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A Thesis Submitted to Tokyo Institute of Technology

Designing Low-order Robust Controllers with Solving Some Fundamental H_∞ Control Problems

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May 1999

Abstract

H_∞ control is one of the most effective methodologies in robust control, and much attention has been paid to it in recent years. So far, the H_∞ controller has been given with a dynamical order that is equivalent to the sum of the orders of a plant and weighting functions. However it is frequently too high to use in practice. Under these circumstances, this thesis considers the problem of designing low-order H_∞ controllers. It considers two kinds of approaches: a *direct* one in which a low-order controller is derived directly in the process of the controller design, and an *indirect* one in which a high-order controller is designed first and then approximated to a lower-order one.

First, this thesis starts with non-standard H_∞ control problems, which have certain key structures for reducing the dynamical order of the controllers. A class of low-order controllers for the non-standard H_∞ problems is derived directly based on an algebraic operation in a class of full-order controllers. This derivation indicates that it is possible to design low-order controllers for problems that are reducible to one of the non-standard H_∞ problems. So far, people have usually avoided treating the non-standard H_∞ problems and they have been solved after being transformed into standard one. This study is a new attempt to utilize the non-standard H_∞ problems.

Second, this thesis also shows that some representative control system design problems are reducible to the non-standard problems, and proposes new formulations for solving these problems. It then proposes methods of designing low-order controllers for two kinds of problems: robust servo controller design and two-degree-of-freedom controller design. It is found that the low-order controllers can be designed based on the new formulations. Besides being used to design low-order controllers, the new formulations make it easier to design controllers than formulations based on the standard H_∞ problems.

Lastly, this thesis also considers an *indirect* approach that treats a specialized solution of the numerically solved H_∞ controller, which is represented by a linear fractional transformation of an optimized Youla parameter. In many cases the numerical approach yields an extremely high-order Youla parameter and the H_∞ controller becomes high order. We propose reducing the order of the H_∞ controller by reducing the order of the Youla parameter. One of advantages of the reduction method is that the resultant reduced-order controller satisfies closed loop properties: internal stability and closed loop pole specification.

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Notation

$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{F}^{m\times n}$	set of $m \times n$ matrices on $\mathbb{F}$, where $\mathbb{F}$ might be $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{C}$.
$\mathrm{Re}(c)$	real part of $c \in \mathbb{C}$
$\mathrm{Im}(c)$	imaginary part of $c \in \mathbb{C}$
I_n	$n \times n$ identity matrix
O	zero matrix
$\mathrm{diag}(a_1, \dots, a_n)$	an $n \times n$ diagonal matrix with a_i as its i-th diagonal element
M^T	transpose of a matrix $M \in \mathbb{R}^{m\times n}$
$M^\dagger$	pseudo inverse matrix of M
$M^\perp$	orthogonal complement of M
$\ker M$	$\{x \in \mathbb{R}^n \quad \| \quad Mx = 0\}$
$\mathrm{range}\, M$	$\{y \in \mathbb{R}^m \quad \| \quad y = Mx, x \in \mathbb{R}^n\}$
$\lambda_i(A)$	an eigenvalue of a square matrix $A \in \mathbb{R}^{n\times n}$, where $i = [1, 2, \cdots, n]$
$\lambda_{max}(A)$	the largest eigenvalue of a matrix A
$\lambda_{min}(A)$	the smallest eigenvalue of a matrix A
$\rho(A)$	spectral radius of a matrix A
$\sigma_{max}(A)$	the largest singular value of a matrix A
$\mathrm{trace}\, A$	trace of a matrix A
$\sim$	equivalent under nonsingular transformations
$\mathcal{RH}_\infty$	set of proper stable real rational transfer functions
$\\|G(s)\\|_\infty$	H_∞ norm of a transfer function $G(s) \in \mathcal{RH}_\infty$
$\mathcal{BH}_\infty$	set of bounded transfer functions in $\mathcal{RH}_\infty$
$\triangleq$	defined as
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	$\begin{cases} \text{a system described with the state space equation } \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \\ \text{a system descrived with the transfer function } D + C(sI - A)^{-1}B \end{cases}$
$Dom(C)$	$\{s \in \mathbb{C} \quad \| \quad \|s + \alpha\| = \beta, \mathrm{Re}(s) < 0\}$
$Dom(D)$	$\{s \in \mathbb{C} \quad \| \quad \|s + \alpha\| < \beta, \mathrm{Re}(s) < 0\}$

Some of transformations that will be used in this thesis are defined as follows.

For a matrix partitioned as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

we define a linear fractinal transformation (LFT), a homographic transformation (HMT), and a dual homographic transformation (DHM) as follows.

- If $\det(I - G_{22}Q) \neq 0$, then the LFT is defined and is denoted as

$$\mathcal{F}_l\,(G, Q) = G_{11} + G_{12}Q(I - G_{22}Q)^{-1}G_{21}.$$

- If $\det(G_{21}Q + G_{22}) \neq 0$, then the HMT is defined and is denoted as

$$\mathcal{HM}\,(G, Q) = (G_{11}Q + G_{12})(G_{21}Q + G_{22})^{-1}.$$

- If $\det(G_{11} + QG_{21}) \neq 0$, then the DHM is defined and is denoted as

$$\mathcal{DHM}\,(G, Q) = (G_{11} + QG_{21})^{-1}(G_{12} + QG_{22}).$$

Those transformations are related with each other by an equation:

$$\begin{aligned} \mathcal{F}_l\,(G, Q) &= \mathcal{HM}\left(\begin{bmatrix} G_{12} - G_{11}G_{21}^{-1}G_{22} & G_{11}G_{21}^{-1} \\ -G_{21}^{-1}G_{22} & G_{21}^{-1} \end{bmatrix}, Q\right) \\ &= \mathcal{DHM}\left(\begin{bmatrix} G_{12}^{-1} & G_{12}^{-1}G_{11} \\ -G_{22}G_{12}^{-1} & G_{21} - G_{22}G_{12}^{-1}G_{11} \end{bmatrix}, Q\right). \end{aligned}$$

Contents

Chapter 1

Introduction

1.1 Overview of the thesis

With the advances in technology, the role of control systems has become more important and they are required to act more precise and to perform robust in the presence of uncertainties of plants. In recent years, a robust control theory has developed and high-performance controllers have become easy to design. On the other hand, the resultant controllers are derived with high order [55], and they are sometimes hard to implement [1]. Thus, designing a low-order controller is an important problem that must be settled [1]. Although the designing of low-order robust controllers has been the subject of many studies, it is still an unsettled problem. In this thesis, we are devoted to the issue of designing the low-order robust controllers.

In the first place, this thesis focuses on a non-standard H_∞ control problem that is a key to the low-order robust controller design. In the non-standard problem, a class of plants for which we consider the controller design includes plants that can be stabilized with minimal-order-observer-based feedback controllers [41, 42, 30]. Hence, it is hoped that approaching the non-standard problem from the viewpoint of the minimal-order-observer-design enables us to develop a method for a wider class of low-order robust controller designs. Thus, the main result of this thesis provides a complete characterization of low-order H_∞ controllers for the non-standard problem.

Although, in this thesis, the non-standard problem is a fundamental topic to give methods for designing the low-order robust controllers, it is too abstract to make the most of practically. Hence, a question "*What sorts of practical problems of robust controller design reduce to the non-standard H_∞ problem ?*" may arise. To answer this question, the class of the practical problems that are reducible to the non-standard problem should be clarified. This thesis investigates the problems of an integral-type robust controller design and a two-degree-of-freedom (TDF) controller design. It then proposes methods for designing the low-order H_∞ controllers in those two kinds of problems.

This thesis also considers an *indirect approach*, in which a model approximation method is used to reduce the order of the controller. In the studies of control application, there are many cases that have adopted the indirect approach. Nevertheless, there are few theoretical results about the reduction of the robust controller. Hence, there is no choice but to rely on the rule of trial and error for reducing the controller. In this thesis, we shall focus on a numerical solution of the H_∞ controller that is derived with the so-called Youla-parameter approach [69, 3, 43], where it is known that a resultant H_∞ controller becomes high-order. However, there is no theoretical result for reducing the order of such a controller. Under the circumstance, this thesis proposes a way to reduce the order of the controller and gives a sufficient condition for the approximated controller satisfies the closed loop properties: internal stability and some closed loop pole specification.

1.2 Studies of the low-order controller design

A problem of designing the low-order controller has been an interest of many researchers for long time, however it is still an open problem. As stated in an earlier monograph [1], there are two approaches to designing low-order controllers for high-order plants. One is a *direct approach*, in which the parameters of the low-order controller are computed by some optimization procedure [2, 33]. The other is an *indirect approach*, in which a high-order controller is designed first and then the dynamical order of the controller is reduced by using some model approximation method [52, 18, 56].

Designing low-order controllers with the *direct approach* [2, 28, 57, 8, 9, 46] is a difficult problem. However, in recent years, a numerical method [3, 16, 32] for control system design has provided a new way for designing low-order controllers directly, and in this way the condition for existence of a fixed-order controller is expressed with an LMI (Linear Matrix Inequality) with a constraint of matrix rank [33, 63, 31]. Nevertheless, it is still difficult to solve the inequality numerically because solving the inequality reduces to a non-convex feasibility problem. Hence the approach via the numerical method requires further studies of the numerical algorithm [57, 8, 4, 46], and it is still important that the way to derive directly the low-order robust controller be studied from another point of view. On the other hand, the present study slightly differs from the earlier studies. It first clarifies a class of problems for which the low-order controllers can be designed and then analyzes some synthesis problems that are reducible to that class.

Essentially, the controller design with evaluation of the closed loop performances and the controller reduction with evaluation of the input-output properties of the controller itself are independent problems [1]. Hence, it is not always true that the good approximation of the controller in a sense of input-output properties means the good approximation of the closed loop properties. Although many model reduction methods [1, 52, 72, 73, 62, 63] are available, there are difficulties in preserving the closed loop performances of an original controller. In recent years, Goddard [20] proposed a method for the controller approximation with preserving closed loop performances. This thesis also considers an *indirect approach* to designing the low-order controller, and it treats the controller order reduction in such a way that the required closed loop performances are assured.

1.3 The non-standard H_∞ problems

This section briefly introduces the non-standard H_∞ problems, which give a key to the designing of the low-order robust controllers, and refers to past studies concerned with the non-standard H_∞ problems.

1.3.1 Problem description

This thesis treats the following linear, time-invariant, finite dimensional system:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{12} u(t) \\ y(t) = C_2 x(t) + D_{21} w(t) \end{cases}, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $z(t) \in \mathbb{R}^{p_1}$ is the controlled output, $y(t) \in \mathbb{R}^{p_2}$ is the measurement output, $w(t) \in \mathbb{R}^{m_1}$ is the exogenous (disturbance) input and $u(t) \in \mathbb{R}^{m_2}$ is the control input. The matrices $A, B_1, B_2, C_1, C_2, D_{12}, D_{21}$ are real, constant and of the appropriate dimensions. The system in (1.1) is called a generalized plant, and the system is represented as follows.

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right] \tag{1.2}$$

The purpose of control is to design a controller $K(s)$ that is connected with the generalized plant as

$$u = K(s)y.$$

A closed loop system composed of the generalized plant and the controller is illustrated in Figure 1.1. Then, the transfer function of the closed loop system is represented as

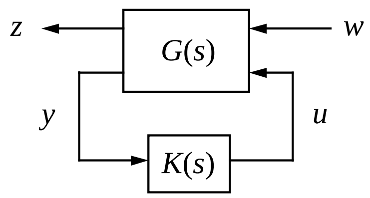

Figure 1.1: A closed loop system

$$T_{zw}(s) \triangleq \mathcal{F}_l\left(G(s), K(s)\right).$$

Remark 1.3.1 *The components of $w(t)$ are all the exogenous inputs: references, disturbances, sensor noise, and so on. The components of $z(t)$ are all the signals we wish to control: tracking errors between reference signals and plant outputs, actuator signals whose values must be kept between certain limits, and so on. The vector $y(t)$ contains the outputs of all sensors. Finally $u(t)$ contains all controlled inputs to the generalized plant.*

The H_∞ control problem is stated as follows.

Definition 1.3.1 *(The H_∞ control problem) Determine whether or not there exists a controller $K(s)$ such that the closed loop system is internally stable and satisfies*

$$\|T_{zw}(s)\|_\infty < 1,$$

where the H_∞ norm of a transfer function $F(s) \in \mathcal{RH}_\infty$ is defined as

$$\|F\|_\infty \triangleq \sup_{\mathrm{Re}(s)>0} \sigma_{max}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left[F(j\omega)\right].$$

Parametrize all such controllers when one exists.

Remark 1.3.2 *Many of the synthesis problems in the control engineering can be reduced to the H_∞ control problem [10, 12]. For example, problems of designing controllers which provides robust stability, robust performances, low-sensitivity, and so on are all reducible to the H_∞ problem.*

1.3.2 Assumptions

In earlier papers [11, 19], the H_∞ problem is considered under the following assumptions with regard to the system (1.1).

A1

1) (A, B_2) is stabilizable.

2) (A, C_2) is detectable.

A2

1) $\mathrm{rank}(D_{12}) = m_2$. ($D_{12}$ is of full column rank.)

2) $\mathrm{rank}(D_{21}) = p_2$. ($D_{21}$ is of full row rank.)

A3

1) the system $G_{12}(s)$ has no invariant zeros on the imaginary axis.

2) the system $G_{21}(s)$ has no invariant zeros on the imaginary axis.

Assumption **A1** concerns the existence of a stabilizing controller for the system (1.1). The two parts of assumption **A2** mean that there are no inputs in u which have the same effect on the output, that the exogenous signals in w affect the states, and that the true states are not directly measurable. Under the assumptions of **A2**, the assumption **A3-1)** is equivalent to the condition such that

$$\operatorname{rank}\left(\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}\right) = n + m_2, \ \forall\omega \in \mathbb{R},$$

and the assumption **A3-2)** is equivalent to the condition such that

$$\operatorname{rank}\left(\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}\right) = n + p_2, \ \forall\omega \in \mathbb{R}.$$

(See appendix A.1.)

The system (1.1) satisfying these assumptions is called **the standard generalized plant** and the corresponding H_∞ problem is called **the standard H_∞ problem**. If one of these assumptions is not satisfied, the system (1.1) is called **the non-standard generalized plant** and the corresponding H_∞ problem is called **the non-standard H_∞ problem**.

1.3.3 Past studies on the non-standard H_∞ problems

Many studies of the H_∞ control treat the standard H_∞ problem [11, 19], however we are sometimes faced with the non-standard problems in which the generalized plant does not satisfy one of the assumptions: **A1**, **A2** and **A3**. This thesis especially treats the non-standard H_∞ control problem where the assumption **A2** is not satisfied, that is, the rank conditions of the direct feed-through terms of the subsystems $G_{12}(s)$ and $G_{21}(s)$ are not satisfied.

The non-standard problems are treated in several papers [35, 51, 60, 50, 64, 65]. Stoorvogel [65] and Sampei et al. [60] studied the non-standard H_∞ control problems by using the Riccati inequalities, and they derived an H_∞ controller of McMillan degree n which is identical with that of the generalized plant. Kimura et al. [35] treated the non-standard H_∞ control problems by using the concept of J-lossless, and they derived the H_∞ controller which is represented with a free parameter. Furthermore, the LMI approach [16, 32, 31] enabled us to solve the controller design problems without being conscious of almost all the assumptions.

The correspondence between the non-standard H_∞ problem and the low-order controller design was first pointed out by Zhang and Hosoe [71], and Stoorvogel, Sabeli and Chen [65] respectively. Zhang et al. derived a low-order controller for the H_∞ problem where the partial states of the real plant are measurable without noise. The discussion in their article is, however, restricted to the plant in which partial states are measurable without noise and is fairly difficult. On the other hand, Stoorvogel et al. derived a low-order controller for the non-standard H_∞ problem. Their discussion bases a matrix inequality approach and the minimal-order observer design, however, the resultant controller is not parametrized.

Recently, Xin et al. [68] studied the designing of the low-order controller for continuous and discrete time singular H_∞ control problem based on LMI. However, the controller is not parametrized with free parameters and its structure is unclear.

Besides the problem of designing the low-order controller, problems in utilizing the non-standard H_∞ control for practical problems have not been tackled in any study, so far. In the application of the standard H_∞ problem, it is known what kind of the control problems is reducible to the standard problem. On the other hand, the problem in the application of the non-standard H_∞ control is an open problem. Thus, first, this thesis aims at constructing a general way of designing the low-order H_∞ controller for the problem of the non-standard H_∞ control, second, it aims at clarifying what kind of practical control problems is reducible to the non-standard problem, then a method of designing the low-order robust controller is to be shown.

1.4 Related issue to the non-standard problem

This section looks into the non-standard H_∞ problem from the viewpoint of the low-order controller design and its applications. It will give us understanding of backgrounds of discussions in later sections.

1.4.1 Fundamental non-standard solution via reduced-order observer design

Consider a generalized plant that is simplified as

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} w + \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} u \\ z = C_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_{12} u \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I_{p_2-m_1} & O \\ O & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} O \\ I_{m_1} \end{bmatrix} w \end{cases}, \tag{1.3}$$

where it is assumed that the triple $\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \begin{bmatrix} I_{p_2-m_1} & O \\ O & C_{22} \end{bmatrix} \right)$ is stabilizable and detectable. Here, it should be noted that the matrix $\begin{bmatrix} O \\ I_{m_1} \end{bmatrix}$ is of full column rank, hence the system is a generalized plant of the non-standard problem where the assumption **A2-2)** is not satisfied. Although this system seems like a special case of the non-standard generalized plant, it can be given by an equivalent transformation of the generalized plant (1.1). In this system, it should be noted that a partial state variable $x_1 \in \mathbb{R}^{p_2-m_1}$ is directly observed through the observation of a measurement output y_1. In order to use information of each state variable, there is therefore no need to estimate the partial state variable x_1 and the dynamics of the observer can be reduced. Actually, since the state variable x_1 and the input u are available, we can describe the system in the following form.

$$\begin{cases} \dot{x}_2 = A_{22}x_2 + B_{12}w + \begin{bmatrix} A_{21} & B_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ u \end{bmatrix} \\ y_2 = C_{22}x_2 + I_{m_1}w \end{cases} \tag{1.4}$$

This system represents a dynamical system of the state variable x_2. The detectability of the pair $\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} I_{p_2-m_1} & O \\ O & C_{22} \end{bmatrix} \right)$ means that the pair (A_{22}, C_{22}) is detectable. Hence, all the state variables in the system (1.3) can be estimated by using a reduced-order observer of the form:

$$\begin{cases} \dot{\hat{x}}_2 = (A_{22} - HC_{22})\,\hat{x}_2 + \begin{bmatrix} A_{21} & H \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + B_{22}u \\ \hat{x} = \begin{bmatrix} O \\ I \end{bmatrix} \hat{x}_2 + \begin{bmatrix} I & O \\ O & O \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{cases}, \tag{1.5}$$

where the matrix $A_{22} - HC_{22}$ is stable. On the other hand, from the stabilizability assumed above, if all the states of the system (1.3) are estimated, the system is stabilized with a state feedback controller of the form:

$$u = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \hat{x},$$

where $\hat{x}$ is an estimate of the state and the feedback gain $\begin{bmatrix} F_1 & F_2 \end{bmatrix}$ stabilizes the matrix $\begin{bmatrix} A_{11} + B_{21}F_1 & A_{12} + B_{21}F_2 \\ A_{21} + B_{22}F_1 & A_{22} + B_{22}F_2 \end{bmatrix}$. From these, it can be verified that a reduced-order-observer-

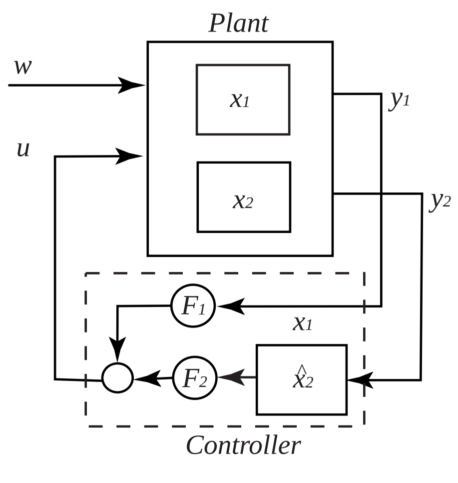

Figure 1.2: A closed loop system

based output feedback controller:

$$\begin{cases} \dot{\hat{x}}_2 = (A_{22} - HC_{22})\,\hat{x}_2 + \begin{bmatrix} A_{21} & H \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + B_{22}u \\ u = F_2\hat{x}_2 + \begin{bmatrix} F_1 & O \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{cases} \tag{1.6}$$

stabilizes the system (1.3), because the A-matrix of the closed loop system composed of (1.3) and (1.6) is represented as

$$
\begin{aligned}
&\begin{bmatrix} A_{11}+B_{21}F_1 & A_{12} & B_{21}F_2 \\ A_{21}+B_{22}F_1 & A_{22} & B_{22}F_2 \\ A_{21}+B_{22}F_1 & HC_{22} & A_{22}-HC_{22}+B_{22}F_2 \end{bmatrix} \\
&\sim \left[\begin{array}{cc|c} A_{11}+B_{21}F_1 & A_{12}+B_{21}F_2 & B_{21}F_2 \\ A_{21}+B_{22}F_1 & A_{22}+B_{22}F_2 & B_{22}F_2 \\ \hline O & O & A_{22}-HC_{22} \end{array}\right],
\end{aligned}
$$

and is verified to be stable. By using the observer design, we can obtain an H_∞ controller [15]. The closed loop system is illustrated in Figure 1.2.

Thus, the non-standard problem corresponds to a problem of the minimal-order-observer-based controller design and is a key problem in designing a low-order robust controller. It is well known that the standard H_∞ controller can be given with an LFT (Linear Fractional Transformation) of a central solution and a free parameter, and that the McMillan degree of the solution is more than that of the generalized plant [11]. A recent paper [65] pointed out that McMillan degree of the non-standard H_∞ controller can be reduced to an order lower than that of the standard controller, and that paper derived a reduced-order H_∞ controller by using Riccati inequality approach. This thesis aims at deriving the class of the reduced-order H_∞ controllers by way of Riccati equation approach.

1.4.2 Integral-type low-order robust controller design

In the controller design, there are many cases where the controller is required to contain integrators. The integral-type robust controller design is indispensable for a synthesis of the robust servo system. However, in general, the framework of the H_∞ control doesn't produce the integral-type controller. The problem of designing the integral-type robust controller has been studied by many researchers and many approaches based on the standard H_∞ control have been proposed. On the other hand, approaches based on the non-standard H_∞ control have not been studied well even though the problem naturally reduces to some non-standard H_∞ problems. One of reasons why the approach based on the non-standard H_∞ control is unpopular is that the solution is complicated compared with the approach of the standard H_∞ control.

Let us consider an SISO system in Figure 1.3, where $P(s)$ is a plant assumed to have neither poles nor zeros on the imaginary axis. This assumption is relaxed in a later chapter. Consider a simple reference tracking problem, where reference signal r is supposed to be the step-formed input. To let the output y track the reference signal r, we require the controller $K(s)$ to contain an integrator. One of the effective ways to meet this requirement is to introduce the integrator into the weight $W_s(s)$ and to let the transfer function $T_{zr}(s) \in \mathcal{RH}_\infty$. This is done by letting $T_{zr}(s)$ satisfy the following inequality:

$$\|T_{zr}(s)\|_\infty < \gamma, \tag{1.7}$$

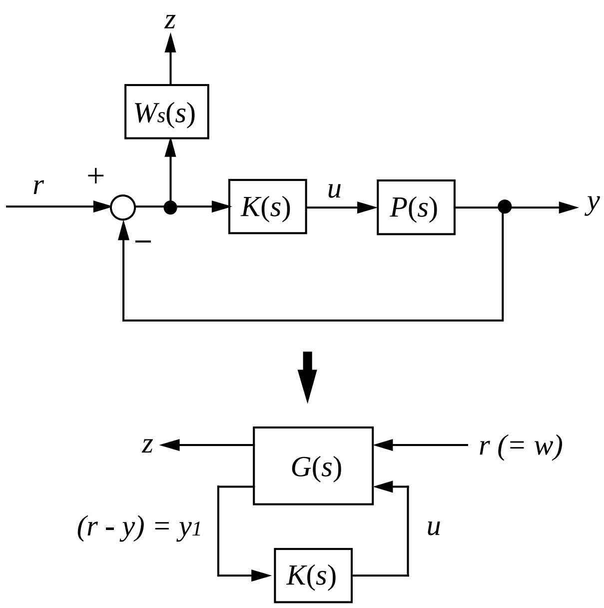

Figure 1.3: A closed loop system

where γ is a positive real number given beforehand. Thus we consider the H_∞ problem stated as: *find internally stabilizing controller $K(s)$ such that inequality (1.7)* is satisfied. Since no states of the weight $W_s(s)$ are observed by $K(s)$, the generalized plant in Figure 1.3 doesn't satisfy the assumptions **A1-2)** nor **A3-2)**, and the problem is thus a non-standard H_∞ problem. For the problems of this type, many approaches based on the conversion to the standard H_∞ problem have been proposed. The book of Zhou, Doyle and Glover [73] lists two types of approaches.

One is based on factorization of the weight $W_s(s)$ into the parts of integrators and another. The factorization makes the problem the standard one. This approach derives an integral-type controller by a series of process: (1) factorizing the weight, (2) solving a problem of the standard H_∞ control, and (3) modifying a controller derived from the problem to an integral type. Thus, this approach requires one to follow several steps. The other is an approximative one, in which the weight $W_s(s)$ is approximated in such a way that the integral part of $W_s(s)$ is replaced as follows:

$$\frac{1}{s+\epsilon} \leftarrow \frac{1}{s},$$

where ϵ is a small positive real number. By the replacement of the integrator in $W_s(s)$, the generalized plant in Figure 1.3 satisfies the assumptions **A1-2)** and **A3-2)**, and the problem reduces to the standard H_∞ problem. This approach has convenience, on the other hand, it derives an incomplete integral-type controller. The other famous one is an approach that uses the generalized H_∞ control [39, 40, 49, 47], where it is required to derive a pseudo-stabilizing solution of ARE. Thus, there are many approaches to solve the problem, however, the solutions are complex or incomplete. This is because the solutions are restricted by the standard problem.

On the other hand, we treat the problem of the integral-type robust controller design as the non-standard H_∞ problem for the generalized plant described in Figure 1.4, where the output from $W_s(s)$ is measured by $K(s)$. In the generalized plant, while the assumptions **A1-**

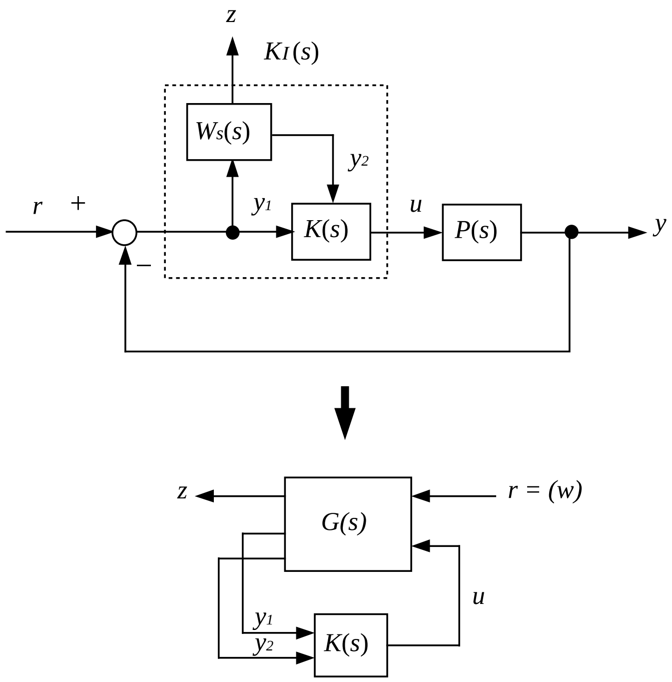

Figure 1.4: An alternative generalized plant

2) and **A3-2)** are satisfied, the assumption **A2-2)** is not satisfied because the dimension of the measurement output is greater than that of the external input. Thus the H_∞ problem for the system in Figure 1.4 is the non-standard problem where the matrix D_{21} is of full column rank, and the problem can be solved by using the solution of the non-standard problem we focus on. Although we can solve the problem of integral-type robust controller design, there is another problem; that is, the resultant integral-type controller $K_I(s)$ becomes high-order than the system $G(s)$ by the dimension of $W_s(s)$. For this problem, we show that our study on the designing of the reduced-order H_∞ controller contributes. Compared with previous works, while the problem description is essentially identical to the description in the works of Zhang, et al. [71] and Hozumi et al. [26], our study takes a different approach to solving the problem of the integral-type H_∞ controller design, and enables us to extend the result for TDF control system design to be stated below.

1.4.3 Low-order TDF controller design

TDF control is one effective way to attain feedback performances and tracking performance simultaneously. Figure 1.5 illustrates the TDF control system where $K(s)$ is a feedback controller and $F(s)$ is a feedforward controller. A past study [66, 44] on the TDF control

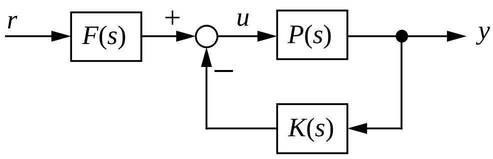

Figure 1.5: TDF control system

has shown a property such that the feedback and tracking performances are determined independently by the feedback and feedforward controllers. Based on the property, the TDF controller is frequently designed in two steps: the feedback controller design and the feedforward controller design. Here, suppose that the output feedback controller is designed in the first step. Then the feedforward controller is designed such that it satisfies the specification:

$$\|W_\rho(s)\,(M(s) - T_{yr}(s))\|_\infty < \gamma,$$

where $M(s)$ is a model of the tracking signal, $W_\rho(s)$ is a weight, and $T_{yr}(s)$ is the closed loop transfer function from the signal r to y. This specification is illustrated in Figure 1.6. In

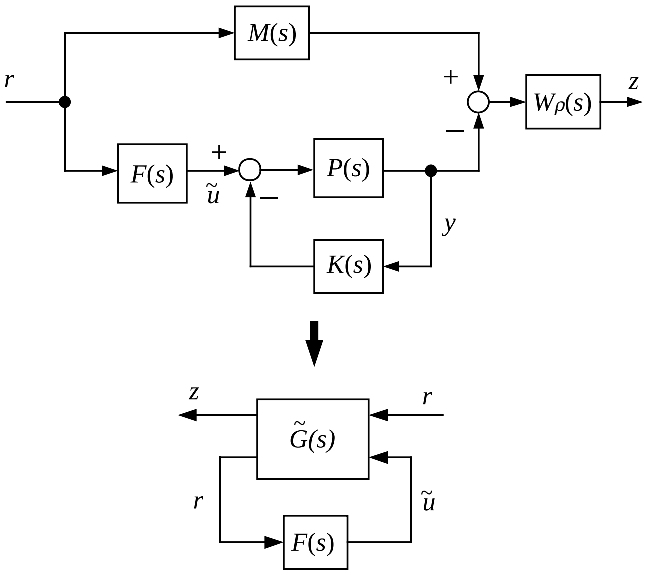

Figure 1.6: A feedforward controller design in the second step

this design, it should be noted that the order of $F(s)$ becomes no less than twice the order of $P(s)$. Hence, the order of the TDF controller becomes not less than 3 times as much as that of the plant model. That is, the two-step design of the TDF controller results in a high-order controller. In the previous studies [17, 5, 6, 24], in order to design a reduced-order TDF controller, the dynamics of the feedback controller which is designed beforehand is used as the dynamics in the feedforward controller.

However, in these studies, it isn't noted that the above property is guaranteed in the only case where the controllers are independently designed. Specifically, the low-order TDF controller design inevitably causes any correlation in the structures of the feedback controller and the feedforward controller. Hence, it is important to analyze the trade-off between these performances of the TDF control system in which the controllers have any correlation. This thesis first analyzes the trade-off between these performances of the low-order TDF control system, in which the controllers share the same dynamics, and shows that the independence of the performances is not maintained in this case.

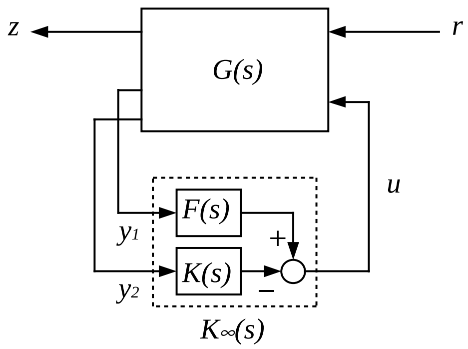

Figure 1.7: Simultaneous design of TDF controller

From the above reason, designing of the TDF controller is considered with a simultaneous approach rather than the two-step approach. Figure 1.7 illustrates a generalized plant for the simultaneous design of the TDF controller. In this formulation, the feedback controller and the feedforward controller are designed simultaneously. It should be noted that the resultant controllers share the common dynamics, and the degree of the controller is not increased unnecessarily. On the other hand, it is interesting to note that the system in Figure 1.7 is the non-standard generalized plant discussed in the previous sections. Hence, it is expected that the order of the controller can be further reduced.

1.5 H_∞ controller reduction

There are many model reduction methods that are applicable for the controller order reduction. However it is difficult to reduce the order of the controller with preserving the closed loop performances of the H_∞ controller. Let us consider the closed loop system illustrated in Figure 1.1, where the controller stabilizes the system and satisfies the constraint such that

$$\|\mathcal{F}_l\,(G(s), K(s))\|_\infty < 1. \tag{1.8}$$

Then we reduce the order of the controller with some model reduction method, if the order of the controller is too high to implement in practical use. However, neither the constraint of (1.8) nor the stability of closed loop system is necessarily satisfied even if the reduction error of the controller is small. This indicates that good approximation in the sense of open loop property doesn't mean good approximation in the sense of closed loop property. Hence, when we evaluate the properties of closed loop systems, the controller approximation should be considered. On the other hand, it is hard to evaluate the degradations in the closed loop properties when we approximate the controller.

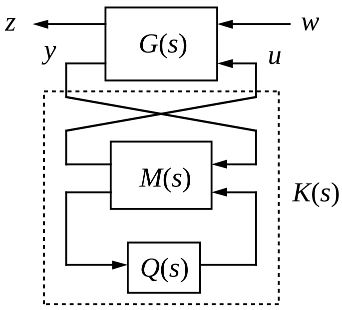

Figure 1.8: Closed loop system

In this thesis, we consider the method for the H_∞ controller reduction by way of the reduction of the Youla parameter $Q(s)$ that is designed such that the closed loop performances are satisfied. We focus on the H_∞ controller that is designed based on a numerical approach where the Youla parameter is optimized under some specification. The closed loop system is illustrated in Figure 1.8. It should be noted that the controller is represented as

$$K(s) = \mathcal{F}_l\left(M(s), Q(s)\right),$$

where the system $M(s)$ is given a priori. It should also be noted that the closed loop transfer function $T_{zw}(s)$ can be represented as

$$T_{zw}(s) = T_1 - T_2(s)Q(s)T_3(s),$$

where $T_1(s), T_2(s)$ and $T_3(s)$ are stable transfer functions which depend on factorization of the plant and control specifications. This representation is useful for evaluating the closed loop performances.

Let us define the optimal Youla parameter as

$$Q^*(s) \triangleq \arg \min_{Q(s)\in\mathcal{RH}_\infty} \|T_{zw}(s)\|_\infty.$$

If we approximate $Q^*(s)$ with $Q_r(s)$, an inequality:

$$\begin{aligned}\min \|T_{zw}(s)\|_\infty &= \left\|T_1 - T_2Q^*(s)T_3(s)\right\|_\infty \\ &\leq \left\|T_1 - T_2Q_r(s)T_3(s)\right\|_\infty + \|T_2\|_\infty\|T_3\|_\infty\|Q^*(s) - Q_r(s)\|_\infty\end{aligned}$$

holds. Letting closed loop transfer functions as

$$\begin{aligned} T^*_{zw}(s) &\triangleq T_1 - T_2 Q^*(s) T_3(s) \\ \tilde{T}_{zw}(s) &\triangleq T_1 - T_2 Q_r(s) T_3(s) \\ \gamma &\triangleq \|T_2\|_\infty \|T_3\|_\infty \|Q^*(s) - Q_r(s)\|_\infty \end{aligned}$$

the inequality is written

$$\|T^*_{zw}(s)\|_\infty \leq \|\tilde{T}_{zw}(s)\|_\infty + \gamma. \tag{1.9}$$

As shown in Figure 1.9, the inequality (1.9) indicates that at every frequency, the graph of

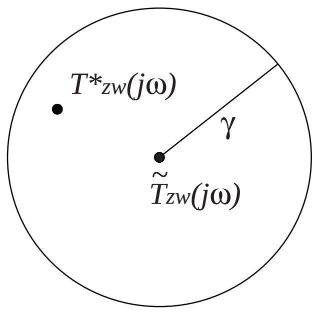

Figure 1.9: An interpretation of inequality 1.9

$T^*_{zw}(j\omega)$ lies in a disk of center $\tilde{T}_{zw}(j\omega)$, radius γ. From the inequality, it can be seen that good approximation of $Q^*(s)$ with $Q_r(s)$ suppresses the degradation in the closed loop performance. Furthermore, it should be noted that the approximation $Q_r(s) \in \mathcal{RH}_\infty$ guarantees the closed loop stability; moreover, the closed loop poles depend directly on the poles of $Q_r(s)$. Hence, if we could obtain an approximation method that preserves the region in which poles of the system are located, we can propose an approximation method of the controller that satisfies a constraint of closed loop pole position.

These observations motivate to approximate the parameter $Q(s)$ in order to approximate the H_∞ controller. Thus, we propose a method of H_∞ controller reduction based on the approximation of the Youla parameter.

1.6 Contribution

The new results and the original points of this thesis are listed as follows.

1. The reduced-order H_∞ controllers are derived in the non-standard H_∞ problems. The class of the reduced-order controllers belongs to a generalized class of the minimal order observer based output feedback controllers. Derivation of the reduced-order controllers is based on an algebraic operation; that is, it is based on the selection of the free parameters in the general representation of the non-standard solutions.

2. As an application of the non-standard problems, a method for designing an integral-type robust controller is proposed. This method makes it possible to design the low-order integral-type robust controller easily and is also applicable to a low-order TDF controller design.

3. In the analysis of the low-order TDF control system, we point out a necessity to consider the trade-off between the feedback performances and the tracking performance. Specifically, in the case of the low-order TDF controller, in which controllers share common dynamics, it is clarified that the independent property among those performances is not maintained.

4. Based on the above analysis, it is pointed out that the low-order TDF controller design problem should be treated as a simultaneous optimization problem. In the synthesis of the low-order TDF control system, an approach based on the non-standard H_∞ problem is proposed. Specifically, the result of the reduced-order H_∞ controller design is applicable to this approach and a method for designing some kind of low-order TDF controller is proposed.

5. An H_∞ controller reduction method based on the reduction of the high-order Youla parameter is proposed. This method makes it easy to take closed loop specifications into account. That is, the reduced-order controller guarantees the closed loop properties such as internal stability and closed loop pole specification, and the degradation in the closed loop performance: the H_∞ norm of the closed loop transfer function can be evaluated.

1.7 Organization

The organization of this thesis is as follows. Chapter 2 gives two-types of reduced-order controllers of the non-standard H_∞ problem. The results given in this chapter are fundamental to chapters of 3, 4 and 5. Chapter 3 considers a synthesis of the low-order robust servo control. It treats the problem as the non-standard H_∞ problem and derives the integral-type H_∞ controller. Chapters 4 and 5 consider the problem of the low-order Two-Degree-of-Freedom controller design. Chapter 4 analyzes the low-order TDF control system and shows that there is a trade-off between the feedback performance and the tracking performance. Based on the analysis, the chapter discusses the formulations of the low-order TDF controller. Chapter 5 proposes a method for designing the low-order TDF controller. It is based on the non-standard H_∞ control. Chapter 6 considers the method for reducing the order of the H_∞ controller, which is a special solution of a numerical approach. Chapter 7 is the conclusion. The structure of the thesis is illustrated in Figure 1.10, where a symbol C indicates a chapter.

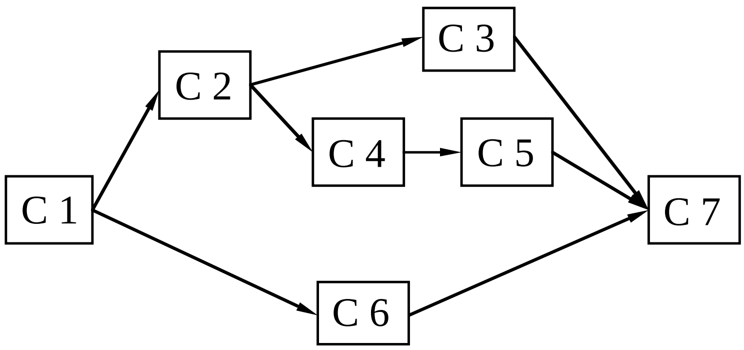

Figure 1.10: Organization of the thesis

Chapter 2

Reduced-order non-standard H_∞ controller design

2.1 Introduction

This chapter treats a fundamental issue to give a reduced-order H_∞ controller. In particular, problems that are focused on are the non-standard H_∞ control problems, where the dimensions of the control inputs are greater than those of the controlled outputs, and the dimensions of the measurement outputs are greater than those of the disturbance inputs. As a special case, those types of the non-standard H_∞ problems include the H_∞ problems of systems with partial state feedback [71] and redundant inputs [59]. It is known that the dynamical order of the observer-based feedback controller for a plant whose partial state variables are measured without noise can be reduced by the number of the partial state variables [30, 64, 65, 68, 70]. We thus hope that we can derive the reduced-order controller in the non-standard H_∞ problems.

Thus, this chapter aims to generalize the reduced-order controller design based on the non-standard H_∞ control problems. It starts from deriving the class of full-order H_∞ controllers [45, 51], in which the controllers are parametrized with free parameters. The remarkable difference between the classes of the H_∞ controllers in the non-standard problem and in the standard problem lies in the parametrization of these controllers. The H_∞ controllers of the non-standard problems are represented with larger number of free parameters than that of the standard problem. The difference in the structure of H_∞ problems reflects the difference in the parametrizations of these controllers. Hence, the free parameters in the H_∞ controllers of the non-standard problems are keys to derive the reduced-order H_∞ controller.

In fact, the approach taken here is based on the selection of the free parameters in the full-order H_∞ controllers. The results given in this chapter are stated as follows. Two types of reduced-order H_∞ controllers are derived. One of the controllers is the minimal order observer type, where the order of the controller is less than that of the generalized plant by the number of the disturbance-free outputs. The other one is the dual version of the aforementioned case, where the order of the controller is less than that of the generalized plant by the number of the redundant inputs. Results given in this chapter will be utilized in the later chapters.

2.2 Formulation

Let us consider the generalized plant described in (1.1) or (1.2). This section considers the class of non-standard H_∞ control problems where the following assumptions are satisfied.

A1 1) (A, B_2) is stabilizable, and 2) (A, C_2) is detectable.

A2' One of the following cases is satisfied.

Case 1 D_{12} and D_{21} are of full row rank.

Case 2 D_{12} and D_{21} are of full column rank.

Case 3 D_{12} is of full row rank and D_{21} is of full column rank.

A3' $\forall\omega \in \mathbb{R}$; $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ and $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ are of full rank.

In this thesis, we call the H_∞ problem, where the assumptions **A1**, **A2'-Case i** and **A3'** hold, the non-standard H_∞ problem of case i.

In the H_∞ problems of case 1 and case 3, we put the assumption such that the matrix $\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}$ is of full column rank. In these cases, B_2 and D_{12} are supposed to be of the form:

$$\left[\begin{array}{c} B_2 \\ \hline D_{12} \end{array}\right] = \left[\begin{array}{cc} B_{21} & B_{22} \\ \hline O & I_{p_1} \end{array}\right], \quad B_{22} \in \mathbb{R}^{n\times p_1}, \tag{2.1}$$

where B_{21} is of full column rank.

In the problems of case 2 and case 3, we put the assumption such that the matrix $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ is of full row rank. In these cases, C_2 and D_{21} are supposed to be of the form:

$$\left[\begin{array}{c|c} C_2 & D_{21} \end{array}\right] = \left[\begin{array}{c|c} C_{21} & O \\ C_{22} & I_{m_1} \end{array}\right], \quad C_{22} \in \mathbb{R}^{m_1\times n}, \tag{2.2}$$

where C_{21} is of full row rank.

These assumptions imposed on the system don't lose the generality in the system (1.1), because an equivalent transformation on the output variable: $\bar{y} = My, |M| \neq 0$, and an input transformation on the input variable: $\bar{u} = N^{-1}u, |N| \neq 0$ in the system (1.1) yield the matrices in the equations (2.1) and (2.2). In this case, if a control low $\bar{u} = K(s)\bar{y}$ is obtained, the real control law is reconstructed as $u = NK(s)My$.

2.3 Preliminaries

2.3.1 Pseudo inverse matrix and orthogonal complement matrix

For a singular and full-rank matrix, a pseudo inverse matrix and an orthogonal complement matrix are defined as follows.

1. Let D be of full column rank, the matrices $D^\dagger$ and $D^\perp$ are defined according to the following equalities:

$$\begin{bmatrix} D^\dagger \\ D^\perp \end{bmatrix} \begin{bmatrix} D & (D^\perp)^T \end{bmatrix} = I,$$
$$DD^\dagger + \left(D^\perp\right)^T D^\perp = I.$$

2. Let D be of full row rank, the matrices $D^\dagger$ and $D^\perp$ are defined according to the following equalities:

$$\begin{bmatrix} D \\ (D^\perp)^T \end{bmatrix} \begin{bmatrix} D^\dagger & D^\perp \end{bmatrix} = I,$$
$$D^\dagger D + D^\perp \left(D^\perp\right)^T = I.$$

We call the matrix $D^\dagger$ pseudo inverse matrix of D, and call the matrix $D^\perp$ orthogonal complement matrix.

2.3.2 Invariant zeros

Let us consider the system

$$T(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where the matrix D is nonsingular and is of full rank. Then the invariant zeros of $T(s)$ are characterized as follows.

Lemma 2.3.1 *1. Let D be of full column rank, then the invariant zeros of $T(s)$ are the unobservable modes of the pair $(A - BD^\dagger C, D^\perp C)$.*

2. Let D be of full row rank, then the invariant zeros of $T(s)$ are the uncontrollable modes of the pair $(A - BD^\dagger C, BD^\perp)$.

Proof. See section A.1. ■

2.3.3 Canonical transformation of the generalized plant

Our subsequent analysis is greatly simplified if the generalized plant is transformed so that the invariant zeros are explicitly represented. Therefore, the generalized plant is transformed to a canonical form:

$$G(s) = \left[\begin{array}{c|cc} T^{-1}AT & T^{-1}B_1 & T^{-1}B_2 \\ \hline C_1T & O & D_{12} \\ C_2T & D_{21} & O \end{array}\right],$$

where the matrix T is supposed to be nonsingular. Then, the following lemma holds.

Lemma 2.3.2 *(1) Assume the generalized plant of case 1 or case 3, i.e., D_{12} is of full row rank, let the stable and unstable zero modes of the subsystem $G_{12}(s)$ be A_- and $A_+ \in \mathbb{R}^{r\times r}$. Then the following transformation is possible*

$$\left[\begin{array}{c:c:c} T^{-1}(A-B_2D_{12}^\dagger C_1)T & T^{-1}B_2D_{12}^\perp & T^{-1}B_2D_{12}^\dagger \end{array}\right] \\ = \left[\begin{array}{cccc:c:c} A_{11} & A_{12} & A_{21} & A_{22} & I_{m_2-p_1} & B_{22uu} \\ A_{13} & A_{14} & A_{23} & A_{24} & O & B_{22ul} \\ O & O & A_- & O & O & B_{22lu} \\ O & O & O & A_+ & O & B_{22ll} \end{array}\right], \tag{2.3}$$

where $\lambda_i(A_{14}) < 0$, both of the pairs (A_{11}, I_{m2-p1}) and (A_+, B_{22ll}) are controllable, and A_+ has no $j\omega$ eigenvalues.

(2) Assume the generalized plant of case 2 or case 3, i.e., D_{21} is of full column rank, let the stable and unstable zero modes of the subsystem $G_{21}(s)$ be $\bar{A}_-$ and $\bar{A}_+ \in \mathbb{R}^{l\times l}$. Then the following transformation is possible

$$\left[\begin{array}{c} \bar{T}^{-1}(A-B_1D_{21}^\dagger C_2)\bar{T} \\ \hdashline D_{21}^\perp C_2\bar{T} \\ \hdashline D_{21}^\dagger C_2\bar{T} \end{array}\right] = \left[\begin{array}{cccc} \bar{A}_{11} & \bar{A}_{12} & O & O \\ \bar{A}_{13} & \bar{A}_{14} & O & O \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_- & O \\ \bar{A}_{33} & \bar{A}_{34} & O & \bar{A}_+ \\ \hdashline I_{p_2-m_1} & O & O & O \\ \hdashline \bar{C}_{22ll} & \bar{C}_{22lr} & \bar{C}_{22rl} & \bar{C}_{22rr} \end{array}\right], \tag{2.4}$$

where $\lambda_i(\bar{A}_{14}) < 0$, both of the pairs $(\bar{A}_{11}, I_{p_2-m_1})$ and $(\bar{A}_+, \bar{C}_{22rr})$ are observable, and $\bar{A}_+$ has no $j\omega$ eigenvalues.

Proof. (1) Choose a basis V for the controllable subspace of $(A - B_2D_{12}^\dagger C_1, B_2D_{12}^\perp)$. Furthermore, choose a basis U such that the transformation matrix

$$M \triangleq \left[\begin{array}{cc} V & U \end{array}\right]$$

is nonsingular. Then the following transformation is possible

$$M^{-1}(A - B_2D_{12}^{\dagger}C_1)M = \begin{bmatrix} A_1 & A_2 \\ O & A_4 \end{bmatrix}$$
$$M^{-1}B_2D_{12}^{\perp} = \begin{bmatrix} \hat{B}_{21u} \\ O \end{bmatrix}$$
$$M^{-1}B_2D_{12}^{\dagger} = \begin{bmatrix} \hat{B}_{22u} \\ \hat{B}_{22l} \end{bmatrix},$$

where $(A_1, \hat{B}_{21u})$ is controllable and $(A_4, \hat{B}_{22l})$ is stabilizable.

By the previous assumption, $\hat{B}_{21u}$ is of full column rank and we put $\hat{B}_{21u}$ as

$$\hat{B}_{21u} = \begin{bmatrix} I_{m_2-p_1} \\ O \end{bmatrix}.$$

Let us define a nonsingular transformation matrix J:

$$J \triangleq \begin{bmatrix} J_1 & O \\ O & J_2 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Then J_1 and J_2 are nonsingular and the following equation holds:

$$J^{-1}M^{-1}(A - B_2D_{12}^{\dagger}C_1)MJ = \begin{bmatrix} J_1^{-1}A_1J_1 & J_1^{-1}A_2J_2 \\ O & J_2^{-1}A_4J_2 \end{bmatrix}$$

Putting J_1 and A_1 as

$$J_1 = \begin{bmatrix} I & X \\ O & I \end{bmatrix}, \; A_1 = \begin{bmatrix} A_{111} & A_{112} \\ A_{121} & A_{122} \end{bmatrix}$$

and substituting these into $J_1^{-1}A_1J_1$, the following equation is obtained

$$J_1^{-1}A_1J_1 = \begin{bmatrix} A_{111} - XA_{121} & A_{111}X - XA_{121}X + A_{112} - XA_{122} \\ A_{121} & A_{121}X + A_{122} \end{bmatrix},$$

where (A_{122}, A_{121}) is controllable, since $(A_1, \hat{B}_{21u})$ is controllable. Hence the arbitrary eigenvalues of $A_{121}X + A_{122}$ can be specified. Choose matrix X such that all the eigenvalues of $A_{121}X + A_{122}$ lie in the left half plane and define the stable matrix A_{14} as follows.

$$A_{121}X + A_{122} \triangleq A_{14}$$

From this definition $J_1^{-1}A_1J_1$ can be represented as

$$J_1^{-1}A_1J_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix}.$$

On the other hand, choosing a matrix J_2 that does permutation and diagonalization, $J_2^{-1}A_4J_2$ becomes

$$J_2^{-1}A_4J_2 = \begin{bmatrix} A_{-} & O \\ O & A_{+} \end{bmatrix},$$

where the matrix A_- is a stable zero-mode of $G_{12}(s)$ and the matrix A_+ is an unstable zero-mode of $G_{12}(s)$. Hence by putting $J_1^{-1}A_2J_2$ as

$$J_1^{-1}A_2J_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix},$$

the following equations are obtained

$$J^{-1}M^{-1}(A - B_2D_{12}^{\dagger}C_1)MJ = \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \\ A_{13} & A_{14} & A_{23} & A_{24} \\ O & O & A_- & O \\ O & O & O & A_+ \end{bmatrix}$$

$$J_1^{-1}\hat{B}_{21u} = \begin{bmatrix} I_{m_2-p_1} \\ O \end{bmatrix}.$$

Partitioning $J^{-1}M^{-1}B_2D_{12}^{\dagger}$ as

$$J^{-1}M^{-1}B_2D_{12}^{\dagger} = \begin{bmatrix} J_1^{-1} & O \\ O & J_2^{-1} \end{bmatrix} \begin{bmatrix} \hat{B}_{22u} \\ \hat{B}_{22l} \end{bmatrix} = \begin{bmatrix} J_1^{-1}\hat{B}_{22u} \\ J_2^{-1}\hat{B}_{22l} \end{bmatrix} = \begin{bmatrix} \hat{B}_{22uu} \\ \hat{B}_{22ul} \\ \hat{B}_{22lu} \\ \hat{B}_{22ll} \end{bmatrix},$$

and putting

$$MJ \triangleq T$$

follow the results in equation (2.4).

(2) This result follows by applying the dual argument of the proof (1). ■

2.4 The non-standard H_∞ problem of case 2

This section derives full-order and reduced-order controllers of the non-standard H_∞ problem of case 2. That is, we consider the H_∞ problem for a generalized plant:

$$G(s) = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ \hdashline C_{21} & O & O \\ C_{22} & I_{m_1} & O \end{array}\right],$$

where the assumptions **A1**, **A2'**-case 2 and **A3'** are satisfied. We put variables as follows.

$$\left[\begin{array}{c:c} C_2 & D_{21} \end{array}\right] = \left[\begin{array}{c:c} C_{21} & O \\ C_{22} & I_{m_1} \end{array}\right]$$

2.4.1 Characterization of zeros in $G_{21}(s)$

Let us consider the non-standard problem of case 2. Since D_{21} is of full column rank, it is assumed that the generalized plant has been put into a basis corresponding to the canonical form in equation (2.4). Hence, the following equations can be assumed without loss of generality

$$\begin{array}{l} A - B_1 D_{21}^{\dagger} C_2 = \left[\begin{array}{c:ccc} A_{11} & A_{12} & O & O \\ \hdashline A_{13} & A_{14} & O & O \\ A_{31} & A_{32} & A_{-} & O \\ A_{33} & A_{34} & O & A_{+} \end{array}\right], A_{11} \in \mathbb{R}^{(p_2-m_1)\times(p_2-m_1)} \\ D_{21}^{\perp} C_2 = \left[\begin{array}{c:ccc} I_{p_2-m_1} & O & O & O \end{array}\right] \\ D_{21}^{\dagger} C_2 = \left[\begin{array}{c:ccc} C_{22ll} & C_{22lr} & C_{22rl} & C_{22rr} \end{array}\right] \triangleq \left[\begin{array}{c:c} C_{22ll} & C_{22r} \end{array}\right], C_{22ll} \in \mathbb{R}^{p_2\times(p_2-m_1)} \end{array}, \quad (2.5)$$

where $\lambda_i(A_{14}) < 0$, both of the pairs $(A_{11}, I_{p_2-m_1})$ and (A_+, C_{22rr}) are observable, and $A_+ \in \mathbb{R}^{l\times l}$ has no $j\omega$ eigenvalues. We call A_- the stable zero mode of $G_{21}(s)$ and A_+ the unstable zero mode of $G_{21}(s)$.

For notational ease, let us partition the following matrices.

$$\begin{array}{l} B_1 = \left[\begin{array}{c} B_{1u} \\ \hdashline B_{1l} \end{array}\right], \ B_{1u} \in \mathbb{R}^{(p_2-m_1)\times m_1} \\ B_2 = \left[\begin{array}{c} B_{2u} \\ \hdashline B_{2l} \end{array}\right], \ B_{2u} \in \mathbb{R}^{(p_2-m_1)\times m_2} \\ C_1 = \left[\begin{array}{c:ccc} C_{1ll} & C_{1lr} & C_{1rl} & C_{1rr} \end{array}\right] \triangleq \left[\begin{array}{c:c} C_{1ll} & C_{1r} \end{array}\right], \ C_{1ll} \in \mathbb{R}^{p_1\times(p_2-m_1)} \end{array} \quad (2.6)$$

Under the above preparations, let us choose L_H such that the observable subspace of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$ is stabilized, and put L_H as follows

$$L_H \triangleq \left[\begin{array}{c} L_{H_1} \\ -A_{13} \\ -A_{31} \\ L_{H_{33}} \end{array}\right] \in \mathbb{R}^{n\times(p_2-m_1)}, \quad (2.7)$$

where $L_{H_1} \in \mathbb{R}^{(p_2-m_1)\times(p_2-m_1)}$ is chosen such that $A_{11} + L_{F_1}$ is stable, and $L_{H_{33}} \in \mathbb{R}^{l\times(p_2-m_1)}$ is an arbitrary matrix. Then we define A_{ZH} as follows:

$$A_{ZH} = A - B_1 D_{21}^{\dagger} C_2 + L_H D_{21}^{\perp} C_2 = \left[\begin{array}{ccc:c} A_{11} + L_{H_1} & A_{12} & O & O \\ O & A_{14} & O & O \\ O & A_{32} & A_{-} & O \\ \hdashline A_{33} + L_{H_{33}} & A_{34} & O & A_{+} \end{array}\right],$$

where the (1,1)-block of the matrix A_{ZH} is stable.

Next, let us choose E_H such that the unstable zero-mode of $G_{21}(s)$ is stabilized without effecting the stable mode of A_{ZH}. This implies that E_H is chosen such that a matrix $A_H \triangleq A_{ZH} + E_H D_{21}^{\dagger} C_2$ is stable under the following constraint:

$$U E_H = O, \quad (2.8)$$

where U is a row-basis of the stable subspace in A_{ZH}, that is, U satisfies

$$U A_{ZH} = \Lambda U,$$

where Λ is a stable mode of A_{ZH}.

Hence, temporarily, we choose E_H as

$$E_H \triangleq \begin{bmatrix} O \\ O \\ O \\ E_{H22} \end{bmatrix}. \tag{2.9}$$

Then A_H is represented as follows

$$\begin{aligned} A_H &= A_{ZH} + E_H D_{21}^{\dagger} C_2 \\ &= \begin{bmatrix} A_{11} + L_{H_1} & A_{12} & O & O \\ O & A_{14} & O & O \\ O & A_{32} & A_{-} & O \\ A_{33} + L_{H_{33}} + E_{H22} C_{22ll} & A_{34} + E_{H22} C_{22lr} & E_{H22} C_{22rl} & A_{+} + E_{H22} C_{22rr} \end{bmatrix}, \end{aligned} \tag{2.10}$$

where E_{H22} can be chosen such that $A_{+} + E_{H22} C_{22rr}$ is stable, because the pair (A_{+}, C_{22rr}) is observable. Then the matrix A_H is stable.

2.4.2 A necessary condition for the solvability

First, we introduce Full Control problem, which will be used to derive a necessary condition for the solvability in the non-standard H_∞ problem of case 2.

Definition 2.4.1 *Let us consider the generalized plant (1.1), where the assumption* **A1** *is satisfied. Suppose that the matrices B_2 and D_{12} are defined as follows*

$$B_2 = \begin{bmatrix} I_n & O \end{bmatrix}, \quad D_{12} = \begin{bmatrix} O & I_{p_1} \end{bmatrix}. \tag{2.11}$$

Then the H_∞ problem for a generalized plant

$$G_{FC}(s) = \left[\begin{array}{c|c|cc} A & B_1 & I_n & O \\ \hline C_1 & O & O & I_{p_1} \\ C_2 & D_{21} & O & O \end{array}\right],$$

is called FC (Full Control) problem.

The following lemma is useful for solving the non-standard H_∞ problems.

Lemma 2.4.1 *Suppose that the H_∞ problem with the assumption* **A1** *is solvable. Then it is necessary that the FC problem is solvable.*

Proof. If the output feedback controller of $u = K(s)y$ solves the H_∞ problem, it is necessary that the controller of the form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} K(s) y$$

solves the FC problem. ■

From Lemma 2.4.1, it is necessary that the non-standard H_∞ problem for $G(s)$ where B_2 and D_{12} are supposed to hold (2.11) is solvable. Hence we consider the non-standard problem for the generalized plant $G_{FC}(s)$ where the assumptions **A1**, **A2'**-case 2 and **A3'** are satisfied.

If we choose an observer gain H and a feedback gain F such that

$$H \triangleq -B_1 D_{21}^\dagger + L_H D_{21}^\perp + E_H D_{21}^\dagger$$
$$F \triangleq \begin{bmatrix} HC_2 \\ -C_1 \end{bmatrix},$$

then $A + HC_2 = A + B_2 F = A_H$ is stable. From Lemma A.2.1, the stabilizing controllers $K(s)$ for $G_{22}(s)$ are parametrized with a free parameter $Q(s) \in \mathcal{RH}_\infty$, and the closed loop transfer function $G_{zw} = \mathcal{F}_l\left(G_{FC}(s), K(s)\right)$ is represented as follows.

$$G_{zw}(s) = \mathcal{F}_l\left(M_2(s), -QD_{21}\right), \quad Q(s) \in \mathcal{RH}_\infty$$
$$M_2(s) = \left[\begin{array}{c|cc} A_H & E_H & O \\ \hline C_1 & O & D_{12} \\ D_{21}^\dagger C_2 & I & O \end{array}\right]$$

From the above discussion, it can be seen that the H_∞ problem for $G_{FC}(s)$ is solvable, if and only if the H_∞ problem for $M_2(s)$ is solvable. Hence we consider the H_∞ problem for $M_2(s)$.

Lemma 2.4.2 *Suppose that the matrices L_H and E_H are selected as in (2.7) and (2.9) such that A_H is stable. Then, if the non-standard H_∞ problem of case 2 is solvable, it is necessary that the ARE*

$$YA_{ZH}^T + A_{ZH}Y + Y\left\{C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right\}Y = O, \tag{2.12}$$

where

$$A_{ZH} \triangleq A - B_1 D_{21}^\dagger C_2 + L_H D_{21}^\perp C_2$$

has a stabilizing solution $Y \geq O$, which stabilizes a matrix

$$A_Y \triangleq A_{ZH} + Y\left\{C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right\}. \tag{2.13}$$

Proof. Consider the H_∞ problem for the generalized plant $M_2(s)$ where A_H is stable. Seeing the matrix D_{12} given in (2.11), it can be verified that $M_2(s)$ holds the assumptions **A1**, **A2** and **A3**, hence the H_∞ problem for $M_2(s)$ is the standard problem. The solvability condition for the standard problem reduces to the condition such that the ARE in (2.12) has a nonnegative definite stabilizing solution Y. ■

Theorem 2.4.1 *Suppose that the matrices L_H and E_H are selected as in (2.7) and (2.9) such that A_H is stable, and that there exists a stabilizing solution $Y \geq O$ in the ARE (2.12). Then the solution Y can be represented as follows:*

$$Y = \begin{bmatrix} O & O \\ O & Y_r \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{2.14}$$

where $Y_r \in \mathbb{R}^{l\times l}$ is a nonnegative definite stabilizing solution of reduced-order ARE:

$$Y_r A_+^T + A_+ Y_r + Y_r \left(C_{1rr}^T C_{1rr} - C_{22rr}^T C_{22rr}\right) Y_r = O. \tag{2.15}$$

Proof. (Necessity:) Suppose that there exists a stabilizing solution $Y \geq O$ in the ARE (2.12). The ARE can be represented as follows.

$$Y A_{ZH}^T + A_Y Y = O \tag{2.16}$$

Let U be a row-basis of the stable subspace of the matrix A_{ZH}. Then the basis U satisfies the following equation:

$$U A_{ZH} = \Lambda U, \tag{2.17}$$

where

$$\Lambda = \begin{bmatrix} A_{11} + L_{H_1} & A_{12} & O \\ O & A_{14} & O \\ O & A_{32} & A_- \end{bmatrix}. \tag{2.18}$$

After post-multiplication by U^T, the equation (2.16) becomes

$$Y A_{ZH}^T U^T + A_Y Y U^T = \left(Y U^T\right)\Lambda^T + A_Y \left(Y U^T\right) = O.$$

Since both Λ and A_Y are stable matrices, one can apply Lemma A.3.1 to this equation and deduce that

$$Y U^T = O. \tag{2.19}$$

Since Y is a symmetric matrix, it must be of the form in (2.14). By substituting this Y into the ARE (2.12), the reduced-order ARE (2.15) can be deduced. Hence it is necessary that the ARE has a nonnegative definite stabilizing solution Y_r.

(Sufficiency:) Suppose that there exists a stabilizing solution $Y_r \geq O$ in the reduced-order ARE (2.15). Then the matrix

$$A_{Y_r} \triangleq A_+ + Y_r \left(C_{1rr}^T C_{1rr} - C_{22rr}^T C_{22rr}\right)$$

is stable. If we select Y as in (2.14), it can be verified that

$$\begin{aligned} A_Y &= A_{ZH} + Y\left\{C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right\} \\ &= \begin{bmatrix} A_{11} + L_{H_1} & A_{12} & O & O \\ O & A_{14} & O & O \\ O & A_{32} & A_- & O \\ \tilde{A}_{33} & \tilde{A}_{34} & \tilde{A}_{43} & A_{Y_r} \end{bmatrix}, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned}\tilde{A}_{33} &= A_{33} + L_{H_{33}} + Y_r\left(C_{1rr}^T C_{1ll} - C_{22rr}^T C_{22ll}\right)\\ \tilde{A}_{34} &= A_{34} + Y_r\left(C_{1rr}^T C_{1lr} - C_{22rr}^T C_{22lr}\right)\\ \tilde{A}_{43} &= Y_r\left(C_{1rr}^T C_{1rl} - C_{22rr}^T C_{22rl}\right)\end{aligned}$$

is a stable matrix, and that the matrix Y satisfies the ARE in (2.12). Hence the ARE has a nonnegative definite stabilizing solution Y. ■

2.4.3 Lossless factorization of $G(s)$

Up to this part, the matrix E_H is fixed as in (2.9). Again, let us choose E_H as follows:

$$E_H = -Y\left(D_{21}^\dagger C_2\right)^T. \tag{2.21}$$

Since $UY = O$, E_H satisfies (2.8). Then the following lemma is stated.

Lemma 2.4.3 *Suppose that the ARE in (2.12) has a nonnegative definite stabilizing solution Y. Then the following statements are equivalent.*

1. *A controller $K(s)$ stabilizes a closed loop system $(G(s), K(s))$, and satisfies*

$$\|\mathcal{F}_l\left(G(s), K(s)\right)\|_\infty < 1.$$

2. *A controller $K(s)$ stabilizes a closed loop system $(G_{tmp}(s), K(s))$, and satisfies*

$$\|\mathcal{F}_l\left(G_{tmp}(s), K(s)\right)\|_\infty < 1,$$

where $G_{tmp}(s)$ is represented as follows:

$$G_{tmp}(s) = \left[\begin{array}{c|cc}\hat{A} & B_1 - E_H & \hat{B}_2\\ \hline C_1 & O & D_{12}\\ C_2 & D_{21} & O\end{array}\right] \tag{2.22}$$

and $\hat{A}$ and $\hat{B}_2$ are

$$\begin{aligned}\hat{A} &= A + YC_1^T C_1\\ \hat{B}_2 &= B_2 + YC_1^T D_{12}.\end{aligned}$$

Proof. We can factorize $G(s)$ as follows:

$$G(s) = \Theta(s) \star G_{tmp}(s),$$

where $\star$ is star-product which is defined in section A.4, and $\Theta(s)$ is represented as follows

$$\Theta(s) = \begin{bmatrix}\Theta_{11}(s) & \Theta_{12}(s)\\ \Theta_{21}(s) & \Theta_{22}(s)\end{bmatrix} = \left[\begin{array}{c|cc}A_\infty & E_H & -YC_1^T\\ \hline C_1 & O & I\\ D_{21}^\dagger C_2 & I & O\end{array}\right] \in \mathcal{RH}_\infty,$$

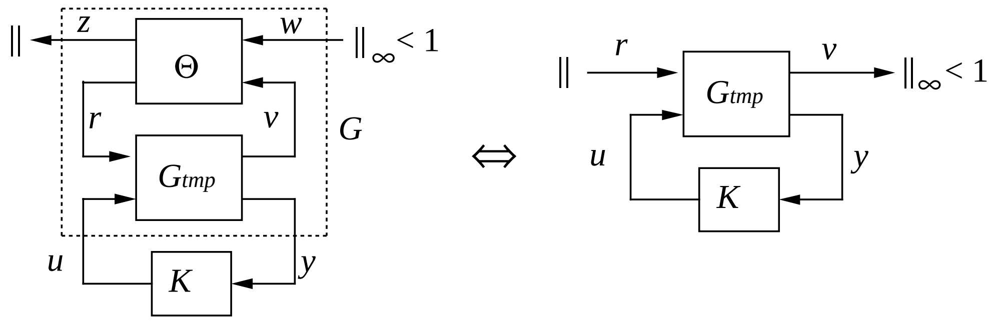

Figure 2.1: Lossless factorization

where

$$\begin{aligned} A_\infty &= A + H_\infty C_2 \\ H_\infty &= -B_1 D_{21}^\dagger + E_H D_{21}^\dagger + L_H D_{21}^\perp. \end{aligned} \tag{2.23}$$

Since the ARE in (2.12) has a nonnegative definite stabilizing solution Y, it can be verified that the following equations hold.

$$\begin{cases} Y A_\infty^T + A_\infty Y + B_\Theta B_\Theta^T = O \\ Y C_\Theta^T + B_\Theta D_\Theta^T = O \\ D_\Theta D_\Theta^T = I \end{cases}$$

Where B_Θ, C_Θ and D_Θ are defined as follows:

$$B_\Theta = \begin{bmatrix} E_H & -Y C_1^T \end{bmatrix}, C_\Theta = \begin{bmatrix} C_1 \\ D_{21}^\dagger C_2 \end{bmatrix}, D_\Theta = \begin{bmatrix} O & I \\ I & O \end{bmatrix}.$$

From the above fact and Lemma A.5.1, we can verify that $\Theta^T(s)$ is inner function. Furthermore, it can be verified that $\Theta_{12}^{-1}(s) \in \mathcal{RH}_\infty$ and $\Theta_{22}(\infty) = O$, hence the system $\Theta^T(s)$ is lossless, where lossless-ness of a system is listed in section A.6.

Denote the system G_{zw} as

$$G_{zw} = \mathcal{F}_l\left(\Theta, \mathcal{F}_l\left(G_{tmp}, K\right)\right).$$

From the definition of the H_∞-norm, it is verified that

$$\|G_{zw}\|_\infty < 1 \quad \Leftrightarrow \quad \left\|G_{zw}^T\right\|_\infty < 1. \tag{2.24}$$

On the other hand, $\|G_{zw}^T\|_\infty$ satisfies

$$\begin{aligned} \left\|G_{zw}^T\right\|_\infty &= \left\|\mathcal{F}_l\left(\Theta, \mathcal{F}_l\left(G_{tmp}, K\right)\right)^T\right\|_\infty \\ &= \left\|\mathcal{F}_l\left(\Theta^T, \mathcal{F}_l\left(G_{tmp}, K\right)^T\right)\right\|_\infty. \end{aligned}$$

Since Θ^T is lossless, from Lemma A.6.1 the following equivalence is verified.

$$\left\|G_{zw}^T\right\|_\infty < 1 \quad \Leftrightarrow \quad \left\|\mathcal{F}_l\left(G_{tmp}, K\right)^T\right\|_\infty < 1 \quad \Leftrightarrow \quad \left\|\mathcal{F}_l\left(G_{tmp}, K\right)\right\|_\infty < 1 \tag{2.25}$$

Hence, from (2.24) and (2.25) it is shown that the statements 1 and 2 are equivalent. ■

2.4.4 Parametrization of full-order H_∞ controller

Under the necessary condition such that the ARE in (2.12) has a nonnegative definite stabilizing solution Y, the H_∞ problem for $G(s)$ is reduced to the H_∞ problem for $G_{tmp}(s)$. Hence, in this section, we consider the H_∞ problem for $G_{tmp}(s)$. First, let us parametrize the stabilizing controllers for $G_{tmp}(s)$.

Suppose that $(\hat{A}, \hat{B}_2)$ is stabilizable. Then we can choose a feedback gain F such that

$$A_F \triangleq \hat{A} + \hat{B}_2 F$$

is stable, and let us choose an observer gain H_∞ as in (2.23), then

$$A_H \triangleq \hat{A} + H_\infty C_2 = A_Y$$

is stable. From Lemma A.2.1, the class of stabilizing controllers is represented as follows.

$$K(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} A_Y + \hat{B}_2 F & -H_\infty & -\hat{B}_2 \\ \hline F & O & -I \\ -C_2 & I & O \end{array}\right], Q(s)\right), Q(s) \in \mathcal{RH}_\infty \tag{2.26}$$

Then the closed loop transfer function is represented as follows

$$\mathcal{F}_l\left(G_{tmp}(s), K(s)\right) = \mathcal{F}_l\left(\tilde{M}_2(s), -Q(s)D_{21}\right),$$

where

$$\tilde{M}_2(s) = \left[\begin{array}{c|cc} \hat{A} + \hat{B}_2 F & B_1 - E_H & \hat{B}_2 \\ \hline C_1 + D_{12}F & O & D_{12} \\ O & I & O \end{array}\right]. \tag{2.27}$$

From (2.27), the solvability conditions of the H_∞ problems for $G_{tmp}(s)$ and for $\tilde{M}_2(s)$ are equivalent, hence we consider the H_∞ problem for $\tilde{M}_2(s)$.

Theorem 2.4.2 *The non-standard H_∞ problem for the generalized plant which satisfies assumptions* **A1**, **A2'***-case 2 and* **A3'** *is solvable if and only if an ARE:*

$$\begin{aligned} &X\left(A - B_2 D_{12}^{\dagger} C_1\right) + \left(A - B_2 D_{12}^{\dagger} C_1\right)^T X \\ &\quad + X\left\{B_1 B_1^T - B_2 D_{12}^{\dagger}\left(B_2 D_{12}^{\dagger}\right)^T\right\} X + \left(D_{12}^{\perp} C_1\right)^T D_{12}^{\perp} C_1 = O \end{aligned} \tag{2.28}$$

and an ARE in (2.15) have stabilizing solutions $X \geq O$ and $Y_r \geq O$ which satisfy $\rho(XY) < 1$, where

$$Y = \begin{bmatrix} O & O \\ O & Y_r \end{bmatrix}.$$

Under such conditions, the class of the H_∞ controllers is represented as:

$$\mathcal{K}_\infty^2 = \left\{K_\infty^2(s) : N(s) \in \mathcal{BH}_\infty, W(s) \in \mathcal{RH}_\infty\right\},$$

where $N(s)$ and $W(s)$ are free parameters, and $K_\infty^2(s)$ is represented as follows

$$K_\infty^2(s) = \mathcal{F}_l\left(\left[\begin{array}{c|c:c} A_Y + \hat{B}_2 F_\infty & H_\infty & -\hat{B}_2\Sigma \\ \hline -F_\infty & O & \Sigma \\ \hdashline D_{21}^\dagger \hat{C}_2 Z & D_{21}^\dagger & O \\ D_{21}^\perp C_2 & D_{21}^\perp & O \end{array}\right], \left[\begin{array}{cc} N(s) & W(s) \end{array}\right]\right) \tag{2.29}$$

and

$$\begin{aligned} A_Y &= A + Y C_1^T C_1 + H_\infty C_2 \\ \hat{B}_2 &= B_2 + Y C_1^T D_{12} \\ \hat{C}_2 &= D_{21} B_1^T X + C_2 \\ F_\infty &= \left\{ -D_{12}^\dagger C_1 - D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T X \right\} Z \\ H_\infty &= -B_1 D_{21}^\dagger - Y \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger + L_H D_{21}^\perp \\ Z &= (I - YX)^{-1} \\ \Sigma &= \left(D_{12}^T D_{12}\right)^{-\frac{1}{2}}. \end{aligned}$$

Proof. The H_∞ problem for $\tilde{M}_2(s)$ in (2.27) is DF (Disturbance Feedforward) problem. In appendix A.7 the solution for the DF problem is listed. By using the solution, we derive the solution for the non-standard H_∞ problem of case 2.

(Solvability condition:) The solvability condition for the DF problem is such that ARE:

$$S\Psi + \Psi^T S + S\Upsilon S + \left(D_{12}^\perp C_1\right)^T D_{12}^\perp C_1 = O,$$

where

$$\begin{aligned} \Psi &\triangleq \hat{A} - \hat{B}_2 D_{12}^\dagger C_1 \\ \Upsilon &\triangleq (B_1 - E_H)(B_1 - E_H)^T - \hat{B}_2 D_{12}^\dagger \left(\hat{B}_2 D_{12}^\dagger\right)^T \end{aligned}$$

has a nonnegative definite stabilizing solution S. By using a definition in appendix A.8, we represent the solvability condition such that

$$\left\{ \begin{array}{l} H_S \in \mathrm{Dom}\,(\mathrm{Ric}) \\ S = \mathrm{Ric}\,(H_S) \geq O \end{array} \right. , \tag{2.30}$$

where

$$H_S \triangleq \left[\begin{array}{cc} \Psi & \Upsilon \\ -\left(D_{12}^\perp C_1\right)^T D_{12}^\perp C_1 & -\Psi^T \end{array}\right].$$

Since H_S satisfies

$$H_X \left[\begin{array}{cc} I & Y \\ O & I \end{array}\right] = \left[\begin{array}{cc} I & Y \\ O & I \end{array}\right] H_S,$$

where $Y \geq O$ is a stabilizing solution of the ARE in (2.12), and H_X is defined as follows:

$$H_X \triangleq \begin{bmatrix} A - B_2 D_{12}^{\dagger} C_1 & B_1 B_1^T - B_2 D_{12}^{\dagger} \left(B_2 D_{12}^{\dagger} \right)^T \\ -\left(D_{12}^{\perp} C_1 \right)^T D_{12}^{\perp} C_1 & -\left(A - B_2 D_{12}^{\dagger} C_1 \right)^T \end{bmatrix},$$

the solvability condition (2.30) is equivalent to

$$\begin{cases} H_X \in \mathrm{Dom}\,(\mathrm{Ric}) \\ X = \mathrm{Ric}\,(H_X) \geq O \\ \rho(XY) < 1 \end{cases}, \tag{2.31}$$

where $Y \geq O$ is a stabilizing solution of the ARE in (2.12). The solvability condition (2.31) implies that the ARE (2.28) has a nonnegative definite stabilizing solution X, and which satisfies $\rho(XY) < 1$, where $Y = \begin{bmatrix} O & O \\ O & Y_r \end{bmatrix}$, and Y_r is a nonnegative definite stabilizing solution of ARE (2.15).

Conversely if the solvability condition (2.31) is satisfied, $H_S \in \mathrm{Dom}\,(\mathrm{Ric})$. Then it can be verified that the matrix

$$\Psi - \hat{B}_2 D_{12}^{\dagger} \left(\hat{B}_2 D_{12}^{\dagger} \right)^T S = \hat{A} + \hat{B}_2 \left\{ -D_{12}^{\dagger} C_1 - D_{12}^{\dagger} \left(B_2 D_{12}^{\dagger} \right)^T S \right\}$$

is stable, and $(\hat{A}, \hat{B}_2)$ is stabilizable.

(Derivation of the controller:) Suppose that the solvability condition (2.31) is satisfied. In the beginning of this subsection, the feedback gain F can be selected from arbitrary matrix which stabilizes $A_F = \hat{A} + \hat{B}_2 F$. Since the condition (2.31) satisfies, we select F such that

$$F = F_\infty \triangleq -D_{12}^{\dagger} C_1 - D_{12}^{\dagger} \left(\hat{B}_2 D_{12}^{\dagger} \right)^T S,$$

where

$$\begin{aligned} S &= XZ \\ Z &= (I - YX)^{-1}. \end{aligned}$$

By using the formula in Lemma A.7, we can derive the controller

$$-Q(s) D_{21} = \mathcal{F}_l \left(\left[\begin{array}{c|cc} \hat{A} + \hat{B}_2 F_\infty & -(B_1 - E_H) & -\hat{B}_2 \Sigma \\ \hline O & O & \Sigma \\ (B_1 - E_H)^T S & I & O \end{array} \right], N(s) \right), \tag{2.32}$$

where $N(s) \in \mathcal{BH}_\infty$, and $\Sigma = \left(D_{12}^T D_{12} \right)^{-\frac{1}{2}}$. The equation (2.32) can be represented as follows:

$$-Q(s) D_{21} = \Sigma N(s) \left(I - M_{22}(s) N(s) \right)^{-1} M_{21}(s), \tag{2.33}$$

where

$$\begin{aligned}
M_{21}(s) &= I - \Phi(s)\,(B_1 - E_H) \\
M_{22}(s) &= -\Phi(s)\hat{B}_2\Sigma \\
\Phi(s) &= (B_1 - E_H)^T S \left(sI - \hat{A} - \hat{B}_2 F_\infty\right)^{-1}.
\end{aligned}$$

In the equation (2.33) D_{21} is of full column rank, hence the solution of $Q(s)$ is represented as follows:

$$Q(s) = -\Sigma N(s)\,(I - M_{22}(s)N(s))^{-1} M_{21}(s) D_{21}^{\dagger} + \hat{W}(s) D_{21}^{\perp},$$

where $\hat{W}(s) \in \mathcal{RH}_\infty^{m_2 \times (p_2 - m_1)}$. Since $(I - M_{22}(s)N(s))^{-1} \in \mathcal{RH}_\infty$, we can replace $\hat{W}(s)$ such that

$$\hat{W}(s) = -\Sigma\,(I - N(s)M_{22}(s))^{-1}\,(W(s) + N(s)\Phi(s)L_H)\,, \quad W(s) \in \mathcal{RH}_\infty^{m_2 \times (p_2 - m_1)}.$$

Then the solution $Q(s)$ is given as

$$\begin{aligned}
Q(s) &= -\Sigma\,(I - N(s)M_{22}(s))^{-1}\left\{N(s)\left(D_{21}^{\dagger} + \Phi(s)H_\infty\right) + W(s)D_{21}^{\perp}\right\} \\
&= \mathcal{F}_l\left(\left[\begin{array}{c|c:c}
\hat{A} + \hat{B}_2 F_\infty & H_\infty & -\hat{B}_2\Sigma \\
\hline
O & O & -\Sigma \\
\hdashline
(B_1 - E_H)^T S & D_{21}^{\dagger} & O \\
O & D_{21}^{\perp} & O
\end{array}\right], \left[\begin{array}{cc} N(s) & W(s) \end{array}\right]\right),
\end{aligned} \tag{2.34}$$

where $N(s) \in \mathcal{BH}_\infty, W(s) \in \mathcal{RH}_\infty^{m_2 \times (p_2 - m_1)}$. By applying this $Q(s)$ into (2.26), we can derive the H_∞ controller in (2.29). ■

Remark 2.4.1

1. *The class of H_∞ controllers is represented with two free parameters, where one of them is a free parameter in $\mathcal{BH}_\infty$ and the other one is a free parameter in $\mathcal{RH}_\infty$. On the other hand, in the standard H_∞ problem, the class of H_∞ controller is represented with only a free parameter in $\mathcal{BH}_\infty$. One of differences between the standard problem and the non-standard problem appears in this point.*

2. *The dynamical order of the central solution equals the dynamical order of the generalized plant, where we call the H_∞ controller, whose free parameters are fixed to zero, a central solution.*

3. *By using the free parameter, we can improve any performance of the central solution.*

2.4.5 Derivation of reduced-order H_∞ controller

This subsection derives reduced-order non-standard H_∞ controller. First, let us represent the full-order H_∞ controller with DHMT as follows:

$$\begin{aligned} K_\infty^2(s) &= \mathcal{DHM}\left(\left[\begin{array}{c|c:c} A_Y & \hat{B}_2 & H_\infty \\ \hline -\Sigma^{-1}F_\infty & \Sigma^{-1} & O \\ \hdashline D_{21}^\dagger \hat{C}_2 Z & O & D_{21}^\dagger \\ D_{21}^\perp C_2 & O & D_{21}^\perp \end{array}\right], \left[\begin{array}{cc} N(s) & W(s) \end{array}\right]\right) \\ &= \left[\begin{array}{c|c} A_Y & \hat{B}_2 \\ \hline C_K(s) & \Sigma^{-1} \end{array}\right]^{-1} \left[\begin{array}{c|c} A_Y & H_\infty \\ \hline C_K(s) & N(s)D_{21}^\dagger + W(s)D_{21}^\perp \end{array}\right], \end{aligned} \tag{2.35}$$

where $C_K(s)$ is defined as follows.

$$C_K(s) \triangleq -\Sigma^{-1}F_\infty + N(s)D_{21}^\dagger \hat{C}_2 Z + W(s)D_{21}^\perp C_2. \tag{2.36}$$

The matrix A_Y is written in the equation (2.20), where $L_{H_{33}} \in \mathbb{R}^{l\times(p_2-m_1)}$ is an arbitrary matrix. We can choose $L_{H_{33}}$ such that $\tilde{A}_{33} = O$ holds. Hence, by letting $L_{H_{33}}$ as

$$L_{H_{33}} = -A_{33} - Y_r\left(C_{1rr}^T C_{1ll} - C_{22rr}^T C_{22ll}\right), \tag{2.37}$$

the matrix A_Y is represented as

$$A_Y = \left[\begin{array}{cccc} A_{11}+L_{H_1} & A_{12} & O & O \\ O & A_{14} & O & O \\ O & A_{32} & A_- & O \\ O & \tilde{A}_{34} & \tilde{A}_{43} & A_{Y_r} \end{array}\right].$$

Then the matrix A_Y satisfies

$$A_Y V_Y = V_Y\left(A_{11} + L_{H_1}\right),$$

where $V_Y = \left[\begin{array}{c} I_{p_2-m_1} \\ O \end{array}\right] \in \mathbb{R}^{n\times(p_2-m_1)}$ is of full column rank. Hence, if $C_K(s)$ satisfies

$$C_K(s)V_Y = O, \tag{2.38}$$

the pair $(A_Y, C_K(s))$ is not observable, hence the dynamical order of the controller $K_\infty^2(s)$ is reduced by the number of $p_2 - m_1$.

From equation (2.36), the matrix $C_K(s)$ is represented with the free parameter $W(s) \in \mathcal{RH}_\infty^{m_2\times(p_2-m_1)}$, and is spanned by the row-basis of $D_{21}^\perp C_2$. Since $D_{21}^\perp C_2 V_Y = I_{p_2-m_1}$ holds, if we choose the free parameter $W(s)$ as

$$W(s) = -\left(-\Sigma^{-1}F_\infty + N(s)D_{21}^\dagger \hat{C}_2 Z\right)V_Y, \tag{2.39}$$

the matrix V_Y satisfies (2.38). Thus we can derive a reduced-order H_∞ controller.

Theorem 2.4.3 *Under the same solvability condition as in Theorem 2.4.2, the class of reduced-order H_∞ controllers is represented as:*

$$\mathcal{K}_\infty^{r2} = \{K_\infty^{r2}(s) : N(s) \in \mathcal{BH}_\infty\},$$

where $N(s)$ is a free parameter, and $K_\infty^{r2}(s)$ is represented as follows

$$K_\infty^{r2}(s) = \mathcal{DHM}\left(\left[\begin{array}{c|cc} \tilde{A}_Y & \tilde{B}_2 & \tilde{H}_\infty \\ \hline -\Sigma^{-1}F_{\infty 2} & \Sigma^{-1} & \Sigma^{-1}F_{\infty 1}D_{21}^{\perp} \\ C_{D_2} & O & D_{21}^{\dagger} - C_{D_1}D_{21}^{\perp} \end{array}\right], N(s)\right), \tag{2.40}$$

and

$$A_Y = \left[\begin{array}{c|ccc} A_{11}+L_{H_1} & A_{12} & O & O \\ \hline O & & & \\ O & & \tilde{A}_Y & \\ O & & & \end{array}\right] \in \mathbb{R}^{n\times n}, \quad \tilde{A}_Y \in \mathbb{R}^{(n-(p_2-m_1))\times(n-(p_2-m_1))}$$

$$\hat{B}_2 = \left[\begin{array}{c} B_{2u} \\ \hline \tilde{B}_2 \end{array}\right] \in \mathbb{R}^{n\times m_2}, \quad \tilde{B}_2 \in \mathbb{R}^{(n-(p_2-m_1))\times m_2}$$

$$H_\infty = \left[\begin{array}{c} -B_{1u}D_{21}^{\dagger} + L_{H_1}D_{21}^{\perp} \\ \hline \tilde{H}_\infty \end{array}\right] \in \mathbb{R}^{n\times p_2}, \quad \tilde{H}_\infty \in \mathbb{R}^{(n-(p_2-m_1))\times p_2}$$

$$F_\infty = \left[\begin{array}{c|c} F_{\infty 1} & F_{\infty 2} \end{array}\right], \quad F_{\infty 1} \in \mathbb{R}^{m_2\times(p_2-m_1)}, F_{\infty 2} \in \mathbb{R}^{m_2\times(n-(p_2-m_1))}$$

$$D_{21}^{\dagger}\hat{C}_2 Z = \left[\begin{array}{c|c} C_{D_1} & C_{D_2} \end{array}\right], \quad C_{D_1} \in \mathbb{R}^{m_1\times(p_2-m_1)}, C_{D_2} \in \mathbb{R}^{m_1\times(n-(p_2-m_1))}.$$

Proof. From the previous argument, the reduced order controller is derived by substituting $L_{H_{33}}$ in (2.37) and $W(s)$ in (2.39) into $K_\infty^2(s)$ in (2.35). Since $\mathcal{K}_\infty^{r2}$ is a subset of $\mathcal{K}_\infty^2$, the reduced order controller is an H_∞ controller. ■

Remark 2.4.2 *The dynamical order of the central solution in $\mathcal{K}_\infty^{r2}$ is $n-(p_2-m_1)$, which is lower than that of the central solution in $\mathcal{K}_\infty^2$ by the number of p_2-m_1. The controller order reduction is analogous to the order reduction in observer-based controllers, where the order of an observer is reduced by the number of independent outputs that are not corrupted by disturbances.*

Remark 2.4.3 *The controller parameters are also represented as follows:*

$$\begin{aligned}
\tilde{A}_Y &= \tilde{A}_r + \tilde{Y}_r\left(C_{1r}^T C_{1r} - C_{22r}^T C_{22r}\right) \\
\tilde{B}_2 &= B_{2l} + \tilde{Y}_r C_{1r}^T D_{12} \\
\tilde{H}_\infty &= -B_{1l}D_{21}^{\dagger} - \tilde{Y}_r C_{22r}^T D_{21}^{\dagger} + L_{H_2}D_{21}^{\perp} \\
F_{\infty 1} &= -D_{12}^{\dagger}\left\{C_{1ll}Z_{11} + C_{1r}Z_{21} + \left(D_{12}^{\dagger}\right)^T\left(B_{2u}^T S_{11} + B_{2l}^T S_{21}\right)\right\} \\
F_{\infty 2} &= -D_{12}^{\dagger}\left\{C_{1ll}Z_{12} + C_{1r}Z_{22} + \left(D_{12}^{\dagger}\right)^T\left(B_{2u}^T S_{12} + B_{2l}^T S_{22}\right)\right\} \\
B_{C_1} &= B_{1u}^T S_{11} + B_{1l}^T S_{21} + C_{22ll}Z_{11} + C_{22r}Z_{21} \\
B_{C_2} &= B_{1u}^T S_{12} + B_{1l}^T S_{22} + C_{22ll}Z_{12} + C_{22r}Z_{22},
\end{aligned}$$

where

$$\tilde{A}_r = \begin{bmatrix} A_{14} & O & O \\ A_{32} & A_- & O \\ A_{34} & O & A_+ \end{bmatrix}, \quad \tilde{Y}_r = \begin{bmatrix} O & O & O \\ O & O & O \\ O & O & Y_r \end{bmatrix} \in \mathbb{R}^{(n-(p_2-m_1))\times(n-(p_2-m_1))}$$

$$S = XZ = \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right], \quad Z = \left[\begin{array}{c|c} Z_{11} & Z_{12} \\ \hline Z_{21} & Z_{22} \end{array}\right], \quad L_H = \left[\begin{array}{c} L_{H_1} \\ \hline L_{H_2} \end{array}\right],$$

and $X \geq O$ is a stabilizing solution of ARE in (2.52) and $Y_r \geq O$ is a stabilizing solution of ARE in (2.15).

2.4.6 The controller structure

This subsection reviews the full-order controller and the reduced-order controller from the viewpoint of the observer based output controller. The central solution in the class of full-order H_∞ controller in (2.29) is represented as

$$\begin{cases} \dot{\hat{x}} = A_Y\hat{x} + \hat{B}_2 u - H_\infty y \\ u = F_\infty \hat{x} \end{cases}, \tag{2.41}$$

where $\hat{x}$ is a state variable of the controller. If we assume an orthogonal condition: $C_1^T D_{12} = O$, these equations can be written as

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B_2 u + H_\infty (C_2\hat{x} - y) + YC_1^T C_1 \hat{x} \\ u = F_\infty \hat{x} \end{cases}. \tag{2.42}$$

From these equations, it can be seen that the controller has the structure of an observer-based controller. In order to interpret the structure of the controller, let us consider the system:

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u + Y\left\{C_1^T C_1 x + \left(D_{21}^\dagger C_2\right)^T w\right\} \\ y = C_2 x + D_{21} w \end{cases}. \tag{2.43}$$

By letting the error between the states of the systems in (2.42) and (2.43) as

$$e \triangleq x - \hat{x},$$

the dynamical equation of the error can be given as follows.

$$\dot{e} = A_Y e \tag{2.44}$$

Since all of the eigenvalues in the matrix

$$A_Y = A + YC_1^T C_1 + H_\infty C_2$$

have negative real parts, as the time intends to ∞, from any initial values of $e(0)$ the error $e(t)$ converges to zero. Thus, the controller in (2.41) consists of an observer for the system in (2.43) and a state feedback controller. Figure 2.2 illustrates the structure of the closed loop system that is constructed with the central controller.

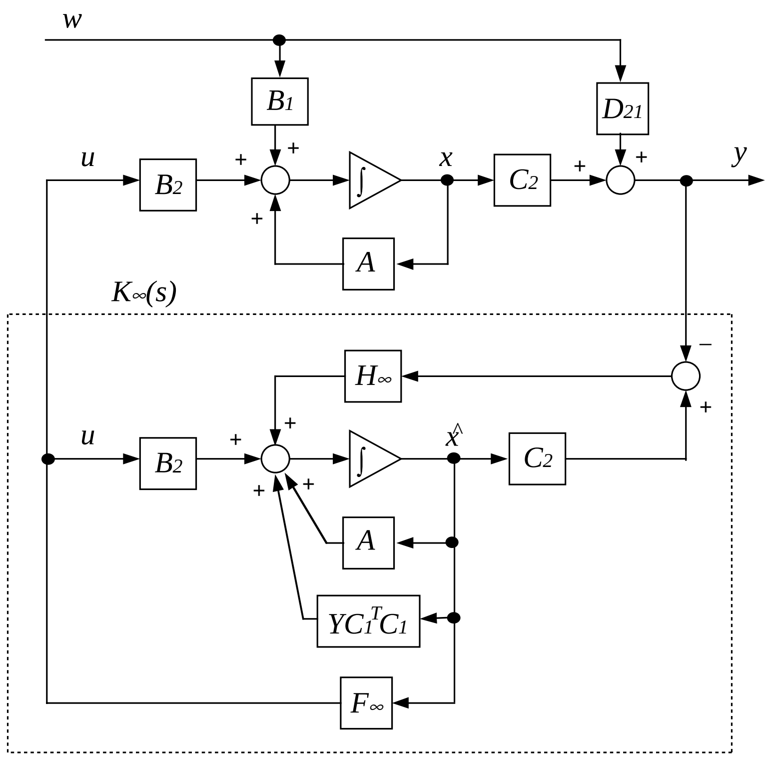

Figure 2.2: The structure of the closed loop system

By representing the class of the H_∞ controllers with the equation (2.35), the structure of the observer can also be seen explicitly in the H_∞ controller which is represented with free parameters. Figure 2.3 illustrates the H_∞ controller with the observer-based representation where the dynamical equation of the observer is expressed explicitly. The size of the matrix A_Y equals the dynamical order of a full-order observer. Since $C_K(s)$ is a function of the free parameters, the selection in the free parameters reduces to yield the unobservable subspace in the pair $(A_Y, C_K(s))$, and the order of the controller can be reduced by the number of the degrees of the subspace. Hence, this type of the reduced-order H_∞ controller can be interpreted as a minimal-order-observer-type H_∞ controller.

2.5 The non-standard H_∞ problem of case 1

This section derives full-order and reduced-order controllers of the non-standard H_∞ problem in case 1. Since the problem of case 1 is the dual problem of case 2, discussions of this section are dual of the previous section, hence details of the procedure in the derivation of the full-order controller are omitted. In this case, we treat a generalized plant:

$$G(s) = \left[\begin{array}{c|c:cc} A & B_1 & B_{21} & B_{22} \\ \hline C_1 & O & O & I_{p_1} \\ \hdashline C_2 & D_{21} & O & O \end{array}\right],$$

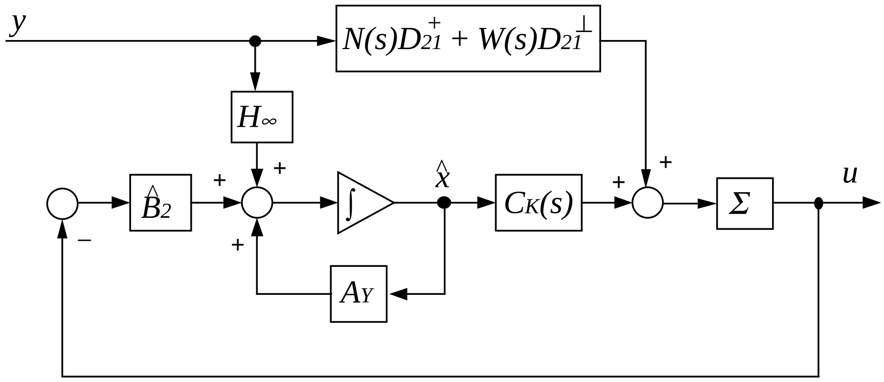

Figure 2.3: The structure of $K_\infty^2(s)$

where the assumptions **A1**, **A2'**-case1 and **A3'** are satisfied. We put variables as follows.

$$\begin{bmatrix} B_2 \\ \hline D_{12} \end{bmatrix} = \begin{bmatrix} B_{21} & B_{22} \\ \hline O & I_{p_1} \end{bmatrix}$$

2.5.1 Characterization of zeros in $G_{12}(s)$

Let us consider the non-standard problem of case 1. Since D_{12} is of full row rank, it is assumed that the generalized plant has been put into a basis corresponding to the canonical form in equation (2.3). Hence the following equations can be assumed without loss of generality

$$\begin{aligned} A - B_2 D_{12}^{\dagger} C_1 &= \left[\begin{array}{c|ccc} A_{11} & A_{12} & A_{21} & A_{22} \\ \hline A_{13} & A_{14} & A_{23} & A_{24} \\ O & O & A_- & O \\ O & O & O & A_+ \end{array}\right], A_{11} \in \mathbb{R}^{(m_2-p_1)\times(m_2-p_1)} \\ B_2 D_{12}^{\perp} &= \begin{bmatrix} I_{m_2-p_1} \\ \hline O \\ O \\ O \end{bmatrix} , \\ B_2 D_{12}^{\dagger} &= \begin{bmatrix} B_{22uu} \\ \hline B_{22ul} \\ B_{22lu} \\ B_{22ll} \end{bmatrix} \triangleq \begin{bmatrix} B_{22uu} \\ \hline B_{22l} \end{bmatrix}, B_{22uu} \in \mathbb{R}^{(m_2-p_1)\times m_2} \end{aligned} \tag{2.45}$$

where $\lambda_i(A_{14}) < 0$, both of the pairs $(A_{11}, I_{m_2-p_1})$ and (A_+, B_{22ll}) are controllable, and A_+ has no $j\omega$ eigenvalues. We call A_- the stable zero mode of $G_{12}(s)$ and A_+ unstable zero mode of $G_{12}(s)$.

For notational ease, let us partition the following matrices.

$$
\begin{aligned}
B_1 &= \begin{bmatrix} B_{1uu} \\ \hdashline B_{1ul} \\ B_{1lu} \\ B_{1ll} \end{bmatrix} \triangleq \begin{bmatrix} B_{1uu} \\ \hdashline B_{1l} \end{bmatrix}, \; B_{1uu} \in \mathbb{R}^{(m_2-p_1)\times m_1} \\
C_1 &= \left[\begin{array}{c:c} C_{1l} & C_{1r} \end{array}\right] \in \mathbb{R}^{p_1\times n}, \; C_{1l} \in \mathbb{R}^{p_1\times(m_2-p_1)} \\
C_2 &= \left[\begin{array}{c:c} C_{2l} & C_{2r} \end{array}\right] \in \mathbb{R}^{p_2\times n}, \; C_{2l} \in \mathbb{R}^{p_2\times(m_2-p_1)}.
\end{aligned}
$$

Under the above preparations, let us choose L_F such that the controllable subspace of the pair $(A - B_2D_{12}^{\dagger}C_1, B_2D_{12}^{\perp})$ is stabilized, and put L_F as follows

$$
L_F \triangleq \begin{bmatrix} L_{F_1} & -A_{12} & -A_{21} & L_{F_{22}} \end{bmatrix} \in \mathbb{R}^{(m_2-p_1)\times n}, \tag{2.46}
$$

where $L_{F_1} \in \mathbb{R}^{(m_2-p_1)\times(m_2-p_1)}$ is chosen such that $A_{11} + L_{F_1}$ is stable, and $L_{F_{22}} \in \mathbb{R}^{(m_2-p_1)\times r}$ is an arbitrary matrix. Then A_{ZF} is defined as follows:

$$
A_{ZF} \triangleq A - B_2D_{12}^{\dagger}C_1 + B_2D_{12}^{\perp}L_F = \left[\begin{array}{ccc:c} A_{11}+L_{F_1} & O & O & A_{22}+L_{F_{22}} \\ A_{13} & A_{14} & A_{23} & A_{24} \\ O & O & A_- & O \\ \hdashline O & O & O & A_+ \end{array}\right],
$$

where the (1,1)-block of A_{ZF} is a stable.

Next, let us choose E_F such that the unstable zero-mode of $G_{12}(s)$ is stabilized without effecting the stable mode of A_{ZF}. This implies that E_F is chosen such that the following constraint:

$$
E_F V = O, \tag{2.47}
$$

where V is a basis of the stable subspace in A_{ZF}, that is, V satisfies

$$
A_{ZF}V = V\Lambda,
$$

where Λ is a stable mode of A_{ZF}.

Hence, temporarily, we choose E_F as

$$
E_F \triangleq \begin{bmatrix} O & O & O & E_{F_{22}} \end{bmatrix}. \tag{2.48}
$$

Then A_F is represented as follows

$$
\begin{aligned}
A_F &= A_{ZF} + B_2D_{12}^{\dagger}E_F \\
&= \begin{bmatrix} A_{11}+L_{F_1} & O & O & A_{22}+L_{F_{22}}+B_{22uu}E_{F22} \\ A_{13} & A_{14} & A_{23} & A_{24}+B_{22ul}E_{F22} \\ O & O & A_- & B_{22lu}E_{F22} \\ O & O & O & A_+ + B_{22ll}E_{F22} \end{bmatrix},
\end{aligned} \tag{2.49}
$$

where E_{F22} can be chosen such that $A_+ + B_{22ll}E_{F22}$ is stable, because the pair (A_+, B_{22ll}) is controllable. Then the matrix A_F is stable.

2.5.2 A necessary condition for the solvability

First, we introduce Full Information problem, which will be used to derive a necessary condition for the solvability in the non-standard H_∞ problem of case 1.

Definition 2.5.1 *Let us consider the generalized plant (1.1), where the assumption* **A1** *is satisfied. Suppose that the matrices C_2 and D_{21} are defined as follows*

$$C_2 = \begin{bmatrix} I_n \\ O \end{bmatrix}, \quad D_{21} = \begin{bmatrix} O \\ I_{m_1} \end{bmatrix}. \tag{2.50}$$

Then the H_∞ problem for a generalized plant

$$G_{FI}(s) = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ \hdashline I_n & O & O \\ O & I_{m_1} & O \end{array}\right],$$

is called FI (Full Information) problem.

The following lemma is useful for solving the non-standard H_∞ problems.

Lemma 2.5.1 *Suppose that the H_∞ problem with the assumption* **A1** *is solvable. Then it is necessary that the FI problem is solvable.*

Proof. Omitted. ■

From Lemma 2.5.1, it is necessary that the non-standard H_∞ problem where C_2 and D_{21} are supposed to hold (2.50) is solvable. Hence we consider the non-standard problem for the generalized plant $G_{FI}(s)$ where the assumptions **A1**, **A2'**-case 1 and **A3'** are satisfied.

If we choose a feedback gain F and an observer gain H such that

$$\begin{aligned} F &\triangleq -D_{12}^{\dagger}C_1 + D_{12}^{\perp}L_F + D_{12}^{\dagger}E_F \\ H &\triangleq \begin{bmatrix} B_2F & -B_1 \end{bmatrix}, \end{aligned} \tag{2.51}$$

then $A + B_2F = A + HC_2 = A_F$ is stable. From Lemma A.2.1, the stabilizing controllers $K(s)$ for $G_{22}(s)$ are parametrized with a free parameter $Q(s) \in \mathcal{RH}_\infty$, and the closed loop transfer function $G_{zw} = \mathcal{F}_l\left(G_{FI}(s), K(s)\right)$ is represented as follows.

$$G_{zw}(s) = \mathcal{F}_l\left(M_1(s), -D_{12}Q\right), \quad Q(s) \in \mathcal{RH}_\infty$$
$$M_1(s) = \left[\begin{array}{c|cc} A_F & B_1 & B_2D_{12}^{\dagger} \\ \hline E_F & O & I \\ O & D_{21} & O \end{array}\right]$$

From the above discussion, the H_∞ problem for $G_{FI}(s)$ is solvable, if and only if the H_∞ problem for $M_1(s)$ is solvable. Hence we consider the H_∞ problem for $M_1(s)$.

Lemma 2.5.2 *Suppose that the matrices L_F and E_F are selected as in (2.46) and (2.48) such that A_F is stable. Then, if the non-standard H_∞ problem of case 1 is solvable, it is necessary that the ARE*

$$XA_{ZF} + A_{ZF}^T X + X\left\{B_1B_1^T - B_2D_{12}^\dagger\left(B_2D_{12}^\dagger\right)^T\right\}X = O, \tag{2.52}$$

where

$$A_{ZF} \triangleq A - B_2D_{12}^\dagger C_1 + B_2D_{12}^\perp L_F$$

has a stabilizing solution $X \geq O$, which stabilizes a matrix

$$A_X \triangleq A_{ZF} + \left\{B_1B_1^T - B_2D_{12}^\dagger\left(B_2D_{12}^\dagger\right)^T\right\}X. \tag{2.53}$$

Proof. Omitted. ■

Theorem 2.5.1 *Suppose that the matrices L_F and E_F are selected as in (2.46) and (2.48) such that A_F is stable, and that there exists a stabilizing solution $X \geq O$ in the ARE (2.52). Then the solution X can be represented as follows:*

$$X = \begin{bmatrix} O & O \\ O & X_r \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{2.54}$$

where $X_r \in \mathbb{R}^{r\times r}$ is a nonnegative definite stabilizing solution of reduced-order ARE:

$$X_rA_+ + A_+^T X_r + X_r\left(B_{1ll}B_{1ll}^T - B_{22ll}B_{22ll}^T\right)X_r = O. \tag{2.55}$$

Proof. Omitted. ■

By using X in (2.54), the matrix A_X is represented as follows:

$$\begin{aligned} A_X &= A_{ZF} + \left\{B_1B_1^T - B_2D_{12}^\dagger\left(B_2D_{12}^\dagger\right)^T\right\}X \\ &= \begin{bmatrix} A_{11}+L_{F_1} & O & O & \tilde{A}_{22} \\ A_{13} & A_{14} & A_{23} & \tilde{A}_{24} \\ O & O & A_- & \tilde{A}_{42} \\ O & O & O & A_{X_r} \end{bmatrix}, \end{aligned} \tag{2.56}$$

where

$$\begin{aligned} \tilde{A}_{22} &= A_{22} + L_{F_{22}} + \left(B_{1uu}B_{1ll}^T - B_{22uu}B_{22ll}^T\right)X_r \\ \tilde{A}_{24} &= A_{24} + \left(B_{1ul}B_{1ll}^T - B_{22ul}B_{22ll}^T\right)X_r \\ \tilde{A}_{42} &= \left(B_{1lu}B_{1ll}^T - B_{22lu}B_{22ll}^T\right) \\ A_{X_r} &= A_+ + \left(B_{1ll}B_{1ll}^T - B_{22ll}B_{22ll}^T\right). \end{aligned}$$

From Theorem 2.5.1, since the matrix A_{X_r} is stable, the matrix A_X is also stable.

2.5.3 Parametrization of full-order H_∞ controller

Theorem 2.5.2 *The non-standard H_∞ problem for the generalized plant which satisfies assumptions* **A1**, **A2'***-case 1 and* **A3'** *is solvable if and only if an ARE in (2.55) has a stabilizing solution $X_+ \geq O$, and an ARE:*

$$\begin{aligned} &Y\left(A - B_1 D_{21}^{\dagger} C_2\right)^T + \left(A - B_1 D_{21}^{\dagger} C_2\right) Y \\ &+Y\left\{C_1^T C_1 - \left(C_2 D_{21}^{\dagger}\right)^T C_2 D_{21}^{\dagger}\right\} Y + B_1 D_{21}^{\perp}\left(B_1 D_{21}^{\perp}\right)^T = O \end{aligned} \tag{2.57}$$

has a stabilizing solution $Y \geq O$ which satisfy $\rho(XY) < 1$, where

$$X = \begin{bmatrix} O & O \\ O & X_r \end{bmatrix}.$$

Under such conditions, the class of the H_∞ controllers is represented as:

$$\mathcal{K}_\infty^1 = \left\{K_\infty^1(s) : N(s) \in \mathcal{BH}_\infty, W(s) \in \mathcal{RH}_\infty\right\},$$

where $N(s)$ and $W(s)$ are free parameters, and $K_\infty^1(s)$ is represented as follows

$$K_\infty^1(s) = \mathcal{F}_l\left(\left[\begin{array}{c|c:cc} A_X + H_\infty \hat{C}_2 & -H_\infty & Z\hat{B}_2 D_{12}^{\dagger} & B_2 D_{12}^{\perp} \\ \hline F_\infty & O & D_{12}^{\dagger} & D_{12}^{\perp} \\ \hdashline -\Sigma \hat{C}_2 & \Sigma & O & O \end{array}\right], \begin{bmatrix} N(s) \\ W(s) \end{bmatrix}\right) \tag{2.58}$$

and

$$\begin{aligned} A_X &= A + B_1 B_1^T X + B_2 F_\infty \\ \hat{B}_2 &= B_2 + Y C_1^T D_{12} \\ \hat{C}_2 &= D_{21} B_1^T X + C_2 \\ F_\infty &= -D_{12}^{\dagger} C_1 + D_{12}^{\perp} L_F - D_{12}^{\dagger}\left(B_2 D_{12}^{\dagger}\right)^T X \\ H_\infty &= Z\left\{-B_1 D_{21}^{\dagger} - Y\left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger}\right\} \\ Z &= (I - YX)^{-1} \\ \Sigma &= \left(D_{21} D_{21}^T\right)^{-\frac{1}{2}}. \end{aligned}$$

Proof. Omitted. ■

Remark 2.5.1

1. *By comparing the controllers of case 1 and case 2, it can be verified that each controller has dual structure of another controller.*
2. *The dynamical order of the central solution equals the dynamical order of the generalized plant, where we call the H_∞ controller, whose free parameters are fixed to zero, a central solution.*
3. *By using the free parameter, we can improve any performance of the central solution.*

2.5.4 Derivation of reduced-order H_∞ controller

By using the free parameter in $\mathcal{K}_\infty^1$, we derive the reduced-order H_∞ controller. First, let us represent the full-order H_∞ controller with a homogeneous transformation as follows:

$$\begin{aligned} K_\infty^1(s) &= \mathcal{HM}\left(\left[\begin{array}{c|cc:c} A_X & Z\hat{B}_2 D_{12}^\dagger & B_2 D_{12}^\perp & -H_\infty \Sigma^{-1} \\ \hline F_\infty & D_{12}^\dagger & D_{12}^\perp & O \\ \hdashline \hat{C}_2 & O & O & \Sigma^{-1} \end{array}\right], \left[\begin{array}{c} N(s) \\ W(s) \end{array}\right]\right) \\ &= \left[\begin{array}{c|c} A_X & B_K(s) \\ \hline F_\infty & D_{12}^\dagger N(s) + D_{12}^\perp W(s) \end{array}\right]\left[\begin{array}{c|c} A_X & B_K(s) \\ \hline \hat{C}_2 & \Sigma^{-1} \end{array}\right]^{-1}, \end{aligned} \tag{2.59}$$

where $B_K(s)$ is defined as

$$B_K(s) \triangleq Z\hat{B}_2 D_{12}^\dagger N(s) + B_2 D_{12}^\perp W(s) - H_\infty \Sigma^{-1}. \tag{2.60}$$

In the equation (2.56), since the matrix $L_{F_{22}} \in \mathbb{R}^{(m_2-p_1)\times r}$ is arbitrary, we can choose $L_{F_{22}}$ such that $\tilde{A}_{22} = O$ holds. Hence let us choose $L_{F_{22}}$

$$L_{F_{22}} = -A_{22} - \left(B_{1uu} B_{1ll}^T - B_{22uu} B_{22ll}^T\right) X_r \tag{2.61}$$

then the matrix A_X is represented as

$$A_X = \left[\begin{array}{cccc} A_{11} + L_{F_1} & O & O & O \\ A_{13} & A_{14} & A_{23} & \tilde{A}_{24} \\ O & O & A_- & \tilde{A}_{42} \\ O & O & O & A_{X_r} \end{array}\right].$$

The matrix A_X satisfies

$$U_X A_X = (A_{11} + L_{F_1}) U_X,$$

where $U_X = \left[\begin{array}{cc} I_{m_2-p_1} & O \end{array}\right] \in \mathbb{R}^{(m_2-p_1)\times n}$. Hence, if $B_K(s)$ satisfies

$$U_X B_K(s) = O, \tag{2.62}$$

the pair $(A_X, B_K(s))$ has an uncontrollable subspace, hence the order of the controller in $\mathcal{K}_\infty^1$ is reduced by the number of $m_2 - p_1$.

From equation (2.60), the matrix $B_K(s)$ is represented with the free parameter $W(s) \in \mathcal{RH}_\infty$, and is spanned by the basis of $B_2 D_{12}^\perp$. Since $U_X B_2 D_{12}^\perp = I_{m_2-p_1}$ holds, if we choose the free parameter $W(s)$ as

$$W(s) = -U_X\left(Z\hat{B}_2 D_{12}^\dagger N(s) - H_\infty \Sigma^{-1}\right), \tag{2.63}$$

the matrix U_X satisfies (2.62). Thus we can derive a reduced-order H_∞ controller.

Theorem 2.5.3 *Under the same solvability condition as in Theorem 2.5.2, the class of reduced-order H_∞ controllers is represented as:*

$$\mathcal{K}_\infty^{r_1} = \{K_\infty^{r_1}(s) : N(s) \in \mathcal{BH}_\infty\},$$

where $N(s)$ is a free parameter, and $K_\infty^{r_1}(s)$ is represented as follows

$$K_\infty^{r_1}(s) = \mathcal{HM}\left(\left[\begin{array}{c|cc} \tilde{A}_X & B_{D_2} & -H_{\infty_2}\Sigma^{-1} \\ \hline \tilde{F}_\infty & D_{12}^\dagger - D_{12}^\perp B_{D_1} & D_{12}^\perp H_{\infty_1}\Sigma^{-1} \\ \tilde{C}_2 & O & \Sigma^{-1} \end{array}\right], N(s)\right), \tag{2.64}$$

and

$$A_X = \left[\begin{array}{c:ccc} A_{11} + L_{F_1} & O & O & O \\ \hdashline A_{13} & & & \\ O & & \tilde{A}_X & \\ O & & & \end{array}\right] \in \mathbb{R}^{n\times n}, \quad \tilde{A}_X \in \mathbb{R}^{(n-(m_2-p_1))\times(n-(m_2-p_1))}$$

$$Z\hat{B}_2 D_{12}^\dagger = \left[\begin{array}{c} B_{D_1} \\ \hdashline B_{D_2} \end{array}\right], \quad B_{D_1} \in \mathbb{R}^{(m_2-p_1)\times p_1}, B_{D_2} \in \mathbb{R}^{(n-(m_2-p_1))\times p_1}$$

$$H_\infty = \left[\begin{array}{c} H_{\infty_1} \\ \hdashline H_{\infty_2} \end{array}\right], \quad H_{\infty_1} \in \mathbb{R}^{(m_2-p_1)\times p_2}, H_{\infty_2} \in \mathbb{R}^{(n-(m_2-p_1))\times p_2}$$

$$F_\infty = \left[\begin{array}{c:c} -D_{12}^\dagger C_{1l} + D_{12}^\perp L_{F_1} & \tilde{F}_\infty \end{array}\right] \in \mathbb{R}^{m_2\times n}, \quad \tilde{F}_\infty \in \mathbb{R}^{m_2\times(n-(m_2-p_1))}$$

$$\hat{C}_2 = \left[\begin{array}{c:c} C_{2l} & \tilde{C}_2 \end{array}\right] \in \mathbb{R}^{p_2\times n}, \quad \tilde{C}_2 \in \mathbb{R}^{p_2\times(n-(m_2-p_1))}.$$

Proof. From the previous argument, the reduced order controller is derived by substituting $L_{F_{22}}$ in (2.61) and $W(s)$ in (2.63) into $K_\infty^1(s)$ in (2.59). Since $\mathcal{K}_\infty^{r_1}$ is a subset of $\mathcal{K}_\infty^1$, the reduced order controller is an H_∞ controller. ■

Remark 2.5.2 *The dynamical order of the central solution in $\mathcal{K}_\infty^{r_1}$ is $n-(m_2-p_1)$, which is lower than that of the central solution in $\mathcal{K}_\infty^1$ by the number of $m_2 - p_1$. The controller order reduction is analogous to the order reduction in dual-observer-based controllers, where the order of the controller is reduced by the number of redundant inputs.*

Remark 2.5.3 *The controller parameters are also represented as follows:*

$$\begin{aligned}
\tilde{A}_X &= \tilde{A}_r + \left(B_{1l}B_{1l}^T - B_{22l}B_{22l}^T\right)\tilde{X}_r \\
\tilde{C}_2 &= C_{2r} + D_{21}B_{1l}^T\tilde{X}_r \\
\tilde{F}_\infty &= -D_{12}^\dagger C_{1r} + D_{12}^\perp L_{F_2} - D_{12}^\dagger B_{22l}^T\tilde{X}_r \\
H_{\infty_1} &= -\left\{Z_{11}B_{1uu} + Z_{12}B_{1l} + \left(S_{11}C_{2l}^T + S_{12}C_{2r}^T\right)\left(D_{21}^\dagger\right)^T\right\}D_{21}^\dagger \\
H_{\infty_2} &= -\left\{Z_{21}B_{1uu} + Z_{22}B_{1l} + \left(S_{21}C_{2l}^T + S_{22}C_{2r}^T\right)\left(D_{21}^\dagger\right)^T\right\}D_{21}^\dagger \\
B_{D_1} &= Z_{11}B_{22uu} + Z_{12}B_{22l} + S_{11}C_{1l}^T + S_{12}C_{1r}^T \\
B_{D_2} &= Z_{21}B_{22uu} + Z_{22}B_{22l} + S_{21}C_{1l}^T + S_{22}C_{1r}^T,
\end{aligned}$$

where

$$\tilde{A}_r = \begin{bmatrix} A_{14} & A_{23} & \tilde{A}_{24} \\ O & A_- & \tilde{A}_{42} \\ O & O & A_{X_r} \end{bmatrix}, \quad \tilde{X}_r = \begin{bmatrix} O & O & O \\ O & O & O \\ O & O & X_r \end{bmatrix} \in \mathbb{R}^{(n-(m_2-p_1))\times(n-(m_2-p_1))}$$

$$S = ZY = \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right], \quad Z = \left[\begin{array}{c|c} Z_{11} & Z_{12} \\ \hline Z_{21} & Z_{22} \end{array}\right], \quad L_F = \left[\begin{array}{c} L_{F_1} \\ \hline L_{F_2} \end{array}\right],$$

and $X_r \geq O$ is a stabilizing solution of ARE in (2.55) and $Y \geq O$ is a stabilizing solution of ARE in (2.57).

2.5.5 The controller structure

This subsection reviews the full-order controller and the reduced-order controller from the viewpoint of the dual-observer based output controller. The central solution in the class of full-order H_∞ controller in (2.58) is represented as

$$\begin{cases} \dot{\hat{x}} = \left(A_X + H_\infty \hat{C}_2\right)\hat{x} + H_\infty y \\ u = -F_\infty \hat{x} \end{cases},$$

where $\hat{x}$ is a state variable of the controller. If we assume a orthogonal condition: $D_{21}B_1^T = O$, these equations can be written as

$$\begin{cases} \dot{\hat{x}} = A_X\hat{x} + \upsilon \\ u = -F_\infty \hat{x} \\ \upsilon = H_\infty \nu \\ \nu = y + C_2\hat{x} \end{cases}. \tag{2.65}$$

From these equations, it can be seen that the controller has the structure of a dual-observer-based controller. In order to interpret the structure of the controller, let us consider the system:

$$\begin{cases} \dot{x} = Ax + ZB_1w + B_2u + \left\{B_1B_1^TXx + ZY\left(D_{21}^\dagger C_2\right)^T w\right\} \\ y = C_2x + D_{21}w \end{cases}. \tag{2.66}$$

Then let us construct a composite system from the systems (2.66) and (2.65). By letting a new state variable of the composite system as

$$\eta = x + \hat{x},$$

the dynamical equation of the composite system is given as follows:

$$\begin{cases} \begin{bmatrix} \dot{\eta} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + B_1B_1^TX & O \\ O & A_X \end{bmatrix}\begin{bmatrix} \eta \\ \hat{x} \end{bmatrix} + \begin{bmatrix} Z\left\{B_1 + Y\left(D_{21}^\dagger C_2\right)^T\right\} \\ O \end{bmatrix} w + \begin{bmatrix} I \\ I \end{bmatrix}\upsilon \\ \nu = \begin{bmatrix} C_2 & O \end{bmatrix}\begin{bmatrix} \eta \\ \hat{x} \end{bmatrix} + D_{21}w \end{cases},$$

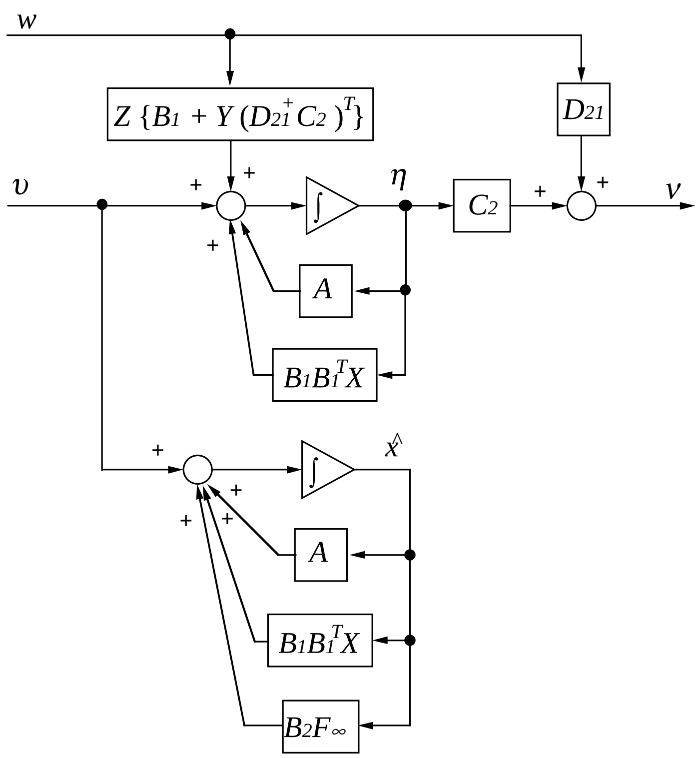

Figure 2.4: The composite system

where it should be noted that $\upsilon \in \mathbb{R}^n$, and that the subsystem of the state variable $\hat{x}$ is unobservable. The composite system is illustrated in Figure 2.4. Hence, the composite system of the state variable η is stabilizable with an output feedback controller, if the pair $\left(A + B_1B_1^TX, C_2\right)$ is observable. In fact, the output feedback controller:

$$\upsilon = H_\infty \nu$$

stabilizes the system as follows:

$$\dot{\eta} = A_X \eta,$$

where A_X is a stable matrix. The size of the matrix A_X equals the dynamical order of a full-order observer.

By representing the class of the H_∞ controllers with the equation (2.64), the structure of the dual-observer can also be seen explicitly in the H_∞ controller which is represented with free parameters. Figure 2.5 illustrates the H_∞ controller with the dual-observer-based representation where the dynamical equation of the dual-observer is expressed explicitly. The size of the matrix A_X equals the dynamical order of a full-order dual-observer. Since $B_K(s)$ is a function of the free parameters, the selection in the free parameters reduces to yield the uncontrollable subspace in the pair $(A_X, B_K(s))$, and the order of the controller can be

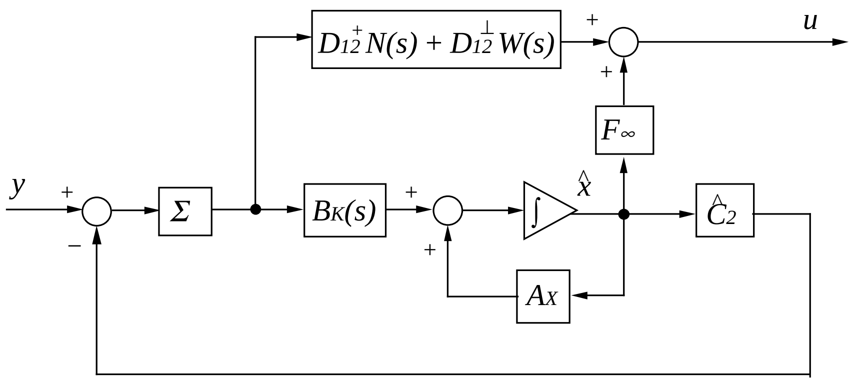

Figure 2.5: The structure of $K_\infty^1(s)$

reduced by the number of the degrees of the subspace. Hence, this type of the reduced-order H_∞ controller can be interpreted as a minimal-order dual-observer-type H_∞ controller.

2.6 The non-standard H_∞ problem of case 3

This section derives full-order and reduced-order controllers in the non-standard H_∞ problem of case 3. Since the problem of case 3 combines problems of cases 1 and 2, the procedure in the derivation of full-order controller is similar to the derivations in the problems of cases 1 and 2. However the reduced-order controller is slightly different from the reduced-order controllers in the problems of cases 1 and 2. Hence, details of the procedure in the derivation of the full-order controller are omitted, and almost all of discussions are devoted for the derivation of the reduced-order controller.

We treat a generalized plant:

$$G(s) = \left[\begin{array}{c|c:cc} A & B_1 & B_{21} & B_{22} \\ \hline C_1 & O & O & I_{p_1} \\ \hdashline C_{21} & O & O & O \\ C_{22} & I_{m_1} & O & O \end{array}\right],$$

where the assumptions **A1**, **A2'**-case 3 and **A3'** are satisfied. We put variables as follows.

$$\left[\begin{array}{c:c} C_2 & D_{21} \end{array}\right] = \left[\begin{array}{c:c} C_{21} & O \\ C_{22} & I_{m_1} \end{array}\right], \left[\begin{array}{c} B_2 \\ \hdashline D_{12} \end{array}\right] = \left[\begin{array}{cc} B_{21} & B_{22} \\ \hdashline O & I_{p_1} \end{array}\right]$$

2.6.1 Parametrization of full-order H_∞ controller

Since the matrix D_{21} is of full column rank, we can derive a necessary condition for the solvability in the problem of case 3 in the same manner that is used in the problem of case 2. Thus we deduce the necessary condition by using the result of the FC problem.

As is seen in the problem of case 2, we can choose a matrix L_H such that the matrix

$$A_{ZH} = A - B_1 D_{21}^{\dagger} C_2 + L_H D_{21}^{\perp} C_2$$

satisfies

$$\left\{ \begin{array}{ll} UA_{ZH} = \Lambda U, & \lambda_i(\Lambda) < O, \forall i \\ \ker U \subset \ker U_o \end{array} \right. ,$$

where U is a row-basis of the stable subspace in A_{ZH}, and U_o is a row-basis of observable subspace of the pair $\left(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2\right)$. Also, we can choose a matrix E_H such that the matrices $A_H = A_{ZH} + E_H D_{21}^{\dagger} C_2$ and E_H satisfy

$$\begin{aligned} &\lambda_i(A_H) < 0, \forall i \\ &UE_H = O. \end{aligned} \tag{2.67}$$

Lemma 2.6.1 *Suppose that the matrices L_H and E_H are chosen such that the matrix A_H is stable. Then, if the non-standard H_∞ problem of case 3 is solvable, it is necessary that the ARE in (2.12) has a stabilizing solution $Y \geq O$.*

Proof. From Lemma 2.4.1, in this case, it is also necessary that the FC problem for $G(s)$ where $B_{21} = I_n, B_{22} = O$ is solvable. The solvability condition is equivalent to the condition such that the H_∞ problem for $M_2(s)$ is solvable. Hence, as is in the problem of case 2, it is necessary that the ARE in (2.12) has a stabilizing solution $Y \geq O$. ■

Suppose that the ARE in (2.12) has a stabilizing solution $Y \geq O$. Then we can choose a matrix E_H as

$$E_H = -Y\left(D_{21}^{\dagger} C_2\right)^T. \tag{2.68}$$

The matrix E_H satisfies (2.67), and stabilizes the matrix A_H. Hence, let us choose E_H as in (2.68). Then, as is in the problem of case 2, Lemma 2.4.3 holds. Under the necessary condition such that the ARE in (2.12) has a nonnegative definite stabilizing solution Y, the H_∞ problem for $G(s)$ is reduced to the H_∞ problem for $G_{tmp}(s)$:

$$G_{tmp}(s) = \left[\begin{array}{c|cc} \hat{A} & B_1 - E_H & \hat{B}_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right],$$

where D_{12} is of full row rank, and

$$\begin{aligned} \hat{A} &= A + YC_1^T C_1 \\ \hat{B}_2 &= B_2 + YC_1^T D_{12}. \end{aligned}$$

Hence, we consider the H_∞ problem for $G_{tmp}(s)$. As is in the problem of case 2, let us parametrize the stabilizing controllers for $G_{tmp}(s)$.

Suppose that $(\hat{A}, \hat{B}_2)$ is stabilizable. Then we can choose a feedback gain F such that

$$A_F \triangleq \hat{A} + \hat{B}_2 F$$

is stable. Here, since D_{12} is of full row rank, as is in the problem of case 1, we choose F as

$$F = -D_{12}^{\dagger}C_1 + D_{12}^{\perp}L_F + D_{12}^{\dagger}E_F.$$

Then the matrix A_F is represented as

$$\begin{aligned} A_F &= A_{ZF} + \hat{B}_2 D_{12}^{\dagger} E_F, \\ A_{ZF} &= A - B_2 D_{12}^{\dagger} C_1 + B_2 D_{12}^{\perp} L_F, \end{aligned}$$

where E_F is an arbitrary matrix which satisfies a constraint:

$$E_F V = O,$$

where V is a basis of the stable subspace in A_{ZF}, and L_F is chosen such that the controllable subspace of the pair $\left(A - B_2 D_{12}^{\dagger} C_1, B_2 D_{12}^{\perp}\right)$ is stabilized.

Next, let us choose an observer gain H_∞ as in (2.23), then

$$A_H \triangleq \hat{A} + H_\infty C_2 = A_Y$$

is stable. From Lemma A.2.1, the class of stabilizing controllers is represented as follows.

$$K(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} A_Y + \hat{B}_2 F & -H_\infty & -\hat{B}_2 \\ \hline F & O & -I \\ -C_2 & I & O \end{array}\right], Q(s)\right), Q(s) \in \mathcal{RH}_\infty \tag{2.69}$$

Then the closed loop transfer function is represented as follows

$$\mathcal{F}_l\left(G_{tmp}(s), K(s)\right) = \mathcal{F}_l\left(M_3(s), -D_{12}Q(s)D_{21}\right), \tag{2.70}$$

where

$$M_3(s) = \left[\begin{array}{c|cc} \hat{A} + \hat{B}_2 F & B_1 - E_H & \hat{B}_2 D_{12}^{\dagger} \\ \hline E_F & O & I \\ O & I & O \end{array}\right]. \tag{2.71}$$

From (2.70), the solvability conditions of the H_∞ problems for $G_{tmp}(s)$ and for $M_3(s)$ are equivalent, hence we consider the H_∞ problem for $M_3(s)$.

Theorem 2.6.1 *The non-standard H_∞ problem for the generalized plant which satisfies assumptions* **A1**, **A2'**-*case 3 and* **A3'** *is solvable if and only if an AREs in (2.52) and (2.12) have stabilizing solutions $X \geq 0$ and $Y \geq O$ which satisfy $\rho(XY) < 1$. Under the conditions, the class of the H_∞ controllers is represented as:*

$$\mathcal{K}_\infty = \left\{K_\infty(s) : N(s) \in \mathcal{BH}_\infty, W_i(s) \in \mathcal{RH}_\infty, i = 1, 2, 3\right\},$$

where $N(s)$ and $W_i(s), \forall i$ are free parameters, and $K_\infty(s)$ is represented as follows

$$K_\infty(s) = \mathcal{F}_l\left(\left[\begin{array}{c|c:cc} \hat{A}+\hat{B}_2F_\infty+H_\infty C_2 & H_\infty & -\hat{B}_2D_{12}^{\dagger} & -B_2D_{12}^{\perp} \\ \hline -F_\infty & O & D_{12}^{\dagger} & D_{12}^{\perp} \\ \hdashline D_{21}^{\dagger}\hat{C}_2Z & D_{21}^{\dagger} & O & O \\ D_{21}^{\perp}\hat{C}_2 & D_{21}^{\perp} & O & O \end{array}\right], \left[\begin{array}{cc} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{array}\right]\right) \tag{2.72}$$

and

$$\begin{aligned} \hat{A} &= A + YC_1^TC_1 \\ \hat{B}_2 &= B_2 + YC_1^TD_{12} \\ \hat{C}_2 &= D_{21}B_1^TX + C_2 \\ F_\infty &= \left\{-D_{12}^{\dagger}C_1 - D_{12}^{\dagger}\left(B_2D_{12}^{\dagger}\right)^TX\right\}Z + D_{12}^{\perp}L_F \\ H_\infty &= -B_1D_{21}^{\dagger} - Y\left(D_{21}^{\dagger}C_2\right)^TD_{21}^{\dagger} + L_HD_{21}^{\perp} \\ Z &= (I-YX)^{-1}. \end{aligned}$$

Proof. The H_∞ problem for $M_3(s)$ in (2.71) is DF (Disturbance Feedforward) problem. In appendix A.7 the solution for the DF problem is listed. By using the result from Lemma A.7.1, we derive the solution for the non-standard H_∞ problem of case 3.

(Solvability condition:) The solvability condition for the DF problem is such that ARE:

$$\tilde{S}A_{ZF} + A_{ZF}^T\tilde{S} + \tilde{S}\Upsilon\tilde{S} = O, \tag{2.73}$$

where

$$\Upsilon \triangleq (B_1 - E_H)(B_1 - E_H)^T - \hat{B}_2D_{12}^{\dagger}\left(\hat{B}_2D_{12}^{\dagger}\right)^T$$

has a nonnegative definite stabilizing solution $\tilde{S}$. By using a definition in appendix A.8, we represent the solvability condition such that

$$\left\{\begin{array}{l} H_{\tilde{S}} \in \mathrm{Dom}\,(\mathrm{Ric}) \\ \tilde{S} = \mathrm{Ric}\left(H_{\tilde{S}}\right) \geq O \end{array}\right., \tag{2.74}$$

where

$$H_{\tilde{S}} \triangleq \left[\begin{array}{cc} A_{ZF} & \Upsilon \\ O & -A_{ZF}^T \end{array}\right].$$

$H_{\tilde{S}}$ satisfies

$$\left[\begin{array}{cc} I & Y \\ O & I \end{array}\right]H_{\tilde{S}} = \left[\begin{array}{cc} A_{ZF} & B_1B_1^T - B_2D_{12}^{\dagger}\left(B_2D_{12}^{\dagger}\right)^T + \delta \\ O & -A_{ZF}^T \end{array}\right]\left[\begin{array}{cc} I & Y \\ O & I \end{array}\right],$$

where $Y \geq O$ is a stabilizing solution of the ARE in (2.12), and δ is defined as

$$\delta = \left(L_H D_{21}^{\perp} C_2 - B_2 D_{12}^{\dagger} L_F\right) Y + Y \left(L_H D_{21}^{\perp} C_2 - B_2 D_{12}^{\dagger} L_F\right)^T.$$

Since a matrix U_o, which is a row-basis of observable subspace of the pair $\left(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2\right)$, satisfies $U_o Y = O$, the matrix $D_{21}^{\perp} C_2$ satisfies

$$D_{21}^{\perp} C_2 Y = O.$$

Hence, the solvability condition (2.74) is equivalent to

$$\left\{\begin{array}{l} \tilde{H}_X \in \mathrm{Dom}\,(\mathrm{Ric}) \\ X = \mathrm{Ric}\left(\tilde{H}_X\right) \geq O \\ \rho(XY) < 1 \end{array}\right. , \tag{2.75}$$

where $\tilde{H}_X$ is defined as follows:

$$H_X \triangleq \left[\begin{array}{cc} A_{ZF} & B_1 B_1^T - B_2 D_{12}^{\dagger}\left(B_2 D_{12}^{\dagger}\right)^T - \left\{B_2 D_{12}^{\dagger} L_F Y + Y\left(B_2 D_{12}^{\dagger} L_F\right)^T\right\} \\ O & -A_{ZF}^T \end{array}\right],$$

and $Y \geq O$ is a stabilizing solution of the ARE in (2.12). The solvability condition (2.75) implies that the ARE:

$$\begin{aligned} X A_{ZF} + A_{ZF}^T X + X &\left\{B_1 B_1^T - B_2 D_{12}^{\dagger}\left(B_2 D_{12}^{\dagger}\right)^T\right\} X \\ &- X\left\{B_2 D_{12}^{\perp} L_F Y + Y\left(B_2 D_{12}^{\perp} L_F\right)^T\right\} X = O \end{aligned} \tag{2.76}$$

has a nonnegative definite stabilizing solution X, and which satisfies $\rho(XY) < 1$.

On the other hand, from the solvability condition in (2.74), it can be verified that the matrix $\tilde{S} = \mathrm{Ric}\left(H_{\tilde{S}}\right)$ satisfies

$$\tilde{S} B_2 D_{12}^{\perp} = O,$$

hence

$$\begin{aligned} X &= \mathrm{Ric}\,(H_X) \\ &= \left(I + \tilde{S} Y\right)^{-1} \tilde{S} \end{aligned}$$

satisfies

$$X B_2 D_{12}^{\perp} = O. \tag{2.77}$$

From the equation in (2.77), the ARE in (2.76) is reduced to the ARE in (2.52). Thus, the solvability condition (2.74) is equivalent to the condition:

$$\left\{\begin{array}{l} H_X \in \mathrm{Dom}\,(\mathrm{Ric}) \\ X = \mathrm{Ric}\,(H_X) \geq O \\ \rho(XY) < 1 \end{array}\right. ,$$

where H_X is given as follows

$$H_X \triangleq \begin{bmatrix} A_{ZF} & B_1B_1^T - B_2D_{12}^\dagger\left(B_2D_{12}^\dagger\right)^T \\ O & -A_{ZF}^T \end{bmatrix}.$$

This implies that the ARE in (2.52) has a stabilizing solution $X \geq O$ where X satisfies $\rho\,(XY) < 1$.

Conversely if the solvability condition (2.75) is satisfied, it can be verified that $H_{\tilde{S}} \in \mathrm{Dom}\,(\mathrm{Ric})$. Hence, the matrix

$$A_{ZF} - \hat{B}_2D_{12}^\dagger\left(\hat{B}_2D_{12}^\dagger\right)^T\tilde{S} = \hat{A} + \hat{B}_2\left\{-D_{12}^\dagger C_1 + D_{12}^\perp L_F - D_{12}^\dagger\left(B_2D_{12}^\dagger\right)^T\tilde{S}\right\}$$

is stable, hence $(\hat{A}, \hat{B}_2)$ is stabilizable.

(Derivation of the controller:) In the beginning of this subsection, the feedback gain F is selected such that $A_F = \hat{A} + \hat{B}_2F$ is stable. Here, we choose F based on the solution of the ARE in (2.52). Suppose that the solvability condition (2.31) is satisfied. Now, we select F such that

$$F = F_\infty \triangleq -D_{12}^\dagger C_1 + D_{12}^\perp L_F - D_{12}^\dagger\left(\hat{B}_2D_{12}^\dagger\right)^T\tilde{S},$$

where

$$\begin{aligned}\tilde{S} &= XZ \\ Z &= (I - YX)^{-1}.\end{aligned}$$

Then $A_F = \hat{A} + \hat{B}_2F_\infty$ is stable. By using the formula in Lemma A.7 we can derive the controller

$$-D_{12}Q(s)D_{21} = \mathcal{F}_l\left(\left[\begin{array}{c|cc} \hat{A} + \hat{B}_2F_\infty & -(B_1 - E_H) & -\hat{B}_2D_{12}^\dagger \\ \hline O & O & I \\ (B_1 - E_H)^T\tilde{S} & I & O \end{array}\right], N(s)\right), \tag{2.78}$$

where $N(s) \in \mathcal{BH}_\infty$. The equation (2.78) can be represented as follows:

$$-D_{12}Q(s)D_{21} = \left(I - N(s)M_{22}(s)\right)^{-1}N(s)M_{21}(s), \tag{2.79}$$

where

$$\begin{aligned}M_{21}(s) &= I - \Phi(s)\,(B_1 - E_H) \\ M_{22}(s) &= -\Phi(s)\hat{B}_2D_{12}^\dagger \\ \Phi(s) &= (B_1 - E_H)^T\tilde{S}\left(sI - \hat{A} - \hat{B}_2F_\infty\right)^{-1}.\end{aligned}$$

In the equation (2.79), D_{12} is of full row rank and D_{21} is of full column rank, hence the solution of $Q(s)$ is derived as follows:

$$Q(s) = -\begin{bmatrix} D_{12}^\dagger & D_{12}^\perp \end{bmatrix}\begin{bmatrix} \left(I - N(s)M_{22}(s)\right)^{-1}N(s)M_{21}(s) & \hat{W}_1(s) \\ \hat{W}_2(s) & \hat{W}_3(s) \end{bmatrix}\begin{bmatrix} D_{21}^\dagger \\ D_{21}^\perp \end{bmatrix},$$

where $\hat{W}_i(s) \in \mathcal{RH}_\infty$. Since $(I - M_{22}(s)N(s))^{-1} \in \mathcal{RH}_\infty$, we can replace $\hat{W}_i(s)$ as

$$\begin{aligned}
\hat{W}_1(s) &= (I - N(s)M_{22}(s))^{-1}(W_1(s) + N(s)\Phi(s)L_H) \\
\hat{W}_2(s) &= W_2(s)\left\{I + M_{22}(s)(I - N(s)M_{22}(s))^{-1}N(s)\right\}M_{21}(s) \\
\hat{W}_3(s) &= W_1(s)M_{22}(s)(I - N(s)M_{22}(s))^{-1}(W_1(s) + N(s)\Phi(s)L_H) \\
&\quad + W_3(s) + W_1(s)\Phi(s)L_H,
\end{aligned}$$

where $W_i(s) \in \mathcal{RH}_\infty$ are free parameters. Then the solution $Q(s)$ is represented as follows:

$$\begin{aligned}
Q(s) = &-\begin{bmatrix} D_{12}^\dagger & D_{12}^\perp \end{bmatrix}\left\{I - \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\begin{bmatrix} M_{22}(s) & -\Phi(s)\hat{B}_2 D_{12}^\perp \\ O & O \end{bmatrix}\right\}^{-1} \\
&\begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\begin{bmatrix} D_{21}^\dagger + \Phi(s)H_\infty \\ D_{21}^\perp \end{bmatrix} \\
= &\mathcal{F}_l\left(\left[\begin{array}{c|c:cc} \hat{A} + \hat{B}_2 F_\infty & -H_\infty & -\hat{B}_2 D_{12}^\dagger & -\hat{B}_2 D_{12}^\perp \\ \hline O & O & D_{12}^\dagger & D_{12}^\perp \\ \hdashline (B_1 - E_H)^T \tilde{S} & -D_{21}^\dagger & O & O \\ O & -D_{21}^\perp & O & O \end{array}\right], \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\right),
\end{aligned} \tag{2.80}$$

where $N(s) \in \mathcal{BH}_\infty, W_i(s) \in \mathcal{RH}_\infty$. By applying this $Q(s)$ into (2.69), we can derive the H_∞ controller in (2.72). ■

2.6.2 Derivation of reduced-order H_∞ controllers based on characterization of zeros in $G_{21}(s)$

This subsection derives the reduced-order non-standard H_∞ controller. Since D_{21} is of full column rank, it is assumed that the generalized plant has been put into a basis corresponding to the canonical form in equation (2.4). Hence the following equations can be assumed without loss of generality

$$\begin{aligned}
A - B_1 D_{21}^\dagger C_2 &= \left[\begin{array}{c:ccc} A_{11} & A_{12} & O & O \\ \hdashline A_{13} & A_{14} & O & O \\ A_{31} & A_{32} & A_- & O \\ A_{33} & A_{34} & O & A_+ \end{array}\right], A_{11} \in \mathbb{R}^{(p_2 - m_1)\times(p_2 - m_1)} \\
D_{21}^\perp C_2 &= \left[\begin{array}{c:ccc} I_{p_2 - m_1} & O & O & O \end{array}\right] \\
D_{21}^\dagger C_2 &= \left[\begin{array}{c:ccc} C_{22ll} & C_{22lr} & C_{22rl} & C_{22rr} \end{array}\right] \triangleq \left[\begin{array}{c:c} C_{22ll} & C_{22r} \end{array}\right], C_{22ll} \in \mathbb{R}^{p_2 \times (p_2 - m_1)}
\end{aligned},$$

where $\lambda_i(A_{14}) < 0$, both of the pairs $(A_{11}, I_{p_2 - m_1})$ and (A_+, C_{22rr}) are observable, and $A_+ \in \mathbb{R}^{l \times l}$ has no $j\omega$ eigenvalues. We call A_- the stable zero mode of $G_{21}(s)$ and A_+ the unstable zero mode of $G_{21}(s)$.

For notational ease, let us partition the following matrices.

$$\begin{aligned}
B_1 &= \begin{bmatrix} B_{1u} \\ \hdashline B_{1l} \end{bmatrix}, \ B_{1u} \in \mathbb{R}^{(p_2 - m_1)\times m_1} \\
B_2 &= \begin{bmatrix} B_{2u} \\ \hdashline B_{2l} \end{bmatrix}, \ B_{2u} \in \mathbb{R}^{(p_2 - m_1)\times m_2} \\
C_1 &= \left[\begin{array}{c:ccc} C_{1ll} & C_{1lr} & C_{1rl} & C_{1rr} \end{array}\right] \triangleq \left[\begin{array}{c:c} C_{1ll} & C_{1r} \end{array}\right], \ C_{1ll} \in \mathbb{R}^{p_1 \times (p_2 - m_1)}
\end{aligned}$$

Under the above preparation, let us represent the full-order H_∞ controller in (2.72) with the dual homogeneous transformation as follows:

$$
\begin{aligned}
K_\infty(s) &= \mathcal{DHM}\left(\left[\begin{array}{c|c:c} A_Y & \hat{B}_2 & H_\infty \\ \hline -\begin{bmatrix} D_{12} \\ \left(D_{12}^{\perp}\right)^T \end{bmatrix} F_\infty & \begin{matrix} D_{12} \\ \left(D_{12}^{\perp}\right)^T \end{matrix} & \begin{matrix} O \\ O \end{matrix} \\ \hdashline D_{21}^{\dagger}\hat{C}_2 Z & O & D_{21}^{\dagger} \\ D_{21}^{\perp} C_2 & O & D_{21}^{\perp} \end{array}\right], \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\right) \\
&= \left[\begin{array}{c|c} A_Y & \hat{B}_2 \\ \hline C_K(s) & \begin{matrix} D_{12} \\ \left(D_{12}^{\perp}\right)^T \end{matrix} \end{array}\right]^{-1} \left[\begin{array}{c|c} A_Y & H_\infty \\ \hline C_K(s) & \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix} \begin{bmatrix} D_{21}^{\dagger} \\ D_{21}^{\perp} \end{bmatrix} \end{array}\right],
\end{aligned}
\tag{2.81}
$$

where $C_K(s)$ is defined as follows.

$$
C_K(s) \triangleq -\begin{bmatrix} D_{12} \\ \left(D_{12}^{\perp}\right)^T \end{bmatrix} F_\infty + \begin{bmatrix} N(s) \\ W_2(s) \end{bmatrix} D_{21}^{\dagger}\hat{C}_2 Z + \begin{bmatrix} W_1(s) \\ W_3(s) \end{bmatrix} D_{21}^{\perp} C_2. \tag{2.82}
$$

In the equation (2.20), since the matrix $L_{H_{33}} \in \mathbb{R}^{l\times(p_2-m_1)}$ is arbitrary, we can choose $L_{H_{33}}$ such that $\tilde{A}_{33} = O$ holds. Hence let us choose $L_{H_{33}}$

$$
L_{H_{33}} = -A_{33} - Y_r\left(C_{1rr}^T C_{1ll} - C_{22rr}^T C_{22ll}\right), \tag{2.83}
$$

then the matrix A_Y is represented as

$$
A_Y = \begin{bmatrix} A_{11} + L_{H_1} & A_{12} & O & O \\ O & A_{14} & O & O \\ O & A_{32} & A_{-} & O \\ O & \tilde{A}_{34} & \tilde{A}_{43} & A_{Y_r} \end{bmatrix}.
$$

The matrix A_Y satisfies

$$
A_Y V_Y = V_Y\left(A_{11} + L_{H_1}\right),
$$

where $V_Y = \begin{bmatrix} I_{p_2-m_1} \\ O \end{bmatrix} \in \mathbb{R}^{n\times(p_2-m_1)}$. Hence, if $C_K(s)$ satisfies

$$
C_K(s) V_Y = O, \tag{2.84}
$$

the pair $(A_Y, C_K(s))$ is not observable, hence the order of the controller $K_\infty(s)$ is reduced by the number of $p_2 - m_1$.

From equation (2.82), the matrix $C_K(s)$ is represented with the free parameter $W_i(s) \in \mathcal{RH}_\infty$, and is spanned by the row-basis of $D_{21}^{\perp}C_2$. Since $D_{21}^{\perp}C_2 V = I_{p_2-m_1}$ holds, if we choose the free parameters $W_1(s)$ and $W_3(s)$ as

$$
\begin{bmatrix} W_1(s) \\ W_3(s) \end{bmatrix} = -\left(-\begin{bmatrix} D_{12} \\ \left(D_{12}^{\perp}\right)^T \end{bmatrix} F_\infty + \begin{bmatrix} N(s) \\ W_2(s) \end{bmatrix} D_{21}^{\dagger}\hat{C}_2 Z\right) V_Y, \tag{2.85}
$$

the matrix V_Y satisfies (2.84). Thus we can derive a reduced-order H_∞ controller.

Theorem 2.6.2 *Under the same solvability condition as in Theorem 2.6.1, the class of reduced-order H_∞ controllers is represented as:*

$$\mathcal{K}_\infty^{r32} = \left\{ K_\infty^{r32}(s) : N(s) \in \mathcal{BH}_\infty, W_2(s) \in \mathcal{RH}_\infty \right\},$$

where $N(s)$ is a free parameter, and $K_\infty^{r32}(s)$ is represented as follows

$$K_\infty^{r32}(s) = \mathcal{DHM}\left(M_\infty^{r32}(s), \begin{bmatrix} N(s) \\ W_2(s) \end{bmatrix} \right), \tag{2.86}$$

and

$$M_\infty^{r32}(s) = \left[\begin{array}{c|c:c} \tilde{A}_Y & \tilde{B}_2 & \tilde{H}_\infty \\ \hline -\begin{bmatrix} D_{12} \\ (D_{12}^\perp)^T \end{bmatrix} F_{\infty 2} & \begin{bmatrix} D_{12} \\ (D_{12}^\perp)^T \end{bmatrix} & \begin{bmatrix} D_{12} \\ (D_{12}^\perp)^T \end{bmatrix} F_{\infty 1} D_{21}^\perp \\ \hdashline C_{D_2} & O & D_{21}^\dagger - C_{D_1} D_{21}^\perp \end{array} \right]$$

$$A_Y = \left[\begin{array}{c:ccc} A_{11} + L_{H_1} & A_{12} & O & O \\ \hdashline O & & & \\ O & & \tilde{A}_Y & \\ O & & & \end{array} \right] \in \mathbb{R}^{n \times n}, \quad \tilde{A}_Y \in \mathbb{R}^{(n-(p_2-m_1)) \times (n-(p_2-m_1))}$$

$$\hat{B}_2 = \left[\begin{array}{c} B_{2u} \\ \hdashline \tilde{B}_2 \end{array} \right] \in \mathbb{R}^{n \times m_2}, \quad \tilde{B}_2 \in \mathbb{R}^{(n-(p_2-m_1)) \times m_2}$$

$$H_\infty = \left[\begin{array}{c} -B_{1u} D_{21}^\dagger + L_{H_1} D_{21}^\perp \\ \hdashline \tilde{H}_\infty \end{array} \right] \in \mathbb{R}^{n \times p_2}, \quad \tilde{H}_\infty \in \mathbb{R}^{(n-(p_2-m_1)) \times p_2}$$

$$F_\infty = \left[\begin{array}{c:c} F_{\infty 1} & F_{\infty 2} \end{array} \right], \quad F_{\infty 1} \in \mathbb{R}^{m_2 \times (p_2 - m_1)}, F_{\infty 2} \in \mathbb{R}^{m_2 \times (n-(p_2-m_1))}$$

$$D_{21}^\dagger \hat{C}_2 Z = \left[\begin{array}{c:c} C_{D_1} & C_{D_2} \end{array} \right], \quad C_{D_1} \in \mathbb{R}^{m_1 \times (p_2-m_1)}, C_{D_2} \in \mathbb{R}^{m_1 \times (n-(p_2-m_1))}.$$

Proof. From the previous argument, the reduced order controller is derived by substituting $L_{H_{33}}$ in (2.83) and $W_1(s)$ and $W_3(s)$ in (2.85) into $K_\infty(s)$ in (2.81). Since $\mathcal{K}_\infty^{r32}$ is a subset of $\mathcal{K}_\infty$, the reduced order controller is an H_∞ controller. ∎

Remark 2.6.1 *The dynamical order of the central solution in $\mathcal{K}_\infty^{r32}$ is $n - (p_2 - m_1)$, which is lower than that of the central solution in $\mathcal{K}_\infty$ by the number of $p_2 - m_1$. The controller order reduction is analogous to the order reduction in observer-based controllers, where the order of the controller is reduced by the number of independent outputs that are not corrupted by disturbances.*

2.6.3 Derivation of reduced-order H_∞ controllers based on characterization of zeros in $G_{12}(s)$

This subsection derives the reduced-order non-standard H_∞ controller.

Corollary 2.6.1 *Under the same condition in Theorem 2.6.1, the class of the H_∞ controllers is also represented as follows:*

$$\mathcal{K}_\infty = \left\{ K_\infty(s) : N(s) \in \mathcal{BH}_\infty, W_i(s) \in \mathcal{RH}_\infty, i = 1, 2, 3 \right\},$$

where $N(s)$ and $W_i(s), \forall i$ are free parameters, and $K_\infty(s)$ is represented as follows

$$K_\infty(s) = \mathcal{F}_l\left(M_\infty^1(s), \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\right) \tag{2.87}$$

and

$$M_\infty^1(s) = \left[\begin{array}{c|c:cc} \hat{A} + B_2F_\infty + H_\infty\hat{C}_2 & -H_\infty & Z\hat{B}_2D_{12}^\dagger & B_2D_{12}^\perp \\ \hline F_\infty & O & D_{12}^\dagger & D_{12}^\perp \\ \hdashline -D_{21}^\dagger\hat{C}_2 & D_{21}^\dagger & O & O \\ -D_{21}^\perp\hat{C}_2 & D_{21}^\perp & O & O \end{array}\right]$$

$$\begin{aligned} \hat{A} &= A + B_1B_1^TX \\ \hat{B}_2 &= B_2 + YC_1^TD_{12} \\ \hat{C}_2 &= D_{21}B_1^TX + C_2 \\ F_\infty &= -D_{12}^\dagger C_1 + D_{12}^\perp L_F - D_{12}^\dagger\left(B_2D_{12}^\dagger\right)^TX \\ H_\infty &= Z\left\{-B_1D_{21}^\dagger - Y\left(D_{21}^\dagger C_2\right)^TD_{21}^\dagger\right\} + L_HD_{21}^\perp \\ Z &= (I - YX)^{-1}. \end{aligned}$$

Proof. Omitted. ∎

Since D_{12} is of full row rank, it is assumed that the generalized plant has been put into a basis corresponding to the canonical form in equation (2.3). Hence the following equations can be assumed without loss of generality

$$\begin{aligned} A - B_2D_{12}^\dagger C_1 &= \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \\ A_{13} & A_{14} & A_{23} & A_{24} \\ O & O & A_- & O \\ O & O & O & A_+ \end{bmatrix}, A_{11} \in \mathbb{R}^{(m_2-p_1)\times(m_2-p_1)} \\ B_2D_{12}^\perp &= \begin{bmatrix} I_{m_2-p_1} \\ O \\ O \\ O \end{bmatrix}, \\ B_2D_{12}^\dagger &= \left[\begin{array}{c} B_{22uu} \\ \hdashline B_{22ul} \\ B_{22lu} \\ B_{22ll} \end{array}\right] \triangleq \begin{bmatrix} B_{22uu} \\ B_{22l} \end{bmatrix}, B_{22uu} \in \mathbb{R}^{(m_2-p_1)\times m_2} \end{aligned}$$

where $\lambda_i(A_{14}) < 0$, both of the pairs $(A_{11}, I_{m_2-p_1})$ and (A_+, B_{22ll}) are controllable, and A_+ has no $j\omega$ eigenvalues. We call A_- the stable zero mode of $G_{12}(s)$ and A_+ unstable zero mode of $G_{12}(s)$.

For notational ease, let us partition the following matrices.

$$
\begin{aligned}
B_1 &= \begin{bmatrix} B_{1uu} \\ \hdashline B_{1ul} \\ B_{1lu} \\ B_{1ll} \end{bmatrix} \triangleq \begin{bmatrix} B_{1uu} \\ B_{1l} \end{bmatrix}, \ B_{1uu} \in \mathbb{R}^{(m_2-p_1)\times m_1} \\
C_1 &= \left[\begin{array}{c:c} C_{1l} & C_{1r} \end{array}\right] \in \mathbb{R}^{p_1 \times n}, \ C_{1l} \in \mathbb{R}^{p_1 \times (m_2-p_1)} \\
C_2 &= \left[\begin{array}{c:c} C_{2l} & C_{2r} \end{array}\right] \in \mathbb{R}^{p_2 \times n}, \ C_2 \in \mathbb{R}^{p_2 \times (m_2-p_1)}.
\end{aligned}
$$

Under the above preparations, let us represent the full-order H_∞ controller in (2.87) with the homogeneous transformation as follows:

$$
\begin{aligned}
K_\infty(s) &= \mathcal{HM}\left(\tilde{M}_\infty^1(s), \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix}\right) \\
&= \left[\begin{array}{c|c} A_X & B_K(s) \\ \hline F_\infty & \begin{bmatrix} D_{12}^\dagger & D_{12}^\perp \end{bmatrix} \begin{bmatrix} N(s) & W_1(s) \\ W_2(s) & W_3(s) \end{bmatrix} \end{array}\right] \left[\begin{array}{c|c} A_X & B_K(s) \\ \hline \hat{C}_2 & \begin{bmatrix} D_{21} & \left(D_{21}^\perp\right)^T \end{bmatrix} \end{array}\right]^{-1},
\end{aligned} \tag{2.88}
$$

where $\tilde{M}_\infty^1(s)$ and $B_K(s)$ are defined as

$$
\begin{aligned}
\tilde{M}_\infty^1(s) &\triangleq \left[\begin{array}{c|cc:cc} A_X & Z\hat{B}_2 D_{12}^\dagger & B_2 D_{12}^\perp & -H_\infty D_{21} & -H_\infty \left(D_{21}^\perp\right)^T \\ \hline F_\infty & D_{12}^\dagger & D_{12}^\perp & O & O \\ \hdashline \hat{C}_2 & O & O & D_{21} & \left(D_{21}^\perp\right)^T \end{array}\right] \\
B_K(s) &\triangleq Z\hat{B}_2 D_{12}^\dagger \begin{bmatrix} N(s) & W_1(s) \end{bmatrix} \\
&\qquad + B_2 D_{12}^\perp \begin{bmatrix} W_2(s) & W_3(s) \end{bmatrix} - H_\infty \begin{bmatrix} D_{21} & \left(D_{21}^\perp\right)^T \end{bmatrix}.
\end{aligned} \tag{2.89}
$$

In the equation (2.56), since the matrix $L_{F_{22}} \in \mathbb{R}^{(m_2-p_1)\times r}$ is arbitrary, we can choose $L_{F_{22}}$ such that $\tilde{A}_{22} = O$ holds. Hence let us choose $L_{F_{22}}$

$$
L_{F_{22}} = -A_{22} - \left(B_{1uu}B_{1ll}^T - B_{22uu}B_{22ll}^T\right) X_r \tag{2.90}
$$

then the matrix A_X is represented as

$$
A_X = \begin{bmatrix} A_{11} + L_{F_1} & O & O & O \\ A_{13} & A_{14} & A_{23} & \tilde{A}_{24} \\ O & O & A_- & \tilde{A}_{42} \\ O & O & O & A_{X_r} \end{bmatrix}.
$$

The matrix A_X satisfies

$$
U_X A_X = (A_{11} + L_{F_1}) U_X,
$$

where $U_X = \begin{bmatrix} I_{m_2-p_1} & O \end{bmatrix} \in \mathbb{R}^{(m_2-p_1)\times n}$. Hence, if $B_K(s)$ satisfies

$$
U_X B_K(s) = O, \tag{2.91}
$$

the pair $(A_X, B_K(s))$ is not controllable, hence the order of the controller in $\mathcal{K}_\infty$ is reduced by the number of $m_2 - p_1$.

From equation (2.89), the matrix $B_K(s)$ is represented with the free parameter $W_i(s) \in \mathcal{RH}_\infty$, and is spanned by the basis of $B_2 D_{12}^\perp$. Since $U_X B_2 D_{12}^\perp = I_{m_2-p_1}$ holds, if we choose the free parameters $W_2(s)$ and $W_3(s)$ as

$$\begin{bmatrix} W_2(s) & W_3(s) \end{bmatrix} = -U_X \left(Z\hat{B}_2 D_{12}^\dagger \begin{bmatrix} N(s) & W_1(s) \end{bmatrix} - H_\infty \begin{bmatrix} D_{21} & (D_{21}^\perp)^T \end{bmatrix} \right), \quad (2.92)$$

the matrix U_X satisfies (2.91). Thus we can derive a reduced-order H_∞ controller.

Theorem 2.6.3 *Under the same solvability condition as in Theorem 2.6.1, the class of reduced-order H_∞ controllers is represented as:*

$$\mathcal{K}_\infty^{r31} = \{K_\infty^{r31}(s) : N(s) \in \mathcal{BH}_\infty, W_1(s) \in \mathcal{RH}_\infty\},$$

where $N(s)$ is a free parameter, and $K_\infty^{r31}(s)$ is represented as follows

$$K_\infty^{r31}(s) = \mathcal{HM}\left(M_\infty^{r31}(s), \begin{bmatrix} N(s) & W_1(s) \end{bmatrix}\right), \quad (2.93)$$

and

$$M_\infty^{r31}(s) = \left[\begin{array}{c|c:c} \tilde{A}_X & B_{D_2} & -H_{\infty 2}\begin{bmatrix} D_{21} & (D_{21}^\perp)^T \end{bmatrix} \\ \hline \tilde{F}_\infty & D_{12}^\dagger - D_{12}^\perp B_{D_1} & D_{12}^\perp H_{\infty 1}\begin{bmatrix} D_{21} & (D_{21}^\perp)^T \end{bmatrix} \\ \hdashline \tilde{C}_2 & O & \begin{bmatrix} D_{21} & (D_{21}^\perp)^T \end{bmatrix} \end{array}\right]$$

$$A_X = \left[\begin{array}{c:ccc} A_{11} + L_{F_1} & O & O & O \\ \hdashline A_{13} & & & \\ O & & \tilde{A}_X & \\ O & & & \end{array}\right] \in \mathbb{R}^{n\times n}, \quad \tilde{A}_X \in \mathbb{R}^{(n-(m_2-p_1))\times(n-(m_2-p_1))}$$

$$Z\hat{B}_2 D_{12}^\dagger = \left[\begin{array}{c} B_{D_1} \\ \hdashline B_{D_2} \end{array}\right], \quad B_{D_1} \in \mathbb{R}^{(m_2-p_1)\times p_1}, B_{D_2} \in \mathbb{R}^{(n-(m_2-p_1))\times p_1}$$

$$H_\infty = \left[\begin{array}{c} H_{\infty 1} \\ \hdashline H_{\infty 2} \end{array}\right], \quad H_{\infty 1} \in \mathbb{R}^{(m_2-p_1)\times p_2}, H_{\infty 2} \in \mathbb{R}^{(n-(m_2-p_1))\times p_2}$$

$$F_\infty = \left[\begin{array}{c:c} -D_{12}^\dagger C_{1l} + D_{12}^\perp L_{F_1} & \tilde{F}_\infty \end{array}\right] \in \mathbb{R}^{m_2\times n}, \quad \tilde{F}_\infty \in \mathbb{R}^{m_2\times(n-(m_2-p_1))}$$

$$\hat{C}_2 = \left[\begin{array}{c:c} C_{2l} & \tilde{C}_2 \end{array}\right] \in \mathbb{R}^{p_2\times n}, \quad \tilde{C}_2 \in \mathbb{R}^{p_2\times(n-(m_2-p_1))}.$$

Proof. From the previous argument, the reduced order controller is derived by substituting $L_{F_{22}}$ in (2.90) and $W_2(s)$ and $W_3(s)$ in (2.92) into $K_\infty(s)$ in (2.88). Since $\mathcal{K}_\infty^{r31}$ is a subset of $\mathcal{K}_\infty$, the reduced order controller is an H_∞ controller. ■

Remark 2.6.2 *The dynamical order of the central solution in $\mathcal{K}_\infty^{r31}$ is $n - (m_2 - p_1)$, which is lower than that of the central solution in $\mathcal{K}_\infty$ by the number of $m_2 - p_1$. The controller order reduction is analogous to the order reduction in dual-observer-based controllers, where the order of the controller is reduced by the number of redundant inputs.*

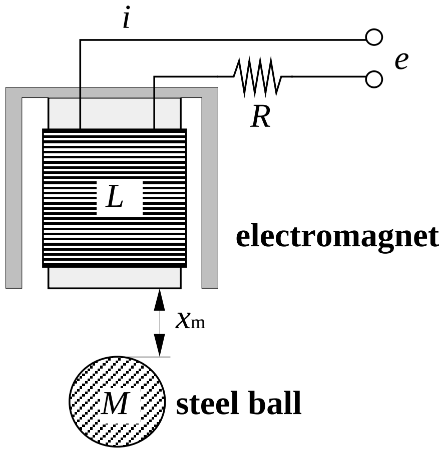

Figure 2.6: The magnetic levitation system

2.7 A numerical example and discussions

2.7.1 Magnetic levitation system

In this thesis, we use a model of a magnetic levitation system to examine the effectiveness of proposed methods by numerical examples. Let us consider the magnetic levitation system[34] shown in the Figure 2.6, where the vertical position $x_m[m]$ of a steel ball is controlled by operating the terminal voltage of an electromagnet e[V]. $R[\Omega]$ is a resistance of the excitation circuit and L[H] is an inductance of the coil and i[A] is an excitation current of the electromagnet. M[kg] is mass of the steel ball. Let us assume that the movement of the ball is restricted to the vertical axis, then we can describe the equation of the motion

$$\begin{cases} M\dfrac{d^2x_m}{dt^2} = Mg + \dfrac{1}{2}i^2\dfrac{\partial L}{\partial x_m} \\ e = Ri + \dfrac{d}{dt}(Li) \end{cases}.$$

Also, let us assume that the inductance L satisfies the equation

$$L(x_m) = \frac{Q}{X + x_m} + L_0,$$

where Q, X, L_0 are some parameters.

The state variables are defined as follows

$$x \triangleq \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} i & \dot{x}_m & x_m \end{bmatrix}^T$$

and the input variable u and the output variable y are defined as follows.

$$u = e, \quad y = x_m$$

Then the state space equation of the magnetic levitation system is written as follows.

$$\begin{cases} \dot{x} = \begin{bmatrix} \frac{\{Qx_2-(X+x_3)^2R\}x_1}{Q(X+x_3)+L_0(X+x_3)^2} \\ g - \frac{Qx_1^2}{2M(X+x_3)^2} \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{X+x_3}{L_0(X+x_3)+Q} \\ 0 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{cases}$$

Thus, the system is represented as a nonlinear state space equation. By expanding the nonlinear equation into Taylor series around an equilibrium point

$$x_{eq} = \begin{bmatrix} x_1^0 & 0 & x_3^0 \end{bmatrix}^T, \quad u_{eq},$$

and by approximating high-order terms, a linear approximation form can be given as follows

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\bar{u} \\ \bar{y} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \bar{x} \end{cases}, \tag{2.94}$$

where A and B are defined as

$$\begin{aligned} A &\triangleq \begin{bmatrix} \frac{-X_cR}{Q+L_0X_c} & \frac{x_3^0Q}{(Q+L_0X_c)X_c} & 0 \\ \frac{-Qx_3^0}{MX_c^2} & 0 & \frac{Q{x_3^0}^2}{MX_c^3} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & 1 & 0 \end{bmatrix} \\ B &\triangleq \begin{bmatrix} \frac{X_c}{Q+L_0X_c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

and $X_c, \bar{x}, \bar{u}, \bar{y}$ are defined as follows

$$\begin{aligned} X_c &\triangleq X + x_1^0 \\ \bar{x} &\triangleq x - x_{eq} \\ \bar{u} &\triangleq u - u_{eq} \\ \bar{y} &\triangleq y - y_{eq}. \end{aligned}$$

By using the Laplace transformation for the equation (2.94), we can obtain a set of transfer functions from the input $\bar{u}$ to the output $\bar{y}$:

$$\mathcal{P}(s; M, R, Q, L_0, X) \triangleq \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} (sI - A)^{-1}B.$$

By substituting the values of the parameters in Table 2.1 into the above equation, we can obtain a transfer function, which we call a nominal model, as follows

$$P(s) = \frac{-67.03}{(s-47.8)(s+46.3)(s+14.9)}. \tag{2.95}$$

Table 2.1: The values of parameters

M	R	Q	L_0	X	x_1^0	x_3^0
0.54	11.6	8.513×10^{-4}	0.789	0.0043	1.144	0.006

2.7.2 An uncertain plant with partial state measurement

For the linear model of the magnetic levitation system, in this chapter we assume that some of the sates are measured without disturbances, and the other states are measured with being effected by disturbances. Thus the model $\tilde{P}(s)$ which we assume here can be written as follows

$$\tilde{P}(s) : \begin{cases} \dot{x} = Ax + Bu \\ y_1 = x_1 \\ y_2 = x_2 \\ y_3 = x_3 + w \end{cases}$$

where, w indicates an input of the disturbance that is caused by the approximation error or measurement noise. The variables $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and u correspond to each variable with upper ver in the equation (2.94).

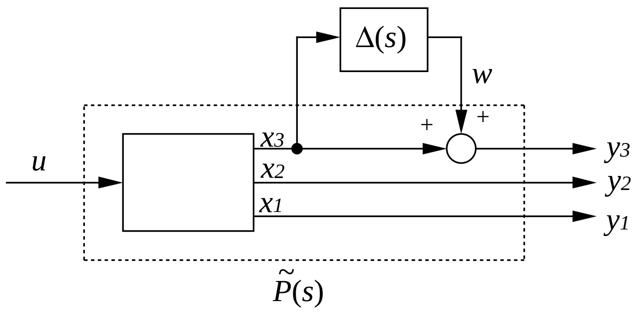

Figure 2.7: The plant model

We assume that the dynamical equation of the disturbance w can be written as follows

$$\Delta(s)P(s) : \begin{cases} \dot{x} = Ax + Bu \\ w = C_w x + d_w u \end{cases},$$

where, $\Delta(s)$ indicates a model of a perturbation that causes the disturbance. Here, it should be noted that both states in the plant and the perturbation are common. Then the whole plant model is illustrated in Figure 2.7. In this chapter, we assume that a transfer function of the error system between the nominal model $P(s)$ and the perturbed model:

$$\begin{cases} \dot{x} = Ax + Bu \\ y_3 = x_3 + w \\ w = C_w x + d_w u \end{cases}$$

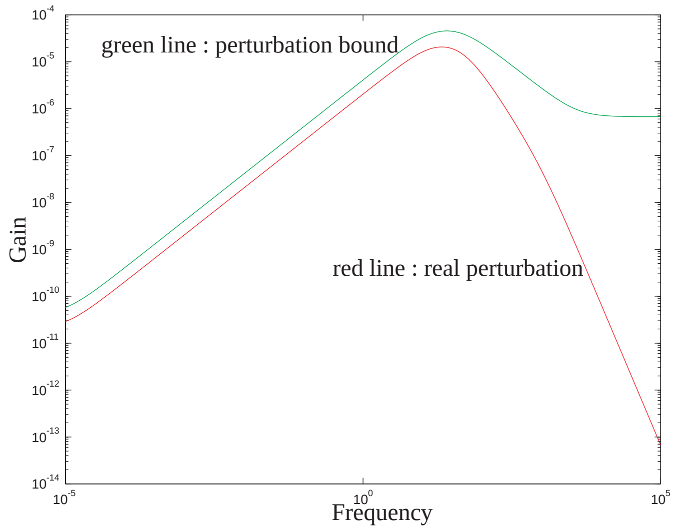

Figure 2.8: Gain plots of the perturbation and its boundary

has already been identified, and is plotted in Figure 2.8 with red line.

For the error system, we set a boundary of the perturbation such that a partial system from the input u to the output y_3 varies in a set

$$\mathcal{P} = \{(1 + \Delta(s))\, P(s) : |\Delta(j\omega)| \leq |W_T(j\omega)|\, , \forall\omega\}\, , \tag{2.96}$$

where $W_T(s)$ is a function of the relative error bound between the perturbed model and $P(s)$. In this thesis, by trial and error, we select $W_T(s)$ as

$$W_T(s) = 0.2 \times 10^{-8}(5.0s^3 + 2.0 \times 10^4 s^2 + 1.0 \times 10^6 s + 10),$$

such that the plant set $\mathcal{P}$ includes the perturbed plant. Figure 2.8 plots the gain of the weighting function $W_T(s)P(s)$ with green line.

2.7.3 Low order robust controller design

We construct a generalized plant $G(s)$ for the robust controller design as shown in Figure 2.9. Then, by solving the problem of finding a stabilizing controller $K(s)$ such that $\|\mathcal{F}_l\,(G(s), K(s))\,\|_\infty < 1$, we can obtain a robust controller. Thus we have reduced the robust controller design problem as an H_∞ problem. By setting the realization of $W_S(s)$ as

$$W_S(s) \triangleq \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & 0 \end{array}\right],$$

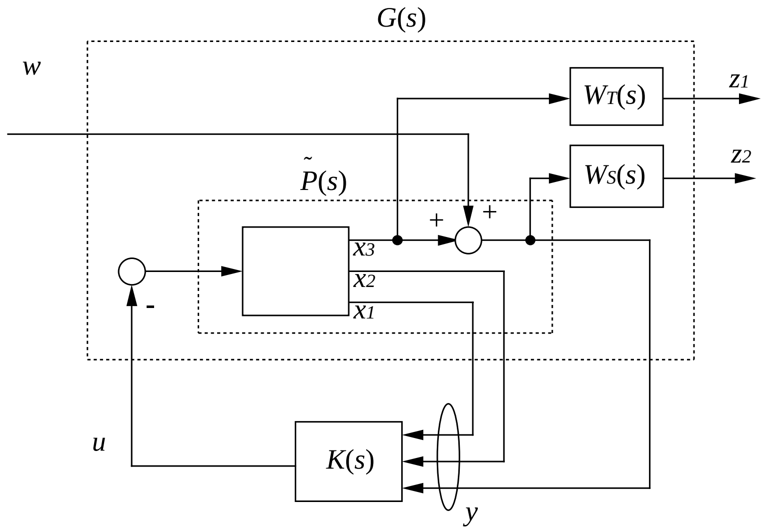

Figure 2.9: The generalized plant

the generalized plant $G(s)$ can be written as follows

$$G(s) = \left[\begin{array}{cccc|c:c} a_{11} & a_{12} & 0 & O & 0 & -b_1 \\ a_{21} & 0 & a_{23} & O & 0 & 0 \\ 0 & 1 & 0 & O & 0 & 0 \\ O & O & B_S & A_S & B_S & O \\ \hline c_{w_1} & c_{w_2} & c_{w_3} & O & 0 & -d_w \\ 0 & 0 & 0 & C_S & 0 & 0 \\ \hdashline 1 & 0 & 0 & O & 0 & 0 \\ 0 & 1 & 0 & O & 0 & 0 \\ 0 & 0 & 1 & O & 1 & 0 \end{array}\right],$$

where, C_w is partitioned as

$$C_w = \left[\begin{array}{ccc} c_{w_1} & c_{w_2} & c_{w_3} \end{array}\right].$$

By substituting each parameter into $G(s)$ and applying an equivalent transformation by $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we can derive the generalized plant as

$$G(s) = \left[\begin{array}{cccc|c:c} -13 & 12 & 0 & 0 & 0 & -3.4 \\ 2400 & 2500 & 2500 & 0 & 0 & 0 \\ -2400 & -2500 & -2500 & 0 & 0 & 3.4 \\ 1.0 & 1.0 & 1.0 & -1.0\times 10^{-5} & 1.0 & 0 \\ \hline 0.098 & 0.10 & 0.099 & 0 & 0 & 6.7\times 10^{-7} \\ 0 & 0 & 0 & 0.50 & 0 & 0 \\ \hdashline 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \\ 1.0 & 1.0 & 1.0 & 0 & 1.0 & 0 \end{array}\right], \tag{2.97}$$

where the weighting function $W_S(s)$ is put as

$$W_S(s) = \frac{0.5}{s + 0.00001}.$$

It can easily be verified that $G(s)$ satisfies the assumptions **A1**, **A2'-Case2** and **A3'**, hence the H_∞ problem is the non-standard H_∞ problem of case 2. Thus, by following the results given in section 2.4 we can derive the low-order H_∞ controller whose order is two. Here, it should be also noted that $G(s)$ in (2.97) satisfies the assumption in (2.5). Thus, by using the result of Theorem 2.4.3, a reduced order H_∞ controller can be given as

$$K(s) = \left[\begin{array}{cc|ccc} -7.9 \times 10^5 & -2.6 \times 10^6 & -7.8 \times 10^5 & -8.0 \times 10^5 & 0 \\ 0 & -1.0 \times 10^{-5} & 0 & 0 & 1.0 \\ \hline -2.3 \times 10^5 & -7.5 \times 10^5 & -2.3 \times 10^5 & -2.3 \times 10^5 & 0 \end{array}\right].$$

The order of the controller is two, and is lower than that of $G(s)$ by second order, which is equal to the dimension of the state measurable without noise.

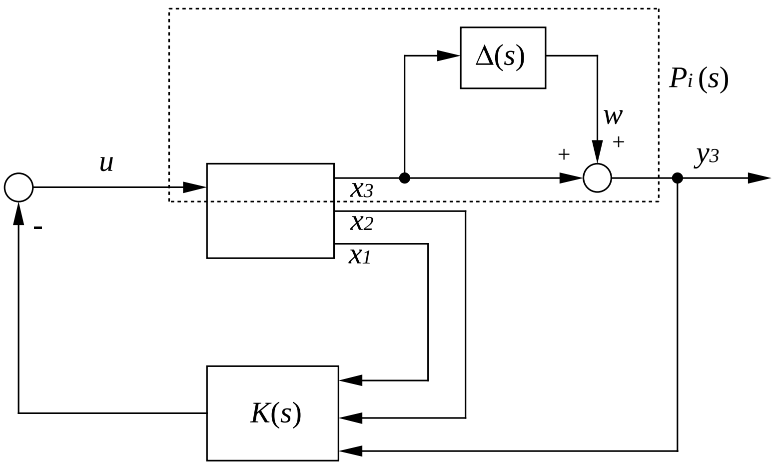

Figure 2.10: The closed loop system
In this figure, $P_i(s)$ indicates a perturbed plant: $(1 + \Delta(s))P(s)$.

In order to examine the performance of the controller, free responses from an initial state $x_0 = \left[\begin{array}{ccc} 0 & 0 & 1.0 \times 10^{-3} \end{array}\right]^T$ are simulated in the closed loop system of Figure 2.10. Figure 2.7.3 illustrates the plots of the free responses, where Figure 2.7.3-(a) is the response of the nominal closed loop system, and Figure 2.7.3-(b) is the response of the perturbed closed loop system. It can be verified that both of the closed loop systems are stable, and that the controller is certainly the robust controller. Thus, we have designed a low-order robust controller by using the method proposed in this chapter.

2.7.4 Discussion

From the example it is verified that the low-order H_∞ controller is certainly derived for the partial state feedback system, where the partial states are measured without disturbances.

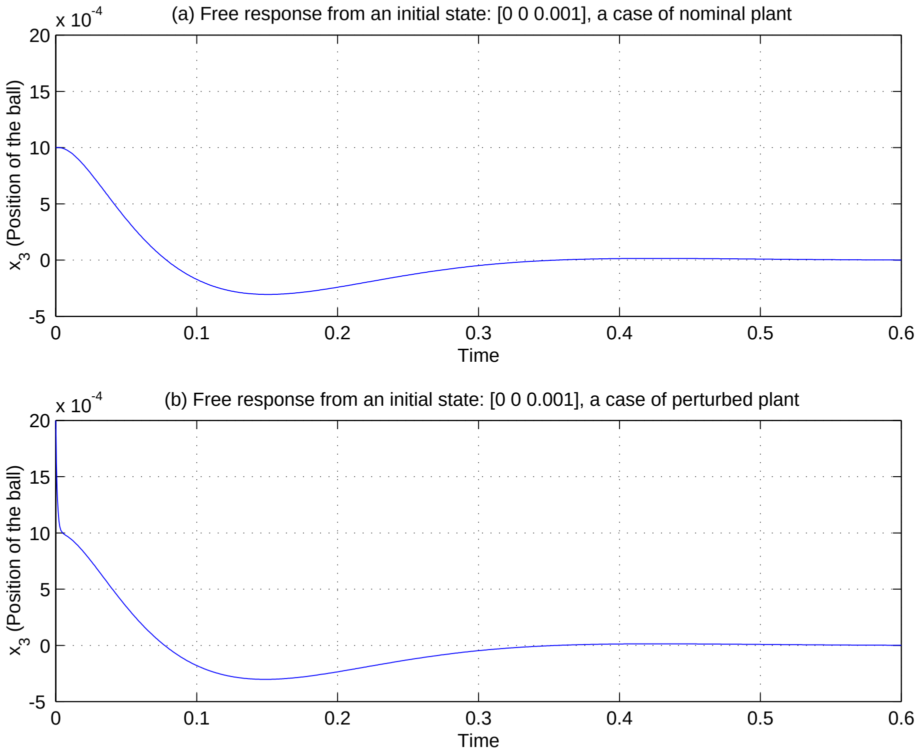

Figure 2.11: The closed loop responses

(a) is an initial response in the closed loop system where the nominal plant $P(s)$ is used. (b) is an initial response in the closed loop system where a perturbed plant $(1 + \Delta(s))P(s)$ is used.

Correctness of the theoretical result that is derived in this chapter is also verified. Although, for the system of the partial state feedback, it is known that a low-order controller which bases its structure on a reduced-order observer can stabilize the system, the method of designing the low-order H_∞ controller, which is derived with so-called ARE approach [11, 47], has not been known except the study of Zhang and Hosoe [71]. On the other hand, as compared with the study of them, our result is more general in the point that we can treat all of the cases of the non-standard problems.

The above numerical example is merely one of instances in which the theoretical result given in this chapter is applicable. There must exist other examples in which the result is also applicable. However, it is hard to find out a non-trivial example that is applicable the result. Hence, it is useful to clarify the class of the applicable problems. The following chapters will make it clear to utilize the result given in this chapter.

2.8 Summary

By using the free parameters in the class of the full order controllers, this chapter has given the reduced-order controllers of the non-standard H_∞ control problems where D_{12} is of full row rank and D_{21} is of full column rank. The controllers are also represented with free parameters; hence, the classes of the reduced-order controllers are derived. The classes are divided into two types: reduced-order observer type and reduced-order dual observer type. The orders of the controllers are listed in Table 2.2. Since the classes of the reduced-order controllers are subclass of the non-standard controllers, the reduced-order controllers satisfy the same H_∞ norm specifications that the full order controllers satisfy. Thus, we have given a fundamental result for designing a low order H_∞ controller, and have clarified one of merits formulating as the non-standard problems. The effectiveness of this method for designing the reduced-order H_∞ controller will be shown with practical problems in later chapters.

Table 2.2: Order of each controller

In this table, n is the order of the generalized plant, p_1 is the number of the controlled outputs, p_2 is the number of the measurement outputs, m_1 is the number of the disturbance inputs and m_2 is the number of the control inputs.

Cases of problems	$\deg(K)$
Case 1	$n-(m_2-p_1)$
Case 2	$n-(p_2-m_1)$
Case 3	$\begin{cases} n-(m_2-p_1) \\ n-(p_2-m_1) \end{cases}$

Chapter 3

A synthesis of low-order integral-type controller

3.1 Introduction

This chapter considers a synthesis problem of a robust servo control system [14] where it is required that controlled outputs track step-formed references in the presence of uncertainties in a plant. It is known that in the case where the references are assumed to be of the step-formed signals, the robust servo controller requires to include integrators in its structure. Hence the problem is formulated as an integral-type H_∞ controller design. This problem naturally reduces to some of the non-standard H_∞ problems in which the non-standard H_∞ problem of case 2 is included.

Up to now, there are many studies concerned about the integral-type H_∞ controller design [23, 27, 26, 25, 49, 47, 39, 40, 73]. In these studies the problem is often solved under a transformation to the standard H_∞ problem. One well known approach is based on an approximation of the integrator with a stable first-order transfer function [73]. This approach enables the H_∞ controller to be designed easily but the controller becomes an approximated integral-type controller. Mita et al. [49, 47] and Liu et al. [39, 40] proposed the concept of pseudo-stabilization of the ARE and solved the problem based on the standard H_∞ control. Hara et al. [23, 27, 26, 25] proposed a method based on the transformation of the generalized plant and the re-construction of the controller. Mita, et al. also proposed a method [48] to solve the problem where the plant has poles on the $j\omega$-axis, and that method requires to factorize the plant into a part having the $j\omega$-poles and another part. Other interesting results for the robust servo controller design include the results by Zhang et al. [71] and Hozumi et al. [26] where the problem is treated as some of the non-standard H_∞ problems.

In this study, we treat the integral-type H_∞ controller design as the non-standard H_∞ problem where a direct feed-through term of the subsystem from the external input to the measurement output (i.e., D_{21}) is column full rank. Although, solving the non-standard H_∞ problem has some advantages, a defect such that the resultant controller becomes high-order

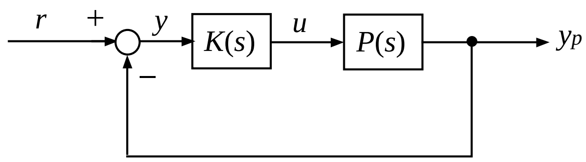

Figure 3.1: A closed loop system

arises. Furthermore, in the case where a plant has the $j\omega$-poles, a solution to the problem becomes more complex. This chapter treats the problem as the non-standard H_∞ problem in both of the cases where the plant has no integrators and the plant has the $j\omega$-poles. Thus the main topic of this study is the derivation of the low-order controller for the problem of the integral-type H_∞ controller design. Another topic is the treatment of the both of the cases where the plant has no integrators and the plant has integrators in a less complex manner.

3.2 Robust servo controller design

3.2.1 Specifications

Consider a feedback system shown in Figure 3.1, where $K(s)$ is a controller and $P(s)$ is a plant which satisfies the following assumptions:

- $P(s) \in \mathbb{C}^{p\times m}$ has no $j\omega$-invariant zeros,
- $m \geq p$.

These assumptions are fundamental for the robust servo system design[14]. In Figure 3.1, $r \in \mathbb{R}^p$ represents the reference signal, $u \in \mathbb{R}^m$ is the signal of control input, $y_p \in \mathbb{R}^p$ is the output from the plant and $y \in \mathbb{R}^p$ which represents the error is the input to the controller. We denote the closed loop system in Figure 3.1 as $(P(s), K(s))$. Specifications of the robust servo system design are

S1 stabilize the closed loop system $(P(s), K(s))$,

S2 let the H_∞ norm of closed loop transfer functions be less than γ, where $\gamma \in \mathbb{R}$ is a positive number fixed *a priori*,

S3 in the presence of the modeling error in $P(s)$ and step-shaped disturbances, let the control outputs asymptotically track the reference inputs which are restricted to the step-shaped signals.

The specification S1 is necessary for the closed loop system to be internally stable. Besides improving the transient responses of the outputs in the closed loop system, the specification

S2 assures the robust stability of the closed loop system in the presence of additive or multiplicative uncertainty in the plant. The specification S3 is the requirement for the system to hold a performance of the robust tracking.

3.2.2 Formulation with the mixed sensitivity problem

In the early many studies, for the above specifications the H_∞ control problem with the generalized plant described in Figure 3.2 is solved. In this figure $W_T(s)$ and $W_S(s)$ represent the weighting functions. This is the generalized plant of the well-known mixed sensitivity problem. Thus, by solving the H_∞ control problem which is formulated as

$$\left\| \begin{array}{c} T_{z_1w}(s) \\ T_{z_2w}(s) \end{array} \right\|_\infty < \gamma,$$

where $T_{z_iw}(s)$ denotes a transfer function from the signal w to the signal z_i, the specifications S1 and S2 are satisfied. In order to satisfy the specification S3, integrators are introduced into the weighting function $W_S(s)$. This is because if a condition

$$T_{z_1w}(s) \in \mathcal{RH}_\infty \tag{3.1}$$

is satisfied for the weight $W_S(s)$ such that

$$W_S(s) = \frac{\tilde{W}_S(s)}{s},\ \tilde{W}_S(s) \in \mathcal{RH}_\infty, \tilde{W}_S(0) \neq 0, \tag{3.2}$$

then the H_∞ controller includes an integrator in the case where $P(s)$ has no integrators, hence in this case the specification S3 is satisfied. Thus, by solving the mixed sensitivity problem with the weighting function in (3.2), the robust servo controller can be designed in the case where $P(s)$ has no integrators. If we give minimal state space representations of $P(s)$, $W_S(s)$ and $W_T(s)$ as

$$P(s) \triangleq \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & O \end{array}\right], A_P \in \mathbb{R}^{n_p \times n_p} \tag{3.3}$$

$$W_S(s) \triangleq \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & O \end{array}\right], A_W \in \mathbb{R}^{n_w \times n_w} \tag{3.4}$$

$$W_T(s)P(s) \triangleq \left[\begin{array}{c|c} A_P & B_P \\ \hline C_T & I_m \end{array}\right], \tag{3.5}$$

the state space representation of the generalized plant in Figure 3.2 is written as follows:

$$\begin{aligned} G_{ms}(s) &= \left[\begin{array}{cc|c:c} A_W & -B_W C_P & B_W & O \\ O & A_P & O & B_P \\ \hline C_W & O & O & O \\ O & C_T & O & I_m \\ \hdashline O & -C_P & I_p & O \end{array}\right] \\ &= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & O \end{array}\right] = \left[\begin{array}{cc} G_{11}(s) & G_{12}(s) \\ \tilde{G}_{21}(s) & \tilde{G}_{22}(s) \end{array}\right]. \end{aligned} \tag{3.6}$$

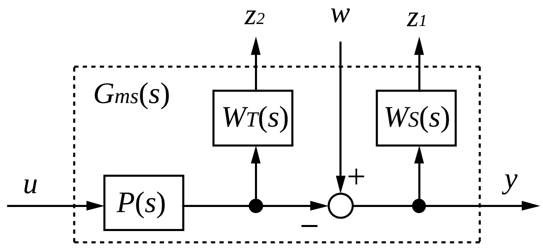

Figure 3.2: The generalized plant for mixed sensitivity problem

However, it should be noted that there are difficulties in solving the mixed sensitivity problem form two points. The first point is that the mixed sensitivity problem with the weight $W_S(s)$ in (3.2) is the non-standard problem where the assumptions **A1-2)** and **A3-2)** are not satisfied. This is easily verified from the generalized plant in (3.6). Especially a deviation from the assumption **A1-2)** makes the problem hard to solve. The other point is that the above formulation is not complete in the case where $P(s)$ has integrators. In this case, the condition (3.1) is satisfied regardless whether the controller has integrators or not, hence there is a possibility such that the solution derives an H_∞ controller which has no integrators. The control system, where the controller has no model of the reference input or the disturbance input, is not robust in the presence of the uncertainty of the plant. This leads to a new formulation for the integral-type H_∞ controller design, where the problem is considered in the two cases where 1) $P(s)$ has no $j\omega$-poles, and 2) $P(s)$ has $j\omega$-poles.

3.3 In the case $P(s)$ has no $j\omega$-poles

3.3.1 Formulation

In this section, it is assumed that the original plant of $P(s)$ has no poles on the $j\omega$-axis. In the first place, let us review the generalized plant of Figure 3.2. In the generalized plant, the state of the weighting function $W_S(s)$, which is unstable transfer function, is not measurable from the output y, hence the system is not detectable and the assumption **A1-2)** is not satisfied. Also in this generalized plant, since the dimensions of the signals w and y are equal, the invariant zeros of the system from w to y include the mode of $W_S(s)$ (See appendix A.1.), hence the assumption **A3-2)** is also not satisfied. Therefore, the H_∞ control problem for the generalized plant of Figure 3.2 is the non-standard H_∞ control problem where the assumptions **A1-2)** and **A3-2)** are not satisfied.

On the other hand, let us consider a generalized plant in Figure 3.3. In the generalized plant the weighting function $W_S(s)$ is also selected such that (3.2) is satisfied. A difference in the generalized plants of Figure 3.2 and Figure 3.3 is that the observed outputs have increased in the case of Figure 3.3, where we assume that the state of $W_S(s)$ is observed from an output

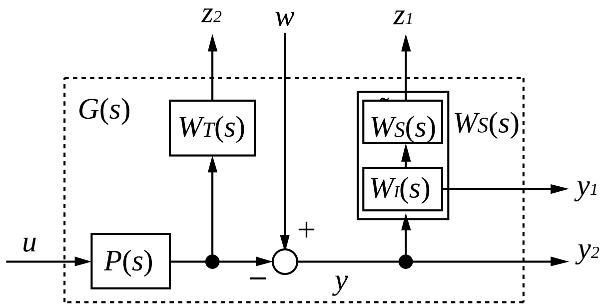

Figure 3.3: The generalized plant for mixed sensitivity problem formulated as the non-standard problem

y_1. Assume that $P(s), W_S(s)$ and $W_T(s)$ have the same state space representations in (3.3) to (3.5), and that $W_I(s)$ has the following stabilizable and detectable state space realization

$$W_I(s) \triangleq \left[\begin{array}{c|c} A_W & B_W \\ \hline I_{n_w} & O \end{array}\right], \tag{3.7}$$

where $W_I(s)$ is a strictly proper rational transfer function such that the poles are located only on the domain $\{s : \mathrm{Re}(s) \le 0\}$. For simplicity $\tilde{W}_s(s) \in \mathcal{RH}_\infty$ is selected as a constant C_W. Then the state space realization of the generalized plant $G(s)$ is represented as follows

$$\begin{aligned} G(s) &= \left[\begin{array}{cc|c:c} A_W & -B_W C_P & B_W & O \\ O & A_P & O & B_P \\ \hline C_W & O & O & O \\ O & C_T & O & I_m \\ \hdashline I_{n_w} & O & O & O \\ O & -C_P & I_p & O \end{array}\right] \\ &= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right] = \left[\begin{array}{cc} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{array}\right]. \end{aligned} \tag{3.8}$$

In this generalized plant, since D_{21} is a matrix of full column rank, the assumption **A2-2)** is not satisfied. On the other hand, the assumption **A1-2)**, i.e., the detectability condition of the pair (A, C_2) is satisfied since the sates of $W_S(s)$ are measurable through the output y_1. Furthermore, since D_{21} is of full column rank, the invariant zeros of $G_{21}(s)$ are unobservable poles of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$, which include the unobservable poles of the pair (A, C_2).* All the modes of $W_S(s)$ are included in the observable subspace of the pair (A, C_2), hence the unstable modes of $W_S(s)$ are not included in the invariant zeros of $G_{21}(s)$. In fact, from the expression such that

$$\left[\begin{array}{c} A - B_1 D_{21}^{\dagger} C_2 \\ \hdashline D_{21}^{\perp} C_2 \end{array}\right] = \left[\begin{array}{cc} A_W & O \\ O & A_P \\ \hdashline I_{n_w} & O \end{array}\right] \tag{3.9}$$

*If λ_i is an unobservable pole of the pair (A, C_2), $\exists v_i \neq O; Av_i = \lambda_i v_i, C_2 v_i = O$. Then λ_i and v_i satisfy $\left(A - B_1 D_{21}^{\dagger} C_2\right) v_i = \lambda_i v_i, D_{21}^{\perp} C_2 v_i = O$. Thus λ_i is an unobservable pole of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$.

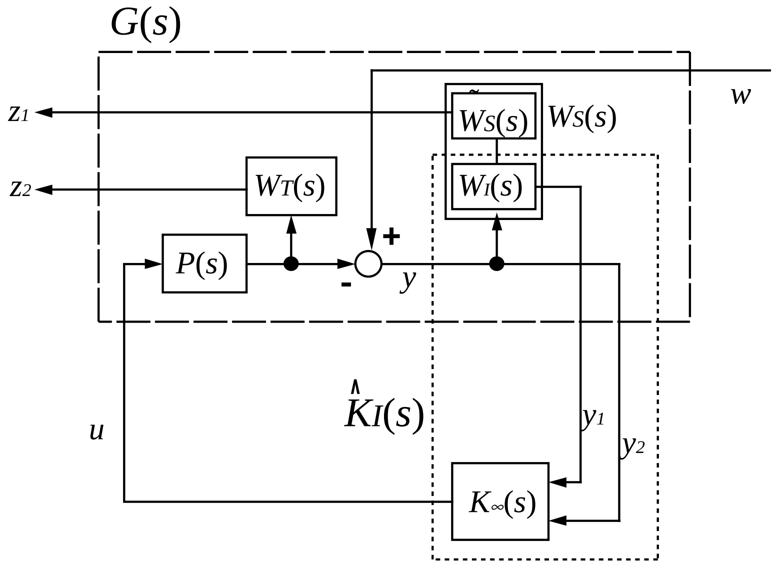

Figure 3.4: The generalized plant and the integral-type H_∞ controller

it is apparently verified that the observable mode of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$ is the mode of $W_S(s)$. On the other hand, it is also verified that the unobservable mode of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$ is the mode of $P(s)$. From the assumption such that $P(s)$ has no $j\omega$ poles, the assumption **A3-2)** is satisfied. Thus, the H_∞ control problem for the generalized plant of Figure 3.4 is the non-standard H_∞ problem where the assumption **A2-2)** is not satisfied.

3.3.2 A high-order controller design

Thus the problem of the robust servo system design is reduced to the non-standard H_∞ problem of case 2. Hence, by following its solution which has already been given in chapter 2, we can obtain an H_∞ controller $K_\infty(s)$. Then as shown in Figure 3.4, by combining a part of the weight $W_S(s)$ and $K_\infty(s)$, an integral-type H_∞ controller $\hat{K}_I(s)$ is constructed as follows.

$$\hat{K}_I(s) = K_\infty(s) \begin{bmatrix} W_I(s) \\ I_p \end{bmatrix} \tag{3.10}$$

Certainly, $\hat{K}_I(s)$ is a controller which satisfies specifications S1, S2 and S3. However, it should be noted that the order of the controller is higher than that of the generalized plant $G(s)$ by the order of $W_I(s)$. Thus, this approach yields a high-order controller, and this is a defect of this approach.

3.3.3 A low-order controller design

This section aims at deriving the integral-type H_∞ controller of low order. This section first derives the controller $K_\infty(s)$ which is the H_∞ controller of the non-standard H_∞ control problem for $G(s)$ in (3.8). Since $G(s)$ fits the generalized plant of the non-standard H_∞ problem of case 2, whose solution has already been discussed in the previous section 2.4. By using the solution we can solve the problem as follows.

Let us define L_H such that the observable mode of the pair $(A - B_1 D_{21}^\dagger C_2, D_{21}^\perp C_2)$ is stabilized with L_H, that is,

$$L_H \triangleq \left[\begin{array}{c} L_{H_1} \\ O \end{array}\right] \in \mathbb{R}^{(n_w+n_p)\times n_w},\ \forall L_{H_1} \in \{L_{H_1} : \mathrm{Re}(\lambda_i(A_W + L_{H_1})) < 0, \forall i\}.$$

Also, let us define A_{ZH} as follows

$$A_{ZH} = A - B_1 D_{21}^\dagger C_2 + L_H D_{21}^\perp C_2. \tag{3.11}$$

Lemma 3.3.1 *If the AREs:*

$$\begin{aligned} &X\left(A - B_2 D_{12}^\dagger C_1\right) + \left(A - B_2 D_{12}^\dagger C_1\right)^T X \\ &\quad + X\left\{B_1 B_1^T - B_2 D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T\right\} X + \left(D_{12}^\perp C_1\right)^T D_{12}^\perp C_1 = O \end{aligned} \tag{3.12}$$

and

$$Y A_{ZH}^T + A_{ZH} Y + Y\left\{C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right\} Y = O \tag{3.13}$$

have stabilizing solutions $X \geq O$ and $Y \geq O$ which satisfy $\rho(XY) < 1$, the class of the H_∞ controllers for $G(s)$ in (3.8) is represented as:

$$\mathcal{K}_\infty = \{K_\infty(s) : N(s) \in \mathcal{BH}_\infty, W(s) \in \mathcal{RH}_\infty\},$$

where $N(s)$ and $W(s)$ are free parameters, and $K_\infty(s)$ is represented as follows

$$K_\infty(s) = \mathcal{F}_l\left(\left[\begin{array}{c|c:c} A_Y + \hat{B}_2 F_\infty & H_\infty & -\hat{B}_2 \\ \hline -F_\infty & O & I_m \\ \hdashline D_{21}^\dagger \hat{C}_2 Z & D_{21}^\dagger & O \\ D_{21}^\perp C_2 & D_{21}^\perp & O \end{array}\right], \left[\begin{array}{cc} N(s) & W(s) \end{array}\right]\right), \tag{3.14}$$

where

$$\begin{aligned} A_Y &= A + Y C_1^T C_1 + H_\infty C_2 \\ \hat{B}_2 &= B_2 + Y C_1^T D_{12} \\ \hat{C}_2 &= D_{21} B_1^T X + C_2 \\ F_\infty &= \left\{-D_{12}^\dagger C_1 - D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T X\right\} Z \\ H_\infty &= -B_1 D_{21}^\dagger - Y\left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger + L_H D_{21}^\perp \\ Z &= (I - YX)^{-1}. \end{aligned}$$

Proof. From Theorem 2.4.2, the result is immediately derived. ■

As we have considered in chapter 2, it is possible to reduce the order of the H_∞ controller derived above. Thus we derive the reduced-order H_∞ controller and obtain the integral-type, reduced-order H_∞ controller. The following lemma is useful for the derivation.

Lemma 3.3.2 *Assume that the ARE in (3.13) has a positive semi-definite stabilizing solution. Then the stabilizing solution of the ARE can be represented as follows:*

$$Y = \begin{bmatrix} O & O \\ O & Y_r \end{bmatrix} \in \mathbb{R}^{(n_p+n_w)\times(n_p+n_w)}, \tag{3.15}$$

where $Y_r \geq O$ is a stabilizing solution of the reduced-order ARE:

$$Y_r A_P^T + A_P Y_r + Y_r \left(C_T^T C_T - C_P^T C_P\right) Y_r = O. \tag{3.16}$$

Proof. The ARE in (3.13) can be represented as follows

$$Y A_Y^T + A_{ZH} Y = O, \tag{3.17}$$

where A_Y is denoted as

$$A_Y = A_{ZH} + Y\left\{C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right\}$$

and is a stable matrix. Let U be a row-basis of the stable subspace of A_{ZH}. Then the matrix U can be selected as $U = \begin{bmatrix} I_{n_w} & O \end{bmatrix}$. Then the matrix U satisfies the following equation:

$$U A_{ZH} = \left(A_W + L_{H_1}\right) U.$$

Pre-multiplying the equation (3.17) by U, the following equation is obtained

$$(UY) A_Y^T + \left(A_W + L_{H_1}\right)(UY) = O.$$

Since $\mathrm{Re}\left(\lambda_i(A_Y)\right) < 0$ and $\mathrm{Re}\left(\lambda_i\left(A_W + L_{H_1}\right)\right) < 0$ hold, the solution Y satisfies $UY = O$ (See appendix A.3.1) and Y must be in the form (3.15). By substituting $Y \geq O$ in (3.15) into the ARE in (3.13), $Y_r \geq O$ must be a solution of the reduced-order ARE in (3.16). ■

In the second place, let us represent the full-order H_∞ controller given in (3.14) with DHMT as follows:

$$\begin{aligned} K_\infty(s) &= \mathcal{DHM}\left(\left[\begin{array}{c|c:c} A_Y & \hat{B}_2 & H_\infty \\ \hline -F_\infty & I_m & O \\ \hdashline D_{21}^\dagger \hat{C}_2 Z & O & D_{21}^\dagger \\ D_{21}^\perp C_2 & O & D_{21}^\perp \end{array}\right], \begin{bmatrix} N(s) & W(s) \end{bmatrix}\right) \\ &= \left[\begin{array}{c|c} A_Y & \hat{B}_2 \\ \hline C_K(s) & I_m \end{array}\right]^{-1} \left[\begin{array}{c|c} A_Y & H_\infty \\ \hline C_K(s) & N(s) D_{21}^\dagger + W(s) D_{21}^\perp \end{array}\right], \end{aligned}$$

where $C_K(s)$ is defined as follows.

$$C_K(s) \triangleq -F_\infty + N(s) D_{21}^\dagger \hat{C}_2 Z + W(s) D_{21}^\perp C_2. \tag{3.18}$$

By using the result of Lemma 3.3.2 the matrix A_Y can be represented as follows

$$A_Y = \begin{bmatrix} A_W + L_{H_1} & O \\ O & A_P + Y_r\left(C_T^T C_T - C_P^T C_P\right) \end{bmatrix},$$

where $Y_r \geq O$ is a stabilizing solution of the reduced-order ARE in (3.16). In the equation (3.18), $C_K(s)$ is explicitly represented with the free parameter $W(s) \in \mathcal{RH}_\infty$. Hence an adequate selection of the free-parameter $W(s)$ yields pole-zero cancellations in the controller, and the order of the controller can be reduced.

Theorem 3.3.1 *Under the same solvability condition as in Lemma 3.3.1, the class of reduced order H_∞ controllers is parametrized as follows:*

$$K_\infty^r(s) = \mathcal{DHM}\left(\left[\begin{array}{c|cc} \tilde{A}_Y & \hat{B}_P & H_{\infty 2} \\ \hline -F_{\infty 2} & I_m & \hat{F}_{\infty 1} D_{21}^\perp \\ C_{D_2} & O & D_{21}^\dagger - \hat{C}_{D_1} D_{21}^\perp \end{array}\right], N(s)\right), \tag{3.19}$$

where $N(s) \in \mathcal{BH}_\infty$ is a free parameter, and $\tilde{A}_Y, \hat{B}_P, H_{\infty 2}, \hat{F}_{\infty 1}, F_{\infty 2}, \hat{C}_{D_1}, C_{D_2}$ are defined as follows:

$$\begin{aligned}
\tilde{A}_Y &= A_P + Y_r\left(C_T^T C_T - C_P^T C_P\right) \\
\hat{B}_P &= B_P + Y_r C_T^T \\
H_{\infty 2} &= \begin{bmatrix} O & Y_r C_P^T \end{bmatrix} \\
F_\infty &= \left[\begin{array}{c:c} F_{\infty 1} & F_{\infty 2} \end{array}\right], \quad F_{\infty 1} \in \mathbb{R}^{m \times n_w}, F_{\infty 2} \in \mathbb{R}^{m \times n_p} \\
D_{21}^\dagger \hat{C}_2 Z &= \left[\begin{array}{c:c} C_{D_1} & C_{D_2} \end{array}\right], \quad C_{D_1} \in \mathbb{R}^{n_w \times n_w}, C_{D_2} \in \mathbb{R}^{n_w \times n_p}.
\end{aligned}$$

Proof. By setting the free parameter as

$$W(s) = -\left(F_\infty + N(s) D_{21}^\dagger \hat{C}_2 Z\right) V,$$

where V is a full column rank matrix defined as

$$V \triangleq \begin{bmatrix} I_{n_w} \\ O \end{bmatrix} \in \mathbb{R}^{(n_w + n_p) \times n_w},$$

the matrices A_Y and $C_K(s)$ satisfy

$$\begin{cases} A_Y V = V\left(A_W + L_{H_1}\right) \\ C_K(s) V = O \end{cases}.$$

Hence the pair $(A_Y, C_K(s))$ is unobservable, and the order of the controller is reduced by the dimension of $\operatorname{rank}(V) = n_w$. Then the reduced order H_∞ controller is derived. ■

Remark 3.3.1 *The McMillan degree of the central solution of the H_∞ controller in (3.14) is reduced by the dimension of* $\operatorname{rank}(V) = n_w$*, and is equal to n_p.*

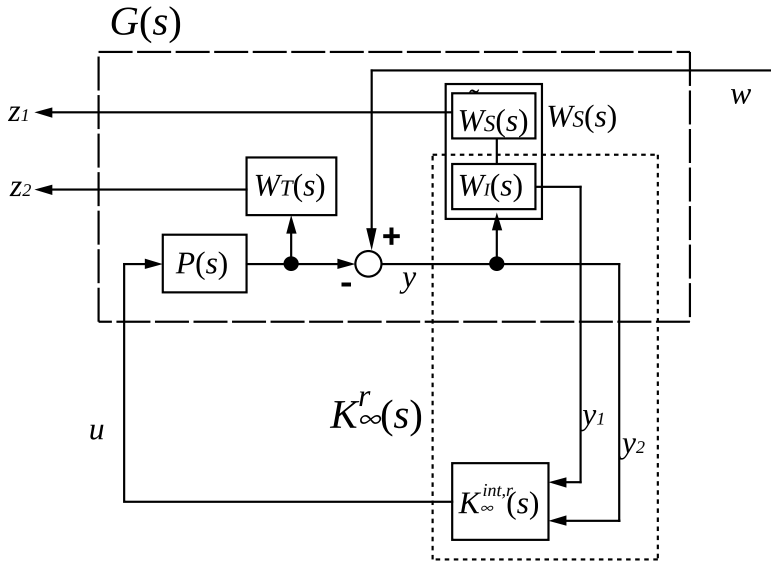

Figure 3.5: The generalized plant and the integral-type, reduced-order H_∞ controller

Remark 3.3.2 *It should be also noted that the class of the reduced-order H_∞ controllers is represented with free parameter $N(s) \in \mathcal{BH}_\infty$.*

Remark 3.3.3 *In this theorem the reduced order TDF controller is represented with DHMT, and this implies that the controller is in a form of coprime factorization over $\mathcal{RH}_\infty$. This form is useful for reducing the order of the controller by approximation, because it is hard to apply approximation methods for unstable systems.*

From the above theorem, a reduced-order controller $K_\infty^r(s)$ is derived. Then as shown in Figure 3.5, by combining a part of the weight $W_S(s)$ and $K_\infty^r(s)$, an integral-type, reduced-order H_∞ controller is constructed as follows.

$$K_\infty^{int,r}(s) = K_\infty^r(s) \begin{bmatrix} W_I(s) \\ I_p \end{bmatrix} \tag{3.20}$$

From Figure 3.5, it can be verified that the controller $K_\infty^{int,r}(s)$ satisfies the specifications S1, S2 and S3. The controller is parametrized with a free parameter $N(s) \in \mathcal{BH}_\infty$. The order of the central solution is $n_w + n_p$, which is lower than that of the controller $\hat{K}_I(s)$ in (3.10). Thus we can derive an integral-type, reduced-order H_∞ controller by way of reconstruction of the controller.

3.3.4 A direct derivation of an integral-type H_∞ controller

In the previous section, the integral-type, reduced-order H_∞ controller is derived indirectly. This section aims at deriving the integral-type, reduced-order H_∞ controller directly. Firstly, we investigate modes of the full-order H_∞ controller in (3.14). The following definition is useful for the investigation.

Definition 3.3.1 (Reduction mode of ARE) *Assume that the ARE in (3.13) has a positive semi-definite stabilizing solution Y. If there exists a full row rank matrix U which satisfies the equations:*

$$UA_{ZH} = \Lambda U \tag{3.21}$$

$$UY = O, \tag{3.22}$$

then we call the matrix Λ **reduction mode** *of the ARE and U* **eigen-subspace** *of the mode.*

Remark 3.3.4 *In the standard H_∞ problem, where D_{21} is of full row rank, the solution is given by solving an ARE:*

$$\begin{aligned} &Y\left(A - B_1 D_{21}^\dagger C_2\right)^T + \left(A - B_1 D_{21}^\dagger C_2\right) Y \\ &\quad + Y\left\{C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right\} Y + B_1 D_{21}^\perp \left(B_1 D_{21}^\perp\right)^T = O. \end{aligned} \tag{3.23}$$

In this case, the reduction mode of the ARE in (3.23) is restricted to the mode of the stable invariant zeros of $G_{21}(s)$, that is, it is restricted to the stable uncontrollable mode of the pair $(A - B_1 D_{21}^\dagger C_2, B_1 D_{21}^\perp)$. This is stated in Lemma A.10.1. On the other hand, in the non-standard H_∞ problem of case 2, not only the mode of the stable invariant zeros of $G_{21}(s)$ but also the stable mode which is obtained by stabilizing the observable mode of the pair $(A - B_1 D_{21}^\dagger C_2, D_{21}^\perp C_2)$ are included in the reduction mode of the ARE. This can be verified from the discussion in section 2.4: for a full row rank matrix U the equations (2.17) and (2.19) hold, where the stable matrix Λ in (2.18) includes the stabilized mode and the stable zero mode. In the case of the non-standard H_∞ problem for the generalized plant in (3.8), the observable mode of the pair $(A - B_1 D_{21}^\dagger C_2, D_{21}^\perp C_2)$ is stabilized with L_{H_1} such that $\Lambda_1 \triangleq A_W + L_{H_1}$ is stable, and Λ_1 is included in the reduction mode of the ARE in addition to the mode of the stable invariant zeros of $G_{21}(s)$.

The following lemma clarifies the relationship between the reduction mode of the ARE and the structure of the non-standard H_∞ controller.

Lemma 3.3.3 *Assume that the non-standard H_∞ controller $K_\infty(s) \in \mathcal{K}_\infty$ in (3.14) is minimal realized. Then if the full row rank matrix U, which is an eigen-subspace of the reduction mode of the ARE in (3.13), satisfies*

$$UB_2 = O, \tag{3.24}$$

the eigenvalues of $\Lambda_1 = A_W + L_{H_1}$ are included in $K_\infty(s) \in \mathcal{K}_\infty 2$ as its real modes.

Proof. Since a full row rank matrix $U \triangleq \begin{bmatrix} I_{n_w} & O \end{bmatrix}$ satisfies the equations (3.21) and (3.22) for $\Lambda = \Lambda_1$, the matrix U holds the equation:

$$UA_Y = \Lambda_1 U,$$

where A_Y is represented as

$$A_Y = A_{ZH} + Y\left\{C_1^T C_1 - \left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger} C_2\right\}, \tag{3.25}$$

hence Λ_1 is a mode of A_Y. If the matrix U satisfies (3.24), the so-called A-matrix of the H_∞ controller in (3.14)

$$A_{K_\infty} \triangleq A_Y + \left(B_2 + YC_1^T D_{12}\right) F_\infty$$

satisfies

$$UA_{K_\infty} = \Lambda_1 U,$$

where U is a matrix of full row rank. Hence, if $K_\infty(s)$ is a minimal form, Λ is a real mode of $K_\infty(s)$. This implies that the eigenvalues of Λ_1 are included as the poles of the H_∞ controller. ■

Remark 3.3.5 *Since in the generalized plant $G(s)$ in (3.8) the matrix U satisfies (3.24), from the Lemma 3.3.3 it follows that if A_W is a stable matrix, that is, $W_S(s)$ is a stable transfer function, L_{H_1} can be selected as zero and the mode of $W_S(s)$ is included in the modes of the controller. Hence the eigenvalues of $W_S(s)$ are included in $K_\infty(s)$ as its eigenvalues if we choose $W_S(s)$ as a stable transfer function, or $W_S(s)$ is regarded as a stable transfer function.*

Based on the above idea, we intend to introduce the $j\omega$-eigenvalues of $W_S(s)$ into the controller. For this purpose the concept of pseudo-stabilization is adopted. The concept is firstly proposed [49] in the problem of the standard H_∞ control.

This study extends the pseudo-stabilizing solution of the ARE in the case of non-standard H_∞ problem of case 2.

Definition 3.3.2 *A pseudo-stabilizing solution of the ARE in (3.13) is defined as a solution that satisfies both of the following items.*

1. *for a full row rank matrix U, the following equations are satisfied*

$$\begin{aligned} UA_{ZH} &= \Lambda U, \quad \mathrm{Re}\left(\lambda_i(\Lambda)\right) \leq 0, \forall i \\ UY &= O, \end{aligned}$$

where A_{ZH} is the matrix which is defined in (3.11) and Y is a positive semi-definite solution of the ARE in (3.13).

2. *real parts of all the eigen values of A_Y in (3.25) are negative except the eigenvalues of Λ*

Remark 3.3.6 *In the standard H_∞ problem, the pseudo-stabilizing solution of ARE is interpreted as the stabilizing solution of the ARE where $j\omega$-invariant zeros of $G_{21}(s)$ are regarded as stable zeros. On the other hand, in the non-standard H_∞ problem of case 2, the pseudo-stabilizing solution of ARE is interpreted as the stabilizing solution of the ARE where not only the $j\omega$-invariant zeros of $G_{21}(s)$ but also $j\omega$-mode which is included in the observable mode of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$ are regarded as stable modes.*

The following theorem enables us to give a pseudo-stabilizing solution of the ARE in (3.13) with solving the stabilizing solution of the ARE.

Theorem 3.3.2 *Let us consider the ARE in (3.13). The stabilizing solution $Y \geq O$, which is obtained with the observable mode of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$ is stabilized, equals the pseudo-stabilizing solution $\tilde{Y} \geq O$ which is obtained by letting $L_H = O$.*

Proof. From (3.9), the full row rank matrix U can be chosen as

$$U = \begin{bmatrix} I_{n_w} & O \end{bmatrix}. \tag{3.26}$$

The stabilizing solution $Y \geq O$ and the pseudo stabilizing solution $\tilde{Y} \geq O$ of the ARE in (3.13) are the symmetric matrices, and they satisfy $UY = O$ and $U\tilde{Y} = O$, hence both of the solutions must be of the form:

$$Y = \begin{bmatrix} O & O \\ O & Y_r \end{bmatrix}, \tag{3.27}$$

$$\tilde{Y} = \begin{bmatrix} O & O \\ O & \tilde{Y}_r \end{bmatrix}. \tag{3.28}$$

Substituting the candidate of the stabilizing solution Y in (3.27) into the ARE in (3.13), a reduced-order ARE :

$$Y_r A_P^T + A_P Y_r + Y_r \left(C_T^T C_T - C_P^T C_P \right) Y_r = O, \tag{3.29}$$

is derived where $Y_r \geq O$ needs to be a stabilizing solution of the reduced-order ARE. On the other hand, substituting the candidate of the pseudo-stabilizing solution $\tilde{Y}$ in (3.28) into the ARE in (3.13), a reduced-order ARE :

$$\tilde{Y}_r A_P^T + A_P \tilde{Y}_r + \tilde{Y}_r \left(C_T^T C_T - C_P^T C_P \right) \tilde{Y}_r = O \tag{3.30}$$

is derived. From the statement in Definition 3.3.2, it is verified that $\tilde{Y}_r \geq O$ needs to be a stabilizing solution of the reduced-order ARE. Thus, it is shown that $Y_r = \tilde{Y}_r$ and that the stabilizing solution Y in (3.27) equals the pseudo-stabilizing solution $\tilde{Y}$ in (3.28). ■

Remark 3.3.7 *The above theorem also states that the pseudo-stabilizing solution of the ARE in (3.13) can be given by solving the stabilizing solution of the reduced-order ARE in (3.30).*

The next theorem shows that by using the pseudo-stabilizing solution, a solution to the non-standard H_∞ problem is given.

Theorem 3.3.3 *Assume that the ARE in (3.12) has a stabilizing solution $X \geq O$ and that the ARE in (3.13) has a pseudo-stabilizing solution $Y \geq O$ and they satisfy an inequality*

$$\rho(XY) < 1. \tag{3.31}$$

Then, the class of the controllers $\{K_\infty(s) : N(s) \in \mathcal{BH}_\infty\}$:

$$K_\infty(s) = \mathcal{F}_l\left(\left[\begin{array}{c|c:c} A_{K_\infty} & H_\infty & -\hat{B}_2 \\ \hline -F_\infty & O & I_m \\ \hdashline D_{21}^\dagger \hat{C}_2 Z & D_{21}^\dagger & O \\ D_{21}^\perp C_2 & D_{21}^\perp & O \end{array}\right], \left[\begin{array}{cc} N(s) & 0 \end{array}\right]\right), \tag{3.32}$$

where $N(s) \in \mathcal{BH}_\infty$ is a free parameter and the other parameters are defined as

$$\begin{aligned} A_{K_\infty} &= A + YC_1^TC_1 + \hat{B}_2F_\infty + H_\infty C_2 \\ \hat{B}_2 &= YC_1^TD_{12} + B_2 \\ \hat{C}_2 &= C_2 + D_{21}B_1^TX \\ F_\infty &= \left(-D_{12}^\dagger C_1 - D_{12}^\dagger \left(B_2D_{12}^\dagger\right)^T X\right) Z \\ H_\infty &= -B_1D_{21}^\dagger + L_HD_{21}^\perp - Y\left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger \\ Z &= (I - YX)^{-1}, \end{aligned}$$

give a subclass of the H_∞ controllers for $G(s)$ in (3.8).

Proof. We show that the controller given by the equation (3.14), where $X \geq O$ is the stabilizing solution for ARE in (3.12) and $Y \geq O$ is the pseudo-stabilizing solution for ARE (3.13), is the H_∞ controller when $W(s) = 0$. As shown in Figure 3.6 the controller in (3.14) has a double-input/single-output structure and this can be represented as follows

$$K_\infty(s) = \left[\begin{array}{cc} K_1(s) & K_2(s) \end{array}\right], K_1(s) \in \mathbb{C}^{m\times n_w}, K_2(s) \in \mathbb{C}^{m\times p}. \tag{3.33}$$

Then the control input u is represented as

$$u = \left[\begin{array}{cc} K_1(s) & K_2(s) \end{array}\right] \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right],$$

where, y_1 is an input signal to the controller and this is yielded from $W_S(s)$, which is a mode of the observable subspace of the pair $(A - B_1D_{21}^\dagger C_2, D_{21}^\perp C_2)$. By letting $L_{H_1} = O$, the following equation

$$H_\infty \left[\begin{array}{c} I_{n_w} \\ O \end{array}\right] = O \tag{3.34}$$

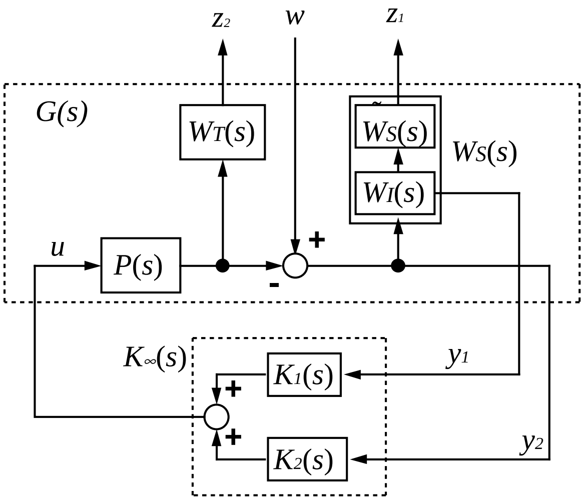

Figure 3.6: Closed loop system

holds. Let us denote the controller in (3.14) as

$$K_\infty(s) = \mathcal{F}_l\left(\begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix}, \begin{bmatrix} N(s) & W(s) \end{bmatrix} \right),$$

where $N(s) \in \mathcal{BH}_\infty$ and $W(s) \in \mathcal{RH}_\infty$ are free parameters. Then the equations

$$K_{11}(s) \begin{bmatrix} I_{n_w} \\ O \end{bmatrix} = 0, \quad K_{21}(s) \begin{bmatrix} I_{n_w} \\ O \end{bmatrix} = \begin{bmatrix} O \\ I_{n_w} \end{bmatrix} \tag{3.35}$$

are satisfied. Therefore, the controller $K_1(s)$ can be represented as follows

$$\begin{aligned} K_1(s) &= K_\infty(s) \begin{bmatrix} I_{n_w} \\ O \end{bmatrix} \\ &= \mathcal{F}_l\left(\left[\begin{array}{c|c:c} A_{K_\infty} & O & -\hat{B}_2 \\ \hline -F_\infty & O & I_m \\ \hdashline D_{21}^\dagger \tilde{C}_2 Z & O & O \\ D_{21}^\perp C_2 & I_{n_w} & O \end{array} \right], \begin{bmatrix} N(s) & W(s) \end{bmatrix} \right) \end{aligned}$$

Hence by letting $W(s) = 0$, $K_1(s)$ becomes $K_1(s) = 0$. Then, $K_2(s)$ becomes

$$\begin{aligned} K_2(s) &= K_\infty(s) \begin{bmatrix} O \\ I_p \end{bmatrix} \\ &= \mathcal{F}_l\left(\left[\begin{array}{c|cc} A_{K_\infty} & H_\infty^O & -\hat{B}_2 \\ \hline -F_\infty & O & I_m \\ \tilde{D}_{21}^\dagger \tilde{C}_2 Z & I_p & O \end{array} \right], N(s) \right) \\ &= \mathcal{F}_l\left(\left[\begin{array}{c|cc} \hat{A}_{K_\infty} & Z H_\infty^0 & -Z\hat{B}_2 \\ \hline -F_\infty^0 & O & I_m \\ \tilde{D}_{21}^\dagger \tilde{C}_2 & I_p & O \end{array} \right], N(s) \right), \end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
\hat{A}_{K\infty} &= A + B_1 B_1^T X + B_2 F_\infty^0 + Z H_\infty^0 \tilde{C}_2 \\
F_\infty^0 &= -D_{12}^\dagger C_1 - B_2^T X \\
H_\infty^0 &= -B_1 - Y \left(\tilde{D}_{21}^\dagger \tilde{C}_2 \right)^T
\end{aligned}$$

On the other hand, by using Lemma A.11.2, it can be verified that the controller in (3.36) equals the H_∞ controller of the standard H_∞ problem for $G_{ms}(s)$ in (3.6). ■

Remark 3.3.8 *The order of the H_∞ controller in (3.36) equals the order of the generalized plant. It is lower than the order of the H_∞ controller in (3.10).*

Thus, a design step of the robust servo controller has been given.

Proposition 3.3.1 *Assume that $P(s)$ has no integrators. The following Design Procedure 1 yields an integral-type H_∞ controller which includes the same number of integrators which are included in $W_S(s)$ and the controller satisfies the specifications S1, S2 and S3.*

Design Procedure 1

STEP 1 *Introduce integrators into $W_S(s)$ as follows:*

$$W_S(s) = \frac{\tilde{W}_S(s)}{s^n}, \ \tilde{W}_S(s) \in \mathcal{RH}_\infty, \tilde{W}_S(0) \neq 0,$$

where n is the number of integrators which are needed to be included in the controller.

STEP 2 *Select an L_H such that $L_{H_1} \in \{L_{H_1} : \mathrm{Re}(\lambda_i(A_W + L_{H_1})) < 0\}$ and obtain the positive semi-definite stabilizing solutions of the AREs in (3.12) and (3.13). Then, verify the inequality (3.31).*

STEP 3 *If the inequality is satisfied, set $L_{H_1} = O$ and the controller (3.32) is a solution.*

Proof. Since it is apparent that the resultant controller is the H_∞ controller from the above discussion, it suffice to show that the controller has integrators. The full row rank matrix U, which satisfies (3.21) and (3.22), can be given as (3.26), hence, from the generalized plant $G(s)$ represented in (3.8), it can be verified that U satisfies (3.24). Using the result of Lemma 3.3.3, it is shown that the controller (3.32) has the integrators, the number of which is same as the number of $j\omega$-eigenvalues of $\Lambda = A_W$. ■

Remark 3.3.9 *It should be noted that, in Design Procedure 1, it is not needed to solve the pseudo-stabilizing solution of the ARE in (3.13).*

Remark 3.3.10 *The order of the resultant controller is equal to that of the generalized plant. Thus, as compared with the controller in (3.10), the order of the controller is reduced by the order of $W_S(s)$.*

3.4 In the case $P(s)$ has $j\omega$-poles

3.4.1 The $j\omega$ invariant zeros of $G_{21}(s)$

Let us consider the problem of the robust servo controller design where $P(s)$ has poles on the $j\omega$-axis. We first show that the $j\omega$-poles of $P(s)$ appear as the invariant zeros of the subsystem from the external input to the measurement output in the generalized plants of Figures 3.2 and 3.3.

First, let us review the generalized plant $G_{ms}(s)$ in Figure 3.2. Since the invariant zeros of $\tilde{G}_{21}(s)$ are all the poles of the matrix

$$A - B_1\tilde{D}_{21}^{-1}\tilde{C}_2 = \begin{bmatrix} A_W & O \\ O & A_P \end{bmatrix}$$

, uncontrollable poles of the pair (A, B_1), i. e., all the modes of $P(s)$ in which the mode on the $j\omega$-axis is included, are included in the zeros of $\tilde{G}_{21}(s)$, hence, in this case, the generalized plant $G_{ms}(s)$ doesn't satisfy the assumption **A3-2)**.

In the case of the generalized plant $G(s)$ in Figure 3.3, since the invariant zeros of $G_{21}(s)$ are included in the unobservable mode of the pair $(A - B_1D_{21}^{\dagger}C_2, D_{21}^{\perp}C_2)$, from the equation (3.9), it is apparently verified that the $j\omega$-poles of $P(s)$ are included in the zeros of $G_{21}(s)$. Hence, in the case of the generalized plant of $G(s)$, the assumption **A3-2)** is not satisfied.

3.4.2 Design

Thus, in the case where $P(s)$ has $j\omega$-poles, the H_∞ control problems for both of the generalized plants $G_{ms}(s)$ and $G(s)$ are the non-standard H_∞ problems in which the assumption **A3-2)** is not satisfied. For the generalized plant of $G_{ms}(s)$, a solution to the H_∞ problem is proposed in a paper [48], where the non-standard H_∞ problem for the generalized plant $G_{ms}(s)$ is transformed to a standard H_∞ problem by separating a $j\omega$ part from $P(s)$. Hence the solution needs a transformation in the generalized plant.

The present study proposes a direct solution to the non-standard H_∞ problem for the generalized plant of $G_{ms}(s)$, and generalizes the solution to the non-standard H_∞ problem for the generalized plant of $G(s)$. Firstly, the following lemma gives a direct solution to the non-standard H_∞ problem for the generalized plant of $G_{ms}(s)$ where $P(s)$ has $j\omega$-poles.

Lemma 3.4.1 *It is assumed that the strictly proper transfer function $P(s)$, which has $j\omega$-poles can be factorized as follows*

$$P(s) = \hat{P}(s)a(s), \tag{3.37}$$

where

$$\hat{P}(j\omega) < \infty, \forall\omega \in \mathbb{R}, \quad a^{-1}(s) \in \mathcal{RH}_\infty.$$

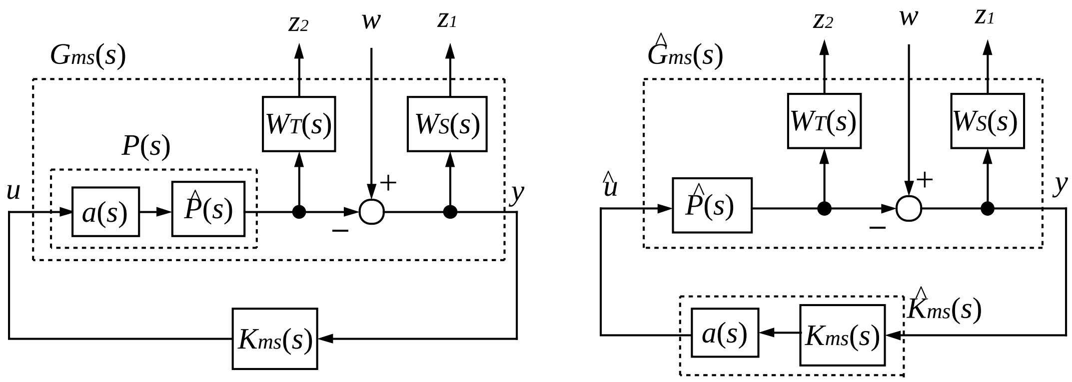

Figure 3.7: A transformation in the closed loop

In the generalized plant of Figure 3.2, let us select $W_S(s)$ as follows

$$W_S(s) = a(s)\hat{W}_s(s), \tag{3.38}$$

where

$$\hat{W}(\infty) = 0, \quad \hat{W}(s) < \infty, \forall s \in \{s : \mathrm{Re}(s) > 0\}.$$

Then, if the ARE in (3.12) has a stabilizing solution $X \geq O$, the ARE in (3.13), where $\tilde{C}_2$ and $\tilde{D}_{21}$† are substituted for C_2 and D_{21}, has a pseudo-stabilizing solution $Y \geq O$ and they satisfy the inequality in (3.31), then by using the solutions X and Y the H_∞ controller for the generalized plant of $G_{ms}(s)$ is given with the expression (A.11) which is a solution of the standard H_∞ problem.

Proof. In the early paper [48], the H_∞ controller for $G_{ms}(s)$ is indirectly given by converting an H_∞ controller for $\hat{G}_{ms}(s)$ which is obtained through a factorization in the closed loop of $G_{ms}(s)$. (See Figure 3.7.) Here, it is shown that the H_∞ controller given by the method coincides with the controller which is directly given by using this theorem. Outline of the proof is as follows.

1. Construct the generalized plant $\hat{G}_{ms}(s)$.

2. It is shown that the solutions of the AREs for $\hat{G}_{ms}(s)$ coincide with the solutions of the AREs for $G_{ms}(s)$.

3. It is shown that the controller $K_{ms}(s)$ which is given by using the theorem equals an H_∞ controller $a^{-1}(s)\hat{K}_{ms}(s)$ which is derived through $\hat{K}_{ms}(s)$ which is an H_∞ controller for $\hat{G}_{ms}(s)$.

†Since $\tilde{D}_{21} = I$, $\tilde{D}_{21}^{\dagger} = I$ and $\tilde{D}_{21}^{\perp} = O$.

Concrete calculations are shown as follows.

1. Let us represent $P(s)$ as follows:

$$P(s) = P_1(s)P_2(s) = \left[\begin{array}{cc|c} A_{P_2} & O & B_{P_2} \\ B_{P_1}C_{P_2} & A_{P_1} & O \\ \hline D_{P_1}C_{P_2} & C_{P_1} & O \end{array}\right] \triangleq \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & O \end{array}\right],$$

where $P_1(s)$ and $P_2(s)$ are defined as

$$P_1(s) = \left[\begin{array}{c|c} A_{P_1} & B_{P_1} \\ \hline C_{P_1} & D_{P_1} \end{array}\right], \quad \mathrm{Re}\left(\lambda_i(A_{P_1})\right) \neq 0, \forall i,$$
$$P_2(s) = \left[\begin{array}{c|c} A_{P_2} & B_{P_2} \\ \hline C_{P_2} & O \end{array}\right], \quad \mathrm{Re}\left(\lambda_i(A_{P_2})\right) = 0, \forall i.$$

Then by letting the factor $a(s)$ as

$$a(s) = \left[\begin{array}{c|c} A_{P_2} & B_{P_2} \\ \hline C_a & I \end{array}\right], \quad \mathrm{Re}\left(\lambda_i(A_{P_2} - B_{P_2}C_a)\right) < 0, \forall i,$$

$\hat{P}(s)$ can be represented as follows

$$\hat{P}(s) = P(s)a^{-1}(s) = \left[\begin{array}{cc|c} A_{P_2} - B_{P_2}C_a & O & B_{P_2} \\ B_{P_1}C_{P_2} & A_{P_1} & O \\ \hline D_{P_1}C_{P_2} & C_{P_1} & O \end{array}\right] = \left[\begin{array}{c|c} A_{\hat{P}} & B_P \\ \hline C_P & O \end{array}\right]. \tag{3.39}$$

Also, by letting $C_T = \left[\, C_{T_1} \quad C_{T_2} \,\right]$, $W_T(s)\hat{P}(s)$ is expressed as follows

$$W_T(s)\hat{P}(s) = \left[\begin{array}{c|c} A_{\hat{P}} & B_P \\ \hline C_{\hat{T}} & I \end{array}\right], C_{\hat{T}} = \left[\, C_{T_1} - C_a \quad C_{T_2} \,\right].$$

For the above plant $\hat{P}(s)$, the generalized plant for the mixed sensitivity problem is represented as follows

$$\begin{aligned} \hat{G}_{ms}(s) &= \left[\begin{array}{cc|c:c} A_W & -B_W C_P & B_W & O \\ O & A_{\hat{P}} & O & B_P \\ \hline C_W & O & O & O \\ O & C_{\hat{T}} & O & I \\ \hdashline O & -C_P & I & O \end{array}\right] \\ &\triangleq \left[\begin{array}{c|cc} \tilde{A} & B_1 & B_2 \\ \hline \tilde{C}_1 & O & D_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & O \end{array}\right]. \end{aligned} \tag{3.40}$$

2. For the generalized plant $\hat{G}_{ms}(s)$, an ARE

$$X\hat{A}_{ZF} + \hat{A}_{ZF}^T X + X\hat{R}X + \left(D_{12}^{\perp}\tilde{C}_1\right)^T D_{12}^{\perp}\tilde{C}_1 = O,$$

where

$$\begin{aligned} &\hat{A}_{ZF} \triangleq \tilde{A} - B_2 D_{12}^{\dagger}\tilde{C}_1 = A_{ZF} \\ &\hat{R} \triangleq B_1 B_1^T - B_2 D_{12}^{\dagger}\left(B_2 D_{12}^{\dagger}\right)^T = R \\ &\left(D_{12}^{\perp}\tilde{C}_1\right)^T D_{12}^{\perp}\tilde{C}_1 = \left(D_{12}^{\perp}C_1\right)^T D_{12}^{\perp}C_1, \end{aligned}$$

coincides with the ARE in (3.12) which is needed to be solved in the mixed sensitivity problem for $G_{ms}(s)$. Hence, the solutions for each AREs coincide with each other if they are solvable.

On the other hand, if an ARE for $\hat{G}_{ms}(s)$

$$Y\hat{A}_{ZH}^T + \hat{A}_{ZH}Y + Y\hat{Q}Y = O, \tag{3.41}$$

where

$$\begin{aligned}\hat{A}_{ZH} &\triangleq \tilde{A} - B_1\tilde{D}_{21}^{\dagger}\tilde{C}_2 \\ \hat{Q} &\triangleq \tilde{C}_1^T\tilde{C}_1 - \left(\tilde{D}_{21}^{\dagger}\tilde{C}_2\right)^T \tilde{D}_{21}^{\dagger}\tilde{C}_2,\end{aligned}$$

has a pseudo-stabilizing solution Y, there exists a full row rank matrix U such that

$$\begin{aligned}&U\hat{A}_{ZH} = \Lambda U, \quad \Lambda = \begin{bmatrix} A_W & O \\ O & A_{P_2} - B_{P_2}C_a \end{bmatrix} \\ &UY = O.\end{aligned}$$

Hence, by applying the equivalent transformation for (3.41)

$$\begin{bmatrix} U \\ (U^{\perp})^T \end{bmatrix} \left(Y\hat{A}_{ZH}^T + \hat{A}_{ZH}Y + Y\hat{Q}Y\right) \begin{bmatrix} U \\ (U^{\perp})^T \end{bmatrix}^T = O \tag{3.42}$$

It appears that the pseudo-stabilizing solution of the ARE in (3.41) is expressed as

$$Y = \begin{bmatrix} O & O & O \\ O & O & O \\ O & O & Y_l \end{bmatrix}, \tag{3.43}$$

where $Y_l \geq O$ is a stabilizing solution of a reduced-order ARE

$$Y_lA_{P_1}^T + A_{P_1}Y_l + Y_l\left(C_{T_2}^TC_{T_2} - C_{P_1}^TC_{P_1}\right)Y_l = O.$$

For $G_{ms}(s)$, in a similar way if an ARE

$$YA_{ZH}^T + A_{ZH}Y + YQY = O, \tag{3.44}$$

where

$$\begin{aligned}A_{ZH} &\triangleq A - B_1\tilde{D}_{21}^{\dagger}\tilde{C}_2 \\ Q &\triangleq C_1^TC_1 - \left(\tilde{D}_{21}^{\dagger}\tilde{C}_2\right)^T \tilde{D}_{21}^{\dagger}\tilde{C}_2,\end{aligned}$$

has a pseudo-stabilizing solution Y, for Y there exists a full row rank matrix U such that

$$\begin{aligned}&UA_{ZH} = \Lambda U, \quad \Lambda = \begin{bmatrix} A_W & O \\ O & A_{P_2} \end{bmatrix} \\ &UY = O.\end{aligned}$$

Hence, by applying the same equivalent transformation as in (3.42) for (3.44), it appears that the pseudo-stabilizing solution of the ARE in (3.44) is also represented as (3.43).

3. By using the parameters of the H_∞ controller for $G_{ms}(s)$ (See appendix A.11), the H_∞ controller for $\hat{G}_{ms}(s)$ can be represented as follows

$$\hat{K}_{ms}(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} \hat{A} & -ZH_\infty & Z\hat{B}_2 \\ \hline \tilde{F}_\infty & O & I \\ -\hat{C}_2 & I & O \end{array}\right], N(s)\right), \tag{3.45}$$

where

$$\tilde{F}_\infty \triangleq F_\infty + \left[\begin{array}{ccc} O & C_a & O \end{array}\right].$$

By calculating $a^{-1}(s)\hat{K}_{ms}(s)$ and transforming its state with a non-singular matrix

$$T = \left[\begin{array}{cccc} I & O & -I & O \\ O & I & O & O \\ O & O & I & O \\ O & O & O & I \end{array}\right],$$

it is shown the following equation holds.

$$a^{-1}(s)\hat{K}_{ms}(s) = K_{ms}(s) \tag{3.46}$$

■

Remark 3.4.1 *From the above proof, it is seen that in the mixed sensitivity problem where $P(s)$ has $j\omega$-poles, the poles appear as $j\omega$-invariant zeros of the controller. Since the invariant zeros of the controller are canceled out by the $j\omega$-poles of $P(s)$, the internal stability of the system is not satisfied. In order to avoid the cancellation, the weighting function $W_S(s)$ is selected as in (3.38).*

The following theorem generalizes the result of Lemma 3.4.1, and gives a direct solution to the non-standard H_∞ problem for the generalized plant of $G(s)$ where $P(s)$ has $j\omega$-poles.

Theorem 3.4.1 *It is assumed that the strictly proper transfer function $P(s)$, which has $j\omega$-poles, can be factorized as (3.37). In the generalized plant of Figure 3.3, let us select $W_S(s)$ as (3.38). Then, if the ARE in (3.12) has a stabilizing solution $X \geq O$, the ARE in (3.13) has a pseudo-stabilizing solution $Y \geq O$ and they satisfy the inequality in (3.31), then using the solutions X and Y, the controller in (3.14) with $W(s) = 0$ is the H_∞ controller for the generalized plant of $G(s)$.*

Proof. In the AREs of the non-standard H_∞ problem for $G(s)$, (3.12) coincides with one of the AREs of the standard H_∞ problem for $G_{ms}(s)$, and by letting $L_{H_1} = O$, (3.13) coincides with the other ARE of the standard H_∞ problem for $G_{ms}(s)$. Hence, from the result of Lemma 3.4.1, using the stabilizing solution for the ARE (3.12) and the pseudo-stabilizing

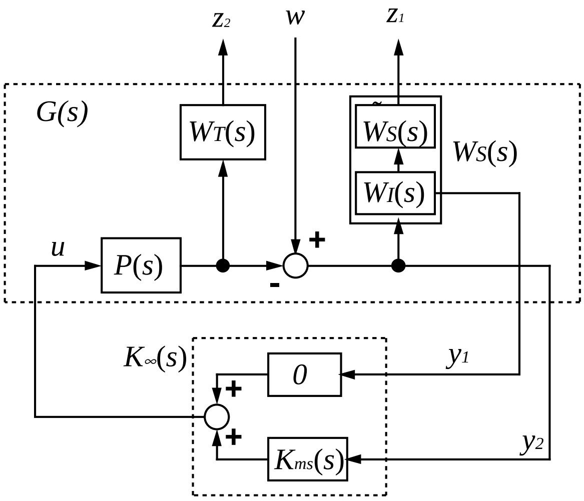

Figure 3.8: A sketch in the proof

solution for the ARE (3.13) with $L_{H_1} = O$, the H_∞ controller for $G_{ms}(s)$, where $P(s)$ has $j\omega$-poles, is given by (A.11).

On the other hand, from the proof of Theorem 3.3.3, it is shown that by letting $W(s) = 0$, the (1,2)-entry of the controller $K_\infty(s)$ in (3.14) coincides with the controller given by (A.11) and the (1,1)-entry of the controller $K_\infty(s)$ is zero. From these, it is apparent that the controller $K_\infty(s)$ in (3.14) with $W(s) = 0$ is the H_∞ controller for the generalized plant of $G(s)$. This is sketched in Figure 3.8. ■

Thus, in the case where $P(s)$ has $j\omega$-poles, a design procedure of the robust servo controller is given.

Proposition 3.4.1 *Let N_P be the number of integrators included in $P(s)$, and let N_W be the number of integrators included in $W_S(s)$. The following procedure yields an integral-type H_∞ controller which includes $N_W - N_P$ integrators and the controller satisfies the specifications S1, S2 and S3.*

Design Procedure 2

STEP 1 *Introduce integrators into $W_S(s)$ as follows:*

$$W_S(s) = \frac{\tilde{W}_S(s)}{s^n}, \ \tilde{W}_S(s) \in \mathcal{RH}_\infty, \tilde{W}_S(0) \neq 0,$$

where $n = N_W$ is the sum of the number of integrators which are needed to be included in the controller and the number of integrators which are included in $P(s)$.

STEP 2 *Select an L_H such that $L_{H_1} \in \{L_{H_1} : \mathrm{Re}\,(\lambda_i(A_W + L_{H_1})) < 0\}$ holds. Then, obtain the positive semi-definite stabilizing solution of the ARE in (3.12) and the* *positive semi-definite pseudo-stabilizing solution of the ARE in (3.13). Then, verify the inequality (3.31).*

STEP 3 *If the inequality (3.31) is satisfied, set $L_{H_1} = O$ and $W(s) = 0$, then the controller (3.32) is a solution.*

Proof. Since it is apparent that the resultant controller is the H_∞ controller from the above discussion, it suffice to show that the controller has integrators. The full row rank matrix U, which satisfies (3.21) and (3.22), can be given as (3.26), hence, from the generalized plant $G(s)$ represented in (3.8), it can be verified that U satisfies (3.24). Using the result of Lemma 3.3.3, it is shown that the controller (3.32) has integrators, the number of which is same as the number of 0-eigenvalues of $\Lambda = A_W$. However, form the relation in (3.46), some of the integrators in the controller are canceled out by zeros on the origin of $a^{-1}(s)$. ■

Remark 3.4.2 *In the above procedure, although, in the case $P(s)$ has $j\omega$-poles, it is still necessary to obtain the pseudo-stabilizing solution of the ARE, it is not necessary to separate the $j\omega$ part from $P(s)$.*

Remark 3.4.3 *The order of the resultant controller is equal to that of the generalized plant. Thus, as compared with the controller in (3.10), the order of the controller is reduced by the order of $W_S(s)$.*

3.5 Numerical examples

This subsection illustrates numerical examples of the robust servo control with simple models.

3.5.1 $P(s)$ has no $j\omega$-poles

Let us consider the following system

$$P(s) = \frac{1}{s+1} = \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right].$$

In order to introduce an integrator into the controller, $W_S(s)$ is selected as follows

$$W_S(s) = \frac{1/\gamma}{s} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1/\gamma & 0 \end{array}\right].$$

Let us adopt $W_T(s)$ as

$$W_T(s)P(s) = \frac{s+0.5}{s+1} = \left[\begin{array}{c|c} -1 & 1 \\ \hline -0.5 & 1 \end{array}\right].$$

Then the problem is solvable if γ satisfies $\gamma \geq 1.2$. Hence, we set $\gamma = 1.2$. In this case, a stabilizing solution of the ARE in (3.12) is given as

$$X = \begin{bmatrix} 25.5 & -25.5 \\ -25.5 & 26.0 \end{bmatrix}.$$

By setting L_{H_1} as $L_{H_1} = -\alpha, \alpha > 0$, the ARE in (3.13) is represented as follows

$$YA_{ZH}^T + A_{ZH}Y + YQY = O,$$

where

$$A_{ZH} = \begin{bmatrix} -\alpha & 0 \\ 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1/\gamma^2 & 0 \\ 0 & -0.75 \end{bmatrix}.$$

Apparently the stabilizing solution of the ARE is $Y = O$. By letting $L_{H_1} = O$, a central controller is derived from (3.32) as

$$K_\infty(s) = \begin{bmatrix} 0 & 25.5\frac{s+1}{s(s+26.5)} \end{bmatrix}.$$

It can be verified that the controller includes an integrator.

3.5.2 $P(s)$ has $j\omega$-poles

Let us consider the following system

$$P(s) = \frac{1}{s} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right],$$

which includes an $j\omega$-pole in $\omega = 0$.

In order to introduce an integrator into the controller $W_S(s)$ is selected as follows

$$W_S(s) = \frac{1/\gamma}{s^2} = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 1/\gamma & 0 \end{array}\right].$$

Let us adopt $W_T(s)$ as

$$W_T(s)P(s) = \frac{s+0.5}{s} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0.5 & 1 \end{array}\right].$$

Then the problem is solvable if γ satisfies $\gamma \geq 3.6$. Hence, $\gamma = 3.6$ is adopted. By obtaining a stabilizing solution of an ARE in (3.12) and obtaining a pseudo stabilizing solution of the ARE in (3.13), an H_∞ controller (a central solution) is derived from (3.32).

$$K_\infty(s) = \begin{bmatrix} 0 & 265.3\frac{s+0.322}{s(s+266.3)} \end{bmatrix}$$

It can also be verified that the controller includes an integrator.

3.6 Summary

In this chapter, we have considered a synthesis of the low-order integral-type H_∞ control system, where two types of plants: integral-type and non-integral-type, are treated. The synthesis problem is formulated as the non-standard H_∞ problem, where a direct feed-through term of the subsystem from the external input to the measurement output is of full column rank. Although the formulation resembles the results by Zhang et al. [71] and Hozumi et al. [26], the solutions are distinctive. This approach to the integral-type robust controller design is based on the non-standard H_∞ control problem. The controller is given with low order by using the solutions of the AREs. In the case where the plant has no integrators, it is not necessary to solve the so-called pseudo stabilizing solution of the ARE. We have extended the result to the case where the plant has the integrators. In this case, although it is still necessary to solve the pseudo stabilizing solution of the ARE, there is no necessity for transforming the non-standard problem to the standard problem and separating the integrators from the plant. Moreover, the results given in this chapter can be extended to the two-degree-of-freedom controller design problem which can also be reduced to the non-standard H_∞ control problem. This problem will be treated in the following chapter.

Chapter 4

Trade-off analysis of a low-order TDF control system

4.1 Introduction

Closed loop stabilization, disturbance elimination, and reference tracking in the presence of uncertainties of a model – these are the main goals of a robust control system design, and TDF control is one of the most effective ways to reach all these goals simultaneously [44, 22, 13, 29]. In recent years, there has been many studies concerned about application of H_∞ and H_2 control theory in the designing of TDF control systems [17, 38, 7, 4, 27]. Most of these studies focused on transformation of the TDF control problems to the standard H_∞ or H_2 control problems. This has resulted in high-performance TDF controllers, but excessive transformation sometimes results in controllers of unacceptably high order. That is, the McMillan degree of the H_∞ controller is no less than that of the generalized plant, especially in the TDF case, it becomes no less than three times as high as that of the plant. Therefore, it is required to study the low-order TDF controller design.

On the basis of such a background, some earlier studies have tackled the problem of designing low-order controllers [5, 6, 24, 7, 38, 17]. These studies are based on the idea of using the same dynamics in both of the feedback controller and the feedforward controller, where some of these studies [5, 6, 24, 17] have adopted a sequential design in which the feedback controller and the feedforward controller are designed separately. In this approach, a low-order TDF controller can be designed. A key feature of the approach is that the zeros and the gain of the feedforward controller are optimized in the second step, which is preceded by designing of the feedback controller. Therefore, it should be noted that, in this approach, the resultant feedforward controller has the same pole position as the feedback controller has, and it is not necessarily optimal for tracking performance. In general, it is well known that the feedback performance and the tracking performance are independently specified in the TDF control system [66, 44], but it is not known whether this is true in the case of the same-dynamic TDF control system. Since the answer to this question affects the optimality

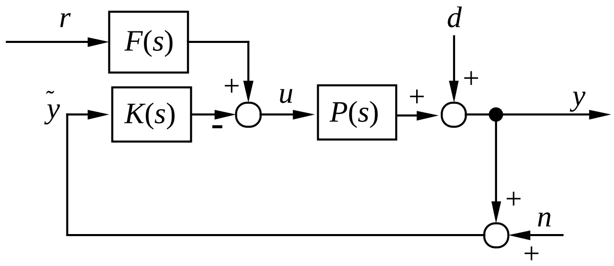

Figure 4.1: Basic structure of TDF control system

in the low-order TDF controller design, the question must be investigated.

Thus, prior to considering the synthesis of the low-order TDF controller, this chapter is devoted to clarifying the relationship between the feedback performance and the feedforward performance in the case where the TDF controller is low order, especially, the feedback controller and the feedforward controller share the common dynamics. Thus, this chapter analyzes the trade-off of the low-order TDF control system. It is shown that the independence of the feedback performance and the tracking performance is not maintained in this case, and that the low-order TDF controller should therefore be constructed with a simultaneous design of the feedback controller and the feedforward controller.

4.2 Basic analysis and design of the TDF control system

In this section, we analyze the feedback properties and the feedforward properties in the TDF control system by representing the class of transfer functions of the TDF control system. Based on the analysis we introduce a basic design of the TDF control system by using the H_∞ control. Then, we evaluate the order of the resultant TDF controller.

4.2.1 Basic analysis of the TDF control system

Consider the TDF control system shown in Figure 4.1. $P(s)$ represents a plant model. $K(s)$ is the feedback controller and $F(s)$ is the feedforward controller. Where, $r \in \mathbb{R}^p$ is the reference input, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurement output and $d, n \in \mathbb{R}^p$ respectively represent disturbance input and observation noise.

Then the feedback performance of the closed loop system is characterized with the transfer functions from d to y and n to y, denoted respectively $T_{yd}(s)$ and $T_{yn}(s)$.

$$
\begin{aligned}
T_{yd}(s) &= (I + P(s)K(s))^{-1} = S(s) && (4.1)\\
T_{yn}(s) &= -(I + P(s)K(s))^{-1}P(s)K(s) = -T(s) && (4.2)
\end{aligned}
$$

Where, the transfer functions $S(s)$ and $T(s)$ are known as the sensitivity function and the complementary sensitivity function, which are defined as follows

$$\begin{aligned} S(s) &\triangleq (I + P(s)K(s))^{-1} \\ T(s) &\triangleq (I + P(s)K(s))^{-1} P(s)K(s). \end{aligned}$$

The tracking performance is characterized with the transfer function

$$T_{yr}(s) = (I + P(s)K(s))^{-1} P(s)F(s). \tag{4.3}$$

Now let us represent $P(s), K(s)$ and $F(s)$ as the coprime factorization [67, 44] over $\mathcal{RH}_\infty$, i.e.,

$$P(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s) \tag{4.4}$$
$$K(s) = N_K(s)M_K^{-1}(s) = \tilde{M}_K^{-1}(s)\tilde{N}_K(s) \tag{4.5}$$
$$F(s) = N_F(s)M_F^{-1}(s) = \tilde{M}_F^{-1}(s)\tilde{N}_F(s) \tag{4.6}$$

where all of the factors are the elements of $\mathcal{RH}_\infty$ and coprime. Then it is known that the internal stability condition of the system in the Figure 4.1 is given as follows.

Lemma 4.2.1 *Let (4.4), (4.5) and (4.6) be coprime factorization over $\mathcal{RH}_\infty$, then the TDF control system in figure 4.1 is internal stable iff*

$$\left\{ \begin{array}{l} K(s) \in \mathcal{K} \\ \tilde{M}_K(s)F(s) \in \mathcal{RH}_\infty \end{array} \right. .$$

Here, $\mathcal{K}$ is the class of the feedback controllers achieving the internal stability, i.e.,

$$\begin{aligned} \mathcal{K} &= \left\{ \left(Y(s) - Q(s)\tilde{N}(s)\right)^{-1} \left(X(s) + Q(s)\tilde{M}(s)\right) \quad | \quad \forall Q(s) \in \mathcal{RH}_\infty \right\} \\ &= \left\{ \left(\tilde{X}(s) + M(s)Q(s)\right) \left(\tilde{Y}(s) - N(s)Q(s)\right)^{-1} \quad | \quad \forall Q(s) \in \mathcal{RH}_\infty \right\}, \end{aligned}$$

where, $X(s), Y(s), \tilde{X}(s), \tilde{Y}(s)$ are the solutions of the Bezout identity

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{M}(s) & \tilde{N}(s) \end{bmatrix} \begin{bmatrix} N(s) & -\tilde{Y}(s) \\ M(s) & \tilde{X}(s) \end{bmatrix} = I.$$

Proof. See reference [66]. ∎

From Lemma 4.2.1, the class of internally stabilizing controllers $K(s), F(s)$ are represented with free-parameters $Q(s), R(s) \in \mathcal{RH}_\infty$.

$$\begin{aligned} \mathcal{K} &= \left\{ \left(Y(s) - Q(s)\tilde{N}(s)\right)^{-1} \left(X(s) + Q(s)\tilde{M}(s)\right) \quad | \quad \forall Q(s) \in \mathcal{RH}_\infty \right\} \\ \mathcal{F} &= \left\{ \left(Y(s) - Q(s)\tilde{N}(s)\right)^{-1} R(s) \quad | \quad \forall R(s) \in \mathcal{RH}_\infty \right\} \end{aligned}$$

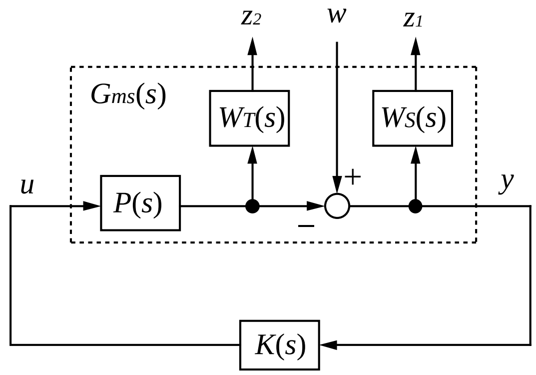

Figure 4.2: Feedback controller design

By substituting controllers $K(s) \in \mathcal{K}$ and $F(s) \in \mathcal{F}$ into (4.1), (4.2) and (4.3), the class of transfer functions $T_{yd}(s), T_{yn}(s)$ and $T_{yr}(s)$ which are achievable in the TDF system are represented as

$$\begin{aligned} \mathcal{T}_{yd} &= \left\{ \left(\tilde{Y}(s) - N(s)Q(s) \right) \tilde{M}(s) \quad | \quad \forall Q(s) \in \mathcal{RH}_\infty \right\} \\ \mathcal{T}_{yn} &= \left\{ N(s) \left(X(s) + Q(s)\tilde{M}(s) \right) \quad | \quad \forall Q(s) \in \mathcal{RH}_\infty \right\} \\ \mathcal{T}_{yr} &= \left\{ N(s)R(s) \quad | \quad \forall R(s) \in \mathcal{RH}_\infty \right\}. \end{aligned} \tag{4.7}$$

Note that $Q(s)$ and $R(s)$ are the independent free-parameters of $\mathcal{K}$ and $\mathcal{F}$. It can be seen that there doesn't exist any trade-off between the feedback performances and the tracking performance, hence these performances are independently specified with those free-parameters. This property affects the way of TDF controller construction, that is, since the TDF performances are optimized with a series of sequential optimization of the feedback controller and the feedforward controller, the TDF controller is frequently designed through two steps: a feedback controller design and a feedforward controller design.

4.2.2 A basic design of the TDF controller

We show a basic approach to designing the TDF control system. This approach of the design is a natural way that can be occurred immediately from the previous analysis. That is, a sequential design of the feedback controller and the feedforward controller can satisfy each of the performances in the TDF control system.

Thus, first of all, let's design the feedback controller by way of solving the mixed sensitivity problem. The generalized plant of the problem is written in Figure 4.2 where the meanings of symbols are identical to that in Figure 3.2. Here, the H_∞ problem of the generalized plant $G_{ms}(s)$ is solved. Since the order of the resultant controller is no less than that of $G_{ms}(s)$, if we denote the order of each system in boxes as deg $(\bullet)$, this design derives H_∞ controllers of

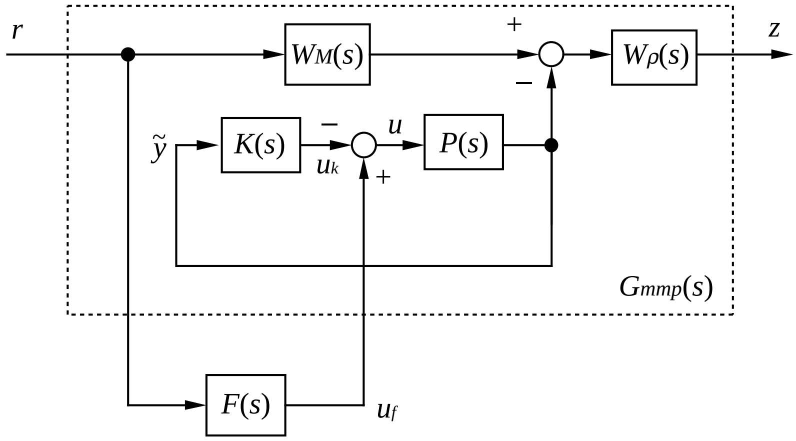

Figure 4.3: Feedforward controller design

order

$$\deg(K(s)) \geq \deg(P(s)) + \deg(W_S(s)) + \deg(W_T(s)).$$

In the second step, let's design the feedforward controller by way of solving the model matching problem. The generalized plant of the problem is written in Figure 4.3 where $W_M(s)$ is a transfer function of an ideal tracking performance, $W_\rho(s)$ is a weighting function of the tracking error. Here, the H_∞ problem of the generalized plant $G_{mmp}(s)$ is solved. Thus, the design derives H_∞ controllers of order

$$\begin{aligned}\deg(F(s)) &\geq \deg(K(s)) + \deg(P(s)) + \deg(W_M(s)) + \deg(W_\rho(s)) \\ &= 2\deg(P(s)) + \deg(W_S(s)) + \deg(W_T(s)) \\ &\quad +\deg(W_M(s)) + \deg(W_\rho(s)).\end{aligned}$$

In order to evaluate the order of the TDF controller, let us represent the state-space equations of the feedback controller and the feedforward controller as follows.

$$F : \begin{cases} \dot{x}_f = A_F x_f + B_F r \\ u_f = C_F x_f + D_F r \end{cases} \qquad K : \begin{cases} \dot{x}_k = A_K x_k + B_K \tilde{y} \\ u_k = C_K x_k + D_K \tilde{y} \end{cases}$$

Then the dynamics of the TDF controller is composed of both of the dynamics of the controllers.

$$\begin{cases} \begin{bmatrix} \dot{x}_f \\ \dot{x}_k \end{bmatrix} = \begin{bmatrix} A_F & O \\ O & A_K \end{bmatrix} \begin{bmatrix} x_f \\ x_k \end{bmatrix} + \begin{bmatrix} B_F \\ O \end{bmatrix} r + \begin{bmatrix} O \\ B_K \end{bmatrix} \tilde{y} \\ u = \begin{bmatrix} C_F & C_K \end{bmatrix} \begin{bmatrix} x_f \\ x_k \end{bmatrix} + D_F r + D_K \tilde{y} \end{cases} \tag{4.8}$$

As shown in (4.8), the McMillan degree of the TDF controller is the sum of the degrees of these controllers. Thus the TDF controller $(F(s), K(s))$ of order

$$\begin{aligned}\deg(F(s), K(s)) &\geq 3\deg(P(s)) + 2\deg(W_S(s)) + 2\deg(W_T(s)) \\ &\quad + \deg(W_M(s)) + \deg(W_\rho(s)).\end{aligned}$$

can be constructed. From this fact, the natural design of the TDF controller yields high-order controller. This is a basic motivation for the low-order TDF controller design.

4.3 An idea of sharing common dynamics

As shown in the previous section, the McMillan degree of the TDF controller that is designed in the natural way becomes very high. Sometimes we are faced with the situation such that the degree of the TDF controller is too high to implement it as hardware. So it is important to consider designing of the low-order TDF controller. One of the effective ways to design the low-order TDF controller is to share common dynamics between these controllers[5, 6, 24]. In this section we define the share of common dynamics in the TDF controller, and show a basic design method of the low-order TDF controller.

4.3.1 Sharing common dynamics between $K(s)$ and $F(s)$

Here, let us define a low-order TDF controller in which the feedback controller and the feedforward controller share common dynamics.

Definition 4.3.1 *The TDF controller which shares common dynamics between $K(s)$ and $F(s)$ is defined as follows.*

$$\left[\begin{array}{c:c} F_s(s) & K_s(s) \end{array}\right] = \left[\begin{array}{c|c:c} A & B_F & B_K \\ \hline C & D_F & D_K \end{array}\right], \tag{4.9}$$

where both of the pairs (A, B_K) and (A, B_F) are controllable, and the pair (A, C) is observable. In the above description, the subscript "s" means "share".

It can be seen that the order of the TDF controller $(F_s(s), K_s(s))$ is

$$\deg\left(F_s(s), K_s(s)\right) = \deg\left(F_s(s)\right) = \deg\left(K_s(s)\right),$$

hence by sharing common dynamics between the feedback controller and the feedforward controller, the order of the TDF controller can be reduced.

In the TDF control system in which the controllers share common dynamics, the internal stability is satisfied if $K_s(s)$ satisfies the internally stability of the feedback loop.

Lemma 4.3.1 *Let us suppose $K_s(s) \in \mathcal{K}$, then the TDF controller given as (4.9) satisfies the stability of the TDF control system.*

Proof. Let us give a coprime factorization of $K_s(s)$ over $\mathcal{RH}_\infty$ as

$$\begin{aligned} K_s(s) &= \tilde{M}_K^{-1}(s)\tilde{N}_K(s) \\ \tilde{M}_K(s) &= \left[\begin{array}{c|c} A+HC & H \\ \hline C & I \end{array}\right] \qquad (4.10) \\ \tilde{N}_K(s) &= \left[\begin{array}{c|c} A+HC & B_K+HD_K \\ \hline C & D_K \end{array}\right], \qquad (4.11) \end{aligned}$$

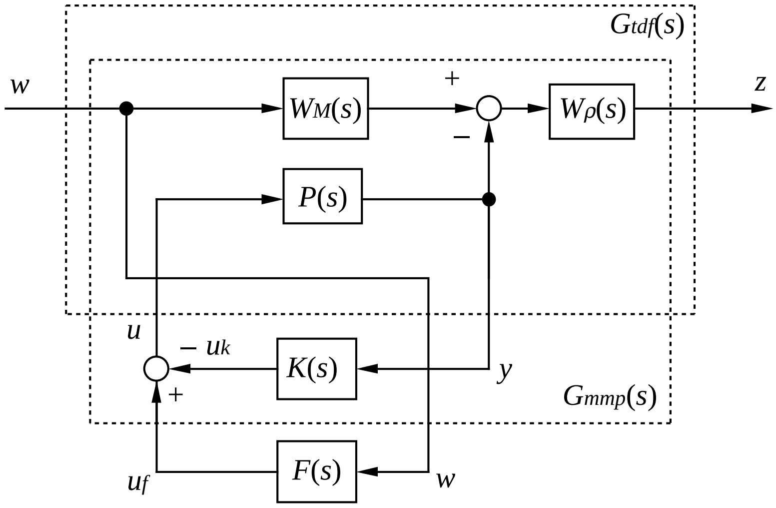

Figure 4.4: A generalized plant for a TDF controller design

where $\tilde{M}_K(s) \in \mathcal{RH}_\infty$ and $\tilde{N}_K(s) \in \mathcal{RH}_\infty$ are left coprime, and $\mathrm{Re}(\lambda_i(A + HC)) < 0, \forall i$ is satisfied. Then $F_s(s)$ can be also factorized over $\mathcal{RH}_\infty$ as follows

$$\begin{aligned} F_s(s) &= \tilde{M}_K^{-1}(s)\tilde{N}_F(s) \\ \tilde{N}_F(s) &= \left[\begin{array}{c|c} A + HC & B_F + HD_F \\ \hline C & D_F \end{array}\right], \end{aligned} \quad (4.12)$$

where $\tilde{M}_K(s) \in \mathcal{RH}_\infty$ and $\tilde{N}_K(s) \in \mathcal{RH}_\infty$ are left coprime. The equation (4.12) implies

$$\tilde{M}_K(s)F_s(s) = \tilde{N}_F(s) \in \mathcal{RH}_\infty,$$

hence from Lemma 4.2.1 the TDF system is internally stable. ■

From Lemma 4.3.1, in order to satisfy the internal stability of the TDF control system in which the controllers share the common dynamics, it is only needed to consider the internal stability of the feedback controller.

4.3.2 A basic design of a low-order TDF controller

Here, we show a basic method for designing the low-order TDF controller where one dynamics is shared by the feedback controller and the feedforward controller. Firstly, let it be designed a feedback controller of the form:

$$K_s(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]. \quad (4.13)$$

Then, construct a generalized plant as in Figure 4.4, where $G_{mmp}(s)$ is the generalized plant of the model matching problem and $G_{tdf}(s)$ is the generalized plant for the TDF controller design. The meaning of each subsystem is same as that in Figure 4.3. The generalized plant

$G_{tdf}(s)$ can be represented as follows:

$$G_{tdf}(s) = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_{2v} & O & O \\ O & I & O \end{array}\right]. \tag{4.14}$$

Since the feedback controller has already been designed, the state-space parameters in (4.13) are fixed. In order to design the feedforward controller which shares the same dynamics with $K_s(s)$ in (4.13), we put the state-space data of the feedforward controller as follows:

$$F_s(s) = \left[\begin{array}{c|c} A_K & B_F \\ \hline C_K & D_F \end{array}\right], \tag{4.15}$$

where B_F and D_F are the parameters which are to be designed.

By using those state-space representations in (4.13), (4.14) and (4.15), the state-space data of the closed loop system $T_{zw}(s)$ is represented as follows:

$$\begin{aligned} T_{zw}(s) &= \mathcal{F}_l\left(G_{tdf}(s), \left[\begin{array}{cc} F_s(s) & K_s(s) \end{array}\right]\right) \\ &= \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right], \end{aligned}$$

where

$$\begin{aligned} A_{cl} &= \left[\begin{array}{cc} A + B_2 D_K C_{2v} & B_2 C_K \\ B_K C_{2v} & A_K \end{array}\right] \\ B_{cl} &= \left[\begin{array}{c} B_1 + B_2 D_F \\ B_F \end{array}\right] = E + J\mathcal{R} \\ C_{cl} &= \left[\begin{array}{cc} C_1 + D_{12} D_K C_{2v} & D_{12} C_K \end{array}\right] \\ D_{cl} &= D_{11} + D_{12} D_F = H + L\mathcal{R}. \end{aligned}$$

Here, it can be seen that the matrices A_{cl} and C_{cl} are composed of the fixed parameters, and both of B_{cl} and D_{cl} include design parameters B_F and D_F. Hence, we put the variable composed of B_F and D_F as

$$\mathcal{R} \triangleq \left[\begin{array}{c} B_F \\ D_F \end{array}\right],$$

and the other fixed parameters are defined as follows:

$$E \triangleq \left[\begin{array}{c} B_1 \\ O \end{array}\right], J = \left[\begin{array}{cc} O & B_2 \\ I & O \end{array}\right], H \triangleq D_{11}, L \triangleq \left[\begin{array}{cc} O & D_{12} \end{array}\right].$$

Under the above preparation, the following result is obtained.

Theorem 4.3.1 *In the TDF control system of Figure 4.4, the closed loop system is stable and it satisfies a condition such that*

$$\|T_{zw}(s)\|_\infty < \gamma,$$

if and only if there exists a positive definite matrix X which satisfies the following inequality:

$$\begin{bmatrix} XA_{cl}^T + A_{cl}X & E + J\mathcal{R} & XC_{cl}^T \\ (E + J\mathcal{R})^T & -\gamma I & (H + L\mathcal{R})^T \\ C_{cl}X & H + L\mathcal{R} & -\gamma I \end{bmatrix} < 0. \tag{4.16}$$

Proof. By using the bounded real lemma, it can be immediately deduced. ■

Remark 4.3.1 *The matrix inequality in (4.16) is a Linear Matrix Inequality, hence by using some convex programming method we can solve the inequality.*

4.4 Trade-off analysis in the low-order TDF control system

In the previous section, a method of the low-order TDF controller is introduced. However, it is unclear whether sharing dynamics between $F(s)$ and $K(s)$ causes a conflict in the TDF performances or not. If the answer to the question is "yes", the trade-off in the performances must be considered in the design of the low-order TDF controller. The two step design is not adequate for balancing the trade-off in the designing of the low-order TDF controller.

4.4.1 Trade-off between the feedback performance and the feedforward performance

In the above section, the TDF controller in which $F(s)$ and $K(s)$ share common dynamics is defined and it is represented in the state-space formula. Then, this section evaluates degradation in the performance of the feedforward controller when the performance of the feedback controller is specified first.

Lemma 4.4.1 *Let us represent $K_s(s) \in \mathcal{K}$ as a right coprime factorization over $\mathcal{RH}_\infty$.*

$$\begin{aligned} &K_s(s) = (Y(s) - Q(s)\tilde{N}(s))^{-1}(X(s) + Q(s)\tilde{M}(s)) \\ &Y(s) - Q(s)\tilde{N}(s) \triangleq \tilde{M}_K(s), \quad X(s) + Q(s)\tilde{M}(s) \triangleq \tilde{N}_K \end{aligned} \tag{4.17}$$

Then all the class of $F_s(s) \in \mathcal{F}$ which shares a dynamics with $K_s(s)$ is given by

$$\mathcal{F}s(s) = \left\{ \tilde{M}_K^{-1}(s)\tilde{R}(s) \quad | \quad \forall Q(s) \in \mathcal{RH}_\infty, \forall B_R \in \mathbb{R}^{n\times p}, \forall D_R \in \mathbb{R}^{m\times p} \right\}, \tag{4.18}$$

where $\tilde{R}(s)$ is represented as follows

$$\tilde{R}(s) = \left[\begin{array}{c|c} A + HC & B_R \\ \hline C & D_R \end{array}\right]$$

and $\tilde{M}_K(s)$ and $\tilde{R}(s)$ are left coprime.

Proof.(necessity:) Let us represent $K(s) \in \mathcal{K}$ and $F(s) \in \mathcal{F}$ with coprime factorizations as follows

$$\begin{aligned} K(s) &= \tilde{M}_K^{-1}(s)\tilde{N}_K(s) \\ F(s) &= \tilde{M}_K^{-1}(s)R(s), \quad R(s) \in \mathcal{RH}_\infty. \end{aligned} \tag{4.19}$$

If $R(s)$ is denoted as $R(s) = \left[\begin{array}{c|c} A_R & B_R \\ \hline C_R & D_R \end{array}\right]$, then by using (4.10), $F(s)$ can be represented as follows

$$\begin{aligned} F(s) &= \left[\begin{array}{cc|c} A & HC_R & HD_R \\ O & A_R & B_R \\ \hline -C & C_R & D_R \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & A + HC_R - A_R & HD_R - B_R \\ O & A_R & B_R \\ \hline -C & -C + C_R & D_R \end{array}\right]. \end{aligned} \tag{4.20}$$

From the equation (4.20), in order to share common dynamics between $F(s)$ and $K(s)$, it is necessary to satisfy $A_R = A + HC, C_R = C$. Hence the feedforward controller and feedback controller share common dynamics only if $F(s)$ is given as follows

$$F(s) = \tilde{M}_K^{-1}(s)\tilde{R}(s). \tag{4.21}$$

(sufficiency:) It is easily verified that F in (4.21) has the same dynamics of $K_s(s)$ and satisfies the internal stability of TDF system from Lemma 4.3.1. ∎

From the above result it is possible to show the class of the transfer functions which can be attained with the TDF controller in (4.9).

Theorem 4.4.1 *(The class of tracking performance) With the TDF controller, which shares the same dynamics, any tracking performance that belongs to the class:*

$$\tilde{\mathcal{T}}_{yr} = \left\{ N(s)\tilde{R}(s) \quad | \quad \forall B_R \in \mathbb{R}^{n \times p}, \forall D_R \in \mathbb{R}^{m \times p} \right\} \tag{4.22}$$

can be attained.

Proof. Substituting (4.18) into (4.3) leads the class. ∎

From equations (4.10), (4.11) and (4.17), it is apparent that the pair $(A+HC, C)$ depends on the parameter $Q(s) \in \mathcal{RH}_\infty$. Hence the class of $\tilde{R}(s)$:

$$\tilde{\mathcal{R}} \triangleq \left\{ \tilde{R}(s) \quad | \quad \forall B_R \in \mathbb{R}^{n \times p}, \forall D_R \in \mathbb{R}^{m \times p} \right\}$$

also depends on $Q(s)$ and the inclusion

$$\tilde{\mathcal{R}} \subset \mathcal{RH}_\infty$$

holds. By comparing (4.7) and (4.22), it is verified that the tracking performance is restricted by the feedback controller design if the TDF controller share the same dynamics.

Let us define the transfer function of the requiring tracking performance as $M_{yr}(s)$ and let $T_{yr}(R(s)) \triangleq N(s)R(s)$ be the tracking performance, then an objective function for the evaluation of the tracking performance can be selected as follows

$$f\left(R(s)\right) \triangleq M_{yr}(s) - T_{yr}\left(R(s)\right).$$

Then the class $\tilde{\mathcal{R}}$ is restricted by the design of the feedback controller. Hence the following inequality

$$\min_{R(s)\in\mathcal{RH}_\infty} \left\| f\left(R(s)\right) \right\| \leq \min_{\tilde{R}(s)\in\tilde{\mathcal{R}}} \left\| f\left(\tilde{R}(s)\right) \right\|,$$

where $\| \bullet \|$ indicates any norm of a transfer function, holds. The above inequality indicates that if the feedback performance is specified in the first time, the tracking performance may be sacrificed.

4.5 Summary

In this chapter, based on the basic analysis and synthesis of the TDF control system, we have motivated to design the low-order TDF controller. Then, we have introduced designing of a low-order TDF controller where a feedback controller and a feedforward controller share common dynamics. Then, we have analyzed the trade-off of the low-order TDF control system. From the analysis it is shown that the independent property of the feedback performance and the feedforward performance in the basic TDF control system is not maintained in the low-order TDF control system. Then, it is pointed out that the low-order TDF controller should be constructed with a simultaneous design of the feedback controller and the feedforward controller. In the next chapter we will introduce the simultaneous design method of the low-order TDF controller.

Chapter 5

A synthesis of low-order TDF controller

5.1 Introduction

The previous chapter has analyzed the trade-off between the feedback performance and the tracking performance in a low-order TDF control system where the controllers share common dynamics. It is clarified that there exists a conflict between those performances, and that the independence in the properties of the feedback controller and the feedforward controller is not maintained in the low-order TDF control system. It is difficult to consider the trade-off in a sequential approach, where a feedback controller and a feedforward controller are designed separately, the low-order TDF controller should be designed with a simultaneous approach, where the feedback controller and the feedforward controller are designed simultaneously.

Based on the analysis, the present chapter proposes a method for designing the low-order TDF controller in a simultaneous approach. Remarkable point of this chapter is the reduction of the low-order TDF control problem to the non-standard H_∞ problem of case 2. This formulation enables the TDF controller to be derived simultaneously. As a result, the feedforward controller and the feedback controller are designed such that the order is lower than that of the controller designed with the sequential approach based on the standard H_∞ problem. Furthermore, a reduced-order controller, which has dynamics of lower-order than the dynamics of a generalized plant, is derived by an algebraic operation using free parameters of the general solution.

5.2 Problem descriptions and comparison

This section describes some problems for the low-order TDF controller design and compares the properties of the problems with each other.

Consider a system shown in Figure 5.1, which is a generalized plant for the TDF control system design. Where $P(s)$ is a plant, $K(s)$ is a feedback controller and $F(s)$ is a feedforward controller. The signal $d \in \mathbb{R}^p$ indicates a disturbance input, $u_f \in \mathbb{R}^m, u_k \in \mathbb{R}^m$ are the

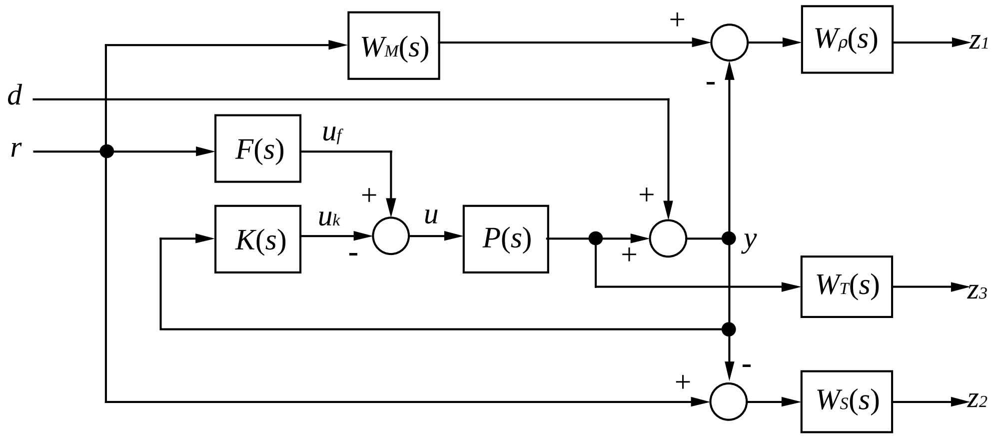

Figure 5.1: A generalized plant

outputs from the feedforward and feedback controllers, and $r \in \mathbb{R}^p, y \in \mathbb{R}^p$ respectively indicate a reference input and control output. The signals $z_i, i = 1, 2, 3$ are used to evaluate the control performances of the TDF control system. Concretely, $z_1 \in \mathbb{R}^p$ is a index of the tracking performance, $z_2 \in \mathbb{R}^p$ and $z_3 \in \mathbb{R}^p$ are the indices of the feedback performances. $W_M(s), W_\rho(s), W_T(s)$ and $W_S(s)$ are the weighting functions for each specification. The generalized plant is arranged and replaced with Figure 5.2.

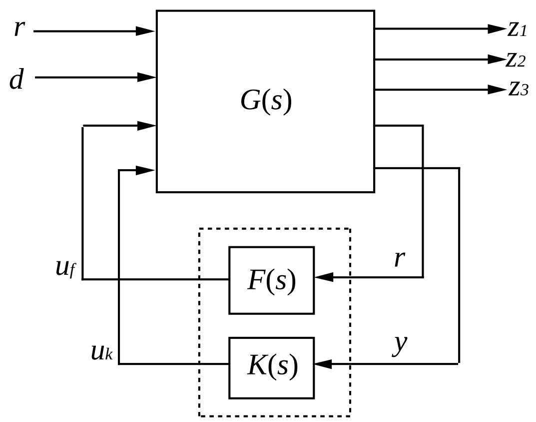

Figure 5.2: A generalized plant

5.2.1 Two-step design

First of all, as seen in the basic TDF controller design, the two-step design is one of the most popular ways to construct the TDF control system.

Problem 5.2.1 (Two-step design I) *Firstly, let's obtain a feedback controller*

$$K^*(s) = \left[\begin{array}{c|c} A_{K^*} & B_{K^*} \\ \hline C_{K^*} & D_{K^*} \end{array}\right] \in \mathcal{K}, A_{K^*} \in \mathbb{R}^{n_k \times n_k}$$

which satisfies an inequality for the feedback performances

$$\left\| T_{\substack{z_2 \\ z_3} d}\left(K(s)\right) \right\|_\infty < \gamma, \tag{5.1}$$

where $\mathcal{K}$ denotes the class of the internally stabilizing feedback controllers, $T_{\substack{z_2 \\ z_3} d}\left(K(s)\right)$ denotes a transfer function matrix:

$$T_{\substack{z_2 \\ z_3} d}\left(K(s)\right) \triangleq \begin{bmatrix} T_{z_2 d}\left(K(s)\right) \\ T_{z_3 d}\left(K(s)\right) \end{bmatrix},$$

and $T_{ab}(s)$ denotes a transfer function from the signal b to a.

Then, find the feedforward controller $F(s)$ which is the solution of the minimization problem described as follows

$$\underset{F(s)\in\mathcal{F}}{\text{minimize}} \quad \left\| T_{z_1 r}\left(F(s), K^*(s)\right) \right\|_\infty,$$

where $\mathcal{F}$ indicates the class of the stabilizing feedforward controllers.

Remark 5.2.1 *Assume that the order of $F(s)$ is n_f, then the maximum order of the TDF controller is $n_k + n_f$. As seen in chapter 4, solving Problem 5.2.1 derives a high-performance but high-order TDF controller.*

In the next problem, we intend to derive a low-order TDF controller.

Problem 5.2.2 (Two-step design II) *Let's obtain a feedback controller*

$$K^*(s) = \left[\begin{array}{c|c} A_{K^*} & B_{K^*} \\ \hline C_{K^*} & D_{K^*} \end{array}\right] \in \mathcal{K}, A_{K^*} \in \mathbb{R}^{n_k \times n_k}$$

which satisfies an inequality in (5.1).

Then, find the feedforward controller $F(s)$ which is the solution of the minimization problem described as follows

$$\underset{F(s)\in\mathcal{F}(K^*(s))}{\text{minimize}} \quad \left\| T_{z_1 r}\left(F(s), K^*(s)\right) \right\|_\infty, \tag{5.2}$$

where the class $\mathcal{F}\left(K^*(s)\right)$ indicates

$$\mathcal{F}\left(K^*(s)\right) \triangleq \left\{ \left[\begin{array}{c|c} A_K^* & B_F \\ \hline C_K^* & D_F \end{array}\right] | B_F \in \mathbb{R}^{n_k \times p}, D_F \in \mathbb{R}^{m \times p} \right\} \subset \mathcal{F}. \tag{5.3}$$

Remark 5.2.2 *The TDF controller that is derived by solving Problem 5.2.2 shares common dynamics between the feedback controller and the feedforward controller. Thus the order of the TDF controller is n_k, which is lower than that of the TDF controller derived by solving Problem 5.2.1.*

Lemma 5.2.1 *The optimal values of Problem 5.2.1 and Problem 5.2.2 satisfy*

$$\min_{F(s)\in\mathcal{F}} \quad \left\| T_{z_1 r}\left(F(s), K^*(s)\right) \right\|_\infty \leq \min_{F(s)\in\mathcal{F}(K^*(s))} \quad \left\| T_{z_1 r}\left(F(s), K^*(s)\right) \right\|_\infty.$$

Proof. From the inclusion in (5.3), the above inequality is immediately verified. ∎

Remark 5.2.3 *By solving Problem 5.2.2, it is possible to design the low-order controller in which common dynamics is shared between $K(s)$ and $F(s)$. From (5.2) and (5.3) it is seen that $A_{K^*} and C_{K^*}$, which are designed in the first step, are the fixed parameters of $F(s) \in \mathcal{F}(K^*(s))$. Thus, the freedom in the designing parameters of $F(s)$ is restricted by designing of the feedback controller, hence the selection of the feedback controller may affect the minimum value of the objective function in (5.2) if the feedback controller which satisfies the restriction (5.1) is not unique. This also holds when the feedforward controller is designed in the first step. Hence, essentially it is important to make a compromise between those performances simultaneously.*

5.2.2 Simultaneous design

Based on the discussion in the above section, this section considers a *simultaneous TDF controller design* where the feedback controller and the feedforward controller are designed at the same time. In the H_∞ control, it is possible to design the multi-input and multi-output controller, hence the problem of the *simultaneous TDF controller design* can be formulated as a double-input and single-output controller design. Then, in the resultant TDF controller the feedback controller and the feedforward controller share common dynamics. Hence the designing parameters of $F(s)$ come to be restricted by the parameters of $K(s)$. Firstly, let's consider the following problem.

Problem 5.2.3 (Constrained optimization problem) *Find $F(s) \in \mathcal{F}$ and $K(s) \in \mathcal{K}$ which achieve the specifications such that*

$$\begin{aligned} \underset{F(s)\in\mathcal{F},K(s)\in\mathcal{K}}{\text{minimize}} \quad & \|T_{z_1 r}(F(s),K(s))\|_\infty \\ \text{subject to} \quad & \left\|T_{\substack{z_2\\z_3}d}(K(s))\right\|_\infty < \gamma. \end{aligned}$$

Remark 5.2.4 *Problem 5.2.3 is a minimization problem with a constraint. In the problem, the feedforward controller and the feedback controller which satisfy the optimal tracking performance subject to the constraint of the feedback performance are simultaneously designed. Thus, in this problem the controllers are designed in a single step, hence the order of the resultant TDF controller isn't so high as the TDF controller obtained in the two-step design of Problem 5.2.1.*

Our interest in this formulation is the improvement of the tracking performance in comparison with the design via Problem 5.2.2.

Lemma 5.2.2 *Let us denote the solutions of each problem $F_i^*(s)$ and $K_i^*(s)$, where subscript i indicates the number of Problem 5.2.i. Then the following inequality holds.*

$$\|T_{z_1 r}(F_3^*(s),K_3^*(s))\|_\infty \le \|T_{z_1 r}(F_2^*(s),K_2^*(s))\|_\infty \tag{5.4}$$

Proof. $(F_i^*(s), K_i^*(s)), i = 2, 3$ are represented as follows

$$\begin{aligned}(F_2^*(s), K_2^*(s)) &= \underset{(F(s),K^*(s))\in\tilde{\Omega}}{\operatorname{argmin}} \|T_{z_1r}(F(s), K^*(s))\|_\infty \\ (F_3^*(s), K_3^*(s)) &= \underset{(F(s),K(s))\in\Omega}{\operatorname{argmin}} \|T_{z_1r}(F(s), K^*(s))\|_\infty,\end{aligned}$$

where classes of $\tilde{\Omega}$ and Ω are defined as

$$\begin{aligned}\tilde{\Omega} &\triangleq \{(F(s), K^*(s)) \,|F(s) \in \mathcal{F}(K^*(s)), K^*(s) : fixed\} \\ \Omega &\triangleq \left\{(F(s), K(s)) \,|F(s) \in \mathcal{F}, K(s) \in \mathcal{K}, \left\|T_{\substack{z_2\\z_3}d}(K(s))\right\|_\infty < \gamma\right\}.\end{aligned}$$

It can be easily verified that the inclusion:

$$\tilde{\Omega} \subset \Omega$$

holds, hence it is also verified that the inequality (5.4) holds. ■

In the H_∞ problem, it is hard to solve Problem 5.2.3 directly. Hence we solve the following problem instead of solving Problem 5.2.3.

Problem 5.2.4 (Satisficsing problem) *Let us give parameters γ_1 and γ_2 a priori. Find $F(s) \in \mathcal{F}$ and $K(s) \in \mathcal{K}$ which satisfy*

$$\begin{aligned}&\|T_{z_1r}(F(s), K(s))\|_\infty < \gamma_1 \\ &\left\|T_{\substack{z_2\\z_3}d}(K(s))\right\|_\infty < \gamma_2.\end{aligned}$$

Remark 5.2.5 *Problem 5.2.4 is a satisficsing problem and this problem reduces to Problem 5.2.3 if the parameter γ_1 is minimized. In this formulation, by adjusting the parameters γ_1 and γ_2, the trade-off between the feedback performances and the feedforward performance can be taken into account.*

Remark 5.2.6 *Problem 5.2.4 is hard to solve directly. Hence we consider solving the stabilizing controller $(F(s), K(s))$ that satisfies*

$$\left\|\begin{array}{c}\frac{\gamma}{\gamma_1} T_{z_1r}(F(s), K(s)) \\ \frac{\gamma}{\gamma_2} T_{\substack{z_2\\z_3}d}(K(s))\end{array}\right\|_\infty < \gamma.$$

If the controller to this problem is obtained, then it is assured that the controller is a solution for Problem 5.2.4.

Thus Problem 5.2.2, Problem 5.2.3 and Problem 5.2.4 aim at deriving the low-order TDF controller in the sense of sharing common dynamics. In these problems, the objective function $\|T_{z_1r}(F(s), K(s))\|_\infty$ is minimized with the constraint such that $F(s)$ and $K(s)$ have common A-matrix.

On the other hand, as seen in chapter 4, if the constraint is not imposed on the controllers, the objective function can be represented as

$$\begin{aligned} T_{z_1 r}\left(F(s), K(s)\right) &= W_\rho(s)\left(W_M(s) - T_{yr}\left(F(s), K(s)\right)\right) \\ &= W_\rho(s)\left(W_M(s) - N(s)R(s)\right), \end{aligned} \tag{5.5}$$

where $N(s) \in \mathcal{RH}_\infty$ is a coprime factor in the coprime factorization of $P(s) = N(s)M^{-1}(s)$, and $R(s) \in \mathcal{RH}_\infty$ is a free parameter. Then, it can be seen that $T_{z_1 r}\left(F(s), K(s)\right)$ is a function of $R(s)$, which is a design parameter of $F(s)$. This indicates that the constraint which is related to the feedback performances doesn't affect the objective function (5.5) if the controllers $F(s)$ and $K(s)$ are not correlated with each other. Hence, solving Problem 5.2.1 accomplishes the derivation of the highest performance-TDF controller at the sacrifice of simplicity of the controller.

5.3 Low-order TDF controller design

In order to design the low-order controller that holds better performances, we must take into account the selection of the problem formulation. It is the point of the discussion in the previous section. In this section, we focus on a point such that " How to obtain the low-order solution to the problem which we have selected ".

5.3.1 Control specification and construction of the generalized plant

Firstly, we describe the specifications of the TDF control system. The feedback performances are specified with making the norms of the sensitivity function and the co-sensitivity function small. The tracking performance is specified with making the norm of the error between the control output and a desirable response of the TDF system small. Let us consider the generalized plant shown in Figure 5.3, where $W_S(s), W_T(s)$ and ρ, η are the weighting functions. In order to evaluate the feedback performances, the controlled variables z_1 and z_2 are selected as follows

$$\begin{aligned} z_1 &= W_S(s)\tilde{S}(s)w \\ z_2 &= W_T(s)\tilde{T}(s)w, \end{aligned}$$

where $\tilde{S}(s)$ and $\tilde{T}(s)$ are defined as:

$$\begin{aligned} \tilde{S}(s) &\triangleq S(s)\left(I + P(s)\left(K(s) - F(s)\right)\right) \\ \tilde{T}(s) &\triangleq T(s)K^{-1}(s)F(s), \end{aligned}$$

and $S(s)$ and $T(s)$ are represented as follows

$$\begin{aligned} S(s) &= \left(I + P(s)K(s)\right)^{-1} \\ T(s) &= \left(I + P(s)K(s)\right)^{-1} P(s)K(s). \end{aligned}$$

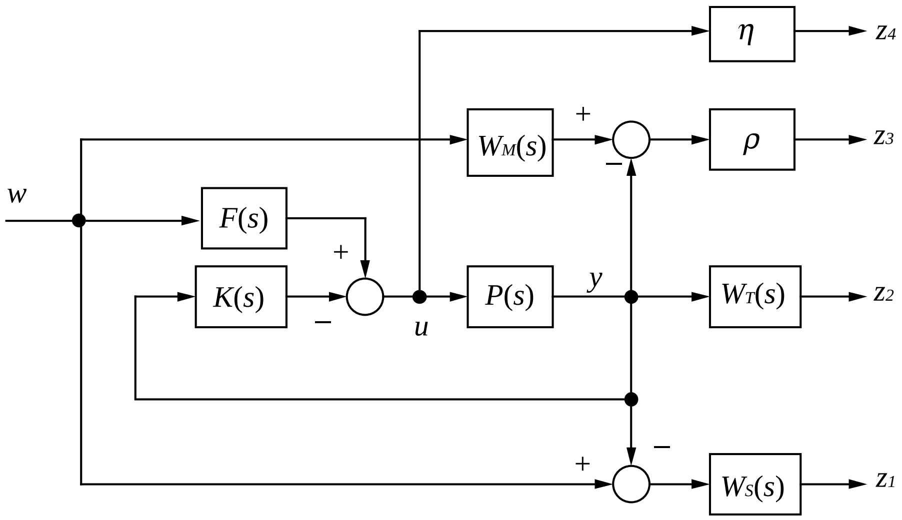

Figure 5.3: A generalized plant

Here it should be noted that $\tilde{S}(s)$ and $\tilde{T}(s)$ satisfy the equation

$$\tilde{S}(j\omega) + \tilde{T}(j\omega) = I, \quad \forall\omega,$$

hence their norm cannot be reduced in the same frequency range. By introducing the weights $W_S(s)$ and $W_T(s)$ into the controlled variable as (5.6) and (5.6) the trade-off between $S(s)$ and $T(s)$ is taken into account.

On the other hand, the controlled variables z_3 and z_4 are represented as follows

$$\begin{aligned} z_3 &= \rho\left(W_M(s) - \tilde{T}(s)\right) w \\ z_4 &= \eta \hat{S}(s) F(s) w, \end{aligned}$$

where

$$\hat{S}(s) \triangleq (I + K(s)P(s))^{-1}$$

and $W_M(s)$ is a transfer function of an ideal tracking performance. These are adopted to specify the tracking performance such that the controlled output y tracks the filtered reference input, which is a desirable and feasible closed loop response under the restriction of control input energy.

In order to introduce integrators into the controller the weighting function $W_S(s)$ is chosen to have the integrators. Since all the modes of weighting function $W_S(s)$ are unobservable from the measured output y, and the assumption **A1-2)** is not satisfied in the generalized plant described in Figure 5.3. As we have discussed in chapter 3, the states of the weighting function are supposed to be observed to satisfy the assumption **A1-2)**. This means that the controller includes the observed mode of the weighting function. Thus, the generalized

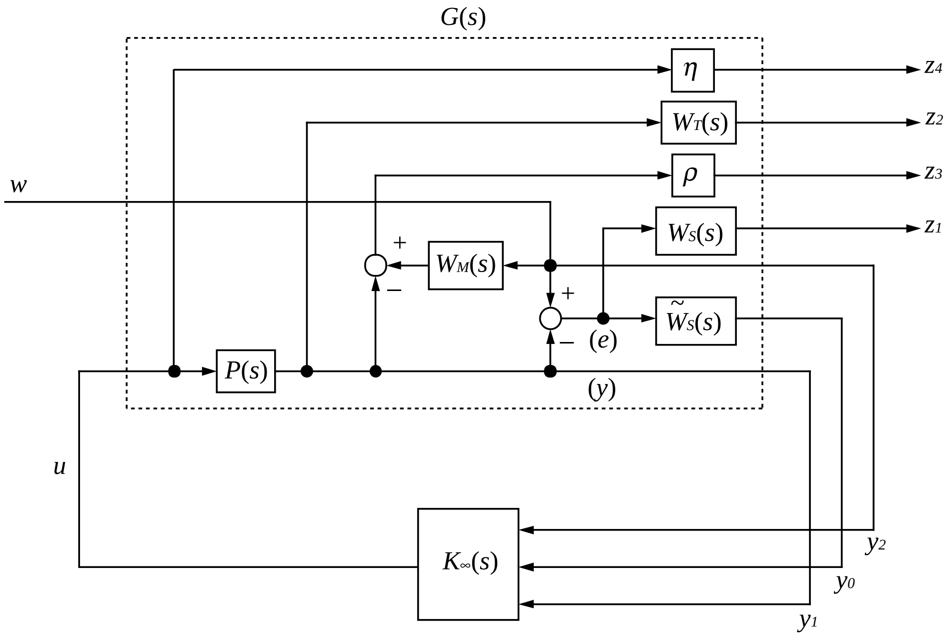

Figure 5.4: A generalized plant

plant described in Figure 5.4 is constructed and the controlled variables $z_i, (i = 1, 2, 3, 4)$ are evaluated simultaneously. By tuning the weighting functions it is possible to consider the trade-off between those specifications. In the generalized plant the H_∞ control problem is formulated such that: find all the internally stabilizing controllers that satisfy

$$\left.\begin{array}{l} \|W_S(s)\tilde{S}(s)\|_\infty < 1 \\ \|W_T(s)\tilde{T}(s)\|_\infty < 1 \\ \|\rho\left(W_M(s) - \tilde{T}(s)\right)\|_\infty < 1 \\ \|\eta\hat{S}(s)F(s)\|_\infty < 1 \end{array}\right\} \tag{5.6}$$

simultaneously. However, this problem is difficult to solve directly, hence let us consider the following criterion alternatively.

$$\left\| \begin{array}{c} W_S(s)\tilde{S}(s) \\ W_T(s)\tilde{T}(s) \\ \rho\left(W_M(s) - \tilde{T}(s)\right) \\ \eta\hat{S}(s)F(s) \end{array} \right\|_\infty < 1 \tag{5.7}$$

If the inequality in (5.7) is satisfied then all the conditions in (5.6) is satisfied.

5.3.2 State-space representation of the generalized plant

Now, let us define the state-space realizations of transfer functions in Figure 5.4, and represent the generalized plant with the state-space form.

The real plant $P(s)$ is assumed to be a system of n_p-dimensional, m-input and p-output system, and is represented as follows:

$$P(s) \triangleq \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & O \end{array}\right], A_P \in \mathbb{R}^{n_p \times n_p}, B_P \in \mathbb{R}^{n_p \times m}, C_P \in \mathbb{R}^{p \times n_p}, \tag{5.8}$$

where we assume without loss of generality that the p outputs of the system are linearly independent – or equivalently that the matrix C_P is of full row rank, and that (A_P, B_P, C_P) is controllable and observable.

The realizations of weighting functions are represented as follows:

$$\begin{aligned}
\left[\begin{array}{c} W_S(s) \\ \hdashline \tilde{W}_S(s) \end{array}\right] &\triangleq \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & O \\ \hdashline \alpha I_{n_w} & O \end{array}\right], A_W \in \mathbb{R}^{n_w \times n_w} \\
W_M(s) &\triangleq \left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & O \end{array}\right], A_M \in \mathbb{R}^{n_m \times n_m} \\
W_T(s)P(s) &\triangleq \left[\begin{array}{c|c} A_P & B_P \\ \hline C_T & D_T \end{array}\right],
\end{aligned}$$

where the parameter α is defined as follows:

$$\alpha \triangleq \begin{cases} 1, & \exists i \in \mathbb{Z},\ \mathrm{Re}(\lambda_i(A_W)) = 0 \\ 0, & \forall i \in \mathbb{Z},\ \mathrm{Re}(\lambda_i(A_W)) < 0 \end{cases}.$$

Then the generalized plant $G(s)$ can be represented as follows.

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right] = \left[\begin{array}{ccc|c:c} A_W & -B_W C_P & O & B_W & O \\ O & A_P & O & O & B_P \\ O & O & A_M & B_M & O \\ \hline C_W & O & O & O & O \\ O & C_T & O & O & D_T \\ O & -\rho C_P & \rho C_M & O & O \\ O & O & O & O & \eta I_m \\ \hdashline \alpha I_{n_w} & O & O & O & O \\ O & C_P & O & O & O \\ O & O & O & I_p & O \end{array}\right] \tag{5.9}$$

The criterion in inequality (5.7) is equal to the following inequality:

$$\left\| \mathcal{F}_l\left(G(s), K_\infty(s)\right) \right\|_\infty < 1. \tag{5.10}$$

Since, in this generalized plant $G(s)$, D_{21} is column full rank and the assumptions A1 and A3 are satisfied, on the other hand, the assumption **A2-2)** is not satisfied, the problem of finding the internally stabilizing controller which satisfies the inequality (5.10) is the aforementioned non-standard H_∞ control problem of case 2.

5.3.3 Parametrization of all H_∞ controllers

Firstly, let us introduce a change of the coordinates in the system $P(s)$ represented with the realization in (5.8). Since C_P is assumed to be a matrix of full row rank, by using a nonsingular matrix:

$$T \triangleq \begin{bmatrix} C_P \\ (C_P^{\perp}) \end{bmatrix}$$

the coordinates can be transformed, and then the state-space realization of $P(s)$ is represented as follows:

$$\begin{aligned}\left[\begin{array}{c|c} \bar{A}_P & \bar{B}_P \\ \hline \bar{C}_P & O \end{array}\right] &\triangleq \left[\begin{array}{c|c} T\bar{A}_P T^{-1} & T\bar{B}_P \\ \hline \bar{C}_P T^{-1} & O \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_{11} & A_{12} & B_{P_1} \\ A_{21} & A_{22} & B_{P_2} \\ \hline I_p & O & O \end{array}\right].\end{aligned}$$

Hence, we assume without generality that the parameters A_P and C_P are in the form

$$\left[\begin{array}{c} A_P \\ \hline C_P \end{array}\right] = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \\ \hline I_p & O \end{array}\right], A_{11} \in \mathbb{R}^{p\times p}, A_{22} \in \mathbb{R}^{(n_p-p)\times(n_p-p)}, \tag{5.11}$$

where it is well known that (A_{22}, A_{12}) is observable [42].

Lemma 5.3.1 *In (5.11), it is assumed without loss of generality that real parts of all the eigenvalues of A_{22} are negative.*

Proof. Since (A_{22}, A_{12}) is observable, there exists a nonsingular matrix:

$$J = \begin{bmatrix} I_p & O \\ L & I_{(n_p-p)} \end{bmatrix}$$

such that J satisfies

$$\left[\begin{array}{c} J^{-1}A_P J \\ \hdashline C_P J \end{array}\right] = \left[\begin{array}{cc} A_{11} + A_{12}L & A_{12} \\ -LA_{11} - LA_{12}L + A_{21} + A_{22}L & -LA_{12} + A_{22} \\ \hdashline I_p & O \end{array}\right],$$

where $\mathbf{Re}\,(\lambda_i(-LA_{12} + A_{22})) < 0, \forall i = 1, 2, \dots, n_p - p$. Hence, in the form of (5.11), it is assumed that real parts of all the eigenvalues of A_{22} are negative. ∎

From the definition in section 2.3, $D_{21}^{\dagger}, D_{21}^{\perp}$ can be given as follows.

$$D_{21}^{\dagger} = \begin{bmatrix} O & O & I_p \end{bmatrix}, D_{21}^{\perp} = \begin{bmatrix} I_{n_w} & O & O \\ O & I_p & O \end{bmatrix}$$

Then, $A - B_1 D_{21}^{\dagger} C_2$ and $D_{21}^{\perp} C_2$ can be represented as follows.

$$\left[\begin{array}{c} A - B_1 D_{21}^{\dagger} C_2 \\ \hdashline D_{21}^{\perp} C_2 \end{array}\right] = \left[\begin{array}{ccc} A_W & -B_W C_P & O \\ O & A_P & O \\ O & O & A_M \\ \hdashline \alpha I_{n_w} & O & O \\ O & C_P & O \end{array}\right]$$

Let us select L_H such that the observable subspace of the pair $(A - B_1 D_{21}^{\dagger} C_2, D_{21}^{\perp} C_2)$ is stabilized by $D_{21}^{\perp} C_2$, i. e., let us select L_H:

$$L_H = \begin{bmatrix} h_1 & B_W \\ O & h_2 \\ O & -A_{21} \\ l_1 & l_2 \end{bmatrix}$$

such that

$$\mathrm{Re}(\lambda_i(A_W + \alpha h_1)) < 0, \forall i = 1, 2, \dots, n_w, \ \mathrm{Re}(\lambda_i(A_{11} + h_2)) < 0, \ \forall i = 1, 2, \dots, p \quad (5.12)$$

are satisfied, and $l_1 \in \mathbb{R}^{n_m \times n_w}, l_2 \in \mathbb{R}^{n_w \times p}$ are arbitrary matrices.

Then, A_{ZH} is represented as follows

$$\begin{aligned} A_{ZH} &= A - B_1 D_{21}^{\dagger} C_2 + L_H D_{21}^{\perp} C_2 \\ &= \left[\begin{array}{c|cc|c} A_W + \alpha h_1 & O & O & O \\ \hline O & A_{11} + h_2 & A_{12} & O \\ O & O & A_{22} & O \\ \hline \alpha l_1 & l_2 & O & A_M \end{array}\right], \end{aligned} \quad (5.13)$$

where $\mathbf{Re}\,(\lambda_i(A_{22})) < 0, \forall i = 1, 2, \dots, n_p - p$.

Based on the above preparation, the H_∞ controller for the generalized plant (5.9) can be obtained from the following lemma.

Lemma 5.3.2 *The H_∞ problem for $G(s)$ in (5.9) is solvable if and only if both of the AREs:*

$$X A_{ZF} + A_{ZF}^T X + XRX + (D_{12}^{\perp} C_1)^T D_{12}^{\perp} C_1 = O \quad (5.14)$$

$$Y A_{ZH}^T + A_{ZH} Y + Y \left\{ C_1^T C_1 - (D_{21}^{\dagger} C_2)^T D_{21}^{\dagger} C_2 \right\} Y = O, \quad (5.15)$$

where A_{ZF} and R are defined as :

$$\begin{aligned} A_{ZF} &\triangleq A - B_2 D_{12}^{\dagger} C_1 \\ R &\triangleq B_1 B_1^T - B_2 D_{12}^{\dagger} (B_2 D_{12}^{\dagger})^T, \end{aligned}$$

and A_{ZH} is represented in (5.13), have stabilizing solutions $X \geq O$ and $Y \geq O$, and which satisfy $\rho\,(XY) < 1$. If the solvability condition is satisfied, the class of the H_∞ controllers are parametrized as follows:

$$K_\infty(s) = \mathcal{F}_l \left(\left[\begin{array}{c|c|c} A_Y + \hat{B}_2 F_\infty & H_\infty & -\hat{B}_2 \Sigma \\ \hline -F_\infty & O & \Sigma \\ \hline D_{21}^{\dagger} \hat{C}_2 Z & D_{21}^{\dagger} & O \\ D_{21}^{\perp} C_2 & D_{21}^{\perp} & O \end{array}\right], \begin{bmatrix} N(s) & W(s) \end{bmatrix} \right), \quad (5.16)$$

where

$$\begin{aligned}
&A_Y = A + YC_1^TC_1 + H_\infty C_2 \\
&\hat{B}_2 = B_2 + YC_1^TD_{12} \\
&\hat{C}_2 = D_{21}B_1^TX + C_2 \\
&F_\infty = \left\{ -D_{12}^\dagger C_1 - D_{12}^\dagger \left(B_2D_{12}^\dagger\right)^T X \right\} Z \\
&H_\infty = -B_1D_{21}^\dagger - Y\left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger + L_HD_{21}^\perp \\
&Z = (I - YX)^{-1} \\
&\Sigma = \left(D_{12}^TD_{12}\right)^{-\frac{1}{2}},
\end{aligned}$$

and $N(s) \in \mathcal{BH}_\infty$ and $W(s) \in \mathcal{RH}_\infty$ are the free parameters.

Proof. By applying the result of Theorem 2.4.2 for $G(s)$ in (5.9), the result is immediately obtained. ■

Remark 5.3.1 *The McMillan degree of the central solution, which is a controller given with the free parameters are set to zero, is equal to the degree of the generalized plant, and is $n = n_w + n_p + n_m$.*

Remark 5.3.2 *It should be also noted that the class of the non-standard H_∞ controllers is represented with two free parameters $N(s) \in \mathcal{BH}_\infty$ and $W(s) \in \mathcal{RH}_\infty$.*

Continuously, we review the H_∞ controller given in Lemma 5.3.2 as an integral-type TDF controller. In this paragraph it is assumed that the H_∞ controller implies the central solution. From Figure 5.4, let us extract the closed loop system composed of $P(s), K_\infty(s)$ and $\tilde{W}_S(s)$. The closed loop system is described in Figure 5.5. Here, it is assumed that $\tilde{W}_S(s)$ has integrators, hence the parameter α is set to 1. If we denote $K_\infty(s)$ according to the dimensions of y_0, y_1 and y_2 as

$$K_\infty(s) = \begin{bmatrix} K_1(s) & K_2(s) & K_3(s) \end{bmatrix}$$

the control input variable u can be represented as follows

$$\begin{aligned}
u &= K_\infty(s) \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \\
&= \begin{bmatrix} K_1(s) & K_2(s) & K_3(s) \end{bmatrix} \begin{bmatrix} \tilde{W}_S(s)\,(r-y) \\ y \\ r \end{bmatrix} \\
&= \begin{bmatrix} F(s) & K(s) \end{bmatrix} \begin{bmatrix} r \\ y \end{bmatrix},
\end{aligned}$$

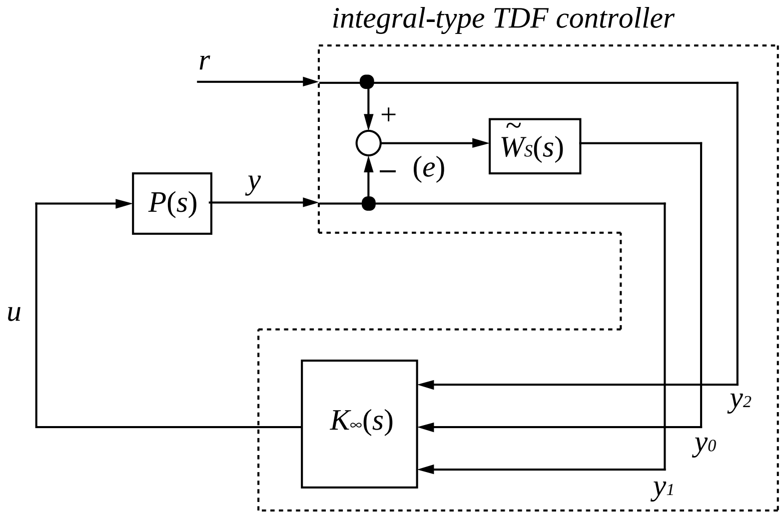

Figure 5.5: Interpretation of the closed loop system as a TDF control system

where $F(s)$ and $K(s)$ are defined as follows

$$F(s) \triangleq K_1(s)\tilde{W}_S(s) + K_3(s)$$
$$K(s) \triangleq K_2(s) - K_1(s)\tilde{W}_S(s).$$

Thus the TDF controller which satisfies prescribed H_∞ performances can be represented. It should be noted that the TDF controller is integral-type. It can be seen that the McMillan degree of the TDF controller is

$$\deg\left(F(s), K(s)\right) = 2n_w + n_p + n_m. \tag{5.17}$$

Compared with the order of the TDF controller evaluated in section 4.2.2, in this synthesis the TDF controller which is also an integral-type can be designed with lower order.

5.3.4 Reduced-order TDF controller design

As we have considered in chapter 2, it is possible to derive the reduced-order H_∞ controller in the non-standard problem, where the dimension of the observed output is greater than that of the disturbance input. This subsection derives the reduced-order TDF controller based on the way that we presented in chapter 2.

The following lemma is useful for later discussion.

Lemma 5.3.3 *Assume that the ARE in (5.15) has a positive semi-definite stabilizing solution. Then the stabilizing solution of the ARE can be represented as follows:*

$$Y = \begin{bmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & Y_r \end{bmatrix} \in \mathbb{R}^{(n_p+n_w+n_m)\times(n_p+n_w+n_m)}, \tag{5.18}$$

where $Y_r \geq O$ is a stabilizing solution of the reduced-order ARE:

$$Y_r A_M^T + A_M Y_r + \rho^2 Y_r C_M^T C_M Y_r = O. \tag{5.19}$$

Proof. The ARE in (5.15) can be represented as follows

$$Y A_Y^T + A_{ZH} Y = O, \tag{5.20}$$

where

$$A_Y = A_{ZH} + Y \left\{ C_1^T C_1 - \left(D_{21}^\dagger C_2 \right)^T D_{21}^\dagger C_2 \right\} \in \mathbb{R}$$

is a stable matrix. Let U be a row-basis of the stable subspace of A_{ZH}. Then the matrix U can be selected as

$$U = \left[\begin{array}{c|cc|c} I_{n_w} & O & O & O \\ \hline O & I_p & O & O \\ O & O & I_{n_p - p} & O \end{array} \right].$$

Then the matrix U satisfies the following equation:

$$U A_{ZH} = \Lambda U,$$

where

$$\Lambda \triangleq \left[\begin{array}{c|cc} A_W + \alpha h_1 & O & O \\ \hline O & A_{11} + h_2 & A_{12} \\ O & O & A_{22} \end{array} \right] \in \mathbb{R}^{(n_w + n_p) \times (n_w + n_p)}$$

is a stable matrix. Pre-multiplying the equation (5.20) by U, the following equation is obtained

$$(UY) A_Y^T + \Lambda (UY) = O.$$

Since $\mathbf{Re}(\lambda_i(\Lambda)) < 0$ and $\mathbf{Re}(\lambda_i(A_Y)) < 0$ hold, the solution $Y \geq O$ satisfies $UY = O$ and Y must be in the form (5.18). By substituting $Y \geq O$ in (5.18) into the ARE in (5.15), $Y_r \geq O$ must be a solution of the reduced-order ARE in (5.19). ■

Let us represent the controller given in (5.16) with DHMT as follows

$$\begin{aligned} K_\infty(s) &= \mathcal{DHM}\left(\left[\begin{array}{c|c|c} A_Y & -\hat{B}_2 & -H_\infty \\ \hline \Sigma^{-1} F_\infty & \Sigma^{-1} & O \\ \hline -D_{21}^\dagger \hat{C}_2 Z & O & D_{21}^\dagger \\ -D_{21}^\perp C_2 & O & D_{21}^\perp \end{array} \right], \left[\begin{array}{cc} N(s) & W(s) \end{array} \right] \right) \\ &= \left[\begin{array}{c|c} A_Y & -\hat{B}_2 \\ \hline C_K(s) & \Sigma^{-1} \end{array} \right]^{-1} \left[\begin{array}{c|c} A_Y & -H_\infty \\ \hline C_K(s) & N(s) D_{21}^\dagger + W(s) D_{21}^\perp \end{array} \right], \end{aligned}$$

where $C_K(s)$ is given as follows

$$C_K(s) \triangleq \Sigma^{-1} F_\infty - N(s) D_{21}^\dagger \hat{C}_2 Z - W(s) D_{21}^\perp C_2. \tag{5.21}$$

By using the result of Lemma 5.3.3 the matrix A_Y can be represented as follows

$$A_Y = \begin{bmatrix} A_W + \alpha h_1 & O & O & O \\ O & A_{11} + h_2 & A_{12} & O \\ O & O & A_{22} & O \\ \alpha l_1 & l_2 - \rho^2 Y_r C_M^T & O & A_M + \rho^2 Y_r C_M^T C_M \end{bmatrix}, \tag{5.22}$$

where $Y_r \geq O$ is a stabilizing solution of the reduced-order ARE in (5.19), and $l_1 \in \mathbb{R}^{n_m \times n_w}$ and $l_2 \in \mathbb{R}^{n_w \times p}$ are free parameters. In the equation (5.21), $C_K(s)$ is explicitly represented with the free parameter $W(s) \in \mathcal{RH}_\infty$. Hence an adequate selection of the free-parameters of l_1, l_2 and $W(s)$ yields pole-zero cancellations in the controller, and the order of the controller can be reduced.

Theorem 5.3.1 *Under the same solvability condition as in Lemma 5.3.2, the class of reduced order H_∞ controllers is parametrized as follows:*

$$K_\infty^r(s) = \mathcal{DHM}\left(\left[\begin{array}{c|cc} \tilde{A}_Y & -\hat{B}_{P_2} & -H_{\infty 2} \\ \hline \Sigma^{-1} F_{\infty 2} & \Sigma^{-1} & \Sigma^{-1} \hat{F}_{\infty 1} D_{21}^{\perp} \\ -C_{D_2} & O & D_{21}^{\dagger} - \hat{C}_{D_1} D_{21}^{\perp} \end{array} \right], N(s) \right), \tag{5.23}$$

where $N(s) \in \mathcal{BH}_\infty$ is a free parameter, and $\tilde{A}_Y, \hat{B}_{P_2}, H_{\infty 2}, \hat{F}_{\infty 1}, F_{\infty 2}, \hat{C}_{D_1}, C_{D_2}$ are defined as follows:

$$\begin{aligned}
\tilde{A}_Y &= \begin{bmatrix} A_{22} & O \\ O & A_M + \rho^2 Y_r C_M^T C_M \end{bmatrix} \in \mathbb{R}^{(n-(n_w+p)) \times (n-(n_w+p))} \\
\hat{B}_{P_2} &= \begin{bmatrix} B_{P_2} \\ O \end{bmatrix} \in \mathbb{R}^{(n-(n_w+p)) \times m} \\
H_{\infty 2} &= \begin{bmatrix} O & -A_{21} & O \\ O & \rho^2 Y_r C_M^T & -B_M \end{bmatrix} \\
\hat{F}_{\infty 1} &= F_\infty \begin{bmatrix} \alpha^{-1} I_{n_w} & O \\ O & I_p \\ O & O \end{bmatrix} \\
F_{\infty 2} &= F_\infty \begin{bmatrix} O \\ I_{n-(n_w+p)} \end{bmatrix} \\
\hat{C}_{D_1} &= D_{21}^{\dagger} \hat{C}_2 Z \begin{bmatrix} \alpha^{-1} I_{n_w} & O \\ O & I_p \\ O & O \end{bmatrix} \\
C_{D_2} &= D_{21}^{\dagger} \hat{C}_2 Z \begin{bmatrix} O \\ I_{n-(n_w+p)} \end{bmatrix}.
\end{aligned}$$

Proof. By setting the free parameters as

$$\begin{cases} l_1 = O \\ l_2 = \rho^2 Y_r C_M^T \\ W(s) = \left(\Sigma^{-1} F_\infty - N(s) D_{21}^{\dagger} \hat{C}_2 Z \right) V \end{cases},$$

where V is a full column rank matrix defined as

$$V \triangleq \begin{bmatrix} \alpha^{-1}I_{n_w} & O \\ O & I_p \\ O & O \\ O & O \end{bmatrix} \in \mathbb{R}^{(n_p+n_w+n_m)\times(n_w+p)},$$

the matrices A_Y and $C_K(s)$ satisfy

$$\begin{cases} A_Y V = V \begin{bmatrix} A_W + \alpha h_1 & O \\ O & A_{11} + h_2 \end{bmatrix} \\ C_K(s)V = O \end{cases}.$$

Hence the pair $(A_Y, C_K(s))$ is unobservable, and the order of the controller is reduced by the dimension of $\operatorname{rank}(V) = n_w + p$. Then the reduced order H_∞ controller is derived. ■

Remark 5.3.3 *The McMillan degree of the central solution of the H_∞ controller in (5.16) is reduced by the dimension of* $\operatorname{rank}(V) = n_w + p$*, and is equal to $n_p - p + n_m$.*

Remark 5.3.4 *It should be also noted that the class of the reduced-order H_∞ controllers is represented with free parameter $N(s) \in \mathcal{BH}_\infty$.*

Remark 5.3.5 *In this theorem the reduced order TDF controller is represented with DHMT, and this implies that the controller is in a form of coprime factorization over $\mathcal{RH}_\infty$. This form is useful for reducing the order of the controller by approximation, because no model approximation methods can be used directly for unstable system.*

We review the reduced-order H_∞ controller given in Theorem 5.3.1 as an integral-type TDF controller. In this paragraph it is assumed that the H_∞ controller implies the central solution. As we have discussed in the previous section, let us extract the closed loop system composed of $P(s), K_\infty^r(s)$ and $\tilde{W}_S(s)$. The closed loop system is shown in Figure 5.6. If we denote $K_\infty^r(s)$ according to the dimensions of y_0, y_1 and y_2 as

$$K_\infty^r(s) = \begin{bmatrix} K_1^r(s) & K_2^r(s) & K_3^r(s) \end{bmatrix}$$

the control input variable u can be represented as follows

$$\begin{aligned} u &= K_\infty^r(s) \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} K_1^r(s) & K_2^r(s) & K_3^r(s) \end{bmatrix} \begin{bmatrix} \tilde{W}_S(s)(r-y) \\ y \\ r \end{bmatrix} \\ &= \begin{bmatrix} F^r(s) & K^r(s) \end{bmatrix} \begin{bmatrix} r \\ y \end{bmatrix}, \end{aligned}$$

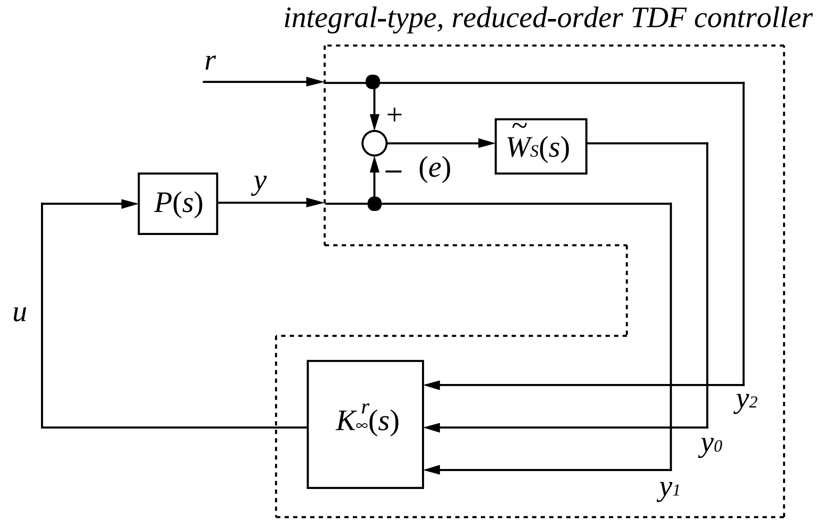

Figure 5.6: Interpretation of the closed loop system as a TDF control system

where $F^r(s)$ and $K^r(s)$ are defined as follows

$$F^r(s) \triangleq K_1^r(s)\tilde{W}_S(s) + K_3^r(s)$$
$$K^r(s) \triangleq K_2^r(s) - K_1^r(s)\tilde{W}_S(s).$$

Thus the reduced-order TDF controller which satisfies prescribed H_∞ performances can be represented. It should be noted that the reduced-order TDF controller is integral-type. It can be seen that the McMillan degree of the reduced-order TDF controller is

$$\deg\left(F^r(s), K^r(s)\right) = n_w + n_p + n_m - p. \tag{5.24}$$

Compared with the order of the TDF controller evaluated in section 4.2.2, in this synthesis the TDF controller which is also integral type can be designed with further lower order. It is also apparent that

$$\deg\left(F(s), K(s)\right) - \deg\left(F^r(s), K^r(s)\right) = n_w + p$$

holds. This order reduction is a benefit which is yielded by the formulation of the non-standard H_∞ control.

5.4 A numerical example

5.4.1 Magnetic levitation system

This section examines the effectiveness of the proposed method by numerical examples. We use the model of the magnetic levitation system that we have used in chapter 2. Let us consider the system shown in the Figure 2.6. The purpose of controlling the magnetic levitation system in this chapter is to stabilize the system regardless of variation of the mass, and

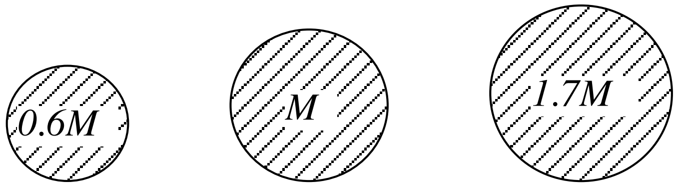

Figure 5.7: Steel balls

besides, to obtain a good tracking property with an integral-type TDF controller. The order of the controller should be obtained with lower order.

5.4.2 Description of perturbed models

In the linear model of equation (2.94), three types of steel balls are assumed. Mass of each steel ball is set as

$$M_1 = 0.6M, \quad M, \quad M_2 = 1.7M.$$

Then the transfer function of each model is obtained as

$$\begin{aligned} P_1(s) &= \frac{-111.71}{(s-61.6)(s+14.8)(s+60.1)}, \text{ where mass is } M_1; \\ P(s) &= \frac{-67.03}{(s-47.8)(s+46.3)(s+14.9)}, \text{ where mass is } M; \\ P_2(s) &= \frac{-39.4}{(s-36.8)(s+15.0)(s+35.1)}, \text{ where mass is } M_2. \end{aligned}$$

We call $P(s)$ nominal model, $P_1(s)$ and $P_2(s)$ perturbed models.

Our purpose of controlling the magnetic levitation system is to stabilize the system regardless of variation of mass with one controller. For this purpose we consider to design a robust controller which stabilizes any system in

$$\mathcal{P}_{real} \triangleq \{P(s;\alpha M) \mid \alpha = [0.6, 1.7]\}. \tag{5.25}$$

For this system we set a boundary of the perturbation such that a partial system from the input u to the output y_3 varies in a set

$$\mathcal{P} = \{(1+\Delta(s))\,P(s) : |\Delta(j\omega)| \le |W_T(j\omega)|\,, \forall\omega\}, \tag{5.26}$$

where $W_T(s)$ is a function of the relative error bound between the perturbed model and $P(s)$. In this thesis, by trial and error, we select $W_T(s)$ as

$$W_T(s) = 0.6 \times 10^{-10}(10^{-2}s^3 + s^2 + 10^8 s + 1),$$

such that the plant set $\mathcal{P}$ includes set of perturbed plants $\mathcal{P}_{real}$. Figure 5.8 plots the gain of the weighting function $W_T(s)P(s)$ with violet line, variations of the models $P_1(s)$ and $P_2(s)$ from the nominal model $P(s)$ with lines of blue and red. Here, it can be verified that the gain of the boundary is not less than that of any variation.

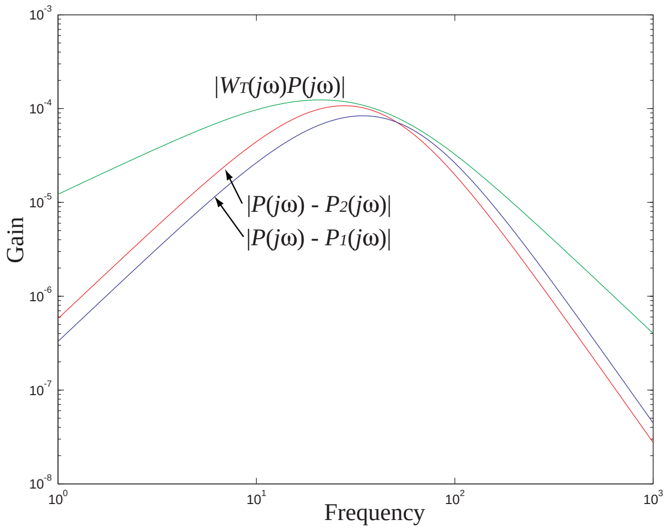

Figure 5.8: Gain plots of perturbations and weighting function

5.4.3 Designing of low-order integral-type TDF controller

We design an integral-type TDF controller based on the method that is proposed in section 5.3. The generalized plant of equation (5.9) is made by using the weighting functions:

$$W_S(s) = \frac{32700}{s(s+1000)} \tag{5.27}$$

$$W_T(s) = 0.6 \times 10^{-10}(10^{-2}s^3 + s^2 + 10^8 s + 1) \tag{5.28}$$

$$W_M(s) = \frac{130^2}{s^2 + 247s + 130^2} \tag{5.29}$$

$$\eta = 1.5 \times 10^{-4} \tag{5.30}$$

$$\rho = 0.2. \tag{5.31}$$

Then 7th order generalized plant is obtained. For the generalized plant, we first obtained a 7th order H_∞ controller whose order is same as that of the generalized plant. By selecting an adequate free parameter $W(s)$ we obtained a 4th order H_∞ controller, and converted the controller into an integral-type. Then we got an integral-type TDF controller of order 6. The frequency response of the TDF controller is shown in Figure 5.9-(a). From the figure it can be seen that the feedback controller and the feedforward controller share common dynamics and they include an integrator. The frequency responses of sensitivity function $S(s)$ and co-sensitivity function $T(s)$ are shown in Figure 5.9-(b). It can be seen that the reduced-order controller satisfies the feedback performances such that

$$\begin{cases} \|W_S(s)S(s)\|_\infty < 1 \\ \|W_T(s)T(s)\|_\infty < 1 \end{cases}.$$

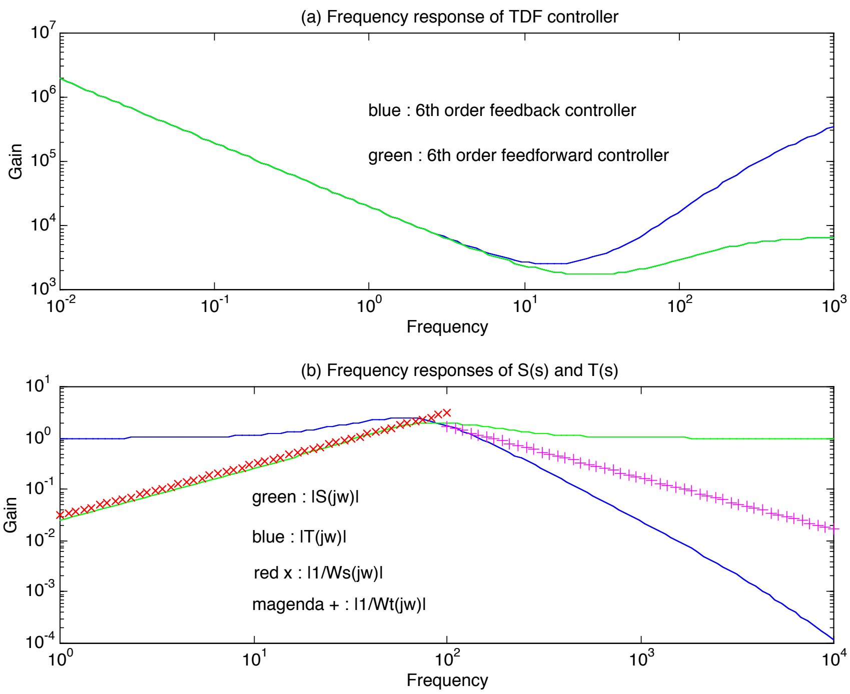

Figure 5.9: Frequency responses of (a) 6th order TDF controller, and (b) closed loop transfer functions.

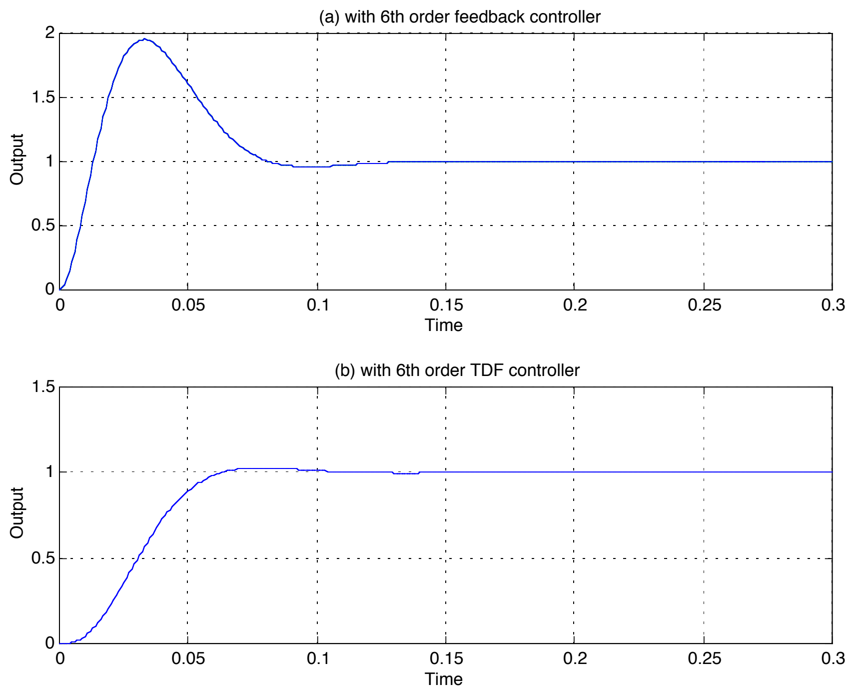

Figure 5.10: Step responses of (a) a feedback control system composed of the nominal plant and the 6th order feedback controller, and (b) a TDF control system composed of the nominal plant and the 6th order TDF controller.

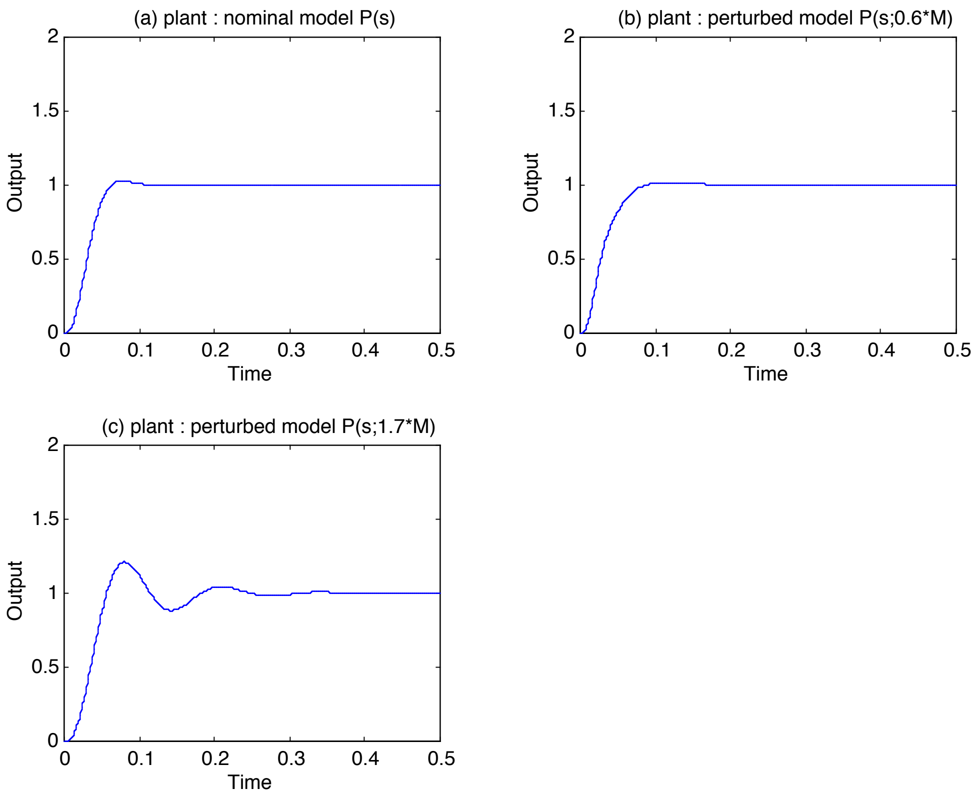

Figure 5.11: Step responses of the perturbed systems

Figure 5.10 illustrates step responses of closed loop systems: (a) a feedback control system composed of the nominal plant and the 6th order feedback controller, (b) a TDF control system composed of the nominal plant and the 6th order TDF controller. From this figure, it can be seen that the tracking performance can be improved by adding the feedforward controller to the feedback control system. In order to verify the robustness of the TDF system, step responses of the closed loop systems, in which the perturbed plants are used, are illustrated in Figure 5.11; in (a) the nominal model is used, in (b) a perturbed model $P_1(s)$ is used, in (c) a perturbed model $P_2(s)$ is used. It can be seen that the closed loop system holds stability within the prescribed variation of the plant.

5.4.4 Comparison with another method

To evaluate the performances of the TDF controller that is obtained with the proposed method, other TDF controllers are also designed with the other method, which is an independent sequential approach in which the feedback controller and the feedforward controller are respectively designed with the standard H_∞ control. First, a feedback controller is obtained by solving the mixed sensitivity problem. The generalized plant of the problem is illustrated in Figure 3.2. Two types of sensitivity weighting functions are adopted. One of the sensitivity functions is type of non-optimal sensitivity function, where the gain of the weight $W_S(s)$ is not so high that there is room for improving the robustness. Here, the weight is used as the following one.

$$W_{S_1}(s) = \frac{4110}{s(s+1000)}$$

We call a mixed sensitivity problem in which $W_{S_1}(s)$ is adopted type1 problem. The other type of the problem is with sensitivity function of maximal gain. The weight is used as

$$W_{S_2}(s) = \frac{482}{s(s+1.0)},$$

which has higher gain than $W_{S_1}(s)$ has in low frequency range. We call this type of problem type 2 problem. On the other hand, the weighting function $W_T(s)$ is used the same one that is used in equation (5.28) in each type problem. Since the order of each generalized plant is five, the order of each resultant feedback controller is also five. The feedback properties: frequency responses of the sensitivity function and co-sensitivity function, which are obtained in those types of problems, are illustrated in Figure 5.12. In (a), the result of the type 1 problem is plotted, in (b) the result of the type 2 problem is plotted.

Second, using each feedback controller, a feedforward controller is designed by solving the model matching problem whose generalized plant is illustrated in Figure 5.13. The weighting functions that are used in the generalized plant are as follows

$$\begin{aligned} W_M(s) &= \frac{130^2}{s^2+247s+130^2} \\ W_\rho(s) &= \frac{2.0}{s} \\ \eta &= 4.0\times 10^{-5}. \end{aligned}$$

Since the order of the generalized plant is 11, an 11th order feedforward controller is obtained in each type of problem. Hence, each resultant TDF controller is 16th order, which is higher by 10th order than that of the TDF controller designed with the proposed method. Then, by using the balanced truncation method [1, 52], each 16th order controller is approximated with lower order TDF controllers. Here, in order to preserve the steady-state property of the feedforward controller $F(s)$, the truncated feedforward controller $\tilde{F}_r(s)$ is scaled as follows.

$$F_r(s) = \tilde{F}_r(s) \times \frac{F(0)}{\tilde{F}_r(0)}$$

The approximated TDF controllers have variation according to the combination of orders of feedback controller and feedforward controller.

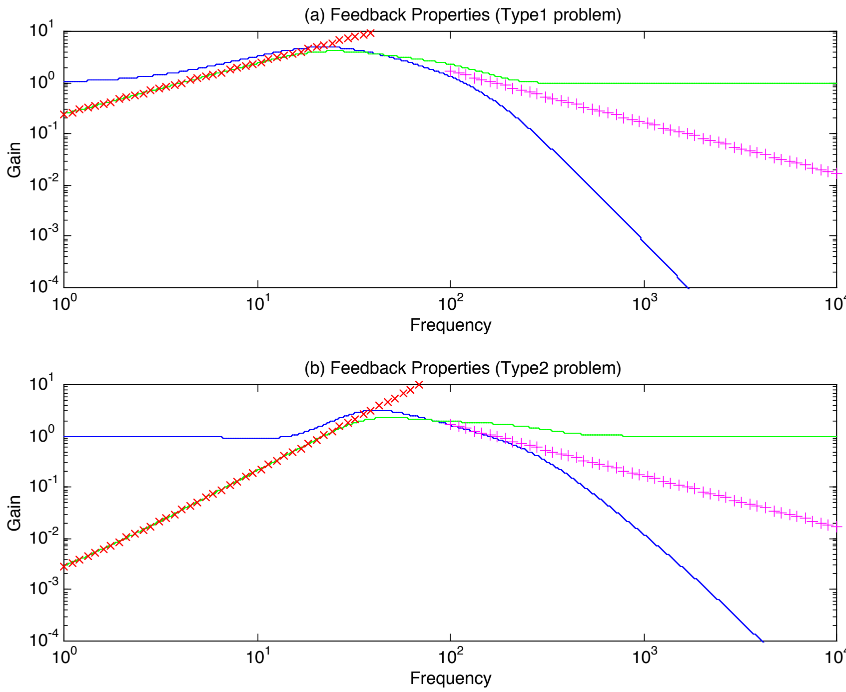

Figure 5.12: The feedback properties of the controllers obtained by solving the problems: (a) Type 1 problem, (b) Type 2 problem. Green line : $|S(j\omega)|$; Blue line : $|T(j\omega)|$; Red $\times$: $|W_S^{-1}(j\omega)|$; Magenta $+$: $|W_T^{-1}(j\omega)|$.

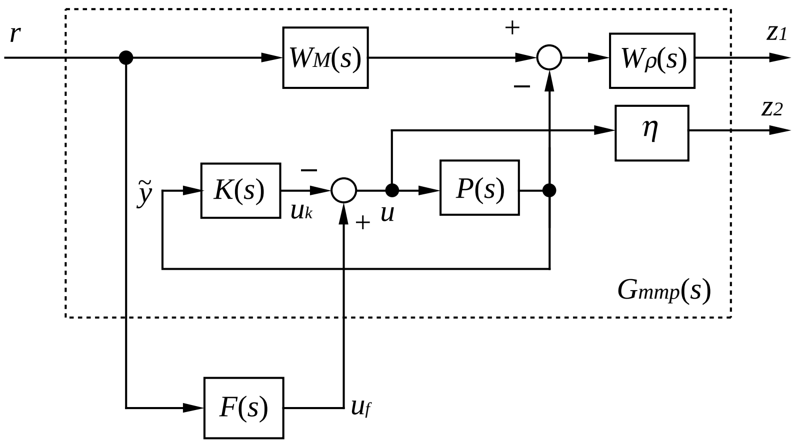

Figure 5.13: The generalized plant of the model matching problem

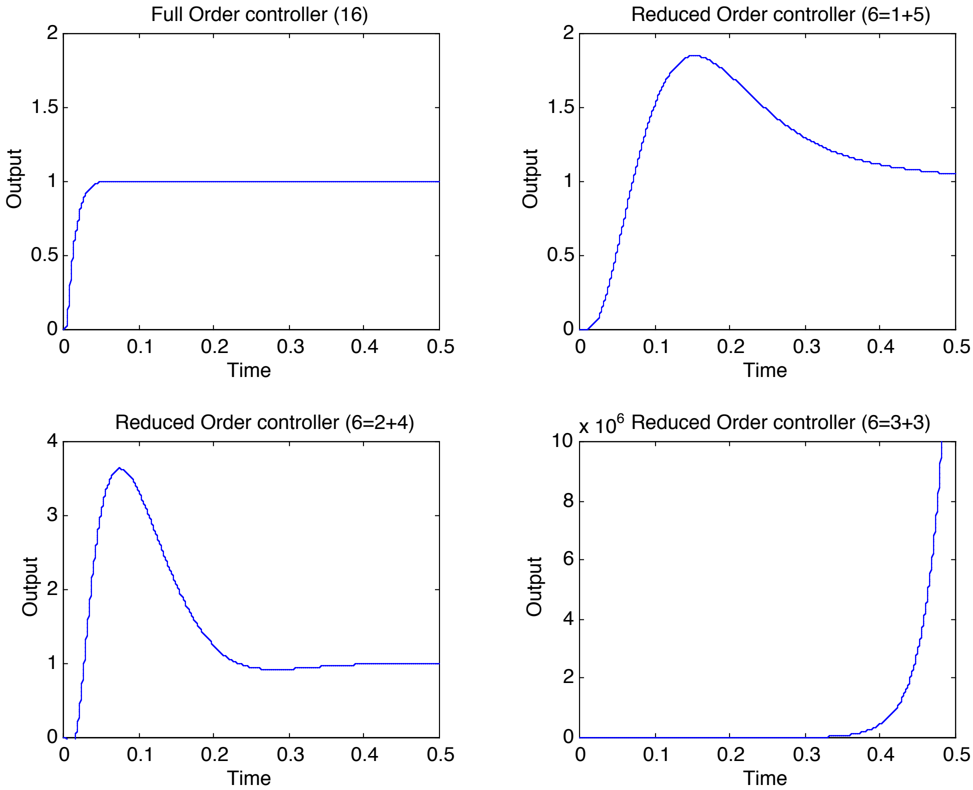

Figure 5.14: The step responses

Step response of each closed loop system, which is composed of the nominal plant and each controller derived from the type 1 problem, are plotted. In (a), a 16th order TDF controller is used. In (b), a 6th order TDF controller, which is given with a first order reduced order feedforward controller and a 5th order feedback controller, is used. In (c), a 6th order TDF controller, which is given with a second order reduced order feedforward controller and a 4th order reduced order feedback controller, is used. In (d), a 6th order TDF controller, which is given with a 3rd order reduced order feedforward controller and a 3rd order reduced order feedback controller, is used.

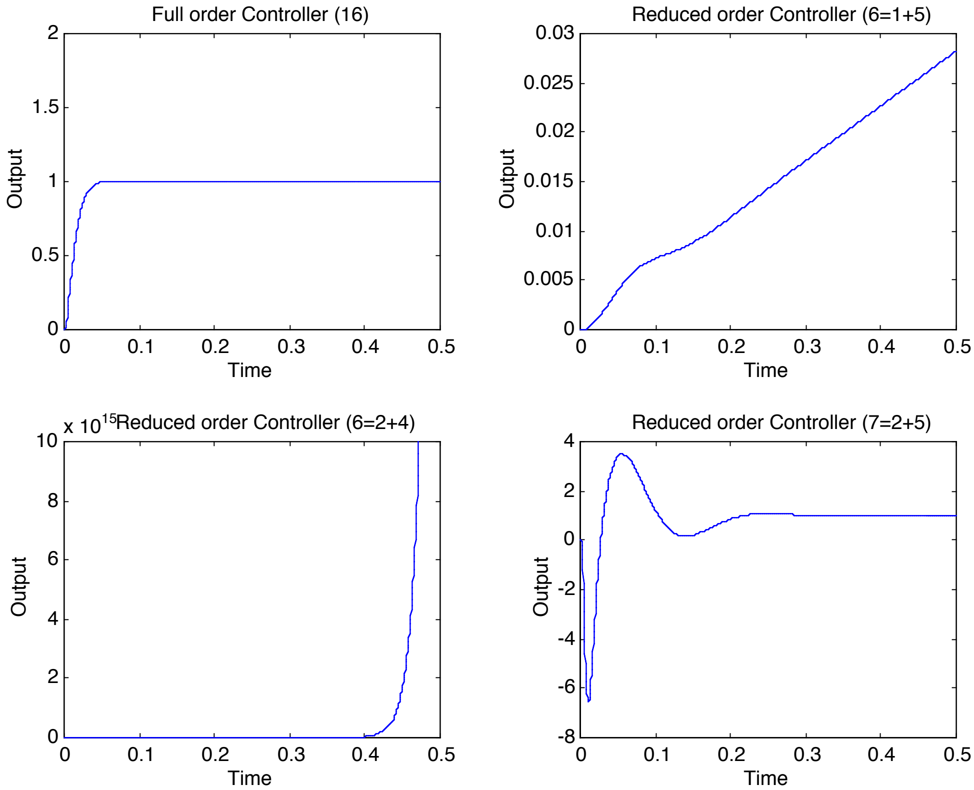

Figure 5.15: The step responses

Step response of each closed loop system, which is composed of the nominal plant and each controller derived from the type 2 problem, are plotted. In (a), a 16th order TDF controller is used. In (b), a 6th order TDF controller, which is given with a first order reduced order feedforward controller and a 5th order feedback controller, is used. In (c), a 6th order TDF controller, which is given with a second order reduced order feedforward controller and a 4th order reduced order feedback controller, is used. In (d), a 7th order TDF controller, which is given with a second order reduced order feedforward controller and a 5th order reduced order feedback controller, is used.

Figure 5.14 plots step responses of closed loop systems in which 16th order TDF controller and its reduced order controllers derived from the type 1 problem are adopted. Figure 5.15 plots step responses of closed loop systems in which 16th order TDF controller and its reduced order controllers derived from the type 2 problem are adopted. From these figures, it can be seen that each 16th order TDF controller has better tracking property than that of the 6th order TDF controller that is obtained with the proposed method. It is pretty natural that a higher order controller performs better than a lower order controller. On the other hand, compared with the 6th order controller obtained with the proposed method, none of the reduced order controllers perform better.

5.5 Summary

In this chapter, we have considered designing a low-order TDF controller. Based on the analysis in the previous chapter, formulations of designing the low-order TDF controller are compared with each other. The result of the comparison has suggested that a method of simultaneously designing the feedback controller and the feedforward controller is the best way to derive the low-order TDF controller. Then we have chosen the way of simultaneous approach. Another reason why we have adopted the simultaneous approach is that the formulation of the simultaneous approach reduces to the non-standard H_∞ problem, where the way of designing the reduced-order controller has just been clarified. Thus we have proposed a method of designing the low-order TDF controller based on the H_∞ control. In this method, we can design the TDF controller whose order is lower than the order of the generalized plant by the sum of the dimensions of the measurement output of the real plant and the order of the weighting function of the sensitivity function. In this method, the TDF controller can be designed with the integral-type by using the result of chapter 3. Effectiveness of the proposed method is verified with a numerical example of controlling the magnetic levitation system.

Chapter 6

H_∞ controller approximation

6.1 Introduction

In the previous chapters, we have considered the *direct* approach to deriving the low-order H_∞ controller. On the other hand, in this chapter we consider the *indirect* approach where an approximation method is used for reducing the order of an H_∞ controller.

In this chapter, we treat a specialized solution of the numerically solved H_∞ controller, which is represented by the linear fractional transformation of an optimized Youla parameter [69] in $\mathcal{RH}_\infty$. The numerical approach [3, 43, 54] is useful for solving the H_∞ problem with a constraint of specifying the closed loop poles. In many cases, the numerical approach derives an extremely high-order Youla parameter and the resultant H_∞ controller becomes high order. However, it is unknown how to reduce the order of the H_∞ controller without deviating the constraint. In this chapter, we consider the possibility of reducing the order of the H_∞ controller by approximating the Youla parameter.

When reducing the order of the high-order H_∞ controller which satisfies the constraint of the closed loop pole location, it is required to reduce the order of the controller such that the specified closed-loop property is preserved. For this requirement, in this study we firstly propose a model approximation method for a system whose poles are located in a specified domain. Since this approximation method assures that the poles of the approximated system locate in the domain in which the poles of the original system locate, it is useful to apply this method for the H_∞ controller reduction problem which we consider in this chapter. Thus we consider the problem of reducing the order of the H_∞ controller based on the approximation. One of advantages of this reduction method is that the reduced-order controller maintains the closed-loop properties of the original controller, such as internal stability and closed-loop pole specification. A sufficient condition for the closed loop transfer function satisfies the constraint of the H_∞ norm is derived. The effectiveness of this H_∞ controller reduction method is verified with a numerical example.

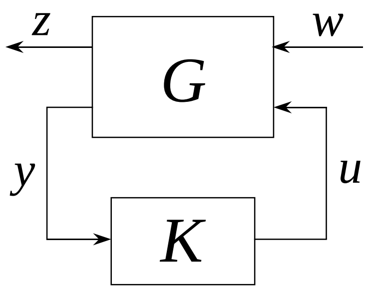

Figure 6.1: Closed loop system

6.2 A numerical approach to H_∞ controller design

6.2.1 Youla parametrization

Consider the following generalized plant $G(s)$ which is a transfer function matrix from $(w^T, u^T)^T$ to $(z^T, y^T)^T$, where vectors w, u, z and y represent the exogenous signals such as perturbation and disturbances, the control input signals from actuators or controllers, the controlled output signals to be evaluated and the measured output signals, respectively.

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

The generalized plant $G(s)$ is composed of the plant and all weighting functions which specify the robustness and the other closed loop specifications. When the controller $K(s)$ is connected between y and u, the closed loop transfer function from w to z is represented by the following equation.

$$T_{zw}(s) = \mathcal{F}_l\left(G(s), K(s)\right)$$

The block diagram of the closed loop system is shown in Figure 6.1.

It is known that if one controller $K_{nom}(s)$ which stabilizes the closed loop system is given, all the controllers which stabilize the closed loop system can be represented by using a stable and proper transfer function $Q(s)$. Assume that a plant $P(s)$ is a minimal scalar system, and is represented with realization

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

If we select a feedback gain F and an observer gain H such that matrices $A_F \triangleq A + BF$ and $A_H \triangleq A + HC$ are stable, then all of the controllers which stabilize $P(s)$ can be represented as follows:

$$K(s) = \frac{X(s) + Q(s)M(s)}{Y(s) - Q(s)N(s)}, \tag{6.1}$$

where $Q(s) \in \mathcal{RH}_\infty$ is a free parameter, and transfer functions $N(s), M(s), X(s)$ and $Y(s)$ are represented as follows

$$\left[\begin{array}{c} N(s) \\ \hdashline M(s) \end{array}\right] = \left[\begin{array}{c|c} A_F & B \\ \hline C + DF & D \\ \hdashline F & 1 \end{array}\right]$$

$$\left[\begin{array}{c:c} X(s) & Y(s) \end{array}\right] = \left[\begin{array}{c|c:c} A_H & H & -B - HD \\ \hline F & 0 & 1 \end{array}\right].$$

This representation of the stabilizing controllers is called Youla parametrization. It should be noted that $K(s)$ in (6.1) stabilizes the closed loop system composed of $P(s)$ and $K(s)$, if and only if $Q(s) \in \mathcal{RH}_\infty$.

In general, it is known that by using the Youla parametrization, arbitrary closed loop transfer function $T_{zw}(s)$ can be represented with the following affine combination of stable transfer functions:

$$T_{zw}(s) \quad = \quad T_1(s) - T_2(s)Q(s)T_3(s) \triangleq F_{affine}\left(Q(s)\right), \tag{6.2}$$

where $T_1(s)$, $T_2(s)$ and $T_3(s)$ are certain stable transfer functions related to the plant, the weighting function and the stabilizing controller, and $Q(s)$ represents a free parameter. The closed loop system is stable, if and only if we choose $Q(s)$ from the set of all real rational stable functions.

For example, let's consider a problem of sensitivity minimization. In this problem the transfer function is selected as

$$T_{zw}(s) = W_S(s)\,\frac{1}{1 + P(s)K(s)},$$

where $W_S(s) \in \mathcal{RH}_\infty$ is a weighting function. If we note the fact that equations:

$$P(s) = \frac{N(s)}{M(s)}, \quad X(s)N(s) + Y(s)M(s) = 1$$

hold, it is verified that the transfer function $T_{zw}(s)$ can be represented as follows

$$T_{zw}(s) = T_1(s) - T_2(s)Q(s)T_3(s),$$

where

$$\begin{aligned} T_1(s) &\triangleq W_S(s)M(s)Y(s) \\ T_2(s) &\triangleq W_S(s)M(s)N(s) \\ T_3(s) &\triangleq 1. \end{aligned}$$

The representation of the closed loop system in (6.2) is useful for designing an H_∞ controller which specifies the location of the closed loop poles. Let us denote $\mathcal{RH}_\infty^D$ as a class of stable proper transfer functions whose poles lie in the disk domain $Dom(D)$. Figure 6.2

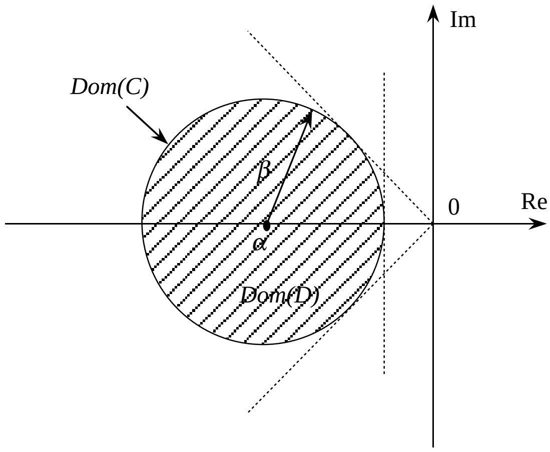

Figure 6.2: Disk domain

illustrates $Dom(D)$. The set of closed loop poles is the union of the poles of the transfer functions $T_1(s)$, $T_2(s), T_3(s)$ and $Q(s)$. Since the poles of transfer functions $T_1(s), T_2(s)$ and $T_3(s)$ can be selected from arbitrary complex numbers on the left half plane by the selection of F, H and weighting functions, adding a constraint such that $Q(s) \in \mathcal{RH}_\infty^D$, the H_∞ controller which locates closed loop poles in a specified domain $Dom(D)$ can be obtained. We call the problem of finding $Q(s) \in \mathcal{RH}_\infty^D$ which satisfies the constraint of H_∞-norm *the H_∞ problem with pole specification.* The problem is formulated as follows.

Problem 6.2.1 The H_∞ problem with pole specification
Suppose that the transfer functions $T_i(s) \in \mathcal{RH}_\infty^D$ are given. Then, find a stable system $Q(s)$ that achieves

$$\|T_1(s) - T_2(s)Q(s)T_3(s)\|_\infty < 1$$

such that

$$Q(s) \in \mathcal{RH}_\infty^D.$$

If such a $Q(s)$ is obtained, the controller in a request is given by substituting $Q(s)$ into the equation in (6.1).

6.2.2 A numerical approach to controller design

In order to solve the problem stated above numerically, we must search a solution over an infinite-dimensional function space $\mathcal{RH}_\infty$. Then, we focus on an approximation stated as follows.

Let us define a sequence of transfer functions $Q_k(s) \in \mathcal{RH}_\infty, k = 1, 2, \ldots, N$, and let us denote transfer functions as follows

$$R_0 \triangleq T_1(s)$$
$$R_k \triangleq T_2(s)Q_k(s)T_3(s).$$

Then, restricting the closed loop transfer function $T_{zw}(s)$ to a finite-dimensional function space as

$$H_N(x) \triangleq R_0 + \sum_{k=1}^{N} x_k R_k, \quad x_k \in \mathbb{R},$$

we optimize a function

$$\phi(x) \triangleq \phi\left(H_N(x)\right),$$

where ϕ is a function which specifies some performance of the control system. This restriction is called Ritz Approximation. In this approximation, it is assured that $\phi(x)$ converges to an optimal value as $N \to \infty$ [3, 54]. To each $x \in \mathbb{R}^N$ there corresponds the controller

$$K(x) = \mathcal{F}_l\left(K_0(s), \sum_{k=1}^{N} x_k Q_k(s)\right)$$

that achieves the closed loop specification $\phi\left(H_N(x)\right) < \gamma$.

Now, in order to solve Problem 6.2.1, we put the free parameter $Q(s)$ as

$$Q(s) = \sum_{k=1}^{N} x_k Q_k(s) \triangleq \sum_{k=1}^{N} x_k \left(\frac{\alpha}{s+\alpha}\right)^k, \tag{6.3}$$

where $\alpha > 0$ is chosen such that $Q_k(s)$ satisfies $Q_k(s) \in \mathcal{RH}_\infty^D$ and $x_k, (k = 1, 2, \ldots, N)$ are finite-dimensional parameters. In this problem the function ϕ indicates the H_∞-norm. Since the function of the H_∞-norm satisfies

$$\|\beta H_a + (1-\beta)H_b\|_\infty \le \beta\|H_a\|_\infty + (1-\beta)\|H_b\|_\infty, \quad \forall\beta \in [0, 1],$$

where H_a and H_b are arbitrary transfer functions in $\mathcal{RH}_\infty$, it is apparent that ϕ is a convex function. Hence, if we put the free parameter $Q(s)$ as (6.3), Problem 6.2.1 can be reduced to a finite-dimensional convex optimization problem.

6.2.3 A defect of the numerical approach

Thus, Problem 6.2.1 can be solved by using a convex optimization method such as ellipsoid algorithm [3]. However, sometimes the order of the free parameter becomes very high, causing the controller also high order. In order to understand this mechanism, let us represent the controller in (6.1) as

$$K(s) = \mathcal{F}_l\left(K_0(s), Q(s)\right),$$

where $K_0(s)$ is defined as

$$K_0(s) \triangleq \begin{bmatrix} X(s)Y^{-1}(s) & M(s) + X(s)Y^{-1}(s)N(s) \\ Y^{-1}(s) & Y^{-1}(s)N(s) \end{bmatrix},$$

and let us introduce the state-space form of $K_0(s)$ and $Q(s)$ as follows

$$K_0(s) = \left[\begin{array}{c|cc} A_k & B_{k_1} & B_{k_2} \\ \hline C_{k_1} & D_{k_{11}} & D_{k_{12}} \\ C_{k_2} & D_{k_{21}} & D_{k_{22}} \end{array}\right]$$
$$Q(s) = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right].$$

Then the state-space form of the controller $K(s) = \mathcal{F}_l\left(K_0(s), Q(s)\right)$ can be represented as follows.

$$K(s) = \left[\begin{array}{cc|c} A_k + B_{k_2}D_qEC_{k_2} & B_{k_2}(I + D_qED_{K_{22}})C_q & B_{k_1} + B_{k_2}D_qED_{k_{21}} \\ B_qEC_{k_2} & A_q + B_qED_{k_{22}}C_q & B_qED_{k_{21}} \\ \hline C_{k_1} + D_{k_{12}}D_qEC_{k_2} & D_{k_{12}}(I + D_qED_{k_{22}})C_q & D_{k_{11}} + D_{k_{12}}D_qED_{k_{21}} \end{array}\right], \tag{6.4}$$

where $E = \left(I - D_{k_{22}}D_q\right)^{-1}$ and $K_0(s)$. From this representation of $K(s)$ it can be seen that the McMillan degree of $K(s)$ can be estimated as follows

$$\deg\left(K(s)\right) \leq \deg\left(K_0(s)\right) + \deg\left(Q(s)\right). \tag{6.5}$$

In the above relation, the inequality holds when pole-zero cancellations occur between $K_0(s)$ and $Q(s)$. In such a case, the order of the controller is reduced to some degree. However, it is not always expected that the pole-zero cancellation reduces the order of the controller to some required order, rather it is ordinary that the pole-zero cancellation doesn't occur. Thus, in this numerical approach the increase in the order of the resultant controller is one of difficulties, hence the controller order reduction is required.

6.3 The controller reduction by the approximation of $Q(s)$

The relation in (6.5) informs us an upper bound of the order of the resultant controller. In this study we consider to reduce the order of the upper bound in order to reduce $\deg\left(K(s)\right)$. In this context, it is useful to reduce the order of $Q(s)$. When approximating $Q(s)$, it is important to consider the preservation of the closed loop specifications such as the closed loop stability, the designation of closed loop pole position and the H_∞ norm constraint. Suppose that $Q(s)$ is approximated by $Q_r(s)$. The conditions of the stability and the pole position are satisfied if $Q_r(s) \in \mathcal{RH}_\infty^D$. Then we have only to consider the constraint on the norm in order to derive the low-order H_∞ controller which satisfies the pole specification. Now, it is required to consider a model approximation method which is closed on $\mathcal{RH}_\infty^D$.

6.3.1 Balanced truncation

Now, let us introduce one of the most useful model approximation method. Balanced truncation [52, 56, 43] is one of the most effective model approximation methods for reducing the order of stable systems. We consider to approximate the system $Q(s)$ based on the Balanced truncation. Consider a linear time invariant stable system $Q(s)$ described by

$$Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where it is assumed that the McMillan degree of $Q(s)$ is n, and the pairs $(A, B), (A, C)$ are controllable and observable. The controllability Gramian of $Q(s)$ is defined as

$$\Sigma_c \triangleq \int_0^\infty e^{At} B B^T e^{A^T t} dt,$$

and is given by the unique positive definite solution to the Lyapunov equation

$$\Sigma_c A^T + A\Sigma_c + BB^T = O.$$

The observability Gramian of $G(s)$ is defined as

$$\Sigma_o \triangleq \int_0^\infty e^{A^T t} C^T C e^{At} dt,$$

and is given by the unique positive definite solution to the Lyapunov equation

$$\Sigma_o A + A^T \Sigma_o + C^T C = O.$$

Positive square roots of eigenvalues of $\Sigma_o \Sigma_c$ are called Hankel singular values of $Q(s)$, and they are invariant under a nonsingular state transformation of $Q(s)$. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ be the Hankel singular values of $Q(s)$. The system is said to be *internally balanced* if the Gramians satisfy the equation

$$\Sigma_c = \Sigma_o = \Sigma,$$

where $\Sigma = \mathrm{diag}\,(\sigma_1, \sigma_2, \cdots, \sigma_n)$. Let us partition Σ accordingly into

$$\Sigma = \begin{bmatrix} \Sigma_a & O \\ O & \Sigma_b \end{bmatrix},$$

where $\Sigma_a = \mathrm{diag}\,(\sigma_1, \sigma_2, \cdots, \sigma_r)$ and $\Sigma_b = \mathrm{diag}\,(\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n)$.

Suppose that $Q(s)$ is *internally balanced.* Let us partition A, B, C conformably with the partitions of Σ.

$$Q(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \quad A_{11} \in \mathbb{R}^{r\times r}, A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$$

Then an approximated system of $Q(s)$ is given by an r-state reduced-order system:

$$Q_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right].$$

Concerned about the approximation the following lemma is known [53].

Lemma 6.3.1 *Let $Q(s)$ be stable and minimal system with n states. Let $r < n$ and let $Q_r(s)$ be a reduced-order system obtained by approximating $Q(s)$ with Balanced truncation. Then the following items hold.*

1) $Q_r(s)$ is stable and minimal system.
2) $\|Q(s) - Q_r(s)\|_\infty \leq 2\mathrm{trace}\Sigma_b$.

Remark 6.3.1 *Balanced truncation is a method to approximate a system in $\mathcal{RH}_\infty$ with a reduced-order system in $\mathcal{RH}_\infty$. If $Q(s) \in \mathcal{RH}_\infty^D$ is approximated by $Q_r(s)$, it is only assured that $Q_r(s) \in \mathcal{RH}_\infty$. Hence the reduced-order system may not satisfy the pole specification of the original system.*

In the following section we will consider a model reduction method which preserves the specified domain in which the poles locate.

6.3.2 Model approximation with constraint of pole position

Besides the internal stability, the closed loop pole specification to a specified domain is important in controller design [21]. The domain is specified to the left hand side divided by the dashed lines which is shown in Figure 6.2. If the closed loop poles are confined to this domain, then the system modes damp asymptotically at desired rates. However, it is difficult to address this domain directly. Hence, we approximate this domain by $Dom(D)$.

This section proposes a model approximation method that preserves the domain in which the poles locate. Here, the domain is limited to a disk domain. Outline of our idea concerned with the approximation method is listed as follows.

1. The coordinate of the original system is transformed using an affine transformation and a bilinear transformation. Then the disk domain on the complex left half plane is transformed to the complex left half plane of the changed coordinate. Poles of the original system are mapped into the left half plane in the new coordinate.

2. The original system in the new coordinate is approximated to a low-order one using Balanced truncation. Then it should be noted that the poles of the reduced-order system in the new coordinate still locate in the left half plane.

3. Again the reduced-order system is transformed using the inverse of the affine transformation and the bilinear transformation. Then the poles of the reduced-order system in the original coordinate locate in the specified disk on the complex left half plane.

Thus, we can approximate a system, which has poles on a specified domain with a reduced-order system, which still has poles on a specified domain. Now let us move on to details.

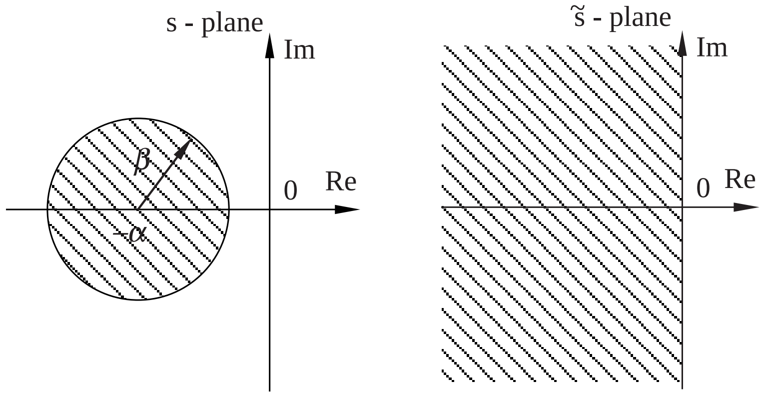

Figure 6.3: s-plane and $\tilde{s}$-plane

Firstly, let us introduce some important transformations. The affine transformation is defined as

$$\frac{s+\alpha}{\beta},$$

where α and β are some real numbers. This transforms $Dom(D)$ to the domain within the unit disk on the complex plane. The bilinear transformation is defined as

$$\tilde{s} = \frac{v-1}{v+1}.$$

This transforms the domain within the unit disk on the complex plane to the complex left half plane. The mixed transformation which transforms $Dom(D)$ to the complex left half plane is described by

$$\tilde{s} = \frac{s+\alpha-\beta}{s+\alpha+\beta}, \tag{6.6}$$

where $\alpha > \beta \neq 0$. This transformation is called a Linear Fractional Transformation (LFT), and is a one-to-one mapping of $\mathcal{S}^2$ onto $\mathcal{S}^2$. LFT has a lot of attractive properties, one of which is given by the following lemma.

Lemma 6.3.2 *Let $Q(s)$ be analytic outside $Dom(D)$. Let the function with the coordinate transformed by LFT be defined as*

$$R(\tilde{s}) \triangleq Q\left(\frac{(-\alpha-\beta)\tilde{s}+\alpha-\beta}{\tilde{s}-1}\right).$$

Then $R(\tilde{s})$ is analytic in the complex right half plane.

Proof. The reader may refer to the Riemann mapping theorem [58]. ∎

Thus, by transforming the coordinate of $Q(s) \in \mathcal{RH}_\infty^D$ with LFT in (6.6) we can obtain another system $R(\tilde{s}) \in \mathcal{RH}_\infty$. In the later discussion, for notational ease $R(\tilde{s})$ is denoted by $R(s)$.

Then we approximate $R(s)$ by a reduced-order system $R_r(s)$. Assume that $R(s)$ is a stable n-th order system. Using Balanced truncation, we can obtain an r-the order reduced-order system. Let $R_r(s)$ be the reduced-order system obtained by Balanced truncation. From Lemma 6.3.1 $R_r(s) \in \mathcal{RH}_\infty$, hence $R_r(s)$ is analytic in the complex right half plane.

Then transform the coordinate of $R_r(s)$ by the inverse of the LFT:

$$s = \frac{(-\alpha - \beta)\tilde{s} + \alpha - \beta}{\tilde{s} - 1}. \tag{6.7}$$

Let the transformed system be $Q_r(s)$. Then the r-the order reduced-order system $Q_r(s)$ is analytic outside $Dom(D)$, i. e., $Q_r(s) \in \mathcal{RH}_\infty^D$. Here, we obtain the following theorem.

Theorem 6.3.1 *Let $Q(s)$ be analytic outside $Dom(D)$. Then the reduced-order system $Q_r(s)$ is analytic outside $Dom(D)$.*

Proof. The system $R(s)$ is analytic in the complex right half plane. The reduced-order system $R_r(s)$ which is obtained by Balanced truncation is also analytic in the complex right half plane. Therefore $Q_r(s)$ obtained by the inverse transformation of LFT in (6.6) is analytic outside $Dom(D)$, because the inverse transformation is also LFT, that is, a one-to-one mapping of $\mathcal{S}^2$ onto $\mathcal{S}^2$. ■

Thus the procedure to approximate $Q(s)$ is listed as follows.

Procedure 6.3.1 *Assume that an original system $Q(s) \in \mathcal{RH}_\infty^D$ is obtained. Then the following procedure derives an approximated system $Q_r(s) \in \mathcal{RH}_\infty^D$.*

step 1: *Transform $Q(s) \in \mathcal{RH}_\infty^D$ to $R(\tilde{s}) \in \mathcal{RH}_\infty$ with LFT in (6.6).*

step 2: *Approximate $R(\tilde{s}) \in \mathcal{RH}_\infty$ by $R_r(\tilde{s}) \in \mathcal{RH}_\infty$ with Balanced truncation.*

step 3: *Transform $R_r(\tilde{s}) \in \mathcal{RH}_\infty$ to $Q_r(s) \in \mathcal{RH}_\infty^D$ with LFT in (6.7).*

Thus, we can reduce the dynamical order of $Q(s) \in \mathcal{RH}_\infty^D$ with preserving the specified domain in which poles locate.

In the second place, we look into the error of the approximation discussed in the above. The following theorem gives an upper bound for the approximation error.

Theorem 6.3.2 *Assume that the H_∞ norm error bound of $R(s) - R_r(s)$ is known to be γ, i.e. the following inequality holds.*

$$\|R(s) - R_r(s)\|_\infty < \gamma \tag{6.8}$$

Then the error bound of $Q(s) - Q_r(s)$ is evaluated with the following inequality.

$$\|Q(s) - Q_r(s)\|_\infty < \gamma \tag{6.9}$$

Proof. Define $E_R(s)$, $E_Q(s)$ as follows.

$$\begin{aligned} E_R(s) &\triangleq R(s) - R_r(s) \\ E_Q(s) &\triangleq Q(s) - Q_r(s) \end{aligned}$$

The systems $Q(s)$ and $Q_r(s)$ are analytic outside $Dom(D)$, hence $E_Q(s)$ is also analytic outside $Dom(D)$. By the maximum modulus principle [58] of the analytical functions we obtain an inequality

$$\sup_{Dom(C)} \{E_Q(s)\} > \sup_{\omega} \{E_Q(j\omega)\}. \tag{6.10}$$

The LFT in (6.6) transforms the value of the function $E_Q(s)$ on $Dom(C)$ to the value of the function $E_R(j\omega)$ on the imaginary axis. Hence the following equation is obtained.

$$\sup_{\omega} \{E_R(j\omega)\} = \sup_{Dom(C)} \{E_Q(s)\}. \tag{6.11}$$

Since $R(s) \in \mathcal{RH}_\infty$ and $R_r(s) \in \mathcal{RH}_\infty$ are analytic in the complex right half plane, $E_R(s)$ is also analytic in the complex right half plane. The H_∞ norm of the function $E_R(s)$ equals the maximal value of the function $E_R(j\omega)$ on the imaginary axis.

$$\|E_R(s)\|_\infty = \sup_{\omega} \{E_R(j\omega)\} \tag{6.12}$$

From equations. (6.11) and (6.12), the following equation is obtained

$$\|E_R(s)\|_\infty = \sup_{Dom(C)} \{E_Q(s)\}. \tag{6.13}$$

If the following inequality

$$\|E_R(s)\|_\infty < \gamma$$

holds, then from equation (6.13), the inequality is obtained

$$\sup_{Dom(C)} \{E_Q(s)\} < \gamma. \tag{6.14}$$

Then from the inequalities (6.10) and (6.14), the following inequality

$$\sup_{\omega} \{E_Q(j\omega)\} = \|E_Q(s)\|_\infty < \gamma$$

holds. ■

Remark 6.3.2 *From Lemma 6.3.1, an upper bound of the reduction error $E_R(s)$ is given by the following inequality.*

$$\|E_R(s)\|_\infty \leq 2\text{trace}, \Sigma_2$$

where Σ_2 *is defined as*

$$\Sigma_2 \triangleq \operatorname{diag}\left(\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n\right)$$

and $\sigma_i, (1 \leq i \leq n)$ *are the Hankel singular values of* $R(s)$. *Hence, using the result of Theorem 6.3.2, the error bound of* $E_Q(s)$ *is evaluated using the Hankel singular values of* $R(s)$ *as follows*

$$\|E_Q(s)\|_\infty < 2\mathrm{trace}\Sigma_2.$$

6.3.3 The H_∞ controller reduction

In this section, by using the approximation method proposed in the previous section we consider to reduce the order of the H_∞ controller which is represented by LFT with a high-order Youla parameter. In order to reduce the order of the H_∞ controller, it is useful to reduce the order of the Youla parameter when it is high order. We apply the approximation method proposed in the above section to reduce the order of the Youla parameter.

Suppose that $Q^*(s) \in \mathcal{RH}_\infty^D$, which is an n-th order solution to Problem 6.2.1 is given, and define the value of the objective function

$$\lambda^* \triangleq \|T_1(s) - T_2(s)Q^*(s)T_3(s)\|_\infty < 1.$$

We approximate $Q^*(s)$ with r-th order system $Q_r(s) \in \mathcal{RH}_\infty^D$ by Procedure 6.3.1, then define the value of the objective function

$$\lambda_r \triangleq \|T_1(s) - T_2(s)Q_r(s)T_3(s)\|_\infty.$$

Then the following lemma holds.

Lemma 6.3.3 *The inequality holds.*

$$\lambda_r - \lambda^* \leq \|T_2(s)\|_\infty \cdot \|Q^*(s) - Q_r(s)\|_\infty \cdot \|T_3(s)\|_\infty \tag{6.15}$$

Proof. It can immediately verified by the following inequality.

$$\begin{aligned}
\lambda_r &= \|T_1(s) - T_2(s)Q^*(s)T_3(s) + T_2(s)\left(Q^*(s) - Q_r(s)\right)T_3(s)\|_\infty \\
&\leq \|T_1(s) - T_2(s)Q^*(s)T_3(s)\|_\infty + \|T_2(s)\left(Q^*(s) - Q_r(s)\right)T_3(s)\|_\infty \\
&\leq \lambda^* + \|T_2(s)\|_\infty \cdot \|Q^*(s) - Q_r(s)\|_\infty \cdot \|T_3(s)\|_\infty
\end{aligned}$$

■

By using the above lemma, we can derive the next result.

Theorem 6.3.3 *Let us denote the Hankel singular values of* $R(s)$ *as* $\sigma_i^R, i = 1, 2, \cdots, n$. *If an inequality*

$$\lambda^* + 2\left\|T_2(s)\right\|_\infty \cdot \mathrm{trace}\ \Sigma_2^R \cdot \|T_3(s)\|_\infty \leq 1, \tag{6.16}$$

where Σ_2^R *implies*

$$\Sigma_2^R \triangleq \mathrm{diag}\left(\sigma_{r+1}^R, \sigma_{r+2}^R, \cdots, \sigma_n^R\right),$$

holds, then $Q_r(s)$ *is a solution to Problem 6.2.1.*

Proof. From Lemma 6.3.1, an inequality

$$\|R(s) - R_r(s)\|_\infty \leq 2 \text{ trace } \Sigma_{R2}$$

holds, hence from Theorem 6.3.2 an inequality

$$\|Q(s) - Q_r(s)\|_\infty < 2 \text{ trace } \Sigma_{R2} \tag{6.17}$$

holds. Then, from Lemma 6.3.3 and (6.17),

$$\lambda_r < \lambda^* + 2\,\|T_2(s)\|_\infty \cdot \text{trace } \Sigma_{R2} \cdot \|T_3(s)\|_\infty$$

holds. Hence, if the inequality (6.16) holds, $\lambda_r < 1$ is satisfied. Then, $Q_r(s) \in \mathcal{RH}_\infty^D$ is a solution to Problem 6.2.1. ■

Substituting $Q_r(s)$ into the representation of the controller in (6.1) derives an approximated controller. Then it is assured that the approximated controller satisfies the constraint of the pole position in the closed loop. A sufficient condition for the approximated controller satisfies the constraint of the H_∞-norm is given in Theorem 6.3.3. Then we have to consider next is to ascertain whether the order of the approximated controller is really reduced or not. In the state space representations of the H_∞ controller in (6.4), if $Q(s)$ is replaced with $Q^*(s)$, the resultant controller is not necessarily minimal, hence the McMillan degree of the controllers may reduce to some order:

$$\deg\left(\mathcal{F}_l\left(K(s), Q^*(s)\right)\right) \leq \deg\left(K_0(s)\right) + \deg\left(Q^*(s)\right).$$

Hence the pole-zero cancellation [61, 36] may cause a case such that the order of the approximated controller is higher than that of the original controller.

In order to avoid such a situation, we look into the pole-zero cancellation. Let us denote the controller $K_0(s)$ as

$$K_0(s) = \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix}.$$

Then it is known the property concerned about the pole-zero cancellation.

Lemma 6.3.4 *Every mode of the unobservable subspace in* $\mathcal{F}_l\left(K_0(s), Q(s)\right)$ *is the invariant zero of* $K_{12}(s)$*. Every mode of the uncontrollable subspace in* $\mathcal{F}_l\left(K_0(s), Q(s)\right)$ *is the invariant zero of* $K_{21}(s)$*.*

Proof. See reference [37, 36].

By using the above lemma, we obtain the following result.

Theorem 6.3.4 *Suppose that sum of the numbers of the invariant zeros of $K_{12}(s)$ and $K_{21}(s)$ is $TZ > 0$. Then, if the inequality*

$$\deg\left(Q^*(s)\right) - \deg\left(Q_r(s)\right) > TZ$$

holds, the following inequality holds.

$$\deg\left(\mathcal{F}_l\left(K_0(s), Q^*(s)\right)\right) > \deg\left(\mathcal{F}_l\left(K_0(s), Q_r(s)\right)\right) \tag{6.18}$$

holds.

Proof. Suppose that the inequality

$$\deg\left(Q^*(s)\right) - \deg\left(Q_r(s)\right) > TZ \tag{6.19}$$

holds. From Lemma 6.3.4, the pole-zero cancellations in $\mathcal{F}_l\left(K_0(s), Q^*(s)\right)$ may happen within its unobservable mode and uncontrollable mode. Hence, an inequality

$$\deg\left(\mathcal{F}_l\left(K_0(s), Q^*(s)\right)\right) \geq \deg\left(K_0(s)\right) + \deg\left(Q^*(s)\right) - TZ \tag{6.20}$$

holds. From (6.19) and (6.20), the following inequality holds.

$$\deg\left(\mathcal{F}_l\left(K_0(s), Q^*(s)\right)\right) > \deg\left(K_0(s)\right) + \deg\left(Q_r(s)\right) \tag{6.21}$$

On the other hand, $\deg\left(\mathcal{F}_l\left(K_0(s), Q_r(s)\right)\right)$ satisfies

$$\deg\left(\mathcal{F}_l\left(K_0(s), Q_r(s)\right)\right) \leq \deg\left(K_0(s)\right) + \deg\left(Q_r(s)\right). \tag{6.22}$$

Hence, from (6.21) and (6.22), (6.18) holds. ■

6.4 A numerical example

Consider the following 17-th order system $Q^*(s)$ which is obtained as a solution to Problem 6.2.1. The pole positions of $Q^*(s)$ are illustrated in Figure 6.4. Let the domain in which the poles are located be denoted $Dom(D)$ with center -100 + j0 and radius 99.5 on the complex plane, hence the parameters α and β are set as $\alpha = 100, \beta = 99.5$. The 12-th order reduced-order system which is obtained by the proposed approximation method is denoted by $Q_{12prop}(s)$ and the 12-th order reduced-order system obtained by Balanced truncation is denoted by $Q_{12BT}(s)$. The pole positions of these systems are illustrated in Figure 6.5 and Figure 6.6, respectively. From these figures, it is apparent that all the poles of $Q_{12prop}(s)$ are located within $Dom(D)$, while some poles of $Q_{12BT}(s)$ are located outside $Dom(D)$. The poles of the closed loop transfer functions $F_{affine}\left(Q_{12prop}(s)\right)$ and $F_{affine}\left(Q_{12BT}(s)\right)$ are

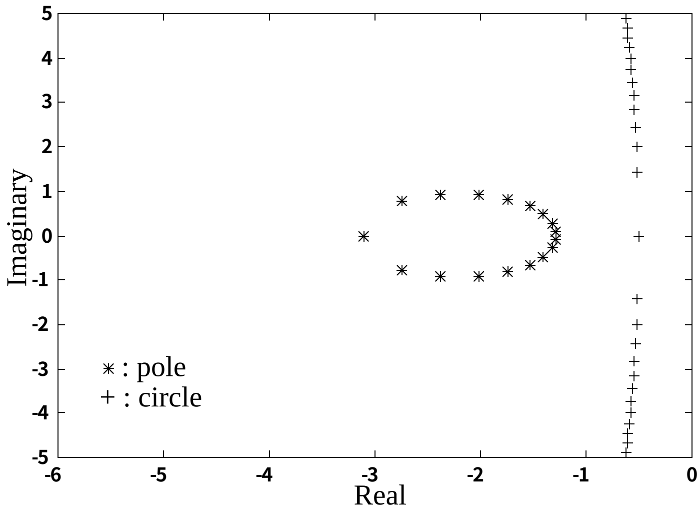

Figure 6.4: Pole position of $Q^*(s)$

illustrated in Figure 6.7 and Figure 6.8, respectively. It is also apparent that the closed loop pole specification is satisfied by using the proposed model reduction method, on the other hand, the specification is not satisfied by using Balanced truncation only. The H_∞ norm of the approximated closed loop transfer function is evaluated by using the following inequality.

$$\begin{aligned} \|F_{affine}\left(Q_{12prop}(s)\right)\|_\infty &= \|T_1(s) - T_2(s)Q(s)T_3(s) + T_2(s)\left(Q(s) - Q_{12prop}(s)\right)T_3(s)\|_\infty \\ &\leq \|T_1(s) - T_2(s)Q(s)T_3(s)\|_\infty + \|T_2(s)\left(Q(s) - Q_{12prop}(s)\right)T_3(s)\|_\infty \\ &\leq \|F_{affine}\left(Q(s)\right)\|_\infty + \|T_2(s)\|_\infty \cdot \|T_3(s)\|_\infty \cdot \|Q(s) - Q_{12prop}(s)\|_\infty \end{aligned} \tag{6.23}$$

We can check the H_∞ norm condition by using the above inequality and the result of Theorem 6.3.2. In fact, the H_∞ norm of the approximated closed loop transfer function is evaluated as

$$\|F_{affine}\left(Q_{12prop}(s)\right)\|_\infty < 0.965. \tag{6.24}$$

Hence, the reduced-order H_∞ controller $\mathcal{F}_l\left(K_0(s), (Q_{12prop}(s)\right)$ which satisfies the pole specification can be obtained.

6.5 Summary and discussion

In this chapter we have proposed a controller reduction method for a H_∞ controller which is designed with the numerical approach based on the optimization of the Youla parameter. This method is based on the approximation of the Youla parameter that is designed to satisfy the constraints of the closed loop system. Thus the parameter holds information of

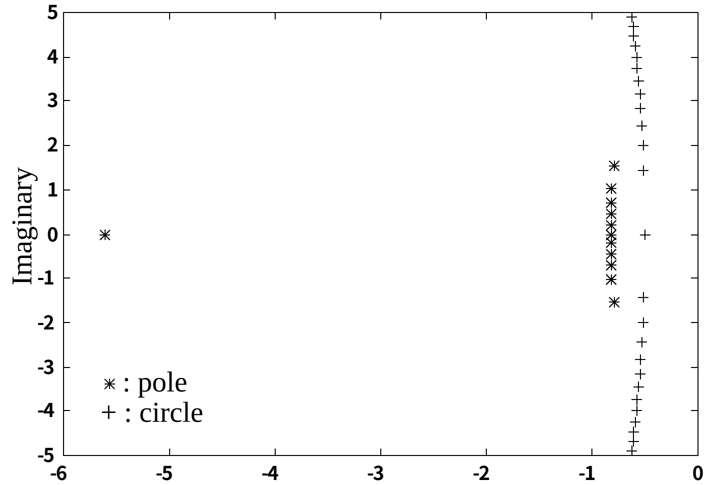

Figure 6.5: Pole position of $Q_{12prop}(s)$

Table 6.1: A summary of deg (Q_r), $\lambda^* + 2\|T_2\|_\infty \cdot \mathrm{trace}\Sigma_2^R \cdot \|T_3\|_\infty$, λ and the McMillan degrees of the controllers

$\deg(Q_r)$	$\lambda^* + 2\|T_2\|_\infty \cdot \mathrm{trace}\Sigma_2^R \cdot \|T_3\|_\infty$	λ	$\deg(\mathcal{F}_l(K_0, Q_r))$
17	–	0.802	20 (19)
16	0.803	0.802	20 (18)
15	0.805	0.802	19 (17)
14	0.814	0.802	18 (16)
13	0.839	0.804	17 (15)
12	0.965	0.807	16 (14)
11	1.21	0.826	15 (13)
10	2.47	1.09	13 (12)

the constraints of the closed loop system, hence it is natural to approximate the parameter without losing the information of the constraints. However there was a difficulty in reducing the order of the controller. Since this method approximates the Youla parameter, the order of the controller is not necessary reduced sufficiently if the order of the parameter is reduced. In order to avoid this situation we have given a sufficient condition for the order of the approximated controller is really reduced.

Thus we have given a H_∞ controller reduction method. The advantages of the method are that the approximated controller certainly satisfies the internal stability and the constraint of the pole position in the closed loop. For the constraint of the H_∞-norm, a sufficient condition for satisfying the constraint is given. The effectiveness of this result was verified with a numerical example.

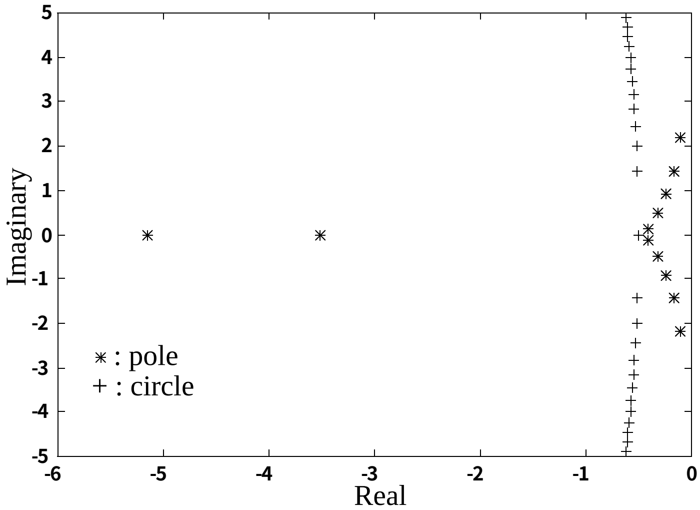

Figure 6.6: Pole position of $Q_{12BT}(s)$

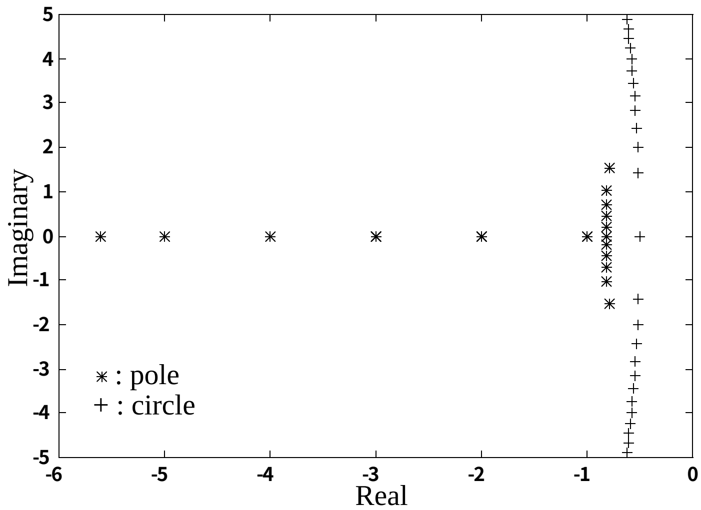

Figure 6.7: Pole position of $F_{affine}\left(Q_{12prop}(s)\right)$

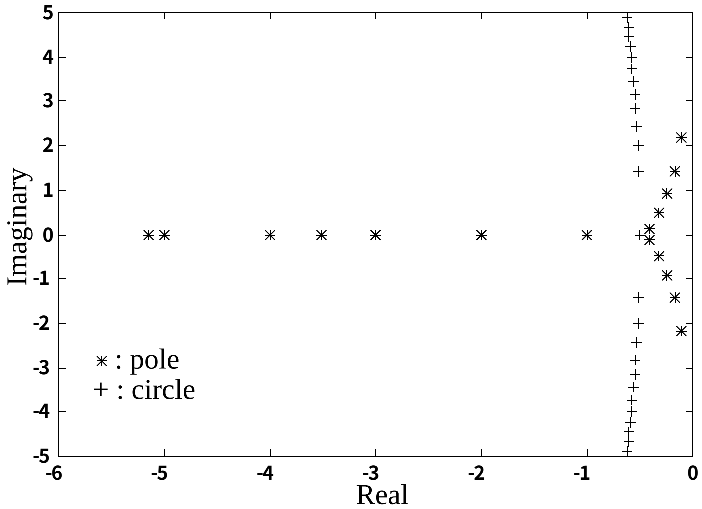

Figure 6.8: Pole position of $F_{affine}\,(Q_{12BT}(s))$

Chapter 7

Conclusion

In general, as the structure of a design problem becomes more complex, the number of choices of formulations to solve the problem increases. As we have seen in chapters 4 and 5, the complexity and the optimality of the solution depend on the formulation chosen. In the designing of robust controllers based on the H_∞ problem, there are many approaches to solving the design problems. However, these problems are usually formulated as the standard H_∞ problem and their solutions sometimes become complex and result in high-order controllers. It is therefore important, not only to solve a problem formulated beforehand but also to consider a new formulation.

This thesis has considered the formulation of the H_∞ control problem in the designing of low-order controllers from several viewpoints. Two types of approaches to the problem are considered: a *direct* approach based on the derivation of the reduced-order H_∞ controllers, and an *indirect* approach based on the approximation of a high-order controller. For each approach, some methods for designing low-order controllers have been proposed.

1. In the *direct* approach, by getting a hint to reduce the dynamical order of a controller from the minimal-order-observer design, we have treated the non-standard H_∞ control problems, where the dimensions of the control inputs are greater than those of the controlled outputs, or the dimensions of the measurement outputs are greater than those of the disturbance inputs. As a result, generalized classes of the reduced-order H_∞ controllers are expressed with free parameters. The reduced-order controllers are classified into two types: a minimal-order-observer-type and a dual type of the minimal order observer. Thus, this study has shown that the low-order controllers can be obtained by reducing some controller design problem to the non-standard H_∞ problem. While the formulation of non-standard H_∞ problem has been avoided in previous studies, it is expected that the non-standard H_∞ problem is positively utilized in the controller designs. This thesis has, in fact, clarified the controller design problems that can be formulated as the non-standard problems and has shown the advantages of formulating them in this way.

This thesis has gone on to clarify some classes of the problems that are reducible to the non-standard H_∞ problem and has proposed methods for deriving robust controllers.

- Based on the non-standard H_∞ problem, a new formulation for designing an integral-type H_∞ controller is proposed, and new method for deriving the low-order controller is shown. The advantages of this formulation are due to the simplicity of the derivation. If the plant doesn't include integrators, it is not necessary to solve the so-called pseudo stabilizing solution of the ARE. On the other hand, if the plant includes the integrators, although it is still necessary to solve the pseudo stabilizing solution, it is not necessary to transform the non-standard problem to the standard problem and to separate the integrators from the plant. Moreover, the results can be extended to TDF controller design which is also reducible to the non-standard H_∞ control problem.
- A TDF robust controller design problem is reduced to the non-standard H_∞ problem, and a new method for designing a low-order TDF controller is proposed. This method has many advantages over conventional design methods of TDF controller. One is that the method makes it possible to design both the feedback controller and the feedforward controller simultaneously, hence the trade-off between the performances of the controllers can be considered. In this thesis, it is pointed out that there exists trade-off between the performances of the controllers if those dynamics are common. The other is that the order of the TDF controller can be reduced to the order which is lower than that of the generalized plant.

2. With regard to the *indirect* approach, this thesis has treated a specialized solution of the H_∞ controller, which is represented with an optimized Youla parameter. Since the order of the parameter is extremely high, the order of the controller is also high. In this thesis, we have proposed a model approximation method for a linear time invariant stable system whose poles are located in a specified disk on the left half plane. This method preserves the domain in which the poles of the original system are located. An upper bound of the model approximation error was evaluated using the Hankel singular values of the original system. By using the approximation method for reducing the order of the Youla parameter, a method for reducing the order of the H_∞ controller, which satisfies a constraint of the closed loop pole position, is proposed.

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Appendix A

Preliminary results

A.1 Invariant zeros

The invariant zeros are characterized by the following lemma.

Lemma A.1.1 (Invariant zeros) *Let us consider the system*

$$T(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

and define a system matrix as follows

$$Q(s) \triangleq \left[\begin{array}{c|c} A - sI & B \\ \hline C & D \end{array}\right].$$

A complex number $z_0 \in \mathbb{C}$ is called an invariant zero of $T(s)$ if it satisfies

$$\mathrm{rank}\,(Q(z_0)) < \textit{normal rank}\,(Q(s))\,,$$

where the normal rank of $Q(s)$ means the maximally possible rank of $Q(s), s \in \mathbb{C}$.

If D is a matrix of full rank, the invariant zeros are defined by the following three cases.

(i) When D is full column rank, the invariant zeros of the system $T(s)$ are the unobservable modes of the pair $(A - BD^{\dagger}C, D^{\perp}C)$.

(ii) When D is full row rank, the invariant zeros of the system $T(s)$ are the uncontrollable modes of the pair $(A - BD^{\dagger}C, BD^{\perp})$.

(iii) When D is a square matrix, the invariant zeros of the system $T(s)$ are the eigenvalues of the matrix $A - BD^{-1}C$.

A.2 Parametrization of stabilizing controllers

Let us consider the system

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right], \tag{A.1}$$

where (A, B_2) is stabilizable and (A, C_2) is detectable.

Lemma A.2.1 *Choose the matrices F and H such that the matrices*

$$A_F = A + B_2F$$
$$A_H = A + HC_2$$

are both stable. Then the class of all the stabilizing controllers for $G_{22}(s)$ is represented as follows:

$$K(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} A + B_2F + HC_2 & -H & -B_2 \\ \hline F & O & -I \\ -C_2 & I & O \end{array}\right], Q(s)\right), \quad Q(s) \in \mathcal{RH}_\infty. \tag{A.2}$$

The class of closed loop transfer functions $G_{zw}(s) = \mathcal{F}_l\left(G(s), K(s)\right)$ is represented as follows:

$$G_{zw}(s) = \mathcal{F}_l\left(\left[\begin{array}{cc|cc} A_F & -B_2F & B_1 & B_2 \\ O & A_H & B_H & O \\ \hline C_F & -D_{12}F & O & D_{12} \\ O & C_2 & D_{21} & O \end{array}\right], -Q(s)\right), \quad Q(s) \in \mathcal{RH}_\infty, \tag{A.3}$$

where B_H and C_F are defined as follows.

$$B_H = B_1 + HD_{21}$$
$$C_F = C_1 + D_{12}F$$

A.3 A matrix equation

Let us define the matrix equation as follows

$$PA_1 + A_2P = B, \tag{A.4}$$

where $A_1 \in \mathbb{R}^{n\times n}$ and $A_2 \in \mathbb{R}^{m\times m}$ are square matrices.

Lemma A.3.1 *If A_1 and A_2 satisfy*

$$\lambda_i(A_1) + \lambda_j(A_2) \neq 0, \ \forall i \in [1, 2, \dots, n], \forall j \in [1, 2, \dots, m] \tag{A.5}$$

the matrix equation in (A.4) has a unique solution. Specifically in the case $B = O$, the unique solution is $P = O$.

A.4 Star product

Let us consider the systems:

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad K(s) = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{array}\right].$$

Then the star product of $G(s)$ and $K(s)$ are defined as

$$G(s) \star K(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right],$$

where

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A + B_2\tilde{R}^{-1}D_{K11}C_2 & B_2\tilde{R}^{-1}C_{K1} \\ B_{K1}R^{-1}C_2 & A_K + B_{K1}R^{-1}D_{22}C_{K1} \end{bmatrix} \\
\bar{B} &= \begin{bmatrix} B_1 + B_2\tilde{R}^{-1}D_{K11}D_{21} & B_2\tilde{R}^{-1}D_{K12} \\ B_{K1}R^{-1}D_{21} & B_{K2} + B_{K1}R_{-1}D_{22}D_{K12} \end{bmatrix} \\
\bar{C} &= \begin{bmatrix} C_1 + D_{12}D_{K11}R^{-1}C_2 & D_{12}\tilde{R}^{-1}C_{K1} \\ D_{K21}R^{-1}C_2 & C_{K2} + D_{K21}R^{-1}D_{22}C_{K1} \end{bmatrix} \\
\bar{D} &= \begin{bmatrix} D_{11} + D_{12}D_{K11}R^{-1}D_{21} & D_{12}\tilde{R}^{-1}D_{K12} \\ D_{K21}R^{-1}D_{21} & D_{K22} + D_{K21}R^{-1}D_{22}D_{K12} \end{bmatrix} \\
R &= I - D_{22}D_{K11}, \quad \tilde{R} = I - D_{K11}D_{22}.
\end{aligned}$$

A.5 Inner function

A transfer function $N(s)$ is inner if $N \in \mathcal{RH}_\infty$ and $N^T(-s)N(s) = I$.

Lemma A.5.1 *Let us consider the system $N(s)$ as follows:*

$$N(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Suppose that D is of full column rank, (A, C) is detectable. Then $N(s)$ is inner if there exists $P \geq O$ such that

$$\begin{cases} PA + A^TP + C^TC = O \\ D^TC + B^TP = O \\ D^TD = I \end{cases}. \tag{A.6}$$

A.6 Lossless system

Let us consider the stable system

$$\Xi(s) = \begin{bmatrix} \Xi_{11}(s) & \Xi_{12}(s) \\ \Xi_{21}(s) & \Xi_{22}(s) \end{bmatrix} \in \mathcal{RH}_\infty^{m\times p}, \quad m \geq p.$$

Suppose that $\Xi(s)$ satisfies

$$\begin{cases} \Xi^T(-s)\Xi(s) = I \\ \Xi_{22}(\infty) = O \\ \det(\Xi_{21}(j\omega)) \neq 0; \forall\omega \end{cases} \tag{A.7}$$

then the system $\Xi(s)$ is called lossless.

Lemma A.6.1 *Suppose that the system $\Xi(s)$ is lossless, and that the system $Q(s)$ is proper. Then the following items are equivalent.*

1. *A closed loop system $(\Xi(s), Q(s))$ is internally stable and $Q(s)$ satisfies*

$$\|\mathcal{F}_l\left(\Xi(s), Q(s)\right)\|_\infty < 1.$$

2. *$Q(s)$ satisfies*

$$\|Q(s)\|_\infty < 1.$$

A.7 Disturbance feedforward problem

Let us consider the standard H_∞ problem for

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right], \tag{A.8}$$

where the assumptions **A1**, **A2** and **A3** are satisfied. If the matrix D_{21} is nonsingular and the matrix $A - B_1 D_{21}^\dagger C_2$ is stable, the H_∞ problem is called DF (Disturbance Feedforward) problem. The solution of DF problem is listed in the following lemma.

Lemma A.7.1 *The DF problem is solvable if and only if an ARE:*

$$\begin{aligned} &X\left(A - B_2 D_{12}^\dagger C_1\right) + \left(A - B_2 D_{12}^\dagger C_1\right)^T X \\ &\quad + X\left\{B_1 B_1^T - B_2 D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T\right\} X + \left(D_{12}^\perp C_1\right)^T D_{12}^\perp C_1 = O \end{aligned}$$

has a nonnegative definite stabilizing solution X. If the condition is satisfied, the H_∞ controller is represented as follows:

$$K(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} A - B_1 D_{21}^{-1} C_2 + B_2 F_\infty & B_1 D_{21}^{-1} & B_2 \Sigma \\ \hline F_\infty & O & \Sigma \\ -\left(D_{21}^{-1} C_2 + B_1^T X\right) & D_{21}^{-1} & O \end{array}\right], N(s)\right), \tag{A.9}$$

where $N(s) \in \mathcal{BH}_\infty$, and F_∞ and Σ are defined as

$$\begin{aligned} F_\infty &= -D_{12}^\dagger C_1 - D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T X \\ \Sigma &= \left(D_{12}^T D_{12}\right)^{-\frac{1}{2}}. \end{aligned}$$

A.8 ARE (Algebraic Riccati Equation)

Let a matrix $A \in \mathbb{R}^{n\times n}$ and a symmetric matrices $R \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ be given. If a solution of the following ARE:

$$XA + A^T X + XRX + Q = O \tag{A.10}$$

stabilizes the matrix

$$A_X = A + RX,$$

then the solution X is called a stabilizing solution of the ARE (A.10).

Definition A.8.1 *Associated with the ARE (A.10), let us define a Hamiltonian matrix:*

$$H_X \triangleq \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n}. \tag{A.11}$$

Lemma A.8.1 *If H_X doesn't have poles on the imaginary axis, then H_X has n stable eigenvalues and n unstable eigenvalues.*

Suppose that H_X doesn't have poles on the imaginary axis. Then let $\Lambda_- \in \mathbb{R}^{n\times n}$ be a matrix whose eigenvalues are all of stable eigenvalues of H_X. A full column rank matrix of $V = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ satisfies

$$H_X V = V\Lambda_-, \tag{A.12}$$

where V is a basis of the mode Λ_-. Then if $\det(X_1) \neq 0$, we denote

$$H_X \in \mathrm{Dom}\,(\mathrm{Ric}) \tag{A.13}$$

$$\mathrm{Ric}\,(H_X) = X_2 X_1^{-1}. \tag{A.14}$$

Lemma A.8.2 *The stabilizing solution of the ARE (A.10) is given by $X = \mathrm{Ric}\,(H_X)$.*

A.9 Solution to Full control problem

Lemma A.9.1 *In order that the FC problem is solvable, it is necessary that the following ARE:*

$$\begin{aligned} &Y\left(A - B_1 D_{21}^{\dagger} C_2\right)^T + \left(A - B_1 D_{21}^{\dagger} C_2\right) Y \\ &\quad + Y\left\{C_1^T C_1 - \left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger} C_2\right\} Y + B_1 D_{21}^{\perp}\left(B_1 D_{21}^{\perp}\right)^T = O \end{aligned} \tag{A.15}$$

has a nonnegative definite stabilizing solution Y. By stabilizing, we mean that the following matrix is stable.

$$A_Y \triangleq A - B_1 D_{21}^{\dagger} C_2 + Y\left\{C_1^T C_1 - \left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger} C_2\right\}$$

Proof. See the paper [11]. ■

A.10 Reduction mode of an ARE for the standard problem

Let us consider an ARE:

$$\begin{aligned} & Y\left(A - B_1 D_{21}^{\dagger} C_2\right)^T + \left(A - B_1 D_{21}^{\dagger} C_2\right) Y \\ & \quad + Y\left\{C_1^T C_1 - \left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger} C_2\right\} Y + B_1 D_{21}^{\perp}\left(B_1 D_{21}^{\perp}\right)^T = O, \end{aligned} \tag{A.16}$$

which is required to be solved in the standard H_∞ problem. For this ARE, the following lemma holds.

Lemma A.10.1 *Assume that the ARE in (A.16) has a positive definite stabilizing solution Y, i.e., there exists a solution $Y > O$ for the ARE and the solution stabilizes the matrix:*

$$A_{Y_s} \triangleq A - B_1 D_{21}^{\dagger} C_2 + Y\left\{C_1^T C_1 - \left(D_{21}^{\dagger} C_2\right)^T D_{21}^{\dagger} C_2\right\}.$$

Then it is shown that

$$\begin{aligned} & \forall U, \det\left(UU^T\right) \neq 0; \begin{cases} U\left(A - B_1 D_{21}^{\dagger} C_2\right) = \Lambda_- U, \lambda_i\left(\Lambda_-\right) < 0, \forall i \\ U B_1 D_{21}^{\perp} = O \end{cases} \\ & \ker\left(Y^T\right) = \text{range}\left(U^T\right). \end{aligned}$$

Proof. Let U be a full row rank matrix which satisfies $UY = O$. Pre-multiplying by U and post-multiplying by U^T, the ARE is written as

$$UYA_Y^T U^T + U\left(A - B_1 D_{21}^{\dagger} C_2\right) Y U^T + U B_1 D_{21}^{\perp}\left(U B_1 D_{21}^{\perp}\right)^T = O,$$

hence U satisfies

$$U B_1 D_{21}^{\perp} = O.$$

Again, pre-multiplying the ARE by U, it can be seen that U satisfies

$$U\left(A - B_1 D_{21}^{\dagger} C_2\right) Y = O,$$

hence $U\left(A - B_1 D_{21}^{\dagger} C_2\right)$ can be represented as

$$U\left(A - B_1 D_{21}^{\dagger} C_2\right) = \Lambda_s U,$$

where Λ_s is an appropriate nonsingular matrix. Then, for the stable matrix A_Y the matrix U satisfies

$$UA_Y = \Lambda_s U.$$

From this equation, it can be seen that the matrix Λ_s is a stable matrix because the matrix U is of full row rank. Thus, the lemma has been proved. ∎

A.11 Solutions to the mixed-sensitivity problem

The solutions to the mixed-sensitivity problems, which are the H_∞ control problems for the generalized plant $G_{ms}(s)$ in (3.6) described in Figure 3.2, are listed according to some conditions.

Lemma A.11.1 (A case where $W_s(s) \in \mathcal{RH}_\infty$ and $P(s)$ includes no $j\omega$-poles) *Assume that $W_s(s) \in \mathcal{RH}_\infty$ and $P(j\omega) \neq \infty, \forall\omega \in \mathbb{R}$ hold. If the AREs:*

$$\begin{aligned} &X\left(A - B_2 D_{12}^\dagger C_1\right) + \left(A - B_2 D_{12}^\dagger C_1\right)^T X \\ &\quad + X\left\{B_1 B_1^T - B_2 D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T\right\} X + \left(D_{12}^\perp C_1\right)^T D_{12}^\perp C_1 = O \end{aligned} \tag{A.17}$$

$$\begin{aligned} &Y\left(A - B_1 \tilde{D}_{21}^\dagger \tilde{C}_2\right)^T + \left(A - B_1 \tilde{D}_{21}^\dagger \tilde{C}_2\right) Y \\ &\quad + Y\left\{C_1^T C_1 - \left(\tilde{D}_{21}^\dagger \tilde{C}_2\right)^T \tilde{D}_{21}^\dagger \tilde{C}_2\right\} Y + B_1 \tilde{D}_{21}^\perp \left(B_1 \tilde{D}_{21}^\perp\right)^T = O \end{aligned} \tag{A.18}$$

have the stabilizing solutions $X \geq O$ and $Y \geq O$ and they satisfy an inequality $\rho(XY) < 1$, then the class of the H_∞ controllers is given as follows.

$$K_{ms}(s) = \mathcal{F}_l\left(\left[\begin{array}{c|cc} \hat{A} & -ZH_\infty & Z\hat{B}_2\left(D_{12}^T D_{12}\right)^{-\frac{1}{2}} \\ \hline F_\infty & O & \left(D_{12}^T D_{12}\right)^{-\frac{1}{2}} \\ -\left(\tilde{D}_{21}\tilde{D}_{21}^T\right)^{-\frac{1}{2}}\hat{C}_2 & \left(\tilde{D}_{21}\tilde{D}_{21}^T\right)^{-\frac{1}{2}} & O \end{array}\right], N\right), \tag{A.19}$$

where $N(s) \in \mathcal{BH}_\infty$ is a free parameter, and the other parameters are defined as

$$\begin{aligned} \hat{A} &= A + B_1 B_1^T X + B_2 F_\infty + ZH_\infty \hat{C}_2 \\ \hat{B}_2 &= B_2 + YC_1^T D_{12} \\ \hat{C}_2 &= \tilde{C}_2 + \tilde{D}_{21} B_1^T X \\ F_\infty &= -D_{12}^\dagger C_1 - D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T X \\ H_\infty &= -B_1 \tilde{D}_{21}^\dagger - Y\left(\tilde{D}_{21}^\dagger \tilde{C}_2\right)^T \tilde{D}_{21}^\dagger \\ Z &= (I - YX)^{-1}. \end{aligned}$$

Lemma A.11.2 (A case where $W_s(s)$ includes $j\omega$-poles and $P(s)$ includes no $j\omega$-poles) *[49] If the ARE in (A.17) has a stabilizing solution $X \geq O$, the ARE in (A.18) has a pseudo-stabilizing solution $Y \geq O$ and they satisfy the inequality $\rho(XY) < 1$, then by using these solutions X and Y the class of the H_∞ controllers is given with the expression in (A.19).*

Lemma A.11.3 (A case where $P(s)$ includes $j\omega$-poles) *[48] Assume that $P(s)$ has $j\omega$-poles, and can be factorized as follows*

$$P(s) = \hat{P}(s)a(s),$$

where

$$\hat{P}(j\omega) < \infty, \forall \omega \in \mathbb{R}, \quad a^{-1}(s) \in \mathcal{RH}_\infty.$$

In Figure 3.2, let us select $W_s(s)$ *as follows*

$$W_S(s) = a(s)\hat{W}_s(s),$$

where $\hat{W}(s)$ *has* $j\omega$*-poles which are needed to be included in the controller, and satisfies*

$$\hat{W}(\infty) = 0, \quad \hat{W}(s) < \infty, \forall s \in \{s : \mathrm{Re}(s) > 0\}.$$

Then, by factorizing $P(s)$ *as in Figure 3.7 let us construct the generalized plant* $\hat{G}_{ms}(s)$.

If there exists $\hat{K}_{ms}(s)$ *which is an* H_∞ *controller for* $\hat{G}_{ms}(s)$*, the* H_∞ *controller for the original plant* $G_{ms}(s)$ *exists and is given by* $a^{-1}(s)\hat{K}_{ms}(s)$.

Appendix B

Publications

Journal Papers

1. T. Watanabe, K. Yasuda and R. Yokoyama: A reduction method of H_∞ controller based on truncation of Youla parameter, *Trans. of SICE*, Vol. 31,No. 8, pp. 1080-1088 (1995) (in Japanese).

2. T. Watanabe, K. Yasuda and R. Yokoyama: Reduced order solution of non-standard H_∞ control problem, *Trans. of SICE*, Vol. 32, No. 1, pp. 16-25 (1996) (in Japanese).

3. T. Watanabe, S. Akimoto and K. Yasuda: Simultaneous design of Two-Degree-of-Freedom controller via linear matrix inequality approach, *Trans. IEE of Japan*, Vol. 118, No. 3, pp. 310-319 (1998) (in Japanese).

4. T. Watanabe and K. Yasuda: Trade-off analysis of TDF control systems sharing common dynamics and simultaneous design of low-order TDF controller via non-standard H_∞ control, *Trans. IEE of Japan*, Vol. 118, No. 6, pp. 917-926 (1998) (in Japanese).

5. T. Watanabe and K. Yasuda: A synthesis of integral type control systems based on non-standard H_∞ control, *Trans. of SICE*, Vol. 34, No. 12, pp. 1822-1830 (1998) (in Japanese).

6. T. Watanabe and K. Yasuda: Reduced order design of integral-type, TDF controller based on a non-standard H_∞ control, submitted to *International Journal of Control* (1998)

Conference Papers (reviewed)

1. T. Watanabe, K. Yasuda and R. Yokoyama: Balanced truncation preserving poles in a specified disk, *Proc. of the 34th SICE Annual Conf.*, International Session Papers, pp. 1387-1390 (1995)

2. T. Watanabe, K. Yasuda and R. Yokoyama: Balanced truncation preserving poles in a specified disk and its application to the reduction of H_∞ controller, *Proc. of 22nd IEEE International Conference on Industrial Electronics, Control and Instrumentation*, pp. 1359-1364, Taipei (1996)

3. T. Watanabe, K. Yasuda: Reduced order solutions to nonstandard H_2 and H_∞ control problems, *Proc. of 35th IEEE Conf. on Decision and Control*, pp. 715-720, Kobe (1996)

Domestic Conference Papers (not reviewed)

1. T. Watanabe, K. Yasuda and R. Yokoyama: A fundamental study on robust control for power systems –A design method of robust servo systems –, *Proc. of the Fourth Annual Conf. of Power and Energy Society, IEE of Japan*, II-No. 102, pp. 3-4 (1993) (in Japanese).

2. T. Watanabe, K. Yasuda and R. Yokoyama: Robust performance of servo systems, *Preprints of the 36th Japan Joint Automatic Control Conf.*, No. 1018, pp. 47-48 (1993) (in Japanese).

3. T. Watanabe, K. Yasuda and R. Yokoyama: A model reduction method of H_∞ controller, *Preprints of the 16th SICE Symposium on Dynamical System Theory*, pp. 35-38 (1993) (in Japanese).

4. T. Watanabe, K. Yasuda and R. Yokoyama: Order reduction of a sub-sptimal H_∞ controller, *National Convention Records, IEE of Japan*, No. 1710, pp. 35-36 (1994) (in Japanese).

5. T. Watanabe, K. Yasuda and R. Yokoyama: A reduction method of H_∞ controller, *Preprints of the 23rd SICE Symposium on Control Theory*, pp. 125-130 (1994) (in Japanese).

6. T. Watanabe, K. Yasuda and R. Yokoyama: Balanced truncation preserving the poles in a specified disk, *National Convention Records, IEE of Japan*, No. 617, pp. 141-142 (1995) (in Japanese).

7. T. Watanabe, K. Yasuda and R. Yokoyama: Parametrization on reduced order singular H_∞ controller, *National Convention Records, IEE of Japan*, No. 622, pp. 150-151 (1995) (in Japanese).

8. T. Watanabe, K. Yasuda and R. Yokoyama: Parametrization and reduction of the nonstandard H_∞ controller, *Proc. of National Conf. on Systems and Control (IEEJ)*, SC-95-19, pp. 19-28 (1995) (in Japanese).

9. T. Watanabe, K. Yasuda and R. Yokoyama: Reduced order solutions of nonstandard H_∞ control problems and parametrization of reduced order nonstandard H_∞ controller, *Preprints of the 24th SICE Symposium on Control Theory*, pp. 45-50 (1995) (in Japanese).

10. T. Watanabe, K. Yasuda and R. Yokoyama: On the algebraic riccati equation in the nonstandard H_∞/H_2 control problems, *National Convention Records, IEE of Japan*, No. 676, pp. 216-217 (1996) (in Japanese).

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12. T. Watanabe and K. Yasuda: A Synthesis of Integral Type Control Systems Based on Non-standard H_2, H_∞ Controls, *Proc. of the Electronics, Information and Systems Conference (IEEJ)*, pp. 323-328 (1996) (in Japanese).

13. R. Kasuga, T. Watanabe, K. Yasuda and R. Yokoyama: Frequency separate control of plant with redundant actuators by nonstandard H_2 control, *Proc. of the 35th SICE Annual Conference Domestic Session*, pp.77-78, (1996) (in Japanese).

14. T. Watanabe and K. Yasuda: A Synthesis of Integral Type Two-Degree-of-Freedom Controller via Non-standard H_∞ Control, *Proc. of National Conf. on Systems and Control (IEEJ)*, SC-97-27, pp. 37-44 (1997) (in Japanese).

15. T. Watanabe and K. Yasuda: Simultaneous Design of Low Order Two-Degree-of-Freedom Controller via Non-standard H_∞ Control, *Proc. of National Conf. on Industorial Instrument and Control (IEEJ)*, IIC-97-30, pp. 25-33 (1997) (in Japanese).

16. T. Watanabe and K. Yasuda: Simultaneous Design of Two-Degree-of-Freedom Controller via Non-standard H_∞ Control, *National Convention Records (IEEJ)*, No. 657, pp. 170-171 (1997) (in Japanese).

17. T. Watanabe, K. Yasuda: Simultaneous Design of Feedforward and Feedback Controllers via Non-standard H_∞ Control, *Proc. of 26th SICE Symposium on Control Theory*, pp. 99-102 (1997)

18. T. Watanabe and K. Yasuda: A Synthesis of Integral Type Control Systems Based on Non-standard H_∞ Control, *Proc. of 26th SICE Symposium on Control Theory*, pp.189-194 (1997) (in Japanese).

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