{a_n\}$ is a sequence in R^+. Then, for each $n \in \boldsymbol{N}$ and $\lambda \in [0,1]$,

$$\frac{1}{n}\sum_{i=1}^{n}\|\lambda x_{i+1} + (1-\lambda)y_{i+1} - T(\lambda x_i + (1-\lambda)y_i)\| \leq L\gamma^{-1}\Big(\frac{R}{n} + (L-1)R + 2a_n\Big) + a_n,$$

where $R = diam\, C$.

Lemma 3.7 *Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \in \boldsymbol{N}$ such that for every $T \in Lip(C, 1+\delta)$ and $\{x_n\}$ in C satisfying $\|x_{n+1} - Tx_n\| \leq \delta$ for all $n \in \boldsymbol{N} \cup \{0\}$, there holds $\frac{1}{n}\sum_{i=0}^{n-1} x_i \in F_\varepsilon(T)$ for every $n \geq N_0$.*

Lemma 3.8 *Let C be a nonempty compact convex subset of a strictly convex Banach space. Then, for each $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \in \boldsymbol{N}$ such that for every $l \in \boldsymbol{N}$ and mapping T from C into itself satisfying $T^l \in Lip(C, 1+\delta)$, there holds*

$$\Big\| \frac{1}{m}\sum_{i=0}^{m-1} T^i x - T^l\Big(\frac{1}{m}\sum_{i=0}^{m-1} T^i x\Big) \Big\| \leq \varepsilon$$

for all $m \in \boldsymbol{N}$ with $m - 1 \geq lN_0$ and $x \in C$.

As in the proof of [10], we have the following lemma. However, for the sake of completeness, we give the proof.

Corollary 3.9 *Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \mid t \geq 0\}$ be an asymptotically nonexpansive semigroup on C. Then,*

$$\limsup_{l\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| = 0.$$

Proof. Let $\{k(t) \mid t \geq 0\}$ be Lipschitz constants of $\mathcal{S}$. Let $\varepsilon > 0$. There exist $\delta > 0$ and $N_0 \in \mathbf{N}$ which satisfy the condition in Lemma 3.8. From $\limsup_{l\to\infty} k(l) \leq 1$, there exists $l_0 \geq 0$ such that $k(l) < 1 + \delta$ for every $l \geq l_0$. Let $l > l_0$. Then, there exists $t_l > 0$ such that $\frac{1}{N_0} \geq \frac{l}{t}$ for all $t \geq t_l$. Let $t \geq t_l$. For each $n \in \mathbf{N}$, let j_n be the nonnegative integer which satisfies $t \cdot \frac{j_n}{n} \leq l < t \cdot \frac{j_n+1}{n}$. Then, $n \geq j_n N_0$ for every $n \in \mathbf{N}$ and by $l > l_0$, there exists $n_0 \in \mathbf{N}$ such that $t \cdot \frac{j_n}{n} \geq l_0$ for all $n \geq n_0$. Hence, from Lemma 3.8 we get

$$\left\| \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x - S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x\Big) \right\| < \varepsilon$$

for every $n \geq n_0$ and $x \in C$. So, we have

$$\begin{aligned}
&\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \\
&\leq \left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x \right\| \\
&\quad + \left\| \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x - S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x\Big) \right\| \\
&\quad + \left\| S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x\Big) - S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \\
&\quad + \left\| S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \\
&\leq (2+\delta) \left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - \frac{1}{n+1}\sum_{i=0}^{n} S\Big(\frac{t}{n}i\Big)x \right\| \\
&\quad + \varepsilon + \left\| S\Big(\frac{t}{n}j_n\Big)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\|
\end{aligned}$$

for every $n \geq n_0$ and $x \in C$. Tending n to infinity, we get

$$\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big) \right\| \leq \varepsilon$$

for every $x \in C$. So,we have

$$\limsup_{l\to\infty}\limsup_{t\to\infty}\sup_{x\in C}\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(\tau)x\,d\tau\Big)\right\| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion. □

Remark 3.10 *We can obtain $F(\mathcal{S}) \neq \emptyset$. In fact, let $x \in C$ and put $x_t = \frac{1}{t}\int_0^t S(\tau)x\,d\tau$ for every $t > 0$. Since C is compact, there exists a subnet $\{x_{t_\alpha}\}$ of $\{x_t\}$ such that x_{t_α} converges strongly to some x_0 in C. So, we have*

$$\begin{aligned} 0 &= \limsup_{l\to\infty}\limsup_{t\to\infty}\|x_t - S(l)x_t\| \\ &= \limsup_{l\to\infty}\limsup_{\alpha}\|x_{t_\alpha} - S(l)x_{t_\alpha}\| = \limsup_{l\to\infty}\|x_0 - S(l)x_0\| \end{aligned}$$

and hence

$$\begin{aligned} \|x_0 - S(s)x_0\| &\le \limsup_{l\to\infty}\|x_0 - S(l)x_0\| + \limsup_{l\to\infty}\|S(l)x_0 - S(s)x_0\| \\ &\le 0 + k(s)\cdot 0 = 0 \end{aligned}$$

for every $s \ge 0$. Therefore $x_0 \in F(\mathcal{S})$.

3.3 Strong ergodic theorem

The following is crucial to prove our theorem.

Lemma 3.11 *Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t)\,|\,t \ge 0\}$ be an asymptotically nonexpansive semigroup on C. Let $x \in C$. Then, there exists a net $\{i_t\}_{t\ge 0} \subset R^+$ such that $\lim_{t\to\infty}\left\| \frac{1}{t}\int_0^t S(\tau + i_t)x\,d\tau - z\right\|$ exists for every $z \in F(\mathcal{S})$.*

Proof. We use the methods employed in Atsushiba and Takahashi [1, 2]. From Lemma 3.3, there exist nets $\{i_t\}_{t\ge 0}$ in R^+ and $\{l_t\}_{t\ge 0}$ in R^+ such that

$$\left\| \frac{1}{t}\int_0^t S(l+i+\tau)x\,d\tau - S(l)\Big(\frac{1}{t}\int_0^t S(i+\tau)x\,d\tau\Big)\right\| < \frac{1}{t} \tag{3.5}$$

for every $t > 0$, $i \ge i_t$ and $l \ge l_t$. Let $z \in F(\mathcal{S})$. For every $s, t > 0$, consider

$$I = \left\| \frac{1}{s}\int_0^s S(i_s + i_t + \tau)x\,d\tau - z\right\|$$

$$\begin{aligned}
&= \left\| \frac{1}{s}\int_0^s \Big(\frac{1}{t}\int_0^t S(\tau+\sigma+i_s+i_t)x\,d\sigma\Big)\,d\tau \right. \\
&\qquad \left. + \frac{1}{st}\int_0^t (t-\tau)\{S(\tau+i_s+i_t)x - S(s+\tau+i_s+i_t)x\}\,d\tau - z \right\|, \\
I_1 &= \left\| \frac{1}{st}\int_0^t (t-\tau)\{S(\tau+i_s+i_t)x - S(s+\tau+i_s+i_t)x\}\,d\tau \right\|, \\
I_2 &= \left\| \frac{1}{s}\int_0^s \Big(\frac{1}{t}\int_0^t S(\tau+\sigma+i_s+i_t)x\,d\sigma\Big)\,d\tau \right. \\
&\qquad \left. - \frac{1}{s}\int_0^s S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big)\,d\tau \right\|
\end{aligned}$$

and

$$I_3 = \left\| \frac{1}{s}\int_0^s S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big)\,d\tau - z \right\|.$$

Then, we have $I \leq I_1 + I_2 + I_3$. Fix $t > 0$ and put $R = diam\, C$. We have

$$I_1 \leq \frac{1}{st}\int_0^t (t-\tau)R\,d\tau = \frac{t}{2s}R$$

for every $s > 0$. It follows from (3.5) that

$$\begin{aligned}
I_2 &\leq \frac{1}{s}\int_0^s \left\| \frac{1}{t}\int_0^t S(\tau+\sigma+i_s+i_t)x\,d\sigma - S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big) \right\| d\tau \\
&\leq \frac{1}{s}\int_0^s \frac{1}{t}\,d\tau = \frac{1}{t}
\end{aligned}$$

for every $s > 0$ with $i_s \geq l_t$. By $z \in F(\mathcal{S})$, we obtain

$$\begin{aligned}
I_3 &\leq \frac{1}{s}\int_0^s \left\| S(\tau+i_s)\Big(\frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma\Big) - z \right\| d\tau \\
&\leq \frac{1}{s}\int_0^s k(\tau+i_s) \left\| \frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma - z \right\| d\tau \\
&= \left\{\frac{1}{s}\int_0^s k(\tau+i_s)\,d\tau\right\}\cdot \left\| \frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma - z \right\|
\end{aligned}$$

for every $s > 0$, where $\{k(t) \mid t \geq 0\}$ is Lipschitz constants of $\mathcal{S}$. Therefore, since $\lim_{s\to\infty} I_1 = 0$ and $\{k(t) \mid t \geq 0\}$ is Lipschitz constants of $\mathcal{S}$, we have

$$\begin{aligned}
&\limsup_{s\to\infty} \left\| \frac{1}{s}\int_0^s S(\tau+i_s)x\,d\tau - z \right\| \\
&= \limsup_{s\to\infty} \left\| \frac{1}{s}\int_0^s S(\tau+i_s+i_t)x\,d\tau - z \right\| \\
&= \limsup_{s\to\infty} I \leq \limsup_{s\to\infty}(I_1+I_2+I_3) \\
&\leq \frac{1}{t} + \left\| \frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma - z \right\| \cdot \limsup_{s\to\infty} \frac{1}{s}\int_0^s k(\tau+i_s)\,d\tau \\
&\leq \frac{1}{t} + \left\| \frac{1}{t}\int_0^t S(\sigma+i_t)x\,d\sigma - z \right\|
\end{aligned}$$

for every $t > 0$. So, we get

$$\limsup_{s\to\infty} \left\| \frac{1}{s}\int_0^s S(\tau + i_s)x\,d\tau - z \right\| \le \liminf_{t\to\infty} \left\| \frac{1}{t}\int_0^t S(\sigma + i_t)x\,d\sigma - z \right\|$$

Hence, $\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t S(\tau + i_t)x\,d\tau - z \right\|$ exists. □

Remark 3.12 *In Lemma 3.11, take a net $\{i'_t\}_{t\ge 0}$ in R^+ such that $i'_t \ge i_t$ for every $t \ge 0$. Then, we can get*

$$\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t S(\tau + i_t)x\,d\tau - z \right\| = \lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t S(\tau + i'_t)x\,d\tau - z \right\|$$

for every $z \in F(\mathcal{S})$.

Theorem 3.13 *Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S} = \{S(t) \mid t \ge 0\}$ be an asymptotically nonexpansive semigroup on C. Let $x \in C$. Then, $\frac{1}{t}\int_0^t S(\tau + h)x\,d\tau$ converges strongly to a common fixed point of $\mathcal{S}$ uniformly in $h \ge 0$. In this case, if $Qx = \lim_{t\to\infty} \frac{1}{t}\int_0^t S(\tau)x\,d\tau$ for every $x \in C$, then Q is a nonexpansive mapping from C onto $F(\mathcal{S})$ such that $QS(t) = S(t)Q = Q$ for every $t \ge 0$ and $Qx \in \overline{co}\{S(t)x \mid t \ge 0\}$ for every $x \in C$.*

Proof. From Lemma 3.11, there exists a net $\{i_t\}_{t\ge 0}$ in R^+ such that

$$\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t S(\tau + i_t)x\,d\tau - z \right\| \tag{3.6}$$

exists for every $z \in F(\mathcal{S})$. Set $\Phi_t = \frac{1}{t}\int_0^t S(\tau + i_t)x\,d\tau$. As in the Remark 3.10, there exists a subnet $\{\Phi_{t_\alpha}\}$ of $\{\Phi_t\}$ such that Φ_{t_α} converges strongly to a common fixed point y_0 of $\mathcal{S}$. So it follows from (3.6) that

$$\lim_{t\to\infty} \|\Phi_t - y_0\| = \lim_{\alpha} \|\Phi_{t_\alpha} - y_0\| = 0.$$

This implies that $\Phi_t \longrightarrow y_0$. Next we prove that $\frac{1}{t}\int_0^t S(\tau + i_t + h)x\,d\tau$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in $h \ge 0$. Take a net $\{i'_t\}_{t\ge 0}$ in R^+ such that $i'_t \ge i_t$ for every $t \ge 0$. Then, from Remark 3.12, we have $\frac{1}{t}\int_0^t S(\tau + i'_t)x\,d\tau \longrightarrow y_0 \in F(\mathcal{S})$. Since $\{i'_t\}_{t\ge 0}$ is any net in R^+ such that

$i'_t \geq i_t$ for every $t \geq 0$, it follows that $\frac{1}{t}\int_0^t S(\tau+i_t+h)x\,d\tau$ converges strongly to y_0 uniformly in $h \geq 0$. Let $\varepsilon > 0$. Then, there exists $t_0 \geq 0$ such that

$$\left\| \frac{1}{t}\int_0^t S(\tau+i_t+h)x\,d\tau - y_0 \right\| < \varepsilon$$

for every $t \geq t_0$ and $h \geq 0$. So, we have

$$\begin{aligned}
&\left\| \frac{1}{t}\int_0^t S(\tau+h)x\,d\tau - y_0 \right\| \\
&= \left\| \frac{1}{t}\int_0^t \Big(\frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma\Big)\,d\tau \right. \\
&\qquad \left. + \frac{1}{ts}\int_0^s (s-\tau)\{S(\tau+h)x - S(t+\tau+h)x\}\,d\tau - y_0 \right\| \\
&\leq \frac{1}{t}\left\| \int_0^t \Big\{\frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0\Big\}\,d\tau \right\| \\
&\qquad + \frac{1}{ts}\int_0^s (s-\tau)\|S(\tau+h)x - S(t+\tau+h)x\|\,d\tau \\
&= \frac{1}{t}\left\| \int_0^{i_s} \Big\{\frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0\Big\}\,d\tau \right. \\
&\qquad \left. + \int_{i_s}^t \Big\{\frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0\Big\}\,d\tau \right\| \\
&\qquad + \frac{1}{ts}\int_0^s (s-\tau)\|S(\tau+h)x - S(t+\tau+h)x\|\,d\tau \\
&\leq \frac{1}{t}\int_0^{i_s} \left\| \frac{1}{s}\int_0^s S(\tau+h+\sigma)x\,d\sigma - y_0 \right\|\,d\tau \\
&\qquad + \frac{1}{t}\int_0^{t-i_s} \left\| \frac{1}{s}\int_0^s S(\tau+i_s+h+\sigma)x\,d\sigma - y_0 \right\|\,d\tau \\
&\qquad + \frac{1}{ts}\int_0^s (s-\tau)\|S(\tau+h)x - S(t+\tau+h)x\|\,d\tau \\
&\leq \frac{i_s}{t}R + \frac{t-i_s}{t}\varepsilon + \frac{s}{2t}R
\end{aligned}$$

for every $s \geq t_0$, $t \geq i_s$ and $h \geq 0$, where $R = diam\, C$. Since $\varepsilon > 0$ is arbitrary, it follows that $\frac{1}{t}\int_0^t S(\tau+h)x\,d\tau$ converges strongly to $y_0 \in F(\mathcal{S})$ uniformly in $h \geq 0$. If $Qx = \lim_{t\to\infty}\frac{1}{t}\int_0^t S(\tau)x\,d\tau$ for every $x \in C$, then Q is a nonexpansive mapping from C onto $F(\mathcal{S})$. In fact, let $\{k(t)\,|\,t \geq 0\}$ be Lipschitz constants of $\mathcal{S}$. Then, we get

$$\left\| \frac{1}{t}\int_0^t S(\tau)x\,d\tau - \frac{1}{t}\int_0^t S(\tau)y\,d\tau \right\| \leq \|x-y\| \cdot \frac{1}{t}\int_0^t k(\tau)\,d\tau\,,$$

which implies $\|Qx - Qy\| \leq \|x - y\|$ for every $x, y \in C$.
Moreover, we have $QS(t) = S(t)Q = Q$ for every $t \geq 0$ and $Qx \in \overline{co}\{S(t)x \mid t \geq 0\}$ for every $x \in C$. □

Bibliography

[1] S.Atsushiba and W.Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domains*, Math. Japonica, **52**(2000), 183-195.

[2] S.Atsushiba and W.Takahashi, *Strong convergence theorems for one-parameter nonexpansive semigroups with compact domains*, to appear in Nonlinear Analysis and Its Applications (S.P.Singh and Bruce Watson, Eds.), Marcel Dekker Inc..

[3] J.B.Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér.A-B, **280**(1975), 1511-1514.

[4] J.B.Baillon and H.Brezis, *Une remarque sur le comportement asymptotique des semigroupes non lineaires*, Houston J. Math., **2**(1976), 5-7.

[5] R.E.Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., **32**(1979), 107-116.

[6] R.E.Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math., **38**(1981), 304-314.

[7] C.M.Dafermos and M.Slemrod, *Asymptotic behavior of nonlinear contraction semigroups*, J. Funct. Anal., **13**(1973), 97-106.

[8] N.Hirano and W.Takahashi, *Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces*, Kodai Math. J., **2**(1979), 11-25.

[9] K.Nakajo and W.Takahashi, *A nonlinear strong ergodic theorem for asymptotically nonexpansive mappings with compact domains*, to appear in Dynam. Cont. Disc. Impul. Syst.

[10] N.Shioji and W.Takahashi, *Strong convergence theorems for continuous semigroups in Banach spaces*, Math. Japonica, **50**(1999), 57-66.

[11] K.K.Tan and H.K.Xu, *An ergodic theorem for nonlinear semigroups of Lipschitzian mappings in Banach spaces*, Nonlinear Anal. **19**(1992), 805-813.

Chapter 4

Strong Convergence Theorems for Nonexpansive Mappings and Nonexpansive Semigroups

4.1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space and let T be a nonexpansive mapping from C into itself, that is, $\|Tx - Ty\| \leq \|x - y\|$ holds for every $x, y \in C$. We denote by $\mathbf{N}$ the set of all positive integers. Halpern [3] introduced an iteration procedure as follows:

$$x_0 = x \in C, \qquad x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, 1]$. Wittmann [12] proved that $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$ when $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, where $F(T) = \{z \in C \,|\, Tz = z\}$ and $P_{F(T)}(\cdot)$ is the metric projection onto $F(T)$.

The purpose of this chapter is to make another method of strong convergence. Motivated by Solodov and Svaiter [10], we consider the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n\,, \\ C_n = \{z \in C \,|\, \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \,|\, (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{4.1}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. Then, we show that $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$ by the hybrid method in the mathematical programming. By this method, we also study the proximal point algorithm [4, 5, 7, 12]. Finally, we obtain a strong convergence theorem for a family of nonexpansive mappings in a Hilbert space.

4.2 Preliminaries and lemma

Throughout this chapter, let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. We know that H satisfies Opial's condition [6], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. We also know that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$ holds. Further, let $\{x_n\}$ be a sequence of H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then, there holds $x_n \to x$. Let C be a nonempty closed convex subset of H. We denote by $P_C(\cdot)$ the metric projection onto C. It is known that for $z \in C$, $z = P_C(x)$ is equivalent to $(z - y, x - z) \geq 0$ for every $y \in C$. Let T be a nonexpansive mapping from C into itself. It is known that $F(T)$ is closed and convex. A family $\mathcal{S} = \{T(s) \mid 0 \leq s < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C$, $s \longmapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \bigcap_{0\leq s<\infty} F(T(s))$. It is known that $F(\mathcal{S})$ is closed and convex. An operator $A \subset H \times H$ is said to be monotone if $(x_1 - x_2, y_1 - y_2) \geq 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. A monotone operator A is said to be maximal if the graph of A is not properly contained in the graph of any other monotone operator. Let A be a monotone operator. It is known that A is maximal iff $R(I + rA) = H$ for every $r > 0$, where $R(I + rA) = \cup\{z + rAz \mid z \in H, Az \neq \emptyset\}$. It is also known that A is maximal iff for $(u, v) \in H \times H$, $(x - u, y - v) \geq 0$ for every $(x, y) \in A$ implies $v \in Au$. For a maximal monotone operator A, we know that $A^{-1}0 = \{x \in H \mid 0 \in Ax\}$ is closed and convex. If A is monotone, then we can define, for each $r > 0$, a nonexpansive mapping

$J_r : R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$, where $D(A) = \{z \in H \mid Az \neq \emptyset\}$. J_r is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$. We also have $F(J_r) = A^{-1}0$ for each $r > 0$, where $F(J_r) = \{z \in D(A) \mid J_r z = z\}$; see [11] for more details.

The following lemma was proved by Shimizu and Takahashi [8]; see also [1, 2, 9].

Lemma 4.1 *Let C be a nonempty bounded closed convex subset of H and let $\mathcal{S} = \{T(s) \mid 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Then, for any $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right)\right\| = 0\,.$$

4.3 Strong convergence theorems for nonexpansive mappings

Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. We consider the sequence $\{x_n\}$ generated by (4.1).

Lemma 4.2 *$\{x_n\}$ is well defined and $F(T) \subset C_n \cap Q_n$ for every $n \in \mathbb{N} \cup \{0\}$.*

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n \in \mathbf{N} \cup \{0\}$. It follows that C_n is convex for every $n \in \mathbf{N} \cup \{0\}$ because $\|y_n - z\| \leq \|x_n - z\|$ is equivalent to

$$\|y_n - x_n\|^2 + 2(y_n - x_n, x_n - z) \leq 0\,.$$

So, $C_n \cap Q_n$ is closed and convex for every $n \in \mathbf{N} \cup \{0\}$. Let $u \in F(T)$. Then from

$$\begin{aligned}\|y_n - u\| &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - u\| \\ &\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\|Tx_n - u\| \\ &\leq \|x_n - u\|\,,\end{aligned}$$

we have $u \in C_n$ for each $n \in \mathbf{N} \cup \{0\}$. So, we have $F(T) \subset C_n$ for all $n \in \mathbf{N} \cup \{0\}$.

Next, we show by mathematical induction that $\{x_n\}$ is well defined and

$F(T) \subset C_n \cap Q_n$ for each $n \in \mathbf{N} \cup \{0\}$. For $n = 0$, we have $x_0 = x \in C$ and $Q_0 = C$, and hence $F(T) \subset C_0 \cap Q_0$. Suppose that x_k is given and $F(T) \subset C_k \cap Q_k$ for some $k \in \mathbf{N} \cup \{0\}$. There exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x_0)$. From $x_{k+1} = P_{C_k \cap Q_k}(x_0)$, there holds

$$(x_{k+1} - z, x_0 - x_{k+1}) \geq 0$$

for each $z \in C_k \cap Q_k$. Since $F(T) \subset C_k \cap Q_k$, we get $F(T) \subset Q_{k+1}$. Therefore we have $F(T) \subset C_{k+1} \cap Q_{k+1}$. This completes the proof. □

Lemma 4.3 *$\{x_n\}$ is bounded.*

Proof. Since $F(T)$ is a nonempty closed convex subset of C, there exists a unique element $z_0 \in F(T)$ such that $z_0 = P_{F(T)}(x_0)$. From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in C_n \cap Q_n$. As $z_0 \in F(T) \subset C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\| \tag{4.2}$$

for each $n \in \mathbf{N} \cup \{0\}$. This implies that $\{x_n\}$ is bounded. □

Lemma 4.4 $\|x_{n+1} - x_n\| \to 0$.

Proof. As $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \in \mathbf{N} \cup \{0\}$. Therefore, by Lemma 4.3 a sequence $\{\|x_n - x_0\|\}$ is bounded and nondecreasing. So there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+1} \in Q_n$, we have $(x_n - x_{n+1}, x_0 - x_n) \geq 0$ and hence

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) - (x_{n+1} - x_0)\|^2 \\
&= \|x_n - x_0\|^2 - 2(x_n - x_0, x_{n+1} - x_0) + \|x_{n+1} - x_0\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2(x_n - x_{n+1}, x_0 - x_n) \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2
\end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. This implies that $\|x_{n+1} - x_n\| \to 0$. □

Theorem 4.5 *$x_n \to z_0$, where $z_0 = P_{F(T)}(x_0)$.*

Proof. Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. It follows from $x_{n+1} \in C_n$ that

$$\begin{aligned} \|Tx_n - x_n\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\le \frac{1}{1-\alpha_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\ &\le \frac{2}{1-\alpha_n}\|x_{n+1} - x_n\| \end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. By Lemma 4.4, we get

$$\|Tx_n - x_n\| \to 0\,. \tag{4.3}$$

Suppose that $w_0 \neq Tw_0$. From Opial's condition and (4.3), we have

$$\begin{aligned} \liminf_{i\to\infty} \|x_{n_i} - w_0\| &< \liminf_{i\to\infty} \|x_{n_i} - Tw_0\| \\ &\le \liminf_{i\to\infty}(\|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - w_0\|) \\ &= \liminf_{i\to\infty} \|x_{n_i} - w_0\|\,. \end{aligned}$$

This is a contradiction. Hence, we get

$$w_0 \in F(T)\,. \tag{4.4}$$

If $z_0 = P_{F(T)}(x_0)$, it follows from (4.2), (4.4) and the lower semicontinuity of norm that

$$\begin{aligned} \|x_0 - z_0\| &\le \|x_0 - w_0\| \le \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \\ &\le \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \le \|x_0 - z_0\|\,. \end{aligned}$$

Thus, we obtain $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|x_0 - w_0\| = \|x_0 - z_0\|$. This implies

$$x_{n_i} \to w_0 = z_0.$$

Therefore, we have $x_n \to z_0$. □

We apply this method to the proximal point algorithm [4, 5, 7, 12] and get the following theorem.

Theorem 4.6 *Let $A \subset H \times H$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let J_r be a resolvent of A, where $r > 0$. Define a sequence*

$\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in H\,, \\ y_n = J_{r_n}(x_n + f_n)\,, \\ C_n = \{z \in H \mid \|y_n - z\| \le \|x_n + f_n - z\|\}\,, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \ge 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{4.5}$$

for every $n \in N \cup \{0\}$, where $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} \|f_n\| = 0$. Then, $x_n \to z_0 = P_{A^{-1}0}(x_0)$.

Proof. As in the proof of Lemma 4.2, $\{x_n\}$ is well defined and $A^{-1}0 \subset C_n \cap Q_n$ for every $n \in \mathbf{N} \cup \{0\}$ because J_{r_n} is nonexpansive and $A^{-1}0 = \{z \in H \mid J_{r_n}z = z\}$ for every $n \in \mathbf{N} \cup \{0\}$. Results in Lemmas 4.3 and 4.4 hold because $A^{-1}0$ is nonempty, closed and convex. We also have from $\lim_{n\to\infty} \|f_n\| = 0$ that $\{y_n\}$ is bounded. Next, we suppose that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. It follows from $x_{n+1} \in C_n$ that

$$\begin{aligned} \|y_n - x_n\| &\le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le \|x_n + f_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le 2\|x_{n+1} - x_n\| + \|f_n\| \end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|f_n\| = 0$, we obtain $\|y_n - x_n\| \to 0$. This implies that

$$y_{n_i} \rightharpoonup w_0\,. \tag{4.6}$$

On the other hand, since A is monotone, we have, for every $i \in \mathbf{N}$ and $(u,v) \in A$,

$$(y_{n_i} - u, \frac{1}{r_{n_i}}(x_{n_i} + f_{n_i} - y_{n_i}) - v) \ge 0$$

and hence

$$(y_{n_i} - u, -v) \ge -\frac{1}{r_{n_i}} \|y_{n_i} - u\| \cdot \|y_{n_i} - (x_{n_i} + f_{n_i})\|\,.$$

By the boundedness of $\left\{ \frac{1}{r_{n_i}} \|y_{n_i} - u\| \right\}$, $\|y_{n_i} - (x_{n_i} + f_{n_i})\| \to 0$ and (4.6), we have $(w_0 - u, -v) \ge 0$ for every $(u,v) \in A$. Therefore, we get $w_0 \in A^{-1}0$ as A is maximal. If $z_0 = P_{A^{-1}0}(x_0)$, as in the proof of Theorem 4.5, we have

$$\begin{aligned} \|z_0 - x_0\| &\le \|w_0 - x_0\| \le \liminf_{i\to\infty} \|x_{n_i} - x_0\| \\ &\le \limsup_{i\to\infty} \|x_{n_i} - x_0\| \le \|z_0 - x_0\|\,. \end{aligned}$$

We obtain $\lim_{i\to\infty} x_{n_i} = w_0 = z_0$. Therefore, we get $\lim_{n\to\infty} x_n = z_0$. □

4.4 Strong convergence theorem for nonexpansive semigroups

Let C be a nonempty closed convex subset of H and $\mathcal{S} = \{T(s) \mid 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Note that $F(\mathcal{S})$ is closed and convex. Consider a sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(s)x_n\,ds\,, \\ C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{4.7}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\}$ is a sequence in $[0, a]$ for some $a \in [0, 1)$ and $\{t_n\}$ is a positive real divergent sequence. Using Lemma 4.1, we get the following theorem.

Theorem 4.7 $x_n \to z_0 = P_{F(\mathcal{S})}(x_0)$.

Proof. Since we have, for every $u \in F(\mathcal{S})$ and $n \in \mathbf{N} \cup \{0\}$,

$$\begin{aligned} \|y_n - u\| &\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - u \right\| \\ &\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} \|T(s)x_n - u\|\,ds \\ &\leq \alpha_n\|x_n - u\| + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} \|x_n - u\|\,ds \\ &= \alpha_n\|x_n - u\| + (1-\alpha_n)\|x_n - u\| \\ &= \|x_n - u\|\,, \end{aligned}$$

it follows that $F(\mathcal{S}) \subset C_n$ for every $n \in \mathbf{N} \cup \{0\}$. As in the proof of Lemma 4.2, we get that $\{x_n\}$ is well defined and $F(\mathcal{S}) \subset C_n \cap Q_n$ for each $n \in \mathbf{N} \cup \{0\}$. Since $F(\mathcal{S})$ is nonempty and $z_0 = P_{F(\mathcal{S})}(x_0)$, as in the proofs of Lemmas 4.3 and 4.4, we get that $\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$ for each $n \in \mathbf{N} \cup \{0\}$, $\{x_n\}$ is bounded and $\|x_{n+1} - x_n\| \to 0$. We assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. We have

$$\begin{aligned} \|T(s)x_n - x_n\| \leq\ & \left\| T(s)x_n - T(s)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\Big) \right\| \\ & + \left\| T(s)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\Big) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds \right\| \end{aligned}$$

$$
\begin{aligned}
&\quad + \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n \, ds - x_n \right\| \\
&\leq 2 \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n \, ds - x_n \right\| \qquad (4.8) \\
&\quad + \left\| T(s)\Big(\frac{1}{t_n} \int_0^{t_n} T(s)x_n \, ds\Big) - \frac{1}{t_n} \int_0^{t_n} T(s)x_n \, ds \right\|
\end{aligned}
$$

for every $0 \leq s < \infty$ and $n \in \mathbf{N} \cup \{0\}$. On the other hand, from $x_{n+1} \in C_n$, we have that

$$
\begin{aligned}
\left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n \, ds - x_n \right\| &= \frac{1}{1-\alpha_n} \|y_n - x_n\| \\
&\leq \frac{1}{1-\alpha_n} (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\
&\leq \frac{2}{1-\alpha_n} \|x_{n+1} - x_n\| \qquad (4.9)
\end{aligned}
$$

for every $n \in \mathbf{N} \cup \{0\}$. Let $X = \{z \in C \mid \|z - z_0\| \leq 2\|z_0 - x_0\|\}$. Then, X is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s \in [0, \infty)$ and contains $\{x_n\}$. By Lemma 4.1, we get

$$
\lim_{n \to \infty} \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n \, ds - T(h)\Big(\frac{1}{t_n} \int_0^{t_n} T(s)x_n \, ds\Big) \right\| = 0 \qquad (4.10)
$$

for every $h \in [0, \infty)$. By (4.8), (4.9), (4.10) and $\|x_{n+1} - x_n\| \to 0$, we obtain

$$
\|T(s)x_n - x_n\| \to 0
$$

for each $0 \leq s < \infty$. This implies that

$$
w_0 \in F(\mathcal{S})
$$

by Opial's condition. As in the proof of Theorem 4.5, we have $x_{n_i} \to w_0 = z_0$. Therefore, we get $x_n \to z_0$. □

Bibliography

[1] S.Atsushiba and W.Takahashi, *A weak convergence theorem for nonexpansive semigroups by the Mann iteration process in Banach spaces*, in Nonlinear Analysis and Convex Analysis (W.Takahashi and T.Tanaka Eds.), World Scientific, Singapore, 102-109.

[2] R.E.Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math.,38 (1981), 304-314.

[3] B.Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc.,73(1967), 957-961.

[4] S.Kamimura and W.Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory, 106(2000), 226-240, doi:10.1006/jath. 2000.3493.

[5] B.Martinet, *Regularisation d'inequations variationnelles par approximations successives*, Rev. Franc. Inform. Rech. Opér.,4(1970), 154-159.

[6] Z.Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc.,73(1967), 591-597.

[7] R.T.Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim.,14(1976), 877-898.

[8] T.Shimizu and W.Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl.,211(1997), 71-83, doi:10.1006/jmaa. 1997.5398.

[9] N.Shioji and W.Takahashi, *Strong convergence theorems for continuous semigroups in Banach spaces*, Math. Japonica, 50(1999), 57-66.

[10] M.V.Solodov and B.F.Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Programming Ser. A, 87(2000), 189-202.

[11] W.Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[12] R.Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., 58(1992), 486-491.

Chapter 5

Strong and Weak Convergence Theorems by an Improved Splitting Method

5.1 Introduction

Let H be a real Hilbert space and let $A : H \longrightarrow 2^H$, $B : H \longrightarrow 2^H$ and $A + B : H \longrightarrow 2^H$ be maximal monotone operators such that $D(B) \subset D(A)$ and $(A + B)^{-1}0 \neq \emptyset$, where $D(A)$ is the domain of A. Let C be a nonempty closed convex subset of H and let ∂i_C denote the subdifferential of the indicator function of C. We denote by $\mathbf{N}$ the set of all positive integers. As one of the methods of finding an element of $(A + B)^{-1}0$, there is the following splitting method (5.1) that was introduced by Passty [14] and by Lions and Mercier [8]:

$$\begin{cases} x_1 = x \in D(A)\,, \\ x_{n+1} = J_{\lambda_n}^B (x_n - \lambda_n w_n) \end{cases} \tag{5.1}$$

for every $n \in \mathbf{N}$, where $w_n \in Ax_n$, $\{\lambda_n\} \subset (0, \infty)$ and $J_{\lambda_n}^B = (I + \lambda_n B)^{-1}$. If $B = \partial i_C$, it is the projection method for variational inequalities by Brézis and Sibony [2] (see also Sibony [16]). Further, if A is the gradient of a continuously Fréchet differentiable convex functional on H, it is the gradient projection method by Goldstein [7]. Later, the splitting method was widely studied by Gabay [6] and several authors [4, 11, 19]. Let $\alpha > 0$. A single valued operator $A : H \longrightarrow H$ is said to be α-inverse-strongly-monotone (see [1, 3, 5, 9, 21]) if $(x - y, Ax - Ay) \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in D(A)$. Gabay [6] proved that the sequence $\{x_n\}$ generated by (5.1) converges weakly to some $z \in (A + B)^{-1}0$ when A is α-inverse-strongly-monotone and $\lambda_n = \lambda$

(constant) with $0 < \lambda < 2\alpha$.

In this chapter, we prove a strong convergence theorem by combining the splitting method with the hybrid method in the mathematical programming (see [12, 17]). Further, using an iteration of Mann's type [10]:

$$\begin{cases} x_1 = x \in D(A), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{\lambda_n}^B (I - \lambda_n A) x_n \end{cases} \tag{5.2}$$

for every $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$, we obtain a weak convergence theorem which generalizes the result of Gabay [6].

5.2 Preliminaries and lemma

Throughout this chapter, let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. It is known that H satisfies Opial's condition [13], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. We also know that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$ holds. Further, let $\{x_n\}$ be a sequence of H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then, there holds $x_n \to x$. Let C be a nonempty closed convex subset of H. We denote by $P_C(\cdot)$ the metric projection onto C. We know that for $z \in C$, $z = P_C(x)$ is equivalent to $(z-y, x-z) \geq 0$ for every $y \in C$. Let A be a mapping of C into H. Then, an element x in C is a solution of the variational inequality if $(y - x, Ax) \geq 0$ for all $y \in C$. An operator $A : H \longrightarrow 2^H$ is said to be monotone if $(x_1 - x_2, y_1 - y_2) \geq 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. A monotone operator A is said to be maximal if the graph of A is not properly contained in the graph of any other monotone operator. It is known that a monotone operator A is maximal iff $R(I + rA) = H$ for every $r > 0$, where $R(I + rA) = \cup\{z + rAz \mid z \in H, Az \neq \emptyset\}$. It is also known that a monotone operator A is maximal iff for $(u,v) \in H \times H$, $(x - u, y - v) \geq 0$ for every $(x,y) \in A$ implies $v \in Au$. For a maximal monotone operator A, we know that $A^{-1}0 = \{x \in H \mid 0 \in Ax\}$ is closed and convex. If A is monotone, then we can define, for each $\lambda > 0$, a nonexpansive mapping $J_\lambda^A : R(I+\lambda A) \longrightarrow D(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$, where $D(A) = \{z \in H \mid Az \neq \emptyset\}$. J_λ^A is called the resolvent of A. We also define the Yosida approximation A_λ by $A_\lambda = (I - J_\lambda^A)/\lambda$. It is known that the resolvent J_λ^A of A is a firm contraction, i.e. it satisfies $\|J_\lambda^A x - J_\lambda^A y\|^2 \leq (J_\lambda^A x - J_\lambda^A y, x - y)$ for every $x, y \in R(I + \lambda A)$. We also have $F(J_\lambda^A) = A^{-1}0$ for each $\lambda > 0$, where $F(J_\lambda^A) = \{z \in D(A) \mid J_\lambda^A z = z\}$. And we know that $A_\lambda x \in AJ_\lambda^A x$ for all

$x \in R(I + \lambda A)$; see [18] for more details.

We have the following lemma for inverse-strongly-monotone operators.

Lemma 5.1 *Let $\alpha > 0$. Let $A : H \longrightarrow H$ be a single valued α-inverse-strongly-monotone operator with $D(A) = H$ amd let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A + B)^{-1}0 \neq \emptyset$. Then the following hold:*

(i) A is maximal monotone;

(ii) $A + B$ is maximal monotone and $(A + B)^{-1}0$ is closed and convex;

(iii) for every $\lambda \in [0, 2\alpha]$, $I - \lambda A : H \longrightarrow H$ is nonexpansive;

(iv) for every $\lambda \in (0, \infty)$, $T_\lambda \equiv J_\lambda^B(I - \lambda A)$ is well defined and $(A+B)^{-1}0 = F(T_\lambda)$, where $F(T_\lambda)$ is the set of the fixed points of T_λ;

(v) for every $\lambda \in (0, 2\alpha]$, T_λ is nonexpansive.

Proof. (i) Since A is α-inverse-strongly-monotone, we have

$$\alpha\|Ax - Ay\|^2 \leq (x - y, Ax - Ay) \leq \|x - y\| \cdot \|Ax - Ay\|$$

for every $x, y \in H$. So, A is monotone and Lipschitz continuous. Therefore, it follows that A is maximal monotone (see e.g. [20]).
(ii) Since A is maximal monotone with $D(A) = H$ and B is maximal monotone, $A+B$ is maximal monotone (see e.g. [15]). Hence, $(A+B)^{-1}0$ is closed and convex.
(iii) Since A is α-inverse-strongly-monotone and λ is in $[0, 2\alpha]$, we have

$$\begin{aligned}
&\|(I - \lambda A)x - (I - \lambda A)y\|^2 \\
&\quad = \|(x - y) - \lambda(Ax - Ay)\|^2 \\
&\quad = \|x - y\|^2 - 2\lambda(x - y, Ax - Ay) + \lambda^2\|Ax - Ay\|^2 \\
&\quad \leq \|x - y\|^2 + (\lambda^2 - 2\alpha\lambda)\|Ax - Ay\|^2 \\
&\quad \leq \|x - y\|^2
\end{aligned}$$

for every $x, y \in H$ (see [5]). So, $I - \lambda A$ is nonexpansive.
(iv) Since B is maximal monotone, it follows that $R(I + \lambda B) = H$ for every $\lambda \in (0, \infty)$ and hence $T_\lambda : H \longrightarrow H$ is well defined. Let $\lambda > 0$. Then, we have

$$\begin{aligned}
u \in F(T_\lambda) &\iff T_\lambda u = u \iff J_\lambda^B(u - \lambda Au) = u \\
&\iff u - \lambda Au \in u + \lambda Bu \iff -Au \in Bu \\
&\iff 0 \in (A + B)u \iff u \in (A + B)^{-1}0 .
\end{aligned}$$

Therefore, we get $F(T_\lambda) = (A+B)^{-1}0$ for every $\lambda \in (0, \infty)$.
(v) Since the resolvent J_λ^B and $I - \lambda A$ are nonexpansive, T_λ is nonexpansive. □

5.3 Strong convergence

Motivated by Gabay's theorem [6], we show the following strong convergence theorem by combining the splitting method and the hybrid method in the mathematical programming [12, 17].

Theorem 5.2 *Let $\alpha > 0$. Let $A : H \longrightarrow H$ be a single valued α-inverse-strongly-monotone operator with $D(A) = H$ and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = J_{\lambda_n}^B (I - \lambda_n A) x_n\,, \\ C_n = \{z \in H \mid \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \boldsymbol{N} \cup \{0\}$, where $\{\lambda_n\} \subset [a, 2\alpha]$ for some $a \in (0, 2\alpha)$. Then, $\{x_n\}$ converges strongly to $P_{(A+B)^{-1}0}(x_0)$.

Proof. We give the proof with four steps (a)-(d).
(a) It is obvious that C_n is closed and Q_n is closed and convex for every $n \in \mathrm{N} \cup \{0\}$. We also have that C_n is convex for every $n \in \mathrm{N} \cup \{0\}$ because

$$\|y_n - z\| \leq \|x_n - z\| \Longleftrightarrow \|y_n - x_n\|^2 + 2(y_n - x_n, x_n - z) \leq 0\,.$$

So, $C_n \cap Q_n$ is closed and convex for every $n \in \mathrm{N} \cup \{0\}$. Let $u \in (A+B)^{-1}0$. By Lemma 5.1 (iv), we have $J_{\lambda_n}^B (I - \lambda_n A) u = u$ for every $n \in \mathrm{N} \cup \{0\}$. So we get

$$\|y_n - u\| = \|J_{\lambda_n}^B (I - \lambda_n A) x_n - J_{\lambda_n}^B (I - \lambda_n A) u\| \leq \|x_n - u\|$$

by Lemma 5.1 (v) and hence $u \in C_n$. Therefore, we have $(A+B)^{-1}0 \subset C_n$ for every $n \in \mathrm{N} \cup \{0\}$. Next, we show by mathematical induction that $\{x_n\}$ is well defined and $(A+B)^{-1}0 \subset C_n \cap Q_n$ for every $n \in \mathrm{N} \cup \{0\}$. For $n = 0$, we have $x_0 = x \in H$. And y_0 is well defined by Lemma 5.1 (iv) and $Q_0 = H$. Hence $(A+B)^{-1}0 \subset C_0 \cap Q_0$. Suppose that x_k is given and $(A+B)^{-1}0 \subset C_k \cap Q_k$ for $k \in \mathrm{N} \cup \{0\}$. It follows from Lemma 5.1 (iv) that y_k is well defined. As $(A+B)^{-1}0$ is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of H. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x_0)$. And there holds $(x_{k+1} - z, x_0 - x_{k+1}) \geq 0$ for every $z \in C_k \cap Q_k$. Since $(A+B)^{-1}0 \subset C_k \cap Q_k$, we get $(A+B)^{-1}0 \subset Q_{k+1}$. Therefore, we have $(A+B)^{-1}0 \subset C_{k+1} \cap Q_{k+1}$.

(b) By Lemma 5.1 (ii), $(A+B)^{-1}0$ is closed and convex and hence $P_{(A+B)^{-1}0}(x_0)$ is well defined. Let $z_0 = P_{(A+B)^{-1}0}(x_0)$. From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and $z_0 \in (A+B)^{-1}0 \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\| \tag{5.3}$$

for every $n \in \mathbf{N} \cup \{0\}$. Therefore, $\{x_n\}$ is bounded. By $z_0 = J^B_{\lambda_n}(I - \lambda_n A)z_0$ and Lemma 5.1 (v), we have

$$\|y_n - z_0\| = \|J^B_{\lambda_n}(I - \lambda_n A)x_n - J^B_{\lambda_n}(I - \lambda_n A)z_0\| \leq \|x_n - z_0\|$$

for every $n \in \mathbf{N} \cup \{0\}$. So, $\{y_n\}$ is bounded.

(c) As $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}(x_0)$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$$

for every $n \in \mathbf{N} \cup \{0\}$. Therefore, a sequence $\{\|x_n - x_0\|\}$ is bounded and nondecreasing. So there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+1} \in Q_n$, we have $(x_n - x_{n+1}, x_0 - x_n) \geq 0$ and hence

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0\|^2 + \|x_n - x_0\|^2 + 2(x_{n+1} - x_0, x_0 - x_n) \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2(x_n - x_{n+1}, x_0 - x_n) \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2
\end{aligned}$$

for every $n \in \mathbf{N} \cup \{0\}$. This implies that

$$\|x_{n+1} - x_n\| \to 0\,. \tag{5.4}$$

(d) Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. It follows from $x_{n+1} \in C_n$ that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|$$

for every $n \in \mathbf{N} \cup \{0\}$. From (5.4), we get

$$\|y_n - x_n\| \to 0\,. \tag{5.5}$$

By (5.5), we have

$$y_{n_i} \rightharpoonup w_0\,. \tag{5.6}$$

On the other hand, we have $\frac{x_n - y_n}{\lambda_n} - Ax_n \in By_n$. Since B is monotone, we get, for every $i \in \mathbf{N}$ and $(u, v) \in A + B$,

$$\left(y_{n_i} - u, \frac{x_{n_i} - y_{n_i}}{\lambda_{n_i}} - Ax_{n_i} - (v - Au)\right) \geq 0\,.$$

So, we have

$$\begin{aligned}
(y_{n_i} - u, -v) &\geq \left(y_{n_i} - u, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + (Ax_{n_i} - Au)\right) \\
&= \frac{1}{\lambda_{n_i}}(y_{n_i} - u, (I - \lambda_{n_i}A)y_{n_i} - (I - \lambda_{n_i}A)x_{n_i}) + (y_{n_i} - u, Ay_{n_i} - Au) \\
&\geq \frac{1}{\lambda_{n_i}}(y_{n_i} - u, (I - \lambda_{n_i}A)y_{n_i} - (I - \lambda_{n_i}A)x_{n_i}) \\
&\geq -\frac{1}{\lambda_{n_i}}\|y_{n_i} - u\| \cdot \|(I - \lambda_{n_i}A)y_{n_i} - (I - \lambda_{n_i}A)x_{n_i}\| \\
&\geq -\frac{1}{\lambda_{n_i}}\|y_{n_i} - u\| \cdot \|y_{n_i} - x_{n_i}\|
\end{aligned}$$

because A is monotone and $I - \lambda_{n_i}A$ is nonexpansive by Lemma 5.1 (iii). By the boundedness of $\left\{\frac{1}{\lambda_{n_i}}\|y_{n_i} - u\|\right\}$, (5.5) and (5.6), we get

$$(w_0 - u, -v) \geq 0$$

for every $(u, v) \in A + B$. Therefore, we have

$$w_0 \in (A + B)^{-1}0 \tag{5.7}$$

because $A + B$ is maximal monotone.
From (5.3), (5.7) and $z_0 = P_{(A+B)^{-1}0}(x_0)$, we have

$$\|z_0 - x_0\| \leq \|w_0 - x_0\| \leq \liminf_{i \to \infty} \|x_{n_i} - x_0\| \leq \limsup_{i \to \infty} \|x_{n_i} - x_0\| \leq \|z_0 - x_0\|\,.$$

Thus, we obtain

$$\lim_{i \to \infty} \|x_{n_i} - x_0\| = \|x_0 - w_0\| = \|x_0 - z_0\|\,.$$

This implies $x_{n_i} \to w_0 = z_0$. Therefore, we have $x_n \to z_0$. □

We apply this result to the problem of the variational inequality and we get the following.

Corollary 5.3 *Let C be a nonempty closed convex subset of H and let $\alpha > 0$. Let $A : H \longrightarrow H$ be a single valued α-inverse-strongly-monotone operator such that $D(A) = H$ and $\{x \in C \,|\, (y - x, Ax) \geq 0\, (\forall y \in C)\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = P_C(x_n - \lambda_n A x_n)\,, \\ C_n = \{z \in C \,|\, \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \,|\, (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \boldsymbol{N} \cup \{0\}$, where $\{\lambda_n\} \subset [a, 2\alpha]$ for some $a \in (0, 2\alpha)$. Then, $\{x_n\}$ converges strongly to the element z_0 in the set $\{x \in C \,|\, (y - x, Ax) \geq 0\, (\forall y \in C)\}$ nearest to x_0.

Proof. Putting $B = \partial i_C$ in Theorem 5.2, we have $(A + B)^{-1}0 = \{x \in C \,|\, (y - x, Ax) \geq 0\, (\forall y \in C)\}$ and $J^B_{\lambda_n} = P_C$ for every $n \in \mathbf{N} \cup \{0\}$, where ∂i_C is the subdifferential of the indicator function i_C of C. □

5.4 Weak convergence

Using an iteration of Mann's type [10], we obtain the following weak convergence theorem which generalizes the result of Gabay [6].

Theorem 5.4 *Let α, A and B be as in Theorem 5.2. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in H\,, \\ y_n = J^B_{\lambda_n}(I - \lambda_n A) x_n\,, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n \end{cases}$$

for every $n \in \boldsymbol{N}$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ with $a < b$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in (0, 1)$. Then, $\{x_n\}$ converges weakly to some $z \in (A + B)^{-1}0$.

Proof. Let $u \in (A+B)^{-1}0$. By Lemma 5.1 (iv), we have $J^B_{\lambda_n}(I - \lambda_n A)u = u$ for every $n \in \mathbf{N}$. So, there holds

$$\begin{aligned} \|y_n - u\|^2 &= \|J^B_{\lambda_n}(I - \lambda_n A)x_n - J^B_{\lambda_n}(I - \lambda_n A)u\|^2 \\ &\leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)u\|^2 \\ &\leq \|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2 \\ &\leq \|x_n - u\|^2 + a(b - 2\alpha)\|Ax_n - Au\|^2 \\ &\leq \|x_n - u\|^2 \end{aligned}$$

for every $n \in \mathbf{N}$ because $\lambda_n(\lambda_n - 2\alpha) \leq a(b-2\alpha) < 0$ holds. It follows that

$$\begin{aligned}\|x_{n+1} - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\ &\leq \|x_n - u\|^2 + (1-\alpha_n)a(b-2\alpha)\|Ax_n - Au\|^2 \\ &\leq \|x_n - u\|^2 + (1-c)a(b-2\alpha)\|Ax_n - Au\|^2 \\ &\leq \|x_n - u\|^2 .\end{aligned}$$

Therefore, there exists the limit of $\|x_n - u\|$ and hence

$$\|Ax_n - Au\| \to 0 \tag{5.8}$$

because $0 \leq (c-1)a(b-2\alpha)\|Ax_n - Au\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2$ holds. And $\{x_n\}$ and $\{y_n\}$ are bounded. For $u \in (A+B)^{-1}0$, we have

$$\begin{aligned}\|y_n - u\|^2 &= \|J^B_{\lambda_n}(I - \lambda_n A)x_n - J^B_{\lambda_n}(I - \lambda_n A)u\|^2 \\ &\leq (y_n - u, (I-\lambda_n A)x_n - (I - \lambda_n A)u) \\ &= \frac{1}{2}\{\|y_n - u\|^2 + \|(I-\lambda_n A)x_n - (I-\lambda_n A)u\|^2 \\ &\qquad - \|(y_n - u) - \{(I-\lambda_n A)x_n - (I-\lambda_n A)u\}\|^2\} \\ &\leq \frac{1}{2}\{\|y_n - u\|^2 + \|x_n - u\|^2 - \|(y_n - x_n) + \lambda_n(Ax_n - Au)\|^2\}\end{aligned}$$

because $J^B_{\lambda_n}$ is a firm contraction. So, we have

$$\begin{aligned}\|y_n - u\|^2 &\leq \|x_n - u\|^2 - \|(y_n - x_n) + \lambda_n(Ax_n - Au)\|^2 \\ &\leq \|x_n - u\|^2 - \|y_n - x_n\|^2 \\ &\qquad -2\lambda_n(y_n - x_n, Ax_n - Au) - {\lambda_n}^2\|Ax_n - Au\|^2\end{aligned}$$

which implies

$$\begin{aligned}\|x_{n+1} - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\ &\leq \|x_n - u\|^2 - (1-\alpha_n)\|y_n - x_n\|^2 \\ &\qquad - 2(1-\alpha_n)\lambda_n(y_n - x_n, Ax_n - Au) - (1-\alpha_n){\lambda_n}^2\|Ax_n - Au\|^2\end{aligned}$$

for every $n \in \mathbf{N}$. Therefore, there holds

$$\begin{aligned}0 &\leq (1-c)\|y_n - x_n\|^2 \leq (1-\alpha_n)\|y_n - x_n\|^2 \\ &\leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\ &\qquad -2(1-\alpha_n)\lambda_n(y_n - x_n, Ax_n - Au) - (1-\alpha_n){\lambda_n}^2\|Ax_n - Au\|^2 .\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - u\|^2 = \lim_{n\to\infty}\|x_{n+1} - u\|^2$ and $\lim_{n\to\infty}(y_n - x_n, Ax_n - Au) = 0$ from (5.8), we get

$$\|y_n - x_n\| \to 0 . \tag{5.9}$$

As $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to w_0. As in the proof of Theorem 5.2, we obtain

$$w_0 \in (A+B)^{-1}0\,. \tag{5.10}$$

So, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w_1$ and $x_{n_j} \rightharpoonup w_2$. By (5.10), we have

$$w_1, w_2 \in (A+B)^{-1}0\,.$$

Assume $w_1 \neq w_2$. From Opial's condition, we get

$$\begin{aligned}\lim_{i\to\infty} \|x_{n_i} - w_1\| &< \lim_{i\to\infty} \|x_{n_i} - w_2\| = \lim_{n\to\infty} \|x_n - w_2\| = \lim_{j\to\infty} \|x_{n_j} - w_2\| \\ &< \lim_{j\to\infty} \|x_{n_j} - w_1\| = \lim_{n\to\infty} \|x_n - w_1\| = \lim_{i\to\infty} \|x_{n_i} - w_1\|\,.\end{aligned}$$

This is a contradiction. Thus, we get $w_1 = w_2$. This implies

$$x_n \rightharpoonup z \in (A+B)^{-1}0.$$

□

Putting $\alpha_n = 0$ and $\{\lambda_n\} = \lambda$ (constant) for all $n \in \mathbf{N}$ in Theorem 5.4, we obtain Gabay's Theorem [6].

Bibliography

[1] J.B.Baillon and G.Haddad, *Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones*, Israel J. Math., 26(1977), 137-150.

[2] H.Brézis and M.Sibony, *Méthodes d'approximation et d'itération pour les opérateurs monotones*, Arch. Rational Mech. Anal., 27(1969), 59-82.

[3] F.E.Browder and W.V.Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., 20(1967), 197-228.

[4] G.H-G.Chen and R.T.Rockafellar, *Convergence rates in forward-backward splitting*, SIAM J. Optim., 7(1997), 421-444.

[5] J.C.Dunn, *Convexity, monotonicity, and gradient processes in Hilbert space*, J. Math. Anal. Appl., 53(1976), 145-158.

[6] D.Gabay, *Applications of the method of multipliers to variational inequalities*, in Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary-Value Problems (M.Fortin and R.Glowinski Eds.), Studies in Mathematics and Its Applications, North Holland, Amsterdam, Holland, Vol.15, 299-331, 1983.

[7] A.A.Goldstein, *Convex programming in Hilbert space*, Bull. Amer. Math. Soc., 70(1964), 709-710.

[8] P.L.Lions and B.Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM J. Numer. Anal., 16(1979), 964-979.

[9] F.Liu and M.Z.Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal., 6(1998), 313-344.

[10] W.R.Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., 4(1953), 506-510.

[11] A.Moudafi and M.Théra, *Finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl., 94(1997), 425-448.

[12] K.Nakajo and W.Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, submitted.

[13] Z.Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., 73(1967), 591-597.

[14] G.B.Passty, *Ergodic convergence to a zero of the sum of monotone operators in Hilbert space*, J. Math. Anal. Appl., 72(1979), 383-390.

[15] R.T.Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., 149(1970), 75-88.

[16] M.Sibony, *Méthodes itératives pour les équations et inéquations aux dérivées partielles nonlinéares de type monotone*, Calcolo, 7(1970), 65-183.

[17] M.V.Solodov and B.F.Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Programming Ser. A, 87(2000), 189-202.

[18] W.Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[19] P.Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., 38(2000), 431-446.

[20] E.Zeidler, *Nonlinear Functional Analysis and its Applications: Nonlinear Monotone Operators, II/B* , Springer-Verlag, New York, 1990.

[21] D.L.Zhu and P.Marcotte, *Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Optim., 6(1996), 714-726.