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Essential surfaces in three-dimensional manifolds

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Contents

Introduction	5
Acknowledgments	11
Preliminary	13
Chapter 1. Boundary curves of essential surfaces	17
1. A construction of an immersed surface	19
2. A sufficient condition to be essential	21
3. Manifolds decomposed into truncated tetrahedra	25
4. Manifolds decomposed into truncated icosahedra	32
5. Boundary of surfaces with bounded Euler number	35
Chapter 2. Framed link presentations of surface bundles	37
1. A construction of a framed link presentation	37
2. Examples of framed links	41
Chapter 3. Knot complements without geodesic surfaces	47
1. Partial settlements	48
2. Criteria	53
Chapter 4. Accidental surfaces in knot complements	57
1. Uniqueness of the slope	59
2. A characterization of accidental surfaces	61
3. A necessary condition to be accidental	62
4. A conjecture on accidental slopes	64
Bibliography	67

Introduction

The main subject of this thesis is the *essential surfaces* in 3-manifolds. An embedded or immersed surface in a 3-manifold is called *essential* if

- (1) it does not come from a simply-connected surface,
- (2) the embedding or immersion induces injective maps of the fundamental groups and of the sets of homotopy classes of proper arcs, and
- (3) it is not freely homotopic into the boundary of the 3-manifold.

The essential surface has played a very important role in the study of 3-manifolds. For example, Waldhausen [64] proved that irreducible 3-manifolds with embedded essential surfaces have the topological rigidity; *a relative homotopy equivalence induces a homeomorphism*. Moreover, it was proved for such manifolds that Thurston's Geometrization Conjecture; *every compact orientable 3-manifold has a canonical decomposition by geometric pieces*, is true [47, 59, 60, 61, 63].

Throughout the thesis, we consider the questions; *how many essential surfaces exist in 3-manifolds* and *what properties do essential surfaces have, in particular, in hyperbolic 3-manifolds?* Being concerned with these questions, we study essential surfaces in four different situations, which correspond to Chapter 1, 2, 3 and 4.

Chapter 1: Boundary curves of essential surfaces.

To know how many essential surfaces exist in a 3-manifold with non-empty boundary, it seems useful to study the boundary curves of such surfaces. We set up the question: *Which closed curve on the boundary of a 3-manifold is contained in an immersed essential surface as a boundary component?*

An essential surface can be homotoped so that it does not intersect the compressing disks of the boundary of a 3-manifold. Hence, the focus of our interest is on irreducible 3-manifolds with incompressible boundary. The first step might be to study the question for simple pieces of the Jaco-Shalen-Johannson decomposition. In a *trivial or twisted I-bundle over a surface*, every non-trivial closed curve on the boundary is contained in a vertical annulus as a boundary component. On the other hand, it is known that the boundary curves of essential surfaces in *Seifert fibered 3-manifolds* are severely restricted. Detailed study was done in [57].

The remaining pieces are *acylindrical* ones, that is to say, the irreducible and ∂ -irreducible 3-manifolds containing no essential tori and annuli. In this chapter, which is based on [25], we give explicit examples of acylindrical 3-manifolds having the property that every non-trivial closed curve on the boundary can appear as a boundary component of an immersed essential surface. The examples will be hyperbolic 3-manifolds with totally geodesic boundary, which are known to be acylindrical. We also show that the immersed surfaces in some particular class of our examples are virtually embedded, namely, it is lifted in a finite cover of the 3-manifold as an embedded one.

This result seems to indicate that there are *sufficiently many* immersed essential surfaces in 3-manifolds with non-empty boundary. In fact, the following conjecture is a motivation of this study, see [5, 6] for survey; *a closed irreducible 3-manifold with infinite fundamental group (A) contains an immersed essential surface, and (B) is finitely covered by a 3-manifold with embedded essential surfaces*. By recent works [13, 16], it is proved that (B) implies that Thurston's Geometrization Conjecture is true for such 3-manifolds.

On the other hand, recently, it is shown in [18, 19] that only finitely many *simple* closed curves on the boundary are realized as the boundaries of immersed essential surfaces of bounded genus for an acylindrical 3-manifold. Concerning these results, we show that only finitely many closed curves, up to homotopy, can appear as boundary components of

essential surfaces with bounded Euler number on the boundary of an acylindrical 3-manifold.

Chapter 2: Framed link presentations of surface bundles.

A fiber surface in a 3-manifold fibering over the circle is a particular example of an embedded essential surface. It is well-known that every closed orientable 3-manifold has a framed link presentation, meaning that it is obtained by a framed surgery on some framed link in the 3-sphere [33]. Moreover such a framed link can be taken as a fibered link with some framing [8].

In this chapter, which is based on [24], we show that every closed orientable 3-manifold fibering over the circle is represented by a fibered link with the framings induced from the fibration of its complement.

Not every 3-manifold admits a fibration over the circle. However, Thurston conjectured [61] that every hyperbolic 3-manifold is finitely covered by a 3-manifold fibering over the circle. The first explicit example of a 3-manifold which contains no embedded essential surfaces and is finitely covered by a 3-manifold fibering over the circle is obtained in [54]. The result of this chapter is a byproduct of the study of this example.

The latter half of the thesis is based on the joint works [26] and [27] with Dr. Makoto Ozawa. In Chapter 3 and Chapter 4, we are mainly concerned with the geometric behavior of embedded essential surfaces in knot complements in the 3-sphere.

The classification of knots in the 3-sphere is reduced to the homeomorphism problem of the knot complements by the solution of a long standing Knot Complement Conjecture in [17]. Hence, essential surfaces must play an important role there, see [22] for example. Also many other problems in knot theory, in particular concerning Dehn surgery, are closely related with essential surfaces in their complements [30].

The geometric behavior of essential surfaces in hyperbolic 3-manifolds is of interest not only in Topology. An essential surface in a hyperbolic 3-manifold gives a surface subgroup in the holonomy image of the fundamental group of the manifold. Such subgroups are important subjects and have been well studied in Kleinian group theory.

Chapter 3: Knot complements without geodesic surfaces.

A large number of works have been done on essential surfaces in knot complements, in particular on the existence of embedded ones. See [21, 36, 37, 38, 39, 42, 49] for examples. On the geometric behavior of embedded essential surfaces in hyperbolic knot complements, it was conjectured [43] that the complements of knots in the 3-sphere have no closed embedded totally geodesic surfaces.

In this chapter, we show that the conjecture is true for hyperbolic 3-bridge knots and hyperbolic double torus knots in the 3-sphere. Besides, we give a few topological criteria for deciding whether a closed embedded essential surface in knot complements fails to be totally geodesic. Roughly speaking, we show that sufficiently “complicated” surfaces fail to be totally geodesic.

The conjecture seems to be meaningful only for knots in the 3-sphere. In fact, there is a *link* in the 3-sphere whose complement contains a closed embedded totally geodesic surface [43]. The complements of the figure-eight knot [55] and some other knots [7] in the 3-sphere contains *immersed* totally geodesic surfaces. It is known [48] that there are cusped hyperbolic 3-manifolds other than knot complements in the 3-sphere which contain closed embedded totally geodesic surfaces.

Chapter 4: Accidental surfaces in knot complements.

Consider an embedded essential surface in a 3-manifold with toral boundary. A non-trivial simple closed curve on the surface is called an

accidental peripheral if there is an annulus running from the curve into the boundary of the 3-manifold. We call this annulus an *accidental annulus*, and such a surface containing an accidental peripheral an *accidental surface*. It is known for many knots in the 3-sphere that all closed essential surfaces in their exteriors are accidental.

In this chapter, we first show that all accidental annulus for an accidental surface determine the unique slope on the boundary of the knot exterior. Hence let us call this slope the *accidental slope* for the accidental surface. Moreover, in the case that this slope is not meridional, we show that all accidental peripherals are mutually isotopic on the surface.

Next, we give a characterization of accidental surfaces. As an application, we give a necessary and sufficient condition for knots to bound a totally knotted Seifert surface.

Finally, we give some criteria for checking whether a given closed essential surface is accidental or not. We also give some corollaries used to determine the accidental slope.

There are two motivations to study accidental surfaces.

One is concerned with the conjecture considered in the previous chapter. When the interior of the 3-manifold admits a hyperbolic metric of finite volume, Thurston showed [60] that a closed embedded essential surface is accidental if and only if corresponding surface subgroup is not quasi-Fuchsian. Hence accidental surfaces fail to be totally geodesic. It is still unsolved which knots in the 3-sphere have non-accidental surfaces in their exteriors.

The other is concerned with Dehn surgery. The question is whether a closed essential surface in a knot complement remains essential after Dehn surgeries on the knot. It is shown in [12, Theorem 2.4.3] that an accidental surface with an integral accidental slope γ remains essential after γ' -Dehn surgery if and only if the geometric intersection number of γ and γ' is just one. Concerning this result, it is a natural question how many accidental slopes exist for an accidental surface, and our result answers this question.

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Preliminary

Let us prepare notations and definitions. See [23, 28] for 3-manifold theory, [11, 56] for knot theory, and [53, 62] for hyperbolic geometry as a general references.

Throughout the thesis, all manifolds will be supposed to be connected and orientable unless otherwise stated. It is known that every 3-manifold has a triangulation which is unique up to subdivision [45], and such a piecewise linear structure gives rise to a smooth structure in an essentially unique way. Hence we will suppose that all manifolds have such a piecewise linear structure and such a smooth structure.

The following notations will be used. The n -dimensional Euclidean space, sphere, disk and hyperbolic space are denoted by $\mathbb{R}^n$, S^n , $\mathbb{D}^n$ and $\mathbb{H}^n$, respectively. The number of connected components of a space X is denoted by $|X|$, the interior of X by $\text{Int}X$, and the boundary of X by ∂X . The regular neighborhood of a subspace Y in X is denoted by $N(Y; X)$, or simply by $N(Y)$, and the exterior of Y in X , namely the closure of $X \setminus N(Y)$, by $E(Y)$. Let M and N be manifolds with non-empty boundary. A map $f : M \rightarrow N$ is called *proper* if $f(\partial M) \subset \partial N$.

Let N be a manifold with boundary and B a subset of ∂N . The manifold obtained by the following operation is called the *double* of N with respect to B , and denoted by $D_B(N)$. Identify $x \in B \subset N$ with $(x, 0) \in B \times [0, 1]$ and $x' \in B \subset N'$ with $(x', 1) \in B \times [0, 1]$, where N' is a copy of N . If B is equal to ∂N , then we simply denote the double of N by $D(N)$. Note that $D_B(N)$ is homeomorphic to the manifold obtained by removing $(\partial N \setminus B) \times (0, 1)$ from $D(N)$.

A *knot* K in 3 3-manifold M means an embedded circle in M , and a *link* L means a disjoint union of knots in M . A *Seifert surface* for L is defined as an orientable connected surface embedded in M with

boundary L . We call a non-trivial closed curve on $\partial N(K)$ bounding a disk in $N(K)$ a *meridian* of K . If K is a knot in particular in S^3 , there is a simple closed curve on $\partial N(K)$ which is homologically trivial in $E(K)$. We call this curve the *preferred longitude* of K . Otherwise stated, we always fix the meridian and the preferred longitude of K as a generator system of the torus $\partial N(K)$.

Let F be an immersed surface in a 3-manifold M . That is, the image of a proper immersion f of a connected 2-manifold S . The *boundary* of F , denoted by ∂F , is defined as the image of ∂S . The surface F is said to be compact or orientable if the domain S is compact or orientable, respectively. An immersed surface in a 3-manifold is called *virtually embedded* if it is lifted in a finite cover of the 3-manifold as an embedded one. In the following definitions, we suppose that S is not simply-connected. If f induces an injection $\pi_1(S) \rightarrow \pi_1(M)$, then we call F π_1 -*injective*. If f induces an injection $\pi_1(S, \partial S) \rightarrow \pi_1(M, \partial M)$, then F is called ∂ - π_1 -*injective*, where $\pi_1(S, \partial S)$ and $\pi_1(M, \partial M)$ means the sets of homotopy classes of proper arcs in S and M , respectively. We define F as being *essential* if it is π_1 -injective, ∂ - π_1 -injective, and F is not freely homotopic into ∂M .

The original definition of the essentiality for *embedded* surfaces is more geometric one as follows.

Let F be an embedded surface in a 3-manifold M . A 2-disk embedded in M is called *compressing disk* for F if $D \cap F = \partial D$ and ∂D is non-trivial in F . If there are no compressing disk for F , then F is called *incompressible*. By Dehn's Lemma (see [23] for a proof), an embedded surface in a 3-manifold is π_1 -injective if and only if it is incompressible. Analogously, a 2-disk D embedded in M is called a ∂ -*compressing disk* for F if $\partial D = a \cup b$ where a is a not- ∂ -parallel arc in F and b is an arc in ∂M not homotopic into ∂F relative endpoints. If there are no ∂ -compressing disk for F , then F is called ∂ -*incompressible*. Also by Dehn's Lemma, a properly embedded surface in a 3-manifold is ∂ - π_1 -injective if and only if it is ∂ -incompressible. Embedded surfaces F and F' are said to be *parallel* if they cobound a product region in M .

In particular, when F is parallel to a subsurface of ∂M , it is called ∂ -parallel. As a result, an embedded surface is essential if and only if it is incompressible, ∂ -incompressible and not ∂ -parallel.

From a surface F with a compressing disk D , a new surface is obtained by cutting F along D and pasting the resultant surface with two parallel copies of D . Such a operation is called a *compression* of F along D . In the same way, we define a ∂ -*compression* of a properly embedded surface along a ∂ -compressing disk for it.

In a hyperbolic 3-manifold, the geometric behavior of essential surfaces were classified by Thurston [47, 61, 63]. See [53, 62] for more detail. Let F be an essential surface in a 3-manifold M with the interior admitting a hyperbolic structure of finite volume. Since F is essential, $\pi_1(F)$ is isomorphic to a subgroup Γ of $\pi_1(M)$. The fundamental group $\pi_1(M)$ of M acts the universal covering space of $\text{Int}M$, which is isometric to $\mathbb{H}^3$, as isometries. Hence Γ is also regarded as a group of isometries, and its limit point set $\Lambda(\Gamma)$ lies on the sphere at infinity S_∞^2 of $\mathbb{H}^3$.

Thurston showed in [63] that $\Lambda(\Gamma)$ coincides with the whole of S_∞^2 if and only if F is a *virtual fiber*, that is, there is a finite-index covering space of M which fibers over S^1 with the lift of F corresponding to a fiber. If Γ is a quasi-Fuchsian group, that is, $\Lambda(\Gamma)$ is a simple closed curve in S_∞^2 , F is called *quasi-Fuchsian*. Thurston also showed that F is quasi-Fuchsian if and only if F is not a virtual fiber and does not contain accidental parabolic elements, namely, loops on F not homotopic into ∂F , but homotopic into ∂M in M . Consequently, we have the following trichotomy. That is, an essential surface in a hyperbolic 3-manifold either is a virtual fiber, is quasi-Fuchsian, or contains an accidental parabolic element.

A surface F in a hyperbolic 3-manifold M is called *totally geodesic* if every geodesic in M tangent to F is entirely contained in F . In this case, the preimage of F in $\mathbb{H}^3$ must be a hyperplane. Therefore, such a surface is automatically essential and quasi-Fuchsian.

Let M be a 3-manifold and $L = l_1 \cup l_2 \cup \cdots \cup l_n$ a link in M . If an embedding $f_i : l_i \times \mathbb{D}^2 \rightarrow M$ avoids the other components and is the identity on $c \times \{0\}$, then f_i is called a *framing* of l_i . A *framed link* is defined as being a link with framings. A simple closed curve $\gamma_i = f_i(l_i \times \{x\})$, where $x \in \partial\mathbb{D}^2$, is called a *longitude* of l_i . Then a framed link can be regarded as a link with longitudes since the choice of a longitude determines the isotopy class of a framing. When a link L with longitudes $\{\gamma_i\}_{1 \leq i \leq n}$ in M is given, the following operation is called a *framed surgery* on it. Remove $\text{Int}N(L)$ from M and glue the solid tori back so that each longitude bounds a disk. We denote the 3-manifold obtained by a framed surgery on a link L with $\{\gamma_i\}_{1 \leq i \leq n}$ from M by $M(L, \{\gamma_i\}_{1 \leq i \leq n})$. If a 3-manifold is obtained by framed surgery on some framed link in S^3 , then we call it a *framed link presentation* of the manifold.

Let M be a 3-manifold, K a knot in M and γ a simple closed curve on $\partial N(K)$. The following operation, which is an extension of the framed surgery, is called a *Dehn surgery* on K with a surgery curve γ : Remove $\text{Int}N(K)$ and glue the solid tori back so that γ bounds a disk.

An isotopy class of simple closed curves on a torus T is called a *slope*. If we fix a system of generators of $\pi_1(T) \cong \mathbb{Z}^2$, then a slope γ is represented by an element (p, q) of $\pi_1(T)$, where p and q are coprime integers, up to the sign. Thus, by assigning p/q to γ , slopes on T are parameterized by $\mathbb{Q} \cup \{\infty\}$.

By using the meridian and the preferred longitude of a knot K in S^3 , we regard a framed link in S^3 as a link with integers. For a Dehn surgery on a knot in S^3 , the rational number corresponding to a surgery curve is called a *surgery slope*. Hence a framed surgery on a framed knot in S^3 can be regarded as a Dehn surgery on the knot with integral surgery slope.

CHAPTER 1

Boundary curves of essential surfaces

It is known that boundary curves of embedded essential surfaces in a 3-manifold are quite restricted [15, 20]. In contrast, Oertel [50] showed that there exists a 3-manifold M with the boundary ∂M such that every simple closed curves on ∂M is realized as the boundary of an immersed essential surface. That is, for each simple closed curve c on ∂M , there is an immersed essential surface whose boundary components are all simple closed curves isotopic to c . The first explicit example of such manifolds was found by Baker and Cooper [9], and later Maher [41] extended it to more wider class.

In this chapter, we set up the question; *Which closed curves on the boundary of a 3-manifold is contained in an immersed essential surface as a boundary component?* Specifically, we ask which 3-manifolds have the following property.

PROPERTY $(\star)$. *For each non-trivial closed curve on the boundary, some immersed essential surface containing it as a boundary component.*

Remark that we does not assume the simplicity of closed curves.

Here, we show the following by giving the explicit examples in Section 3.

THEOREM 1.1. *There are acylindrical 3-manifolds with non-empty boundary having Property $(\star)$.*

Remark that Property $(\star)$ is invariant under finite coverings. That is, if a 3-manifold has the property, then its all finite covers also has the property. In fact, a closed curve $\tilde{c}$ in a finite cover of a 3-manifold M covers some closed curve c in M , and if c is a boundary component

of an essential surface F in M , then $\tilde{c}$ is a boundary component of the essential surface appearing as a preimage of F in the cover. As an application, it is shown that no Seifert fibered 3-manifolds have the property of Theorem 1.1. Since a Seifert fibered 3-manifold is finitely covered by a trivial circle bundle over a surface, which has only vertical or horizontal essential surfaces.

An immersed surface in a 3-manifold is called *virtually embedded* if it is lifted in a finite cover of the 3-manifold as an embedded one. In general, it is not known whether the immersed surfaces in the examples given in Section 3 are virtually embedded or not. Hence, we are interested in the stronger property:

PROPERTY ($\star\star$). *For each non-trivial closed curve on the boundary, there is some virtually embedded essential surface containing it as a boundary component.*

As for this property, we give the other examples which have Property ($\star\star$) in Section 4.

A hyperbolic 3-manifold with totally geodesic boundary, that is, a 3-manifold admitting a Riemannian metric locally modeled on a subspace of the hyperbolic 3-space $\mathbb{H}^3$ bounded by a geodesic plane, is shown to be acylindrical. We will prove that if a hyperbolic 3-manifold with totally geodesic boundary admits a certain geometric truncated polyhedral decomposition, then it has Property ($\star$).

On the other hand, recently, it is shown in [18, 19] that only finitely many *simple* closed curves on the boundary are realized as the boundaries of immersed essential surfaces of bounded genus for an acylindrical 3-manifold. Then, we show

THEOREM 1.2. *Let M be an acylindrical 3-manifold with boundary ∂M . Then only finitely many homotopy classes of closed curves are represented by the boundary components of immersed essential surfaces with bounded Euler number.*

This chapter is organized as follows. In Section 1, we give a construction of an immersed surface containing the given closed curve as a boundary component. Section 2 is devoted to give a sufficient condition for certain immersed surface to be essential in a hyperbolic 3-manifold with totally geodesic boundary. We show in Section 3 that the surfaces obtained by our construction is essential in a hyperbolic 3-manifold decomposed into certain regular truncated tetrahedra. In Section 4, we also show that the surfaces obtained by our construction is essential in a hyperbolic 3-manifold decomposed into right-angled regular truncated icosahedra, and is virtually embedded in some class of such manifolds. In Section 5, we give a proof of Theorem 1.2.

1. A construction of an immersed surface

The following quick construction of an immersed surface containing a given non-trivial closed curve as a boundary component was suggested by D. Cooper. For a non-trivial simple closed curve on ∂M , one can easily find two curves such that these three together bound a pair of pants on ∂M . This construction can be applied to a non-simple closed curve by passing a finite sheeted cover, because the fundamental group of totally geodesic boundary of a hyperbolic 3-manifold is separable by a finite group [35]. However, the surfaces obtained by this construction are freely homotopic into ∂M . Since we would like to require the essentiality, we develop other construction which is not of this kind.

Let Γ be a finite graph, and we regard it as a 1-dimensional cellular complex. Namely, a vertex and an edge are regarded as a 0-cell and a closed 1-cell, respectively.

DEFINITION 1. An *immersed graph* G in a 2-dimensional cellular complex X is defined as the image of a map of Γ into X satisfying the following; each vertex of Γ is mapped into a 1-cell of X , and each edge of Γ is mapped to an arc connecting distinct 1-cells in a closed 2-cell of X .

We say that G is connected if Γ is connected. The image of a vertex of Γ is said to be a *vertex* of G . An *edge* and a *connected component* of G are defined in the same way.

Now, let us explain our construction, which consists of three steps.

Let S be a spine of a 3-manifold M with boundary ∂M , namely, a 2-dimensional cellular complex embedded in M such that S is a strong deformation retract of M . Without loss of generality, we may suppose that each 1-cell of S is shared by at least three closed 2-cells. The retraction of M into S is denoted by r . For convenience, we give an orientation to each 2-cell of S .

Suppose that a non-trivial closed curve α on ∂M is given. We will construct a connected immersed graph G in S which contains $r(\alpha)$ and satisfy the following condition:

Vertex consistency; there is a neighborhood N of each vertex of G in S such that every connected component of the intersection of N and 2-cells of S meets just one edge of G including the vertex.

By virtue of the vertex consistency, the preimage $r^{-1}(G)$ is regarded as an immersed surface, and it includes α as a boundary component.

Step 1. We homotope α such that $r(\alpha)$ is in the general position to the 1-skeleton of S and is regarded as an immersed graph in S .

Assume that $r(\alpha)$ includes n vertices $x_1, \dots, x_n$. Since each 1-cell of S is shared by at least three closed 2-cells, there exist 2-cells $C_{i,1}, \dots, C_{i,n_i}$ of S such that each closure $\overline{C_{i,j}}$ contains x_i but $C_{i,j} \cap N(x_i; S)$ does not intersect edges of $r(\alpha)$ including x_i , where $N(x_i; S)$ denotes some small neighborhood of x_i in S . Let e_i be the 1-cell of S including x_i , where it might be that $e_i = e_j$ for $i \neq j$.

Then add new edges $E_{i,j}$ to $r(\alpha)$ on $\overline{C_{i,j}}$ such that $E_{i,j}$ connects x_i and the new degree one vertex on the *furthest* 1-cell from e_i , and we get an extended immersed graph G_0 . This means that if $\overline{C_{i,j}}$ contains n 1-cells, then $E_{i,j}$ connects e_i and the $[n/2]$ -th 1-cell from e_i in $\overline{C_{i,j}}$ in cyclic order.

Furthermore, we parallelize G_0 , and get an immersed graph G_1 with two connected components.

Step 2. One can independently construct an immersed graph G' as follows.

(1) Put the two vertices on each 1-cell of S . (2) Connect each vertex and the *furthest* one with respect to cyclic order of vertices by an edge in each 2-cell of S . This means that if a closed 2-cell includes n 1-cells, then an edge connects one vertex and the n -th one in cyclic order.

It is remarked that G' satisfies the vertex consistency.

Step 3. Let $y_{i,j}$ be the vertex of G_1 included in $E_{i,j}$ other than x_i and $e_{i,j}$ the 1-cell of S containing $y_{i,j}$. Again, there exist 2-cells $C_{i,j,1}, \dots, C_{i,j,n_{i,j}}$ of S such that each closure $\overline{C_{i,j,k}}$ contains $y_{i,j}$ but $C_{i,j,k} \cap N(y_{i,j}; S)$ does not intersect edges of $r(\alpha)$ including $y_{i,j}$, where $N(y_{i,j}; S)$ denotes some small neighborhood of $y_{i,j}$ in S .

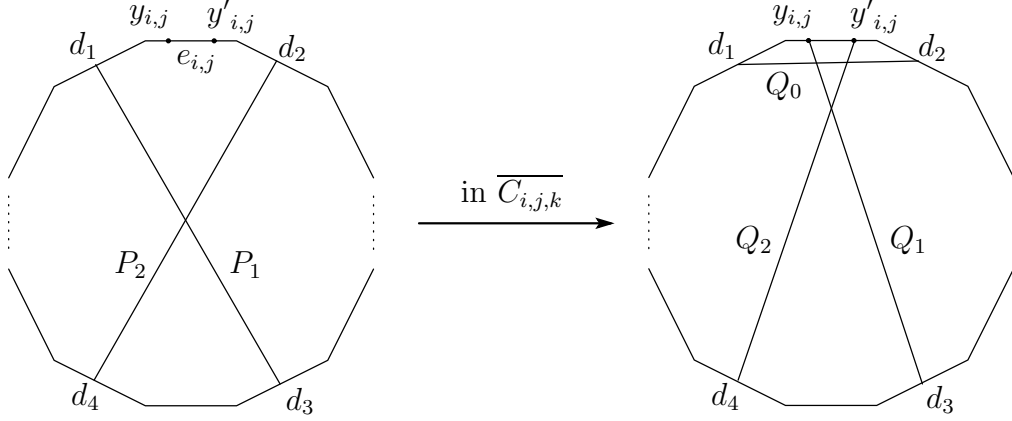
We prepare $\sum_i^n \sum_{j=1}^{n_i} n_{i,j}$ parallel copies of the immersed graph G' , and label them $G_{i,j,k}$ corresponding $C_{i,j,k}$.

Let $y'_{i,j}$ be the vertex of G_1 on $e_{i,j}$ included in the edge parallel to $E_{i,j}$. In each $C_{i,j,k}$, we perform the following operation. Let d_1 be one of the 1-cells adjacent to $e_{i,j}$, d_2 another one, d_3 the furthest one from d_1 , and d_4 the furthest one from d_2 in $\overline{C_{i,j,k}}$. There are the edges P_1 (respectively P_2) of $G_{i,j,k}$ connecting d_1 and d_3 (respectively d_2 and d_4). Then the new immersed graph G_2 is obtained from G_1 and $G_{i,j,k}$ by exchanging P_1, P_2 for Q_0, Q_1, Q_2 , where Q_0 is the arc connecting $P_1 \cap d_1$ and $P_2 \cap d_2$, Q_1 is the one connecting $y_{i,j}$ and $P_1 \cap d_3$ and Q_2 is the one connecting $y'_{i,j}$ and $P_2 \cap d_4$.

As a result, by taking the connected component including $r(\alpha)$ of G_2 , we obtain a connected immersed graph G in S satisfying the vertex consistency and including $r(\alpha)$.

2. A sufficient condition to be essential

The aim of this section is to give a sufficient condition for an immersed surface obtained by our construction to be essential.

FIGURE 1.1. The operation in $\overline{C_{i,j,k}}$

2.1. From now on, let M always denote a hyperbolic 3-manifold with totally geodesic boundary ∂M . The *cut locus* L_M with respect to ∂M is defined as the set of points in M each of which admits at least two distinct shortest paths to ∂M . The stratification defined by the number of shortest paths to ∂M gives L_M a cellular decomposition naturally [31], and then it is regarded as a spine of M . This cellular decomposition is geometric one in the sense that each 1-cell is geodesic in M and each 2-cell is totally geodesic in M . The geometric dual of L_M gives a cellular decomposition of M , which is called a canonical decomposition of M [31]. A retraction $r_M : M \rightarrow L_M$ is defined by assigning to each point x in M the point $r_M(x)$ in L_M so that the shortest path from $r_M(x)$ to ∂M goes through x . Let $\widetilde{M}$ be the universal covering space of M . Since $\widetilde{M}$ is also a hyperbolic 3-manifold with totally geodesic boundary, the cut locus $L_{\widetilde{M}}$ and the projection $r_{\widetilde{M}}$ are defined in the same way.

Let F be an immersed surface in M , namely F is the image of a proper immersion f of a 2-manifold S into M . Let $\widetilde{S}$ be the covering space of S corresponding the kernel of the induced homomorphism $f_* : \pi_1(S) \rightarrow \pi_1(M)$. Then the map $f : \widetilde{S} \rightarrow M$ is lifted to the map $\widetilde{f} : \widetilde{S} \rightarrow \widetilde{M}$. We denote the image $\widetilde{f}(\widetilde{S})$ by $\widetilde{F}$.

Let DM be the double of M . Namely, DM is a closed hyperbolic 3-manifold obtained from two copies of M by identifying their boundaries. Let DF be the subset of DM obtained from two copies of F . The map $f_D : DS \rightarrow DM$ is induced from f naturally such that $DF = f_D(DS)$, where DS is the double of S . The universal cover of DM is identified with the 3-dimensional hyperbolic space $\mathbb{H}^3$, and we regard $\widetilde{M}$ as a subset of $\mathbb{H}^3$. Let $\widetilde{DS}$ be the covering space of DS corresponding the kernel of the induced homomorphism $f_{D*} : \pi_1(DS) \rightarrow \pi_1(DM)$. Then the map $f_D : \widetilde{DS} \rightarrow M$ is lifted to the map $\widetilde{f_D} : \widetilde{DS} \rightarrow \mathbb{H}^3$ whose image $\widetilde{f_D}(\widetilde{DS})$, denoted by $\widetilde{DF}$, includes $\widetilde{F}$.

LEMMA 1.3. *If $\widetilde{DS}$ is simply connected, then F is essential.*

PROOF. Since ∂M is incompressible, S is not a 2-disk, and so DS is not simply connected. Hence, if $\widetilde{DS}$ is simply connected, the map $\pi_1(DS) \rightarrow \pi_1(DM)$ is injective. Therefore, DF is essential in DM , and so is F . $\square$

2.2. A *right distorted prism* P in $\mathbb{H}^3$ is defined as a non-compact convex polyhedron in $\mathbb{H}^3$, which is the intersection of several half spaces bounded by hyperplanes such that each one orthogonally intersects just two of the others with ultra-parallel lines. The intersection of P and each hyperplane is said to be a *face* of P .

Let G be a connected immersed graph in L_M . In $L_{\widetilde{M}}$, the preimage of G by the covering projection is a connected immersed graph, which is denoted by $\widetilde{G}$.

CONDITION 1. One can take a point on each edge of G such that it is not contained in the geometric dual of the 2-cell including it and the following holds. For each vertex $\widetilde{v}$ of $\widetilde{G}$, the hyperplanes in $\mathbb{H}^3$, each one of which contains the lift of the point on each edge of $\widetilde{G}$ including $\widetilde{v}$ and contains the geometric dual of the 2-cell of $L_{\widetilde{M}}$ including the lift, together with the components of $\partial \widetilde{M}$ containing $r_{\widetilde{M}}^{-1}(\widetilde{v})$ bound a right distorted prism.

The following proposition is the goal of this section.

PROPOSITION 1.4. *If G satisfies the vertex consistency and the Condition 1, then F is essential.*

PROOF. If G satisfies the vertex consistency, then F is regarded as an immersed surface, and so let us use the notations as in Subsection 2.1. Then, by Lemma 1.3, it suffices to show that $\widetilde{DS}$ is simply connected.

We remark that if G satisfies Condition 1, the intersection of $\widetilde{F}$ and each right distorted prism is homeomorphic to the 2-disk. Therefore, a cellular decomposition of $\widetilde{F}$ is defined by setting such 2-disks as closed 2-cells. Naturally, a cellular decomposition of $\widetilde{DF}$ is induced from that of $\widetilde{F}$, and a cellular decomposition of $\widetilde{DS}$ is induced from that of $\widetilde{DF}$.

We fix a closed 2-cell Q_0 of $\widetilde{DF}$, and define the set of closed 2-cells $\mathcal{Q}_n$ inductively as follows. Set $\mathcal{Q}_0$ as Q_0 . Let Q be a closed 2-cell not included in $\mathcal{Q}_i$, $i < n$. The cell Q is included in $\mathcal{Q}_n$ if there is some 2-cell Q' included in $\mathcal{Q}_{n-1}$ such that the cells in $\widetilde{DS}$ corresponding Q and Q' share a closed 1-cell or a 0-cell in $\widetilde{DS}$. Let $\mathcal{P}_n$ be the union of the prisms corresponding to the closed 2-cells included $\mathcal{Q}_n$.

Let us show that $\bigcup_{i=0}^n \mathcal{P}_i$ is also a right distorted prism for every n by induction. It is clear for $n = 0$. Suppose that $\bigcup_{i=0}^{n-1} \mathcal{P}_i$ is a right distorted prism for $n \geq 1$. We denote it by $\overline{\mathcal{P}}_{n-1}$, and suppose that it has m faces $\Lambda_1, \dots, \Lambda_m$ labeled in cyclic order. Consider the prisms contained in $\mathcal{P}_n$ having a face included in Λ_1 . Since each pair of them are disjoint or share one common face, these union also become a right distorted prism P_1 . The prisms P_1 and $\overline{\mathcal{P}}_{n-1}$ have one common face, and so their union $\overline{\mathcal{P}}_{n-1} \cup P_1$ is again a right distorted prism. In the same way, we see that a right distorted prism P_2 is formed by the prisms in $\mathcal{P}_n$ having a face included in the face of $\overline{\mathcal{P}}_{n-1} \cup P_1$ containing Λ_2 , and again $\overline{\mathcal{P}}_{n-1} \cup P_1 \cup P_2$ is a right distorted prism. Inductively, we proceed the same argument, and as a result $\overline{\mathcal{P}}_{n-1} \cup P_1 \cup \dots \cup P_m = \overline{\mathcal{P}}_{n-1} \cup \mathcal{P}_n$ is a right distorted prism.

This fact implies that the projection from $\bigcup_{i=0}^n \mathcal{Q}_i$ into the sphere at infinity of $\mathbb{H}^3$ along $\bigcup_{i=0}^n \mathcal{P}_i$ is a homeomorphism. Thus $\bigcup_{i=0}^n \mathcal{Q}_i$ is

an embedded 2-disk in $\mathbb{H}^3$ for every n , and $\widetilde{DF}$ is an embedded plane in $\mathbb{H}^3$. $\square$

REMARK 1.1. It follows from the above proof that the surface DF is quasi-Fuchsian. That is, the holonomy image of the subgroup corresponding DF is quasi-Fuchsian.

3. Manifolds decomposed into truncated tetrahedra

A *regular truncated tetrahedron* in $\mathbb{H}^3$ is defined as a polyhedron obtained from an ideal regular tetrahedron centered at the origin of the projective model of $\mathbb{H}^3$ by the following process. Move the vertices simultaneously away from the origin beyond the sphere at infinity and truncate a neighborhood of each vertex by its dual plane. Faces appearing by truncation are regular triangles and the other faces are right-angled hexagons. The dihedral angles formed by hexagonal faces equals each other, and they are called *internal dihedral angles*. We use Δ_n to denote a regular truncated tetrahedron with internal dihedral angles π/n , where n must be greater than three. We say that M admits a Δ_n -decomposition if M is decomposed by polyhedra each one of which is isometric to Δ_n so that their triangular faces form ∂M . We remark that M admits a Δ_n -decomposition if and only if each piece of the canonical decomposition of M is isometric to Δ_n [31].

The following is the main theorem in this section.

THEOREM 1.5. *If a hyperbolic 3-manifold M with totally geodesic boundary admits a Δ_n -decomposition and $n \geq 5$, then M has Property $(\star)$.*

Miyamoto showed that if M admits a Δ_n -decomposition and ∂M is connected, then n must be greater than or equal to six [40]. Hence, a hyperbolic 3-manifold with connected totally geodesic boundary admitting a Δ_n -decomposition has Property $(\star)$.

An explicit example of the 3-manifold admitting such a decomposition was given in [62]. It is the exterior of the knotted theta curve in the 3-sphere illustrated in Figure 1.2.

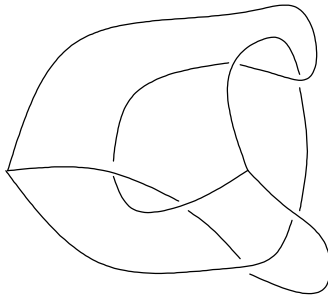


FIGURE 1.2. the knotted theta curve

It is shown that this manifold has the minimal volume among hyperbolic 3-manifolds with totally geodesic boundary [32]. Further, Miyamoto [40] proved that for each fixed $g \geq 2$, a hyperbolic 3-manifold with connected totally geodesic boundary of genus g having the minimal volume among such 3-manifolds admits a Δ_{3g} -decomposition. Therefore, a hyperbolic 3-manifold with connected totally geodesic boundary of genus g having the minimal volume among such manifolds has Property $(\star)$.

PROOF (Proof of Theorem 1.5.). Let M be a hyperbolic 3-manifold with totally geodesic boundary ∂M admitting a Δ_n -decomposition, where $n \geq 5$. Let L_M be the cut locus of M , and r_M the retraction of M onto L_M defined in the previous section.

Suppose that a non-trivial closed curve α on ∂M is given. We homotope α to be geodesic on ∂M , and then apply the construction given in Section 1. Let G denote the immersed graph in L_M obtained by that construction, and then it satisfies the vertex consistency. Let us show that G also satisfies Condition 1.

Homotope G such that each vertex lies on the midpoint on the 1-cell and each edge of G is geodesic in the 2-cell of L_M . We consider the midpoints of the edges of G . Let $\tilde{v}$ be a lift on $\tilde{G}$ of a vertex v of G . There are three edges of $\tilde{G}$ including $\tilde{v}$, since each 1-cell of L_M is shared by three 2-cells and G satisfies the vertex consistency. For $i = 1, 2, 3$, let m_i be the midpoint of the edges including $\tilde{v}$, C_i the 2-cell of $L_{\tilde{M}}$ including m_i , Φ_i the hyperplane containing m_i and the geometric

dual of C_i , and Ψ_i the components of $\partial\widetilde{M}$ containing $r_{\widetilde{M}}^{-1}(\widetilde{v})$ but not containing $r_{\widetilde{M}}^{-1}(m_i)$.

In order to prove that Φ_i 's and Ψ_i 's bound a right distorted prism, it is sufficient to show that

- (1) Φ_1 and Ψ_1 , (respectively Φ_2 and Ψ_2 , Φ_3 and Ψ_3) are ultra-parallel, and
- (2) Φ_1 and Φ_2 , (respectively Φ_2 and Φ_3 , Φ_3 and Φ_1) are ultra-parallel.

(1). We show that Φ_1 and Ψ_1 are ultra-parallel. The other cases can be proved in the same way.

Let E_2 be the edge of $\widetilde{G}$ containing m_2 . Note that Φ_1 bisects E_2 orthogonally in $\mathbb{H}^3$, since the closed 2-cell $\overline{C_2}$ is isometric to a totally geodesic regular $2n$ -gon in $\mathbb{H}^3$.

Now, M has a decomposition by polyhedra, each one of which is isometric to Δ_n . Let Δ be the one of the polyhedra containing v , and $\widetilde{\Delta}$ the lift of Δ containing $\widetilde{v}$. Since Φ_1 contains the geometric dual of $\overline{C_2}$, Φ_1 contains the edge of $\widetilde{\Delta}$ shared by two hexagonal faces, one of which contains $\widetilde{v}$ and the other is disjoint from Ψ_1 . Moreover, if E_2 connects $\widetilde{v}$ and a point on the j -th edge in $\overline{C_2}$, then the angle between Φ_2 and the face of $\widetilde{\Delta}$ containing $\widetilde{v}$ is $j\pi/2n$.

Then, the following claim assures that Φ_1 and Ψ_1 are ultra-parallel.

CLAIM 1. *Let F be a triangular face of Δ_n in $\mathbb{H}^3$, F' a hexagonal face intersecting F , e the edge of F' in the opposite side of $F \cap F'$, and H the hyperplane containing e . If the angle between F' and H is greater than or equal to $\pi/2n$ and $n \geq 4$, then H is ultra-parallel to the hyperplane containing F .*

PROOF. Suppose that the angle between F' and H is equal to $\theta \geq \pi/2n$.

Let x be the length of the edges of F . Then by the hyperbolic cosine law, we have

$$\cosh x = \frac{\cos^2 \pi/n + \cos \pi/n}{\sin^2 \pi/n} = \frac{\cos \pi/n}{1 - \cos \pi/n}.$$

Let y be the length of the geodesic arc on F' connecting the midpoints of $F \cap F'$ and e . This arc decomposes F' into two congruent right-angled pentagons. Then we have

$$\sinh y = \frac{\cosh x}{\sinh x/2}.$$

Suppose that H intersects the hyperplane containing F , and let θ' be the angle between them. By using the hyperbolic cosine law and the above equations, the square of the cosine of θ' is calculated as follows when $n \geq 4$.

$$\sin^2 \theta' \cosh^2 y \geq \sin^2 \frac{\pi}{2n} \cosh^2 y = \frac{3 \cos \pi/n - 1}{2(2 \cos \pi/n - 1)}.$$

However, the value is greater than 1 for $n \geq 4$. This is a contradiction. $\square$

(2). Since Ψ_3 is isometric to $\mathbb{H}^2$ and Φ_1 and Φ_2 intersect Ψ_3 orthogonally, it is enough to show that $\Phi_1 \cap \Psi_3$ and $\Phi_2 \cap \Psi_3$ are ultra-parallel on Ψ_3 , since the other cases can be proved in the same way.

There are two cases that (2A) both m_1 and m_2 lie on the lift of $r_M(\alpha)$ or its parallel part, or (2B) otherwise. We first prove the case (2B).

(2B) We prepare a geometric cellular decomposition of Ψ_3 induced from that of $L_{\widetilde{M}}$ by setting the preimage of cells of $L_{\widetilde{M}}$ by $r_{\widetilde{M}}$ as cells of Ψ_3 . Such a 2-cell of Ψ_3 is isometric to the interior of a regular $2n$ -gon with internal angles $2\pi/3$ in the hyperbolic plane $\mathbb{H}^2$. Hence the cellular decomposition of Ψ_3 gives a tiling $\mathcal{T}$ of Ψ_3 by such polygons.

By the construction of G together with the fact that Φ_i bisects the edge of $\widetilde{G}$ at m_i orthogonally, the next claim assures that $\Phi_1 \cap \Psi_3$ and $\Phi_2 \cap \Psi_3$ are ultra-parallel on Ψ_3 for $n \geq 5$.

Let T and T' be two tiles of $\mathcal{T}$ with one common edge e_0 . Let e_i (respectively e'_j) be the i -th (respectively j -th) edge of T (respectively T') in a counterclockwise (respectively clockwise) order from e_0 , where $1 \leq j \leq i \leq 2n - 1$ and $j \leq n$.

CLAIM 2. Let a_i (respectively b_j) be the bisector of the geodesic arc which connects the midpoint of e_0 and the midpoint of e_i (respectively e'_j) in T (respectively T'). Then a_i and b_j are ultra-parallel if and only if (i, j) is neither $(1, 1)$, $(2, 1)$, $(2, 2)$, $(3, 1)$ nor $(4, 1)$.

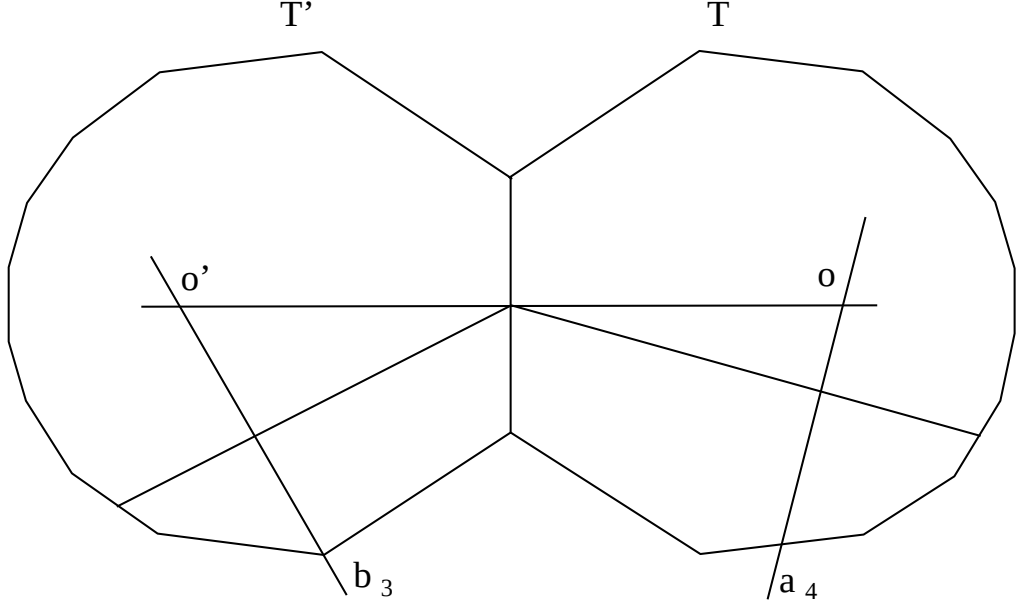


FIGURE 1.3. a_4 and b_3 in T and T'

PROOF. Let o (respectively o') be the center of T (respectively T'). Consider the triangle whose vertices are o , o' and an endpoint of e_0 . The hyperbolic cosine law gives

$$\cosh oo' = \frac{\cos^2 \pi/2n + \cos 2\pi/3}{\sin^2 \pi/2n} = \frac{\cos \pi/n}{1 - \cos \pi/n}.$$

If a_i , b_j and oo' forms a triangle or an ideal triangle, then the cosine of the angle between a_i and b_j is calculated by the hyperbolic cosine law as

$$\frac{\cos \pi/n}{1 - \cos \pi/n} \sin \frac{i\pi}{2n} \sin \frac{j\pi}{2n} - \cos \frac{i\pi}{2n} \cos \frac{j\pi}{2n}.$$

Let $X_{i,j}$ be the value of the formula. The followings are shown immediately. If $j \leq i \leq i' \leq n$, then $X_{i,j} \leq X_{i',j}$. If $n \leq i \leq i' \leq 2n$,

then $X_{i,j} \geq X_{i',j}$. If $1 \leq j \leq j' \leq n$, then $X_{i,j} \leq X_{i,j'}$. Moreover, we have

$$\begin{aligned} X_{4,1} &= \cos \frac{\pi}{2n}, \\ X_{5,1} &= \cos \frac{\pi}{n} + \frac{1}{2}, \\ X_{2,2} &= \cos \frac{\pi}{n}, \\ X_{3,2} &= 2 \cos \frac{\pi}{n} \cos \frac{\pi}{2n}, \\ X_{2n-1,1} &= \cos \frac{\pi}{n} + \frac{1}{2}. \end{aligned}$$

Now, n is greater than five. Hence $|X_{4,1}| < 1$, $|X_{5,1}| > 1$, $|X_{2,2}| < 1$, $|X_{3,2}| > 1$ and $|X_{2n-1,1}| > 1$ hold. This completes the proof. $\square$

(2A) Suppose that both m_1 and m_2 lie on the lift of $r_M(\alpha)$ or its parallel part. Recall that we have homotoped α to be geodesic on ∂M . Then, the next claim together with Claim 2 shows that $\Phi_1 \cap \Psi_3$ and $\Phi_2 \cap \Psi_3$ are ultra-parallel on Ψ_3 for $n \geq 4$. In the following, the notations are the same as for Claim 2.

CLAIM 3. *Let l be a geodesic line on Ψ_3 intersecting e_0 transversally. If l intersects e_1 , then it has no intersection with $e'_1, \dots, e'_4$. If l intersects e_2 then it has no intersection with e'_1, e'_2 .*

PROOF. First, suppose that l intersects e_1 . Let f_1 be the geodesic line including e_1 . This f_1 goes through the vertex shared by e'_n and e'_{n+1} , and separates T' into two components. Since l and f_1 do not intersect in T' , l does not intersect with $e'_1, \dots, e'_4$.

Next, suppose that l intersects e_2 . Let f_2 be the geodesic line which goes through the vertices $e_1 \cap e'_1$ and $e_2 \cap e_3$. Then f_2 has no intersection with e'_1 and e'_2 . To show this, let us compute the angle formed by f_2 and e_0 . Let x be the length of e_1 . Then

$$\cosh x = \frac{\cos^2 \pi/3 + \cos \pi/n}{\sin^2 \pi/3} = \frac{1 + 4 \cos \pi/n}{3}.$$

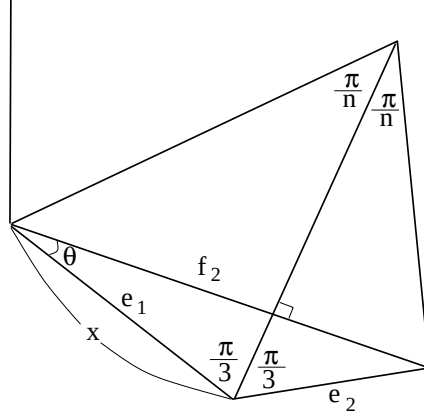


FIGURE 1.4

Let θ be the angle formed by e_1 and f_2 . Considering the triangle formed by e_1 , e_2 and f_2 , we obtain

$$\cosh x = \frac{\cos \theta \cos 2\pi/3 + \cos \theta}{\sin 2\pi/3 \sin \theta} = \frac{1}{\sqrt{3} \tan \theta}.$$

It follows from these equations that

$$\tan \theta = \frac{\sqrt{3}}{1 + 4 \cos \pi/n}.$$

The angle between f_2 and e_0 is $2\pi/3 - \theta$. Hence, we have

$$\tan \left(\frac{2\pi}{3} - \theta \right) = \frac{\tan 2\pi/3 - \tan \theta}{1 + \tan 2\pi/3 \tan \theta} = \frac{\sqrt{3}(2 \cos \pi/n + 1)}{1 - 2 \cos \pi/n}.$$

This value is negative for $n \geq 4$, so that the angle formed by f_2 and e_0 is greater than $\pi/2$. This implies that f_2 has no intersection with e'_1 and e'_2 and then l has no intersection with e'_1 and e'_2 . $\square$

Consequently, the immersed graph G in L_M obtained by the construction given Section 1 is homotoped to satisfy Condition 1. Since G contains $r(\alpha)$, the immersed graph $r^{-1}(G)$ contains α as a boundary component and is essential by Proposition 1.4. $\square$

4. Manifolds decomposed into truncated icosahedra

A *regular truncated icosahedron* in $\mathbb{H}^3$ is defined in the same way as a regular truncated tetrahedron. It looks like an angular soccer ball. Faces appearing by truncation are regular pentagons and the other faces are right-angled hexagons. We say that a regular truncated icosahedron is *right-angled* if the dihedral angles formed by its hexagonal faces are all $\pi/2$, and we denote it by Σ .

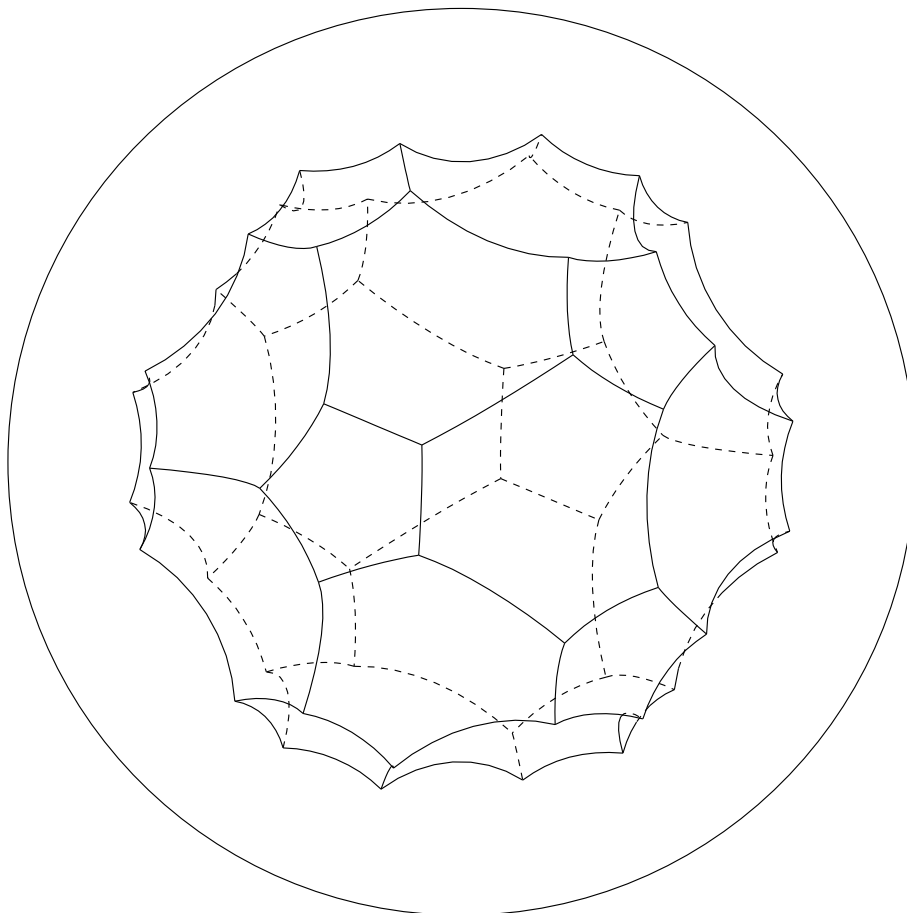


FIGURE 1.5. A right-angled regular truncated icosahedron in the Poincaré ball model of $\mathbb{H}^3$

The following theorem is proved in the similar way as Theorem 1.5.

THEOREM 1.6. *Let M be a hyperbolic 3-manifold with totally geodesic boundary ∂M decomposed into the polyhedra, each of which is*

isometric to Σ , so that their pentagonal faces form ∂M . Then M has Property $(\star)$.

PROOF. Again, let L_M be the cut locus of M , and r_M the retraction of M onto L_M defined in Section 3. Suppose that a non-trivial closed curve α on ∂M is given.

We introduce a geometric cellular decomposition of ∂M induced from that of L_M by setting the preimage of cells of L_M by r_M as cells of ∂M . Such a 2-cell of ∂M is isometric to the interior of a square with internal angles $2\pi/5$ in $\mathbb{H}^2$.

First, homotope α to be geodesic on ∂M . In addition, if α passes the three 1-cells of ∂M sharing one 0-cell successively, then we homotope α as Figure 1.6.

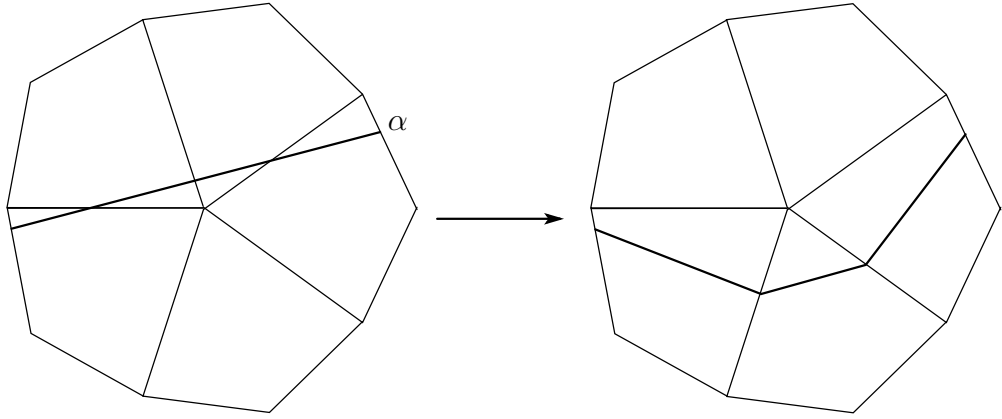


FIGURE 1.6

After these homotopies, we construct the immersed graph G using by our method. With the same strategy for Theorem 1.5, the following two claims guarantee that G satisfies Condition 1.

CLAIM 4. *Let F be a pentagonal face of Σ , F' a hexagonal face intersecting F , e the edge of F' in the opposite side of $F \cap F'$, and H the hyperplane containing e . If the angle between F' and H is greater than or equal to $\pi/4$, then H is ultra-parallel to the hyperplane containing F .*

In the similar way as in the proof of Theorem 1.5, the geometric cellular decomposition of $\partial\widetilde{M}$ induced from that of $L_{\widetilde{M}}$ gives a tiling $\mathcal{T}'$ of $\partial\widetilde{M}$ by the squares with internal angles $2\pi/5$. Let T and T' be two tiles of $\mathcal{T}'$ with one common edge e_0 . Let e_i (respectively e'_j) be the i -th (respectively j -th) edge of T (respectively T') in a counterclockwise (respectively clockwise) order from e_0 , where $1 \leq i, j \leq 3$.

CLAIM 5. *Let a_i (respectively b_j) be the bisector of the geodesic arc which connects the midpoint of e_0 and the midpoint of e_i (respectively e'_j) in T (respectively T'). Then a_i and b_j are ultra-parallel if and only if (i, j) is neither $(1, 1)$ nor $(3, 3)$.*

The above two claims are proved by the similar calculation as Claim 1 and Claim 2, and hence, we omit the proofs.

Consequently, the immersed graph G in L_M is homotoped to satisfy Condition 1. Since G contains $r(\alpha)$, the immersed graph $r^{-1}(G)$ contains α as a boundary component and is essential by Proposition 1.4. $\square$

By the Poincaré polyhedron theorem, the group generated by reflections in the faces of Σ is discrete, and we denote it by $G(\Sigma)$.

THEOREM 1.7. *Let M be the same as in Theorem 1.6, and Γ the holonomy image of the fundamental group of the double DM of M . If Γ is a finite index subgroup of $G(\Sigma)$, then M has Property $(\star\star)$.*

PROOF. By Theorem 1.6, for a non-trivial closed curve on ∂M , an immersed essential surface F containing it as a boundary component is constructed. Let DF be the immersed surface in DM obtained from two copies of F . As we remarked before, the holonomy image of the surface subgroup corresponding DF is a quasi-Fuchsian group. If Γ is a finite index subgroup of $G(\Sigma)$, then every geometrically finite subgroup is separable [4, Theorem 3.1] in Γ . Hence, DF is virtually embedded, and so is F . $\square$

5. Boundary of surfaces with bounded Euler number

Here, we give a proof of Theorem 1.2. The proof is essentially due to the arguments in [18, 19].

PROOF (Theorem 1.2.). Let M be an acylindrical 3-manifold with non-empty boundary ∂M . Let $\partial^S M$ be the union of the non-toral components of ∂M . Let M_0 the manifold obtained from M removing the toral components of ∂M . By Thurston's Uniformization theorem, we may suppose that M_0 is a hyperbolic 3-manifold with totally geodesic boundary of finite volume. We take the horotori disjoint each other in M_0 , and denote their union by T .

Let F be an immersed essential surface of Euler number χ . By [18, Theorem 4.4], F is homotoped in M_0 so that F has least area among its homotopy class. Let $c_1, \dots, c_k$ be the closed curves appearing as $F \cap T$, and denote by $L(c_i)$ the length of c_i on the horotori. Correspondingly, let $c'_1, \dots, c'_l$ be the closed curves appearing as $F \cap \partial^S M$, and denote by $L(c'_j)$ the length of c'_j on $\partial^S M$. Then, by [18, Theorem 4.3] and [19, Theorem 1], we have

$$\sum_{i=1}^k L(c_i) + h \sum_{j=1}^l L(c'_j) \leq \text{Area}(F) \leq 2\pi\chi,$$

where h is some constant given in [19].

Consequently, the length of each one of the curves are bounded. Hence, up to homotopy, the number of the boundary curves of the essential surface with fixed Euler number in M is bounded. $\square$

CHAPTER 2

Framed link presentations of surface bundles

A *surface bundle* is defined as an orientable closed fiber bundle over S^1 with an orientable surface as a fiber.

A link L in a 3-manifold M is called *fibred* if the complement fibers over S^1 such that its fiber is the interior of a Seifert surface for L . In the complement of a fibred link L , closed curves appear as the intersection of the fiber and $\partial N(L)$, and framings of L is defined by setting these curves as longitudes. This framings are said to be induced by the fibration of the complement. The fibration of the complement of a fibred link is extended to that of the closed 3-manifold obtained by the framed surgery on the link with induced framings. This means that every fibred link in S^3 with induced framings represents a surface bundle. Then we will show the following.

THEOREM 2.1. *Every surface bundle is represented by some fibred link in S^3 with induced framings.*

It is well-known that every closed orientable 3-manifold has a framed link presentation [33]. Moreover such a framed link can be taken as a *fibred* link in S^3 with some framings [8]. Our result shows that if a closed 3-manifold fibers over the circle, then such framings are taken to be the induced framings.

The proof will be given in Section 1. In Section 2, we will give a construction of a framed link presentation of the surface bundle with given monodromy.

1. A construction of a framed link presentation

At first, let us prepare definitions and a lemma. Throughout this chapter, F_g denotes a closed surface of genus g .

Every surface bundle with fiber F_g can be constructed by identifying $(x, 0)$ with $(\varphi(x), 1)$ in $F_g \times [0, 1]$, where φ is an orientation-preserving self-homeomorphism on F_g . We call this identifying map φ the *monodromy* of the bundle, and denote the surface bundle with monodromy φ by M_φ . Remark that if a self-homeomorphism φ_1 on F_g is isotopic to another one φ_2 , then M_{φ_1} is homeomorphic to M_{φ_2} , see [11].

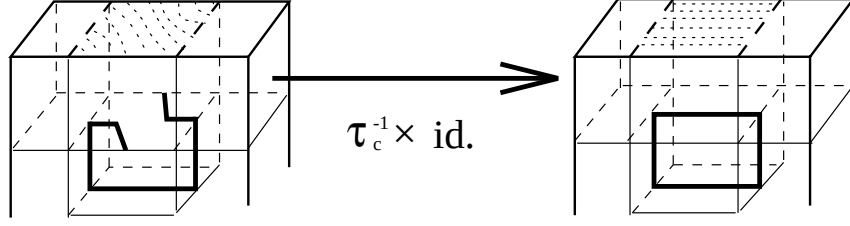
Let c be a simple closed curve on a surface F_g . The regular neighborhood $N(c)$ in F_g is parameterized by $\{(r, \theta) \mid 1 \leq r \leq 2, \theta \in \mathbb{R} \bmod 2\pi\}$. A *Dehn twist* along c is defined as a self-homeomorphism of F_g which is the identity on $F_g \setminus N(c)$ and is the map $(r, \theta) \mapsto (r, \theta - 2\pi r)$ on $N(c)$. It is denoted by τ_c .

The surface bundle with monodromy τ_c is obtained by identifying $(x, \frac{1}{2})$ in $F_g \times [0, \frac{1}{2}]$ with $(\tau_c(x), \frac{1}{2})$ in $F_g \times [\frac{1}{2}, 1]$, and $(x, 0)$ in $F_g \times [0, \frac{1}{2}]$ with $(x, 1)$ in $F_g \times [\frac{1}{2}, 1]$. Let N_c denote this manifold.

LEMMA 2.2. *Let c' be a simple closed curve $c \times \{\frac{1}{2} + \delta\}$ in $F_g \times [\frac{1}{2}, 1] \subset N_c$, where δ is a positive constant. Let γ be the longitude of c' obtained by twisting one of the curves appearing as $\partial N(c') \cap F_g \times \{\frac{1}{2} + \delta\}$ once in the counterclockwise direction. Then $N_c(c', \gamma)$ is homeomorphic to $F_g \times S^1$.*

PROOF. Let $N(c)$ be a regular neighborhood of c such that τ_c is the identity on $F_g \setminus N(c)$. We choose positive constants $\varepsilon, \varepsilon'$ such that $N(c) \times [\frac{1}{2} + \varepsilon, \frac{1}{2} + \varepsilon']$ is a regular neighborhood of c' in N_c . D denotes this subset of N_c . Then, an embedding $\tau_c^{-1} \times id : F_g \times [\frac{1}{2}, \frac{1}{2} + \varepsilon] \subset N_c \rightarrow F_g \times [\frac{1}{2}, \frac{1}{2} + \varepsilon] \subset F_g \times S^1$ is extended to an embedding $h : N_c \setminus \text{Int} D \rightarrow F_g \times S^1$. The complement of the image is homeomorphic to an open solid torus. The preimage of a meridian of this solid torus is isotopic to γ . Therefore $N_c(c', \gamma)$ is homeomorphic to $F_g \times S^1$. $\square$

Now we suppose that a surface bundle M_φ with monodromy φ is given. In the following, let us denote the trivial surface bundle $F_g \times S^1$ by E_g .

FIGURE 2.1. $\tau_c^{-1} \times \text{id.}$ on $F_g \times [\frac{1}{2}, \frac{1}{2} + \varepsilon]$

PROPOSITION 2.3. *There is a link $L = l_0 \cup \dots \cup l_n$ in M_φ with longitudes $\{\gamma_i\}_{0 \leq i \leq n}$ such that each l_i intersects each fiber at only one point and $M_\varphi(L, \{\gamma_i\}_{0 \leq i \leq n})$ is homeomorphic to S^3 .*

PROOF. we construct such a framed link in M_φ step by step.

At first, we construct a link L_1 in M_φ and longitudes $\{\gamma'_j\}_{1 \leq j \leq m}$ such that $M_\varphi(L_1, \{\gamma'_j\}_{1 \leq j \leq m})$ is homeomorphic to E_g .

By [34], any orientation-preserving self-homeomorphism on F_g is isotopic to a composition of a finite number of Dehn twists. Thus, we can choose simple closed curves $c_1, c_2, \dots, c_m$ on F_g such that φ is isotopic to $\tau_{c_1}^{\varepsilon_1} \circ \tau_{c_2}^{\varepsilon_2} \circ \dots \circ \tau_{c_m}^{\varepsilon_m}$, where $\varepsilon_i = \pm 1$, $1 \leq i \leq m$. Then M_φ can be represented as follows.

- $(x, \frac{j}{m+1})$ in $F_g \times [\frac{j-1}{m+1}, \frac{j}{m+1}]$ is identified with $(\tau_{c_j}^{-\varepsilon_j}(x), \frac{j}{m+1})$ in $F_g \times [\frac{j}{m+1}, \frac{j+1}{m+1}]$, $1 \leq j \leq m$.
- $(x, 1)$ in $F_g \times [\frac{m}{m+1}, 1]$ is identified with $(x, 0)$ in $F_g \times [0, \frac{1}{m+1}]$.

As in Lemma 2.2, simple closed curves l'_j and longitudes γ'_j of l'_j , $1 \leq j \leq m$, are specified so that $M_\varphi(L_1, \{\gamma'_j\}_{1 \leq j \leq m})$ is homeomorphic to E_g , where L_1 is the link $l'_1 \cup \dots \cup l'_m$. We remark that each l'_j is on a fiber of M_φ .

Next, we construct a link L_2 in E_g and longitudes $\{\gamma''_k\}_{0 \leq k \leq 2g}$ such that $E_g(L_2, \{\gamma''_k\}_{0 \leq k \leq 2g})$ is homeomorphic to S^3 .

Fix a point x on F_g . Then $F_g \setminus \text{Int}N(x)$ is homeomorphic to the surface obtained by plumbing $2g$ copies of annuli. Let S_g denote this surface. Note that $E_g = F_g \times S^1 \setminus \text{Int}N(l''_0)$ is homeomorphic to $S_g \times S^1$, where l''_0 is the curve $\{x\} \times S^1 \subset E_g$. We decompose $\partial(S_g \times [0, 1])$ into

two parts, $B_0 = S_g \times \{0, 1\}$ and $B_1 = \partial S_g \times (0, 1)$. Then $E_g \setminus \text{Int} N(l''_0)$ is homeomorphic to $D_{B_0}(S_g \times [0, 1])$. Since this $S_g \times [0, 1]$ is homeomorphic to the handlebody of genus $2g$, the double of the handlebody of genus $2g$ is homeomorphic to the connected sum of $2g$ copies of $S^2 \times S^1$. Hence $D_{B_0}(S_g \times [0, 1])$ is homeomorphic to the 3-manifold which is obtained by removing $B_1 \times (0, 1)$ from the connected sum of $2g$ copies of $S^2 \times S^1$. As B_1 is homeomorphic to an open annulus, $B_1 \times (0, 1)$ is homeomorphic to an open solid torus. A meridian of this solid torus corresponds to $D(\{y\} \times [0, 1])$, $y \in \partial S_g = \partial N(x)$. Let γ''_0 be this simple closed curve in $\partial N(l''_0)$. Then γ''_0 is a longitude of l''_0 such that $E_g(l''_0, \gamma''_0)$ is homeomorphic to the connected sum of $2g$ copies of $S^2 \times S^1$. In k th connected summand of that manifold, which is a copy of $S^2 \times S^1$, let l''_k be a simple closed curve running once in S^1 direction, and γ''_k a longitude of l''_k , $1 \leq k \leq 2g$. Then S^3 is obtained by framed surgery on $l''_1 \cup \dots \cup l''_{2g}$ with $\{\gamma''_k\}_{1 \leq k \leq 2g}$ from the connected sum of $2g$ copies of $S^2 \times S^1$. We regard $l''_1 \cup \dots \cup l''_{2g}$ with $\{\gamma''_k\}_{1 \leq k \leq 2g}$ as being in E_g , and let L_2 be a link $l''_0 \cup l''_1 \cup \dots \cup l''_{2g}$ in E_g . As a result, $E_g(L_2, \{\gamma''_k\}_{0 \leq k \leq 2g})$ is homeomorphic to S^3 . We remark that each l''_k is regarded as being on $S_g \times \{t\} \subset S_g \times [0, 1]$, where $1 \leq k \leq 2g$. This means that l''_k are regarded as being on fibers of E_g .

Finally, we regard E_g as $M_\varphi(L_1, \{\gamma'_j\}_{1 \leq j \leq m})$. L_2 can be regarded as a link in M_φ . Note that $M_\varphi(L_1 \cup L_2, \{\gamma'_1, \dots, \gamma'_m, \gamma''_0, \gamma''_1, \dots, \gamma''_{2g}\})$ is homeomorphic to S^3 . As we remarked before, each component of L_1 is on a fiber of M_φ and $l''_1 \cup \dots \cup l''_{2g} \subset L_2$ are regarded as being on fibers of M_φ . We replace each component of $l'_1 \cup \dots \cup l'_m \cup l''_1 \cup \dots \cup l''_{2g}$ and $\gamma'_1 \cup \dots \cup \gamma'_m \cup \gamma''_1 \cup \dots \cup \gamma''_{2g}$ by the band sum of itself and γ''_0 . The 3-manifold represented by the new link is homeomorphic to that represented by the original one (see [29]). Since γ''_0 intersects a fiber at only one point, we can slightly isotope the link obtained by this move such that each component intersects a fiber of M_φ at only one point.

Let L be this modified link in M_φ and $\{\gamma_i\}_{0 \leq i \leq n}$ be these modified longitudes, where $n = m + 2g$. By construction, this is a framed link which we want. $\square$

PROOF (Theorem 2.1). Let $L^* = l_0^* \cup \cdots \cup l_n^*$ be the union of cores of the solid tori attached by framed surgery on L with $\{\gamma_i\}_{0 \leq i \leq n}$. $S^3 \setminus L^*$ is homeomorphic to $M_\varphi \setminus L$ and it admits a fibration over S^1 . Moreover, a meridian of $N(l_i^*)$ is identified with γ_i , and the intersection of γ_i with a fiber of M_φ is only one point by construction. Hence, the intersection of a fiber of M_φ with a meridian of $N(l_i^*)$ is only one point. This implies that L^* is a fibered link in S^3 . $\square$

2. Examples of framed links

The aim of this section is to give a construction of a framed link presentation of the surface bundle with given monodromy.

Let L be a link in a 3-manifold M and $\{\gamma_i\}_{1 \leq i \leq n}$ longitude of L . The union of cores of solid tori attached by a framed surgery on L along $\{\gamma_i\}_{1 \leq i \leq n}$ is regarded as a link in $M(L, \{\gamma_i\}_{1 \leq i \leq n})$. Longitudes of this link are induced from the curves on $\partial N(L) \subset M$ bounding 2-disks in $N(K)$. We call such a link with longitudes in $M(L, \{\gamma_i\}_{1 \leq i \leq n})$ the *dual framed link* of $(L, \{\gamma_i\}_{1 \leq i \leq n})$.

EXAMPLE 2.1. The simplest example might be $F_g \times S^1$. Although a framed link presentation of this surface bundle is described in [46], we give it here by a different construction.

By tracing the proof of Theorem 2.1, we can illustrate the core curves of the solid tori attached by the framed surgery on $(L_2, \{\gamma'_j\}_{0 \leq j \leq 2g})$, where this is a framed link in $F_g \times S^1$ such that $\{F_g \times S^1\}(L_2, \{\gamma'_j\}_{0 \leq j \leq 2g})$ is homeomorphic to S^3 .

By the construction, these link represents $F_g \times S^1$ with some framings. Such framings are determined by the following way. $l'_0 \subset L_2$ intersects with a fiber of $F_g \times S^1$ at only one point. Since the other component are regarded as being on fibers of $F_g \times S^1$, there is a fiber $F_g \times \{t\}$ which $F_g \times \{t\} \cap L_2 = F_g \times \{t\} \cap l'_0$. This implies that the longitude of l'_0 which we want bounds a surface in $S^3 \setminus L_2$. Therefore, we label 0 to l'^*_0 . The other component are also labeled by 0. Because each of them with the framing which we want is a framed link presentation of $S^2 \times S^1$.

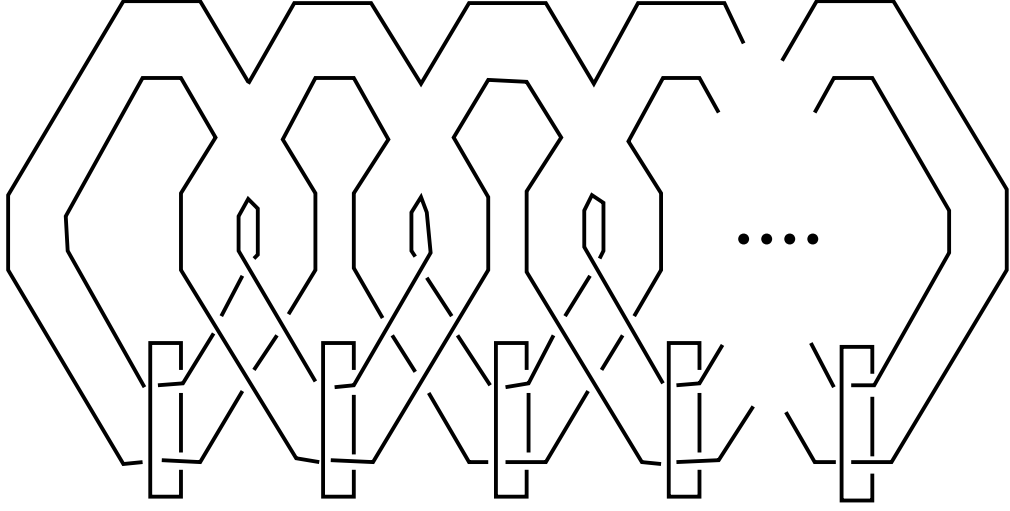


FIGURE 2.2. the dual framed link of $(L_2, \{\gamma'_j\}_{0 \leq j \leq 2g})$

EXAMPLE 2.2. Let A_1, A_2, B_1, B_2, C_1 be Dehn twists along the closed curves a_1, a_2, b_1, b_2, c_1 illustrated in Figure 2.3.

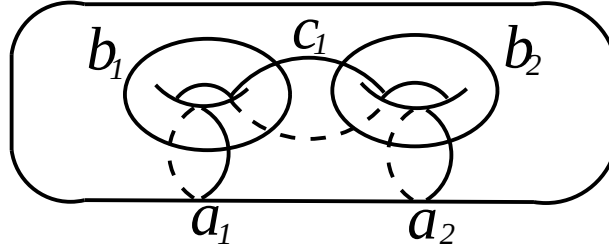


FIGURE 2.3

We regard the curve b_1 illustrated in Figure 2.3 as being on a fiber of M_{B_1} . The first step of the proof of Theorem 2.1 shows that $M_{B_1}(b_1, \gamma)$ is homeomorphic to $F_2 \times S^1$ where γ is the longitude of b_1 specified in Lemma 2.2. By the proof of Lemma 2.2, we may suppose that the core curve b_1^* of the solid torus attached by the framed surgery on (b_1, γ) , lies

on a fiber of $F_2 \times S^1$. This b_1^* is regarded as in S^3 since S^3 is obtained by framed surgery from $F_2 \times S^1$. We can determine the framing of b_1^* by tracing the proof of Lemma 2.2. Hence we get a framed link presentation of M_{B_1} .

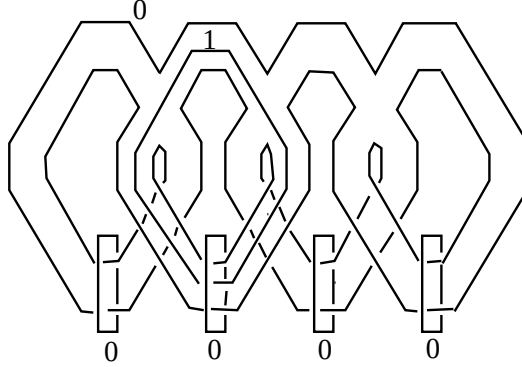


FIGURE 2.4. A framed link which represents M_{B_1}

We can find framed links which represents M_{A_1} , M_{B_2} , M_{C_1} , similarly.

EXAMPLE 2.3. Let φ be a self-homeomorphism of F_2 such that $\varphi = A_1 C_1 B_1 B_2$. Then a framed link which represents M_φ in our construction is illustrated in Figure 2.5.

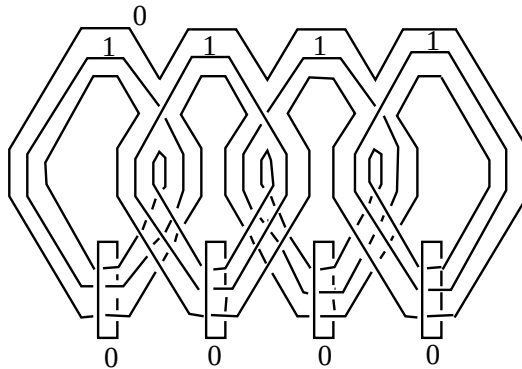


FIGURE 2.5. a framed link which represents M_φ

This can be modified by Kirby moves (see [29], [14]) to the $(2,5)$ torus knot. In the same way, if $\varphi = A_1 B_1 \cdots B_g C_1 \cdots C_{g-1}$ for F_g , a framed link presentation of M_φ in our construction is modified by Kirby moves to a $(2, 2g + 1)$ torus knot labeled 0.

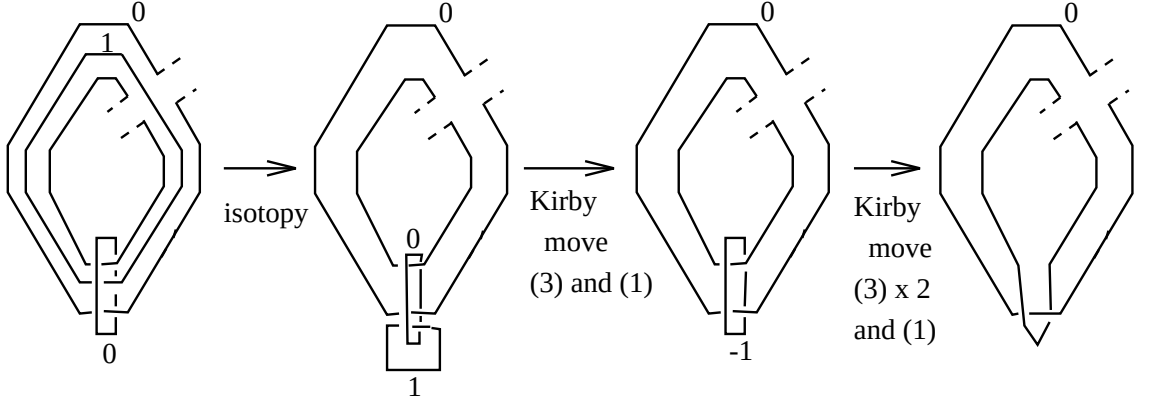
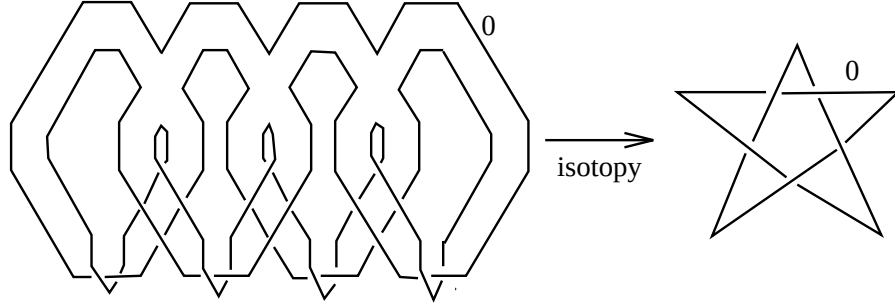


FIGURE 2.6. local picture

FIGURE 2.7. a framed link which represents M_φ

EXAMPLE 2.4. Let φ be a self-homeomorphism of F_2 such that $\varphi = A_1^{-1} A_2^2 C_1 B_1 B_2$. We obtain a framed link which represents M_φ illustrated in Figure 2.8.

This is modified by Kirby moves to a framed knot in S^3 .

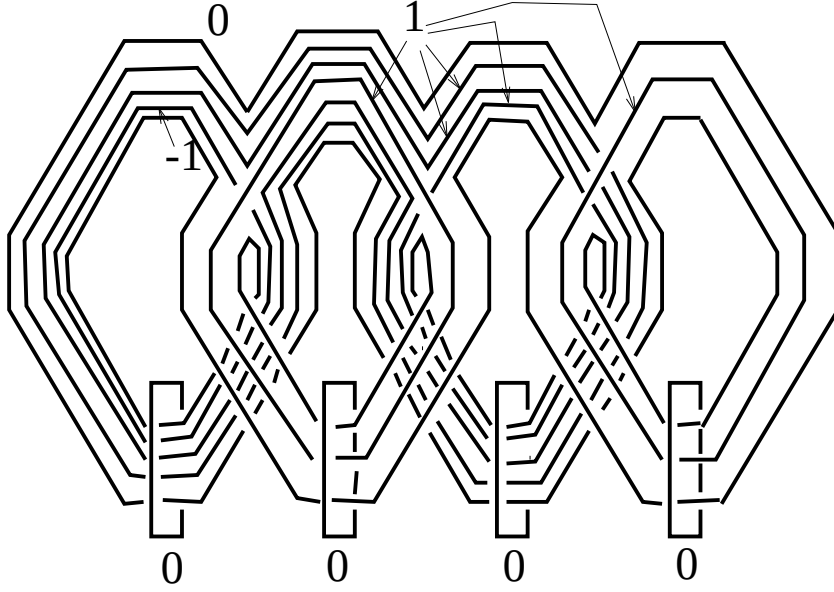


FIGURE 2.8

Next proposition easily follows Example 2.4. Let A_i be a Dehn twist along a_i , B_j a Dehn twist along b_j , and C_k a Dehn twist along c_k , where a_i, b_j, c_k are the curves on F illustrated in Figure 2.10, ($1 \leq i, j \leq g, 1 \leq k \leq g-1$).

PROPOSITION 2.4. *If a self-homeomorphism φ of F is isotopic to $A_1^{\lambda_1} \circ A_2^{\lambda_2} \circ B_1^{\varepsilon_1} \circ \dots \circ B_g^{\varepsilon_g} \circ C_1^{\delta_1} \circ \dots \circ C_{g-1}^{\delta_{g-1}}$, where $\varepsilon_i, \delta_j = \pm 1, (1 \leq i \leq g), (1 \leq j \leq g-1), \lambda_1 = -\delta_1$ and λ_2 is an arbitrary integer. Then the surface bundle with monodromy φ is represented by a fibered knot in S^3 labeled 0.*

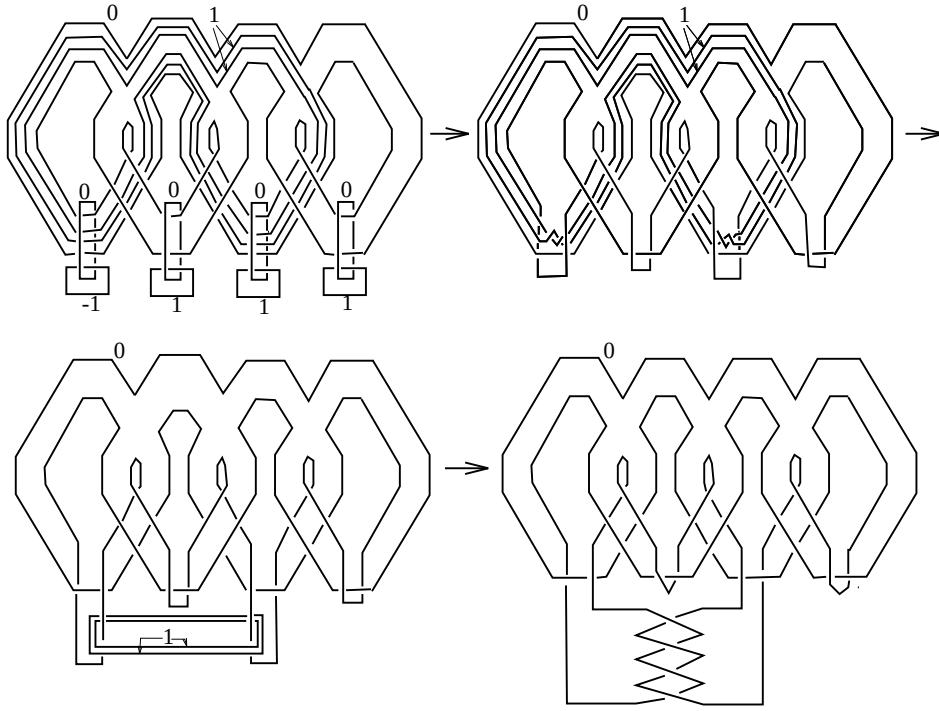


FIGURE 2.9

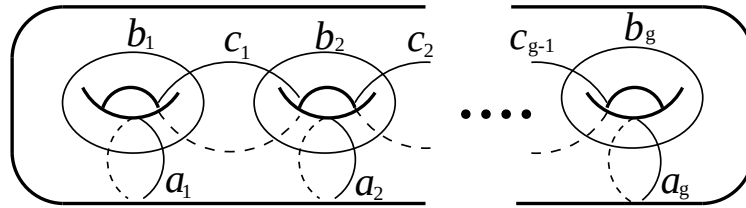


FIGURE 2.10

CHAPTER 3

Knot complements without geodesic surfaces

The following conjecture was proposed by Menasco and Reid in [43].

CONJECTURE 3.1. ([43, Conjecture 1], [30, Problem 1.76]) *A hyperbolic knot in S^3 does not have a closed totally geodesic surface embedded in its complement.*

This conjecture was solved for alternating knots ([42]), almost alternating knots ([2]), toroidally alternating knots ([1]), closed 3-braids ([37]), Montesinos knots ([49]) and tunnel number one knots ([43]). In fact, except for tunnel number one knots, any closed incompressible surface in these knot complements is meridionally compressible. This implies that there exists an accidental parabolic element, and so the surface is not even quasi-Fuchsian. For tunnel number one knots, the conjecture was solved algebraically in [43].

In Section 1, we will give another topological settlement of the conjecture for tunnel number one knot. The main result of this chapter is the following.

THEOREM 3.2. *Hyperbolic 3-bridge knots and double torus knots do not have a closed totally geodesic surface embedded in their complements.*

Moreover, slightly stronger results for free genus one knots will be given.

Next, we give in Section 2 two topological criteria for checking whether a closed essential surface fails to be totally geodesic in the complement of a hyperbolic knot K in S^3 .

For a closed surface S in $E(K)$, we define the *order* of S for K as follows [51]. Let $i_* : H_1(S) \rightarrow H_1(S^3 - K)$ be the homomorphism

induced by the inclusion map $i : S \rightarrow S^3 - K$. Since $\text{Im}(i_*)$ is a subgroup of $H_1(S^3 - K) = \mathbb{Z}\langle \text{meridian} \rangle$, there is an integer m such that $\text{Im}(i_*) = m\mathbb{Z}$. Then we define the *order* of S as $|m|$.

THEOREM 3.3. *Let K be a hyperbolic knot in S^3 of genus g , S a closed essential surface in $S^3 - K$ having order o . If $o > 3g - 3$, then S is not totally geodesic.*

The *canonical genus* of a knot is defined as the minimum among genera of Seifert surfaces built by Seifert's algorithm on all diagrams of the knot. Let V_1 denote the volume of a regular ideal tetrahedron in $\mathbb{H}^3$ and $V_{g'}$ that of a regular truncated tetrahedron with dihedral angles $\pi/3g'$ in $\mathbb{H}^3$.

THEOREM 3.4. *Let K be a hyperbolic knot in S^3 of canonical genus g , S a closed essential surface of genus g' in $S^3 - K$. If $g'V_{g'} \geq 60gV_1$, then S is not totally geodesic.*

1. Partial settlements

We start with preparing some definitions to state our results.

Let K be a knot in S^3 , S a closed essential surface in $E(K)$, and M_1 and M_2 denote the closure of components of $S^3 - S$, where M_1 contains K . We call S *accidental* if there is an embedded annulus A in $E(K)$ running from a non-trivial loop on S to a loop on $\partial E(K)$. We call this annulus A an *accidental annulus*, and such an essential surface S an *accidental surface*. If S is accidental, then we can construct an essential annulus A_1 in $M_1 - K$ by pasting two copies of an accidental annulus A for S and an annulus $\partial N(K) - A$. But, the converse does not hold generally. An example is given in [3, Figure 2].

The following lemma gives us a topological necessary condition for a closed essential surface isotopic to be a totally geodesic surface.

LEMMA 3.5. *If K is hyperbolic and S is totally geodesic in $S^3 - K$, then S is not accidental and neither $M_1 - K$ nor M_2 contains an essential annulus.*

REMARK 3.1. This lemma is well-known, but we give a proof for the convenience of the reader. We remark that this necessary condition is rather far from complete, see [3].

PROOF. Suppose that S is totally geodesic. Then the representation of $\pi_1(S)$ induced by a faithful discrete representation of $\pi_1(S^3 - K)$ is Fuchsian. Therefore it does not contain an accidental parabolic element. Assume $M_1 - K$ or M_2 contains an essential annulus. Then its double contains an essential torus. However, when S is totally geodesic, both doubles of $M_1 - K$ and M_2 are hyperbolic, and hence atoroidal. It is a contradiction. $\square$

For tunnel number one knots, Conjecture 3.1 was solved in [43, Corollary 4] by an algebraic method. The next theorem gives us a purely topological proof.

THEOREM 3.6. *If K is a tunnel number one knot in S^3 , S a closed essential surface in $E(K)$, M_1 and M_2 the closure of components of $S^3 - S$, where M_1 contains K , then S is accidental or M_2 contains an essential annulus.*

PROOF. Let $V_1 \cup V_2$ be a genus two Heegaard splitting of $E(K)$, where V_1 is a compression body and V_2 is a handlebody. By an isotopy of S , we may assume that $S \cap V_1$ consists of mutually parallel essential separating disks and non-separating disks in V_1 , where the labels are consecutive in V_1 . Suppose that $|S \cap V_1|$ is minimal among all surfaces isotopic to S . Note that $|S \cap V_1| \neq 0$ because of the incompressibility of S . If $S \cap V_2$ is a single disk, then S is a 2-sphere and hence compressible. Otherwise, since $S \cap V_2$ is ∂ -compressible in V_2 , a ∂ -compression of $S \cap V_2$ yields a band b in V_1 . By the minimality of $|S \cap V_1|$, b forms an incompressible non- ∂ -parallel annulus A in V_1 together with some disk of $S \cap V_1$. If A is non-separating in V_1 , then there is an accidental annulus for A in V_1 . If A is separating in V_1 , then either there is an accidental annulus for A in V_1 , or a sub-annulus A_0 of $\partial V_1 - A$ gives an essential annulus in M_1 or M_2 . In particular, if A_0 is in M_1 , then

the next ∂ -compression of $S \cap V_2$ yields a separating annulus parallel to A , and there is an essential annulus in M_2 . $\square$

It follows from Lemma 3.5 and the next theorem that hyperbolic 3-bridge knot complements do not contain a closed embedded totally geodesic surface.

THEOREM 3.7. *Let K be a 3-bridge knot in S^3 , S a closed essential surface in $E(K)$, M_1 and M_2 the closure of components of $S^3 - S$, where M_1 contains K . Then S is meridionally compressible or M_2 contains an essential annulus.*

PROOF. Suppose that S is meridionally incompressible. Let $(B_1, T_1) \cup (B_2, T_2)$ be a 3-bridge tangle decomposition of (S^3, K) , D_i ($i = 1, 2$) a disjoint union of disks in B_i which are cobounded by the strings of T_i and arcs in ∂B_i . By an isotopy of S , we may assume that $S \cap D_1 = \emptyset$, $S \cap B_1$ consists of essential disks in $B_1 - T_1$, $S \cap D_2$ consists of essential arcs on $S \cap B_2$, and $(S \cap B_2) - D_2$ consists of open disks. Under the above conditions, we assume that $|S \cap B_1|$ is minimal among all surfaces isotopic to S . By the incompressibility of S , $|S \cap B_1| \geq 1$. Moreover, since S separates S^3 into $M_1 (\supset K)$ and M_2 , $|S \cap B_1|$ is an even integer greater than 1. We perform a ∂ -compression of $S \cap B_2$ along an outermost disk in D_2 , and obtain an incompressible annulus A_1 in $B_1 - T_1$ which consists of a disk of $S \cap B_1$ and a resultant band b . It follows from the meridionally incompressibility of S that $S \cap B_1$ consists of mutually parallel disks which divide (B_1, T_1) into the trivial 1-string tangle (B_{11}, T_{11}) and the trivial 2-string tangle (B_{22}, T_{22}) , and the band b runs over (B_{22}, T_{22}) . Note that $|S \cap B_1|$ is an even integer greater than 1 since S separates S^3 into $M_1 (\supset K)$ and M_2 . If we perform the ∂ -compression of $S \cap B_2$ along the next outermost arc, then we obtain an incompressible annulus A_2 in $B_1 - T_1$ which is parallel to A_1 . Hence, there exists an annulus A_0 in $B_1 - T_1$ connecting A_1 and A_2 . We note that A_0 is contained in M_2 . If A_0 is ∂ -parallel in M_2 , then it contradicts the minimality of $|S \cap B_1|$. Since $S \cap D_2$ are essential

arcs in $S \cap B_2$, A_i ($i = 1, 2$) is incompressible in M_2 . Therefore, A_0 is incompressible in M_2 , thus, A_0 is an essential annulus in M_2 . $\square$

A knot K in S^3 is said to be a *double torus* if K is contained in a genus two Heegaard surface of S^3 . It is known that 2-bridge knots, pretzel knots, tunnel number one knots, and free genus one knots are double torus. Combined with previous results, the next theorem completes the proof of Theorem 3.3.

THEOREM 3.8. *Let K be a non-cable double torus knot with respect to a genus two Heegaard splitting $(F; V_1, V_2)$ of S^3 , S a closed essential surface in $E(K)$. Then S is accidental or $M_1 - K$ or M_2 contains an essential annulus.*

REMARK 3.2. The condition “non-cable” cannot be omitted. It is easy to see that cable knots with tunnel number one companion knots are double torus. In general, an accidental surface for the companion knot is not accidental for a cable knot if it is disjoint from its companion solid torus.

PROOF. We divide the proof into two cases, whether $F - K$ is incompressible in $S^3 - K$ or not.

Suppose that $F - K$ is compressible in $S^3 - K$ and let D be a compressing disk for $F - K$ in $S^3 - K$. We may assume that by exchanging it if necessary, D is a non-separating disk in V_1 . Put $V'_1 = cl(V_1 - N(D))$ and $F' = \partial V'_1$. Then V'_1 is a solid torus and its core, say γ , is a tunnel number one knot. Therefore K is a tunnel number one knot or a cable knot with a companion knot γ . In the former case, Theorem 3.6 gives the conclusion. In the latter case, if γ is non-trivial, then this contradicts the assumption of K . Hence γ is trivial and K is a torus knot. Since any incompressible closed surface in a torus knot complement must be isotopic to a peripheral torus, this case does not occur.

Next, suppose that $F - K$ is incompressible in $S^3 - K$, and assume that $|S \cap F|$ is minimal up to isotopy of S . Then each loop of $S \cap F$ is essential in both S and $F - K$. If there are mutually parallel

loops of $S \cap F$ in F or there is a loop of $S \cap F$ which is parallel to K in F , then M_i contains an essential annulus or S is accidental. Otherwise, $S \cap F$ consists of a single loop which separates F into two once punctured tori $F_i (\subset M_i)$, and K is non-separating in F_i . Since the loop $S \cap F$ is essential in S , F_1 is incompressible in M_1 . Let D be a compressing disk for S in S^3 , hence in M_1 . We may assume that by modifying D if necessary, $D \cap F_1$ consists of essential arcs in F_1 . Note that by the incompressibility of S in $S^3 - K$, $D \cap F_1 \neq \emptyset$. Let α be an outermost arc of $D \cap F_1$ in D , and δ the corresponding outermost disk in D . We perform a ∂ -compression of F_1 along δ in M_1 , and get an annulus F'_1 disjoint from F_1 . Since F_1 is incompressible in M_1 , F'_1 is also incompressible in M_1 , hence in $M_1 - K$. If F'_1 is essential in $M_1 - K$, then we have the desired conclusion. So, suppose that F'_1 is parallel to a subannulus A_1 in ∂M_1 , and let V be the solid torus cobounded by F'_1 and A_1 . Now, to recover F_1 , we attach a band b corresponding to the ∂ -compressing disk δ , to F'_1 . If b is contained in V , then we can find a compressing disk for F_1 in M_1 , a contradiction. Otherwise, F_1 is parallel to a subsurface in ∂M_1 , thus in S . This implies that K is isotopic to a loop in S , hence S is accidental. $\square$

A Seifert surface F for K in S^3 is said to be *free* if the fundamental group $\pi_1(S^3 - F)$ is free. We define the *free genus* of K as the minimal genus over all free Seifert surfaces for K . For free genus one knots, also we obtain

THEOREM 3.9. *If K is a free genus one knot in S^3 , S a closed essential surface in $E(K)$, M_1 and M_2 the closure of components of $S^3 - S$, where M_1 contains K , then S is accidental or M_2 contains an essential annulus.*

PROOF. Let F be a genus one free Seifert surface for K . We may assume that $F \cap S$ consists of essential loops in both F and S , and assume that $|F \cap S|$ is minimal. Since $\pi_1(S^3 - F)$ is a free group, $F \cap S \neq \emptyset$. If there is a loop of $F \cap S$ which is parallel to K in F , then S is accidental. Otherwise, all loops of $F \cap S$ are non-separating

and mutually parallel in F . Since S separates S^3 , hence F , there is an annulus component A of $F - S$ in M_2 . Then, the minimality of $|F \cap S|$ assures us that A is essential in M_2 . $\square$

2. Criteria

Theorem 3.3 follows from the next theorem and Lemma 3.5.

THEOREM 3.10. *Let K be a knot in S^3 of genus g , S a closed essential surface in $E(K)$ having order o . If $o > 3g - 3$, then S is accidental or $M_1 - K$ or M_2 contains an essential annulus.*

PROOF. Suppose that $o > 3g - 3$ holds. Let F be a minimal genus Seifert surface for K . We assume that $|S \cap F|$ is minimal among all minimal genus Seifert surfaces. Then each curve of $S \cap F$ is essential in both S and F .

Let us show that $|S \cap F| \geq o$. To do this, we construct a finite connected graph, say G_S , by corresponding a connected component of $S - (S \cap F)$ with a vertex, and a curve of $S \cap F$ with an edge. We fix an orientation of each edge of G_S induced from orientations of S , F and S^3 . Note that G_S is not a tree. Because if so, each edge is a cutting edge, and so each curve of $S \cap F$ is separating on S . But this contradicts the fact that $o > 0$. Hence G_S has at least one cycle. Let c be a cycle of G_S with an arbitrary orientation. We give the weight $+1$ to an edge of c when the orientation of the edge coincides with the orientation of c , and -1 otherwise. Then $|S \cap F| = |E(G_S)| \geq |o(c)|$ holds, where $o(c)$ is the sum of the weights of edges of c . On the other hand, a loop on S can be constructed for c such that the linking number of it and K is equal to $o(c)$. This implies that $|o(c)| \geq o$, and hence $|S \cap F| \geq o$.

By the above argument and the assumption, $|S \cap F| \geq o > 3g - 3$ holds. In the case that $n := |S \cap F| > 3g - 2$, we find an accidental annulus for S or a subannulus bounded by two curves of $S \cap F$ on F , since there are at most $3g - 2$ mutually non-parallel and non- ∂ -parallel simple closed curves on F . This subannulus is essential in M_i , since otherwise, we can reduce $|S \cap F|$. In the remaining case, that is,

$n = 3g - 2$, we consider the graph G_F constructed from F in the same way as G_S . Suppose that the curves of $S \cap F$ are mutually non-parallel and non- ∂ -parallel on F . Then that curves give a pants decomposition of F . This decomposition admits a checkerboard-coloring, since each pair of pants is contained in M_1 or M_2 . Let us note that G_F has the following property:

- 1) There is a vertex $v_0 \in V(G_F)$ having degree 2.
- 2) Every $v \in V(G_F) - \{v_0\}$ has degree 3.

Then the next lemma gives us a contradiction. Consequently, $S \cap F$ has some mutually parallel curves or a ∂ -parallel curve in F , and hence we can find an accidental annulus for S or an essential annulus in M_i . $\square$

LEMMA 3.11. *Let G be a finite graph satisfying the following two properties:*

- (1) *There is a vertex $v_0 \in V(G)$ having degree 2.*
- (2) *Every $v \in V(G) - \{v_0\}$ has degree 3.*

Then G is not a bipartite graph.

PROOF. Suppose that there is a 2-coloring $f : V(G) \rightarrow \{B, W\}$, and assume that $f(v_0) = B$. Let V_w be the set $\{v \in V(G) \mid f(v) = W\}$ and let V_b be the set $\{v \in V(G) \mid f(v) = B\}$. Then the number of edges which are incident to the vertices of V_b (respectively V_w) is $3(|V_b| - 1) + 2$ (respectively $3|V_w|$). However, since every edge of G connects a vertex of V_b and one of V_w , these values must be equal. Hence a contradiction. $\square$

REMARK 3.3. In the proof of Theorem 3.10, we note that $|S \cap F| = o > 0$ if and only if $S \cap F$ consists of mutually parallel non-separating loops in S .

Finally we prove Theorem 3.4.

PROOF (Theorem 3.4.). On one hand, it was shown in [10] that the volume of the complement of a hyperbolic knot is less than $120gV_1$,

where g is the canonical genus of the knot and V_1 is the volume of a regular ideal tetrahedron in $\mathbb{H}^3$. On the other hand, it was shown in [44] that the minimal volume of complete hyperbolic 3-manifolds with a totally geodesic boundary of genus g' is $g'V_{g'}$, where $V_{g'}$ is the volume of a regular truncated tetrahedron with dihedral angles $\pi/3g'$ in $\mathbb{H}^3$. Recall that K is a hyperbolic knot in S^3 of canonical genus g and S is a closed essential surface of genus g' in $E(K)$. Now, suppose that S is totally geodesic in $S^3 - K$. Let M_1 and M_2 be the closure of components of $S^3 - S$, where M_1 contains K . Then both $M_1 - K$ and M_2 are hyperbolic 3-manifolds with totally geodesic boundary, and so $\text{vol}(S^3 - K) = \text{vol}(M_1 - K) + \text{vol}(M_2)$. As a consequence, we get $60gV_1 > g'V_{g'}$. $\square$

REMARK 3.4. Since V_1 is approximately 1.01494 and $V_{g'}$ is computed by using the formula given in [44], we can see that the inequality $60gV_1 > g'V_{g'}$ fails if $g = 1$ and $g' > 16$. Therefore, there is no closed embedded totally geodesic surface of genus $g' > 16$ in a hyperbolic knot complement of canonical genus one. Note that $V_{g'} \nearrow V_0$ as $g' \rightarrow \infty$, where V_0 is the volume of a regular ideal octahedron in $\mathbb{H}^3$ and is approximately 3.6639.

CHAPTER 4

Accidental surfaces in knot complements

In this chapter, we study accidental surfaces in knot exteriors. Let S be a closed embedded essential surface in a knot exterior $E(K)$. A non-trivial simple closed curve l on S is called an *accidental peripheral* when there is an annulus A in $E(K)$ running from l to a loop on $\partial E(K)$. We call this annulus A an *accidental annulus* for l , and such an essential surface S with an accidental peripheral an *accidental surface*.

We first consider the accidentality of the essential surfaces. In Section 1, we will show the following.

THEOREM 4.1. (Uniqueness of the accidental slope) *Let S be an accidental surface in $E(K)$. On $\partial E(K)$, boundary curves of all accidental annuli for all accidental peripherals on S are mutually isotopic. Moreover, in the case that these curves are non-meridional, all accidental peripherals are mutually isotopic on S , and all accidental annuli for these are mutually isotopic relative to $\partial E(K)$ and S .*

As a result, boundary curves of accidental annuli for S determine the unique slope on $\partial E(K)$. Let us call this slope the *accidental slope* for S .

In Section 2, we will give a characterization of the accidental surfaces. Let F be a surface with non-empty boundary properly embedded in $E(K)$. We say that F is *strongly essential* if F is essential in $E(K)$ and at least one of the components of the exterior of F is ∂ -irreducible. The *boundary slope* of F is defined as the slope represented by ∂F on $\partial N(K)$. Then, we will prove

THEOREM 4.2. *The following are equivalent.*

- (1) *There exists an accidental surface S with an accidental slope γ in $E(K)$.*

- (2) *There exists a strongly essential surface F with a boundary slope γ in $E(K)$.*

Moreover if one of the conditions holds, the surface in another condition can be taken disjoint from the prescribed one.

As an application, we consider totally knotted knots. A Seifert surface F for a knot K is called *totally knotted* if $F \cap E(K)$ is strongly essential in $E(K)$. A knot K is called *totally knotted*, if there is a strongly essential Seifert surface for K . (Note that our definition differs from that of [58] slightly.) We will show

THEOREM 4.3. (Separating case) *The following are equivalent.*

- (1) *a knot K is totally knotted.*
- (2) *There exists an accidental surface in $E(K)$ with a separating accidental peripheral.*

In Section 3, we will give some criteria for checking whether the given essential surface S is accidental or not. The main tool is

THEOREM 4.4. *Let S be a closed embedded essential surface in $E(K)$. If there exists an essential, but not strongly essential surface with one or two boundary curves disjoint from S , then S is not accidental.*

In Section 4, we will show

THEOREM 4.5. *Let S be an accidental surface and F an essential surface with non-empty boundary disjoint from S in $E(K)$. Then the accidental slope of S and the boundary slope of F are coincident.*

From this result, it will be obtained that mutually disjoint accidental surfaces have the same accidental slope.

A motivation to study accidental surfaces is concerned with Menasco-Reid's conjecture considered in the previous chapter. When K is a hyperbolic knot, it is shown [60] that a closed embedded essential surface is accidental if and only if corresponding surface subgroup is not quasi-Fuchsian. Hence accidental surfaces fail to be totally geodesic.

The first explicit example of a closed embedded essential, but not accidental surface in a knot complement was found in [3], which is proved not to become totally geodesic. It is still unsolved which knots have accidental surfaces in their complements.

1. Uniqueness of the slope

In this section, we give a proof to show that Theorem 4.1.

PROOF (Theorem 4.1). Let A_1 and A_2 be accidental annuli for accidental peripherals l_1 and l_2 on S respectively. By isotopies of A_1 and A_2 , we may assume that $A_1 \cap A_2$ consists of essential arcs and loops in both A_1 and A_2 , and assume that $|A_1 \cap A_2|$ is minimal up to isotopy of A_1 and A_2 . If $A_1 \cap A_2$ has no arc, then the slopes determined by A_1 and A_2 coincide. Otherwise, let δ be a rectangle shaped disk region of $A_2 - A_1$. By the minimality of $|A_1 \cap A_2|$, δ meets A_1 in its both sides. Therefore, there is an accidental annulus A_0 for S coming from the disk δ and one of the two disks that $\partial\delta$ cuts A_1 into. This A_0 intersects A_1 in a single essential arc. Then the loop $\partial N((A_1 \cup A_0) \cap S; S)$ bounds a disk Δ in $E(K)$ such that $\Delta \cap S = \partial\Delta$. Since S is incompressible in $E(K)$, Δ shows that S is a peripheral torus, a contradiction.

Hereafter, suppose that the accidental slope for S is non-meridional. Again let A_1 and A_2 be accidental annuli for accidental peripherals l_1 and l_2 on S respectively. By the above argument, we may assume that $A_1 \cap A_2$ consists of only loops. First, we consider the case that $|A_1 \cap A_2| = 0$. Take a compressing disk D for S in S^3 so that $|D \cap K|$ is minimal. Then, we obtain an essential planar surface $P = D \cap E(K)$. By an isotopy of P , we may assume that $P \cap (A_1 \cup A_2)$ consists of essential arcs in all of A_1 , A_2 and P , and that $|P \cap (A_1 \cup A_2)|$ is minimal. Then, there exists a rectangle shaped disk region δ of $P - (A_1 \cup A_2)$ whose boundary is a union of an essential arc of A_1 , an arc α on S connecting l_1 and l_2 , an essential arc of A_2 , and an arc β in $\partial E(K)$ connecting $A_1 \cap \partial E(K)$ and $A_2 \cap \partial E(K)$. This β is an essential arc in an annulus A_0 of $\partial E(K) - (\partial A_1 \cup \partial A_2)$. By pasting A_1 , A_2 , two copies of δ and a disk $A_0 - \beta$, we can construct a disk Δ in $E(K)$ such

that $\Delta \cap S = \partial\Delta$. Since S is incompressible in $E(K)$, $\partial\Delta$ bounds a disk Δ' in S which does not contain l_1 , l_2 and α . This implies that l_1 is isotopic to l_2 in S . By irreducibility of a knot complement, $\Delta \cup \Delta'$ bounds a 3-ball in $E(K)$. Hence A_1 and A_2 are mutually isotopic with respect to S and $\partial E(K)$.

Next, suppose that $|A_1 \cap A_2| \geq 1$, and let A'_2 be a component of $A_2 - A_1$ which meets $\partial E(K)$. Pasting a subannulus of A_1 with A'_2 , we obtain an accidental annulus A''_2 disjoint from A_1 . By the above argument, A''_2 is parallel to A_1 , and this shows that A'_2 is parallel to a subannulus of A_1 since S is not a peripheral torus. Therefore we can reduce $|A_1 \cap A_2|$ by isotopy, and eventually we have $|A_1 \cap A_2| = 0$. This completes the proof. $\square$

REMARK 4.1. By [12, Lemma 2.5.3], we know that any accidental slope is either meridional or integral. An accidental surface with the meridional accidental slope is called *meridionally compressible*. In this case, there is a compressing disk for the surface in S^3 meeting K at just one point, which is called a *meridionally compressing disk*. It is known that for alternating knots ([42]), almost alternating knots ([2]), toroidally alternating knots ([1]), closed 3-braids ([37]), and Montesinos knots ([49]), every closed incompressible surface in their complements is meridionally compressible.

A motivation of Theorem 4.1 is the question whether a closed embedded essential surface in a knot exterior remains essential after Dehn surgeries on the knot. Wu proved in [66] that an essential, but not accidental surface become inessential after at most three surgeries. In the case of meridionally compressible surfaces, it is shown in [42] that if a meridionally compressible surface has two non-parallel accidental annuli, then it remains essential after all non-trivial Dehn surgeries. We remark that many knots have meridionally compressible surfaces which become compressible after all integral Dehn surgeries, see [65, Theorem 4.8]. In the case that the accidental slope γ is integral, it remains essential after γ' -Dehn surgery if and only if the geometric intersection number of γ and γ' is greater than one [12, Theorem 2.4.3].

Concerning these results, it is a natural question how many accidental slopes exist for an accidental surface. Our Theorem 4.1 answers this question.

2. A characterization of accidental surfaces

In this section, we give a characterization of accidental surface and totally knotted knot using strongly essential surfaces.

PROOF (Theorem 4.2). $1 \Rightarrow 2$. Let l be an accidental peripheral for S , and A the corresponding accidental annulus determining the accidental slope γ . Cutting S along l and pasting two copies of A , we construct a properly embedded surface F with non-empty boundary in $E(K)$. Since $S - l$ is incompressible in $E(K) - A$, F is incompressible in $E(K)$. The incompressibility of S in $E(K)$ implies that at least one of the closure of components of $E(K) - F$ is ∂ -irreducible. If F is ∂ -parallel in $E(K)$, then S would be a peripheral torus, a contradiction. Thus F is strongly essential in $E(K)$. By the construction, the boundary slope of F is the same as the accidental slope for S , and we can take F disjoint from S .

$2 \Rightarrow 1$. Let M be the closure of a component of $E(K) - F$ which is ∂ -irreducible. By pushing ∂M into $\text{Int}E(K)$, we obtain a closed surface S in $E(K)$. Since M is ∂ -irreducible and F is incompressible in $E(K)$, S is incompressible in $E(K)$. If S is a peripheral torus, then F is ∂ -parallel in $E(K)$, a contradiction. Finally, by the construction, S is accidental with the accidental slope γ and disjoint from F . $\square$

The following corollary is suggested by H. Matsuda. This follows from Remark 4.1 and Theorem 4.2 immediately.

COROLLARY 4.6. *The boundary slope of a strongly essential surface is either meridional or integral.* $\square$

As an application, we give a characterization of a totally knotted knot.

PROOF (Theorem 4.3). (1) $\Rightarrow$ (2). Suppose that K is totally knotted, and let F be a totally knotted Seifert surface for K . Then $\partial N(F)$ gives an accidental surface with a separating accidental peripheral.

(2) $\Rightarrow$ (1). Suppose that there exists an accidental surface S with a separating accidental peripheral l , and let A be the corresponding accidental annulus for S . Then cut S along l and paste the resultant surfaces and A . Since l separates S , each of obtained surfaces is a Seifert surface F disjoint from S . From the incompressibility of S in $E(K)$, In the same way as the proof of Theorem 4.2, F is shown to be strongly essential, and K is totally knotted. $\square$

REMARK 4.2. By the above proof, it follows that two Seifert surfaces obtained by cutting S along l and pasting A are strongly essential.

Also compare the result of [51] that there exists an order zero incompressible closed surface if and only if K bounds a non-free incompressible Seifert surface.

3. A necessary condition to be accidental

At first, we give a proof of Theorem 4.4. Remark that we do not assume in the proof that F is orientable.

PROOF (Theorem 4.4). Let S be an accidental surface with boundary slope γ , A the corresponding peripheral annulus, and F an essential surface with non-empty boundary such that $|\partial F| \leq 2$ and F is disjoint from S . Since A is incompressible in $E(K) - S$ and F is essential in $E(K)$, we may assume that $A \cap F$ consists of essential arcs and loops in both A and F . But, since F does not meet $S \cap A$, it follows that $A \cap F = \emptyset$ or $A \cap F$ consists of only loops. Hence, the boundary slope of F is γ .

Now, suppose that F is not strongly essential in $E(K)$. At first, we consider the case that $A \cap F = \emptyset$. By assumption, the closure M of the component of $E(K) - F$ containing A is ∂ -reducible. Let D be a compressing disk for ∂M in M . We may assume that $D \cap S = \emptyset$ because S is incompressible in $E(K)$. Note that ∂D intersects $A \cap E(K)$ since

$|\partial F| \leq 2$ and F is incompressible in $E(K)$. Since A is incompressible in $E(K) - S$, we may assume that $D \cap A$ consists of arcs whose ends are in $A \cap \partial E(K)$. Then, by cutting and pasting D along an outermost disk in A , we have a new compressing disk D' for ∂M in M with $|D' \cap A| < |D \cap A|$. Repeating this operation, we can get a compressing disk D for ∂M in M with $D \cap A = \emptyset$. This gives a compression disk for F , a contradiction.

In the case that $A \cap F$ consists of only loops, we consider the subannulus A' of $A - F$ nearest to $\partial E(K)$. Since F is not strongly essential, the closure M of the exterior of F containing A' is ∂ -reducible. By the same argument, we have a compressing disk D for ∂M in M such that $D \cap A'$ consists of essential arcs in A' . Then an outermost disk on D gives a ∂ -compressing disk for A' in M . This implies that A' is parallel to a subannulus in F with respect to $\partial E(K)$ since F is incompressible in $E(K)$. Thus we can reduce $|A \cap F|$ and eventually have $A \cap F = \emptyset$. $\square$

We have a few corollaries which can be used to check whether a given closed incompressible surface S is accidental or not. The following one is easily obtained from Theorem 4.4.

COROLLARY 4.7. *Let K be a non-trivial knot, S a closed incompressible surface in $S^3 - K$, and F a closed surface in S^3 such that $K \subset F$ and $F \cap S = \emptyset$. If $F - K$ is incompressible in $S^3 - K$ but $F \cap E(K)$ is not strongly essential in $E(K)$, then S is not accidental.* $\square$

As we stated, the first explicit examples of quasi-Fuchsian surfaces in the knot complement were given in [3]. The example of Hatcher in [3, Figure 4] gives a knot K in the genus two handlebody H such that $\text{Int}(H - N(K))$ is hyperbolic and $\partial(H - N(K))$ contains no accidental parabolic elements ([3, Lemma 2.3]). An observation gives a genus three closed surface F in $\text{Int}H$ containing K such that $F - K$ is incompressible in $H - K$ but $F - \text{Int}N(K)$ is not strongly essential in $H - \text{Int}N(K)$. Hence, by our corollary, ∂H is not accidental.

COROLLARY 4.8. *If K is composite, then every closed incompressible surface in its complement is either meridionally compressible or not accidental.*

PROOF. Let F be a decomposing sphere for K . We may assume that $F \cap S$ consists of essential loops in $F - K$. If $F \cap S \neq \emptyset$, then an innermost disk on F gives a meridionally compressing disk for S . If $F \cap S = \emptyset$ and S is accidental, then Theorem 4.4 says the accidental slope of S is meridional. $\square$

Let K be a knot and F a closed surface in S^3 . Then K is called *alternating with respect to F* if it can be isotoped into a neighborhood $F \times [-1, 1]$ of F so that it has an alternating diagram $\pi(K)$ on F with respect to a projection $\pi : F \times [-1, 1] \rightarrow F$. A diagram $\pi(K)$ on F is said to be *prime* if for any simple closed curve γ in F intersecting $\pi(K)$ in two non-double points, γ bounds a disk in F which intersects $\pi(K)$ in an arc.

THEOREM 4.9. ([52]) *Let K be an alternating knot with respect to F . Suppose that K has a prime alternating diagram $\pi(K)$ on F . Then, there exist a closed surface R in $F \times [-1, 1]$ and an isotopy of K so that $R \supset K$, $R - K$ is connected, incompressible in $S^3 - K$, and R bounds a handlebody in $F \times [-1, 1]$.*

By combining this and Theorem 4.4, we have

COROLLARY 4.10. *If K is alternating with respect to a closed incompressible surface S in $S^3 - K$ such that $\pi(K)$ is prime, then S is not accidental.*

This corollary seems to be a kind of generalization of non-triviality of alternating knots.

4. A conjecture on accidental slopes

We first give a proof of Theorem 4.5.

PROOF (Theorem 4.5). Let F be an essential surface with boundary slope γ , S an accidental surface disjoint from F , and A the accidental

annulus for S . By the essentiality of F and the incompressibility of A in $E(K) - S$, A can be isotoped so that $F \cap A = \emptyset$ or $F \cap A$ consists of loops. Thus S has the accidental slope γ . $\square$

We have the following corollary.

COROLLARY 4.11. *Mutually disjoint accidental surfaces have the same accidental slope.*

PROOF. Suppose that there are mutually disjoint accidental surfaces in $E(K)$ and let S be the nearest one to K . By Theorem 4.2, there exists a strongly essential surface F disjoint from S such that its boundary slope equals the accidental slope for S . Since all other surfaces are disjoint from F , the accidental slopes for those are all coincide with that of S by Theorem 4.5. $\square$

This corollary lets us expect the next property of accidental surfaces.

CONJECTURE 4.12. *All integral accidental slopes of accidental surfaces in a knot complement are coincident.*

The assumption “integral” cannot be omitted. There is a genus one totally knotted knot which admits an essential 2-string tangle decomposition (See Figure 4.1). It can be shown that the complement of this knot contains a closed, incompressible, meridionally compressible surface by tubing operations on the 2-string tangle decomposing sphere. Thus there are two accidental surfaces with slopes 0 and ∞ respectively.

However, for integral slopes, we have a partial solution.

PROPOSITION 4.13. *Let K be a cable knot in S^3 . Then any accidental surface in $E(K)$ has the accidental slope same as the cabling annulus for K .*

PROOF. Let T be a torus containing K , and S an accidental surface in $E(K)$ with an accidental annulus A . Let V be a solid torus bounded by T in S^3 . We assume that $|S \cap T|$ is minimal up to isotopy of S .

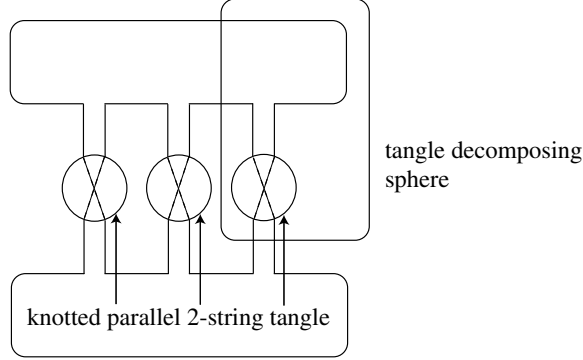


FIGURE 4.1. genus one totally knotted knot which has an essential 2-string tangle decomposition

If $|S \cap T| = 0$, then by the incompressibility of S , $S \subset S^3 - V$. Further, we may assume that $A \cap T$ consists of essential loops in both A and $T - K$. By pasting the subannulus of $A - T$ meeting S and a subannulus in T , we obtain an accidental annulus A' such that $A' \cap (T \cap E(K)) = \emptyset$. Thus S is also an accidental surface for a companion knot of K . But, this is a contradiction since the accidental slope of S is neither meridional nor integral.

Otherwise, a subannulus of T gives an accidental annulus for S . Then, Theorem 4.1 assures Proposition 4.13. $\square$

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