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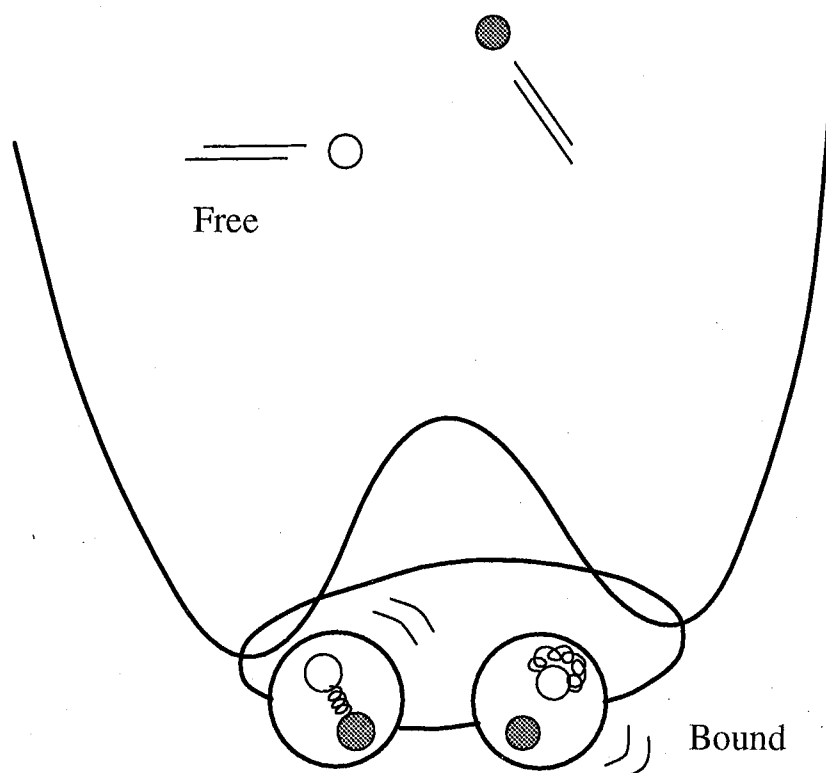
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*Dynamical Chiral Symmetry Breaking
in Light Quark Theory*

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March 2, 1998



Dissertation for the Degree of Doctor of Science

Abstract

We study pseudo-scalar mesons as the Nambu–Goldstone boson for the dynamical chiral symmetry breaking. We use effective models of QCD described by light quark fields in order to explain the derivation of the Bethe–Salpeter equation as a non-perturbative analysis which describes relativistic bound states. We find that the Noether current in the effective model, which is called effective currents in this thesis, is complicated and different from that in QCD. Introducing the effective currents, the modifications of the Ward–Takahashi identity for axial-vector current and the PCAC (partially conserved axial vector current) relation are derived.

Using the Cornwall–Jackiw–Tomboulis effective action formulation, the Schwinger–Dyson equation and the Bethe–Salpeter equation are derived. We find that the modified Ward–Takahashi identity for axial-vector current and the PCAC relation still hold even in the rainbow-ladder approximation if one employs a suitable definition of the decay constant. For this purpose we derive the systematic formula of the decay constant. Our approach is an image of the Noether’s theorem and the Nambu–Goldstone theorem on the classical action to the CJT action.

As an example of our formula, numerical result of the pion decay constant is reported. The mass of the η meson is also calculated.

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1 Introduction

1.1 Introduction

Nowadays it is believed that the origin of the strong interaction is a non-abelian gauge theory called Quantum Chromodynamics(QCD).

In the high energy region the superb methods of perturbation theory are available and there are successful results such as the deep inelastic scattering of hadrons. This is due to the fact that at high energies QCD has a property of asymptotic freedom and the perturbation theory and perturbative renormalization group equation(RGE) work well.

In the low energy region, however, it seems hard to study the hadron properties, as they are essentially determined by the non-perturbative dynamics. The lattice theory is in fact the most prominent in the non-perturbative regime, but it suffers from several serious problems such as ambiguities in chiral Fermions. It has exhausted the computer power.

One of the powerful ways to study the hadron properties qualitatively is to use the symmetry of the system. Especially the chiral symmetry plays an important role. It is believed that the light pseudo-scalar mesons π, K, η are pseudo Nambu–Goldstone(NG) bosons appearing with the spontaneous chiral symmetry breaking(SCSB) which is well described by the wine bottle potential shown in Fig.1-1. The small masses of them must be an evidence of explicit chiral symmetry breaking(ECSB). From experimental data, these light mesons should be not elementary particles but the dynamical bound states of elementary particles quarks. The SCSB

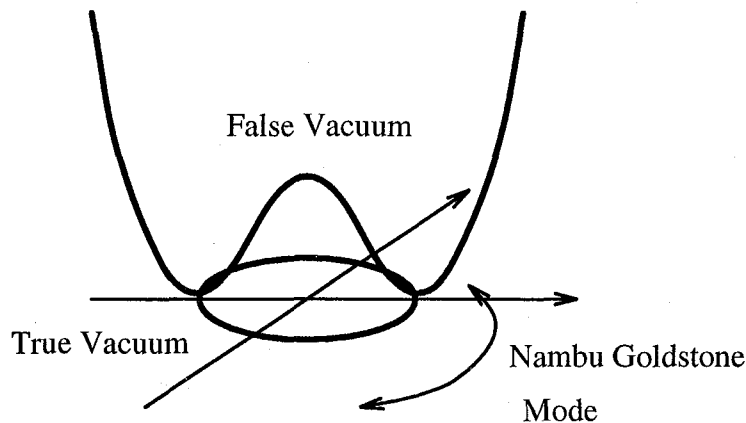


Fig.1-1: Wine Bottle Poential

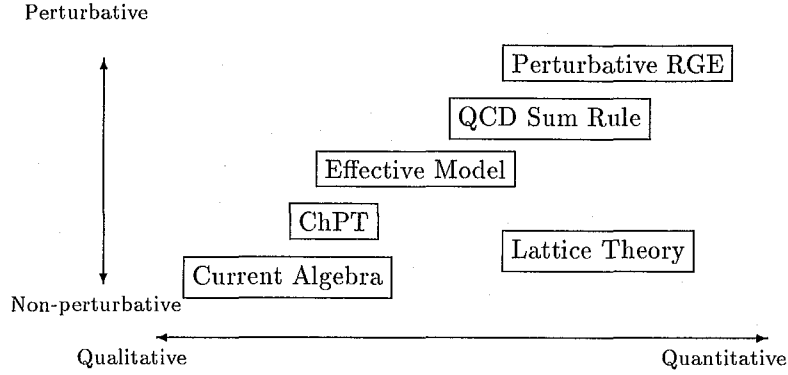


Fig.1-2: Personal View of Strong Interaction

which occurs due to the dynamics is called dynamical chiral symmetry breaking(DCSB). Several methods such as current algebra, chiral perturbation theory(ChPT) etc. have been developed to study the properties of hadrons from the chiral symmetry viewpoint.

On the other hand, to study the dynamics in the low energy region quantitatively, it is popular to use effective models for light quarks. There are various types of effective models which respect the chiral symmetry. A few examples are the Nambu–Jona-Lasinio(NJL) model[1], the ‘improved ladder’ Kyoto Group(KG) model[2], and so on. These models violate the gauge symmetry but are simpler than QCD. Some of the non-perturbative analyses are available for them. My personal view of the strong interaction is shown in Fig.1-2. We suppose that the effective model approach connects the quark-gluon description of QCD and the mesonic description of the low energy effective theories. Our aim is to study the dynamics at the low energy with effective models in order to improve the hadron theories.

As the non-perturbative analysis we concentrate on the method of the Bethe–Salpeter equation(BSE). The case of the non-dynamical SCSB, one can use perturbation methods after translation of the local field $\phi \rightarrow \tilde{\phi} := \varphi + \langle \phi \rangle$. Then the picture of the SCSB can be understood clearly. But this is not applied for the DCSB where non-perturbative methods are indispensable. The analysis of the DCSB must be relativistic, because the Nambu–Goldstone theorem holds under the assumption the Lorentz invariance of the system. The BSE was proposed by many authors almost fifty years ago and is one of the most popular methods as ‘ladder model’ in order to describe relativistic bound states.[3] Although it suffered from mathematical and

numerical difficulties for first decade of its history, the Cornwall, Jakiw and Tomboulis effective action formulation[4, 5, 6] produces a systematic derivation of BSE in mathematical viewpoint and the recent development of the computer technology overcomes a numerical difficulty.

But it is not yet trivial to calculate the physical quantities consistently with the chiral symmetry. One difficulty is that the chiral symmetric property of the effective models may be different from QCD even if the effective model has a global chiral symmetry. In fact, we started our study by estimating the pion mass and the decay constant in the KG model in order to confirm the Gell-Mann, Oakes and Renner(GMOR) mass formula

$$M_\pi^2 f_\pi^2 \simeq 2m_0 \langle \bar{\psi}\psi \rangle \quad (1.1)$$

for a finite bare quark mass m_0 . The GMOR formula holds when both the ECSB and the DCSB occur and plays an important role in ChPT. But the numerical result implied that the GMOR formula is violated even in the chiral limit. We found that this strange result is caused by large violation of the Ward–Takahashi identity for axial vector current and the PCAC(partially conserved axial vector current) relation.

Another difficulty is that one must carry out a non-perturbative analysis so as to preserve the chiral symmetric properties of the effective models. In the BSE approach, it is important to keep the mutual consistency between the Schwinger–Dyson equation(SDE) and BSE.

Furthermore in constructing an effective model or in the non-perturbative analysis we need a systematic procedure to avoid divergences of Feynman loop integrals. Such regularization scheme may violate the chiral symmetry and modify the chiral symmetric properties.

The light quark theory in the title of this paper means the method of the coupled system of the SDE and the BSE for the effective models described by light quarks. We study the DCSB in this light quark theory at mainly mathematical viewpoint.

In Sec.2 we construct the effective models. In Sec.3 we study the chiral properties of the effective models. In Sec.4 we review the CJT action formulation. In Sec.5 we study the SDE-BSE approach and derive a formula of the modified decay constant which is our main result. In Sec.6 we prove that our regularization procedure works well. In Sec.7 we collect the related topics. In Sec.8 we give an application. In Sec.9 we show the numerical result of Sec.8. In Sec.10 we conclude our study.

1.2 Chiral Symmetry in QCD

The classical QCD lagrangian (density) is given by

$$\mathcal{L}_{\text{cl}}[\psi, \bar{\psi}, A] := \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} \quad (1.2)$$

where

$$F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \quad (1.3)$$

$$D_\mu \psi := \partial_\mu \psi - igT^a A_\mu^a \psi. \quad (1.4)$$

The full QCD lagrangian is given by

$$\mathcal{L}_{\text{full}}[\psi, \bar{\psi}, A, B, c, \bar{c}] := \mathcal{L}_{\text{cl}}[\psi, \bar{\psi}] + \mathcal{L}_{\text{GF+FP}}[A, B, c, \bar{c}] \quad (1.5)$$

where the second term denotes the gauge fixing and Faddeev–Popov ghost term. Except for the mass term, each QCD lagrangian is ‘formally’ invariant under the (chiral) vector $U(N_F)_V$ transformation defined by

$$\psi(x) \mapsto \psi'(x) := \exp\left\{i\frac{\lambda^\alpha}{2}\theta^\alpha\right\}\psi(x) \quad (1.6)$$

and under the (chiral) axial $U(N_F)_A$ transformation defined by

$$\psi(x) \mapsto \psi'(x) := \exp\left\{i\gamma_5\frac{\lambda^\alpha}{2}\theta^\alpha\right\}\psi(x) \quad (1.7)$$

where λ^α is a generator of flavor $U(N_F)$. We wrote “formally” since, as is known, in quantum gauge theories the $U(1)_A$ symmetry is violated by the Adler–Bell–Jackiw (ABJ) anomaly.

Then QCD incorporates the $SU(N_F)_L \times SU(N_F)_R \times U(1)_V$ symmetry.

1.3 Theorems

Theorem 1 (Noether) *Let us consider a one-parameter set of the continuous transformations under which the lagrangian is invariant. Then it is associated a local conserved current $J_\mu^\alpha(x)$. Integrating the time component $J_0^\alpha(x)$ of this current over three-space generates a conserved ‘charge’*

$$Q^\alpha := \int d\mathbf{x} J_0^\alpha(x).$$

Furthermore this charge Q^α is a generator of the transformation.

The conserved current $J_\mu^\alpha(x)$ and the charge Q^α are called Noether current and Noether charge respectively.

Theorem 2 (Nambu–Goldstone) *Let us consider a Lorentz and translational invariant local field theory with conserved currents J_μ relating to a Lie group G . Let us assume that this symmetry is spontaneously broken so that there is some local fields $\phi_k(x)$ with*

$$\langle 0|[iQ^\alpha, \phi_k(x)]|0\rangle \neq 0$$

where Q^α denotes Noether charge. Then there are massless spinless particles which couple to both the currents J_μ^α and fields ϕ_k .

The massless spinless particle is called Nambu–Goldstone boson.

1.4 Notation

Here we summarize notations of this thesis.

' $:=$ ' and ' $=:$ ' denote the definition of left hand side(LHS) and right hand side(RHS), respectively.

' $\equiv$ ' denotes the identity in function space.

The metric tensor is $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

' $\{, \}$ ' denotes anti commutator. i.e. $\{A, B\} = AB + BA$.

For the Dirac indices we use $a, b, \dots$. For the color indices we use $i, j, \dots$. For the flavor indices we use $f, g, \dots$. The combined indices $m, n, \dots$ are introduced such as $m = (a, i, f)$, $n = (b, j, g), \dots$.

Tr denotes the trace in the coordinate space and the Dirac, color, flavor space. tr denotes the trace in the Dirac, color, flavor space. $\text{tr}^{(\text{DC})}$ denotes the trace in the Dirac and color space. $\text{tr}^{(\text{D})}$ denotes the trace in the Dirac space. $\det^{(\text{F})}$ denotes the determinant in the flavor space.

For the Lorentz indices we use $\mu, \nu, \rho, \sigma, \tau, \dots$.

T^a is a generator of color $SU(N_C)$ with $a = 1, \dots, N_C^2 - 1$. We use the same index a as the Dirac index. In the $N_C = 3$ case, T^a is a half of Gell-Mann matrix:

$$\begin{aligned} T^1 &:= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^2 &:= \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T^3 &:= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ T^4 &:= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & T^5 &:= \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & T^6 &:= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ T^7 &:= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & T^8 &:= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \quad (1.8)$$

λ^α is a matrix of flavor $SU(N_F)$ space. We does not use the index α, β as the Lorentz index. For $\alpha = 1, \dots, N_F^2 - 1$, they are generators of $SU(N_F)$ and for $\alpha = 0$ it is given by

$$\lambda^0 := \sqrt{\frac{2}{N_F}} \text{diag}(1, \dots, 1). \quad (1.9)$$

Moreover we define the $\lambda^\pi, \lambda^{qq}, \lambda^{ss}$. In the $N_F = 3$ case

$$\lambda^{\pi^+} := \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{\pi^0} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{\pi^-} := \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (1.10)$$

$$\lambda^{qq} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^{ss} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \quad (1.11)$$

The naive axial current $J_{5\mu}^\alpha(x)$ is defined by Eq.(3.7) model independently. The effective axial current owing to the regularization $I_{5\mu}^\alpha(x)$ is defined by Eq.(3.10) which depends on only the regularization function $f(x)$. The effective axial current $\tilde{J}_{5\mu}^\alpha(x)$ is a generalized Noether current which depends on a full lagrangian of a model.

The symbol $*$ in Eqs.(2.13),(2.15),(2.17),(2.35),(3.13),(5.37),(5.51),(8.6),(8.13),(8.17) and (8.18) means $x_1, x_2, \dots, y_1, y_2, \dots \rightarrow x$ after all derivatives are operated.

We often omit the comma in the argument.

For the integral of the momentum variable we use $\int_q := \int \frac{d^4 q}{(2\pi)^4}$.

The Fourier transformation of the quark field $\psi(x)$ is given by $\psi(x) =: \int_q e^{-iqx} \psi(q)$.

We sometimes omit the integral if it is obvious. An example is

$$\lim_{P \rightarrow P_B} \frac{1}{P^2} \bar{\chi}_{nm}(x, y; P_B) \frac{\delta \partial^\mu \tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](0)}{\delta S_{Fnm}(x, y)} \quad (1.12)$$

where we omit $\int d^4x d^4y$.

2 Construction of Effective Models

2.1 Preliminary

In this section we construct effective models which are described by light quark fields.

2.2 QED Case

The QED lagrangian is given by

$$\mathcal{L}[\psi, \bar{\psi}, A] := \bar{\psi}(i\rlap{\not{\partial}} + eA - m_0)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial A)^2, \quad (2.1)$$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.2)$$

QED is a prototype of the gauge theory and the effective model can be constructed systematically as following orthodox procedure. This lagrangian is written as

$$\mathcal{L}[\psi, \bar{\psi}, A] = \frac{1}{2}A^\mu M_{\mu\nu}(-i\partial)A^\nu + e\bar{\psi}A\psi + \bar{\psi}(i\rlap{\not{\partial}} - m_0)\psi, \quad (2.3)$$

$$M_{\mu\nu}(p) := -p^2 g_{\mu\nu} + (1 - \frac{1}{\alpha})p_\mu p_\nu. \quad (2.4)$$

Defining the translated photon field $\tilde{A}^\mu$ as

$$\tilde{A}^\mu := A^\mu + (M^{-1})^{\mu\nu}(-i\partial)e\bar{\psi}\gamma_\nu\psi, \quad (2.5)$$

the lagrangian Eq.(2.3) is rewritten as

$$\begin{aligned} \mathcal{L}[\psi, \bar{\psi}, A] &= \frac{1}{2}\tilde{A}^\mu M_{\mu\nu}(-i\partial)\tilde{A}^\nu \\ &\quad + \bar{\psi}(i\rlap{\not{\partial}} - m_0)\psi - \frac{1}{2}e^2\bar{\psi}\gamma_\mu\psi(M^{-1})^{\mu\nu}(-i\partial)\bar{\psi}\gamma_\nu\psi. \end{aligned} \quad (2.6)$$

Eq.(2.6) is quadratic in $\tilde{A}$. Using the measure invariance $\mathcal{D}A = \mathcal{D}\tilde{A}$ we can integrate the photon field

$$\begin{aligned} &\int \mathcal{D}A \exp i \int d^4x \mathcal{L}[\psi, \bar{\psi}, A](x) \\ &= \int \mathcal{D}\tilde{A} \exp i \int d^4x \left(\frac{1}{2}\tilde{A}^\mu M_{\mu\nu}(-i\partial)\tilde{A}^\nu + \bar{\psi}(i\rlap{\not{\partial}} - m_0)\psi \right. \\ &\quad \left. - \frac{1}{2}e^2\bar{\psi}\gamma_\mu\psi(M^{-1})^{\mu\nu}(-i\partial)\bar{\psi}\gamma_\nu\psi \right)(x) \\ &= \exp i \int d^4x \left(\bar{\psi}(i\rlap{\not{\partial}} - m_0)\psi - \frac{1}{2}e^2\bar{\psi}\gamma_\nu\psi(M^{-1})^{\mu\nu}(-i\partial)\bar{\psi}\gamma_\nu\psi \right)(x) \\ &=: \exp i \int d^4x \mathcal{L}[\psi, \bar{\psi}](x). \end{aligned} \quad (2.7)$$

$$\quad (2.8)$$

Here we use

$$\int \mathcal{D}\tilde{A} \exp i \int d^4x \frac{1}{2} \tilde{A}^\mu M_{\mu\nu} (-i\partial) \tilde{A}^\nu(x) = 1 \quad (2.9)$$

in the second equality.

Then we obtain the effective fermion lagrangian

$$\mathcal{L}[\psi, \bar{\psi}] = \bar{\psi}(i\cancel{\partial} - m_0)\psi - \frac{1}{2}e^2 \bar{\psi}\gamma_\mu\psi(M^{-1})^{\mu\nu}(-i\partial)\bar{\psi}\gamma_\nu\psi. \quad (2.10)$$

Here M^{-1} means the photon propagator. In fact the bare propagator of the photon is defined by

$$D_{\mu\nu}^{-1}(x, y) := i \frac{\delta^2 S[\psi, \bar{\psi}, A]}{\delta A^\mu(x) \delta A^\nu(y)}, \quad S[\psi, \bar{\psi}, A] := \int d^4x \mathcal{L}[\psi, \bar{\psi}, A](x) \quad (2.11)$$

and its Fourier transformation is given by

$$D^{\mu\nu}(p) := \int \frac{d^4(x-y)}{(2\pi)^4} e^{ip(x-y)} D^{\mu\nu}(x, y) = -i(M^{-1})^{\mu\nu}(p). \quad (2.12)$$

For the later convenience we write the lagrangian in a symmetric form:

$$\begin{aligned} \mathcal{L}[\psi, \bar{\psi}](x) &= \bar{\psi}(i\cancel{\partial} - m_0)\psi(x) \\ &\quad - \frac{1}{2}e^2 i D^{\mu\nu} \left(\frac{-i\partial^{x_2} + i\partial^{y_1}}{2} - \frac{i\partial^{x_1} - i\partial^{y_2}}{2} \right) \bar{\psi}(y_1)\gamma_\mu\psi(y_2)\bar{\psi}(x_1)\gamma_\nu\psi(x_2) \Big|_*, \end{aligned} \quad (2.13)$$

$$iD^{\mu\nu}(p) = \frac{-p^2 g^{\mu\nu} + (1-\alpha)p^\mu p^\nu}{p^4} \quad (2.14)$$

or

$$\begin{aligned} \mathcal{L}[\psi, \bar{\psi}](x) &= \bar{\psi}(i\cancel{\partial} - m_0)\psi(x) \\ &\quad - \frac{1}{2}K^{mn, n'm'} \left(\frac{-i\partial^{x_2} + i\partial^{y_1}}{2}, \frac{i\partial^{x_1} - i\partial^{y_2}}{2} \right) \bar{\psi}_{m'}(y_1)\psi_m(y_2)\bar{\psi}_n(x_1)\psi_{n'}(x_2) \Big|_*, \end{aligned} \quad (2.15)$$

$$K^{mn, n'm'}(p, k) := e^2 i D^{\mu\nu}(p-k)(\gamma_\mu)^{m'm}(\gamma_\nu)^{nn'}. \quad (2.16)$$

Here the symbol $*$ in Eq.(2.15) means $x_1, x_2, y_1, y_2 \rightarrow x$ after all derivatives are operated and $m, n, \dots$ are Dirac indices.

We use Eq.(2.15) in the next subsection.

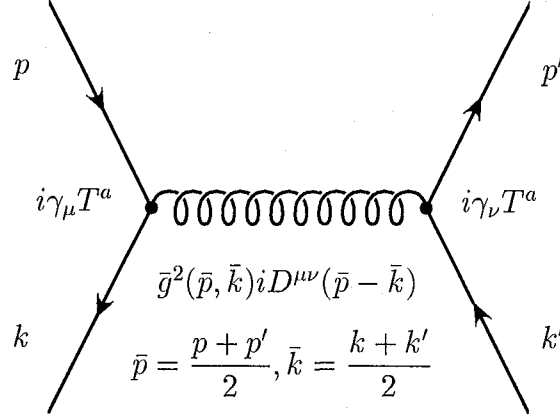


Fig.2-1: Gluon Exchange Interaction

2.3 QCD Case

In QCD, the gluon field in the lagrangian is not in a quadratic form. It is impossible to average the gluon field like Eq.(2.7). Instead we assume the effective quark lagrangian as

$$\mathcal{L}[\psi, \bar{\psi}](x) = \bar{\psi}(i\not{\partial} - m_0)\psi(x) - \frac{1}{2}K^{mn,n'm'} \left(\frac{-i\partial^{x_2} + i\partial^{y_1}}{2}, \frac{i\partial^{x_1} - i\partial^{y_2}}{2} \right) \bar{\psi}_{m'}(y_1)\psi_m(y_2)\bar{\psi}_n(x_1)\psi_{n'}(x_2) \Big|_*, \quad (2.17)$$

$$K^{mn,n'm'}(p, k) := \bar{g}^2(p, k) iD^{\mu\nu}(p - k) (\gamma_\mu T^a)^{m'm} (\gamma_\nu T^a)^{nn'}. \quad (2.18)$$

$$iD^{\mu\nu}(p) = \frac{-p^2 g^{\mu\nu} + (1 - \alpha)p^\mu p^\nu}{p^4} \quad (2.19)$$

Here $m, n, \dots$ are combined indices $m := (a, i, f)$, $n := (b, j, g), \dots$ with Dirac indices $a, b, \dots$ and color indices $i, j, \dots$ and flavor indices $f, g, \dots$. T^a is a generator of color $SU(N_C)$ whose normalization is $\text{tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$. This interaction is shown by Fig.2-1.

This is an analogy to QED case. Of course there is no theoretical guarantee that the assumption Eq.(2.17) is valid. We take only the effect of non-abelian gauge theory in the running coupling constant.[7] In this paper we use gluon propagator Eq.(2.19) although some people use more complicated gluon propagators as shown in 7.9.

There is an ambiguity in the choice of the coupling constant $\bar{g}^2(p, k)$. Some examples are

$$\bar{g}^2(p, k) = \theta(p_E^2 - k_E^2)g^2(p_E^2) + \theta(k_E^2 - p_E^2)g^2(k_E^2), \quad (2.20)$$

$$\begin{aligned}\bar{g}^2(p, k) &= g^2((p_E - k_E)^2), \\ &\vdots\end{aligned}\tag{2.21}$$

where p_E denotes an Euclid momentum i.e. $p_E^2 = -p^2$. Here $g(\mu^2)$ is a running coupling constant which should be consistent with the perturbative QCD analysis in the high energy region. That is to say, this lagrangian contains the effect of non-abelian gauge in the high energy region.

The choice Eq.(2.20) was known as Higashijima–Miransky approximation and the Bethe–Salpeter equation with Eq.(2.20) was introduced by Aoki, Kugo, Bando, Mitchard and Nakatani.[2] In this paper we call this model as Kyoto Group(KG) model.

We comment that when one use the Higashijima–Miransky approximation Eq.(2.20) the interaction is not invariant under local chiral transform. This results in the violation of usual Ward–Takahashi identity for axial vector current. We will discuss this issue in Secs.3 and 5.

2.4 Comparison with NJL model

When we make a replacement

$$\bar{g}(p, k)iD^{\mu\nu}(p - k) \mapsto \frac{g^2}{M^2}g^{\mu\nu},\tag{2.22}$$

in Eq.(2.18) we obtain an effective quark lagrangian

$$\mathcal{L}[\psi, \bar{\psi}] = \bar{\psi}(i\not{\partial} - m_0)\psi - \frac{g^2}{2M^2}(\bar{\psi}\gamma_\mu T^a \psi)(\bar{\psi}\gamma^\mu T^a \psi).\tag{2.23}$$

After the Fierz transformation

$$\begin{aligned}(\gamma_\mu)^{a'a}(\gamma^\mu)^{bb'} &= -\delta^{a'b'}\delta^{ba} + (\gamma_5)^{a'b'}(\gamma_5)^{ba} + \frac{1}{2}(\gamma_\mu)^{a'b'}(\gamma^\mu)^{ba} \\ &\quad + \frac{1}{2}(\gamma_5\gamma_\mu)^{a'b'}(\gamma_5\gamma^\mu)^{ba},\end{aligned}\tag{2.24}$$

$$(T^a)^{i'i}(T^a)^{j'j} = \frac{1}{2}\delta^{i'j'}\delta^{ji} - \frac{1}{N_C}\delta^{i'i}\delta^{j'j}\tag{2.25}$$

$$\delta^{f'f}\delta^{gg'} = \frac{1}{2}(\lambda^\alpha)^{f'g'}(\lambda^\alpha)^{gf},\tag{2.26}$$

Eq.(2.23) is rewritten as

$$\begin{aligned}\mathcal{L}[\psi, \bar{\psi}] &= \bar{\psi}(i\not{\partial} - m_0)\psi + G_S \left[\left(\bar{\psi} \frac{\lambda^\alpha}{2} \psi \right)^2 + \left(\bar{\psi} i\gamma_5 \frac{\lambda^\alpha}{2} \psi \right)^2 \right] \\ &\quad + G_V \left[\left(\bar{\psi} \gamma_\mu \frac{\lambda^\alpha}{2} \psi \right)^2 + \left(\bar{\psi} \gamma_5 \gamma_\mu \frac{\lambda^\alpha}{2} \psi \right)^2 \right] - \frac{1}{2N_C}(\bar{\psi}\gamma_\mu \psi)(\bar{\psi}\gamma^\mu \psi),\end{aligned}\tag{2.27}$$

$$G_S := \frac{g^2}{2M^2}, \quad G_V := \frac{-g^2}{4M^2} \quad (2.28)$$

where λ^α is a flavor matrix with normalization condition $\text{tr}[\lambda^\alpha \lambda^\beta] = 2\delta^{\alpha\beta}$.

If we neglect the last term in RHS of Eq.(2.27) and treat G_S, G_V as free parameters, Eq.(2.27) is nothing but the NJL model.

Note that the second, third and fourth terms in RHS of Eq.(2.27) are chiral invariant independently. Furthermore in the third term,

$$\left[\left(\bar{\psi} \gamma_\mu \frac{\lambda^0}{2} \psi \right)^2 + \left(\bar{\psi} \gamma_5 \gamma_\mu \frac{\lambda^0}{2} \psi \right)^2 \right] \text{ and } \sum_{\alpha=1}^{N_F} \left[\left(\bar{\psi} \gamma_\mu \frac{\lambda^\alpha}{2} \psi \right)^2 + \left(\bar{\psi} \gamma_5 \gamma_\mu \frac{\lambda^\alpha}{2} \psi \right)^2 \right] \quad (2.29)$$

are chiral invariant independently. Altogether, we have four local 4-fermi interactions that are independent and chiral invariant.

2.5 Cut Off Reguralization

In the above subsections we construct effective models. But there is a problem about the renormalization. Since the NJL model is not a renormalizable theory, we must introduce a momentum scale Λ_{UV} which cuts off the diverging quantum effects in the high energy region. In the chiral limit $m_0 = 0$, asymptotically free models like KG model can avoid this problem because the running coupling constant has been already renormalized with quantum effects of the high energy region.[7, 2] But when one introduces the quark bare mass term (which is not running but constant), the divergence appears again and a scale must be introduced at which the quark masses are evaluated or renormalized.[8, 9] A simple answer for this problem is to introduce the cut off of loop momentums in the Feynman diagrams. But it is not systematic because the values of the divergent loop diagrams depend on the choice of the loop momentum variables for which we make the cut off. There is ambiguity how to cut the momentum diagram by diagram.

In order to avoid this ambiguity, we introduce a formulation with a cut off function in the lagrangian.[10] We replace the free part of the lagrangian

$$\bar{\psi}(i\partial - m_0)\psi \mapsto \bar{\psi}f(\partial^2)(i\partial - m_0)\psi. \quad (2.30)$$

Here the function $f(x)$ of $x = \partial^2$ may be any function if it satisfies $f(x) = 1$ for small x and $f(x) \rightarrow \infty$ for $x \gg \Lambda_{UV}^2$. A few examples of the cut off function are

$$f(x) = 1 + \frac{x}{\Lambda_{UV}^2}, \quad (2.31)$$

$$f(x) = 1 + M\theta(x - \Lambda_{UV}^2), \quad M \rightarrow \infty. \quad (2.32)$$

This cut off regularization function, however, modifies the chiral property of the effective model. This is discussed in Secs.3 and 5.

The reason why we employ the cut off regularization scheme is that it is the simplest way and match for numerical studies. The dimensional regularization scheme is a best way but too difficult for the numerical calculation. Although the Pauli–Villars–Gupta regularization scheme is a nice method, there is a problem about the accuracy of the numerical integration since the integral region is infinite. The cut off scheme employing Eq.(2.32) requests only the finite integral region in the Euclidean space.

2.6 Flavor Mixing Term

The flavor mixing interaction suggested by Kobayashi, Maskawa and 't Hooft(KMT) is

$$\mathcal{L}_{\text{FM}}[\psi, \bar{\psi}] := G_D \left\{ \det^{(\text{F})} [\bar{\psi}(1 + \gamma_5)\psi] + \det^{(\text{F})} [\bar{\psi}(1 - \gamma_5)\psi] \right\} \quad (2.33)$$

where $\det^{(\text{F})}$ denotes the determinant in the flavor space.[11, 12] This interaction is invariant under the $SU(N_F)_L \times SU(N_F)_R \times U(1)_V$ but not under the $U(1)_A$.

In the $N_F = 3$ case, it is written as

$$\begin{aligned} \mathcal{L}_{\text{FM}}[\psi, \bar{\psi}] = & \frac{1}{3} G_D \epsilon^{f_1 f_2 f_3} \epsilon^{g_1 g_2 g_3} \left\{ [\bar{\psi}_{g_1} \psi_{f_1}] [\bar{\psi}_{g_2} \psi_{f_2}] [\bar{\psi}_{g_3} \psi_{f_3}] \right. \\ & \left. + 3 [\bar{\psi}_{g_1} \psi_{f_1}] [\bar{\psi}_{g_2} \gamma_5 \psi_{f_2}] [\bar{\psi}_{g_3} \gamma_5 \psi_{f_3}] \right\}. \end{aligned} \quad (2.34)$$

In Ref.[13, 14] the NJL model with this flavor mixing interaction is studied.

This flavor mixing interaction corresponds to the existence of the instanton solution of the gluon field. That is to say, this interaction contains some effect of non-abelian gauge theory in the low energy region. This must vanish in the high energy region. But it is not trivial to estimate how it depends on momenta. Generally we can introduce the weight function as

$$\mathcal{L}_{\text{FM}}[\psi, \bar{\psi}](x) = \frac{1}{3} G_D \epsilon^{f_1 f_2 f_3} \epsilon^{g_1 g_2 g_3} \bar{w}(i\partial_{x_1}, i\partial_{y_1}, i\partial_{x_2}, i\partial_{y_2}, i\partial_{x_3}, i\partial_{y_3})$$

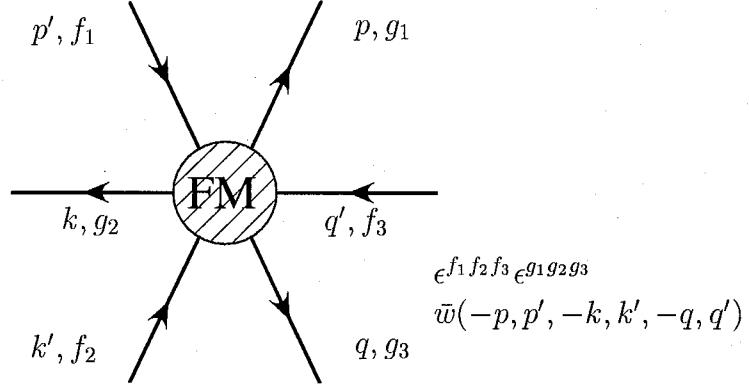


Fig.2-2: Flavor Mixing Interaction

$$\begin{aligned} & \times \left\{ [\bar{\psi}_{g_1}(x_1)\psi_{f_1}(y_1)][\bar{\psi}_{g_2}(x_2)\psi_{f_2}(y_2)][\bar{\psi}_{g_3}(x_3)\psi_{f_3}(y_3)] \right. \\ & \left. + 3[\bar{\psi}_{g_1}(x_1)\psi_{f_1}(y_1)][\bar{\psi}_{g_2}(x_2)\gamma_5\psi_{f_2}(y_2)][\bar{\psi}_{g_3}(x_3)\gamma_5\psi_{f_3}(y_3)] \right\} \Big|_*. \end{aligned} \quad (2.35)$$

The symbol $*$ in Eq.(2.35) means $x_1, x_2, x_3, y_1, y_2, y_3 \rightarrow x$ after all derivatives are operated.

There is an ambiguity in the argument of the function. Some examples are

$$\bar{w}(\dots) = w \left(\left(\frac{(\partial_{x_1} - \partial_{y_1}) + (\partial_{x_2} - \partial_{y_2}) + (\partial_{x_3} - \partial_{y_3})}{2\sqrt{3}} \right)^2 \right), \quad (2.36)$$

$$\bar{w}(\dots) = w \left(\frac{\partial_{x_1}^2}{2} \right) w \left(\frac{\partial_{y_1}^2}{2} \right) w \left(\frac{\partial_{x_2}^2}{2} \right) w \left(\frac{\partial_{y_2}^2}{2} \right) w \left(\frac{\partial_{x_3}^2}{2} \right) w \left(\frac{\partial_{y_3}^2}{2} \right), \quad (2.37)$$

$\vdots$

where $w(x)$ is a scalar weight function which satisfies $w(x) = 1$ for small x and $w(x) \rightarrow 0$ for $x \gg \Lambda_{\text{FM}}^2$.

This generalization from local form Eq.(2.34) to non-local form Eq.(2.35) may bring a violation of local chiral invariance at the interaction.

However, when one chooses Eq.(2.36) the interaction $\mathcal{L}_{\text{FM}}[\psi, \bar{\psi}]$ is locally chiral invariant. If one chooses Eq.(2.37) the numerical calculation becomes much simpler than employing Eq.(2.36), but the interaction is not locally chiral invariant. We will use this flavor mixing interaction in Sec.8.

3 Chiral Property of the Effective Models

3.1 Preliminary

The effective models may have their own properties which are different from QCD. In this section we re-consider the feature of the effective models especially the KG model carefully and examine the well-known relations which hold in the full QCD. The main purpose is to make a formulation which is appropriate for the study of the chiral properties of the effective models. We then propose a modified axial current, modified Ward–Takahashi identity in the effective models.

Let us start with the system of massive fermionic effective model like

$$\mathcal{L}[\psi, \bar{\psi}] := \mathcal{L}_{\text{free}}[\psi, \bar{\psi}] + \mathcal{L}_{\text{int}}[\psi, \bar{\psi}], \quad (3.1)$$

$$\mathcal{L}_{\text{free}}[\psi, \bar{\psi}] := \bar{\psi} f(\partial^2)(i\not{\partial} - m_0)\psi. \quad (3.2)$$

($:=$ denotes definition)

Here the function $f(x)$ of $x = \partial^2$ is introduced as a cutoff regularization function to avoid the ultraviolet divergences of the fermion loops and determines a scale at which the fermion bare mass m_0 is evaluated. $f(x)$ may be any function if it satisfies $f(x) = 1$ for small x and $f(x) \rightarrow \infty$ for $x \gg \Lambda_{\text{UV}}^2$. Usually m_0 is a diagonal flavor matrix i.e. $m_0 = \text{diag}(m_u, m_d, m_s)$ in $N_f = 3$ case. But in this paper we deal only with a flavor independent mass except for other specified. The generalization to flavor dependent masses can be done in a systematic way, although there may be some technical complexity^{*1} in choosing the center of mass of bound states.

Although in the chiral limit ($m_0 \rightarrow 0$) this regularization function maintains the global chiral symmetry, it changes the property of the local chiral symmetry of the kinetic term. And even if the regularization or interaction keeps the global chiral symmetry, the modification of the local chiral symmetry changes some important equalities held in QCD, such as the PCAC relation, the Ward–Takahashi identities.

To see this, let us consider the infinitesimal local chiral transformation

$$\psi(x) \rightarrow \psi'(x) := \left(1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x)\right) \psi(x), \quad (3.3)$$

^{*1}This complexity occurs in kaon state. In this thesis we do not discuss the kaon state.

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) := \bar{\psi}(x)(1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x)) \quad (3.4)$$

where λ^α is a hermitian flavor matrix with the normalization $\text{tr}[\lambda^\alpha \lambda^\beta] = 2\delta^{\alpha\beta}$. Throughout this paper we treat λ^α as hermitian. The generalization to non-hermitian flavor matrix with $\text{tr}[\lambda^{\alpha\dagger} \lambda^\beta] = 2\delta^{\alpha\beta}$ is straightforward.

At the moment we concentrate on the kinetic term and assume that the interaction term $\mathcal{L}_{\text{int}}$ is locally chiral invariant.

If we do not introduce the cut off function, under this transformation the change of the action $S[\psi, \bar{\psi}] := \int d^4x \mathcal{L}[\psi, \bar{\psi}](x)$ is

$$\begin{aligned} \Delta_5 S[\psi, \bar{\psi}] &:= S[\psi', \bar{\psi}'] - S[\psi, \bar{\psi}] \\ &\equiv \int d^4x \left\{ \frac{\delta S}{\delta \psi_m(x)} \left(i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x) \psi(x) \right)_m + \left(\bar{\psi}(x) i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x) \right)_n \frac{\delta S}{\delta \bar{\psi}_n(x)} \right\} \\ &\equiv \int d^4x \theta^\alpha(x) \left\{ \partial^\mu (\bar{\psi} \gamma_\mu \gamma_5 \frac{\lambda^\alpha}{2} \psi) - 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi \right\}(x). \end{aligned} \quad (3.5)$$

Here $\equiv$ implies that this is an identity in function space. (i.e. This holds for any function $\psi, \bar{\psi}$.) $\frac{\delta}{\delta \psi}$ means right functional differential, $\frac{\delta}{\delta \bar{\psi}}$ means left functional differential.

Using the equation of motion $\frac{\delta S}{\delta \psi} = \frac{\delta S}{\delta \bar{\psi}} = 0$, we can see from Eq.(3.5) that the axial current $J_{5\mu}^\alpha$ satisfies the well-known PCAC relation

$$\partial^\mu J_{5\mu}^\alpha = 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi, \quad (3.6)$$

$$J_{5\mu}^\alpha := \bar{\psi} \gamma_\mu \gamma_5 \frac{\lambda^\alpha}{2} \psi. \quad (3.7)$$

This axial current is the same as in QCD. In various effective models, however, we will see that the form of the axial current is modified. For conventions, we call $J_{5\mu}^\alpha$ defined by Eq.(3.7) the *naive axial current*.

3.2 Cut off regularization

If we introduce the cut off function, under the local chiral transformation the change of the action is

$$\begin{aligned} \Delta_5 S[\psi, \bar{\psi}] &\equiv \int d^4x \theta^\alpha(x) \left\{ \bar{\psi}(\vec{\partial}) f(\vec{\partial}^2) \gamma_5 \frac{\lambda^\alpha}{2} + f(\partial^2) \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \right\} \psi \\ &\quad - 2m_0 i \bar{\psi} \frac{f(\vec{\partial}^2) + f(\partial^2)}{2} \gamma_5 \frac{\lambda^\alpha}{2} \psi \Big\}(x). \end{aligned} \quad (3.8)$$

Since the kinetic term $\bar{\psi}f(\partial^2)i\not{p}\psi$ is globally chiral invariant, the first term in the braces in the right hand side of Eq.(3.8) can be written in the form $\partial^\mu I_{5\mu}^\alpha$ with the new current $I_{5\mu}^\alpha$ which satisfies the PCAC relation:

$$\partial^\mu I_{5\mu}^\alpha = 2m_0 i \bar{\psi} \frac{f(\partial^2) + f(\partial^2)}{2} \gamma_5 \frac{\lambda^\alpha}{2} \psi \quad (3.9)$$

$$I_{5\mu}^\alpha(x) := \int_{p,q} \bar{\psi}(p) \left(\gamma_\mu f_1(-p^2, -q^2) + (p-q)_\mu (\not{p} - \not{q}) f_2(-p^2, -q^2) \right) \gamma_5 \frac{\lambda^\alpha}{2} \psi(q) e^{-i(p+q)x} \quad (3.10)$$

with

$$f_1(-p^2, -q^2) := \frac{f(-p^2) + f(-q^2)}{2}, \quad f_2(-p^2, -q^2) := \frac{f(-p^2) - f(-q^2)}{2(p^2 - q^2)} \quad (3.11)$$

where $\int_q$ denotes $\int \frac{d^4 q}{(2\pi)^4}$. Formally all the effects of the cut off function are confined in the replacement

$$J_{5\mu}^\alpha \mapsto I_{5\mu}^\alpha, \quad \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi \mapsto \bar{\psi} \frac{f(\partial^2) + f(\partial^2)}{2} \gamma_5 \frac{\lambda^\alpha}{2} \psi. \quad (3.12)$$

Thus new current $I_{5\mu}^\alpha$ defined by Eq.(3.10) is conserved in the chiral limit and called a generalized Noether current in the appendix of Ref.[10]. [10, 6] In this paper we call this current $I_{5\mu}^\alpha$ the *effective* or *modified axial current owing to the regularization*.

This current $I_{5\mu}^\alpha$ looks dangerous because $f(x) \rightarrow \infty$ for $x \gg \Lambda_{UV}^2$. One must be careful about it in using in the concrete equations. We will return to this problem in Sec.6.

3.3 Locally chiral variant interaction

Let us next consider the KG model introduced by Aoki, Kugo, Bando, Mitchard and Nakatani.[2] This model incorporates the asymptotic free property according to QCD and also gives a charge universality (or the vector Ward identity $Z_1 = Z_2(= 1)$) at the rainbow SDE level with the Landau gauge condition. The latter is proved in 8.4. However the interaction is not locally chiral invariant and the axial Ward-Takahashi identity should be modified as we will see. This was shown in Ref.[15] at the ladder BSE level. In this subsection we do not introduce the cut off function $f(x)$ for simplicity.

This model with the Landau gauge condition has the following interaction form:

$$\mathcal{L}_{\text{int}}[\psi, \bar{\psi}](x) := -\frac{1}{2} \tilde{K}^{mn, n_2 m_2} \left(\frac{-i\partial^{x_2} + i\partial^{y_1}}{2}, \frac{i\partial^{x_1} - i\partial^{y_2}}{2} \right) \times \bar{\psi}_{m_2}(y_1) \psi_m(y_2) \bar{\psi}_n(x_1) \psi_{n_2}(x_2) \Big|_* \quad (3.13)$$

$$\tilde{K}^{mn, n_2 m_2}(p, k) := \bar{g}^2(p, k) iD^{\mu\nu}(p - k) (\gamma_\mu T^a)^{m_2 m} (\gamma_\nu T^a)^{nn_2} \quad (3.14)$$

$$iD^{\mu\nu}(p) := \frac{-g^{\mu\nu} p^2 + p^\mu p^\nu}{p^4} \quad (3.15)$$

$$\bar{g}^2(p, k) := \theta(p_E^2 - k_E^2) g^2(p_E^2) + \theta(k_E^2 - p_E^2) g^2(k_E^2) \quad (3.16)$$

$$g^2(p_E^2) := \frac{3(4\pi)^2}{11N_C - 2N_F} \left(\theta(t_{\text{IF}} - t) \frac{1}{t_{\text{IF}} + 1} + \theta(t - t_{\text{IF}}) \frac{1}{t + 1} \right) \quad (3.17)$$

$$t := \ln \frac{p_E^2}{\Lambda_{\text{QCD}}^2} - 1 \quad (3.18)$$

Here the symbol $*$ in Eq.(3.13) means $x_1, x_2, y_1, y_2 \rightarrow x$ after all derivatives are operated. p_E and k_E denote Euclidian momentum, (i.e. $p_E^2 = -p^2$), and the indices $m, n, \dots$ are combined indices $m := (a, i, f)$, $n := (b, j, g), \dots$ with Dirac indices $a, b, \dots$ and color indices $i, j, \dots$ and flavor indices $f, g, \dots$. T^a is a generator of color $SU(N_C)$.

This interaction is globally chiral invariant, but it is not invariant^{*2} under the local chiral transformation as follows.

The change of the action under the infinitesimal chiral transformation becomes

$$\begin{aligned} \Delta_5 S[\psi, \bar{\psi}] \equiv & \int d^4x \theta^\alpha(x) \left\{ \partial^\mu (\bar{\psi} \gamma_\mu \gamma_5 \frac{\lambda^\alpha}{2} \psi) - 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi \right. \\ & - i \int_{p, q, k, l} \left(\bar{g}^2 \left(p + \frac{q+k}{2}, -q + \frac{q+k}{2} \right) - \bar{g}^2 \left(p - \frac{p+l}{2}, -q - \frac{p+l}{2} \right) \right) \\ & \times iD^{\mu\nu}(p+q) [\bar{\psi}(p) \gamma_\mu T^a \psi(q)] [\bar{\psi}(k) \gamma_\nu \gamma_5 T^a \frac{\lambda^\alpha}{2} \psi(l)] e^{-i(p+q+k+l)x} \Big\}. \end{aligned} \quad (3.19)$$

The third term in the braces should be written down in the form $\partial^\mu K_\mu^\alpha$ with a current K_μ^α because the interaction is globally chiral invariant. In fact, K_μ^α is given by

$$\begin{aligned} K_\mu^\alpha = & \int_{p, q, k, l} G_\mu(p, q, k, l) \\ & \times iD^{\rho\sigma}(p+q) [\bar{\psi}(p) \gamma_\rho T^a \psi(q)] [\bar{\psi}(k) \gamma_\sigma \gamma_5 T^a \frac{\lambda^\alpha}{2} \psi(l)] e^{-i(p+q+k+l)x}, \end{aligned} \quad (3.20)$$

^{*2}We consider whether the interaction term is invariant or not under the local chiral transformation. We do not consider the local chiral invariance of the total lagrangian. Each effective model we treat in this paper is not a chiral gauge theory, the total lagrangian is always locally chiral variant.

$$G_\mu(p, q, k, l) := \frac{(p + q + k + l)_\mu}{(p + q + k + l)^2} \times \left\{ \bar{g}^2 \left(p + \frac{q + k}{2}, -q + \frac{q + k}{2} \right) - \bar{g}^2 \left(p - \frac{p + l}{2}, -q - \frac{p + l}{2} \right) \right\}. \quad (3.21)$$

Again using the equation of motion we obtain the modified PCAC relation

$$\partial^\mu J_{5\mu}^\alpha = 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi + \partial^\mu K_\mu^\alpha. \quad (3.22)$$

We notice the naive axial current $J_{5\mu}^\alpha$ is not conserved even in the chiral limit $m_0 \rightarrow 0$. If we define the current $\tilde{J}_{5\mu}^\alpha := J_{5\mu}^\alpha - K_\mu^\alpha$, it is conserved in the chiral limit

$$\partial^\mu \tilde{J}_{5\mu}^\alpha = 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi. \quad (3.23)$$

This new current $\tilde{J}_{5\mu}^\alpha$ is also a generalized Noether current as is the effective axial current owing to the regularization $I_{5\mu}^\alpha$ in the previous subsection. We call this current $\tilde{J}_{5\mu}^\alpha$ is the *effective* or *modified axial current*. When the cut off function is employed the definition of the effective axial current is $\tilde{J}_{5\mu}^\alpha := I_{5\mu}^\alpha - K_\mu^\alpha$ instead. This approach can be used for other effective models which has a locally chiral variant interaction.

3.4 Low energy relations

Now we get the (partially) conserved current $\tilde{J}_{5\mu}^\alpha := J_{5\mu}^\alpha - K_\mu^\alpha$. When the cut off function is employed the definition is $\tilde{J}_{5\mu}^\alpha := I_{5\mu}^\alpha - K_\mu^\alpha$ instead^{*3}. In this subsection, however, we do not employ the cut off function $f(x)$ without other specified.

Once we get the (partially) conserved current $\tilde{J}_{5\mu}^\alpha$, the Ward–Takahashi identities can be derived. In the path integral form, the Green function is given by

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \mathcal{N} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(iS[\psi, \bar{\psi}]) \psi(x) \bar{\psi}(y) \quad (3.24)$$

$$\begin{aligned} &= \mathcal{N} \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' \exp(iS[\psi', \bar{\psi}']) \psi'(x) \bar{\psi}'(y) \\ &= \mathcal{N} \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' \exp(iS[\psi, \bar{\psi}] + i\Delta_5 S[\psi, \bar{\psi}]) \\ &\quad \times (1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x)) \psi(x) \bar{\psi}(y) (1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(y)) \end{aligned} \quad (3.25)$$

^{*3}To avoid the confusion, we use the following notations. The naive axial current $J_{5\mu}^\alpha(x)$ is defined by Eq.(3.7) model independently. The effective axial current owing to the regularization $I_{5\mu}^\alpha(x)$ is defined by Eq.(3.10) which depends on only the regularization function $f(x)$. The effective axial current $\tilde{J}_{5\mu}^\alpha(x)$ is a (partially) conserved current which depends on a full lagrangian of a model.

where $\mathcal{N}$ is a normalization factor. Throughout this paper we do not consider the anomaly and assume the measure invariance $\mathcal{D}\psi\mathcal{D}\bar{\psi} = \mathcal{D}\psi'\mathcal{D}\bar{\psi}'$.

Using

$$i\Delta_5 S[\psi, \bar{\psi}] \equiv i \int d^4x \theta^\alpha(x) \left\{ \partial^\mu \tilde{J}_{5\mu}^\alpha - 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi \right\}(x) \quad (3.26)$$

and comparing Eq.(3.24) with Eq.(3.25), we obtain the Ward–Takahashi identity for the modified axial current

$$\begin{aligned} \langle 0 | T^* i \left(\partial^\mu \tilde{J}_{5\mu}^\alpha(0) - 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi(0) \right) \psi(x) \bar{\psi}(y) | 0 \rangle \\ + i \gamma_5 \frac{\lambda^\alpha}{2} \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \delta^4(x) + \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle i \gamma_5 \frac{\lambda^\alpha}{2} \delta^4(y) = 0 \end{aligned} \quad (3.27)$$

or

$$\begin{aligned} -i \partial^\mu \langle 0 | T \psi(x) \bar{\psi}(y) \tilde{J}_{5\mu}^\alpha(0) | 0 \rangle = 2m_0 \langle 0 | T \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi(0) \psi(x) \bar{\psi}(y) | 0 \rangle \\ + i \gamma_5 \frac{\lambda^\alpha}{2} \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \delta^4(x) + \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle i \gamma_5 \frac{\lambda^\alpha}{2} \delta^4(y). \end{aligned} \quad (3.28)$$

We note here that the above derivation does not use the axial charge $\tilde{Q}_5 := \int d^3x \tilde{J}_{5\mu=0}^\alpha(x)$ which may be the generator of the transformation. The specific form of the (partially) conserved current in the effective model makes the discussion difficult using canonical comutation relations of the field operator. But anyway, when the spontaneous symmetry breaking occurs the charge $\tilde{Q}_5$ is an ill defined object. The above treatment, however, does work safely. We will comment on this topic at the end of this section.

The pion decay constant can be defined in two ways,

$$J_{5\mu}^\alpha =: -f_\pi \partial_\mu \pi + \dots, \quad (3.29)$$

$$\tilde{J}_{5\mu}^\alpha =: -\tilde{f}_\pi \partial_\mu \pi + \dots \quad (3.30)$$

where $\dots$ means the higher state contributions. f_π is the conventional definition, and $\tilde{f}_\pi$ is defined for the modified axial current. We will see that the modified decay constant $\tilde{f}_\pi$ plays the role of f_π in the effective models.

In the momentum space, the LHS of Eq.(3.28) is given by

$$\text{LHS of (3.28)} = P^\mu \langle 0 | T \psi \bar{\psi} | \mathbf{P} \rangle \frac{i}{P^2 - M_\pi^2} \langle \mathbf{P} | \tilde{J}_{5\mu}^\alpha | 0 \rangle + \dots \quad (3.31)$$

$$= \frac{P^2}{P^2 - M_\pi^2} e^{-iP_B X} \tilde{f}_\pi(P, P_B) \chi(q; P_B) + \dots \quad (3.32)$$

where $\chi(q; P_B)$ is the Bethe–Salpeter amplitude defined by

$$\chi(q; P_B) := e^{-iP_B X} \int d^4(x-y) e^{iq(x-y)} \langle 0 | T \psi(x) \bar{\psi}(y) | \mathbf{P} \rangle \quad (3.33)$$

where $X := \frac{x+y}{2}$ is the position of center of mass of the bound state $|\mathbf{P}\rangle$ with normalization condition $\langle \mathbf{P} | \mathbf{P}' \rangle = (2\pi)^3 2P_{B0} \delta^3(\mathbf{P} - \mathbf{P}')$ and $P_B := (\sqrt{M_\pi^2 + \mathbf{P}^2}, \mathbf{P})$ is the on-shell momentum corresponding to $P := (P_0, \mathbf{P})$. $\tilde{f}_\pi(P, P_B)$ is defined by

$$\tilde{f}_\pi(P, P_B) := \frac{iP^\mu}{P^2} \langle \mathbf{P} | \tilde{J}_{5\mu}^\alpha(0) | 0 \rangle \quad (3.34)$$

and satisfies $\lim_{P \rightarrow P_B} \tilde{f}_\pi(P, P_B) = \tilde{f}_\pi$.

In the chiral limit the first term of the RHS of Eq.(3.28) disappears. If $M_\pi \neq 0$ the LHS diverges in the on-shell limit $P^2 \rightarrow M_\pi^2$. This contradicts with the RHS, which converges. Therefore we conclude $M_\pi = 0$, and the Nambu–Goldstone theorem holds. Furthermore in the soft limit $\mathbf{P} \rightarrow 0$, all the higher state contributions can be neglected in Eq.(3.32). So one obtains

$$\text{LHS of (3.28)} \rightarrow \tilde{f}_\pi \chi(q; 0). \quad (3.35)$$

On the other hand in the momentum space the RHS of Eq.(3.28) in the same limit should be

$$\text{RHS of (3.28)} \rightarrow \{i \frac{\lambda^\alpha}{2} \gamma_5, S_F(q)\} \quad (3.36)$$

where $S_F(q) = \text{F.T.} \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle$. Then one can get the Nambu–Goldstone solution of the Bethe–Salpeter amplitude from Eqs.(3.35) and (3.36)

$$\chi(q; 0) = \frac{1}{\tilde{f}_\pi} \{i \frac{\lambda^\alpha}{2} \gamma_5, S_F(q)\}. \quad (3.37)$$

This is a realization of the Nambu–Goldstone theorem. The dynamical mass generation

$$\{i \gamma_5 \frac{\lambda^\alpha}{2}, S_F(q)\} \neq 0 \quad (3.38)$$

indicates the existence of the NG boson. The normalization constant is $\tilde{f}_\pi^{-1}$, not f_π^{-1} .

It is known that the solution (3.37) of the BSE contains only the information of the soft limit of the massless particle and therefore is not enough to calculate $\tilde{f}_\pi$, for instance. We need to solve the BSE explicitly for $M_\pi \neq 0$, because $\tilde{f}_\pi$ is related to the derivative by M_π (or P_B) of the BS amplitude (in the chiral limit). On the other hand, Pagels and Stokar proposed a formula[16] to calculate the decay constant approximately using Eq.(3.37), which is shown in 7.6.

Finally we give the Gell-Mann, Oakes and Renner's mass formula. Operating $\langle \mathbf{P} |$ and $|0\rangle$ to the equality

$$\partial^\mu \tilde{J}_{5\mu}^\alpha = -\tilde{f}_\pi \square \pi = 2m_0 i \bar{\psi} \gamma_5 \frac{\lambda^\alpha}{2} \psi, \quad (3.39)$$

one obtains

$$M_\pi^2 \tilde{f}_\pi = -2m_0 i \int_q \text{tr}[\bar{\chi}(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}]. \quad (3.40)$$

In this paper we call this equality PCAC relation too. Then approximating the Bethe-Salpeter amplitude of the RHS of Eq.(3.40) by Eq.(3.37) we obtain

$$M_\pi^2 \tilde{f}_\pi^2 \simeq -2m_0 \langle \bar{\psi} \psi \rangle_0, \quad \langle \bar{\psi} \psi \rangle_0 := - \int_q \text{tr}[S_F(q)_{m_0=0}] \quad (3.41)$$

where $S_F(q)_{m_0=0}$ is the full propagator in the chiral limit. Indeed this is the GMOR mass formula.

Three remarks follow.

The first is that when the cut off function $f(x)$ is introduced, we should use the current $\tilde{J}_{5\mu}^\alpha := I_{5\mu}^\alpha - K_\mu^\alpha$ and replace Eq.(3.40) by

$$M_\pi^2 \tilde{f}_\pi = -2m_0 i \int_q \frac{f(-q_+^2) + f(-q_-^2)}{2} \text{tr}[\bar{\chi}(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}], \quad (3.42)$$

$$q_+ := q + \frac{P_B}{2}, \quad q_- := q - \frac{P_B}{2}. \quad (3.43)$$

Similarly we replace the definition of pair condensation in Eq.(3.41) by

$$\langle \bar{\psi} \psi \rangle_0 := - \int_q f(-q^2) \text{tr}[S_F(q)_{m_0=0}] \quad (3.44)$$

The second is that the formal definition such as Eq.(3.30) is not sufficient for a concrete calculation. Because the modified current $\tilde{J}_{5\mu}^\alpha$ contains a non-local 4-fermi interaction term, we can't evaluate the modified decay constant $\tilde{f}_\pi$ directly even if the Bethe-Salpeter amplitude is obtained. In Sec.5 we will give a systematic procedure to calculate $\tilde{f}_\pi$ for the approximated SDE-BSE system.

The third is whether the modified decay constant $\tilde{f}_\pi$ is really a decay constant in the meaning of weak decay $\pi^+ \rightarrow \mu^+ + \nu_\mu$ or not. This is a difficult problem because our formulation is only for a strong interaction. We discuss this problem in 7.8.

3.5 How about the Noether Charge?

Up to present we did not discuss the Noether charge Q . We believe that the Noether charge should be the generator of the transformation, but it is not trivial to see this.

In the true theory, the *classical* lagrangian $\mathcal{L}_{\text{cl}}(\phi, \partial\phi)$ is a starting point of all. After a suitable quantization procedure, one obtains the Feynman's formula for the generating function

$$Z[J] = \mathcal{N}_0 \int \mathcal{D}\pi \mathcal{D}\phi \exp \frac{i}{\hbar} \int d^4x [\pi \dot{\phi} - \mathcal{H}(\pi, \phi)] \quad (3.45)$$

$$= \mathcal{N} \int \mathcal{D}\phi \exp \frac{i}{\hbar} \int d^4x [\mathcal{L}_q(\phi, \partial\phi) + J(x)\phi(x)], \quad (3.46)$$

$$\pi := \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \quad \mathcal{H}(\pi, \phi) := \pi \dot{\phi} - \mathcal{L}_{\text{cl}}(\phi, \partial\phi) \quad (3.47)$$

by which various Green functions are derived. Usually the *quantum* lagrangian $\mathcal{L}_q$ in Eq.(3.46) is same as the *classical* one $\mathcal{L}_{\text{cl}}$ according to the Matthews theorem.[17]

In effective models, we treat lagrangians which contain higher derivative terms. Then the *Quantum* lagrangian may be different from the *classical* lagrangian or the quantization procedure may be obscure.

For example, in Ref.[10] where the NJL model with cut off function $f(x) = 1 + \frac{x}{\Lambda_{\text{UV}}^2}$ was studied, the original fermion field is divided into three component fields for which the higher derivative terms disappear and the conventional quantization is possible. In this case, the *quantum* lagrangian rewritten by the original fermion field is same as the *classical* lagrangian, but again the conjugate momentum field of the original fermion field is obscure.

We here employ an 'interpretation' that the lagrangian of the effective model is only a dummy which imitates the *quantum* lagrangian in Eq.(3.46). The lagrangian of the effective model revives a same feature of the true theory to some extent, but it is a different object. The integral of the conjugate momentum in Eq.(3.45) has been already performed, and the gauge degrees of freedom has been averaged. We do not know how to come from the classical theory through a quantization but we can only go to the classical theory in the limit $\hbar \rightarrow 0$.

In fact, as far as quantum effects are calculated with Feynman rules derived from formula Eq.(3.46), every result respects the property of 'classical action' of the *quantum* lagrangian (or the lagrangian of effective model).

Therefore our approach can't confirm whether the Noether charge is a generator of the transformation or not directly. Nevertheless the fact that the Ward-Takahashi identity does

hold and the current algebra is based only on the symmetry may imply that the Noether charge is a generator of the transformation.

4 CJT Effective Action Formulation

4.1 Preliminary

The Cornwall, Jackiw and Tomboulis's (CJT) effective action formulation is one of the most powerful methods to derive the Schwinger–Dyson equation and the Bethe–Salpeter equation consistently.[18, 19]

Especially the CJT formulation is fitted for a DCSB problem. Non-dynamical SCSB is well-described by the effective action $\Gamma[\phi]$ for a local field ϕ . As mentioned in 1.1, the picture that the wine bottle potential brings a SCSB is so clear. But the DCSB is more complicated. The CJT action $\Gamma^{\text{CJT}}[S_F]$ is the generalization of effective action for local field, and is an effective action for bilocal Green function $S_F(x, y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$.

In this section, we review the CJT formulation[6, 5] and derive the Bethe–Salpeter equation systematically.

4.2 Generating Functionals

The definition of the generating functional of the Green function is

$$Z[J] := \mathcal{N} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ iS[\psi, \bar{\psi}] + i\text{Tr}[\psi \bar{\psi} J] \right\}. \quad (4.1)$$

Here we use the notation $\text{Tr}[\psi \bar{\psi} J] := \int d^4x d^4y \text{tr}[\psi(y)\bar{\psi}(x)J(x, y)]$ and $J_{nm}(x, y)$ denotes a bilocal external source.

$Z[J]$ can be expanded as power series of the external source

$$\begin{aligned} Z[J] &= iG^{(1)}(xy)J(yx) + \frac{i^2}{2!}G^{(2)}(x_1y_1; x_2y_2)J(y_1x_1)J(y_2x_2) \\ &\quad + \frac{i^3}{3!}G^{(3)}(x_1y_1; x_2y_2; x_3y_3)J(y_1x_1)J(y_2x_2)J(y_3x_3) + \cdots, \end{aligned} \quad (4.2)$$

$$G^{(n)}(x_1y_1; \cdots; x_ny_n) := \langle 0|T\psi(x_1)\bar{\psi}(y_1) \cdots \psi(x_n)\bar{\psi}(y_n)|0\rangle \quad (4.3)$$

where we omit the color, flavor, Dirac indices for simplicity.

The definition of the generating functional for the connected Green function is

$$W[J] := -i \ln Z[J]. \quad (4.4)$$

$W[J]$ can be expanded as power series of the external source

$$\begin{aligned} iW[J] &= iG_C^{(1)}(xy)J(yx) + \frac{i^2}{2!}G_C^{(2)}(x_1y_1; x_2y_2)J(y_1x_1)J(y_2x_2) \\ &\quad + \frac{i^3}{3!}G_C^{(3)}(x_1y_1; x_2y_2; x_3y_3)J(y_1x_1)J(y_2x_2)J(y_3x_3) + \dots \end{aligned} \quad (4.5)$$

Comparing Eq.(4.2) and Eq.(4.5), we obtain

$$G_C^{(1)}(xy) := \frac{1}{Z}G^{(1)}(xy), \quad (4.6)$$

$$\begin{aligned} G_C^{(2)}(x_1y_1; x_2y_2) &:= \frac{1}{Z}G^{(2)}(x_1y_1; x_2y_2) - \frac{1}{Z^2}G^{(1)}(x_1y_1)G^{(1)}(x_2y_2), \\ &\vdots \end{aligned} \quad (4.7)$$

where $Z := Z[J]|_{J=0}$. It is clear that $G_C^{(n)}(x_1y_1; \dots; x_ny_n)$ denotes the connected Green function indeed.

From now on we use the notation $S_F(x, y) := G^{(1)}(x, y)$ i.e.

$$S_F(x, y) := \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \frac{\delta W[J]}{\delta J(y, x)}. \quad (4.8)$$

The effective action is introduced as a Legendre transform of $W[J]$:

$$\Gamma[S_F] := W[J] - \text{Tr}[S_F J]. \quad (4.9)$$

From this definition, the conjugate relation

$$\frac{\delta \Gamma[S_F]}{\delta S_{Fmn}(y, x)} = -J_{nm}(x, y) \quad (4.10)$$

holds. This equation is called the Schwinger–Dyson equation(SDE).

From a trivial relation $\frac{\delta J}{\delta S_F} \frac{\delta S_F}{\delta J} = 1$, the equation

$$\frac{1}{i} \frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{Fn'm'}(x', y')} G_{C; m'm'm''n''}^{(2)}(x'y'; y''x'') = \delta_{m''m} \delta_{nn''} \delta(y'' - y) \delta(x - x'') \quad (4.11)$$

holds. This equation is called the inhomogeneous Bethe–Salpeter equation(BSE). The operator in the LHS of Eq.(4.11)

$$i\Gamma_{mn'n'm'}^{(2)}(yx, x'y') := \frac{1}{i} \frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{Fn'm'}(x', y')} \quad (4.12)$$

is often called the BS operator.

Throughout this paper, we treat SDE and BSE only when the external source $J_{nm}(x, y)$ vanishes.

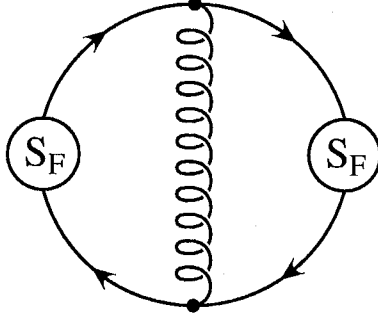


Fig.4-1: Feynman diagram in $i\Gamma_{\text{loop}}[S_F]$

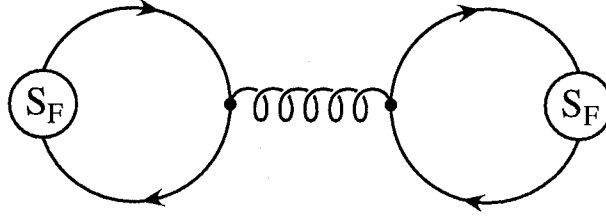


Fig.4-2: Feynman diagram in $i\Gamma_{\text{loop}}[S_F]$

4.3 The CJT action

The CJT action is the representation of $\Gamma[S_F]$ and given by

$$\Gamma^{\text{CJT}}[S_F] := i\text{TrLn}[S_F] - i\text{Tr}[S_0^{-1}S_F] + \Gamma_{\text{loop}}[S_F] \quad (4.13)$$

where S_0 means the bare propagator defined by

$$iS_0^{-1}(x, y) := \frac{\delta^2 S[\psi, \bar{\psi}]}{\delta \bar{\psi}(x) \delta \psi(y)} \Big|_{\psi=\bar{\psi}=0} = \int_p f(-p^2)(\not{p} - m_0)e^{-i(x-y)} \quad (4.14)$$

where $f(x)$ is a cut off regularization function introduced in 2.5.

Here after we write $\Gamma[S_F]$ as the CJT action simply.

The last term of Eq.(4.13) is the residual term. Multiplying a factor i , $i\Gamma_{\text{loop}}[S_F]$ is given by the sum of all the Feynman amplitudes of 2-loop or higher loop 2-particle irreducible vacuum diagrams in which every bare Fermion propagator $S_{0nm}(x, y)$ is replaced by the full one $S_{Fnm}(x, y)$.

A few examples of the Feynman diagrams in $i\Gamma_{\text{loop}}[S_F]$ are shown in Figs.4-1, 4-2, 4-3. In Fig.4-3 'FM' implies a flavor mixing term introduced in 2.6.

The proof of the derivation of Eq.(4.13) is given by Cornwall, Jackiw and Tomboulis.[4]

Instead we prove Eq.(4.13) in the diagrammatic viewpoint.

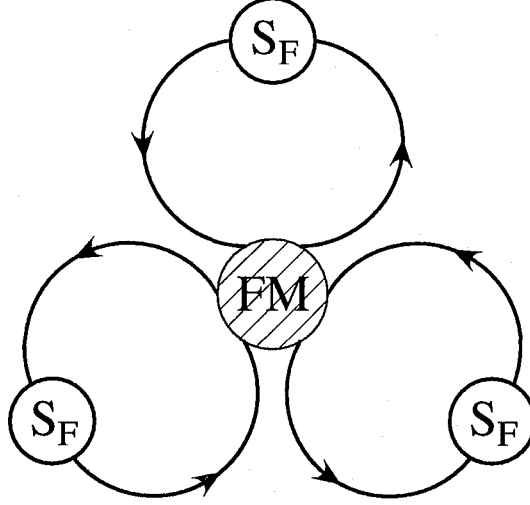


Fig.4-3: Feynman diagram in $i\Gamma_{\text{loop}}[S_F]$

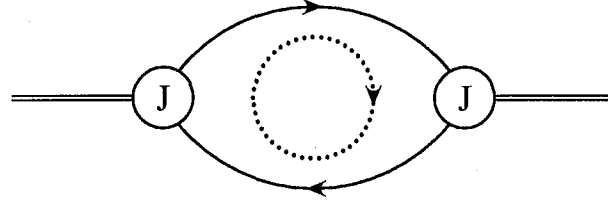


Fig.4-4: Feynman diagram in $iW[J]$

Some examples of the Feynman diagrams which consist of the generating functional $iW[J]$ are Figs.4-4,4-5,4-6. More complicated examples are Figs.4-7,4-8. In this paper, we call the quark 1-loop which connects separated two or more parts of a diagram as 'articulation loop'. In Figs.4-4, 4-5, 4-6 the dotted lines imply the articulation loops. We call the bare quark propagators on an articulation loop the 'legs of the articulation loop' if two of them connect the separated 2 parts of the diagram. In Figs.4-4, 4-5 every bare quark propagator is a leg of an articulation loop. In Fig.4-6 each bare quark propagator connected to an external source is

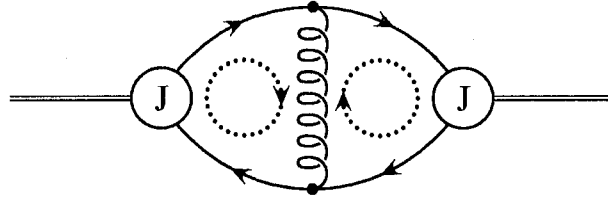


Fig.4-5: Feynman diagram in $iW[J]$

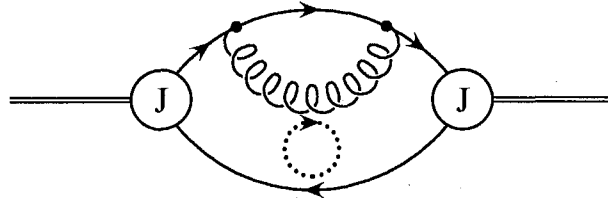


Fig.4-6: Feynman diagram in $iW[J]$

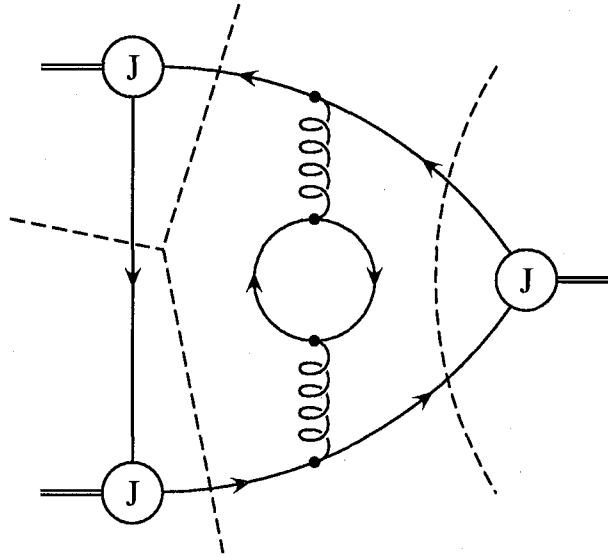


Fig.4-7: Complicated example in $iW[J]$

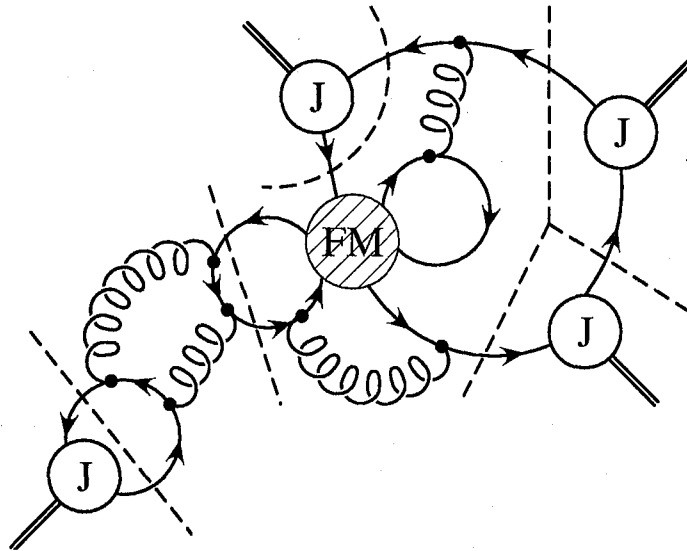


Fig.4-8: Complicated example in $iW[J]$

	n_a	n_b	n_c	n_d	n
Fig.4-4	1	-2	2	0	1
Fig.4-5	2	-4	2	1	1
Fig.4-6	1	-3	2	1	1
Fig.4-7	2	-5	3	1	1
Fig.4-8	4	-9	4	2	1

Table 4-1: Counting Number

a leg of an articulation loop. In Fig.4-7 each bare quark propagator connected to an external source is a leg but two bare quark propagators in the gluon vacuum polarization are not legs. For an articulation loop, one can draw the lines from the center of the articulation loop to each leg. We call these lines for an articulation loop as 'separation tree'. In Figs.4-7,4-8, the separation trees are shown by dashed lines.

Of course the number of the articulation loops is equal to number of the separation trees.

For each diagram in $iW[J]$, we introduce counting numbers n_a, n_b, n_c, n_d as following.

n_a is the number of articulation loops. That is to say, n_a specifies how many times the diagram is contained in $-\text{TrLn}[S_F]$. For Figs.4-4, 4-6, $n_a = 1$, for Figs.4-5, 4-7, $n_a = 2$ and for Fig.4-8, $n_a = 4$.

$(-n_b)$ is the sum of all the legs in all articulation loops. That is to say, $(-n_b)$ is the number how often the diagram is contained in $-\text{Tr}[S_0^{-1}S_F]$. For Fig.4-4, $n_b = -2$, for Fig.4-5, $n_b = -3$, for Fig.4-6, $n_b = -4$, for Fig.4-7, $n_b = -5$ and for Fig.4-8, $n_b = -9$.

n_c is the number of external sources. That is to say, n_c represents how often the diagram is contained in $i\text{Tr}[S_F J]$. For Figs.4-4, 4-5, 4-6, $n_c = 2$, for Fig.4-7, $n_c = 3$ and for Fig.4-8, $n_c = 4$.

n_d is a number how many times the diagram is contained in $i\Gamma_{\text{loop}}[S_F]$. For Fig.4-4, $n_d = 0$, for Figs.4-5, 4-6, 4-7 $n_d = 1$ (Figs.4-5, 4-6 are contained in Fig.4-1.) and for Fig.4-8 $n_d = 2$.

Table 4-1 shows these counting numbers.

These counting numbers n_a, n_b, n_c, n_d are different from diagram to diagram. But the total number

$$n := n_a + n_b + n_c + n_d \tag{4.15}$$

equals to 1 for any diagram.

We conclude

$$iW[J] = -\text{TrLn}[S_F] + \text{Tr}[S_0^{-1}S_F] + i\text{Tr}[S_F J] + \Gamma_{\text{loop}}[S_F]. \quad (4.16)$$

From the definition Eq.(4.9), the validity of Eq.(4.13) holds. $\blacksquare$

4.4 Homogeneous BSE

When the external source $J_{nm}(x, y)$ vanishes, the connected Green function $G_{C;nmn'}^{(2)}(xy; y'x')$ is rewritten by

$$\begin{aligned} G_{C;nmn'}^{(2)}(xy; y'x') &= \langle 0|T\psi_n(x)\bar{\psi}_m(y)\psi_{m'}(y')\bar{\psi}_{n'}(x')|0\rangle \\ &\quad - \langle 0|T\psi_n(x)\bar{\psi}_m(y)|0\rangle\langle 0|T\psi_{m'}(y')\bar{\psi}_{n'}(x')|0\rangle \\ &= \frac{\delta^2 W[J]}{\delta J_{mn}(y, x)\delta J_{n'm'}(x', y')}\Big|_{J=0}. \end{aligned} \quad (4.17)$$

This $G_{C;nmn'}^{(2)}(xy; y'x')$ can be written down with one particle physical state as

$$\begin{aligned} G_{C;nmn'}^{(2)}(xy; y'x') &= \int_P e^{-i(P-P_B)X+i(P-P_B)X'} \\ &\quad \times \sum_r i \frac{\chi_{r;nm}(xy; P_B)\bar{\chi}_{r;m'n'}(y'x'; P_B)}{P^2 - M^2 + i\epsilon} + R_{nmn'}(xy; y'x'), \end{aligned} \quad (4.18)$$

$$\chi_{r;nm}(xy; P_B) := \langle 0|T\psi_n(x)\bar{\psi}_m(y)|\mathbf{P}, r\rangle, \quad (4.19)$$

$$\bar{\chi}_{r;m'n'}(y'x'; P_B) := \langle \mathbf{P}, r|T\psi_{m'}(y')\bar{\psi}_{n'}(x')|0\rangle, \quad (4.20)$$

$$\langle \mathbf{P}|\mathbf{P}'\rangle = (2\pi)^3 2P_{B0}\delta^3(\mathbf{P} - \mathbf{P}'), \quad (4.21)$$

$$X := \frac{x+y}{2}, \quad X' := \frac{x'+y'}{2}, \quad (4.22)$$

$$P = (P_0, \mathbf{P}), \quad P_B = (\sqrt{M^2 + \mathbf{P}^2}, \mathbf{P}). \quad (4.23)$$

Here X, X' denotes center of mass coordinate, M deontes the mass of the bound state. The regular term $R_{nmn'}(xy; y'x')$ denotes the contributions from higher physical states. $\chi_{r;nm}(xy; P_B)$ and $\bar{\chi}_{r;m'n'}(y'x'; P_B)$ are BS amplitude and its conjugate respectively. For simplicity we often omit the symbol r which distinguishes degenerate one particle physical states. The validity and remark of Eq.(4.18) are discussed in 7.2.

Substituting Eq.(4.18) into inhomogeneous BSE (4.11), we obtain the homogeneous BSE

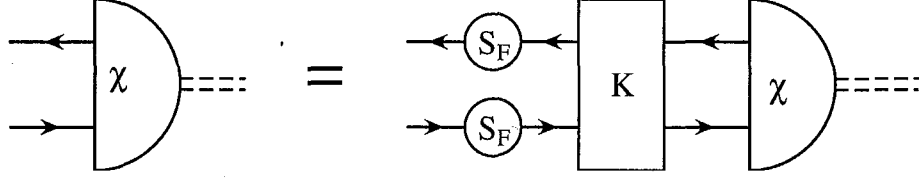


Fig.4-9: Homogeneous Bethe-Salpeter Equation

$$\frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{Fn'm'}(x', y')} \chi_{n'm'}(x', y'; P_B) = 0. \quad (4.24)$$

It is useful to write the homogeneous BSE (4.24)

$$S_{Fnn'}^{-1}(x, x') \chi_{n'm'}(x' y'; P_B) S_{Fn'm}^{-1}(y' y) + K_{mnn'm'}(yx; x' y') \chi_{n'm'}(x' y'; P_B) = 0 \quad (4.25)$$

where the operator defined by

$$K_{mnn'm'}(yx; x' y') := i \frac{\delta^2 \Gamma_{\text{loop}}[S_F]}{\delta S_{Fmn}(y, x) \delta S_{Fn'm'}(x', y')} \quad (4.26)$$

is called BS kernel. Fig.4-9 shows this homogeneous BSE (4.25).

4.5 Normalization of BS Amplitude

In this subsection we review the normalization condition of BS amplitudes. Firstly we define the Fourier transformation of the connected Green function $G_{C;m'n'nm}^{(2)}(y'x'; xy)$, the BS kernel $K_{n'm'mn}(x'y'; yx)$, the BS amplitude $\chi_{nm}(y'x'; P_B)$ and its conjugate $\bar{\chi}_{nm}(xy; P_B)$.

$$G_{C;n'm'mn}^{(2)}(x'y'; yx) =: \int_{p,q,P} e^{-i\{p(x'-y')+q(y-x)+P(X'-X)\}} G_{C;n'm'mn}^{(2)}(p, q; P), \quad (4.27)$$

$$K_{m'n'nm}(y'x'; xy) =: \int_{p,q,P} e^{-i\{p(x'-y')+q(y-x)+P(X'-X)\}} K_{m'n'nm}(p, q; P), \quad (4.28)$$

$$\chi_{nm}(xy; P_B) =: e^{-iP_B X} \int_q e^{-iq(x-y)} \chi_{nm}(q; P_B). \quad (4.29)$$

$$\bar{\chi}_{m'n'}(y'x'; P_B) =: e^{iP_B X'} \int_q e^{-iq(y'-x')} \bar{\chi}_{m'n'}(q; P_B), \quad (4.30)$$

The inhomogeneous BSE (4.11) in momentum space becomes

$$\begin{aligned} & \int_{p',q'} G_{C;nmm_1n_1}^{(2)}(p, p'; P) D_{m_1n_1n_2m_2}(p', q'; P) G_{C;n_2m_2m_3n_3}^{(2)}(q', q; P) \\ & = - (2\pi)^{12} G_{C;nmm_3n_3}^{(2)}(p, q; P), \end{aligned} \quad (4.31)$$

$$D_{m_1 n_1 n_2 m_2}(p', q'; P) := S_{F_{n_1 n_2}}^{-1}(p' + \frac{P}{2}) S_{F_{m_2 m_1}}^{-1}(p' - \frac{P}{2}) \delta^4(p' - q') + \frac{1}{(2\pi)^4} K_{m_1 n_1 n_2 m_2}(p', q'; P). \quad (4.32)$$

Substituting the Fourier transformation of Eq.(4.18)

$$G_{C;nm m' n'}^{(2)}(p, q; P) = i \frac{\chi_{nm}(p; P_B) \bar{\chi}_{m' n'}(q; P_B)}{P^2 - M^2 + i\epsilon} + R_{nm m' n'}(p, q; P) \quad (4.33)$$

into Eq.(4.31), we obtain

$$\begin{aligned} & \int_{p', q'} \left[i \frac{\chi_{nm}(p; P_B) \bar{\chi}_{m_1 n_1}(p'; P_B)}{P^2 - M^2 + i\epsilon} + R_{nm m_1 n_1}(p, p'; P) \right] D_{m_1 n_1 n_2 m_2}(p', q'; P) \\ & \quad \times \left[i \frac{\chi_{n_2 m_2}(q'; P_B) \bar{\chi}_{m_3 n_3}(q; P_B)}{P^2 - M^2 + i\epsilon} + R_{n_2 m_2 m_3 n_3}(q', q; P) \right] \\ & = - (2\pi)^{12} \left[i \frac{\chi_{nm}(p; P_B) \bar{\chi}_{m_3 n_3}(q; P_B)}{P^2 - M^2 + i\epsilon} + R_{nm m_3 n_3}(p, q; P) \right]. \end{aligned} \quad (4.34)$$

Here we omit the index r , which specifies the degeneracy of the one particle state, in order to avoid the unnecessary complexity.

From Eq.(4.34) we extract the dominant term in the on-shell limit $P \rightarrow P_B$. Using the homogeneous BSE

$$\lim_{P \rightarrow P_B} \int_{p'} \bar{\chi}_{m_1 n_1}(p'; P_B) D_{m_1 n_1 n_2 m_2}(p', q'; P) = 0, \quad (4.35)$$

$$\lim_{P \rightarrow P_B} \int_{q'} D_{m_1 n_1 n_2 m_2}(p', q'; P) \chi_{n_2 m_2}(q'; P_B) = 0, \quad (4.36)$$

we obtain the normalization condition

$$\lim_{P \rightarrow P_B} \int_{p', q'} i \frac{\bar{\chi}_{m_1 n_1}(p'; P_B) D_{m_1 n_1 n_2 m_2}(p', q'; P) \chi_{n_2 m_2}(q'; P_B)}{P^2 - M^2 + i\epsilon} = -(2\pi)^{12} \quad (4.37)$$

or equivalently

$$\lim_{P \rightarrow P_B} \int_{p', q'} i \bar{\chi}_{m_1 n_1}(p'; P_B) \left[\frac{\partial}{\partial P^\mu} D_{m_1 n_2 n_2 m_2}(p', q'; P) \right] \chi_{n_2 m_2}(q'; P_B) = -(2\pi)^{12} 2P_{B\mu}. \quad (4.38)$$

We introduce the norm of $\chi(q; P_B)$ defined by

$$\begin{aligned} \|\chi(q; P_B)\|^2 &:= -\frac{1}{(2\pi)^{12}} \lim_{P \rightarrow P_B} \frac{1}{2P^2} \\ & \quad \times \int_{p', q'} i \bar{\chi}_{m_1 n_1}(p'; P_B) P^\mu \left[\frac{\partial}{\partial P^\mu} D_{m_1 n_2 n_2 m_2}(p', q'; P) \right] \chi_{n_2 m_2}(q'; P_B). \end{aligned} \quad (4.39)$$

The normalization condition is rewritten by

$$||\chi(q; P_B)|| = 1. \quad (4.40)$$

Note that the norm Eq.(4.39) is defined only for the solution of the BSE. The norm may be 0 or negative for the function which does not satisfy the BSE. For example, let us consider Eq.(5.20) in **5.2**. In the chiral limit ($m_0 = 0$ and $M_\pi = 0$), neglecting the higher state contributions $R_{m''n''nm}(p, q; P)$ and taking the on-shell limit $P \rightarrow P_B$ we obtain

$$\chi_{mn}(q; P_B) = \frac{1}{f_\pi} \left(i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P_B}{2}) \right)_{mn} + \left(S_F(p + \frac{P_B}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{mn}. \quad (4.41)$$

This $\chi_{mn}(q; P_B)$ does not satisfy the BSE and its norm equals to 0. The reason is that generally one can not neglect the higher state contributions, except for the soft limit $\mathbf{P} \rightarrow 0$ after the on-shell limit $P \rightarrow P_B$.

5 Approximation of SDE-BSE System

5.1 Preliminary

In this section we check whether the CJT action formulation derived in Sec.4 preserves the chiral properties of the effective model given in Sec.3. Simultaneously we confirm that the results given in Sec.3 are correct within the SDE-BSE approach. And we will derive a formula to calculate the decay constant of NG boson which is our main result. Although the approximation may violate the property of the chiral symmetry, our formula still preserves it. This is due to the systematic approximation in the CJT action formulation.

The CJT action is given by

$$\Gamma[S_F] := i\text{TrLn}[S_F] - i\text{Tr}[S_0^{-1}S_F] + \Gamma_{\text{loop}}[S_F] \quad (5.1)$$

The last term of Eq.(5.1) is the residual term. Multiplying a factor i , $i\Gamma_{\text{loop}}[S_F]$ is given by the sum of all Feynman amplitudes of 2-loop or higher-loop 2-particle irreducible vacuum diagrams in which every bare Fermion propagator S_0 is replaced by the full one S_F . It resembles to the usual perturbative expansion, but in this case there is no small expansion parameter (i.e. coupling constant in the perturbation theory.) For this reason it is impossible to check the Ward-Takahashi identity order by order. We will show that the truncation of the loop expansion preserves properties of the chiral symmetry.

The SDE and the inhomogeneous BSE introduced in 4.2 are given by

$$\frac{\delta\Gamma[S_F]}{\delta S_F(x, y)} = 0, \quad (5.2)$$

$$\frac{1}{i} \frac{\delta^2\Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{Fm'n'}(x', y')} G_{C;n'm'm''n''}^{(2)}(x'y'; y''x'') = \delta_{m''m} \delta_{nn''} \delta(y'' - y) \delta(x - x'') \quad (5.3)$$

respectively where $G_{C;n'm'm''n''}^{(2)}(x'y'; y''x'')$ is defined by

$$\begin{aligned} G_{C;nmm'n'}^{(2)}(xy; y'x') &:= \langle 0 | T \psi_n(x) \bar{\psi}_m(y) \psi_{m'}(y') \bar{\psi}_{n'}(x') | 0 \rangle \\ &\quad - \langle 0 | T \psi(x) \bar{\psi}_m(y) | 0 \rangle \langle 0 | T \psi_{m'}(y') \bar{\psi}_{n'}(x') | 0 \rangle \\ &= \left. \frac{\delta^2 W[J]}{\delta J_{mn}(y, x) \delta J_{n'm'}(x', y')} \right|_{J=0}. \end{aligned} \quad (5.4)$$

The $G_{C;n'n'm''n''}^{(2)}(x'y';y''x'')$ can be written down by one particle physical state as

$$G_{C;nmm'n'}^{(2)}(x, y; y', x') = \int_P e^{-i(P-P_B)X+i(P-P_B)X'} \times i \frac{\chi_{nm}(xy; P_B) \bar{\chi}_{m'n'}(y'x'; P_B)}{P^2 - M_\pi^2 + i\epsilon} + R_{nmm'n'}(xy; y'x') \quad (5.5)$$

$$X := \frac{x+y}{2}, \quad X' := \frac{x'+y'}{2}, \quad (5.6)$$

$$P = (P_0, \mathbf{P}), \quad P_B = (\sqrt{M_\pi^2 + \mathbf{P}^2}, \mathbf{P}). \quad (5.7)$$

Here X, X' denotes the center of mass coordinate, M_π denotes the mass of the bound state. The regular term $R_{nmm'n'}(xy; y'x')$ denotes the contributions from higher physical states. We omit the index r for simplicity.

The homogeneous BSE is given by

$$\frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{F'n'm'}(x', y')} \chi_{n'm'}(x', y'; P_B) = 0. \quad (5.8)$$

The question is whether the chiral properties of the effective model are preseved when one truncates the sum in $\Gamma_{\text{loop}}[S_F]$. To answer this question we use the Munczek's technique.[18]

Let us first consider the infinitesimal global chiral transformation of $S_F(x, y)$.

$$S_F(x, y) \rightarrow S'_F(x, y) := (1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha) S_F(x, y) (1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha) \quad (5.9)$$

If the lagrangian of the effective model is globally chiral invariant, both the CJT action and each term of the sum in $\Gamma_{\text{loop}}[S_F]$ are also globally chiral invariant. Then the change of the CJT action under this global transformation is

$$\begin{aligned} \Delta_5 \Gamma[S_F] &:= \Gamma[S'_F] - \Gamma[S_F] \\ &\equiv \frac{\delta \Gamma[S_F]}{\delta S_{F'n'm'}(x', y')} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F\}_{n'm'}(x', y') \theta^\alpha \end{aligned} \quad (5.10)$$

$$\equiv 0. \quad (5.11)$$

Here $\equiv$ denotes that this is an identity in the function space. (i.e. This holds for any S_F) Differentiating this identity by $S_{Fmn}(y, x)$ one obtains an identity

$$\begin{aligned} &\frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{F'n'm'}(x', y')} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F\}_{n'm'}(x', y') \\ &+ \frac{\delta \Gamma[S_F]}{\delta S_{F'n'm'}(y, x)} \left((i\gamma_5 \frac{\lambda^\alpha}{2})_{n'm} \delta_{nm'} + \delta_{n'm} (i\gamma_5 \frac{\lambda^\alpha}{2})_{nm'} \right) \equiv 0. \end{aligned} \quad (5.12)$$

Using the SDE (5.2) the second term of the left hand side of Eq.(5.12) vanishes. We get the equation^{*1}

$$\frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{F'n'm'}(x', y')} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F\}_{n'm'}(x', y') = 0, \quad (5.13)$$

and comparing it with homogeneous BSE (5.8) we obtain the Nambu–Goldstone solution

$$\chi_{n'm'}(x, y; P_B = 0) := N \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F\}_{n'm'}(x, y) \quad (5.14)$$

where N is a normalization constant. Here we use the fact that the right hand side of Eq.(5.14) depends only on $x - y$, the left hand side must also depend only on $x - y$, which implies $P_B = 0$. So we conclude that even if one truncates the sum in $\Gamma_{\text{loop}}[S_F]$, the dynamical chiral symmetry breaking produces a solution of BSE for a massless particle. We remark that though the global chiral symmetry breaking implies the existence of pseudo-scalar Nambu–Goldstone particle in this way, the normalization constant can't be determined.

Let us next consider a local infinitesimal chiral transformation of $S_F(x, y)$.

$$S_F(x, y) \rightarrow S'_F(x, y) := (1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x)) S_F(x, y) (1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(y)) \quad (5.15)$$

Under this local transformation, the change of the CJT action is

$$\Delta_5 \Gamma[S_F] \equiv \frac{\delta \Gamma[S_F]}{\delta S_{Fm'n'}(y', x')} \{i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha, S_F\}_{m'n'}(y', x'). \quad (5.16)$$

The following equation holds.

$$\begin{aligned} & G_{C;m''n''nm}^{(2)}(y''x''; xy) \frac{\delta(\Delta_5 \Gamma[S_F])}{\delta S_{Fnm}(x, y)} \\ & \equiv G_{C;m''n''nm}^{(2)}(y''x''; xy) \frac{\delta^2 \Gamma[S_F]}{\delta S_{Fnm}(xy) \delta S_{Fm'n'}(y'x')} \{i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha, S_F\}_{m'n'}(y'x') \\ & \quad + \frac{\delta \Gamma[S_F]}{\delta S_{Fm'n'}(y'x')} \left\{ G_{C;m''n''m'l}^{(2)}(y''x''; y'x') (i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x'))_{ln'} \right. \\ & \quad \left. + (i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(y'))_{m'l} G_{C;m''n''ln'}^{(2)}(y''x''; y'x') \right\} \\ & = i \{i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha, S_F\}_{m''n''}(y''x''). \end{aligned} \quad (5.17)$$

Here to derive the last form we use the SDE (5.2) and the inhomogeneous BSE (5.3). Eq.(5.17) is a starting equation from which we examine the SDE-BSE system.

^{*1}Eq.(5.13) is not an identity but an equation, because it holds only for the solution of SDE (5.2).

5.2 Analysis for the CJT action

In this subsection we consider the case that the cut off function is not introduced and the interaction term is locally chiral invariant.

In this case each term of the $\Gamma_{\text{loop}}[S_F]$ is also locally chiral invariant. Then only the second term $-i\text{Tr}[S_0^{-1}S_F]$ in Eq.(5.1) is locally or globally variant in the CJT action, Eq.(5.17) reduces in momentum space

$$\begin{aligned} & \int_q G_{C;m''n''nm}^{(2)}(p, q; P) \left(i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) \right)_{mn} \\ &= i \left(i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P}{2}) \right)_{m''n''} + i \left(S_F(p + \frac{P}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{m''n''}. \end{aligned} \quad (5.18)$$

Here the Fourier transformation of $G_{C;m'n'nm}^{(2)}(y'x'; xy)$ is given by

$$G_{C;m'n'nm}^{(2)}(y'x'; xy) = \int_{p,q,P} e^{-i\{p(y'-x')+q(x-y)+P(X'-X)\}} G_{C;m'n'nm}^{(2)}(p, q; P). \quad (5.19)$$

Using Eq.(5.5), Eq.(5.18) becomes

$$\begin{aligned} & \int_q \left[i \frac{\chi_{m''n''}(p; P_B) \bar{\chi}_{nm}(q; P_B)}{P^2 - M_\pi^2 + i\epsilon} + R_{m''n''nm}(p, q; P) \right] \left(i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) \right)_{mn} \\ &= i \left(i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P}{2}) \right)_{m''n''} + i \left(S_F(p + \frac{P}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{m''n''}, \end{aligned} \quad (5.20)$$

$$\bar{\chi}_{nm}(xy; P_B) = e^{iP_B X} \int_q e^{-iq(x-y)} \bar{\chi}_{nm}(q; P_B), \quad (5.21)$$

$$\chi_{m'n'}(y'x'; P_B) = e^{-iP_B X'} \int_q e^{-iq(y'-x')} \chi_{m'n'}(q; P_B). \quad (5.22)$$

The Chiral Limit

In the chiral limit $m_0 \rightarrow 0$, the CJT action is globally chiral invariant. Then a massless Nambu-Goldstone particle will appear because if $M_\pi \neq 0$ the left hand side of Eq.(5.20) diverges in the limit $P^2 \rightarrow M_\pi^2$, while the right hand side converges. In this case the left hand side of Eq.(5.20) becomes

$$\begin{aligned} \text{LHS of Eq.(5.20)} &= \int_q \frac{i}{P^2} \text{tr}[\bar{\chi}(q; P_B) i\gamma_5 \frac{\lambda^\alpha}{2} \not{P}] \chi_{m''n''}(p; P_B) \\ &+ \int_q R_{m''n''nm}(p, q; P) \left(i\gamma_5 \frac{\lambda^\alpha}{2} \not{P} \right)_{mn}. \end{aligned} \quad (5.23)$$

If we take in the soft limit $\mathbf{P} \rightarrow 0$ after taking the on-shell limit $P \rightarrow P_B$ (i.e. $P_0 \rightarrow \sqrt{\mathbf{P}^2}$), the second term of Eq.(5.23) vanishes and the first term becomes

$$\text{1st term of Eq.(5.23)} \rightarrow i f_\pi \chi_{m''n''}(p; 0), \quad (5.24)$$

$$f_\pi = \lim_{P \rightarrow P_B} \frac{1}{P^2} \int_q \text{tr}[\bar{\chi}(q; P_B) i \gamma_5 \frac{\lambda^\alpha}{2} \not{P}]. \quad (5.25)$$

Here we use the definition of the decay constant given by Eq.(3.29). On the other hand the right hand side of Eq.(5.20) becomes

$$\text{RHS of Eq.(5.20)} \rightarrow i \left\{ i \gamma_5 \frac{\lambda^\alpha}{2}, S_F(p) \right\}_{m''n''} \quad (5.26)$$

in the same limit. Then we obtain the equation

$$\chi_{mn}(q; 0) = \frac{1}{f_\pi} \left\{ i \gamma_5 \frac{\lambda^\alpha}{2}, S_F(q) \right\}_{mn}. \quad (5.27)$$

This is the same equation as Eq.(3.37) which was derived from the axial Ward–Takahashi identity. Note that to derive Eq.(5.27) we used only SDE and BSE.

For a Finite Quark Mass

When $m_0 > 0$, the first term in the brackets of Eq.(5.20) will diverge in the on-shell limit $P \rightarrow P_B$. Then the equation

$$\int_q \text{tr}[\bar{\chi}(q; P_B) i \gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}_B)] = 0 \quad (5.28)$$

must be satisfied and we obtain

$$M_\pi^2 f_\pi = -2m_0 i \int_q \text{tr}[\bar{\chi}(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}] \quad (5.29)$$

where f_π is given by Eq.(5.25). This equation is again same as Eq.(3.40). The GMOR relation is now re-derived

$$M_\pi^2 f_\pi^2 \simeq -2m_0 \langle \bar{\psi} \psi \rangle_0, \quad \langle \bar{\psi} \psi \rangle_0 := - \int_q \text{tr}[S_F(q)_{m_0=0}]. \quad (5.30)$$

We again insist that we use only SDE and BSE to derive Eqs.(5.29), (5.30). Thus we have proved that taking finite number of terms of $\Gamma_{\text{loop}}[S_F]$ preserves the low energy property of the effective model.

5.3 Cut off Regularization

When the cut off function is introduced, one can derive the low energy formulae using only SDE and BSE in the same way. From Eq.(5.17) we must replace

$$i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) \quad (5.31)$$

$$\mapsto i\gamma_5 \frac{\lambda^\alpha}{2} \left(2m_0 \frac{f(-q_-^2) + f(-q_+^2)}{2} - f(-q_-^2) \not{q}_- + f(-q_+^2) \not{q}_+ \right), \quad (5.32)$$

$$q_- := q - \frac{P}{2}, \quad q_+ := q + \frac{P}{2} \quad (5.33)$$

in the previous subsection. The definition of the decay constant f_π is now modified

$$f_\pi = \lim_{P \rightarrow P_B} \frac{1}{P^2} \int_q \text{tr} \left[\bar{\chi}(q; P_B) i\gamma_5 \frac{\lambda^\alpha}{2} \left\{ \frac{f(-q_-^2) + f(-q_+^2)}{2} \not{P} + (f(-q_+^2) - f(-q_-^2)) \not{q} \right\} \right] \quad (5.34)$$

as Eq.(3.29) with replacement $J_{5\mu}^\alpha \mapsto I_{5\mu}^\alpha$. As mentioned in 3.2, one must be careful about Eq.(5.34) because $f(x) \rightarrow \infty$ for $x \gg \Lambda_{UV}^2$.

Therefore the PCAC relation Eq.(5.29) is modified

$$M_\pi^2 f_\pi = -2m_0 i \int_q \frac{f(-q_-^2) + f(-q_+^2)}{2} \text{tr} [\bar{\chi}(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}] \quad (5.35)$$

and the definition of pair condensation is modified

$$\langle \bar{\psi} \psi \rangle_0 = - \int_q f(-q^2) \text{tr} [S_F(q)_{m_0=0}]. \quad (5.36)$$

Under these modifications, Eq.(5.27) and Eq.(5.30) hold.

5.4 Locally chiral variant interaction

Here we consider the KG model as an example of locally chiral variant interaction. For simplicity we omit the cut off function $f(x)$.

In this case the effective axial current $\tilde{J}_{5\mu}^\alpha$ is different from the naive axial current $J_{5\mu}^\alpha$ shown in Sec.3. In the same way, the conserved current for the framework of the approximated SDE and BSE equations must be modified from the naive form. For simplicity we employ the lowest order(lowest loop) approximation of $\Gamma_{\text{loop}}[S_F]$ such that

$$\begin{aligned} \Gamma_{\text{loop}}[S_F] &= -\frac{1}{2} \int d^4x \tilde{K}^{mn, n_2 m_2} \left(\frac{i\partial^{x_2} - i\partial^{y_1}}{2}, \frac{-i\partial^{x_1} + i\partial^{y_2}}{2} \right) \\ &\quad \times [S_{Fm m_2}(y_2, y_1) S_{Fn_2 n}(x_2, x_1) - S_{Fmn}(y_2, x_1) S_{Fn_2 m_2}(x_2, y_1)]|_*. \end{aligned} \quad (5.37)$$

Note that this approximation corresponds to the rainbow-ladder approximation shown by Figs.4-1 and 4-2.

In this case Eq.(5.17) reduces in momentum space to

$$\begin{aligned} & \int_q G_{C;m''n''nm}^{(2)}(p, q; P) \left(i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) + E^\alpha(q; P) \right)_{mn} \\ &= i \left(i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P}{2}) \right)_{m''n''} + i \left(S_F(p + \frac{P}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{m''n''}, \end{aligned} \quad (5.38)$$

$$\begin{aligned} E_{mn}^\alpha(q; P) &:= \int_k \left[\left\{ \tilde{K}_{nmm'n'}(k + \frac{P}{2}, q) - \tilde{K}_{nmm'n'}(k, q - \frac{P}{2}) \right\} (i\gamma_5 \frac{\lambda^\alpha}{2} S_F(k))_{m'n'} \right. \\ &\quad \left. + \left\{ \tilde{K}_{nmm'n'}(k - \frac{P}{2}, q) - \tilde{K}_{nmm'n'}(k, q + \frac{P}{2}) \right\} (S_F(k) i\gamma_5 \frac{\lambda^\alpha}{2})_{m'n'} \right]. \end{aligned} \quad (5.39)$$

One sees that $E^\alpha(q; P) \rightarrow 0$ in the limit $P \rightarrow 0$.

Using Eq.(5.5), Eq.(5.38) becomes

$$\begin{aligned} & \int_q \left[i \frac{\chi_{m''n''}(p; P_B) \bar{\chi}_{nm}(q; P_B)}{P^2 - M_\pi^2 + i\epsilon} + R_{m''n''nm}(p, q; P) \right] \left(i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) + E^\alpha(q; P) \right)_{mn} \\ &= i \left(i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P}{2}) \right)_{m''n''} + i \left(S_F(p + \frac{P}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{m''n''}. \end{aligned} \quad (5.40)$$

Chiral Limit

In the chiral limit, the global chiral symmetry breaking implies the existence of the Nambu-Goldstone particle again. Then we obtain

$$\begin{aligned} \text{LHS of Eq.(5.40)} &= \int_q \frac{i}{P^2} \text{tr}[\bar{\chi}(q; P_B) (i\gamma_5 \frac{\lambda^\alpha}{2} \not{P} + E^\alpha(q; P))] \chi_{m''n''}(p; P_B) \\ &\quad + \int_q R_{m''n''nm}(p, q; P) (i\gamma_5 \frac{\lambda^\alpha}{2} \not{P})_{mn}. \end{aligned} \quad (5.41)$$

holds. Taking the soft limit $\mathbf{P} \rightarrow 0$ after the on-shell limit $P \rightarrow P_B$ (i.e. $P_0 \rightarrow \sqrt{\mathbf{P}^2}$) the second term of Eq.(5.41) vanishes and the first term becomes

$$\text{1st term of Eq.(5.41)} \rightarrow i \tilde{f}_\pi \chi_{m''n''}(p; 0), \quad (5.42)$$

$$\tilde{f}_\pi := \lim_{P \rightarrow P_B} \frac{1}{P^2} \int_q \text{tr}[\bar{\chi}(q; P_B) (i\gamma_5 \frac{\lambda^\alpha}{2} \not{P} + E^\alpha(q; P))]. \quad (5.43)$$

We note that this definition of $\tilde{f}_\pi$ may not necessarily coincide with that from Eq.(3.30).

On the other hand the right hand side of Eq.(5.40) becomes

$$\text{RHS of Eq.(5.40)} \rightarrow i \left\{ i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(p) \right\}_{m''n''} \quad (5.44)$$

in the same limit. Then we get the equation

$$\chi_{mn}(q; 0) = \frac{1}{\tilde{f}_\pi} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(q)\}_{mn}. \quad (5.45)$$

This is the same equation as Eq.(3.37) which was derived from the axial Ward–Takahashi identity. Therefore we conclude the definition of $\tilde{f}_\pi$ in Eq.(5.43) is equivalent to the definition Eq.(3.30).

Finite Quark Mass

When $m_0 > 0$, the first term in the brackets of Eq.(5.40) diverges in the on-shell limit $P \rightarrow P_B$. Then the equation

$$\int_q \text{tr}[\bar{\chi}(q; P_B)(i\gamma_5 \frac{\lambda^\alpha}{2}(2m_0 + \not{P}_B) + E^\alpha(q; P_B))] = 0 \quad (5.46)$$

must hold. We obtain

$$M_\pi^2 \tilde{f}_\pi = -2m_0 i \int_q \text{tr}[\bar{\chi}(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}] \quad (5.47)$$

where $\tilde{f}_\pi$ is given by Eq.(5.43). This equation again coincides with Eq.(3.40) and implies that our choice of $\tilde{f}_\pi$ in Eq.(5.43) is consistent with Eq.(3.30). The GMOR relation is written as

$$M_\pi^2 \tilde{f}_\pi^2 \simeq -2m_0 \langle \bar{\psi}\psi \rangle_0, \quad \langle \bar{\psi}\psi \rangle_0 := - \int_q \text{tr}[S_F(q)_{m_0=0}]. \quad (5.48)$$

Thus we have proved that the truncation of $\Gamma_{\text{loop}}[S_F]$ preserves the low energy property of the effective model if one use the suitable formula of the decay constant like Eq.(5.43).

Up to now we have treated the lowest order (rainbow-ladder) approximation. When we use the higher order approximation, the definition Eq.(5.43) should be modified to an appropriate form. Then we can prove that truncation of $\Gamma_{\text{loop}}[S_F]$ preserves the low energy property of the effective model with locally variant interaction if one uses the appropriate formula of the decay constant like Eq.(5.43).

5.5 Formula of the decay constant

We summarize this section briefly and derive our main result, a formula of the decay constant.

Under the infinitesimal local chiral transformation, the change of the classical action can be written down formally as

$$\Delta_5 S[\psi, \bar{\psi}] \equiv \int d^4 x \theta^\alpha(x) \left(\partial^\mu \tilde{J}_{5\mu}^\alpha[\psi, \bar{\psi}](x) - M^\alpha[\psi, \bar{\psi}](x) \right). \quad (5.49)$$

Here $M^\alpha[\psi, \bar{\psi}](x)$ comes from the globally variant terms, such as the fermion mass term and $\tilde{J}_{5\mu}^\alpha[\psi, \bar{\psi}](x)$ is a effective axial current.

The change of the CJT action corresponding to the classical action must be written down formally as

$$\Delta_5 \Gamma[S_F] \equiv \int d^4 x \theta^\alpha(x) \left(\partial^\mu \tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](x) - \mathcal{M}^\alpha[S_F](x) \right). \quad (5.50)$$

Again $\mathcal{M}^\alpha[S_F](x)$ comes from the globally variant terms and $\tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](x)$ is a proxy of the effective axial current in the CJT action. Both Eq.(5.49) and Eq.(5.50) hold by the same mechanism.

When the effective axial current $\tilde{J}_{5\mu}^\alpha[\psi, \bar{\psi}](x)$ has a non-local interaction term, the exact proxy $\tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](x)$ is a sum of infinite Feynman amplitudes which come from the expansion of $\Gamma_{\text{loop}}[S_F]$. If one truncates the expansion of $\Gamma_{\text{loop}}[S_F]$, the approximated proxy $\tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](x)$ is built of a sum of finite Feynman amplitudes.

For example, for the KG model with cut off function $f(x)$, at the lowest loop approximation Eq.(5.37), we obtain

$$\begin{aligned} \partial^\mu \tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](z) = & \int d^4 x \int_p e^{-ip(z-x)} f(p^2) \left\{ \text{tr}[\not{p} i \gamma_5 \frac{\lambda^\alpha}{2} S_F(z, x)] - \text{tr}[\not{p} S_F(x, z) i \gamma_5 \frac{\lambda^\alpha}{2}] \right\} \\ & - \int d^4 x \int_k e^{ikz} \tilde{K}^{mn, n_2 m_2} \left(\frac{i \partial^{x_2} - i \partial^{y_1}}{2}, \frac{-i \partial^{x_1} + i \partial^{y_2}}{2} \right) \left[\right. \\ & S_{Fm m_2}(y_2, y_1) \left\{ e^{-ikx_2} i \gamma_5 \frac{\lambda^\alpha}{2} S_F(x_2, x_1) + e^{-ikx_1} S_F(x_2, x_1) i \gamma_5 \frac{\lambda^\alpha}{2} \right\}_{n_2 n} \\ & \left. - S_{Fmn}(y_2, x_1) \left\{ e^{-ikx_2} i \gamma_5 \frac{\lambda^\alpha}{2} S_F(x_2, x_1) + e^{-iky_1} S_F(x_2, y_1) i \gamma_5 \frac{\lambda^\alpha}{2} \right\}_{n_2 m_2} \right] \Big|_*, \quad (5.51) \end{aligned}$$

$$\mathcal{M}^\alpha[S_F](z) = -m_0 \int d^4 x \int_p e^{-ip(z-x)} f(p^2) \left\{ \text{tr}[i \gamma_5 \frac{\lambda^\alpha}{2} S_F(z, x)] + \text{tr}[S_F(x, z) i \gamma_5 \frac{\lambda^\alpha}{2}] \right\}. \quad (5.52)$$

Then to calculate the decay constant, we derive the formula

$$\begin{aligned} \tilde{f}_\pi &:= \lim_{P \rightarrow P_B} \frac{i P^\mu}{P^2} \langle \mathbf{P} | \tilde{J}_{5\mu}^\alpha[\psi, \bar{\psi}](0) | 0 \rangle \\ &= \lim_{P \rightarrow P_B} \frac{i P^\mu}{P^2} \bar{\chi}_{nm}(x, y; P_B) \frac{\delta \tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](0)}{\delta S_{Fnm}(x, y)} \quad (5.53) \end{aligned}$$

$$= \lim_{P \rightarrow P_B} \frac{1}{P^2} \bar{\chi}_{nm}(x, y; P_B) \frac{\delta \partial^\mu \tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](0)}{\delta S_{Fnm}(x, y)}. \quad (5.54)$$

Therefore systematically approximated $\tilde{f}_\pi$ can be obtained for any approximation of $\Gamma_{\text{loop}}[S_F]$. Eq.(5.53) is perfectly consistent with both the chiral symmetric feature of effective models and the approximation methods of the nonperturbative SDE-BSE approach.

Simultaneously we have proven that the formal analyses for the effective models in Sec.3 hold. In Sec.8 we will introduce an ExKG model. The explicit formula of the decay constant for the ExKG model is given in **8.10**.

6 Regularized Equations

6.1 Preliminary

In this section, using the cut off regularization function $f(\partial^2)$ we show the integral equations which avoid the divergence and are well-defined. Although this approach is general, we concentrate on the KG model with a finite quark mass within the rainbow ladder approximated SDE-BSE system as an example. In such a model with the asymptotic free property, the running coupling constant g has a function of regularization to some extent. But the convergence is too slow, and every result contains an order $(g(\Lambda_{UV}^2))^\alpha$ error (α is a suitable positive constant) which goes to zero in the limit $\Lambda_{UV} \rightarrow \infty$. [20]

We use the notation $P = (\sqrt{M_\pi^2 + \mathbf{P}^2}, \mathbf{P})$ for simplicity instead of P_B in the previous sections.

A few examples of the cut off function are

$$f(x) = 1 + \frac{x}{\Lambda_{UV}^2}, \quad (6.1)$$

$$f(x) = 1 + M\theta(x - \Lambda_{UV}^2), \quad M \rightarrow \infty. \quad (6.2)$$

In the cut off function a scale parameter Λ_{UV} is needed. The fermion mass m_0 is evaluated in this scheme and at this scale.

The first choice Eq.(6.1) makes a smooth cut off and is useful for the non-renormalizable model like the NJL model.

The second choice Eq.(6.2) makes a sharp cut off and is useful for a numerical calculation of asymptotically free models like the KG model. In this scheme, every integral region (in Euclid space) is finite. The second term of the left hand side of Eq.(6.13) in the next subsection disappears.

6.2 Regularized Equations

The Schwinger–Dyson equation (5.2) is

$$S_{Fmn}^{-1}(q) = S_{0mn}^{-1}(q) + i \int_p \tilde{K}^{nm, m'n'}(q, p) S_{Fm'n'}(p), \quad (6.3)$$

$$S_0(q) := \frac{1}{f(-q^2)} \frac{i}{\not{q} - m_0}. \quad (6.4)$$

The regularized propagators are defined by

$$S_0^R(q) := f(-q^2)S_0(q) = \frac{i}{\not{q} - m_0}, \quad (6.5)$$

$$S_F^R(q) := f(-q^2)S_F(q). \quad (6.6)$$

Then Eq.(6.3) can be rewritten with regularized propagators

$$S_{Fmn}^{R-1}(q) = S_{0mn}^{R-1}(q) + \frac{1}{f(-q^2)} i \int_p \tilde{K}^{nm,m'n'}(q,p) \frac{S_{Fm'n'}^R(p)}{f(-p^2)}. \quad (6.7)$$

The cut off function $f^{-1}(-p^2)$ in the integrand can control the integral to converge. The other relations are obtained in the same way.

The pair condensation Eq.(3.44) is given in terms of S_F^R by

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_0 &= - \int_q f(-q^2) \text{tr}[S_F(q)_{m_0=0}] \\ &= - \int_q \text{tr}[S_F^R(q)_{m_0=0}]. \end{aligned} \quad (6.8)$$

We note that this $\langle \bar{\psi}\psi \rangle_0$ is defined only in the chiral limit. Eq.(6.8) is well-defined even in the NJL like model if a suitable cut off function $f(-q^2)$ like Eq.(6.1) is chosen. Because for a factor $f^{-1}(-q^2)$ in front of the integral of the right hand side of Eq.(6.7), the integrand in Eq.(6.8) becomes small as q grows.

The homogeneous Bethe-Salpeter equation (5.8) is given by

$$S_{Fnn_1}^{-1}(q_+) \chi_{n_1 m_1}(q; P) S_{Fm_1 m}^{-1}(q_-) = -i \int_k \tilde{K}^{mn,n'm'}(q,k) \chi_{n'm'}(k; P). \quad (6.9)$$

with $q_+ := q + P/2$, $q_- := q - P/2$.

We define the regularized Bethe-Salpeter amplitude

$$\chi_{nm}^R(q; P) := f(q_+^2) \chi_{nm}(q; P) f(q_-^2). \quad (6.10)$$

Then Eq.(6.9) can be rewritten as

$$S_{Fnn_1}^{R-1}(q_+) \chi_{n_1 m_1}^R(q; P) S_{Fm_1 m}^{R-1}(q_-) = -i \int_k \tilde{K}^{mn,n'm'}(q,k) \frac{\chi_{n'm'}^R(k; P)}{f(k_+^2) f(k_-^2)}. \quad (6.11)$$

The normalization condition of the Bethe-Salpeter amplitude is derived from the inhomogeneous Bethe-Salpeter equation (5.3) as

$$i \int_q \chi_{n_1 m_1}(q; P) \bar{\chi}_{m_2 n_2}(q; P) \frac{\partial}{\partial P_\mu} \left(S_{F n_2 n_1}^{-1}(q_+) S_{F m_1 m_2}^{-1}(q_-) \right) = -2P_\mu. \quad (6.12)$$

This condition is now rewritten with the regularized function

$$\begin{aligned}
& i \int_q \frac{1}{f(-q_+^2)f(-q_-^2)} \chi_{n_1 m_1}^R(q; P) \bar{\chi}_{m_2 n_2}^R(q; P) \frac{\partial}{\partial P^\mu} (S_{F n_2 n_1}^{R-1}(q_+) S_{F m_1 m_2}^{R-1}(q_-)) \\
& + i \int_q \frac{-(q_+)_\mu f'(-q_+^2) f(-q_-^2) - (q_-)_\mu f'(-q_-^2) f(-q_+^2)}{f^2(-q_+^2) f^2(-q_-^2)} \\
& \times \chi_{n_1 m_1}^R(q; P) S_{F m_1 m_2}^{R-1}(q_-) \chi_{m_2 n_2}^R(q; P) S_{F n_2 n_1}^{R-1}(q_+) = -2P_\mu.
\end{aligned} \tag{6.13}$$

The modified decay constant $\tilde{f}_\pi$ (see Eqs.(5.34), (5.43)) is given by

$$\begin{aligned}
\tilde{f}_\pi &= \frac{1}{P^2} \int_q \bar{\chi}_{mn}(q; P) \left[i\gamma_5 \frac{\lambda^\alpha}{2} \left\{ \frac{f(-q_-^2) + f(-q_+^2)}{2} \not{P} \right. \right. \\
& \quad \left. \left. + (f(-q_+^2) - f(-q_-^2)) \not{q} \right\} + E^\alpha(q; P) \right]_{nm}
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
&= \frac{1}{P^2} \int_q \bar{\chi}_{mn}^R(q; P) \left[i\gamma_5 \frac{\lambda^\alpha}{2} \left\{ \left(\frac{1}{f(-q_+^2)} + \frac{1}{f(-q_-^2)} \right) \frac{\not{P}}{2} \right. \right. \\
& \quad \left. \left. + \left(\frac{1}{f(-q_-^2)} - \frac{1}{f(-q_+^2)} \right) \not{q} \right\} + \frac{1}{f(-q_-^2)f(-q_+^2)} E^\alpha(q; P) \right]_{nm}.
\end{aligned} \tag{6.15}$$

$E_{mn}^\alpha(q; P)$ defined by Eq.(5.39) is also written with regularized propagator

$$\begin{aligned}
E_{mn}^\alpha(q; P) &:= \int_k \frac{1}{f(-k^2)} \left[\left\{ \tilde{K}_{nmm'n'}(k + \frac{P}{2}, q) - \tilde{K}_{nmm'n'}(k, q - \frac{P}{2}) \right\} (i\gamma_5 \frac{\lambda^\alpha}{2} S_F^R(k))_{m'n'} \right. \\
& \quad \left. + \left\{ \tilde{K}_{nmm'n'}(k - \frac{P}{2}, q) - \tilde{K}_{nmm'n'}(k, q + \frac{P}{2}) \right\} (S_F^R(k) i\gamma_5 \frac{\lambda^\alpha}{2})_{m'n'} \right].
\end{aligned} \tag{6.16}$$

The quantity which appears in the PCAC relation Eq.(5.35) is

$$\mathcal{E}_\pi := -i \int_q \frac{f(-q_-^2) + f(-q_+^2)}{2} \text{tr}[\bar{\chi}(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}] \tag{6.17}$$

$$= -i \int_q \frac{1}{2} \left(\frac{1}{f(-q_+^2)} + \frac{1}{f(-q_-^2)} \right) \text{tr}[\bar{\chi}^R(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}]. \tag{6.18}$$

All these integrals are finite due to the cut off function. The conserved current Eq.(3.10) looks dangerous because it contains a divergent function $f(q^2)$. However, as in Eq.(6.15), $\tilde{f}_\pi$ is finite and therefore the conserved current Eq.(3.10) is appropriate.

7 Related Topics

7.1 Preliminary

In this section, we discuss some topics concerning our study.

7.2 Extraction of Pole Contribution

In Eqs.(4.18) and (5.5) we write down the connected Green function $G_{C;n'n'm''n''}^{(2)}(x'y';y''x'')$ in terms of one particle physical state. In the momentum space, it is given by

$$G_{C;nmm'n'}^{(2)}(p, q; P) = i \frac{\chi_{nm}(p; P_B) \bar{\chi}_{m'n'}(q; P_B)}{P^2 - M^2 + i\epsilon} + R_{nmm'n'}(p, q; P), \quad (7.1)$$

$$\langle \mathbf{P} | \mathbf{P}' \rangle = (2\pi)^3 2P_{B0} \delta^3(\mathbf{P} - \mathbf{P}'), \quad (7.2)$$

$$P = (P_0, \mathbf{P}), \quad P_B = (\sqrt{M^2 + \mathbf{P}^2}, \mathbf{P}). \quad (7.3)$$

Here we prove Eq.(7.1).

The complete set is

$$\int \frac{d\mathbf{K}}{(2\pi)^3 2K_{B0}} |\mathbf{K}\rangle \langle \mathbf{K}| + \sum_{\lambda} |\lambda\rangle \langle \lambda| = 1 \quad (7.4)$$

where $|\lambda\rangle$ denotes the two or more particle states.

Using Eq.(7.4), we obtain

$$\begin{aligned} [G_{C;nmm'n'}^{(2)}(p, q; P)]_{\text{pole}} &= \int dz_1 dz_2 dz_3 e^{i(pz_1 + qz_2 + Pz_3)} \int \frac{d\mathbf{K}}{(2\pi)^3 2K_{B0}} \\ &\quad \left\{ \theta(z_{30}) \langle 0 | T \psi_n(x) \bar{\psi}_m(y) | \mathbf{K} \rangle \langle \mathbf{K} | T \psi_{m'}(y') \bar{\psi}_{n'}(x') | 0 \rangle \right. \\ &\quad \left. + \theta(-z_{30}) \langle 0 | T \psi_{m'}(y') \bar{\psi}_{n'}(x') | \mathbf{K} \rangle \langle \mathbf{K} | T \psi_n(x) \bar{\psi}_m(y) | 0 \rangle \right\} \\ &= \int dz_3 e^{iPz_3} \int \frac{d\mathbf{K}}{(2\pi)^3 2K_{B0}} \\ &\quad \times \left\{ \theta(z_{30}) e^{-iK_B X} \chi_{nm}(p; K_B) \bar{\chi}_{m'n'}(q; K_B) e^{iK_B X'} \right. \\ &\quad \left. + \theta(-z_{30}) e^{-iK_B X'} \chi_{m'n'}(q; K_B) \bar{\chi}_{nm}(p; K_B) e^{iK_B X} \right\} \\ &= \int dz_3 \int \frac{d\mathbf{K}}{(2\pi)^3 2K_{B0}} \left\{ \theta(z_{30}) e^{i(P-K_B)z_3} \chi_{nm}(p; K_B) \bar{\chi}_{m'n'}(q; K_B) \right. \\ &\quad \left. + \theta(-z_{30}) e^{i(P+K_B)z_3} \chi_{m'n'}(q; K_B) \bar{\chi}_{nm}(p; K_B) \right\} \end{aligned}$$

$$\begin{aligned}
&= \int dz_{30} \frac{1}{2P_{B0}} \left\{ \theta(z_{30}) e^{i(P_0 - P_{B0})z_{30}} \chi_{nm}(p; P_B) \bar{\chi}_{m'n'}(q; P_B) \right. \\
&\quad \left. + \theta(-z_{30}) e^{i(P_0 + P_{B0})z_{30}} \chi_{m'n'}(q; (P_B)_P) \bar{\chi}_{nm}(p; (P_B)_P) \right\}
\end{aligned} \tag{7.5}$$

where

$$X := \frac{x+y}{2}, \quad X' := \frac{x'+y'}{2}, \tag{7.6}$$

$$z_1 := x - y, \quad z_2 := y' - x', \quad z_3 := X - X', \tag{7.7}$$

$$(P_B)_P := (\sqrt{M^2 + \mathbf{P}^2}, -\mathbf{P}). \tag{7.8}$$

Substituting

$$\theta(z_{30}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{e^{iz_{30}s}}{s - i\epsilon} \tag{7.9}$$

into Eq.(7.5) we obtain

$$\begin{aligned}
&\left[G_{C;nm m'n'}^{(2)}(p, q; P) \right]_{\text{pole}} = \int dz_{30} \frac{1}{2P_{B0}} \frac{1}{2\pi i} \int ds \frac{1}{s - i\epsilon} \\
&\quad \times \left\{ e^{i(s+P_0-P_{B0})z_{30}} \chi_{nm}(p; P_B) \bar{\chi}_{m'n'}(q; P_B) \right. \\
&\quad \left. + e^{i(-s+P_0+P_{B0})z_{30}} \chi_{m'n'}(q; (P_B)_P) \bar{\chi}_{nm}(p; (P_B)_P) \right\} \\
&= \frac{1}{2P_{B0}} \frac{1}{i} \left\{ \frac{1}{-P_0 + P_{B0} - i\epsilon} \chi_{nm}(p; P) \bar{\chi}_{m'n'}(q; P) \right. \\
&\quad \left. + \frac{1}{P_0 + P_{B0} - i\epsilon} \chi_{m'n'}(q; (P_B)_P) \bar{\chi}_{nm}(p; (P_B)_P) \right\}.
\end{aligned} \tag{7.10}$$

The second term in the braces in RHS of Eq.(7.10) is finite in the on-shell limit $P \rightarrow P_B$. Then

$$G_{C;nm m'n'}^{(2)}(p, q; P) = i \frac{P_0 + P_{B0}}{2P_{B0}} \frac{\chi_{nm}(p; P_B) \bar{\chi}_{m'n'}(q; P_B)}{P^2 - M^2 + i\epsilon} + \tilde{R}_{nm m'n'}(p, q; P) \tag{7.11}$$

$$= i \frac{\chi_{nm}(p; P_B) \bar{\chi}_{m'n'}(q; P_B)}{P^2 - M^2 + i\epsilon} + R_{nm m'n'}(p, q; P) \tag{7.12}$$

holds. ■

Note that the regular term in Eq.(7.12) is different in Eq.(7.11). It is clear from Eq.(7.11) that when $P_{B0} = \sqrt{\mathbf{P}^2 + M^2} = 0$ this pole extraction is not valid. One must be careful to treat the massless $M = 0$ boson. For the analyses of massless boson, we must not take the soft limit $\mathbf{P} \rightarrow 0$ before taking the on-shell limit $P \rightarrow P_B$. For the numerical study, it should be noted that one calculates every physical quantity with a temporary mass $M_{\text{tmp}} \neq 0$ and then make extrapolation for the real mass M .

7.3 Property of BS Amplitude

Here we consider the property of BS amplitude $\chi_r(q; P)$ for the pseudo-scalar boson. In this subsection we use the notation $|P, r\rangle$ or $|P_P, r\rangle$ for

$$P := (\sqrt{M^2 + \mathbf{P}^2}, \mathbf{P}), \quad (7.13)$$

$$P_P := (\sqrt{M^2 + \mathbf{P}^2}, -\mathbf{P}). \quad (7.14)$$

Index r distinguishes the mass degeneration and introduce an index $\bar{r}$ denotes anti particle state of r .

Generally BS amplitude for (pseudo) scalar boson can be expressed as

$$\chi_r(q; P) = (I_r + J_r^\mu \gamma_\mu + K_r^{\mu\nu} \gamma_{\mu\nu}) + (S_r + V_r^\mu \gamma_\mu + W_r^{\mu\nu} \gamma_{\mu\nu}) \gamma_5, \quad (7.15)$$

$$\gamma_{\mu\nu} := \frac{\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu}{2} \quad (7.16)$$

with

$$I_r := I_r(q^2, P^2, qP), \quad (7.17)$$

$$J_r^\mu := q^\mu J_r^A(q^2, P^2, qP) + P^\mu J_r^B(q^2, P^2, qP), \quad (7.18)$$

$$K_r^{\mu\nu} := (P^\mu q^\nu - q^\mu P^\nu) K_r(q^2, P^2, qP), \quad (7.19)$$

$$S_r := S_r(q^2, P^2, qP), \quad (7.20)$$

$$V_r^\mu := q^\mu V_r^A(q^2, P^2, qP) + P^\mu V_r^B(q^2, P^2, qP), \quad (7.21)$$

$$W_r^{\mu\nu} := (P^\mu q^\nu - q^\mu P^\nu) W_r(q^2, P^2, qP). \quad (7.22)$$

Parity Transformation

Let's first consider the parity transformation

$$\psi(x) \mapsto \mathcal{P}\psi(x)\mathcal{P}^{-1} = \gamma_0\psi(x_P), \quad (7.23)$$

$$\bar{\psi}(x) \mapsto \mathcal{P}\bar{\psi}(x)\mathcal{P}^{-1} = \bar{\psi}(x_P)\gamma_0, \quad (7.24)$$

$$x := (t, \mathbf{x}), x_P := (t, -\mathbf{x}). \quad (7.25)$$

Under this transformation the pseudo-scalar boson state $|P\rangle$ transforms

$$|P, r\rangle \mapsto \mathcal{P}|P, r\rangle = -|P_P, r\rangle. \quad (7.26)$$

Then we derive the property of BS amplitude

$$\chi_r(x, y; P) := \langle 0 | T \psi(x) \bar{\psi}(y) | P, r \rangle \quad (7.27)$$

$$= \langle 0 | \mathcal{P}^{-1} \mathcal{P} T \psi(x) \bar{\psi}(y) \mathcal{P}^{-1} \mathcal{P} | P, r \rangle \quad (7.28)$$

$$= -\gamma_0 \chi_r(x_P, y_P; P_P) \gamma_0 \quad (7.29)$$

or in momentum space

$$\chi_r(q; P) = -\gamma_0 \chi_r(q_P; P_P) \gamma_0. \quad (7.30)$$

Substituting Eq.(7.15) into Eq.(7.30), we obtain

$$I_r = J_r^A = J_r^B = K_r = 0. \quad (7.31)$$

Charge Conjugation

Next consider the charge conjugation

$$\psi(x) \mapsto \mathcal{C} \psi(x) \mathcal{C}^{-1} = \mathcal{C} \bar{\psi}^T(x), \quad (7.32)$$

$$\bar{\psi}(x) \mapsto \mathcal{C} \bar{\psi}(x) \mathcal{C}^{-1} = -\psi^T(x) \mathcal{C}^\dagger \quad (7.33)$$

where $\mathcal{C}$ denotes the charge conjugation matrix which satisfies

$$\mathcal{C}^\dagger \mathcal{C} = 1, \mathcal{C}^T = -\mathcal{C}. \quad (7.34)$$

(In both the chiral representation and the Dirac representation $\mathcal{C} = i\gamma^2\gamma^0$.)

Under this transformation the pseudo-scalar boson state $|P, r\rangle$ transforms as

$$|P, r\rangle \mapsto \mathcal{C} |P, r\rangle = |P, \bar{r}\rangle \quad (7.35)$$

where $r, \bar{r}$ denote anti particle states each other.

Then we derive the property of BS amplitude

$$\begin{aligned} \chi_r(x, y; P) &:= \langle 0 | T \psi(x) \bar{\psi}(y) | P, r \rangle \\ &= \langle 0 | \mathcal{C}^{-1} \mathcal{C} T \psi(x) \bar{\psi}(y) \mathcal{C}^{-1} \mathcal{C} | P, r \rangle \end{aligned} \quad (7.36)$$

$$\begin{aligned} &= -\mathcal{C} \langle 0 | T \bar{\psi}^T(x) \psi^T(y) | P, \bar{r} \rangle \mathcal{C}^\dagger, \\ &= \mathcal{C} \chi_{\bar{r}}^T(y, x; P) \mathcal{C}^\dagger \end{aligned} \quad (7.37)$$

or in momentum space

$$\chi_r(q; P) = C \chi_{\bar{r}}^T(-q; P) C^\dagger. \quad (7.38)$$

Substituting Eq.(7.15) and Eq.(7.31) into Eq.(7.38) we obtain

$$S_r(q^2, P^2, qP) = S_{\bar{r}}(q^2, P^2, -qP), \quad (7.39)$$

$$V_r^A(q^2, P^2, qP) = -V_{\bar{r}}^A(q^2, P^2, -qP), \quad (7.40)$$

$$V_r^B(q^2, P^2, qP) = V_{\bar{r}}^B(q^2, P^2, -qP), \quad (7.41)$$

$$W_r(q^2, P^2, qP) = W_{\bar{r}}(q^2, P^2, -qP). \quad (7.42)$$

If the symmetry exists among the quark and anti-quark composing a bound state, these scalar functions can be written by

$$S_r(q^2, P^2, qP) = \tilde{S}_r(q^2, P^2, (qP)^2), \quad (7.43)$$

$$V_r^A(q^2, P^2, qP) = (qP) \tilde{V}_r^A(q^2, P^2, (qP)^2), \quad (7.44)$$

$$V_r^B(q^2, P^2, qP) = \tilde{V}_r^B(q^2, P^2, (qP)^2), \quad (7.45)$$

$$W_r(q^2, P^2, qP) = \tilde{W}_r(q^2, P^2, (qP)^2). \quad (7.46)$$

7.4 Conjugate BS Amplitude

In this subsection we derive the relation between the BS amplitude and its conjugate.

We use the symbol Θ as CPT transformation. Θ is an anti unitary operator which satisfies

$$(i) \quad \Theta(\alpha|A\rangle + \beta|B\rangle) = \alpha^* \Theta|A\rangle + \beta^* \Theta|B\rangle,$$

$$(ii) \quad \langle A|\Theta^\dagger|B\rangle = \langle \Theta A|B\rangle^* = \langle B|\Theta A\rangle,$$

$$(iii) \quad \Theta^{-1} = \Theta^\dagger.$$

For a state $|\mathbf{P}, r\rangle$

$$\Theta|\mathbf{P}, r\rangle = |\mathbf{P}, \bar{r}\rangle \quad (7.47)$$

holds where an index r is introduced in 7.3 and for a Dirac field $\psi(x)$

$$\Theta\psi(x)\Theta^\dagger = -i\gamma_5\gamma_0\bar{\psi}^T(-x) \quad (7.48)$$

holds.

Let us consider the conjugate BS amplitude.

$$\begin{aligned}
\bar{\chi}_{r;mn}(y, x; P) &= \langle \mathbf{P}, r | T\psi_m(y)\psi_{n'}^\dagger(x) | 0 \rangle \gamma_{n'n}^0 \\
&= \langle 0 | (T\psi_m(y)\psi_{n'}^\dagger(x))^\dagger | \mathbf{P}, r \rangle^* \gamma_{n'n}^0 \\
&= \langle 0 | \bar{T}\psi_{n'}(x)\psi_m^\dagger(y) | \mathbf{P}, r \rangle^* \gamma_{n'n}^0 \\
&= \gamma_{mm'}^{0\dagger} \langle 0 | \bar{T}\psi_{n'}(x)\bar{\psi}_{m'}(y) | \mathbf{P}, r \rangle^* \gamma_{n'n}^0 \\
&= \gamma_{mm'}^0 \langle 0 | \bar{T}\psi_{n'}(x)\bar{\psi}_{m'}(y) | \mathbf{P}, r \rangle^* \gamma_{n'n}^0
\end{aligned} \tag{7.49}$$

$$\begin{aligned}
\langle 0 | \bar{T}\psi_{n'}(x)\bar{\psi}_{m'}(y) | \mathbf{P}, r \rangle^* &= \langle 0 | \Theta^\dagger \Theta (\bar{T}\psi_{n'}(x)\bar{\psi}_{m'}(y)) \Theta^\dagger \Theta | \mathbf{P}, r \rangle^* \\
&= \langle 0 | \Theta (\bar{T}\psi_{n'}(x)\bar{\psi}_{m'}(y)) \Theta^\dagger | \mathbf{P}, \bar{r} \rangle \\
&= -(\gamma_5 \gamma_0)_{n'n''} \langle 0 | T\psi_{m''}^T(-y)\bar{\psi}_{n''}^T(-x) | \mathbf{P}, \bar{r} \rangle (\gamma_5 \gamma_0)_{m''m'} \\
&= -(\gamma_5 \gamma_0)_{n'n''} \chi_{\bar{r};m''n''}(-y, -x; P) (\gamma_5 \gamma_0)_{m''m'}
\end{aligned} \tag{7.50}$$

Then we obtain

$$\bar{\chi}_{r;mn}(y, x; P) = (\gamma_5)_{mm'} \chi_{\bar{r};m'n'}(-y, -x; P) (\gamma_5)_{n'n} \tag{7.51}$$

or in momentum space

$$\bar{\chi}_{r;mn}(q; P) = (\gamma_5)_{mm'} \chi_{\bar{r};m'n'}(-q; P) (\gamma_5)_{n'n} = \chi_{\bar{r};mn}(q; -P). \tag{7.52}$$

7.5 Representation of BS Amplitude

From 7.3 and 7.4, we can write down the BS amplitude and its conjugate as

$$\begin{aligned}
\chi_{nm}(q; P) &= \delta_{ji} \frac{(\lambda^\alpha)_{gf}}{\sqrt{2}} \left[(\phi_S^\alpha(q; P) + \phi_P^\alpha(q; P) \not{P} + \phi_Q^\alpha(q; P) \not{P} \right. \\
&\quad \left. + \frac{1}{2} \phi_T^\alpha(q; P) (\not{P} \not{P} - \not{P} \not{P})) \frac{\gamma_5}{\sqrt{2}} \right]_{ba},
\end{aligned} \tag{7.53}$$

$$\begin{aligned}
\bar{\chi}_{mn}(q; P) &= \delta_{ij} \frac{(\lambda^\alpha)_{fg}^T}{\sqrt{2}} \left[(\phi_S^\alpha(q; P) - \phi_P^\alpha(q; P) \not{P} + \phi_Q^\alpha(q; P) \not{P} \right. \\
&\quad \left. + \frac{1}{2} \phi_T^\alpha(q; P) (\not{P} \not{P} - \not{P} \not{P})) \frac{\gamma_5}{\sqrt{2}} \right]_{ba}.
\end{aligned} \tag{7.54}$$

7.6 Pagels–Stokar Formula

Pagels and Stokar proposed a useful formula for the decay constant f_π in the chiral limit in term of the constituent quark mass function $B(q^2)$ of the Schwinger–Dyson equation. It is

$$f_\pi^2 = \frac{N_C}{8\pi^2} \int_0^\infty dq_E^2 q_E^2 \frac{B(-q_E^2) [2B(-q_E^2) + q_E^2 B'(-q_E^2)]}{(q_E^2 + B^2(-q_E^2))^2} \quad (7.55)$$

where $B(-q_E^2)$ is defined by

$$S_F(q) = \frac{i}{\not{q} - B(q^2)}. \quad (7.56)$$

This formula can be derived from an approximation to the BS amplitude in the following way.

As we mentioned in 3.4, the BS amplitude is given by

$$\chi_{nm}(q; 0) = \frac{1}{f_\pi} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(q)\}_{nm} \quad (7.57)$$

from (modified) axial vector Ward–Takahashi identity in the chiral limit, but this solution is not enough to calculate $\tilde{f}_\pi$, because $\tilde{f}_\pi$ is related to the derivative of the BS amplitude with respect to the total momentum P .

The amputated BS amplitude(or BS vertex) $\hat{\chi}_{nm}(q; P)$ is defined by

$$\hat{\chi}_{nm}(q; P) := S_{Fnn'}^{-1}(q + \frac{P}{2}) \chi_{n'm'}(q; P) S_{Fm'm}^{-1}(q - \frac{P}{2}), \quad (7.58)$$

and in the chiral limit, we obtain from Eq.(7.57)

$$\hat{\chi}_{nm}(q; 0) = \frac{1}{f_\pi} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F^{-1}(q)\}_{nm}. \quad (7.59)$$

Let's consider an ansatz for the BS amplitude as

$$\chi_{nm}^{\text{PS}}(q; P) = S_{Fnn'}(q + \frac{P}{2}) \hat{\chi}_{n'm'}(q; 0) S_{Fm'm}(q - \frac{P}{2}) \quad (7.60)$$

$$= \frac{1}{f_\pi} S_{Fnn'}(q + \frac{P}{2}) \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F^{-1}(q)\}_{n'm'} S_{Fm'm}(q - \frac{P}{2}). \quad (7.61)$$

In this paper we call this the Pagels–Stokar ansatz. This has a dependence on the total momentum P . Therefore using this ansatz one can estimate the (modified) decay constant without solving the BS equation, for instance using our formula Eq.(5.54). But the validity of this ansatz Eq.(7.61) is questionable.

Let us next substitute Eq.(7.61) to Eq.(5.25). Neglecting the effect of the cut off regularization function and assuming the SD solution as Eq.(7.56)^{★1}, one obtains

$$f_\pi \tilde{f}_\pi = \frac{N_C}{8\pi^2} \int_0^\infty dq_E^2 \frac{B(-q_E^2) [2B(-q_E^2) + q_E^2 B'(-q_E^2)]}{(q_E^2 + B^2(-q_E^2))^2}. \quad (7.62)$$

In the case $\tilde{f}_\pi = f_\pi$ or assuming it, Eq.(7.62) reduces to Eq.(7.55). So the original Pagels-Stokar formula holds on the three assumptions: (i) the BS amplitude is approximately given by the ansatz Eq.(7.61), (ii) the SD solution is given by the form as Eq.(7.56) (iii) the naive axial current $J_{5\mu}^\alpha$ is close to the effective axial current $\tilde{J}_{5\mu}$.

7.7 A problem of the Normalization of the Quark Fields

We use the cut off function $f(x)$ for the regularization. So there are no divergences and the system seems to be well-defined. But it gives a problem of the normarization of the quark fields concerning the chiral properties.

To see this we consider a simple chiral symmetric lagrangian

$$\mathcal{L}[\psi, \bar{\psi}] = \bar{\psi} f(\partial^2) (i\not{\partial} - m_0) \psi + G [\bar{\psi} \gamma_\mu \psi] [\bar{\psi} \gamma^\mu \psi] \quad (7.63)$$

for simplicity. Under the infinitesimal local chiral transformation

$$\psi(x) \rightarrow \psi'(x) := (1 + i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x)) \psi(x) \quad (7.64)$$

the change of the action is

$$\Delta_5 S[\psi, \bar{\psi}] \equiv \int d^4x \left\{ \frac{\delta S}{\delta \psi_m(x)} \left(i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x) \psi(x) \right)_m + \left(\bar{\psi}(x) i\gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x) \right)_n \frac{\delta S}{\delta \bar{\psi}_n(x)} \right\} \quad (7.65)$$

$$\equiv \int d^4x \theta^\alpha(x) \left\{ \partial^\mu I_{5\mu}^\alpha - 2m_0 i \bar{\psi} \frac{f(\partial^2) + f(\partial^2)}{2} \gamma_5 \frac{\lambda^\alpha}{2} \psi \right\}(x), \quad (7.66)$$

$$I_{5\mu}^\alpha(x) = \int_{p,q} \bar{\psi}(p) (\gamma_\mu f_1(-p^2, -q^2) + (p-q)_\mu (\not{p} - \not{q}) f_2(-p^2, -q^2)) \gamma_5 \frac{\lambda^\alpha}{2} \psi(q) e^{-i(p+q)x}, \quad (7.67)$$

$$f_1(-p^2, -q^2) = \frac{f(-p^2) + f(-q^2)}{2}, \quad f_2(-p^2, -q^2) = \frac{f(-p^2) - f(-q^2)}{2(p^2 - q^2)} \quad (7.68)$$

as mentioned in **3.2**. The PCAC relation

^{*1}The general form of the SD solution is

$$S_F(q) = \frac{i}{\not{q} A(q^2) - B(q^2)}.$$

$$\partial^\mu I_{5\mu}^\alpha = 2m_0 i \bar{\psi} \frac{f(\overleftarrow{\partial^2}) + f(\partial^2)}{2} \gamma_5 \frac{\lambda^\alpha}{2} \psi \quad (7.69)$$

is derived from Eq.(7.66).

On the other hand, we consider the new variable

$$\psi_r(x) := \sqrt{f(\partial^2)} \psi(x). \quad (7.70)$$

The lagrangian Eq.(7.63) is rewritten with ψ_r to

$$\begin{aligned} \mathcal{L}[\psi_r, \bar{\psi}_r] &= \bar{\psi}_r (i \not{\partial} - m_0) \psi_r \\ &+ G[(f^{-1/2}(\partial^2) \bar{\psi}_r) \gamma_\mu (f^{-1/2}(\partial^2) \psi_r)] [(f^{-1/2}(\partial^2) \bar{\psi}_r) \gamma^\mu (f^{-1/2}(\partial^2) \psi_r)]. \end{aligned} \quad (7.71)$$

Under the infinitesimal local chiral transformation for new variable

$$\psi_r(x) \rightarrow \psi'_r(x) := (1 + i \gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha) \psi_r(x) \quad (7.72)$$

the change of the action is

$$\begin{aligned} \Delta_5 S[\psi_r, \bar{\psi}_r] &\equiv \int d^4x \left\{ \frac{\delta S}{\delta(\psi_r)_m(x)} \left(i \gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x) \psi_r(x) \right)_m \right. \\ &\quad \left. + \left(\bar{\psi}_r(x) i \gamma_5 \frac{\lambda^\alpha}{2} \theta^\alpha(x) \right)_n \frac{\delta S}{\delta(\bar{\psi}_r)_n(x)} \right\} \end{aligned} \quad (7.73)$$

$$\equiv \int d^4x \theta^\alpha(x) \{ \partial^\mu I_{5\mu}^{r\alpha} - 2m_0 i \bar{\psi}_r \gamma_5 \frac{\lambda^\alpha}{2} \psi_r \}(x) \quad (7.74)$$

where

$$\begin{aligned} I_{5\mu}^{r\alpha}(x) &:= \bar{\psi}_r \gamma_\mu \gamma_5 \frac{\lambda^\alpha}{2} \psi_r(x) + 2G \int_{p,q,k,l} \frac{i(p+q+k+l)_\mu}{(p+q+k+l)^2} f^{-1/2}(-p^2) f^{-1/2}(-q^2) \\ &\times \left\{ f^{-1/2}(-k^2) f^{-1/2}(-(p+q+k)^2) - f^{-1/2}(-(p+q+l)^2) f^{-1/2}(-l^2) \right\} \\ &\times [\bar{\psi}_r(p) \gamma_\mu \psi_r(q)] [\bar{\psi}_r(k) \gamma^\mu i \gamma_5 \frac{\lambda^\alpha}{2} \psi_r(l)] e^{-i(p+q+k+l)x}. \end{aligned} \quad (7.75)$$

Then the PCAC relation

$$\partial^\mu I_{5\mu}^{r\alpha} = 2m_0 i \bar{\psi}_r \gamma_5 \frac{\lambda^\alpha}{2} \psi_r \quad (7.76)$$

is derived from Eq.(7.74).

The two PCAC relations Eqs.(7.69) and (7.76) are different. The current $I_{5\mu}^\alpha$ is rewritten with ψ to

$$\begin{aligned} I_{5\mu}^\alpha(x) &= \bar{\psi} f(\partial^2) \gamma_\mu \gamma_5 \frac{\lambda^\alpha}{2} \psi(x) + 2G \int_{p,q,k,l} \frac{i(p+q+k+l)_\mu}{(p+q+k+l)^2} f^{1/2}(-k^2) f^{1/2}(-l^2) \\ &\times \left\{ f^{-1/2}(-k^2) f^{-1/2}(-(p+q+k)^2) - f^{-1/2}(-(p+q+l)^2) f^{-1/2}(-l^2) \right\} \\ &\times [\bar{\psi}(p) \gamma_\mu \psi(q)] [\bar{\psi}(k) \gamma^\mu i \gamma_5 \frac{\lambda^\alpha}{2} \psi(l)] e^{-i(p+q+k+l)x} \end{aligned} \quad (7.77)$$

which is different from the current $I_{5\mu}^\alpha$. This is a problem of the normalization of the quark fields. This stemmed from the difference of the two chiral transformation Eq.(7.64) and Eq.(7.72). Therefore the two definitions of the decay constant

$$I_{5\mu}^\alpha = -f_\pi \partial_\mu \pi + \dots \quad (7.78)$$

$$I_{5\mu}^{\alpha r} = -f_\pi^r \partial_\mu \pi + \dots \quad (7.79)$$

are possible. To choose the correct decay constant, one must consider the connection of the decay constant and the physical phenomena, for example the pion weak decay process.

Although there is such ambiguity in the definition of the current or the decay constant, the same regularized SDE and BSE are derived from both lagrangians (7.63) and (7.71).

7.8 Gauging of the Effective Model

In this thesis we concentrate the strong interaction. It is a natural question how quark field of the effective models interacts with the gauge field photon.

To see this, we consider a simple chiral symmetric lagrangian

$$\mathcal{L}[\psi, \bar{\psi}] = \bar{\psi} f(\partial^2) (i \not{\partial} - m_0) \psi + G [\bar{\psi} \gamma_\mu \psi] [\bar{\psi} \gamma^\mu \psi] \quad (7.80)$$

same as in 7.7. Under the infinitesimal local $U(1)$ phase transformation

$$\psi(x) \rightarrow \psi'(x) := (1 + i\theta(x))\psi(x) \quad (7.81)$$

the change of the action defined by

$$S[\psi, \bar{\psi}] := \int d^4x \mathcal{L}[\psi, \bar{\psi}](x) \quad (7.82)$$

is

$$\Delta S[\psi, \bar{\psi}] \equiv \int d^4x \left\{ \frac{\delta S}{\delta \psi_m(x)} i\theta(x) \psi_m(x) - \bar{\psi}_n(x) i\theta(x) \frac{\delta S}{\delta \bar{\psi}_n(x)} \right\} \quad (7.83)$$

$$\equiv \int d^4x \theta(x) \partial^\mu \tilde{J}_\mu(x) \quad (7.84)$$

where

$$\tilde{J}_\mu(x) = \int_{p,q} \bar{\psi}(p) (\gamma_\mu f_1(-p^2, -q^2) + (p-q)_\mu (\not{p} - \not{q} + 2m_0) f_2(-p^2, -q^2)) \psi(q) e^{-i(p+q)x}, \quad (7.85)$$

$$f_1(-p^2, -q^2) = \frac{f(-p^2) + f(-q^2)}{2}, \quad f_2(-p^2, -q^2) = \frac{f(-p^2) - f(-q^2)}{2(p^2 - q^2)}. \quad (7.86)$$

We call this conserved current $\tilde{J}_\mu(x)$ ^{*2} the *effective electromagnetic (EM) current*. If we do not introduce the cut off function $f(x)$, the effective EM current $\tilde{J}_\mu(x)$ reduces to the *naive EM current* $J_\mu(x)$ defined by

$$J_\mu(x) := \bar{\psi} \gamma_\mu \psi(x). \quad (7.87)$$

Let us introduce the gauge field $A_\mu(x)$ and consider the infinitesimal $U(1)_{EM}$ gauge transformation

$$\psi(x) \rightarrow \psi'(x) := (1 + i\theta(x))\psi(x), \quad (7.88)$$

$$A_\mu(x) \rightarrow A'_\mu(x) := A_\mu(x) + \frac{1}{e} \partial_\mu \theta(x). \quad (7.89)$$

Under this transformation, whether the total action^{*3} defined by

$$S[\psi, \bar{\psi}, A] := S[\psi, \bar{\psi}] + e \int d^4x A^\mu(x) \tilde{J}_\mu(x) \quad (7.90)$$

is invariant or not?

If we do not introduce the cut off function, the naive EM current $J_\mu(x)$ is invariant under the local $U(1)_{EM}$ transformation. In this case the total action Eq.(7.90) is invariant. But in general, the effective EM current $\tilde{J}_\mu(x)$ is not invariant but transforms as

$$\tilde{J}_\mu(x) \rightarrow \tilde{J}'_\mu(x) := \tilde{J}_\mu(x) + \int d^4y \theta(y) \partial_y^\nu \tilde{J}_{\mu\nu}(x, y) \quad (7.91)$$

^{*2}If we employ the locally $U(1)_{EM}$ variant interaction like KG model, $\tilde{J}_\mu(x)$ depends on not only the cut off regularization function but also the interaction term. The generalization to such a case is straight forward.

^{*3}Here we omit the kinetic term $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ of the gauge field for simplicity.

where $\tilde{J}_{\mu\nu}(x, y)$ satisfies conservation laws $\partial_x^\mu \tilde{J}_{\mu\nu}(x, y) = 0$ and $\partial_y^\nu \tilde{J}_{\mu\nu}(x, y) = 0$. In this case the change of the total action Eq.(7.90) is

$$\Delta_{EM} S[\psi, \bar{\psi}, A] = e \int d^4x d^4y \theta(y) A^\mu(x) \partial_y^\nu \tilde{J}_{\mu\nu}(x, y). \quad (7.92)$$

If $\tilde{J}_{\mu\nu}(x, y)$ is invariant under the gauge transformation, the total action defined by

$$S[\psi, \bar{\psi}, A] := S[\psi, \bar{\psi}] + e \int d^4x A^\mu(x) \tilde{J}_\mu(x) + \frac{e^2}{2} \int d^4x d^4y A^\mu(x) A^\nu(y) \tilde{J}_{\mu\nu}(x, y) \quad (7.93)$$

instead of Eq.(7.90) is invariant. But in general, $\tilde{J}_{\mu\nu}(x, y)$ is not invariant but transforms as

$$\tilde{J}_{\mu\nu}(x, y) \rightarrow \tilde{J}'_{\mu\nu}(x, y) := \tilde{J}_{\mu\nu}(x, y) + \int d^4z \theta(z) \partial_z^\lambda \tilde{J}_{\mu\nu\lambda}(x, y, z). \quad (7.94)$$

Then the action Eq.(7.93) is not yet invariant.

This is a feature of the effective model. Generally the gauge invariant action is an infinite sum of the interaction between the matter field and the gauge field as

$$\begin{aligned} S[\psi, \bar{\psi}, A] &:= S[\psi, \bar{\psi}] \\ &+ e \int d^4x A^\mu(x) \tilde{J}_\mu(x) \\ &+ \frac{e^2}{2} \int d^4x d^4y A^\mu(x) A^\nu(y) \tilde{J}_{\mu\nu}(x, y) \\ &+ \frac{e^3}{3!} \int d^4x d^4y d^4z A^\mu(x) A^\nu(y) A^\lambda(z) \tilde{J}_{\mu\nu\lambda}(x, y, z) \\ &+ \dots \end{aligned} \quad (7.95)$$

For the pion electromagnetic decay $\pi^0 \rightarrow 2\gamma$, one has to take account of the third term of the RHS in Eq.(7.95).

Up to now we consider only the $U(1)_{EM}$ gauge transformation. The same situation occurs for the electro-weak $SU(2)_W \times U(1)_Y$ gauge transformation. In this case, the quark-anti-quark-gauge interaction is

$$\begin{aligned} \mathcal{L}_{\text{int}} &= e A^\mu \tilde{J}_\mu^{EM} + \frac{g_W}{2\sqrt{2}} (W^{\mu\dagger} \tilde{J}_\mu^W + W^\mu \tilde{J}_\mu^{W\dagger}) + \frac{g_W}{\cos \theta_W} Z^\mu \tilde{J}_\mu^Z \\ &+ [\text{higher terms}] \end{aligned} \quad (7.96)$$

where θ_W is the Weinberg angle. Introducing six flavors and using the notation

$$\psi := (u, d, c, s, t, b)^T, \quad (7.97)$$

$$\psi_u := (u, c, t)^T, \quad (7.98)$$

$$\psi_d := (d, s, b)^T, \quad (7.99)$$

the effective currents $\tilde{J}_\mu^{EM}$, $\tilde{J}_\mu^W$, $\tilde{J}_\mu^Z$ are given^{*4} by

$$\tilde{J}_\mu^{EM}(x) := \int_{p,q} \bar{\psi}(p) Q \tilde{\Gamma}_\mu(p, q) \psi(q) e^{-i(p+q)x} \quad (7.100)$$

$$\tilde{J}_\mu^W(x) := \int_{p,q} \bar{\psi}_d(p) U^\dagger (\tilde{\Gamma}_\mu(p, q) - \tilde{\Gamma}_{5\mu}(p, q)) \psi_u(q) e^{-i(p+q)x} \quad (7.101)$$

$$\begin{aligned} \tilde{J}_\mu^Z(x) := & \frac{1}{4} \int_{p,q} \left\{ \bar{\psi}_u(p) (\tilde{\Gamma}_\mu(p, q) - \tilde{\Gamma}_{5\mu}(p, q)) \psi_u(q) \right. \\ & \left. - \bar{\psi}_d(p) U^\dagger (\tilde{\Gamma}_\mu(p, q) - \tilde{\Gamma}_{5\mu}(p, q)) U \psi_d(q) \right\} e^{-i(p+q)x} \\ & - \sin^2 \theta_W \tilde{J}_\mu^{EM}(x) \end{aligned} \quad (7.102)$$

where

$$\Gamma_\mu(p, q) := \gamma_\mu f_1(-p^2, -q^2) + (p - q)_\mu (\not{p} - \not{q} + 2m_0) f_2(-p^2, -q^2), \quad (7.103)$$

$$\Gamma_{5\mu}(p, q) := (\gamma_\mu f_1(-p^2, -q^2) + (p - q)_\mu (\not{p} - \not{q}) f_2(-p^2, -q^2)) \gamma_5. \quad (7.104)$$

The symbol Q in Eq.(7.100) denotes the diagonal charge matrix defined by

$$Q := \begin{cases} = \frac{2}{3} & \text{for } u, c, t \\ = -\frac{1}{3} & \text{for } d, s, b \end{cases} \quad (7.105)$$

and the symbol U in Eqs.(7.101) and (7.102) denotes the Kobayashi–Maskawa mixing matrix defined by

$$U := \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} \quad (7.106)$$

$$c_i := \cos \theta_i, \quad s_i := \sin \theta_i \quad (i = 1, 2, 3). \quad (7.107)$$

For this interaction form, the pion weak decay mode $\pi^+ \rightarrow \mu^+ + \nu_\mu$ through one weak boson W exchange is parametrized by the modified decay constant $\tilde{f}_\pi$ defined by Eq.(3.30) even in the framework of the effective model. Therefore it seems to be natural to call $\tilde{f}_\pi$ the modified decay constant.

All the above treatments are in calssical. Quantization of them is a future problem.

^{*4}As mentioned in the footnote of p. 63, these currents are modified when we use the more complicated interaction.

7.9 The color confinement

In the non-abelian gauge theory, it is believed that all the physical states are color singlet. In fact, there found no colored states experimentally. However, the mechanism of the color confinement is very difficult. In this subsection, we review some studies of the color confinement in effective model approach.

Asymptotically Free Property

The first sign of the color confinement is an asymptotically free property of the running coupling constant. Asymptotically free property is a result of the perturbative renormalization group equation. Since the running coupling constant is so strong in the low energy region, the force between the quark and anti-quark is strong at the long distance. Effective models like KG model equip the running coupling constant. There is a possibility of the confinement property to some extent in them.

Quark Mass Pole

In the NJL model, the solution of the SDE is a constant solution. The quark propagator possesses a mass pole. This is not the case in effective models with asymptotically free property. Fig.7-1 shows the typical solution of the SDE in KG model. The solid line is a solution of the SDE in the space like region. The dashed line is a on-shell line $p^2 = B^2(p^2)$ in the time like region. It seems that the solution of the SDE does not contact the on-shell line. If it is the case, the quark can't become on-shell. It may be a sign of the confinement of the quark.

Linear Confining Force

If the color flux is squeezed, the potential $V(r)$ between the quark and anti-quark may be

$$V(r) \simeq kr + \dots \quad (7.108)$$

phenomenologically at a large distance r . [23] In this case, the corresponding gluon propagator is

$$D^{\mu\nu}(p) = -g^{\mu\nu} \frac{8\pi k}{p^4} + \dots \quad (7.109)$$

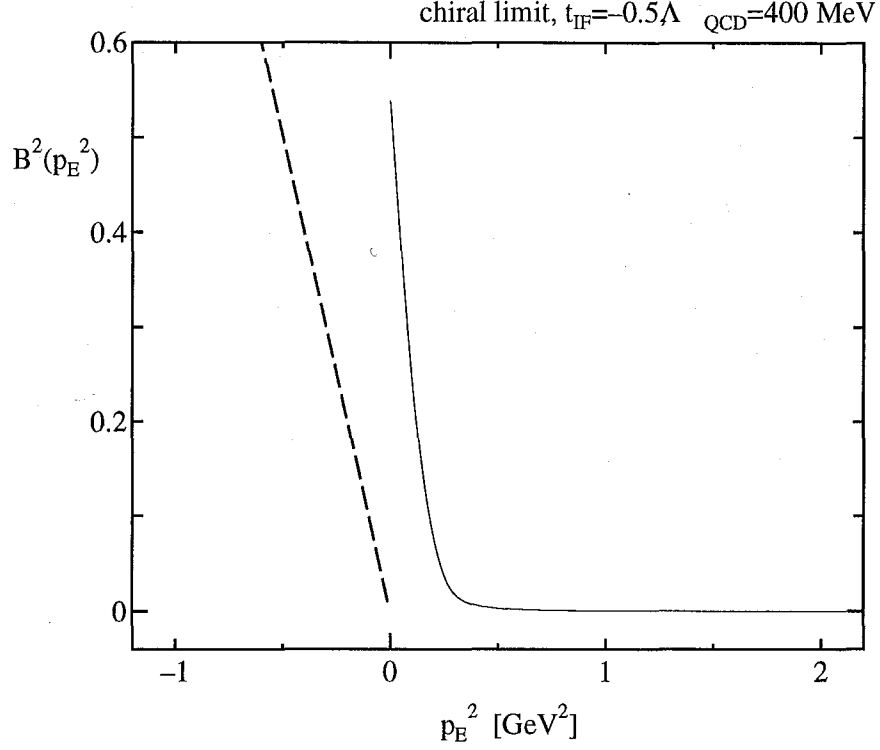


Fig.7-1: SD solution

at small p^2 . This type of the gluon propagator is called *confining gluon propagator*.

The studies of the SDE for the gluon propagator (and ghost propagator) seem to suggest the confining gluon propagator.[24, 25] The studies of the chiral symmetry breaking in the effective model with confining gluon propagator are reported in Ref.[26, 27, 28].

Gluon Confinement

For the confinement of the gluon itself, one of the ideas is that the gluon propagator has no pole or less singular than $1/p^2$ for $p^2 \rightarrow 0$. For example, the gluon propagators

$$D^{\mu\nu}(p) = (g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) \frac{p^2}{p^4 + b^2}, \quad (7.110)$$

$$D^{\mu\nu}(p) = (g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) \left(\frac{1}{p^2} \right)^{1-f(p^2)} \quad \text{with} \quad 0 < f(p^2) < 1 \text{ for } p^2 \sim 0, \quad (7.111)$$

$$D^{\mu\nu}(p) = D_{\text{free}}^{\mu\nu}(p)(1 - e^{-\frac{p^2}{\Lambda^2}}) \quad (7.112)$$

satisfy this property. These gluon propagators are called *confined gluon propagator*. But it is reported that above three examples of the confined gluon propagators can't confine the quark. (Here the confinement of the quark means that there are no linear confining force.)[24] It is challenging to see how gluon propagator possesses both the confined and confining properties. In Ref.[33] a phenomenological gluon propagator which satisfies both properties is presented.

Confinement of Color Octet Meson

All the mesons in real world are color singlet. It is a natural question whether a color octet meson state (i.e. its color structure is T^a) exists or not as the solution of the Bethe-Salpeter equation. To study it, we consider a simple colored model

$$\int d^4x \mathcal{L}[\psi, \bar{\psi}](x) := \int_q \bar{\psi}(-q) \not{p} \psi(q) + \frac{1}{2} \int_{p,q,k} iD^{\mu\nu}(q-k) \left[\bar{\psi}(-k + \frac{p}{2}) \gamma_\mu T^a \psi(q - \frac{p}{2}) \right] \left[\bar{\psi}(-q + \frac{p}{2}) \gamma_\nu T^a \psi(k + \frac{p}{2}) \right]. \quad (7.113)$$

Using the standard derivation, the ladder Bethe-Salpeter equation is written down by

$$\chi(q; P) = \int_k S_F(q_+) \gamma_\nu T^a \chi(k; P) \gamma_\mu T^a S_F(q_-) D^{\nu\mu}(k-q) \quad (7.114)$$

with $q_\pm := q \pm \frac{P}{2}$, $k_\pm := k \pm \frac{P}{2}$. If the color structure of the BS amplitude $\chi(q; P)$ is a singlet form δ_{ij} , Eq.(7.114) reduces to

$$\chi(q; P) = \frac{N_C^2 - 1}{2N_C} \int_k S_F(q_+) \gamma_\nu \chi(k; P) \gamma_\mu S_F(q_-) D^{\nu\mu}(k-q) \quad (7.115)$$

owing to

$$(T^a T^a)_{ij} = \frac{N_C^2 - 1}{2N_C} \delta_{ij}. \quad (7.116)$$

If the color structure of the BS amplitude is a octet form T^b , Eq.(7.114) reduces to

$$\chi(q; P) = -\frac{1}{2N_C} \int_k S_F(q_+) \gamma_\nu \chi(k; P) \gamma_\mu S_F(q_-) D^{\nu\mu}(k-q) \quad (7.117)$$

owing to

$$(T^a T^b T^a)_{ij} = -\frac{1}{2N_C} T_{ij}^a. \quad (7.118)$$

Then we see the relative sign between the Eqs.(7.115) and (7.117) is different. So there is no color octet solutions in the ladder Bethe-Salpeter equation level.

Confinement of Diquark

The diquark is a (bound) state of two quarks and is a color anti-triplet state. Although it should not appear physically, there exists often a diquark state in the framework of the ladder Bethe-Salpeter equation. Bender and Roberts have reported in Ref.[30] that this solution vanishes if one proceeds beyond the ladder approximation. To review its argument, we again consider the simple colored model Eq.(7.113). The color structure of the diquark BS amplitude $\Lambda(q; P)$ is $(Y^h)_{ij} := \epsilon^{hij}$ where ϵ denotes a totally anti-symmetric tensor. (We treat the $N_C = 3$ case although it seems for a general N_C in the following argument.) The ladder Bethe-Salpeter equation is

$$\Lambda(q; P) = \int_k S_F(q_+) \gamma_\nu T^a \Lambda(k; P) \gamma_\mu^T (T^a)^T S_F^T(-q_-) D^{\nu\mu}(k - q) \quad (7.119)$$

derived through the similar method for a meson case. This can be rewritten by

$$\Lambda^C(q; P) = - \int_k S_F(q_+) \gamma_\nu T^a \Lambda^C(k; P) \gamma_\mu (T^a)^T S_F(q_-) D^{\nu\mu}(k - q) \quad (7.120)$$

with $\Lambda^C(q; P) := \Lambda(q; P)C$ where C denotes the charge conjugation matrix. For the color structure of the $\Lambda(q; P)$, this reduces to

$$\Lambda^C(q; P) = \frac{N_C + 1}{2N_C} \int_k S_F(q_+) \gamma_\nu \Lambda^C(k; P) \gamma_\mu S_F(q_-) D^{\nu\mu}(k - q) \quad (7.121)$$

owing to

$$(T^a Y^h (T^a)^T)_{ij} = -\frac{N_C + 1}{2N_C} (Y^h)_{ij}. \quad (7.122)$$

Comparing Eqs.(7.115) and (7.121) with $N_C = 3$, the strength of the BS kernel for a diquark is a half for a meson. Then there is a possibility of the existence of the diquark state.

Then we go beyond the ladder approximation. Including the Feynman diagram shown in Fig.7-2 in the BS kernel for the next to ladder approximation, the BS equation for a color singlet meson is given by

$$\begin{aligned} \chi(q; P) = & \frac{N_C^2 - 1}{2N_C} \int_k S_F(q_+) \gamma_\nu \chi(k; P) \gamma_\mu S_F(q_-) D^{\nu\mu}(k - q) \\ & - \frac{N_C^2 - 1}{4N_C^2} \int_{k,l} S_F(q_+) \gamma_\nu S_F(q_+ + l) \gamma_\rho S_F(k_+ + l) \gamma_\sigma \chi(k; P) \gamma_\mu S_F(q_-) D^{\rho\sigma}(l) D^{\nu\mu}(k - q) \\ & - \frac{N_C^2 - 1}{4N_C^2} \int_{k,l} S_F(q_+) \gamma_\nu S_F\left(\frac{q_+ + k_+}{2} + l\right) \gamma_\rho \chi(k; P) \gamma_\mu S_F\left(\frac{q_- + k_-}{2} - l\right) \gamma_\sigma S_F(q_-) \end{aligned} \quad (7.123)$$

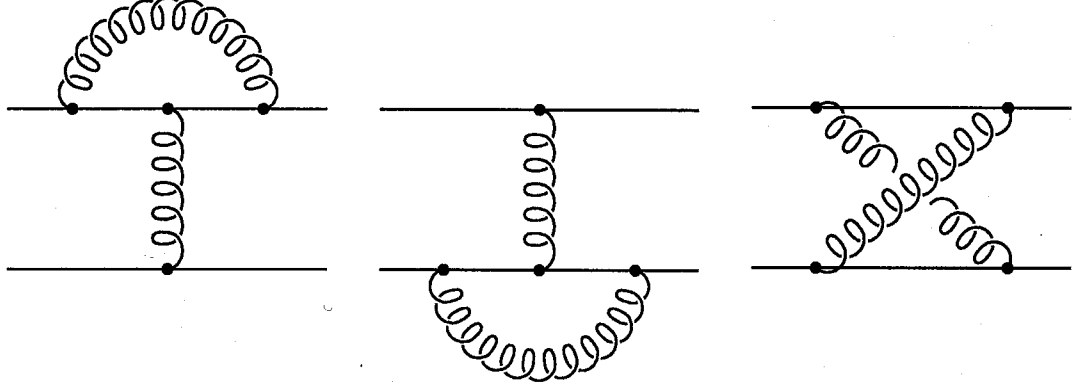


Fig.7-2: BS Kernel next to the ladder

$$\begin{aligned} & \times D^{\nu\mu}(\frac{k-q}{2} - l) D^{\rho\sigma}(\frac{k-q}{2} + l) \\ & - \frac{N_C^2 - 1}{4N_C^2} \int_{k,l} S_F(q_+) \gamma_\nu \chi(k; P) \gamma_\sigma S_F(k_- + l) \gamma_\mu S_F(q_- + l) \gamma_\rho S_F(q_-) D^{\rho\sigma}(l) D^{\nu\mu}(k - q) \end{aligned}$$

owing to

$$(T^a T^b T^a T^b)_{ij} = -\frac{N_C^2 - 1}{4N_C^2}. \quad (7.124)$$

On the other hand, the BS equation for a diquark is given by

$$\begin{aligned} \Lambda^C(q; P) &= \frac{N_C + 1}{2N_C} \int_k S_F(q_+) \gamma_\nu \Lambda^C(k; P) \gamma_\mu S_F(q_-) D^{\nu\mu}(k - q) \\ & - \frac{N_C + 1}{4N_C^2} \int_{k,l} S_F(q_+) \gamma_\nu S_F(q_+ + l) \gamma_\rho S_F(k_+ + l) \gamma_\sigma \Lambda^C(k; P) \gamma_\mu S_F(q_-) D^{\rho\sigma}(l) D^{\nu\mu}(k - q) \\ & - \frac{N_C^3 - 2N_C - 1}{4N_C^2} \int_{k,l} S_F(q_+) \gamma_\nu S_F(\frac{q_+ + k_+}{2} + l) \gamma_\rho \Lambda^C(k; P) \gamma_\mu S_F(\frac{q_- + k_-}{2} - l) \gamma_\sigma S_F(q_-) \\ & \times D^{\nu\mu}(\frac{k-q}{2} - l) D^{\rho\sigma}(\frac{k-q}{2} + l) \\ & - \frac{N_C + 1}{4N_C^2} \int_{k,l} S_F(q_+) \gamma_\nu \Lambda^C(k; P) \gamma_\sigma S_F(k_- + l) \gamma_\mu S_F(q_- + l) \gamma_\rho S_F(q_-) D^{\rho\sigma}(l) D^{\nu\mu}(k - q) \end{aligned} \quad (7.125)$$

owing to

$$(T^a T^b T^a Y^h (T^b)^T)_{ij} = \frac{N_C + 1}{4N_C^2} (Y^h)_{ij}, \quad (7.126)$$

$$(T^a T^b Y^h (T^a)^T (T^b)^T)_{ij} = \frac{-N_C^3 + 2N_C + 1}{4N_C^2} (Y^h)_{ij}. \quad (7.127)$$

We compare the Eqs.(7.123) and (7.125) with $N_C = 3$. The strength of the BS kernel for a diquark is again a half for a meson except for the third term. The third term of the BS kernel

for a diquark is $\frac{5}{2}$ times larger than that for a meson. Bender et. al. showed that only the third term in the above BS kernel is repulsive in the $0^-, 1^-, 0^+, 1^+$ channels in both meson and diquark states. Then the large contribution of this term can eliminates diquark bound states.

We assume that the same argument can be applied to the present model.

8 Application

8.1 Preliminary

In this section, we study a concrete model called extended Kyoto group(ExKG) model.

8.2 Lagrangian for ExKG model

The ExKG model is defined by the KG model plus a KMT flavor mixing interaction. Its lagrangian is the following:

$$\mathcal{L}[\psi, \bar{\psi}] := \mathcal{L}_{\text{free}}[\psi, \bar{\psi}] + \mathcal{L}_{\text{GE}}[\psi, \bar{\psi}] + \mathcal{L}_{\text{FM}}[\psi, \bar{\psi}], \quad (8.1)$$

$$\psi := (u, d, s)^T. \quad (8.2)$$

We treat the case of three colors, $N_C = 3$, and three flavors, $N_F = 3$.

$\mathcal{L}_{\text{free}}[\psi, \bar{\psi}]$ is the free part, given by

$$\mathcal{L}_{\text{free}}[\psi, \bar{\psi}] := \bar{\psi} f(\partial^2)(i\partial - m_0)\psi, \quad (8.3)$$

$$f(x) = 1 + M\theta(x - \Lambda_{\text{UV}}^2), \quad M \rightarrow \infty, \quad (8.4)$$

$$m_0 = \text{diag}(m_q, m_q, m_s). \quad (8.5)$$

$f(x)$ is a regularization function discussed in **2.5, 3.2, 5.3** and **6.1**. The bare mass matrix m_0 is diagonal in flavor space and s quark bare mass m_s is different from u, d quark bare mass $m_q := m_u = m_d$.

$\mathcal{L}_{\text{GE}}[\psi, \bar{\psi}]$ is a gluon exchange interaction with Landau gauge condition:

$$\begin{aligned} \mathcal{L}_{\text{GE}}[\psi, \bar{\psi}](x) := & -\frac{1}{2} \tilde{K}^{mn, n_2 m_2} \left(\frac{-i\partial^{x_2} + i\partial^{y_1}}{2}, \frac{i\partial^{x_1} - i\partial^{y_2}}{2} \right) \\ & \times \bar{\psi}_{m_2}(y_1) \psi_m(y_2) \bar{\psi}_n(x_1) \psi_{n_2}(x_2) \Big|_*, \end{aligned} \quad (8.6)$$

$$\tilde{K}^{mn, n_2 m_2}(p, k) := \bar{g}^2(p, k) iD^{\mu\nu}(p - k) (\gamma_\mu T^a)^{m_2 m} (\gamma_\nu T^a)^{nn_2}, \quad (8.7)$$

$$iD^{\mu\nu}(p) := \frac{-g^{\mu\nu} p^2 + p^\mu p^\nu}{p^4}, \quad (8.8)$$

$$\bar{g}^2(p, k) := \theta(p_E^2 - k_E^2) g^2(p_E^2) + \theta(k_E^2 - p_E^2) g^2(k_E^2), \quad (8.9)$$

$$g^2(p_E^2) := \begin{cases} \frac{1}{\beta_0} \frac{1}{1+t} & \text{for } t_{\text{IF}} \leq t \\ \frac{1}{2\beta_0} \frac{1}{(1+t_{\text{IF}})^2} \left[3t_{\text{IF}} - t_0 + 2 - \frac{(t-t_0)^2}{t_{\text{IF}}-t_0} \right] & \text{for } t_0 \leq t \leq t_{\text{IF}} \\ \frac{1}{2\beta_0} \frac{3t_{\text{IF}}-t_0+2}{(1+t_{\text{IF}})^2} & \text{for } t \leq t_0 \end{cases}, \quad (8.10)$$

$$t := \ln \frac{p_E^2}{\Lambda_{\text{QCD}}^2} - 1, \quad (8.11)$$

$$\beta_0 := \frac{1}{(4\pi)^2} \frac{11N_C - 2N_f}{3}. \quad (8.12)$$

Here the symbol $*$ in Eq.(8.6) means $x_1, x_2, y_1, y_2 \rightarrow x$ after all derivatives are operated. p_E and k_E denote Euclidian momentum.(i.e. $p_E^2 = -p^2$), and the indices $m, n, \dots$ are combined indices $m := (a, i, f)$, $n := (b, j, g), \dots$ with Dirac indices $a, b, \dots$ and color indices $i, j, \dots$ and flavor indices $f, g, \dots$. T^a is a generator of color $SU(N_C)$. This gluon exchange interaction are discussed in **2.3** and **5.4**.

The behavior of the running coupling constant is slightly different from that of **3.3**. In this section, the infrared behavior of the running coupling constant is parameterized by t_0 and t_{IF} .

$\mathcal{L}_{\text{FM}}[\psi, \bar{\psi}]$ is a flavor mixing interaction with weight function:

$$\begin{aligned} \mathcal{L}_{\text{FM}}[\psi, \bar{\psi}](x) &= \frac{1}{3} G_D \epsilon^{f_1 f_2 f_3} \epsilon^{g_1 g_2 g_3} w \left(\frac{\partial_{x_1}^2 + \partial_{y_1}^2 + \partial_{x_2}^2 + \partial_{y_2}^2 + \partial_{x_3}^2 + \partial_{y_3}^2}{2} \right) \\ &\times \left\{ [\bar{\psi}_{g_1}(x_1) \psi_{f_1}(y_1)] [\bar{\psi}_{g_2}(x_2) \psi_{f_2}(y_2)] [\bar{\psi}_{g_3}(x_3) \psi_{f_3}(y_3)] \right. \\ &\left. + 3 [\bar{\psi}_{g_1}(x_1) \psi_{f_1}(y_1)] [\bar{\psi}_{g_2}(x_2) \gamma_5 \psi_{f_2}(y_2)] [\bar{\psi}_{g_3}(x_3) \gamma_5 \psi_{f_3}(y_3)] \right\} \Big|_*, \end{aligned} \quad (8.13)$$

$$w(\mu^2) := \exp(-\kappa \mu^2). \quad (8.14)$$

This interaction was introduced in **2.6**. Note that for our choice of the Gaussian weight function the association rule

$$w(-p^2 - q^2 - k^2) = w(-p^2) w(-q^2) w(-k^2) \quad (8.15)$$

holds.

The lagrangian $\mathcal{L}[\psi, \bar{\psi}]$ is invariant under the $SU(N_F)_L \times SU(N_F)_R \times U(1)_V$ but not under the $U(1)_A$. There are 5 parameters

QCD scale parameter	:	Λ_{QCD} ,
Infrared parameters	:	t_0, t_{IF} ,
Flavor mixing parameters	:	G_D, κ

in this model other than quark bare masses and UV cut off Λ_{UV} .

There is an additional comment. This ExKG model is well-defined in space like region but perhaps is inappropriate in time like region where the values of the running coupling constant $\bar{g}^2(-p^2)$ and the weight function $w(-p^2)$ are not reliable. Therefore in this section we use the traditional Euclidean formulation and do not discuss whether the Euclid formulation is justified or not.

8.3 CJT Action for ExKG model

We take the 2-loop diagrams with gluon exchange interaction Γ_{GE} and 3-loop diagrams with flavor mixing interaction Γ_{FM} as the expansion term $\Gamma_{loop}[S_F]$ of CJT action. Then the CJT action is

$$\Gamma[S_F] = i\text{TrLn}[S_F] - i\text{Tr}[S_0^{-1}S_F] + \Gamma_{GE}[S_F] + \Gamma_{FM}[S_F], \quad (8.16)$$

$$\begin{aligned} \Gamma_{GE}[S_F] = & -\frac{1}{2} \int d^4x \tilde{K}^{mn, n_2 m_2} \left(\frac{i\partial^{x_2} - i\partial^{y_1}}{2}, \frac{-i\partial^{x_1} + i\partial^{y_2}}{2} \right) \\ & \times [S_{Fm m_2}(y_2, y_1) S_{Fn_2 n}(x_2, x_1) - S_{Fmn}(y_2, x_1) S_{Fn_2 m_2}(x_2, y_1)] \Big|_*, \end{aligned} \quad (8.17)$$

$$\begin{aligned} \Gamma_{FM}[S_F] = & \frac{G_D}{3} \int d^4x \epsilon^{f_1 f_2 f_3} \epsilon^{g_1 g_2 g_3} w \left(\frac{\partial_{x_1}^2 + \partial_{y_1}^2 + \partial_{x_2}^2 + \partial_{y_2}^2 + \partial_{x_3}^2 + \partial_{y_3}^2}{2} \right) \\ & \times \left\{ -\text{tr}^{(DC)}[S_{Ff_1 g_1}(y_1, x_1)] \text{tr}^{(DC)}[S_{Ff_2 g_2}(y_2, x_2)] \text{tr}^{(DC)}[S_{Ff_3 g_3}(y_3, x_3)] \right. \\ & + 3\text{tr}^{(DC)}[S_{Ff_1 g_2}(y_1, x_2) S_{Ff_2 g_1}(y_2, x_1)] \text{tr}^{(DC)}[S_{Ff_3 g_3}(y_3, x_3)] \\ & - 2\text{tr}^{(DC)}[S_{Ff_1 g_2}(y_1, x_2) S_{Ff_2 g_3}(y_2, x_3) S_{Ff_3 g_1}(y_3, x_1)] \\ & - 3\text{tr}^{(DC)}[S_{Ff_1 g_1}(y_1, x_1)] \text{tr}^{(DC)}[\gamma_5 S_{Ff_2 g_2}(y_2, x_2)] \text{tr}^{(DC)}[\gamma_5 S_{Ff_3 g_3}(y_3, x_3)] \\ & + 6\text{tr}^{(DC)}[S_{Ff_1 g_2}(y_1, x_2) \gamma_5 S_{Ff_2 g_1}(y_2, x_1)] \text{tr}^{(DC)}[\gamma_5 S_{Ff_3 g_3}(y_3, x_3)] \\ & + 3\text{tr}^{(DC)}[S_{Ff_1 g_1}(y_1, x_1)] \text{tr}^{(DC)}[\gamma_5 S_{Ff_2 g_3}(y_2, x_3) \gamma_5 S_{Ff_3 g_2}(y_3, x_2)] \\ & \left. - 6\text{tr}^{(DC)}[S_{Ff_1 g_2}(y_1, x_2) \gamma_5 S_{Ff_2 g_3}(y_2, x_3) \gamma_5 S_{Ff_3 g_1}(y_3, x_1)] \right\} \Big|_*. \end{aligned} \quad (8.18)$$

Since the flavor mixing term $\Gamma_{FM}[S_F]$ is still complicated, we take only the dominant term about $1/N_C$ expansion.

$$\Gamma_{FM}[S_F] = \frac{G_D}{3} \int d^4x \epsilon^{f_1 f_2 f_3} \epsilon^{g_1 g_2 g_3} w \left(\frac{\partial_{x_1}^2 + \partial_{y_1}^2 + \partial_{x_2}^2 + \partial_{y_2}^2 + \partial_{x_3}^2 + \partial_{y_3}^2}{2} \right) \quad (8.19)$$

$$\times \left\{ -\text{tr}^{(\text{DC})}[S_{Ff_1g_1}(y_1, x_1)]\text{tr}^{(\text{DC})}[S_{Ff_2g_2}(y_2, x_2)]\text{tr}^{(\text{DC})}[S_{Ff_3g_3}(y_3, x_3)] \right. \\ \left. - 3\text{tr}^{(\text{DC})}[S_{Ff_1g_1}(y_1, x_1)]\text{tr}^{(\text{DC})}[\gamma_5 S_{Ff_2g_2}(y_2, x_2)]\text{tr}^{(\text{DC})}[\gamma_5 S_{Ff_3g_3}(y_3, x_3)] \right\}$$

In this approximation, the global chiral symmetry is again preserved. In the rest of this section we use Eq.(8.19) instead of Eq.(8.18).

8.4 Schwinger–Dyson Equation

The SDE introduced in 4.2

$$\frac{\delta\Gamma[S_F]}{\delta S_{Fmn}(y, x)} = 0 \quad (8.20)$$

can be written in momentum space by

$$iS_{Fmn}^{-1}(q) - iS_{0mn}^{-1}(q) + \int_p \tilde{K}^{nm, m_2 n_2}(q, p) S_{Fm_2 n_2}(p) \\ - G_D \delta_{ij} \delta_{ab} \epsilon^{ff_1 f_2} \epsilon^{gg_1 g_2} \int_{p, k} w(-q^2 - p^2 - k^2) \\ \times \text{tr}^{(\text{DC})}[S_{Fg_1 f_1}(p)] \text{tr}^{(\text{DC})}[S_{Fg_2 f_2}(k)] = 0. \quad (8.21)$$

Generally the quark propagator is written by

$$S_{Fh}(p) = \frac{i}{\not{p}A_h(p^2) - B_h(p^2)} \quad (8.22)$$

where h is the flavor index and we omit the Dirac and color indices. Both $A_h(p^2)$ and $B_h(p^2)$ are scalar functions.

For these scalar functions, SDE becomes

$$q^2 A_q(q^2) - q^2 + iC_F \int_p \bar{g}^2(q, p) iD^{\mu\nu}(q - p) \frac{(pq)g_{\nu\mu} - q_\nu p_\mu - p_\nu q_\mu}{p^2 A_q^2(p^2) - B_q^2(p^2)} A_q(p^2) = 0, \quad (8.23)$$

$$q^2 A_s(q^2) - q^2 + iC_F \int_p \bar{g}^2(q, p) iD^{\mu\nu}(q - p) \frac{(pq)g_{\nu\mu} - q_\nu p_\mu - p_\nu q_\mu}{p^2 A_s^2(p^2) - B_s^2(p^2)} A_s(p^2) = 0 \quad (8.24)$$

and

$$B_q(q^2) = m_q + iC_F \int_p \bar{g}^2(q, p) \frac{1}{(p - q)^2} \frac{-3B_q(p^2)}{p^2 A_q^2(p^2) - B_q^2(p^2)} \quad (8.25)$$

$$+ 32G_D N_C^2 \int_{p, k} w(-q^2 - p^2 - k^2) \frac{B_q(p^2)}{p^2 A_q^2(p^2) - B_q^2(p^2)} \frac{B_s(k^2)}{k^2 A_s^2(k^2) - B_s^2(k^2)},$$

$$B_s(q^2) = m_s + iC_F \int_p \bar{g}^2(q, p) \frac{1}{(p - q)^2} \frac{-3B_s(p^2)}{p^2 A_s^2(p^2) - B_s^2(p^2)} \quad (8.26)$$

$$+ 32G_D N_C^2 \int_{p, k} w(-q^2 - p^2 - k^2) \frac{B_q(p^2)}{p^2 A_q^2(p^2) - B_q^2(p^2)} \frac{B_q(k^2)}{k^2 A_q^2(k^2) - B_q^2(k^2)} \quad (8.27)$$

where C_F denotes the second Casimir constant for color $SU(N_C)$

$$C_F := \frac{\text{tr}[T^a T^a]}{N_C} = \frac{N_C^2 - 1}{2N_C} \quad (8.28)$$

Note that in Eqs.(8.23), (8.24) the flavor mixing term does not appear. Here we use the Wick rotation. The third term of Eq.(8.23) or Eq.(8.24) vanishes after the angular integration in the Euclid space. This is a result of the Higashijima–Miransky approximation under the Landau gauge condition. Then we obtain

$$A_q(-q_E^2) = A_s(-q_E^2) = 1. \quad (8.29)$$

This implies the charge universality or the vector Ward identity($Z_1 = Z_2$). Eq.(8.29) is obtained in the Euclidean space. The analytic continuation gives that $A_q(-s)$, $A_s(-s)$ equal to identically 1 also in the time like region $s < 0$. We use this result following subsections. Strictly speaking, however, since the Higashijima–Miransky approximation is not analytic, the validity of the analytic continuation is obscure.

Using Eq.(8.29), SD equations (8.25) and (8.26) become in the Euclid space

$$\begin{aligned} B_q(-s) = & m_q + \frac{3C_F}{16\pi^2} \int_0^\infty dr \bar{g}^2(s, r) \frac{B_q(-r)}{r + B_q^2(-r)} \left\{ \theta(r - s) + \frac{r}{s} \theta(s - r) \right\} \\ & - \frac{G_D N_C^2 w(s)}{8\pi^4} \int_0^\infty dr w(r) \frac{r B_q(-r)}{r + B_q^2(-r)} \int_0^\infty dl w(l) \frac{l B_s(-l)}{l + B_s^2(-l)}, \end{aligned} \quad (8.30)$$

$$\begin{aligned} B_s(-s) = & m_s + \frac{3C_F}{16\pi^2} \int_0^\infty dr \bar{g}^2(s, r) \frac{B_s(-r)}{r + B_s^2(-r)} \left\{ \theta(r - s) + \frac{r}{s} \theta(s - r) \right\} \\ & - \frac{G_D N_C^2 w(s)}{8\pi^4} \int_0^\infty dr w(r) \frac{r B_q(-r)}{r + B_q^2(-r)} \int_0^\infty dl w(l) \frac{l B_q(-l)}{l + B_q^2(-l)}. \end{aligned} \quad (8.31)$$

These integral equations are solvable numerically.

8.5 Homogeneous Bethe–Salpeter Equation

The homogeneous BSE introduced in 4.2

$$\frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(y, x) \delta S_{Fn'm'}(x', y')} \chi_{n'm'}(x', y'; P_B) = 0 \quad (8.32)$$

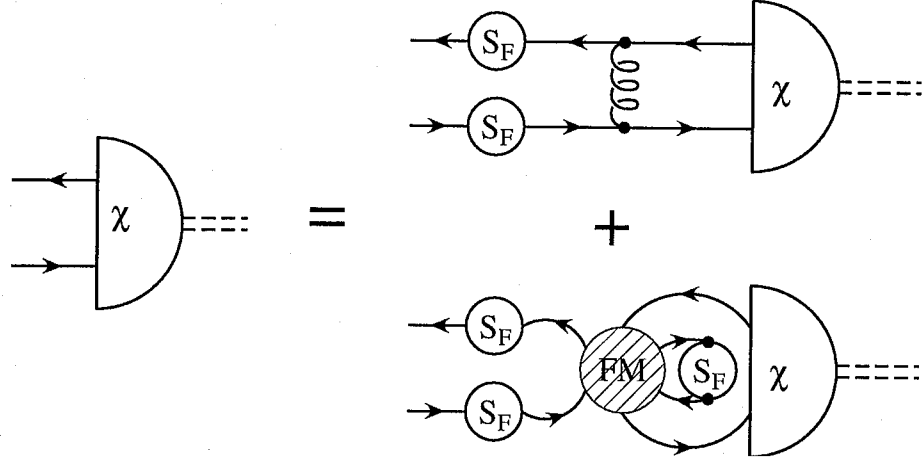


Fig.8-1: Homogeneous BSE in ExKG model

for the pion or the η meson is given in momentum space by

$$\begin{aligned}
& S_{Fnn_1}^{-1}\left(q + \frac{P}{2}\right) \chi_{n_1 m_1}(q; P) S_{Fm_1 m}^{-1}\left(q - \frac{P}{2}\right) \\
&= -i \int_k \tilde{K}^{mn, n_2 m_2}(k, q) \chi_{n_2 m_2}(k; P) \\
&\quad + 2i G_D \epsilon^{ghf'} \epsilon^{fh'g'} \delta_{ji} \int_{p,k} w \left(-p^2 - q^2 - k^2 - \frac{P^2}{2} \right) \\
&\quad \times \text{tr}^{(\text{DC})}[S_{Fh'h}(p)] \left\{ (\gamma_5)_{ba} \text{tr}^{(\text{DC})}[\gamma_5 \chi_{g'f'}(k; P)] + 1_{ba} \text{tr}^{(\text{DC})}[\chi_{g'f'}(k; P)] \right\}. \quad (8.33)
\end{aligned}$$

This is shown by Fig.8-1 graphically. For the kaon one must introduce the center of mass parameter and the BSE is more complicated.

The second term in the braces of RHS of (8.33) vanishes for the property of the BS amplitude for a pseudo-scalar particle.

Let us substitute

$$\begin{aligned}
\chi_{nm}(k; P) &= \delta_{ji} \frac{(\lambda^\alpha)_{gf}}{2} \left[\left(\phi_S^\alpha(k; P) + \phi_P^\alpha(k; P) \not{k} \right. \right. \\
&\quad \left. \left. + \phi_Q^\alpha(k; P) \not{P} + \frac{1}{2} \phi_T^\alpha(k; P) (\not{P} \not{k} - \not{k} \not{P}) \right) \gamma_5 \right]_{ba} \quad (8.34)
\end{aligned}$$

$$=: \delta_{ji} \frac{(\lambda^\alpha)_{gf}}{2} \left[\left(\phi_S^\alpha(k; P) + \gamma_\mu \phi_V^{\alpha;\mu}(k; P) + \gamma_{\mu\nu} \phi_W^{\alpha;\mu\nu}(k; P) \right) \gamma_5 \right]_{ba}, \quad (8.35)$$

$$S_{Fh'h}(p) = \frac{i \delta_{h'h}}{\not{p} - B_h(p^2)} \quad (8.36)$$

into Eq.(8.33). Here we use SD solution $A_h(p^2) = 1$ from Eq.(8.29) and the scalar functions form Eq.(7.53) of the BS amplitude.

Multiplying γ_5 , $\gamma_5\gamma^\rho$, $\gamma_5\gamma^{\rho\sigma}$ and taking traces for Dirac indices, respectively we obtain

$$\begin{aligned}
& (\lambda^\alpha)_{gf} \left[(-q^2 + \frac{P^2}{4} + B_f B_g) \phi_S^\alpha - \frac{B_f + B_g}{2} (P \phi_V^\alpha) - (B_f - B_g) (q \phi_V^\alpha) + 2q_\mu P_\nu \phi_W^{\alpha;\mu\nu} \right] \\
& = -iC_F(\lambda^\alpha)_{gf} \int_k \bar{g}^2(q, k) \frac{-3}{(q-k)^2} \phi_S^\alpha(k; P) \\
& \quad + 32G_D N_C^2 \epsilon^{ghf'} \epsilon^{fh'g'} \delta_{h'h} (\lambda^\alpha)_{g'f'} \int_{p,k} \\
& \quad \times w(-p^2 - q^2 - k^2 - \frac{1}{2}P^2) \frac{B_h(p^2)}{p^2 - B_h^2(p^2)} \phi_S^\alpha(k; P),
\end{aligned} \tag{8.37}$$

$$\begin{aligned}
& (\lambda^\alpha)_{gf} \left[(q^2 - \frac{P^2}{4} + B_f B_g) \phi_V^{\alpha;\rho} - (\frac{B_f + B_g}{2} P^\rho + (B_f - B_g) q^\rho) \phi_S^\alpha \right. \\
& \quad \left. - 2(\frac{B_f - B_g}{2} P_\mu + (B_f + B_g) q_\mu) \phi_W^{\alpha;\mu\rho} + \frac{P^\rho}{2} (P \phi_V^\alpha) - 2q^\rho (q \phi_V^\alpha) \right] \\
& = -iC_F(\lambda^\alpha)_{gf} \int_k \bar{g}^2(q, k) \left\{ \frac{\phi_V^{\alpha;\rho}}{(q-k)^2} + \frac{2(q-k)^\rho (q-k)_\sigma \phi_V^{\alpha;\sigma}}{(q-k)^4} \right\},
\end{aligned} \tag{8.38}$$

$$\begin{aligned}
& (\lambda^\alpha)_{gf} \left[2(q^2 - \frac{P^2}{4} + B_f B_g) \phi_W^{\pi;\sigma\rho} + (P^\rho q^\sigma - q^\rho P^\sigma) \phi_S^\alpha \right. \\
& \quad \left. - \frac{B_f - B_g}{2} (P^\sigma \phi_V^{\alpha;\rho} - P^\rho \phi_V^{\alpha;\sigma}) - (B_f + B_g) (q^\sigma \phi_V^{\alpha;\rho} - q^\rho \phi_V^{\alpha;\sigma}) \right] \\
& = -i(\lambda^\alpha)_{gf} C_F \int_k \bar{g}^2(q, k) \\
& \quad \times \left\{ \frac{-2\phi_W^{\alpha;\rho\sigma}}{(q-k)^2} + \frac{4(q-k)_\tau}{(q-k)^4} ((q-k)^\rho \phi_W^{\alpha;\tau\sigma} + (q-k)^\sigma \phi_W^{\alpha;\rho\tau}) \right\}
\end{aligned} \tag{8.39}$$

with

$$B_g = B_g((q + \frac{1}{2}P)^2), \quad B_f = B_f((q - \frac{1}{2}P)^2). \tag{8.40}$$

8.6 Nambu–Goldstone solution

We multiply $\gamma_5 \frac{\lambda^\alpha}{2}$ from left and $\gamma_5 \frac{\lambda^\alpha}{2}$ from right to SDE (8.21) and take their sum, we obtain

$$\begin{aligned}
& \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F^{-1}(q)\}_{nm} - \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_0^{-1}(q)\}_{nm} = -i \int_p \tilde{K}^{mn, n_2 m_2}(q, p) \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(p)\}_{n_2 m_2} \\
& \quad + G_D \delta_{ji} \delta_{ba} \left(\frac{(\lambda^\alpha)_{gg'}}{2} \epsilon^{g'f_1 f_2} \epsilon^{fg_1 g_2} + \epsilon^{gf_1 f_2} \epsilon^{f'g_1 g_2} \frac{(\lambda^\alpha)_{f'f}}{2} \right) \\
& \quad \times \int_{p,k} w(-q^2 - p^2 - k^2) \text{tr}^{(\text{DC})}[S_{F_{g_1 f_1}}(p)] \text{tr}^{(\text{DC})}[S_{F_{g_2 f_2}}(k)].
\end{aligned} \tag{8.41}$$

In the chiral limit the second term of LHS vanishes and furthermore if λ^α is an octet matrix i.e. $\text{tr}[\lambda^\alpha] = 0$,

$$\begin{aligned} & \left(\frac{(\lambda^\alpha)_{gg'}}{2} \epsilon^{g'f_1f_2} \epsilon^{fg_1g_2} + \epsilon^{gf_1f_2} \epsilon^{f'g_1g_2} \frac{(\lambda^\alpha)_{f'f}}{2} \right) \delta_{g_1f_1} \delta_{g_2f_2} \\ &= -4\epsilon^{gf_1f_2} \epsilon^{fg_1g_2} \delta_{g_1f_1} \frac{(\lambda^\alpha)_{g_2f_2}}{2} \end{aligned} \quad (8.42)$$

holds. In this case, Eq.(8.41) becomes

$$\begin{aligned} & \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F^{-1}(q)\}_{nm} - \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_0^{-1}(q)\}_{nm} = -i \int_p \tilde{K}^{mn, n_2 m_2}(q, p) \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(p)\}_{n_2 m_2} \\ & + 2iG_D \delta_{ji} \delta_{ba} \epsilon^{gf_1f_2} \epsilon^{fg_1g_2} \int_{p,k} w(-q^2 - p^2 - k^2) \\ & \times \text{tr}^{(\text{DC})}[S_{Fg_1f_1}(p)] \text{tr}^{(\text{DC})}[\gamma_5 \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(k)\}_{g_2f_2}]. \end{aligned} \quad (8.43)$$

Comparing homogeneous BSE (8.33) in which $P = 0$ and Eq.(8.43), we obtain the Nambu–Goldstone solution

$$\chi_{nm}(q; P = 0) = N \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(q)\}_{nm} \quad (8.44)$$

where N is a normalization constant. This is a same equation as Eq.(5.14). Therefore in the framework of this model, we verified the realization of the Nambu–Goldstone theorem. Eq.(8.44) does not hold for a singlet matrix λ^α i.e. $\text{tr}[\lambda^\alpha] \neq 0$.

8.7 Bethe–Salpeter Equation for the Pion

For a pion state, only $\alpha = \pi$ component in Eq.(8.34) exists.

We introduce the scalar amplitudes $\phi_X^\pi(q; P)$, $\phi_Y^\pi(q; P)$ and $\phi_W^\pi(q; P)$ defined by

$$\begin{aligned} \phi_V^{\pi, \mu}(q; P) &= \frac{\frac{B_q(q-) + B_q(q+)}{2} P^\mu + (B_q(q-) - B_q(q+)) q^\mu}{\frac{P^2}{4} - q^2 + B_q(q-) B_q(q+)} \phi_S^\pi(q; P) \\ &+ q^\mu \phi_X^\pi(q; P) + P^\mu \phi_Y^\pi(q; P), \end{aligned} \quad (8.45)$$

$$\begin{aligned} \phi_W^{\pi, \mu\nu}(q; P) &= \frac{-P^\mu q^\nu + q^\mu P^\nu}{2 \left(\frac{P^2}{4} - q^2 + B_q(q-) B_q(q+) \right)} \phi_S^\pi(q; P) \\ &+ (P^\mu q^\nu - q^\mu P^\nu) \phi_Z^\pi(q; P) \end{aligned} \quad (8.46)$$

where

$$B_q(q+) := B_q((q + \frac{P}{2})^2), \quad B_q(q-) := B_q((q - \frac{P}{2})^2). \quad (8.47)$$

Then we obtain the BSE for pion written by four scalar amplitudes $\phi_S^\pi(q; P)$, $\phi_X^\pi(q; P)$, $\phi_Y^\pi(q; P)$ and $\phi_Z^\pi(q; P)$.

$$\begin{aligned}
& \left[-q^2 + \frac{P^2}{4} + B_q(q-)B_q(q+) + \frac{1}{\frac{P^2}{4} - q^2 + B_q(q-)B_q(q+)} \left\{ \right. \right. \\
& \quad - \frac{1}{4} \left(B_q(q-) + B_q(q+) \right)^2 P^2 - \left(B_q^2(q-) - B_q^2(q+) \right) (Pq) \\
& \quad \left. \left. - \left(B_q(q-) - B_q(q+) \right)^2 q^2 - (Pq)^2 + P^2 q^2 \right\} \right] \phi_S^\pi(q; P) \\
& = -iC_F \int \frac{d^4 k}{(2\pi)^4} \bar{g}^2(q, k) \frac{-3}{(q-k)^2} \phi_S^\pi(k; P) \\
& \quad - 32G_D N_C^2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} w(-p^2 - q^2 - k^2 - \frac{P^2}{2}) \frac{B_s(p^2)}{p^2 - B_s^2(p^2)} \phi_S^\pi(k; P) \\
& \quad + \left(\frac{B_q(q-) + B_q(q+)}{2} (Pq) + \left(B_q(q-) - B_q(q+) \right) q^2 \right) \phi_X^\pi(q; P) \\
& \quad + \left(\frac{B_q(q-) + B_q(q+)}{2} P^2 + \left(B_q(q-) - B_q(q+) \right) (Pq) \right) \phi_Y^\pi(q; P) \\
& \quad + 2(P^2 q^2 - (Pq)^2) \phi_Z^\pi(q; P) \tag{8.48}
\end{aligned}$$

$$\begin{aligned}
\phi_X^\pi(q; P) & = \frac{1}{\Delta^\pi} \left\{ P^2 B_q^2(q-) B_q^2(q+) - \frac{P^6}{16} + P^2 q^4 + \frac{P^2}{2} (Pq)^2 - 2q^2 (Pq)^2 \right. \\
& \quad \left. - \left(B_q^2(q-) + B_q^2(q+) \right) (P^2 q^2 - (Pq)^2) \right\} D_1^\pi(q; P) \\
& \quad + \frac{1}{\Delta^\pi} \left\{ - \frac{B_q^2(q-) - B_q^2(q+)}{2} (P^2 q^2 - (Pq)^2) \right. \\
& \quad \left. + \left(\frac{P^4}{16} - \frac{P^2 q^2}{2} + q^4 - B_q^2(q-) B_q^2(q+) \right) (Pq) \right\} D_2^\pi(q; P) \\
& \quad + \frac{1}{\Delta^\pi} \left\{ \frac{B_q(q-) - B_q(q+)}{2} \left(P^2 B_q(q-) B_q(q+) + \frac{P^4}{4} + P^2 q^2 - 2(Pq)^2 \right) \right. \\
& \quad \left. + \left(B_q(q-) + B_q(q+) \right) (Pq) \left(B_q(q-) B_q(q+) + \frac{P^2}{4} - q^2 \right) \right\} D_3^\pi(q; P) \tag{8.49}
\end{aligned}$$

$$\begin{aligned}
\phi_Y^\pi(q; P) & = \frac{1}{\Delta^\pi} \left\{ \left(\frac{P^4}{16} - \frac{P^2 q^2}{2} + q^4 - B_q^2(q-) B_q^2(q+) \right) (Pq) \right. \\
& \quad \left. - \frac{1}{2} \left(B_q^2(q-) - B_q^2(q+) \right) (P^2 q^2 - (Pq)^2) \right\} D_1^\pi(q; P) \\
& \quad + \frac{1}{\Delta^\pi} \left\{ - \frac{1}{4} \left(B_q^2(q-) + B_q^2(q+) \right) (P^2 q^2 - (Pq)^2) \right. \\
& \quad \left. + \left(\frac{P^4}{16} - q^4 + B_q^2(q-) B_q^2(q+) \right) q^2 + \left(-\frac{P^2}{8} + \frac{q^2}{2} \right) (Pq)^2 \right\} D_2^\pi(q; P)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Delta^\pi} \left\{ (B_q(q-) + B_q(q+)) \left(-B_q(q-)B_q(q+)q^2 + \frac{1}{4}P^2q^2 + q^4 - \frac{(Pq)^2}{2} \right) \right. \\
& \left. + \frac{1}{2}(B_q(q-) - B_q(q+))(Pq) \left(-B_q(q-)B_q(q+) - \frac{P^2}{4} + q^2 \right) \right\} D_3^\pi(q; P) \quad (8.50)
\end{aligned}$$

$$\begin{aligned}
\phi_Z^\pi(q; P) &= \frac{1}{\Delta^\pi} \left\{ \frac{B_q(q-) + B_q(q+)}{2} (Pq) \left(B_q(q-)B_q(q+) + \frac{P^2}{4} - q^2 \right) \right. \\
&+ (B_q(q-) - B_q(q+)) \left(\frac{P^2}{4} (B_q(q-)B_q(q+) + \frac{P^2}{4} + q^2) - \frac{1}{2}(Pq)^2 \right) \Big\} D_1^\pi(q; P) \\
&+ \frac{1}{\Delta^\pi} \left\{ -\frac{B_q(q-) + B_q(q+)}{2} \left(B_q(q-)B_q(q+)q^2 - \frac{P^2}{4}q^2 - q^4 + \frac{1}{2}(Pq)^2 \right) \right. \\
&+ \frac{1}{4}(B_q(q-) - B_q(q+))(Pq) \left(q^2 - \frac{P^2}{4} - B_q(q-)B_q(q+) \right) \Big\} D_2^\pi(q; P) \\
&+ \frac{1}{2\Delta^\pi} \left(B_q^2(q-)B_q^2(q+) - \frac{P^4}{16} - \frac{P^2q^2}{2} - q^4 + (Pq)^2 \right) D_3^\pi(q; P) \quad (8.51)
\end{aligned}$$

$$\begin{aligned}
\Delta^\pi &:= -\left(q^2 - \frac{P^2}{4} - B_q(q-)B_q(q+) \right) \left(q^2 + \frac{P^2}{4} - B_q^2(q-) - (Pq) \right) \\
&\times \left(q^2 + \frac{P^2}{4} - B_q^2(q+) + (Pq) \right) (P^2q^2 - (Pq)^2) \quad (8.52)
\end{aligned}$$

$$\begin{aligned}
D_1^\pi(q; P) &:= -iC_F \int \frac{d^4k}{(2\pi)^4} \bar{g}^2(-q^2, -k^2) \\
&\times \left[\frac{1}{(q-k)^2} \left\{ \frac{B_q(k-) + B_q(k+)}{2} (Pq) + (B_q(k-) - B_q(k+))(qk) \right. \right. \\
&\quad \left. \frac{P^2}{4} - k^2 + B_q^{(k-)} B_q^{(k+)} \phi_S^\pi(k; P) \right. \\
&\quad \left. + (qk) \phi_X^\pi(k; P) + (Pq) \phi_Y^\pi(k; P) \right\} \\
&+ \frac{2(q^2 - qk)}{(q-k)^4} \left\{ \frac{B_q(k-) + B_q(k+)}{2} (Pq - Pk) + (B_q(k-) - B_q(k+))(qk - k^2) \right. \\
&\quad \left. \frac{P^2}{4} - k^2 + B_q(k-)B_q(k+) \phi_S^\pi(k; P) \right. \\
&\quad \left. + (qk - k^2) \phi_X^\pi(k; P) + (Pq - Pk) \phi_Y^\pi(k; P) \right\} \Big] \quad (8.53)
\end{aligned}$$

$$\begin{aligned}
D_2^\pi(q; P) &:= -iC_F \int \frac{d^4k}{(2\pi)^4} \bar{g}^2(-q^2, -k^2) \\
&\times \left[\frac{1}{(q-k)^2} \left\{ \frac{B_q(k-) + B_q(k+)}{2} P^2 + (B_q(k-) - B_q(k+))(Pk) \right. \right. \\
&\quad \left. \frac{P^2}{4} - k^2 + B_q(k-)B_q(k+) \phi_S^\pi(k; P) \right. \\
&\quad \left. + (Pk) \phi_X^\pi(k; P) + P^2 \phi_Y^\pi(k; P) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(Pq - Pk)}{(q - k)^4} \left\{ \right. \\
& \quad \frac{\frac{B_q(k-) + B_q(k+)}{2}(Pq - Pk) + (B_q(k-) - B_q(k+))(qk - k^2)}{\frac{P^2}{4} - k^2 + B_q(k-)B_q(k+)} \phi_S^\pi(k; P) \\
& \quad \left. + (qk - k^2)\phi_X^\pi(k; P) + (Pq - Pk)\phi_Y^\pi(k; P) \right\} \quad (8.54)
\end{aligned}$$

$$\begin{aligned}
D_3^\pi(q; P) &:= -iC_F \int \frac{d^4k}{(2\pi)^4} \bar{g}^2(-q^2, -k^2) \\
& \times \frac{1}{(q - k)^4} \left\{ (-2q^2 + 2k^2)((Pq)(Pk) - (qk)P^2) \right. \\
& \quad \left. + 4(Pq - Pk)(q^2(Pk) - (qk)(Pq)) - 4(qk - k^2)(P^2q^2 - (Pq)^2) \right\} \\
& \times \left\{ \frac{-1}{2(\frac{P^2}{4} - k^2 + B_q(k-)B_q(k+))} \phi_S^\pi(k; P) + \phi_Z^\pi(k; P) \right\} \quad (8.55)
\end{aligned}$$

8.8 Bethe-Salpeter Equation for the η meson

For the η meson, $\alpha = 0$ and 8 components are mixed. Then we represent the BS amplitude as

$$\chi_{nm}^\eta(q; P) = \delta_{ji} \frac{(\lambda^{qq})_{gf}}{\sqrt{2}} \chi_{ba}^{qq}(q; P) + \delta_{ji} \frac{(\lambda^{ss})_{gf}}{\sqrt{2}} \chi_{ba}^{ss}(q; P) \quad (8.56)$$

$$= \delta_{ji} \begin{pmatrix} \frac{1}{\sqrt{2}} \chi_{ba}^{qq}(q; P) & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \chi_{ba}^{qq}(q; P) & 0 \\ 0 & 0 & \chi_{ba}^{ss}(q; P) \end{pmatrix}_{gf}, \quad (8.57)$$

$$\begin{aligned}
\chi_{ba}^{hh}(q; P) &:= \left[\left(\phi_S^{hh}(q; P) + \phi_P^{hh}(q; P) \not{P} + \phi_Q^{hh}(q; P) \not{P} \right. \right. \\
& \quad \left. \left. + \frac{1}{2} \phi_T^{hh}(q; P) (\not{P} \not{P} - \not{P} \not{P}) \right) \frac{\gamma_5}{\sqrt{2}} \right]_{ba}. \quad (8.58)
\end{aligned}$$

$$=: \left[\left(\phi_S^{hh}(k; P) + \gamma_\mu \phi_V^{hh;\mu}(k; P) + \gamma_{\mu\nu} \phi_W^{hh;\mu\nu}(k; P) \right) \gamma_5 \right]_{ba} \quad (8.59)$$

where $h = q, s$. The following equation are obtained in the same way as pion BSE.

$$\begin{aligned}
\phi_V^{hh;\mu}(q; P) &= \frac{\frac{B_h(q-) + B_h(q+)}{2} P^\mu + (B_h(q-) - B_h(q+)) q^\mu}{\frac{P^2}{4} - q^2 + B_h(q-)B_h(q+)} \phi_S^{hh}(q; P) \\
& \quad + q^\mu \phi_X^{hh}(q; P) + P^\mu \phi_Y^{hh}(q; P) \quad (8.60)
\end{aligned}$$

$$\begin{aligned}
\phi_W^{hh;\mu\nu}(q; P) &= \frac{-P^\mu q^\nu + q^\mu P^\nu}{2 \left(\frac{P^2}{4} - q^2 + B_h(q-)B_h(q+) \right)} \phi_S^{hh}(q; P) \\
& \quad + (P^\mu q^\nu - q^\mu P^\nu) \phi_Z^{hh}(q; P) \quad (8.61)
\end{aligned}$$

$$\begin{aligned}
& \left[-q^2 + \frac{P^2}{4} + B_q(q-)B_q(q+) + \frac{1}{\frac{P^2}{4} - q^2 + B_q(q-)B_q(q+)} \left\{ \right. \right. \\
& \quad - \frac{1}{4}(B_q(q-) + B_q(q+))^2 P^2 - (B_q^2(q-) - B_q^2(q+))(Pq) \\
& \quad \left. \left. - (B_q(q-) - B_q(q+))^2 q^2 - (Pq)^2 + P^2 q^2 \right\} \right] \phi_S^{qq}(q; P) \\
& = -iC_F \int \frac{d^4 k}{(2\pi)^4} \bar{g}^2(q, k) \frac{-3}{(q-k)^2} \phi_S^{qq}(k; P) \\
& \quad + 32G_D N_C^2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} w(-p^2 - q^2 - k^2 - \frac{P^2}{2}) \frac{B_s(p^2)}{p^2 - B_s^2(p^2)} \phi_S^{qq}(k; P) \\
& \quad + 32\sqrt{2}G_D N_C^2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} w(-p^2 - q^2 - k^2 - \frac{P^2}{2}) \frac{B_q(p^2)}{p^2 - B_q^2(p^2)} \phi_S^{ss}(k; P) \\
& \quad + \left(\frac{B_q(q-) + B_q(q+)}{2} (Pq) + (B_q(q-) - B_q(q+)) q^2 \right) \phi_X^{qq}(q; P) \\
& \quad + \left(\frac{B_q(q-) + B_q(q+)}{2} P^2 + (B_q(q-) - B_q(q+))(Pq) \right) \phi_Y^{qq}(q; P) \\
& \quad + 2(P^2 q^2 - (Pq)^2) \phi_Z^{qq}(q; P) \tag{8.62}
\end{aligned}$$

$$\begin{aligned}
& \left[-q^2 + \frac{P^2}{4} + B_s(q-)B_s(q+) + \frac{1}{\frac{P^2}{4} - q^2 + B_s(q-)B_s(q+)} \left\{ \right. \right. \\
& \quad - \frac{1}{4}(B_s(q-) + B_s(q+))^2 P^2 - (B_s^2(q-) - B_s^2(q+))(Pq) \\
& \quad \left. \left. - (B_s(q-) - B_s(q+))^2 q^2 - (Pq)^2 + P^2 q^2 \right\} \right] \phi_S^{ss}(q; P) \\
& = -iC_F \int \frac{d^4 k}{(2\pi)^4} \bar{g}^2(q, k) \frac{-3}{(q-k)^2} \phi_S^{ss}(k; P) \\
& \quad + 32\sqrt{2}G_D N_C^2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} w(-p^2 - q^2 - k^2 - \frac{P^2}{2}) \frac{B_q(p^2)}{p^2 - B_q^2(p^2)} \phi_S^{qq}(k; P) \\
& \quad + \left(\frac{B_s(q-) + B_s(q+)}{2} (Pq) + (B_s(q-) - B_s(q+)) q^2 \right) \phi_X^{ss}(q; P) \\
& \quad + \left(\frac{B_q(q-) + B_q(q+)}{2} P^2 + (B_q(q-) - B_q(q+))(Pq) \right) \phi_Y^{ss}(q; P) \\
& \quad + 2(P^2 q^2 - (Pq)^2) \phi_Z^{ss}(q; P) \tag{8.63}
\end{aligned}$$

$$\begin{aligned}
\phi_X^{hh}(q; P) & = \frac{1}{\Delta^{hh}} \left\{ P^2 B_h^2(q-) B_h^2(q+) - \frac{P^6}{16} + P^2 q^4 + \frac{P^2}{2} (Pq)^2 - 2q^2 (Pq)^2 \right. \\
& \quad \left. - (B_h^2(q-) + B_h^2(q+)) (P^2 q^2 - (Pq)^2) \right\} D_1^{hh}(q; P) \\
& \quad + \frac{1}{\Delta^{hh}} \left\{ - \frac{B_h^2(q-) - B_h^2(q+)}{2} (P^2 q^2 - (Pq)^2) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{P^4}{16} - \frac{P^2 q^2}{2} + q^4 - B_h^2(q-) B_h^2(q+) \right) (Pq) \Big\} D_2^{hh}(q; P) \\
& + \frac{1}{\Delta^{hh}} \left\{ \frac{B_h(q-) - B_h(q+)}{2} \left(P^2 B_h(q-) B_h(q+) + \frac{P^4}{4} + P^2 q^2 - 2(Pq)^2 \right) \right. \\
& \left. + (B_h(q-) + B_h(q+)) (Pq) \left(B_h(q-) B_h(q+) + \frac{P^2}{4} - q^2 \right) \right\} D_3^{hh}(q; P) \tag{8.64}
\end{aligned}$$

$$\begin{aligned}
\phi_Y^{hh}(q; P) &= \frac{1}{\Delta^{hh}} \left\{ \left(\frac{P^4}{16} - \frac{P^2 q^2}{2} + q^4 - B_h^2(q-) B_h^2(q+) \right) (Pq) \right. \\
& - \frac{1}{2} (B_h^2(q-) - B_h^2(q+)) (P^2 q^2 - (Pq)^2) \Big\} D_1^{hh}(q; P) \\
& + \frac{1}{\Delta^{hh}} \left\{ -\frac{1}{4} (B_h^2(q-) + B_h^2(q+)) (P^2 q^2 - (Pq)^2) \right. \\
& + \left(\frac{P^4}{16} - q^4 + B_h^2(q-) B_h^2(q+) \right) q^2 + \left(-\frac{P^2}{8} + \frac{q^2}{2} \right) (Pq)^2 \Big\} D_2^{hh}(q; P) \\
& + \frac{1}{\Delta^{hh}} \left\{ (B_h(q-) + B_h(q+)) \left(-B_h(q-) B_h(q+) q^2 + \frac{1}{4} P^2 q^2 + q^4 - \frac{(Pq)^2}{2} \right) \right. \\
& \left. + \frac{1}{2} (B_h(q-) - B_h(q+)) (Pq) \left(-B_h(q-) B_h(q+) - \frac{P^2}{4} + q^2 \right) \right\} D_3^{hh}(q; P) \tag{8.65}
\end{aligned}$$

$$\begin{aligned}
\phi_Z^{hh}(q; P) &= \frac{1}{\Delta^{hh}} \left\{ \frac{B_h(q-) + B_h(q+)}{2} (Pq) \left(B_h(q-) B_h(q+) + \frac{P^2}{4} - q^2 \right) \right. \\
& + (B_h(q-) - B_h(q+)) \left(\frac{P^2}{4} (B_h(q-) B_h(q+) + \frac{P^2}{4} + q^2) - \frac{1}{2} (Pq)^2 \right) \Big\} D_1^{hh}(q; P) \\
& + \frac{1}{\Delta^{hh}} \left\{ -\frac{B_h(q-) + B_h(q+)}{2} \left(B_h(q-) B_h(q+) q^2 - \frac{P^2}{4} q^2 - q^4 + \frac{1}{2} (Pq)^2 \right) \right. \\
& + \frac{1}{4} (B_h(q-) - B_h(q+)) (Pq) \left(q^2 - \frac{P^2}{4} - B_h(q-) B_h(q+) \right) \Big\} D_2^{hh}(q; P) \\
& + \frac{1}{2\Delta^{hh}} \left(B_h^2(q-) B_h^2(q+) - \frac{P^4}{16} - \frac{P^2 q^2}{2} - q^4 + (Pq)^2 \right) D_3^{hh}(q; P) \tag{8.66}
\end{aligned}$$

$$\begin{aligned}
\Delta^{hh} &:= - \left(q^2 - \frac{P^2}{4} - B_h(q-) B_h(q+) \right) \left(q^2 + \frac{P^2}{4} - B_h^2(q-) - (Pq) \right) \\
&\quad \times \left(q^2 + \frac{P^2}{4} - B_h^2(q+) + (Pq) \right) (P^2 q^2 - (Pq)^2) \tag{8.67}
\end{aligned}$$

$$\begin{aligned}
D_1^{hh}(q; P) &:= -i C_F \int \frac{d^4 k}{(2\pi)^4} \bar{g}^2(-q^2, -k^2) \\
&\quad \times \left[\frac{1}{(q-k)^2} \left\{ \frac{B_h(k-) + B_h(k+)}{2} (Pq) + (B_h(k-) - B_h(k+)) (qk) \right\} \frac{\phi_S^{hh}(k; P)}{\frac{P^2}{4} - k^2 + B_h^{(k-)} B_h^{(k+)}} \right]
\end{aligned}$$

$$\begin{aligned}
& + (qk)\phi_X^{hh}(k; P) + (Pq)\phi_Y^{hh}(k; P) \Big\} \\
& + \frac{2(q^2 - qk)}{(q - k)^4} \Big\{ \\
& \quad \frac{\frac{B_h(k-) + B_h(k+)}{2}(Pq - Pk) + (B_h(k-) - B_h(k+))(qk - k^2)}{\frac{P^2}{4} - k^2 + B_h(k-)B_h(k+)} \phi_S^{hh}(k; P) \\
& \quad + (qk - k^2)\phi_X^{hh}(k; P) + (Pq - Pk)\phi_Y^{hh}(k; P) \Big\} \Big] \tag{8.68}
\end{aligned}$$

$$\begin{aligned}
D_2^{hh}(q; P) &:= -iC_F \int \frac{d^4k}{(2\pi)^4} \bar{g}^2(-q^2, -k^2) \\
&\times \Big[\frac{1}{(q - k)^2} \Big\{ \frac{\frac{B_h(k-) + B_h(k+)}{2}P^2 + (B_h(k-) - B_h(k+))(Pk)}{\frac{P^2}{4} - k^2 + B_h(k-)B_h(k+)} \phi_S^{hh}(k; P) \\
&\quad + (Pk)\phi_X^{hh}(k; P) + P^2\phi_Y^{hh}(k; P) \Big\} \\
&+ \frac{2(Pq - Pk)}{(q - k)^4} \Big\{ \\
&\quad \frac{\frac{B_h(k-) + B_h(k+)}{2}(Pq - Pk) + (B_h(k-) - B_h(k+))(qk - k^2)}{\frac{P^2}{4} - k^2 + B_h(k-)B_h(k+)} \phi_S^{hh}(k; P) \\
&\quad + (qk - k^2)\phi_X^{hh}(k; P) + (Pq - Pk)\phi_Y^{hh}(k; P) \Big\} \Big] \tag{8.69}
\end{aligned}$$

$$\begin{aligned}
D_3^{hh}(q; P) &:= -iC_F \int \frac{d^4k}{(2\pi)^4} \bar{g}^2(-q^2, -k^2) \\
&\times \frac{1}{(q - k)^4} \Big\{ (-2q^2 + 2k^2)((Pq)(Pk) - (qk)P^2) \\
&\quad + 4(Pq - Pk)(q^2(Pk) - (qk)(Pq)) - 4(qk - k^2)(P^2q^2 - (Pq)^2) \Big\} \\
&\times \Big\{ \frac{-1}{2(\frac{P^2}{4} - k^2 + B_h(k-)B_h(k+))} \phi_S^{hh}(k; P) + \phi_Z^{hh}(k; P) \Big\} \tag{8.70}
\end{aligned}$$

8.9 Normalization

The normalization of the BS amplitude $\chi(q; P)$ discussed in 4.5 is given by

$$\begin{aligned}
\|\chi(q; P)\|^2 &:= \frac{N_C}{P^2} \sum_{\alpha} i \int \frac{d^4q}{(2\pi)^4} \Big[\\
&\quad \left\{ (Pq)\kappa_1 + \frac{P^2}{2}\kappa_2 + \frac{P^2}{2} \right\} \phi_S^{\alpha} \phi_S^{\alpha} \\
&\quad + \left\{ q^2(Pq)\kappa_1 + \frac{P^2q^2}{2}\kappa_2 - \frac{1}{2}(P^2q^2 - 2(Pq)^2) \right\} \phi_P^{\alpha} \phi_P^{\alpha} \\
&\quad + \left\{ P^2(Pq)\kappa_1 + \frac{P^4}{2}\kappa_2 + \frac{P^4}{2} \right\} \phi_Q^{\alpha} \phi_Q^{\alpha} \Big]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ - (Pq)\kappa_1 - \frac{P^2}{2}\kappa_2 + \frac{P^2}{2} \right\} ((Pq)^2 - P^2q^2) \phi_T^\alpha \phi_T^\alpha \\
& + \left\{ (Pq)^2\kappa_1 + \frac{P^2}{2}(Pq)\kappa_2 + \frac{P^2}{2}(Pq) \right\} (2\phi_P^\alpha \phi_Q^\alpha) \\
& + \left\{ - \frac{(Pq)}{2}\kappa_3 + (q^2 - \frac{P^2}{4})(Pq)\kappa_5 - \frac{1}{2}\kappa_6((Pq)^2 - P^2q^2) \right\} (2\phi_S^\alpha \phi_P^\alpha) \\
& + \left\{ - \frac{P^2}{2}\kappa_3 + ((Pq)^2 - \frac{P^4}{4})\kappa_5 \right\} (2\phi_S^\alpha \phi_Q^\alpha) \\
& + ((Pq)^2 - P^2q^2) (2\phi_S^\alpha \phi_T^\alpha) \\
& + \left\{ - \frac{1}{2}\kappa_4 - \frac{P^2}{4}\kappa_6 - \frac{Pq}{2}\kappa_5 \right\} ((Pq)^2 - P^2q^2) (2\phi_P^\alpha \phi_T^\alpha) \\
& + \left\{ - (Pq)\kappa_6 - \frac{P^2}{2}\kappa_5 \right\} ((Pq)^2 - P^2q^2) (2\phi_Q^\alpha \phi_T^\alpha) \Big] = 1
\end{aligned} \tag{8.71}$$

where

$$\kappa_1 := B'_h(q_+^2)B_h(q_-^2) - B_h(q_+^2)B'_h(q_-^2), \tag{8.72}$$

$$\kappa_2 := B'_h(q_+^2)B_h(q_-^2) + B_h(q_+^2)B'_h(q_-^2), \tag{8.73}$$

$$\kappa_3 := B_h(q_+^2) + B_h(q_-^2), \tag{8.74}$$

$$\kappa_4 := B_h(q_+^2) - B_h(q_-^2), \tag{8.75}$$

$$\kappa_5 := B'_h(q_+^2) + B'_h(q_-^2), \tag{8.76}$$

$$\kappa_6 := B'_h(q_+^2) - B'_h(q_-^2), \tag{8.77}$$

$$q_+ := q + \frac{P}{2}, \quad q_- := q - \frac{P}{2}. \tag{8.78}$$

For the pion, the summation is only $\alpha = \pi$ and the flavor index $h = q$. For the eta, the summation is $\alpha = qq, ss$ and the flavor index $h = q, s$ respectively.

8.10 PCAC and the decay constant

Corresponding the flavor mixing term, the conserved current of the ExKG model is more difficult than that of KG model. We introduce

$$\begin{aligned}
F_{mn}^\alpha(q; P) &:= 2G_D(\gamma_5)_{ab}\delta_{ij} \int_{k,l} \text{tr}^{(\text{DC})}[S_{F_{g_2 f_2}}(k)] \\
&\times \left[\left\{ w \left(- \left(q - \frac{P}{2} \right)^2 - k^2 - l^2 \right) - w \left(- q^2 - \frac{3}{4}P^2 - k^2 - l^2 + Pl \right) \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& \times \epsilon^{ff_2f_3} \epsilon^{gg_2g_3} \text{tr}^{(\text{DC})}[(S_F i \frac{\lambda^\alpha}{2})_{g_3f_3}(l)] \\
& + \left\{ w\left(-\left(q + \frac{P}{2}\right)^2 - k^2 - l^2\right) - w\left(-q^2 - \frac{3}{4}P^2 - k^2 - l^2 - Pl\right) \right\} \\
& \times \epsilon^{ff_2f_3} \epsilon^{gg_2g_3} \text{tr}^{(\text{DC})}[(i \frac{\lambda^\alpha}{2} S_F)_{g_3f_3}(l)],
\end{aligned} \tag{8.79}$$

$$\begin{aligned}
A_{mn}^\alpha(q; P) &:= -3\delta^{\alpha 0} G_D(\gamma_5)_{ab} \delta_{ij} \int_{k,l} \text{tr}^{(\text{DC})}[S_F g_2 f_2(k)] \\
& \times \left[w\left(-\left(q - \frac{P}{2}\right)^2 - k^2 - l^2\right) \epsilon^{ff_2f_3} \epsilon^{gg_2g_3} \text{tr}^{(\text{DC})}[(S_F i \frac{\lambda^\alpha}{2})_{g_3f_3}(l)] \right. \\
& \left. + w\left(-\left(q + \frac{P}{2}\right)^2 - k^2 - l^2\right) \epsilon^{ff_2f_3} \epsilon^{gg_2g_3} \text{tr}^{(\text{DC})}[(i \frac{\lambda^\alpha}{2} S_F)_{g_3f_3}(l)] \right].
\end{aligned} \tag{8.80}$$

For a factor $\delta^{\alpha 0}$, $A_{mn}^\alpha(q; P)$ is non-zero only when $\alpha = 0$. In the limit $P \rightarrow 0$, $E_{mn}^\alpha(q; P)$ vanishes but $A_{mn}^\alpha(q; P)$ does not. $A_{mn}^\alpha(q; P)$ corresponds to the lack of the global $U(1)_A$ symmetry.

We obtain

$$\begin{aligned}
& \int_q G_{C;m''n''nm}^{(2)}(p, q; P) \left(i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) + E^\alpha(q; P) + F^\alpha(q; P) + A^\alpha(q; P) \right)_{mn} \\
& = i \left(i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P}{2}) \right)_{m''n''} + i \left(S_F(p + \frac{P}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{m''n''}
\end{aligned} \tag{8.81}$$

where $E_{mn}^\alpha(q; P)$ is given by Eq.(5.39).

The decay constant is given by

$$\tilde{f}_\pi = f_\pi + \lim_{P \rightarrow \bar{P}_B} \frac{1}{2P^2} \int_q \left\{ \text{tr}[\bar{\chi}(q; P) E^\alpha(q; P)] + \text{tr}[\bar{\chi}(q; P) F^\alpha(q; P)] \right\} \tag{8.82}$$

where f_π is given by Eq.(5.34).

The PCAC relation is

$$M_\pi^2 \tilde{f}_\pi + \mathcal{A}_\pi = -2[m_0 \mathcal{E}_\pi], \tag{8.83}$$

$$\mathcal{A}_\pi := \lim_{P \rightarrow \bar{P}_B} \int_q \text{tr}[\bar{\chi}(q; P) A^\alpha(q; P)], \tag{8.84}$$

$$[m_0 \mathcal{E}_\pi] := \lim_{P \rightarrow \bar{P}_B} i \int_q \frac{f(-q_-^2) + f(-q_+^2)}{2} \text{tr}[\bar{\chi}(q; P) m_0 \gamma_5 \frac{\lambda^\alpha}{2}], \tag{8.85}$$

$$q_+ := q - \frac{P}{2}, \quad q_- := q + \frac{P}{2}. \tag{8.86}$$

9 Numerical Results

9.1 Preliminary

In this section we show our numerical results for the solutions of SDE and BSE for pseudoscalar mesons, π and η in ExKG model. We use the regularization procedure discussed in Sec.6.

Our aim in this section is to verify the discussion in the previous sections numerically. Therefore we do not adjust the model parameters so that the experimental data are reproduced.

9.2 Numerical Result

In the numerical calculation, we set the off-shell soft limit momentum $P_E = (M_E, 0, 0, 0)$. The integral equation

$$\chi(q; M_E) = \int_k K(q, k; M_E^2) \chi(k; M_E) \quad (9.1)$$

has no solution for a real M_E because M_E is an Euclid mass. (Note that a solution should exist for $M_E^2 = -M_\pi^2 < 0$).

Instead we solve an eigenvalue equation

$$\lambda(M_E^2) \chi(q; M_E) = K(q, k; M_E^2) \chi(k; M_E) \quad (9.2)$$

and plot the $\lambda(M_E^2)$ as a function of M_E^2 and extrapolate it to the physical region $M_E^2 = -M_\pi^2 < 0$. In the present calculation, we extract only the maximum eigenvalue using the iteration procedure. The solution of Eq.(9.1) corresponds to $M_E^2 = -M_\pi^2$ where $\lambda(-M_\pi^2) = 1$.

Fig.9-1 shows this state.

We calculate $f_\pi(M_E^2)$, $\tilde{f}_\pi(M_E^2)$, $\mathcal{E}_\pi(M_E^2)$ and $\mathcal{R}_\pi(M_E^2)$ as functions of M_E^2 . Here $\mathcal{E}_\pi(M_E^2)$ and $\mathcal{R}_\pi(M_E^2)$ are defined by

$$\mathcal{E}_\pi(M_E^2) := i \int_q \frac{f(-q_-^2) + f(-q_+^2)}{2} \text{tr}[\bar{\chi}(q; P_B) \gamma_5 \frac{\lambda^\alpha}{2}] \quad (9.3)$$

and

$$\mathcal{R}_\pi(M_E^2) := \frac{-M_E^2 \tilde{f}_\pi(M_E^2)}{2m_q \mathcal{E}(M_E^2)} \quad (9.4)$$

respectively. $\mathcal{E}_\pi$ appears in the RHS of the PCAC relation Eqs.(3.40), (3.42), (5.29) and (5.47). $\mathcal{R}_\pi$ should be equal to 1 for a on-shell mass $M_E^2 = -M_\pi^2$ due to the PCAC relation. We fix $t_0 = -3.0$. The quantities m_0 , $\langle \bar{q}q \rangle$ and $\mathcal{E}_\pi$ depend on the renormalization point which equals to Λ_{UV} in this paper. Each physical quantity is extrapolated to the time like region $M_\pi^2 > 0$ independently. The validity of the extrapolation depends on the property of the functions. Table.9-1 shows the physical quantities for various m_q with $G_D = 0$. M_π grows as m_q increases. M_π^2 is almost proportional to m_q . This result indicates the GMOR mass formula holds almost for a finite quark mass. This is not trivial result. Since both $\tilde{f}_\pi$ and $\mathcal{E}_\pi$ grows as m_q increases, the ratio of M_π^2/m_q is almost constant due to the PCAC relation ($\mathcal{R}_\pi = 1$). Comparing f_π and $\tilde{f}_\pi$, one notices that $\tilde{f}_\pi$ is smaller than f_π near the chiral limit and $\tilde{f}_\pi$ depends on m_q more strongly than f_π .

$\mathcal{R}_\pi$ is close to unity as far as m_q is small. This implies that our approach in the previous sections works well. The small violation of the PCAC relation(5%—10%) seems mainly due to the extrapolation to the time like region. The extrapolation is more difficult for larger m_q . Indeed the error grows as m_q increases.

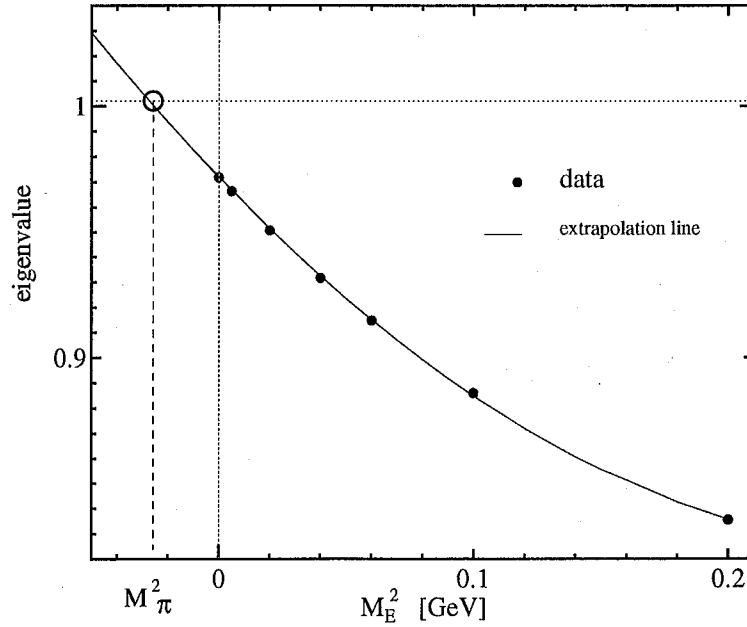


Fig.9-1: EIGEN VALUE

The masses of the pion and the η meson are shown in table 9-3 for various i_G which is defined by

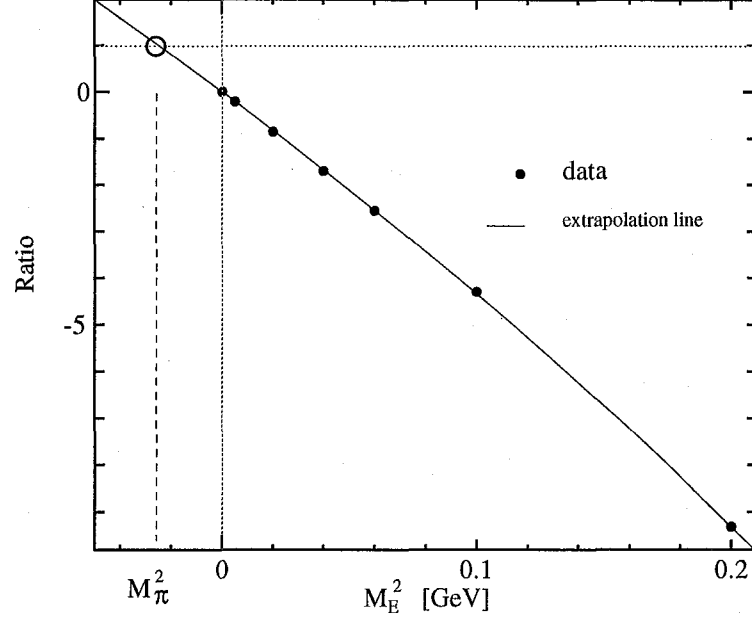


Fig.9-2: The ratio $\mathcal{R}_\pi(M_E^2)$

$$\Lambda_{\text{QCD}} = 400 \text{ [MeV]}, t_{\text{IF}} = -0.5, \Lambda_{\text{UV}} = 2.0 \text{ [GeV]}$$

$m_{q \text{ 2GeV}}$ [MeV]	M_π [MeV]	f_π [MeV]	$\tilde{f}_\pi$ [MeV]	$\langle \bar{q}q \rangle_{\text{2GeV}}$ [MeV ³]	$\mathcal{E}_{\pi \text{ 2GeV}}$ [MeV ²]	$\mathcal{R}_\pi$	M_π^2/m_q [GeV]
0.0	—	98.5	57.3	-0.753×10^7 $-(196)^3$	-1.33×10^5	—	—
5.0	157 $(25 \times 10^3)^{\frac{1}{2}}$	100	60.0	—	-1.38×10^5	1.04	5.0
20.0	320 $(102 \times 10^3)^{\frac{1}{2}}$	104	69.0	—	-1.51×10^5	1.06	5.1
40.0	467 $(218 \times 10^3)^{\frac{1}{2}}$	108	84.4	—	-1.61×10^5	1.11	5.4
70.0	662 $(438 \times 10^3)^{\frac{1}{2}}$	122	122	—	-2.09×10^5	1.14	6.2
100.0	830 $(690 \times 10^3)^{\frac{1}{2}}$	152	198	—	-2.28×10^5	1.22	8.3

Table 9-1: The physical values for some m_q .

$$\Lambda_{\text{QCD}} = 400 \text{ [MeV]}, t_{\text{IF}} = -0.5, \Lambda_{\text{UV}} = 2.0 \text{ [GeV]}$$

$$m_{s2\text{GeV}} = 150[\text{MeV}], G_D = -30 \times \Lambda_{\text{QCD}}^5[\text{MeV}^5]$$

$m_{q2\text{GeV}}$ [MeV]	M_π [MeV]	f_π [MeV]	$\tilde{f}_\pi$ [MeV]	$\langle \bar{q}q \rangle_{2\text{GeV}}$ [MeV ³]	$\mathcal{E}_{\pi2\text{GeV}}$ [MeV ²]	$\mathcal{R}_\pi$	M_π^2/m_q [GeV]
0.0	—	98.5	59.6	-0.764×10^7 $-(197)^3$	-1.33×10^5	—	—
5.0	153 $(23.5 \times 10^3)^{\frac{1}{2}}$	100	62.5	—	-1.38×10^5	1.03	4.7
20.0	312 $(97.4 \times 10^3)^{\frac{1}{2}}$	104	71.5	—	-1.50×10^5	1.05	4.9
40.0	467 $(206 \times 10^3)^{\frac{1}{2}}$	108	86.6	—	-1.59×10^5	1.10	5.0

Table 9-2: The physical values for some m_q .

$$i_G := -\frac{G_D}{\Lambda_{\text{QCD}}^5}. \quad (9.5)$$

We fix $\kappa = 6.25[\text{MeV}^{-2}]$. The extrapolation method to obtain table 9-3 is slightly different from that for table 9-1. For this reason the pion masses in these tables are slightly different.

The mass difference of the pion and the η meson grows as i_G increases. But both f_π and $\tilde{f}_\pi$ are almost independent of i_G .

$\Lambda_{UV}=2.0$ [GeV], radius $\times$ angle = 80×55
 $m_q = 5$ [MeV], $m_s = 150$ [MeV], $\Lambda_{QCD} = 400$ [MeV], $t_{IF} = -0.5$

i_G	M_π [MeV]	M_η/M_π	f_π [MeV]	$\tilde{f}_\pi$ [MeV]	$\mathcal{E}_\pi$ [MeV ²]
0	158	1.00	100	60.0	-1.38×10^5
5	158	1.08	100	60.4	-1.38×10^5
10	157.7	1.16	100	60.9	-1.38×10^5
20	156.5	1.31	100	61.7	-1.38×10^5
30	155.6	1.45	100	62.5	-1.38×10^5
50	153.7	1.71	100	64.2	-1.38×10^5
100	149.3	2.28	100	68.2	-1.38×10^5

Table 9-3: pion and η meson masses

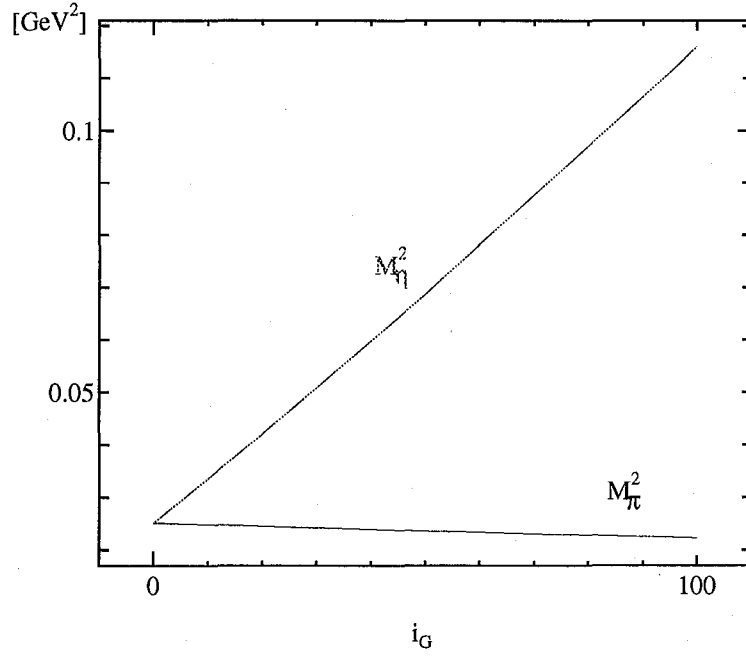


Fig.9-3: pion and η meson masses

10 Conclusion

In this thesis we study the DCSB. The picture of the non-dynamical SCSB is well-described by wine bottle potential. So the non-dynamical SCSB is clearly understood. However the DCSB is more complicated. At the DCSB the dynamical mass generation of a quark field and the appearance of the Nambu–Goldstone boson as a composite field occur simultaneously. The former is described by the SDE, and the latter is described by the BSE respectively. One must be careful with the consistency between the SDE and the BSE. One of the powerful ways to keep this consistency is the CJT action formulation. The CJT action is a generalization of the 1-particle irreducible effective action for a local field to a 2 point Green function. We apply this method to effective models. Our main works are in Secs.3, 5, 8 and 9.

In Sec.3 we examine the chiral properties of the effective models. Under the local chiral (axial) transformation, a conserved current (effective axial current) $\tilde{J}_{5\mu}^\alpha(x)$ introduced by

$$\Delta_5 S[\psi, \bar{\psi}] = \int d^4x \theta^\alpha(x) \left\{ \partial^\mu \tilde{J}_{5\mu}^\alpha[\psi, \bar{\psi}](x) - M^\alpha[\psi, \bar{\psi}](x) \right\} \quad (10.1)$$

is often different from the naive axial current $J_{5\mu}^\alpha(x)$ defined by

$$J_{5\mu}^\alpha(x) = \bar{\psi} \gamma_\mu \gamma_5 \frac{\lambda^\alpha}{2} \psi(x) \quad (10.2)$$

which is conserved in QCD. There are two reasons. One is the cut off procedure. Even if one uses the cut off procedure which respects the global chiral symmetry, the local chiral property is changed and the conserved current is modified. The other is the locally chiral variant interaction. We study the KG model as an example. So we found that the conserved current is modified from the naive axial current due to Higashijima–Miransky approximation of the running coupling constant. When the conserved current is modified, the Ward–Takahashi identity, the PCAC relation and the GMOR mass formula should be also modified. We found that introducing the modified decay constant $\tilde{f}_\pi$ as

$$\tilde{J}_{5\mu}^\alpha(x) = -\tilde{f}_\pi \partial_\mu \pi(x) + \dots \quad (10.3)$$

each relation can hold as the modified version. In Sec.5 we study the properties of the chiral symmetry in the framework of the CJT action. We found that the truncation of the CJT action still preserves the chiral relations of effective models discussed in Sec.3. Under the local chiral transformation of the propagator $S_F(x, y)$, the CJT action changes as

$$\Delta_5 \Gamma[S_F] = \int d^4x \theta^\alpha(x) \{ \partial^\mu \tilde{\mathcal{J}}_{5\mu}^\alpha[S_F] - \mathcal{M}^\alpha(x)[S_F] \}(x). \quad (10.4)$$

In this equation, we call $\tilde{\mathcal{J}}_{5\mu}^\alpha[S_F]$ a proxy of the effective axial current. Our formula of the decay constant is

$$\tilde{f}_\pi = \lim_{P \rightarrow P_B} \frac{iP^\mu}{P^2} \langle \mathbf{P} | T \psi_n(x) \bar{\psi}_m(y) | 0 \rangle \frac{\delta \tilde{\mathcal{J}}_{5\mu}^\alpha[S_F](0)}{\delta S_{Fnm}(x, y)}. \quad (10.5)$$

This is a main result of this thesis. This formula is useful for an effective model with a locally chiral variant interaction. In this case, although the conserved current is complicated form, our formula works well.

In Sec.8 is an application. We consider the ExKG model which is composed of the KG model and the KMT flavor mixing interaction. This model has a $SU(3)_L \times SU(3)_R \times U(1)_V$ chiral symmetry just like QCD and a possibility to study the η' meson problem to some extent perhaps. We verify the existence of 8 Nambu–Goldstone solutions due to the dynamical chiral symmetry breaking to $SU(3)_V \times U(1)_V$. Applying the methods in Sec.5, the PCAC relation is derived.

In Sec.9 we report the numerical result. Although the numerical result is not enough in accuracy yet, we think that our formulations are justified numerically. We hope our further numerical study and want to go to the next stage where more quantitative discussions are available.

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