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RANK TWO AMPLE VECTOR BUNDLES ON SOME SMOOTH RATIONAL SURFACES

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PREFACE

Ample vector bundles are playing important roles in the field of algebraic geometry. For example, we point out Mori's solution of the Hartshorne conjecture:

Theorem ([Mo]). *Let X be an n-dimensional smooth projective variety. If X has ample tangent bundle $\mathcal{T}_X$, then $X \simeq \mathbb{P}^n$.*

This theorem is a piece of evidence that ample vector bundles impose restrictions on their base spaces. Recently Lanteri and Maeda have obtained the following

Theorem ([LMa]). *Let X be an n-dimensional compact complex manifold and $\mathcal{E}$ an ample vector bundle of rank $r \geq 2$ on X with the property that*

($*$) *there exists a section $s \in H^0(X, \mathcal{E})$ whose zero locus $Z := (s)_0$ is a smooth submanifold of X of the expected dimension $n - r$.*

If $Z \simeq \mathbb{P}^{n-r} (n - r \geq 1)$, then $(X, \mathcal{E})$ is one of the following:

(P1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus r})$;
(P2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$;
(P3) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)})$, *where $\mathbb{Q}^n$ is a smooth hyperquadric in $\mathbb{P}^{n+1}$;*
(P4) $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ *for some vector bundle $\mathcal{F}$ of rank n on $\mathbb{P}^1$, and $\mathcal{E} = \oplus_{j=1}^{n-1}(H(\mathcal{F}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, where $H(\mathcal{F})$ stands for the tautological bundle and $\pi : X \to \mathbb{P}^1$ is the projection.*

If $Z \simeq \mathbb{Q}^{n-r} (n - r \geq 2)$, then $(X, \mathcal{E})$ is one of the following:

(Q1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(r-1)})$;
(Q2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus r})$;
(Q3) $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ *and $\mathcal{E} = \oplus_{j=1}^{n-2}(H(\mathcal{F}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, where $\mathcal{F}$ is the same as that in* (P4).

Here we remark that

(1) if $\mathcal{E}$ is spanned (by global sections), then $\mathcal{E}$ satisfies the condition ($*$);
(2) when $r = n$, we have $Z \simeq \mathbb{P}^{n-r}$ if and only if $c_n(\mathcal{E}) = 1$, and $Z \simeq \mathbb{Q}^{n-r}$ if and only if $c_n(\mathcal{E}) = 2$;
(3) when $r = n$ and $\mathcal{E}$ is spanned, Wiśniewski [W] has shown that $c_n(\mathcal{E}) = 1$ if and only if $(X, \mathcal{E}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus n})$, and Noma [N] has obtained a classification of $\mathcal{E}$ with $c_n(\mathcal{E}) = 2$.

In the thesis we consider ample vector bundles of rank two on Hirzebruch surfaces, and on Del Pezzo surfaces. Without spannedness, we classify these bundles by using their Chern numbers c_1^2 and c_2. As a corollary we obtain a classification of these bundles with small c_2.

Further we present two articles as the addenda to the thesis. Both of them treat ample line bundles of sectional genus three. The former gives partial classification of the line bundles; the latter gives almost complete classification of the line bundles under the condition that they are spanned.

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RANK TWO AMPLE VECTOR BUNDLES ON SOME SMOOTH RATIONAL SURFACES

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ABSTRACT. Several classification results for ample vector bundles of rank two on Hirzebruch surfaces, and on Del Pezzo surfaces, are obtained. In particular, we classify rank two ample vector bundles with c_2 less than seven on Hirzebruch surfaces, and with c_2 less than four on Del Pezzo surfaces.

Introduction.

In recent years ample and spanned vector bundles with small Chern numbers have been studied by several authors. Among them, Lanteri-Sommese [12] proved that $(S, \mathcal{E}) \simeq (\mathbb{P}^2, \mathcal{O}(1)^{\oplus 2})$ when S is a normal surface and $\mathcal{E}$ is an ample and spanned rank two vector bundle with $c_2(\mathcal{E}) = 1$ on S. Ballico-Lanteri [2] and Noma [15] classified ample and spanned rank two vector bundles with $c_2 = 2$ on smooth surfaces. Noma [16] extended the classification to the case of normal Gorenstein surfaces.

Motivated by the results above, we attempt to classify ample vector bundles with small c_2 on surfaces without spannedness. As the first step, we consider rank two ample vector bundles on Hirzebruch surfaces, and on Del Pezzo surfaces, in the present paper. We obtain classification results for rank two ample vector bundles with c_2 less than seven on Hirzebruch surfaces, and with c_2 less than four on Del Pezzo surfaces. Note that we do not treat all smooth rational surfaces because of technical difficulty.

This paper is organized as follows. In Sec. 1 we collect some preliminary results. In Sec. 2 we study ample vector bundles $\mathcal{E}$ of rank two on e-th Hirzebruch surfaces. We see that $c_2(\mathcal{E}) \geq e+2$, and classification results for $\mathcal{E}$ with $e+2 \leq c_2(\mathcal{E}) \leq e+6$ are given. As a corollary, we obtain a classification of $\mathcal{E}$ with $c_2(\mathcal{E}) \leq 6$. In Sec. 3 we study ample vector bundles $\mathcal{E}$ of rank two on Del Pezzo surfaces of degree $d \leq 7$. We see that $c_2(\mathcal{E}) \geq d$, and classification results for $\mathcal{E}$ with $c_2(\mathcal{E}) = d,\ d+1$ are given. A partial classification result for $\mathcal{E}$ with $c_2(\mathcal{E}) = d+2$ is also given. As a corollary, we obtain a classification of $\mathcal{E}$ with $c_2(\mathcal{E}) \leq 3$. In Sec. 4 we study ample vector bundles $\mathcal{E}$ of rank two on $\mathbb{P}^2$. We see that

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$c_2(\mathcal{E}) \geq c_1(\mathcal{E}) - 1$, and classification results for $\mathcal{E}$ with $c_1(\mathcal{E}) - 1 \leq c_2(\mathcal{E}) \leq c_1(\mathcal{E}) + 2$ are given. Then, using the classification of $\mathcal{E}$ with $c_1(\mathcal{E}) \leq 3$, we obtain a classification of $\mathcal{E}$ with $c_2(\mathcal{E}) \leq 6$.

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Notation and Terminology.

Basically we follow the notation and terminology of Hartshorne's text [8]. We work over the complex number field. Vector bundles are identified with the locally free sheaves of their sections, and line bundles are also identified with the linear equivalence classes of Cartier divisors. The tensor products of line bundles are usually denoted additively, while we use multiplicative notation for intersection products. The linear equivalence classes are often denoted by [ ]. We use = (resp. $\equiv$) for linear (resp. numerical) equivalence.

A line bundle L on a variety X is called nef if $LC \geq 0$ for every irreducible curve C in X. For a morphism $f : Y \to X$, we often denote f^*L by L_Y, or sometimes by L, when there is no fear of confusion. For a vector bundle $\mathcal{E}$ on X, we denote by $\mathbb{P}(\mathcal{E})$ the associated projective space bundle and by $H(\mathcal{E})$ the tautological line bundle on $\mathbb{P}(\mathcal{E})$ in the sense of [7]. We say that $\mathcal{E}$ is ample if $H(\mathcal{E})$ is ample. The determinant $\det \mathcal{E}$ of $\mathcal{E}$ and the first Chern class $c_1(\mathcal{E})$ of $\mathcal{E}$ are used interchangeably. The canonical bundle of a smooth surface S is denoted by K_S. For an ample line bundle A on S, the sectional genus $g(S, A)$ (or $g(A)$ for short) of the pair (S, A) is given by the formula $2g(S, A) - 2 = (K_S + A)A$. For a closed subscheme Z of S with the ideal sheaf $\mathcal{I}_Z$, we set $\deg Z := \text{length}(\mathcal{O}_S/\mathcal{I}_Z)$.

1. Preliminaries.

In this section we collect some preliminary results that will be used frequently.

Theorem 1.1 (Lanteri-Palleschi [11, Remark 1.3]). *Let A be an ample line bundle on a smooth surface S. If $K_S + A$ is not nef, then (S, A) is one of the following:*

(1) $(S, A) \simeq (\mathbb{P}^2, \mathcal{O}(1))$ *or* $(\mathbb{P}^2, \mathcal{O}(2))$;
(2) *S is a $\mathbb{P}^1$-bundle over a smooth curve and $A_F = \mathcal{O}_{\mathbb{P}^1}(1)$ for every fiber F of the ruling.*

For the proof of this theorem, Mori's cone theorem [14, Theorem (1.4)] and the classification theorem of extremal rational curves [14, Theorem (2.1)] are essential.

Using these two theorems, we obtain a generalization of (1.1).

Proposition 1.2. *Let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on a smooth surface S.*

If $rK_S + c_1(\mathcal{E})$ is not nef, then we have one of the following:

(1) *$S \simeq \mathbb{P}^2$ and $c_1(\mathcal{E}) = \mathcal{O}(a)$ $(r \leq a < 3r)$;*
(2) *S is a $\mathbb{P}^1$-bundle over a smooth curve and $r \leq c_1(\mathcal{E}) \cdot F < 2r$.*

Proof. Suppose that $rK_S + c_1(\mathcal{E})$ is not nef. By the cone theorem, there exists an extremal rational curve C on S such that $(rK_S + c_1(\mathcal{E})) \cdot C < 0$. By the classification theorem of extremal rational curves, we have one of the following:

(i) $S \simeq \mathbb{P}^2$ and C is a line;
(ii) S is a $\mathbb{P}^1$-bundle over a smooth curve and C is one of its fibers;
(iii) C is a (-1)-curve on S.

Then the case (iii) is excluded and the assertion follows by the next lemma. □

Lemma 1.3 (see, e.g., [6, (1.3)]). *Let S and $\mathcal{E}$ be as above. Then $c_1(\mathcal{E}) \cdot C \geq r$ for every rational curve C on S.*

Theorem 1.4 (Kleiman [9, Theorem 3]). *Let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on a smooth surface. Then we have $0 < c_2(\mathcal{E}) < c_1^2(\mathcal{E})$.*

Theorem 1.5 (Ballico [1, Theorem 0.1]). *Let $\mathcal{E}$ be an ample vector bundle of rank two on a smooth surface. Then we have $c_1^2(\mathcal{E}) \leq (c_2(\mathcal{E}) + 1)^2$.*

Remark 1.6. In fact, Ballico obtained the inequality in a general setting, though (1.5) is enough for our use.

The following theorem is essential for the proof of (1.5).

Theorem 1.7 (Bogomolov [3], see also [17, Theorem 1]). *Let $\mathcal{E}$ be a vector bundle of rank two on a smooth surface S. Then $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$ if and only if there exists an exact sequence*

$$0 \to L \to \mathcal{E} \to \mathcal{I}_Z \otimes M \to 0,$$

with L and M line bundles on S, and Z a zero-dimensional subscheme of S with sheaf of ideals $\mathcal{I}_Z$, such that:

(i) *$(L - M)^2 > 4 \deg Z$;*
(ii) *$(L - M) \cdot A > 0$ for every ample line bundle A on S.*

Remark 1.8. If $\mathcal{E}$ is ample in (1.7), then we see that M is ample in (1.7).

Indeed, the assertion is clear in case $Z = \emptyset$. In case $Z \neq \emptyset$, let $\pi : S' \to S$ be the blowing-up of S with respect to $\mathcal{I}_Z$. We denote by E the exceptional divisor corresponding to the inverse image ideal sheaf $\pi^{-1}\mathcal{I}_Z \cdot \mathcal{O}_{S'}$. Then we have $\deg Z = -E^2$ and an exact sequence

$$0 \to [\pi^* L + E] \to \pi^* \mathcal{E} \to [\pi^* M - E] \to 0$$

that is induced by the exact sequence in (1.7). For each irreducible curve C on S, we denote by C' the strict transform of C under π. Then we have

$$M \cdot C = (\pi^*M) \cdot C' \geq (\pi^*M - E) \cdot C' > 0$$

since $\pi^*M - E$ is a quotient bundle of $\pi^*\mathcal{E}$. We have also $M^2 > \deg Z > 0$ since $(\pi^*M - E)^2 > 0$. Thus we conclude that M is ample.

Remark 1.9. Using (1.8), we consider the equality condition in (1.5).

Suppose that $c_1^2(\mathcal{E}) = (c_2(\mathcal{E}) + 1)^2$ in (1.5). If $c_1^2(\mathcal{E}) \leq 4c_2(\mathcal{E})$, then we get $c_2(\mathcal{E}) = 1$ and $c_1^2(\mathcal{E}) = 4$. If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, then we get an exact sequence

$$0 \to L \to \mathcal{E} \to \mathcal{I}_Z \otimes M \to 0$$

as in (1.7). Note that $c_1(\mathcal{E}) = L + M$ and $c_2(\mathcal{E}) = LM + \deg Z$. We have

(1.9.1)
$$\begin{aligned} 0 &= (c_2(\mathcal{E}) + 1)^2 - c_1^2(\mathcal{E}) \\ &= (LM)^2 - L^2M^2 + (L^2 - (\deg Z + 1))(M^2 - (\deg Z + 1)) + (L + M)^2 \cdot (\deg Z). \end{aligned}$$

Since $\mathcal{E}$ is ample, $c_1(\mathcal{E})$ is also ample and then $0 < (L - M) \cdot c_1(\mathcal{E}) = L^2 - M^2$. Since M is ample by (1.8), we get $L^2M^2 \leq (LM)^2$ from the Hodge index theorem. We get also $M^2 > \deg Z$ from the argument in (1.8). Thus from (1.9.1) we infer that

$$(LM)^2 = L^2M^2, \ M^2 = \deg Z + 1, \text{ and } \deg Z = 0.$$

It follows that $M^2 = 1$ and then $L \equiv tM$ for some integer $t \geq 2$.

On Hirzebruch surfaces and on Del Pezzo surfaces, we get the eqality condition precisely (see (2.12), (3.15), and (4.8)).

2. On Hirzebruch surfaces.

Definition 2.1. A smooth surface S is said to be an *e-th Hirzebruch surface* if $S \simeq \Sigma_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ for some non-negative integer e.

In this section we denote by S an e-th Hirzebruch surface, by H the tautological line bundle $H(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ on S, and by F a fiber of the ruling $\rho : S \to \mathbb{P}^1$.

Let $\mathcal{E}$ be an ample vector bundle of rank two on S. Since $\operatorname{Pic} S \simeq \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot F$, we set $c_1(\mathcal{E}) = aH + bF$ for some integers a and b. We have $a = c_1(\mathcal{E}) \cdot F \geq 2$ and $b - ae = c_1(\mathcal{E}) \cdot H \geq 2$ because of (1.3).

First we consider the relation between $c_1^2(\mathcal{E})$ and e.

Lemma 2.2. *Let S and $\mathcal{E}$ be as above. Then $-K_S \cdot c_1(\mathcal{E}) \geq 2e + 8$.*

Proof. Since $K_S = -2H - (e+2)F$, we have

$$\begin{aligned} -K_S \cdot c_1(\mathcal{E}) &= (2H + (e+2)F)(aH + bF) \\ &= 2(b - ae) + (e+2)a \\ &\geq 2 \cdot 2 + (e+2) \cdot 2 \\ &= 2e + 8. \quad \square \end{aligned}$$

Proposition 2.3. *Let S and $\mathcal{E}$ be as above. If $2K_S + c_1(\mathcal{E})$ is nef, then $c_1^2(\mathcal{E}) \geq 8e + 24$.*

Proof. Suppose that $2K_S + c_1(\mathcal{E})$ is nef. Since $\mathcal{E}$ is ample, $c_1(\mathcal{E})$ is also ample and hence $(2K_S + c_1(\mathcal{E}))c_1(\mathcal{E}) \geq 0$. Since $2K_S + c_1(\mathcal{E}) = (a-4)H + (b - 2e - 4)F$, we have $a \geq 4$ and then

$$-K_S \cdot c_1(\mathcal{E}) = 2(b - ae) + (e+2)a \geq 2 \cdot 2 + (e+2) \cdot 4 = 4e + 12.$$

Thus we get $c_1^2(\mathcal{E}) \geq -2K_S \cdot c_1(\mathcal{E}) \geq 8e + 24$. $\square$

Proposition 2.4. *Let S and $\mathcal{E}$ be as above. If $2K_S + c_1(\mathcal{E})$ is not nef, then $c_1^2(\mathcal{E}) \geq 4e + 8$.*

Proof. Suppose that $2K_S + c_1(\mathcal{E})$ is not nef. From (1.2) we obtain $c_1(\mathcal{E}) \cdot F = 2$ or 3.

In case $c_1(\mathcal{E}) \cdot F = 2$, we have $\mathcal{E}|_F \simeq \mathcal{O}_F(1)^{\oplus 2}$ since $\mathcal{E}$ is ample. Then $(\mathcal{E} \otimes [-H])_F \simeq \mathcal{O}_F^{\oplus 2}$. Hence $\mathcal{G} := \rho_*(\mathcal{E} \otimes [-H])$ is a locally free sheaf of rank two on $\mathbb{P}^1$ and $\rho^*\mathcal{G} \simeq \mathcal{E} \otimes [-H]$. We can set $\mathcal{G} = \mathcal{O}(t_1) \oplus \mathcal{O}(t_2)$ for some integers t_1 and t_2 ($t_1 \leq t_2$). Then $\mathcal{E} \simeq [H + t_1 F] \oplus [H + t_2 F]$. Note that $t_1 > e$ and $t_2 > e$ since $\mathcal{E}$ is ample. We have

$$c_1^2(\mathcal{E}) = (2H + bF)^2 = -4e + 4b \geq -4e + 4(2e+2) = 4e + 8.$$

In case $c_1(\mathcal{E}) \cdot F = 3$, we have $(\mathcal{E} \otimes [-2H])_F \simeq \mathcal{O}_F \oplus \mathcal{O}_F(-1)$. Hence $\mathcal{G} := \rho_*(\mathcal{E} \otimes [-2H])$ is an invertible sheaf on $\mathbb{P}^1$ and the morphism $\rho^*\mathcal{G} \to \mathcal{E} \otimes [-2H]$ is injective. We have

$$\operatorname{Coker}(\rho^*\mathcal{G} \to \mathcal{E} \otimes [-2H]) = \det(\mathcal{E} \otimes [-2H]) - \rho^*\mathcal{G} = -H + (b-t)F,$$

where $t := \deg \mathcal{G}$. Then we get an exact sequence

$$0 \to [2H + tF] \to \mathcal{E} \to [H + (b-t)F] \to 0.$$

Note that $b - t > e$ since $\mathcal{E}$ is ample. We have

$$c_1^2(\mathcal{E}) = (3H + bF)^2 = -9e + 6b \geq -9e + 6(3e+2) = 9e + 12 > 4e + 8. \quad \square$$

Theorem 2.5. *Let S be an e-th Hirzebruch surface and $\mathcal{E}$ an ample vector bundle of rank two on S. Then $c_1^2(\mathcal{E}) \geq 4e+8$, and equality holds if and only if $\mathcal{E} \simeq [H+(e+1)F]^{\oplus 2}$, where H is the tautological line bundle on S and F is a fiber of the ruling.*

Furthermore, if $4e+9 \leq c_1^2(\mathcal{E}) \leq 8e+12$, then $\mathcal{E} \simeq [H+t_1F] \oplus [H+t_2F]$, where t_1, $t_2 \in \mathbb{Z}$, $e+1 \leq t_1 \leq t_2$, and $t_1+t_2 \leq 3e+3$.

Proof. From (2.3) and (2.4) we obtain $c_1^2(\mathcal{E}) \geq 4e+8$ immediately. Suppose that $c_1^2(\mathcal{E}) \leq 8e+12$. Then $2K_S + c_1(\mathcal{E})$ is not nef by (2.3). In view of the argument in (2.4), there are the following two possibilities:

(i) $c_1(\mathcal{E}) \cdot F = 2$, $\mathcal{E} \simeq [H+t_1F] \oplus [H+t_2F]$ $(t_1 \leq t_2)$, $t_1 > e$, $t_2 > e$, and $c_1^2(\mathcal{E}) = -4e+4(t_1+t_2)$;

(ii) $c_1(\mathcal{E}) \cdot F = 3$, $0 \to [2H+tF] \to \mathcal{E} \to [H+(b-t)F] \to 0$ is exact, $b-3e \geq 2$, $b-t > e$, and $c_1^2(\mathcal{E}) = -9e+6b$.

In the case (i) we see that $t_1+t_2 \leq 3e+3$ since $c_1^2(\mathcal{E}) \leq 8e+12$.

In the case (ii) we see that $e=0$ since $9e+12 \leq c_1^2(\mathcal{E}) \leq 8e+12$. Then we have $b=2$ and $c_1(\mathcal{E}) = 3H+2F$. Note that the condition $c_1(\mathcal{E}) = 3H+2F$ is equivalent to the condition $c_1(\mathcal{E}) = 2H+3F$. Hence we infer that $\mathcal{E} \simeq [H+F] \oplus [H+2F]$ from the argument above. □

Using this theorem, we can classify rank two ample vector bundles with small c_1^2 on Hirzebruch surfaces.

Corollary 2.6. *Let S be an e-th Hirzebruch surface. Then rank two ample vector bundles $\mathcal{E}$ with $c_1^2(\mathcal{E}) \leq 16$ on S are the following:*

(i) $c_1^2(\mathcal{E}) = 8$, $e=0$, *and* $\mathcal{E} \simeq [H+F]^{\oplus 2}$;

(ii) $c_1^2(\mathcal{E}) = 12$, $e=0$, *and* $\mathcal{E} \simeq [H+F] \oplus [H+2F]$;

(iii) $c_1^2(\mathcal{E}) = 12$, $e=1$, *and* $\mathcal{E} \simeq [H+2F]^{\oplus 2}$;

(iv) $c_1^2(\mathcal{E}) = 16$, $e=0$, *and* $\mathcal{E} \simeq [H+F] \oplus [H+3F]$;

(v) $c_1^2(\mathcal{E}) = 16$, $e=0$, *and* $\mathcal{E} \simeq [H+2F]^{\oplus 2}$;

(vi) $c_1^2(\mathcal{E}) = 16$, $e=1$, *and* $\mathcal{E} \simeq [H+2F] \oplus [H+3F]$;

(vii) $c_1^2(\mathcal{E}) = 16$, $e=2$, *and* $\mathcal{E} \simeq [H+3F]^{\oplus 2}$.

Proof. Suppose that $c_1^2(\mathcal{E}) \leq 16$. From (2.5) we get $e \leq 2$; moreover, $\mathcal{E}$ is a vector bundle of the type (vii) in case $e=2$, and $\mathcal{E}$ is of the type (iii) or (vi) in case $e=1$.

In case $e=0$, $\mathcal{E}$ is of the type (i) or (ii) if $c_1^2(\mathcal{E}) \leq 12$. In case $e=0$ and $13 \leq c_1^2(\mathcal{E}) \leq 16$, $2K_S + c_1(\mathcal{E})$ is not nef by (2.3). From the proof of (2.4), we infer that $\mathcal{E}$ is of the type (iv) or (v). □

Next we consider the relation between $c_2(\mathcal{E})$ and e.

Proposition 2.7. *Let S be an e-th Hirzebruch surface and $\mathcal{E}$ an ample vector bundle of rank two on S. If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, then $c_2(\mathcal{E}) \geq e+4$.*

Proof. Suppose that $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. From (1.7) we obtain an exact sequence

$$0 \to L \to \mathcal{E} \to \mathcal{I}_Z \otimes M \to 0,$$

where $L, M \in \operatorname{Pic} S$ and $\mathcal{I}_Z$ is the ideal sheaf of a zero-dimensional subscheme Z of S. Note that M is ample (see (1.8)).

(2.7.1) If $K_S + M$ is nef, we have $(-K_S) \cdot M \le M^2 < LM$ since $(K_S + M)M \ge 0$ and $(L - M)M > 0$. We have also $(-K_S) \cdot L \le LM$ since $(K_S + M)L = (K_S + M)(L - M) + (K_S + M)M \ge 0$. Hence we obtain

$$2e + 8 \le (-K_S) \cdot c_1(\mathcal{E}) = (-K_S) \cdot L + (-K_S) \cdot M < 2LM \le 2c_2(\mathcal{E})$$

by (2.2). It follows that $c_2(\mathcal{E}) > e + 4$.

(2.7.2) If $K_S + M$ is not nef, we infer that $MF = 1$ from (1.1) since M is ample. Then we can set $M = H + tF$ and $L = (a - 1)H + (b - t)F$ for integers a, b, and t. Note that $t > e$ since M is ample. We have

$$LM - (e + 4) = ((a - 1)H + (b - t)F)(H + tF) - (e + 4) = (b - ae - 2) + (a - 2)t - 2.$$

Hence we see that $LM \ge e + 4$ if $a \ge 3$ and $t \ge 2$.

In case $a = 2$, we have $L - M = (b - 2t)F$ and hence $b \ge 2t + 1 \ge 2e + 3$. Thus we see that $LM \ge e + 4$ unless $b = 2e + 3$. If $b = 2e + 3$, then $t = e + 1$ and we find that $c_2(\mathcal{E}) = e + 3 + \deg Z$ and $c_1^2(\mathcal{E}) = 4e + 12$. This is a contradiction to the assumption $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$.

In case $t = 1$, we have $e = 0$, and then $LM - (e + 4) = a + b - 6$. This is non-negative because

$$0 < c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) = 2(a - 2)(b - 2) - 4 \deg Z.$$

As a result, we have $LM \ge e + 4$ if $K_S + M$ is not nef. It follows that $c_2(\mathcal{E}) \ge e + 4$. □

Theorem 2.8. *Let S be an e-th Hirzebruch surface and $\mathcal{E}$ an ample vector bundle of rank two on S. Then $c_2(\mathcal{E}) \ge e + 2$, and equality holds if and only if $\mathcal{E} \simeq [H + (e + 1)F]^{\oplus 2}$, where H is the tautological line bundle on S, and F is a fiber of the ruling.*

Furthermore, $c_2(\mathcal{E}) = e + 3$ if and only if $\mathcal{E} \simeq [H + (e + 1)F] \oplus [H + (e + 2)F]$.

Proof. Assume that $c_2(\mathcal{E}) \le e + 3$. From (2.7) we obtain $c_1^2(\mathcal{E}) \le 4c_2(\mathcal{E}) \le 4e + 12$, and hence $2K_S + c_1(\mathcal{E})$ is not nef by (2.3). In view of the argument in (2.4), there are the following two possibilities:

(i) $c_1(\mathcal{E}) \cdot F = 2$, $\mathcal{E} \simeq [H + t_1F] \oplus [H + t_2F]$ $(t_1 \le t_2)$, $t_1 > e$, $t_2 > e$, and $c_2(\mathcal{E}) = t_1 + t_2 - e$;

(ii) $c_1(\mathcal{E}) \cdot F = 3$, $0 \to [2H + tF] \to \mathcal{E} \to [H + (b - t)F] \to 0$ is exact, $b - 3e \ge 2$, $b - t > e$, and $c_2(\mathcal{E}) = 2b - t - 2e$.

In the case (i) we see that $(t_1, t_2) = (e+1, e+1)$ or $(e+1, e+2)$. Hence we have either $\mathcal{E} \simeq [H + (e+1)F]^{\oplus 2}$ and $c_2(\mathcal{E}) = e+2$, or $\mathcal{E} \simeq [H + (e+1)F] \oplus [H + (e+2)F]$ and $c_2(\mathcal{E}) = e+3$. In the case (ii) we see that

$$e + 3 \geq c_2(\mathcal{E}) = 2b - t - 2e > b - e \geq 2e + 2,$$

and hence $e = 0$, $c_2(\mathcal{E}) = 3$, $b = 2$, and $t = 1$. Then we get an exact sequence

$$0 \to [2H + F] \to \mathcal{E} \to [H + F] \to 0.$$

Since $\mathrm{Ext}^1([H+F], [2H+F]) \simeq H^1(S, H) = 0$, we have $\mathcal{E} \simeq [2H+F] \oplus [H+F]$. Hence we obtain that $\mathcal{E} \simeq [H+F] \oplus [H+2F]$. □

Theorem 2.9. *Let S, $\mathcal{E}$, H, and F be as in* (2.8).

(I) $c_2(\mathcal{E}) = e+4$ *if and only if $\mathcal{E}$ is one of the following:*

- (I-i) $\mathcal{E} \simeq [H + (e+1)F] \oplus [H + (e+3)F]$ *or* $[H + (e+2)F]^{\oplus 2}$;
- (I-ii) $e = 0$ *and* $\mathcal{E} \simeq [H + F] \oplus [2H + 2F]$;
- (I-iii) $e = 1$ *and* $\mathcal{E} \simeq [H + 2F] \oplus [2H + 3F]$.

(II) $c_2(\mathcal{E}) = e+5$ *if and only if $\mathcal{E}$ is one of the following:*

- (II-i) $\mathcal{E} \simeq [H + (e+1)F] \oplus [H + (e+4)F]$ *or* $[H + (e+2)F] \oplus [H + (e+3)F]$;
- (II-ii) $e = 0$ *and* $\mathcal{E} \simeq [H + F] \oplus [2H + 3F]$;
- (II-iii) $e = 0$ *and* $\mathcal{E} \simeq [H + 2F] \oplus [2H + F]$;
- (II-iv) $e = 1$ *and* $\mathcal{E} \simeq [H + 2F] \oplus [2H + 4F]$;
- (II-v) $e = 1$ *and there exists a non-split exact sequence* $0 \to [2H + 2F] \to \mathcal{E} \to [H + 3F] \to 0$;
- (II-vi) $e = 2$ *and* $\mathcal{E} \simeq [H + 3F] \oplus [2H + 5F]$.

(III) $c_2(\mathcal{E}) = e+6$ *if and only if $\mathcal{E}$ is one of the following:*

- (III-i) $\mathcal{E} \simeq [H + (e+1)F] \oplus [H + (e+5)F]$ *or* $[H + (e+2)F] \oplus [H + (e+4)F]$ *or* $[H + (e+3)F]^{\oplus 2}$;
- (III-ii) $e = 0$ *and* $\mathcal{E} \simeq [H + F] \oplus [2H + 4F]$;
- (III-iii) $e = 0$ *and* $\mathcal{E} \simeq [H + 2F] \oplus [2H + 2F]$;
- (III-iv) $e = 0$ *and* $\mathcal{E} \simeq [H + F] \oplus [3H + 3F]$;
- (III-v) $e = 0$ *and there exists a non-split exact sequence* $0 \to [2H] \to \mathcal{E} \to [H + 3F] \to 0$;
- (III-vi) $e = 1$ *and* $\mathcal{E} \simeq [H + 2F] \oplus [2H + 5F]$;
- (III-vii) $e = 1$ *and* $\mathcal{E} \simeq [H + 3F] \oplus [2H + 3F]$;
- (III-viii) $e = 1$ *and* $\mathcal{E} \simeq [H + 2F] \oplus [3H + 4F]$;
- (III-ix) $e = 1$ *and there exists a non-split exact sequence* $0 \to [2H + F] \to \mathcal{E} \to [H + 4F] \to 0$;
- (III-x) $e = 2$ *and* $\mathcal{E} \simeq [H + 3F] \oplus [2H + 6F]$;
- (III-xi) $e = 2$ *and there exists a non-split exact sequence* $0 \to [2H + 4F] \to \mathcal{E} \to [H + 4F] \to 0$;
- (III-xii) $e = 3$ *and* $\mathcal{E} \simeq [H + 4F] \oplus [2H + 7F]$.

Proof. Suppose that $e+4 \leq c_2(\mathcal{E}) \leq e+6$. The proof is divided into two parts.

(2.9.1) If $2K_S + c_1(\mathcal{E})$ is nef, then we get

$$c_1^2(\mathcal{E}) \geq -2K_S \cdot c_1(\mathcal{E}) \geq 8e+24 \geq 4c_2(\mathcal{E})$$

from the proof of (2.3). In case $c_1^2(\mathcal{E}) = 4c_2(\mathcal{E})$, we have $e=0$ and $c_1^2(\mathcal{E}) = -2K_S \cdot c_1(\mathcal{E}) = 24$. Then we get $(2K_S + c_1(\mathcal{E}))c_1(\mathcal{E}) = 0$, and hence $c_1(\mathcal{E}) = -2K_S$. It follows that $c_1^2(\mathcal{E}) = 4K_S^2 = 32$, which is a contradiction. Thus we obtain $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. We argue as in the proof of (2.7).

First we get the same exact sequence

$$0 \to L \to \mathcal{E} \to \mathcal{I}_Z \otimes M \to 0$$

as that in (2.7). If $K_S + M$ is nef, then we have

$$4e+12 \leq -K_S \cdot c_1(\mathcal{E}) < 2c_2(\mathcal{E}) \leq 2e+12,$$

a contradiction. Hence $K_S + M$ is not nef. Then we can set $M = H + tF$ and $L = (a-1)H + (b-t)F$ $(a, b, t \in \mathbb{Z})$. We have $t > e$ and

$$0 \leq LM - (e+4) = (b - ae - 2) + (a-2)t - 2.$$

Note that $a \geq 4$, otherwise we have $(2K_S + c_1(\mathcal{E})) \cdot F < 0$.

In case $LM = e+4$, we have $a=4$, $t=1$, $e=0$, and $b=2$. Then $c_1^2(\mathcal{E}) = (4H+2F)^2 = 16$, a contradiction to $c_1^2(\mathcal{E}) \geq 8e+24 = 24$. In case $LM = e+5$, we have $t=1$, $e=0$, and $(a,b) = (4,3)$ or $(5,2)$. Then $c_1^2(\mathcal{E}) = 24$ or 20, a contradiction to $c_1^2(\mathcal{E}) \geq -2K_S \cdot c_1(\mathcal{E}) = 28$.

Thus we obtain $LM = c_2(\mathcal{E}) = e+6$ and $\deg Z = 0$ since $c_2(\mathcal{E}) \leq e+6$. Then we have $t \leq 2$. If $t=1$, then $e=0$ and $a=b=4$ since $2K_S + c_1(\mathcal{E})$ is nef. Hence we get an exact sequence

$$0 \to [3H+3F] \to \mathcal{E} \to [H+F] \to 0.$$

Since $\mathrm{Ext}^1([H+F],[3H+3F]) \simeq H^1(S, 2H+2F) = 0$, we have $\mathcal{E} \simeq [H+F] \oplus [3H+3F]$. This is the case (III-iv). If $t=2$, then $a=4$, $b=4e+2$, and $e \leq 1$. We find that $e=1$ from $8e+24 \leq c_1^2(\mathcal{E}) = (4H+(4e+2)F)^2 = 16e+16$. Hence we get an exact sequence

$$0 \to [3H+4F] \to \mathcal{E} \to [H+2F] \to 0.$$

Since $\mathrm{Ext}^1([H+2F],[3H+4F]) \simeq H^1(S, 2H+2F) = 0$, we have $\mathcal{E} \simeq [H+2F] \oplus [3H+4F]$. This is the case (III-viii).

(2.9.2) If $2K_S + c_1(\mathcal{E})$ is not nef, we argue as in the proof of (2.8). We have $c_1(\mathcal{E}) \cdot F = 2$ or 3. If $c_1(\mathcal{E}) \cdot F = 2$, then we obtain $\mathcal{E} \simeq [H+t_1F] \oplus [H+t_2F]$, where $t_1, t_2 \in \mathbb{Z}$,

$e+1 \leq t_1 \leq t_2$, and $2e+4 \leq t_1+t_2 \leq 2e+6$ since $e+4 \leq c_2(\mathcal{E}) \leq e+6$. These are the cases (I-i), (II-i), and (III-i).

If $c_1(\mathcal{E}) \cdot F = 3$, then there exists an exact sequence

$$0 \to [2H+tF] \to \mathcal{E} \to [H+(b-t)F] \to 0$$

with the property that

$$e+6 \geq c_2(\mathcal{E}) = 2b-t-2e > b-e \geq 2e+2.$$

Hence we have $e \leq 3$. In case $e=3$, we have $c_2(\mathcal{E})=9$, $b=11$, and $t=7$. Then we get an exact sequence

$$0 \to [2H+7F] \to \mathcal{E} \to [H+4F] \to 0.$$

Since $\mathrm{Ext}^1([H+4F],[2H+7F]) \simeq H^1(S, H+3F) = 0$, we have $\mathcal{E} \simeq [H+4F] \oplus [2H+7F]$. This is the case (III-xii).

In case $e=2$, we have $b=8$ or 9. If $b=9$, then $c_2(\mathcal{E})=8$ and $t=6$. Hence we get an exact sequence

$$0 \to [2H+6F] \to \mathcal{E} \to [H+3F] \to 0.$$

Since $\mathrm{Ext}^1([H+3F],[2H+6F]) \simeq H^1(S, H+3F) = 0$, we have $\mathcal{E} \simeq [H+3F] \oplus [2H+6F]$. This is the case (III-x). If $b=8$, then $(c_2(\mathcal{E}), t) = (7,5)$ or $(8,4)$. In the former case we obtain $\mathcal{E} \simeq [H+3F] \oplus [2H+5F]$, which is the case (II-vi). In the latter case we obtain an exact sequence

$$0 \to [2H+4F] \to \mathcal{E} \to [H+4F] \to 0,$$

which is non-split because $2H+4F$ is not ample. This is the case (III-xi).

In case $e=1$, we have $5 \leq b \leq 7$. If $b=7$, then $c_2(\mathcal{E})=7$ and $t=5$. We obtain $\mathcal{E} \simeq [H+2F] \oplus [2H+5F]$, which is the case (III-vi). If $b=6$, then $(c_2(\mathcal{E}), t) = (6,4)$ or $(7,3)$. In the former case we obtain $\mathcal{E} \simeq [H+2F] \oplus [2H+4F]$, which is the case (II-iv). In the latter case we obtain $\mathcal{E} \simeq [H+3F] \oplus [2H+3F]$, which is the case (III-vii). If $b=5$, then $(c_2(\mathcal{E}), t) = (5,3)$, $(6,2)$, or $(7,1)$. In the first case we obtain $\mathcal{E} \simeq [H+2F] \oplus [2H+3F]$, which is the case (I-iii). In the second case we obtain an exact sequence

$$0 \to [2H+2F] \to \mathcal{E} \to [H+3F] \to 0,$$

which is non-split because $2H+2F$ is not ample. This is the case (II-v). In the last case we obtain an exact sequence

$$0 \to [2H+F] \to \mathcal{E} \to [H+4F] \to 0,$$

which is non-split. This is the case (III-ix).

In case $e=0$, we have $2 \leq b \leq 5$. If $b=5$, then $c_2(\mathcal{E})=6$ and $t=4$. We obtain $\mathcal{E} \simeq [H+F] \oplus [2H+4F]$, which is the case (III-ii). If $b=4$, then $(c_2(\mathcal{E}), t) = (5,3)$ or $(6,2)$. In the former case we obtain $\mathcal{E} \simeq [H+F] \oplus [2H+3F]$, which is the case (II-ii). In the latter case we obtain $\mathcal{E} \simeq [H+2F] \oplus [2H+2F]$, which is the case (III-iii). If $b=3$, then $(c_2(\mathcal{E}), t) = (4,2)$, $(5,1)$, or $(6,0)$. In the first case we obtain $\mathcal{E} \simeq [H+F] \oplus [2H+2F]$, which is the case (I-ii). In the second case we obtain $\mathcal{E} \simeq [H+2F] \oplus [2H+F]$, which is the case (II-iii). In the last case we obtain a non-split exact sequence

$$0 \to [2H] \to \mathcal{E} \to [H+3F] \to 0,$$

which is the case (III-v). If $b=2$, then $c_1(\mathcal{E}) = 3H+2F$. Note that the condition $c_1(\mathcal{E}) = 3H+2F$ is equivalent to the condition $c_1(\mathcal{E}) = 2H+3F$. Hence we have already treated this case. □

Remark 2.10. The existence of $\mathcal{E}$ in the cases (II-v) and (III-xi) is shown by Fujisawa [5, Example 3.7]. The existence of $\mathcal{E}$ in the case (III-v) can be shown similarly.

The existence of $\mathcal{E}$ in the case (III-ix) is shown as follows. Let C_0 be the minimal section of $\rho : \Sigma_1 \to \mathbb{P}^1$. We fix a non-split exact sequence

(2.10.1) $$0 \to \mathcal{O}_{C_0}(-1) \to \mathcal{O}_{C_0}(1)^{\oplus 2} \to \mathcal{O}_{C_0}(3) \to 0$$

on C_0. Since $\mathrm{Ext}^1([C_0+4F],[2C_0+F]) \simeq H^1(\Sigma_1, C_0-3F)$ and

$$h^2(\Sigma_1, (C_0-3F)-C_0) = h^2(\Sigma_1, -3F) = h^0(\Sigma_1, -2C_0) = 0,$$

we have a non-trivial extension

$$0 \to [2C_0+F] \to \mathcal{E} \to [C_0+4F] \to 0$$

whose restriction to C_0 is (2.10.1). Then we see that $c_1^2(\mathcal{E}) = 21$ and $c_2(\mathcal{E}) = 7$.

We show that the tautological line bundle $H(\mathcal{E})$ on $\mathbb{P}(\mathcal{E})$ is ample. Note that $H(\mathcal{E})^3 = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) = 14$. The surjection $\mathcal{E} \to [C_0+4F]$ above determines a section Z of the projection $p : \mathbb{P}(\mathcal{E}) \to S$. Then $Z \in |H(\mathcal{E}) - p^*(2C_0+F)|$ and $H(\mathcal{E})|_Z \simeq C_0+4F$ is ample. In addition, $H(\mathcal{E})|_{p^{-1}(C_0)}$ is ample since $\mathcal{E}|_{C_0} = \mathcal{O}_{C_0}(1)^{\oplus 2}$.

Let W be an arbitrary irreducible surface in $\mathbb{P}(\mathcal{E})$ with the property that $W \neq Z$ and $W \neq p^{-1}(C_0)$. Then we infer that $|H(\mathcal{E})|_W| = |[Z]_W + [p^*(2C_0+F)]_W|$ has a non-zero member for some F. Let C be an arbitrary irreducible curve in $\mathbb{P}(\mathcal{E})$ with the property that $C \not\subset Z \cup p^{-1}(C_0)$. Then we see that $H(\mathcal{E}) \cdot C = Z \cdot C + p^*(2C_0+F) \cdot C > 0$.

We thus conclude that $\mathcal{E}$ is ample in view of the Nakai criterion.

Using the results above, we can classify rank two ample vector bundles with small c_2 on Hirzebruch surfaces.

Corollary 2.11. *Let S be an e-th Hirzebruch surface. Then rank two ample vector bundles $\mathcal{E}$ with $c_2(\mathcal{E}) \leq 6$ on S are the following:*

(1) $c_2(\mathcal{E}) = 2$, $e = 0$, *and* $\mathcal{E} \simeq [H+F]^{\oplus 2}$;

(2) $c_2(\mathcal{E}) = 3$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [H+2F]$;
(3) $c_2(\mathcal{E}) = 3$, $e = 1$, *and* $\mathcal{E} \simeq [H+2F]^{\oplus 2}$;
(4) $c_2(\mathcal{E}) = 4$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [H+3F]$;
(5) $c_2(\mathcal{E}) = 4$, $e = 0$, *and* $\mathcal{E} \simeq [H+2F]^{\oplus 2}$;
(6) $c_2(\mathcal{E}) = 4$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [2H+2F]$;
(7) $c_2(\mathcal{E}) = 4$, $e = 1$, *and* $\mathcal{E} \simeq [H+2F] \oplus [H+3F]$;
(8) $c_2(\mathcal{E}) = 4$, $e = 2$, *and* $\mathcal{E} \simeq [H+3F]^{\oplus 2}$;
(9) $c_2(\mathcal{E}) = 5$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [H+4F]$;
(10) $c_2(\mathcal{E}) = 5$, $e = 0$, *and* $\mathcal{E} \simeq [H+2F] \oplus [H+3F]$;
(11) $c_2(\mathcal{E}) = 5$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [2H+3F]$;
(12) $c_2(\mathcal{E}) = 5$, $e = 0$, *and* $\mathcal{E} \simeq [H+2F] \oplus [2H+F]$;
(13) $c_2(\mathcal{E}) = 5$, $e = 1$, *and* $\mathcal{E} \simeq [H+2F] \oplus [H+4F]$;
(14) $c_2(\mathcal{E}) = 5$, $e = 1$, *and* $\mathcal{E} \simeq [H+3F]^{\oplus 2}$;
(15) $c_2(\mathcal{E}) = 5$, $e = 1$, *and* $\mathcal{E} \simeq [H+2F] \oplus [2H+3F]$;
(16) $c_2(\mathcal{E}) = 5$, $e = 2$, *and* $\mathcal{E} \simeq [H+3F] \oplus [H+4F]$;
(17) $c_2(\mathcal{E}) = 5$, $e = 3$, *and* $\mathcal{E} \simeq [H+4F]^{\oplus 2}$;
(18) $c_2(\mathcal{E}) = 6$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [H+5F]$;
(19) $c_2(\mathcal{E}) = 6$, $e = 0$, *and* $\mathcal{E} \simeq [H+2F] \oplus [H+4F]$;
(20) $c_2(\mathcal{E}) = 6$, $e = 0$, *and* $\mathcal{E} \simeq [H+3F]^{\oplus 2}$;
(21) $c_2(\mathcal{E}) = 6$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [2H+4F]$;
(22) $c_2(\mathcal{E}) = 6$, $e = 0$, *and* $\mathcal{E} \simeq [H+2F] \oplus [2H+2F]$;
(23) $c_2(\mathcal{E}) = 6$, $e = 0$, *and* $\mathcal{E} \simeq [H+F] \oplus [3H+3F]$;
(24) $c_2(\mathcal{E}) = 6$, $e = 0$, *and there exists a non-split exact sequence* $0 \to [2H] \to \mathcal{E} \to [H+3F] \to 0$;
(25) $c_2(\mathcal{E}) = 6$, $e = 1$, *and* $\mathcal{E} \simeq [H+2F] \oplus [H+5F]$;
(26) $c_2(\mathcal{E}) = 6$, $e = 1$, *and* $\mathcal{E} \simeq [H+3F] \oplus [H+4F]$;
(27) $c_2(\mathcal{E}) = 6$, $e = 1$, *and* $\mathcal{E} \simeq [H+2F] \oplus [2H+4F]$;
(28) $c_2(\mathcal{E}) = 6$, $e = 1$, *and there exists a non-split exact sequence* $0 \to [2H+2F] \to \mathcal{E} \to [H+3F] \to 0$;
(29) $c_2(\mathcal{E}) = 6$, $e = 2$, *and* $\mathcal{E} \simeq [H+3F] \oplus [H+5F]$;
(30) $c_2(\mathcal{E}) = 6$, $e = 2$, *and* $\mathcal{E} \simeq [H+4F]^{\oplus 2}$;
(31) $c_2(\mathcal{E}) = 6$, $e = 3$, *and* $\mathcal{E} \simeq [H+4F] \oplus [H+5F]$;
(32) $c_2(\mathcal{E}) = 6$, $e = 4$, *and* $\mathcal{E} \simeq [H+5F]^{\oplus 2}$.

Proof. Suppose that $c_2(\mathcal{E}) \leq 6$. Then $e + 2 \leq c_2(\mathcal{E}) \leq e + 6$ by (2.8).

In case $c_2(\mathcal{E}) = e + 2$, we have $0 \leq e \leq 4$ and $\mathcal{E}$ is a vector bundle of the type (1), (3), (8), (17), or (32) by (2.8).

In case $c_2(\mathcal{E}) = e + 3$, we have $0 \leq e \leq 3$ and $\mathcal{E}$ is of the type (2), (7), (16), or (31) by (2.8).

In case $c_2(\mathcal{E}) = e + 4$, we have $0 \leq e \leq 2$ and $\mathcal{E}$ is of the type (4), (5), (6), (13), (14), (15), (29), or (30) by (2.9).

In case $c_2(\mathcal{E}) = e + 5$, we have $0 \leq e \leq 1$ and $\mathcal{E}$ is of the type (9), (10), (11), (12), (25), (26), (27), or (28) by (2.9).

In case $c_2(\mathcal{E}) = e + 6$, we have $e = 0$ and $\mathcal{E}$ is of the type (18), (19), (20), (21), (22), (23), or (24) by (2.9). □

Remark 2.12. We easily see that there are no ample line bundles M with $M^2 = 1$ on Hirzebruch surfaces Σ_e. Hence, on Σ_e, equality does not hold in (1.5) because of (1.9) and (2.11). Furthermore, by an argument similar to that in (1.9), we obtain that $c_1^2(\mathcal{E}) = (c_2(\mathcal{E}) + 1)^2 - 1$ if and only if $e = 0$ and $\mathcal{E} \simeq [H + F]^{\oplus 2}$ for ample vector bundles $\mathcal{E}$ of rank two on Σ_e.

3. On Del Pezzo surfaces (of degree less than eight).

Definition 3.1. A smooth surface S is said to be a *Del Pezzo surface of degree d* if $-K_S$ is ample and $d = (-K_S)^2$.

Proposition 3.2 (see, e.g., [4, p. 27, Théorème 1]). *A Del Pezzo surface S of degree d is one of the following:*

(i) $d = 9$ *and* $S \simeq \mathbb{P}^2$;
(ii) $d = 8$ *and* $S \simeq \Sigma_0$ *or* Σ_1;
(iii) $1 \leq d \leq 7$ *and S is isomorphic to the blowing-up of $\mathbb{P}^2$ at $9 - d$ points, no three of which lie on a line, no six of which lie on a conic, and for $d = 1$ all eight do not lie on a cubic that is singular at one of them.*

Conversely, every surface satisfying the condition (i), (ii), *or* (iii) *is a Del Pezzo surface of the corresponding degree.*

In this section we denote by S a Del Pezzo surface of degree $d \leq 7$ and by $\mathcal{E}$ an ample vector bundle of rank two on S. Note that we have already studied rank two ample vector bundles on Σ_0 or Σ_1 in §2. Rank two ample vector bundles on $\mathbb{P}^2$ are studied in §4.

Sometimes we specify a blowing-up $\rho : S \to \mathbb{P}^2$ at $9 - d$ points $x_1, \ldots, x_{9-d}$ and denote by $E_1, \ldots, E_{9-d}$ the exceptional curves of ρ. Then $\operatorname{Pic} S \simeq \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E_1 \oplus \cdots \oplus \mathbb{Z} \cdot E_{9-d}$.

First we consider the relation between $c_1^2(\mathcal{E})$ and d.

Proposition 3.3. *Let S be a Del Pezzo surface of degree $d \leq 7$ and $\mathcal{E}$ an ample vector bundle of rank two on S. Then $2K_S + c_1(\mathcal{E})$ is nef.*

Proof. This assertion follows from (1.2). □

Corollary 3.4. *Let S and $\mathcal{E}$ be as above. Then $-K_S \cdot c_1(\mathcal{E}) \geq 2d$.*

Proof. Since $-K_S$ is ample, we get $(2K_S + c_1(\mathcal{E}))(-K_S) \geq 0$ by (3.3). Then the assertion easily follows. □

Corollary 3.5. *Let S and $\mathcal{E}$ be as above. Then $c_1^2(\mathcal{E}) \geq 4d$.*

Proof. Since $c_1(\mathcal{E})$ is ample, we get $(2K_S + c_1(\mathcal{E}))c_1(\mathcal{E}) \geq 0$ by (3.3). Then we obtain $c_1^2(\mathcal{E}) \geq -2K_S \cdot c_1(\mathcal{E}) \geq 4d$ by (3.4). □

The following proposition will be used later.

Proposition 3.6. *Let S be a Del Pezzo surface of degree $d \leq 7$ and $\mathcal{E}$ an ample vector bundle of rank two on S. If $c_1^2(\mathcal{E}) \leq 4d+8$, then we have one of the following:*
(i) $c_1(\mathcal{E}) = -2K_S$;
(ii) $d = 1$ *and* $c_1(\mathcal{E}) = -3K_S$;
(iii) $c_1(\mathcal{E}) = -2K_S + C$, *where C is a 0-curve (i.e., $C \simeq \mathbb{P}^1$ and $C^2 = 0$).*

Proof. Suppose that $c_1^2(\mathcal{E}) \leq 4d+8$. From (3.5) we get

$$4d \leq -2K_S \cdot c_1(\mathcal{E}) \leq c_1^2(\mathcal{E}) \leq 4d+8,$$

and hence $2d \leq -K_S \cdot c_1(\mathcal{E}) \leq 2d+4$. Note that $K_S \cdot c_1(\mathcal{E}) + c_1^2(\mathcal{E}) = 2g(\det \mathcal{E}) - 2$ is even, where $g(\det \mathcal{E})$ is the sectional genus of $(S, \det \mathcal{E})$.

If $-K_S \cdot c_1(\mathcal{E}) = 2d$, then $(2K_S + c_1(\mathcal{E}))(-K_S) = 0$. Hence we get $c_1(\mathcal{E}) = -2K_S$ by (3.3).

If $-K_S \cdot c_1(\mathcal{E}) = 2d+4$, then $c_1^2(\mathcal{E}) = 4d+8$ and $(2K_S + c_1(\mathcal{E}))c_1(\mathcal{E}) = 0$. Hence we have $c_1(\mathcal{E}) = -2K_S$ and then $c_1^2(\mathcal{E}) = 4d$, which is a contradiction.

If $-K_S \cdot c_1(\mathcal{E}) = 2d+3$, then $c_1^2(\mathcal{E}) = 4d+7$. We have $(2K_S + c_1(\mathcal{E}))^2 = -5$, a contradiction to (3.3).

If $-K_S \cdot c_1(\mathcal{E}) = 2d+1$, then $c_1^2(\mathcal{E}) = 4d+3$, $4d+5$, or $4d+7$. In case $c_1^2(\mathcal{E}) = 4d+3$, we have $(2K_S + c_1(\mathcal{E}))^2 = -1$, a contradiction. In case $c_1^2(\mathcal{E}) = 4d+5$ or $4d+7$, we note that $((2d+1)K_S + d \cdot c_1(\mathcal{E}))(-K_S) = 0$. By the Hodge index theorem, we get

$$0 \geq ((2d+1)K_S + d \cdot c_1(\mathcal{E}))^2 = d(d \cdot c_1^2(\mathcal{E}) - (2d+1)^2).$$

It follows that $c_1^2(\mathcal{E}) = 4d+5$ and $d = 1$. Then we have $(3K_S + c_1(\mathcal{E}))(-K_S) = 0$ and $(3K_S + c_1(\mathcal{E}))^2 = 0$. Hence we obtain $c_1(\mathcal{E}) = -3K_S$.

If $-K_S \cdot c_1(\mathcal{E}) = 2d+2$, then $c_1^2(\mathcal{E}) = 4d+4$, $4d+6$, or $4d+8$. In case $c_1^2(\mathcal{E}) = 4d+4$ or $4d+6$, we have $(2K_S + c_1(\mathcal{E}))^2 < 0$, a contradiction. Hence we get $c_1^2(\mathcal{E}) = 4d+8$ and $(2K_S + c_1(\mathcal{E}))^2 = 0$. Note that $(2K_S + c_1(\mathcal{E})) - K_S$ is ample by (3.3) and (3.1). By the base point free theorem, there exists a fibration $\varphi : S \to W$ such that $2K_S + c_1(\mathcal{E}) = \varphi^* A$ for some ample line bundle A on W (see [7, (0.4.15)]). From $(2K_S + c_1(\mathcal{E}))^2 = 0$ and $(2K_S + c_1(\mathcal{E}))K_S = -2$, we infer that $\dim W = 1$, $\deg A = 1$, and a general fiber C of φ is a 0-curve. Hence we obtain $c_1(\mathcal{E}) = -2K_S + C$. □

Remark 3.7. We make some comments on (3.6).

In the case of (i): if $d = 1$, by Fujita [6, (2.8)], $c_2(\mathcal{E}) = 1$ and $\mathcal{E} \simeq [-K_S]^{\oplus 2}$, or $c_2(\mathcal{E}) = 3$ and there exists a non-split exact sequence

$$0 \to \mathcal{O}_{S'}(F) \to \pi^*\mathcal{E} \to \pi^*(-2K_S) - \mathcal{O}_{S'}(F) \to 0,$$

where $\pi : S' \to S$ is the blowing-up at (possibly infinitely near) three points y_1, y_2, y_3 and F is the sum of exceptional curves over $\{y_i\}_{i=1}^3$. Moreover, the existence of $\mathcal{E}$ has been proved with $\{y_i\}_{i=1}^3$ in a generic position.

If $d = 2$ and $\mathcal{E}$ is ample and spanned, by Lanteri-Maeda [10, §5], $c_2(\mathcal{E}) = 2$ and $\mathcal{E} \simeq [-K_S]^{\oplus 2}$, or $c_2(\mathcal{E}) = 4$ and $\mathcal{E} \simeq \rho^*\mathcal{F} \otimes [-K_S]$, where $\rho : S \to \mathbb{P}^2$ is a blowing-up of $\mathbb{P}^2$ at seven points and $\mathcal{F}$ is the cokernel of a bundle monomorphism

$$\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to (\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(1))^{\oplus 2}$$

($\Omega^1_{\mathbb{P}^2}$ is the cotangent bundle of $\mathbb{P}^2$). Moreover, an example for $\mathcal{E}$ of the above type is given in [10, §6].

The case $d \geq 3$ is yet to be studied. It is clear that $\mathcal{E} := [-K_S]^{\oplus 2}$ gives an example for each d.

In the case of (ii) and (iii), we obtain classification results if $c_2(\mathcal{E}) \leq d + 2$ (see (3.9), (3.11), and (3.13)).

In the case of (iii), we find that $C \in |H - E_1|$ for some blowing-up $\rho : S \to \mathbb{P}^2$. Indeed, each singular fiber of φ in the proof of (3.6) is the union of two (-1)-curves that intersect at one point. By contracting one (-1)-curve in a singular fiber, we get a Del Pezzo surface of degree $d + 1$ and C is still a 0-curve on it. Thus we may consider only the case $d = 7$, and then the assertion is clear.

Next we consider the relation between $c_2(\mathcal{E})$ and d.

Proposition 3.8. *Let S be a Del Pezzo surface of degree $d \leq 7$ and $\mathcal{E}$ an ample vector bundle of rank two on S. If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, then $c_2(\mathcal{E}) > d$.*

Proof. Suppose that $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. From (1.7) and (1.8), we obtain an exact sequence

$$0 \to L \to \mathcal{E} \to \mathcal{I}_Z \otimes M \to 0,$$

where L, $M \in \operatorname{Pic} S$, M is ample, and $\mathcal{I}_Z$ is the ideal sheaf of a zero-dimensional subscheme Z of S.

Then $K_S + M$ is nef by (1.1). Hence we obtain

$$d = (-K_S)^2 \leq (-K_S) \cdot M \leq M^2 < LM \leq c_2(\mathcal{E})$$

by the fact that $(K_S + M)(-K_S) \geq 0$, $(K_S + M)M \geq 0$, and $(L - M)M > 0$. □

Theorem 3.9. *Let S be a Del Pezzo surface of degree $d \leq 7$ and $\mathcal{E}$ an ample vector bundle of rank two on S. Then $c_2(\mathcal{E}) \geq d$, and equality holds if and only if $\mathcal{E} \simeq [-K_S]^{\oplus 2}$.*

For a proof of this theorem, we need the following lemma.

Lemma 3.10. *Let S be as above and let A be the union of all (-1)-curves on S. If $d \leq 6$, then A is connected and a member of $|-m_d K_S|$, where*

$$m_1 = 240,\ m_2 = 28,\ m_3 = 9,\ m_4 = 4,\ m_5 = 2,\ \text{and}\ m_6 = 1.$$

If $d = 7$, then A is connected and a member of $|H|$, where H is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ by the blowing-up $\rho : S \to \mathbb{P}^2$.

Proof. We fix a blowing-up $\rho : S \to \mathbb{P}^2$ and denote by $\{E_i\}_{i=1}^{9-d}$ the exceptional curves of ρ. The number of (-1)-curves on S are listed in Table 1 (cf., e.g., [4, p. 35, Table 3]).

Table 1

type \ d	1	2	3	4	5	6	7
$(0;-1)$	8	7	6	5	4	3	2
$(1;1^2)$	28	21	15	10	6	3	1
$(2;1^5)$	56	21	6	1	0	0	0
$(3;2,1^6)$	56	7	0	0	0	0	0
$(4;2^3,1^5)$	56	0	0	0	0	0	0
$(5;2^6,1^2)$	28	0	0	0	0	0	0
$(6;3,2^7)$	8	0	0	0	0	0	0
total	240	56	27	16	10	6	3

There a (-1)-curve C is said to be of the type $(a_0; a_1^{n_1}, a_2^{n_2}, \dots)$ if $C \in |a_0 H - \sum_{k=1}^{n_1} a_1 E_{i_k} - \sum_{l=1}^{n_2} a_2 E_{j_l} - \dots|$ ($\{E_{i_k}, E_{j_l}, \dots\}_{k,l,\dots}$ are all distinct).

Then the assertion can be shown by simple computation. □

Proof of Theorem 3.9. We may assume $c_1^2(\mathcal{E}) \leq 4c_2(\mathcal{E})$ because of (3.8). Then we obtain $c_2(\mathcal{E}) \geq d$ by (3.5). Suppose that $c_2(\mathcal{E}) = d$. Then we have $c_1^2(\mathcal{E}) = 4d$, and hence $c_1(\mathcal{E}) = -2K_S$ by (3.6).

Using the Riemann-Roch theorem, we get $\chi(\mathcal{E} \otimes K_S) = 2$. We have $H^2(\mathcal{E} \otimes K_S) = 0$ since $\mathcal{E}$ is ample. Thus there exists a non-zero section $s \in H^0(\mathcal{E} \otimes K_S)$.

Let $(s)_0$ be the scheme of zeros of s. In case $\dim(s)_0 \leq 0$, we have $(s)_0 = \emptyset$ since $c_2(\mathcal{E} \otimes K_S) = 0$. Then the section s induces an exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\cdot s} \mathcal{E} \otimes K_S \to \det(\mathcal{E} \otimes K_S) \to 0.$$

Since $\mathrm{Ext}^1(\det(\mathcal{E} \otimes K_S), \mathcal{O}_S) \simeq H^1(S, \mathcal{O}_S) = 0$, we obtain $\mathcal{E} \simeq [-K_S]^{\otimes 2}$.

We will show that the case $\dim(s)_0 = 1$ cannot occur.

In case $\dim(s)_0 = 1$, we denote by Z the one-dimensional part of $(s)_0$ as a cycle. For every (-1)-curve C, we have $C \subset \operatorname{Supp} Z$ or $C \cap \operatorname{Supp} Z = \emptyset$ since $[\mathcal{E} \otimes K_S]_C \simeq \mathcal{O}_C^{\oplus 2}$. If $d \leq 6$, then the union A of all (-1)-curves on S is an ample connected divisor by (3.10). Since $AZ > 0$, we see that $A \subset \operatorname{Supp} Z$. Then the section s determines a non-zero section $s' \in H^0(\mathcal{E} \otimes K_S \otimes [-A])$. Hence $H^0(\mathbb{P}(\mathcal{E}), H(\mathcal{E}) + p^*[(m_d + 1)K_S]) \neq 0$, where p is the projection $\mathbb{P}(\mathcal{E}) \to S$. Since $\mathcal{E}$ is ample, we have

$$0 < H(\mathcal{E})^2(H(\mathcal{E})+p^*[(m_d+1)K_S]) = c_1^2(\mathcal{E})-c_2(\mathcal{E})+c_1(\mathcal{E})\cdot[(m_d+1)K_S] = (1-2m_d)d < 0,$$

a contradiction.

In case $\dim(s)_0 = 1$ and $d = 7$, let $\rho : S \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at two points x_1 and x_2. Setting $H := \rho^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $E_i := \rho^{-1}(x_i)$ $(i = 1, 2)$, we have $A = E_1 + E_2 + C_{12} \in |H|$, where C_{12} is the strict transform of the line in $\mathbb{P}^2$ passing through x_1 and x_2. Note that $A \subset \operatorname{Supp} Z$ or $A \cap \operatorname{Supp} Z = \emptyset$ since A is connected. If $A \cap \operatorname{Supp} Z = \emptyset$, then $E_1 Z = 0$, $E_2 Z = 0$, and $HZ = AZ = 0$. This is a contradiction since $\operatorname{Pic} S \simeq \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E_1 \oplus \mathbb{Z} \cdot E_2$. Hence $A \subset \operatorname{Supp} Z$ and then s determines a non-zero section $s' \in H^0(\mathcal{E} \otimes K_S \otimes [-A])$. Since $H^0(\mathbb{P}(\mathcal{E}), H(\mathcal{E}) + p^*[K_S - H]) \neq 0$ and

$$H(\mathcal{E})^2(H(\mathcal{E}) + p^*[K_S - H]) = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot [K_S - H] = d - 6 = 1,$$

every member D of $|H(\mathcal{E}) + p^*[K_S - H]|$ is irreducible and reduced. Since $p(D) = S$, we see that $\dim(s')_0 \leq 0$. Furthermore, we find that $c_2(\mathcal{E} \otimes [K_S - H]) = 1$ and hence $(s')_0$ is one reduced point x_0.

Let $\pi : S' \to S$ be the blowing-up of S at x_0 and denote by E_0 the exceptional curve of π. Then π^*s' determines a non-zero section $s'' \in H^0(S', \pi^*(\mathcal{E} \otimes [K_S - H]) \otimes [-E_0])$ such that $(s'')_0 = \emptyset$. Hence we get an exact sequence

$$0 \to \mathcal{O}'_S \xrightarrow{\cdot s''} \pi^*(\mathcal{E} \otimes [K_S - H]) \otimes [-E_0] \to \det(\pi^*(\mathcal{E} \otimes [K_S - H]) \otimes [-E_0]) \to 0,$$

and then the exact sequence

$$0 \to \pi^*(-K_S + H) + \mathcal{O}_{S'}(E_0) \to \pi^*\mathcal{E} \to \pi^*(-K_S - H) - \mathcal{O}_{S'}(E_0) \to 0$$

is induced.

Let C'_{12} be the strict transform of C_{12} by π. Since $\pi|_{C'_{12}} : C'_{12} \to C_{12}$ is an isomorphism, we see that $[\pi^*\mathcal{E}]_{C'_{12}}$ is ample, and hence $[\pi^*(-K_S - H) - \mathcal{O}_{S'}(E_0)]_{C'_{12}}$ is ample. But we have

$$(\pi^*(-K_S - H) - \mathcal{O}_{S'}(E_0))C'_{12} = (-K_S - H)C_{12} - E_0 \cdot C'_{12} \leq 0,$$

a contradiction. Thus the case $\dim(s)_0 = 1$ does not occur. □

Theorem 3.11. *Let S be a Del Pezzo surface of degree $d \leq 7$ and $\mathcal{E}$ be an ample vector bundle of rank two on S. If $c_2(\mathcal{E}) = d+1$, then we have either*

(i) $d = 1$ *and* $\mathcal{E} \simeq [-K_S] \oplus [-2K_S]$, *or*
(ii) $2 \leq d \leq 7$ *and there exists a non-split exact sequence*

$$0 \to \pi^*(-K_S) + \mathcal{O}_{S'}(E_0) \to \pi^*\mathcal{E} \to \pi^*(-K_S) - \mathcal{O}_{S'}(E_0) \to 0,$$

where $\pi : S' \to S$ is the blowing-up of S at one point x_0 and $E_0 := \pi^{-1}(x_0)$.

Proof. Suppose that $c_2(\mathcal{E}) = d+1$. If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, then as in the proof of (3.8), we obtain an exact sequence $0 \to L \to \mathcal{E} \to \mathcal{I}_Z \otimes M \to 0$ and

$$d = (-K_S)^2 \leq (-K_S) \cdot M \leq M^2 < LM \leq c_2(\mathcal{E}) = d+1.$$

From this inequality we get $\deg Z = 0$ and $M = -K_S$ since $K_S + M$ is nef. Then $L = (-K_S) + (2K_S + c_1(\mathcal{E}))$ is ample and $\mathcal{E} \simeq [-K_S] \oplus L$ by $\mathrm{Ext}^1(-K_S, L) = 0$. We find that $c_1^2(\mathcal{E}) = L^2 + 3d + 2$. Then we have $L^2 \geq d+3$ since $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. Thus

$$d(d+3) \leq (-K_S)^2 L^2 \leq (-K_S L)^2 = (d+1)^2,$$

and then $d = 1$ and $c_2(\mathcal{E}) = 2$. Hence we obtain $L = -2K_S$ and $\mathcal{E} \simeq [-K_S] \oplus [-2K_S]$.

If $c_1^2(\mathcal{E}) \leq 4c_2(\mathcal{E})$, then we have $c_1^2(\mathcal{E}) \leq 4d+4$. From (3.6) we obtain that $c_1(\mathcal{E}) = -2K_S$. Then we get $\chi(\mathcal{E} \otimes K_S) = 1$ from $c_2(\mathcal{E}) = d+1$. We get also $h^2(\mathcal{E} \otimes K_S) = 0$, and hence there exists a non-zero section $s \in H^0(\mathcal{E} \otimes K_S)$. Then we have $H^0(\mathbb{P}(\mathcal{E}), H(\mathcal{E}) + p^*K_S) \neq 0$ and

$$0 < H(\mathcal{E})^2 (H(\mathcal{E}) + p^*K_S) = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot K_S = d-1.$$

We infer that $\dim(s)_0 \leq 0$ as in the proof of (3.9). Since $c_2(\mathcal{E} \otimes K_S) = 1$, we see that $(s)_0$ is one reduced point x_0. Let $\pi : S' \to S$ be the blowing-up of S at x_0 and denote by E_0 the exceptional curve of π. Then we obtain an exact sequence

$$0 \to \pi^*(-K_S) + \mathcal{O}_{S'}(E_0) \to \pi^*\mathcal{E} \to \pi^*(-K_S) - \mathcal{O}_{S'}(E_0) \to 0$$

by an argument similar to that in (3.9). This exact sequence is non-split, otherwise we have

$$\mathcal{O}_{E_0}^{\oplus 2} \simeq [\pi^*\mathcal{E}]_{E_0} \simeq [\pi^*(-K_S) + \mathcal{O}_{S'}(E_0)]_{E_0} \oplus [\pi^*(-K_S) - \mathcal{O}_{S'}(E_0)]_{E_0} \simeq \mathcal{O}_{E_0}(-1) \oplus \mathcal{O}_{E_0}(1),$$

a contradiction. We have thus proved the theorem. □

Remark 3.12. The existence of $\mathcal{E}$ in the case (ii) of (3.11) is shown by Fujisawa [5, Example (3.11)].

Proposition 3.13. *Let S be a Del Pezzo surface of degree $d \leq 7$ and $\mathcal{E}$ be an ample vector bundle of rank two on S. If $c_2(\mathcal{E}) = d+2$, then we have one of the following:*

(i) $c_1(\mathcal{E}) = -2K_S$ *(see (3.7));*
(ii) $\mathcal{E} \simeq [-K_S] \oplus [-K_S + C]$, *where C is a 0-curve;*
(iii) $d = 1$ *and* $\mathcal{E} \simeq [-K_S] \oplus [-2K_S + C]$, *where C is a (-1)-curve;*
(iv) $d = 1$ *and* $\mathcal{E} \simeq [-K_S] \oplus [-3K_S]$;
(v) $d = 2$ *and* $\mathcal{E} \simeq [-K_S] \oplus [-2K_S]$.

Proof. Suppose that $c_2(\mathcal{E}) = d+2$. We argue as in the proof of (3.11).

(3.13.1) If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, we get an exact sequence $0 \to L \to \mathcal{E} \to \mathcal{I}_Z \otimes M \to 0$ and

$$d \leq (-K_S) \cdot M \leq M^2 < LM \leq c_2(\mathcal{E}) = d+2.$$

Then we have $\deg Z \leq 1$ and $(-K_S) \cdot M = M^2$ since $K_S \cdot M + M^2 = 2g(M) - 2$ is even. It follows that $M = -K_S$. If $\deg Z = 1$, then $c_1^2(\mathcal{E}) = L^2 + 3d + 2$ and $L^2 \geq d+7$ since $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. It follows that $d(d+7) \leq dL^2 \leq (d+1)^2$, a contradiction. Hence we have $\deg Z = 0$. Then $c_1^2(\mathcal{E}) = L^2 + 3d + 4$ and $L^2 \geq d+5$ since $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. Note that $L^2 - d = 2g(L)$ is even. Thus we get $d(d+6) \leq dL^2 \leq (d+2)^2$, and then $d \leq 2$.

In case $d = 2$, we have $L^2 = 8$. From $K_S^2 L^2 = (K_S L)^2 = 16$, we obtain $L = -2K_S$, and hence $\mathcal{E} \simeq [-K_S] \oplus [-2K_S]$. This is the case (v).

In case $d = 1$, we have $L^2 = 7$ or 9. If $L^2 = 9$, then $K_S^2 L^2 = (K_S L)^2 = 9$. Hence we obtain $L = -3K_S$ and $\mathcal{E} \simeq [-K_S] \oplus [-3K_S]$. This is the case (iv). If $L^2 = 7$, we set $D := L + 2K_S$. We find $\chi(\mathcal{O}_S(D)) = 1$, and we get $h^2(S, D) = h^0(S, K_S - D)$ by the Serre duality. If $h^0(S, K_S - D) > 0$, then the divisor $K_S - D$ is effective and hence $0 < (-K_S)(K_S - D) = -2$, a contradiction. Thus we have $h^2(S, D) = 0$, hence $h^0(S, D) \geq 1$. Since $(-K_S) \cdot D = 1$, every member C of $|D|$ is an irreducible reduced curve. Furthermore, C is a (-1)-curve because $C^2 = -1$. Thus we obtain $L = -2K_S + C$, and hence $\mathcal{E} \simeq [-K_S] \oplus [-2K_S + C]$. This is the case (iii).

(3.13.2) If $c_1^2(\mathcal{E}) \leq 4c_2(\mathcal{E})$, then we have $c_1^2(\mathcal{E}) \leq 4d + 8$. Because of (3.6), there are the following three posibilities: $c_1(\mathcal{E}) = -2K_S$; $d = 1$ and $c_1(\mathcal{E}) = -3K_S$; $c_1(\mathcal{E}) = -2K_S + C$, where C is a 0-curve.

The second case leads to a contradiction as below. Assume that $d = 1$ and $c_1(\mathcal{E}) = -3K_S$. We get $\chi(\mathcal{E} \otimes [2K_S]) = 1$ by Riemann-Roch. We get also $h^2(\mathcal{E} \otimes [2K_S]) = h^0(\mathcal{E}^\vee \otimes [-K_S]) = h^0(\mathcal{E} \otimes [2K_S])$ by Serre duality and the fact that $\mathcal{E} \simeq \mathcal{E}^\vee \otimes \det \mathcal{E}$. (The symbol $^\vee$ stands for the dual.) Thus we have $h^0(\mathcal{E} \otimes [2K_S]) > 0$, and then $h^0(\mathbb{P}(\mathcal{E}), H(\mathcal{E}) + p^*[2K_S]) > 0$, where $p : \mathbb{P}(\mathcal{E}) \to S$ is the projection. Since $\mathcal{E}$ is ample, we have

$$0 < H(\mathcal{E})^2 (H(\mathcal{E}) + p^*[2K_S]) = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot (2K_S) = 0,$$

a contradiction.

In case $c_1(\mathcal{E}) = -2K_S + C$ (C is a 0-curve), we will show that $\mathcal{E} \simeq [-K_S] \oplus [-K_S + C]$. Since $\chi(\mathcal{E} \otimes K_S) = 3$ and $h^2(\mathcal{E} \otimes K_S) = 0$, there exists a non-zero section $s \in H^0(\mathcal{E} \otimes K_S)$. If $\dim(s)_0 \leq 0$, since $c_2(\mathcal{E} \otimes K_S) = 0$, we get an exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\cdot s} \mathcal{E} \otimes K_S \to \det(\mathcal{E} \otimes K_S) \to 0,$$

and then the exact sequence

$$0 \to [-K_S] \to \mathcal{E} \to [-K_S + C] \to 0$$

is induced. We have $\mathrm{Ext}^1([-K_S + C], [-K_S]) \simeq H^1(S, -C)$ and we find $\chi(\mathcal{O}_S(-C)) = 0$. We have also $h^0(S, -C) = 0$ and $h^2(S, -C) = h^0(S, K_S + C) = 0$ since $-K_S$ is ample. Hence we obtain $h^1(S, -C) = 0$, and then $c_1(\mathcal{E}) \simeq [-K_S] \oplus [-K_S + C]$.

If $\dim(s)_0 = 1$, then we denote by Z the one-dimensional part of $(s)_0$ as a cycle. We fix a blowing-up $\rho : S \to \mathbb{P}^2$ for which $C \in |H - E_1|$ (see (3.7)).

Claim. *$Z \in |t(H - E_1)|$ for some positive integer t.*

Proof. Let j be an integer such that $2 \leq j \leq 9 - d$. We denote by C_{1j} the (-1)-curve obtained by the strict transform of the line in $\mathbb{P}^2$ passing through x_1 and x_j. Since $C_{1j} \in |H - E_1 - E_j|$ and $c_1(\mathcal{E}) \cdot C_{1j} = 2$, we have $[\mathcal{E} \otimes K_S]_{C_{1j}} \simeq \mathcal{O}_{C_{1j}}$, and hence $C_{1j} \cap \operatorname{Supp} Z = \emptyset$ or $C_{1j} \subset \operatorname{Supp} Z$. We have also $[\mathcal{E} \otimes K_S]_{E_j} \simeq \mathcal{O}_{E_j}$ since $c_1(\mathcal{E}) \cdot E_j = 2$. Hence there are the following two possibilities:

(a) $E_j \cap \operatorname{Supp} Z = \emptyset$ for every j;
(b) $E_j \subset \operatorname{Supp} Z$ for some j.

In the case (a), each irreducible component Z_1 of Z can be written as $[Z_1] = uH + vE_1$ for some non-negative integers u and v. Since $E_j \cap C_{1j} \neq \emptyset$, we have $C_{1j} \not\subset \operatorname{Supp} Z$, and hence $C_{1j} \cap \operatorname{Supp} Z = \emptyset$. It follows that $0 = C_{1j} \cdot Z_1 = u + v$ and then $Z_1 \in |u(H - E_1)|$. Thus we obtain $Z \in |t(H - E_1)|$ for some positive integer t.

In the case (b), we infer that $C_{1j} \subset \operatorname{Supp} Z$ from the argument above. Note that $E_j + C_{1j} \in |H - E_1|$. Let t_j be the largest integer with the property that the divisor $Z - t_j(E_j + C_{1j})$ is effective. If $E_j \subset \operatorname{Supp}(Z - t_j(E_j + C_{1j}))$, then we have $C_{1j} \subset \operatorname{Supp}(Z - t_j(E_j + C_{1j}) - E_j)$. This contradicts the definition of t_j, and hence we see that $E_j \not\subset \operatorname{Supp}(Z - t_j(E_j + C_{1j}))$ for every j. Thus we obtain that $E_j \cap \operatorname{Supp}(Z - \sum_{k=2}^{9-d} t_k(E_k + C_{1k})) = \emptyset$ for every j. Then the claim follows from an argument similar to that in the case (a). □

Proof of Proposition 3.13, continued. From the claim we infer that s determines a non-zero section $s' \in H^0(\mathcal{E} \otimes [K_S - t(H - E_1)])$ satisfying $\dim(s')_0 \leq 0$. Since $c_2(\mathcal{E} \otimes [K_S - t(H - E_1)]) = 0$, we get an exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{\cdot s} \mathcal{E} \otimes [K_S - t(H - E_1)] \to \det(\mathcal{E} \otimes [K_S - t(H - E_1)]) \to 0,$$

and then the exact sequence

$$0 \to [-K_S + t(H - E_1)] \to \mathcal{E} \to [-K_S + (1-t)(H - E_1)] \to 0$$

is induced. Since $\mathcal{E}$ is ample, $-K_S + (1-t)(H - E_1)$ is also ample, and hence $t = 1$. Then we see that

$$0 \to [-K_S + H - E_1] \to \mathcal{E} \to [-K_S] \to 0$$

is exact and $\mathrm{Ext}^1([-K_S], [-K_S + H - E_1]) = 0$. Hence we obtain $\mathcal{E} \simeq [-K_S] \oplus [-K_S + H - E_1]$. □

Using the results above, we can classify rank two ample vector bundles with small c_2 on Del Pezzo surfaces.

Corollary 3.14. *Let S be a Del Pezzo surface of degree $d \leq 7$. Then rank two ample vector bundles $\mathcal{E}$ with $c_2(\mathcal{E}) \leq 3$ on S are the following:*

(1) $c_2(\mathcal{E}) = 1$, $d = 1$, *and* $\mathcal{E} \simeq [-K_S]^{\oplus 2}$;
(2) $c_2(\mathcal{E}) = 2$, $d = 1$, *and* $\mathcal{E} \simeq [-K_S] \oplus [-2K_S]$;
(3) $c_2(\mathcal{E}) = 2$, $d = 2$, *and* $\mathcal{E} \simeq [-K_S]^{\oplus 2}$;
(4) $c_2(\mathcal{E}) = 3$, $d = 1$, *and there exists a non-split exact sequence* $0 \to \mathcal{O}_{S'}(F) \to \pi^*\mathcal{E} \to \pi^*(-2K_S) - \mathcal{O}_{S'}(F) \to 0$ *as in* (3.7);
(5) $c_2(\mathcal{E}) = 3$, $d = 1$, *and* $\mathcal{E} \simeq [-K_S] \oplus [-K_S + C]$ *where C is a* 0*-curve;*
(6) $c_2(\mathcal{E}) = 3$, $d = 1$, *and* $\mathcal{E} \simeq [-K_S] \oplus [-2K_S + C]$, *where C is a* (-1)*-curve;*
(7) $c_2(\mathcal{E}) = 3$, $d = 1$, *and* $\mathcal{E} \simeq [-K_S] \oplus [-3K_S]$;
(8) $c_2(\mathcal{E}) = 3$, $d = 2$, *and there exists a non-split exact sequence* $0 \to \pi^*(-K_S) + \mathcal{O}_{S'}(E_0) \to \pi^*\mathcal{E} \to \pi^*(-K_S) - \mathcal{O}_{S'}(E_0) \to 0$ *as in* (3.11);
(9) $c_2(\mathcal{E}) = 3$, $d = 3$, *and* $\mathcal{E} \simeq [-K_S]^{\oplus 2}$.

Proof. Suppose that $c_2(\mathcal{E}) \leq 3$. Then $d \leq c_2(\mathcal{E}) \leq d + 2$ by (3.9).

In case $c_2(\mathcal{E}) = d$, we have $d \leq 3$ and $\mathcal{E}$ is a vector bundle of the type (1), (3), or (9) by (3.9).

In case $c_2(\mathcal{E}) = d + 1$, we have $d \leq 2$ and $\mathcal{E}$ is of the type (2) or (8) by (3.11).

In case $c_2(\mathcal{E}) = d + 2$, we have $d = 1$. By (3.13) $\mathcal{E}$ is of the type (5), (6), or (7) unless $c_1(\mathcal{E}) = -2K_S$; if $c_1(\mathcal{E}) = -2K_S$, then $\mathcal{E}$ is of the type (4) in view of (3.7). □

Remark 3.15. On Del Pezzo surfaces S of degree $d \leq 7$, equality holds in (1.5) if and only if $d = 1$ and $\mathcal{E} \simeq [-K_S] \oplus [-tK_S]$ for some positive integer t. In view of (1.9) and (3.14), the assertion follows by an argument similar to that in the proof of (3.11).

4. On $\mathbb{P}^2$.

In this section we consider rank two ample vector bundles $\mathcal{E}$ on $\mathbb{P}^2$. We always denote $\mathcal{O}_{\mathbb{P}^2}$ by $\mathcal{O}$ for simplicity. Since $\mathrm{Pic}\,\mathbb{P}^2 \simeq \mathbb{Z} \cdot \mathcal{O}(1)$, we regard $c_1(\mathcal{E})$ as an integer. We have $c_1(\mathcal{E}) \geq 2$ because of (1.3).

The following theorem is essentially due to Van de Ven [19].

Theorem 4.1 (cf. [19]). *Let $\mathcal{E}$ be an ample vector bundle of rank two on $\mathbb{P}^2$. If $c_1(\mathcal{E}) \leq 3$, then $\mathcal{E}$ is one of the following:*

(i) $c_1(\mathcal{E}) = 2$ *and* $\mathcal{E} \simeq \mathcal{O}(1)^{\oplus 2}$;
(ii) $c_1(\mathcal{E}) = 3$ *and* $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(2)$;
(iii) $c_1(\mathcal{E}) = 3$ *and* $\mathcal{E} \simeq \mathcal{T}_{\mathbb{P}^2}$, *where* $\mathcal{T}_{\mathbb{P}^2}$ *is the tangent bundle of* $\mathbb{P}^2$.

Proof. We have $c_1(\mathcal{E}) \geq 2$. If $c_1(\mathcal{E}) = 2$ (resp. 3), then $\mathcal{E}|_L \simeq \mathcal{O}_L(1)^{\oplus 2}$ (resp. $\mathcal{O}_L(1) \oplus \mathcal{O}_L(2)$) for every line L in $\mathbb{P}^2$. Hence $\mathcal{E}$ is a uniform vector bundle if $c_1(\mathcal{E}) \leq 3$, and then the assertion follows by [19]. □

Remark 4.2. Note that $\mathcal{E}$ is not necessarily a uniform vector bundle if $c_1(\mathcal{E}) \geq 4$. In case $c_1(\mathcal{E}) = 4$, we have $1 \leq c_2(\mathcal{E}) \leq 15$ because of (1.4), and we obtain classification results if $c_2(\mathcal{E}) \leq 6$ (see (4.7)).

By definition, $\mathcal{E}$ is said to be *stable* (resp. *semistable*) if $c_1(\mathcal{E}) > 2t$ (resp. $c_1(\mathcal{E}) \geq 2t$) for every invertible subsheaf $\mathcal{O}(t)$ of $\mathcal{E}$.

Ample and stable vector bundles on $\mathbb{P}^2$ are studied by Le Potier [13]. In particular, these bundles with $c_1 = 4$ and $c_2 = 7, 8$ are studied in detail.

Next we consider the relation between $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$.

Theorem 4.3. *Let $\mathcal{E}$ be an ample vector bundle of rank two on $\mathbb{P}^2$. Then $c_2(\mathcal{E}) \geq c_1(\mathcal{E}) - 1$, and equality holds if and only if $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(t)$ for some positive integer t.*

Proof. From (1.5) we obtain $c_2(\mathcal{E}) \geq c_1(\mathcal{E}) - 1$ immediately. Suppose that $c_2(\mathcal{E}) = c_1(\mathcal{E}) - 1$. Then we obtain $c_1^2(\mathcal{E}) \geq 4c_2(\mathcal{E})$. If $c_1^2(\mathcal{E}) = 4c_2(\mathcal{E})$, we get $c_1(\mathcal{E}) = 2$, and hence $\mathcal{E} \simeq \mathcal{O}(1)^{\oplus 2}$ by (4.1). If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, then from (1.7) we obtain an exact sequence

$$0 \to \mathcal{O}(l) \to \mathcal{E} \to \mathcal{I}_Z(m) \to 0,$$

where $l, m \in \mathbb{Z}$ and $\mathcal{I}_Z$ is the ideal sheaf of a zero-dimensional subscheme Z of $\mathbb{P}^2$. Note that $l > m > 0$. We have

$$0 = c_2(\mathcal{E}) - (c_1(\mathcal{E}) - 1) = (l-1)(m-1) + \deg Z,$$

and hence $m = 1$ and $\deg Z = 0$. Then we get an exact sequence

$$0 \to \mathcal{O}(l) \to \mathcal{E} \to \mathcal{O}(1) \to 0,$$

which splits. Hence we obtain $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(l)$. Conversely, for every positive integer t, we easily see that $\mathcal{E} := \mathcal{O}(1) \oplus \mathcal{O}(t)$ satisfies $c_2(\mathcal{E}) = c_1(\mathcal{E}) - 1$. □

Proposition 4.4. *Let $\mathcal{E}$ be as above. Assume that $c_2(\mathcal{E}) > c_1(\mathcal{E}) - 1$. Then $c_2(\mathcal{E}) \geq 2c_1(\mathcal{E}) - 4$, and equality holds if and only if $\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(t)$ for some integer $t \geq 2$.*

Proof. If $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$, as in the proof of (4.3), we get an exact sequence

$$0 \to \mathcal{O}(l) \to \mathcal{E} \to \mathcal{I}_Z(m) \to 0,$$

where $l, m \in \mathbb{Z}$, $l > m > 0$, and $m^2 > \deg Z$ in view of (1.8). Then we have

$$0 < c_2(\mathcal{E}) - (c_1(\mathcal{E}) - 1) = (l-1)(m-1) + \deg Z \leq (l+m)(m-1),$$

and hence $m \geq 2$. Thus we see that

$$c_2(\mathcal{E}) - (2c_1(\mathcal{E}) - 4) = (l-2)(m-2) + \deg Z \geq 0,$$

and equality holds if and only if $m = 2$ and $\deg Z = 0$. Under these conditions, we obtain $\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(l)$. Conversely, for every integer $t \geq 2$, we see that $\mathcal{E} := \mathcal{O}(2) \oplus \mathcal{O}(t)$ satisfies $c_2(\mathcal{E}) = 2c_1(\mathcal{E}) - 4$.

If $c_1^2(\mathcal{E}) \leq 4c_2(\mathcal{E})$, we have

$$4(c_2(\mathcal{E}) - (2c_1(\mathcal{E}) - 4)) \geq (c_1(\mathcal{E}) - 4)^2 \geq 0,$$

and equality holds if and only if $c_1(\mathcal{E}) = 4$ and $c_2(\mathcal{E}) = 4$. Under these conditions, we find that $\chi(\mathcal{E}(-2)) = 2$, $h^2(\mathcal{E}(-2)) = 0$, and $h^0(\mathcal{E}(-3)) = 0$. Hence there exists a non-zero section $s \in H^0(\mathcal{E}(-2))$ such that $\dim(s)_0 \leq 0$. We have $(s)_0 = \emptyset$ since $c_2(\mathcal{E}(-2)) = 0$. Then the section s induces an exact sequence

$$0 \to \mathcal{O} \xrightarrow{\cdot s} \mathcal{E}(-2) \to \det(\mathcal{E}(-2)) \to 0.$$

Since $\det(\mathcal{E}(-2)) \simeq \mathcal{O}$, we obtain that $\mathcal{E} \simeq \mathcal{O}(2)^{\oplus 2}$. □

Theorem 4.5. *Let $\mathcal{E}$ be an ample vector bundle of rank two on $\mathbb{P}^2$.*

(I) $c_2(\mathcal{E}) = c_1(\mathcal{E})$ *if and only if either*
- (I-i) $\mathcal{E} \simeq \mathcal{O}(2)^{\oplus 2}$, *or*
- (I-ii) $\mathcal{E} \simeq \mathcal{T}_{\mathbb{P}^2}$.

(II) $c_2(\mathcal{E}) = c_1(\mathcal{E}) + 1$ *if and only if either*
- (II-i) $\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(3)$, *or*
- (II-ii) $\mathcal{E}$ *is semistable, but not stable, and there exists an exact sequence* $0 \to \mathcal{O}(2) \to \mathcal{E} \to \mathcal{I}_x(2) \to 0$, *where x is a point of* $\mathbb{P}^2$.

(III) $c_2(\mathcal{E}) = c_1(\mathcal{E}) + 2$ *if and only if $\mathcal{E}$ is one of the following:*
- (III-i) $\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(4)$;
- (III-ii) $\mathcal{E}$ *is not semistable, and there exists an exact sequence* $0 \to \mathcal{O}(3) \to \mathcal{E} \to \mathcal{I}_x(2) \to 0$, *where x is a point of* $\mathbb{P}^2$;
- (III-iii) $\mathcal{E} \simeq \mathcal{T}_{\mathbb{P}^2}(1)$;
- (III-iv) $\mathcal{E}$ *is stable and there exists an exact sequence* $0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{I}_Z(3) \to 0$, *where Z is a zero-dimensional subscheme of $\mathbb{P}^2$ with* $\deg Z = 3$.

Proof. Suppose that $c_1(\mathcal{E}) \leq c_2(\mathcal{E}) \leq c_1(\mathcal{E}) + 2$. From (4.4) we get $c_2(\mathcal{E}) \geq 2c_1(\mathcal{E}) - 4$, and hence $c_1(\mathcal{E}) \leq 6$. If $c_1(\mathcal{E}) = 6$, then $c_2(\mathcal{E}) = 2c_1(\mathcal{E}) - 4 = 8$. Hence we obtain $\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(4)$ by (4.4). This is the case (III-i).

If $c_1(\mathcal{E}) \leq 3$, we obtain $\mathcal{E} \simeq \mathcal{T}_{\mathbb{P}^2}$ by (4.1). This is the case (I-ii).

If $c_1(\mathcal{E}) = 5$, then we get $6 \leq c_2(\mathcal{E}) \leq 7$. In case $(c_1(\mathcal{E}), c_2(\mathcal{E})) = (5,6)$, we have $c_2(\mathcal{E}) = 2c_1(\mathcal{E}) - 4$, and hence $\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(3)$ by (4.4). This is the case (II-i). In case $(c_1(\mathcal{E}), c_2(\mathcal{E})) = (5,7)$, we find that $\chi(\mathcal{E}(-2)) = 3$, $h^2(\mathcal{E}(-2)) = 0$, and $h^0(\mathcal{E}(-4)) = 0$. If $h^0(\mathcal{E}(-3)) > 0$, there exists a non-zero section $s \in H^0(\mathcal{E}(-3))$ such that $\dim(s)_0 \leq 0$. Since $c_2(\mathcal{E}(-3)) = 1$, we see that $(s)_0$ is one reduced point x of $\mathbb{P}^2$. Hence the section s induces an exact sequence

$$0 \to \mathcal{O} \xrightarrow{\cdot s} \mathcal{E}(-3) \to \mathcal{I}_x(-1) \to 0$$

since $c_1(\mathcal{E}(-3)) = -1$. Then tensoring with $\mathcal{O}(3)$ gives an exact sequence

$$0 \to \mathcal{O}(3) \to \mathcal{E} \to \mathcal{I}_x(2) \to 0.$$

This is the case (III-ii). If $h^0(\mathcal{E}(-3)) = 0$, there exists a non-zero section $s \in H^0(\mathcal{E}(-2))$ such that $\dim(s)_0 \leq 0$. Since $c_2(\mathcal{E}(-2)) = 1$, we see that $(s)_0$ is one reduced point x of $\mathbb{P}^2$. Let $\pi : \Sigma_1 \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at x and E the exceptional curve of π. Then π^*s determines a non-zero section $s' \in H^0(\Sigma_1, \pi^*(\mathcal{E}(-2)) \otimes [-E])$ such that $(s')_0 = \emptyset$. Hence we get an exact sequence

$$0 \to \mathcal{O}_{\Sigma_1} \xrightarrow{\cdot s'} \pi^*(\mathcal{E}(-2)) \otimes [-E] \to \det(\pi^*(\mathcal{E}(-2)) \otimes [-E]) \to 0$$

and then the exact sequence

$$0 \to [\pi^*\mathcal{O}(2) + E] \to \pi^*\mathcal{E} \to [\pi^*\mathcal{O}(3) - E] \to 0$$

is induced. This exact sequence is non-split, otherwise we have

$$\mathcal{O}_E^{\oplus 2} \simeq [\pi^*\mathcal{E}]_E \simeq [\pi^*\mathcal{O}(2) + E]_E \oplus [\pi^*\mathcal{O}(3) - E]_E \simeq \mathcal{O}_E(-1) \oplus \mathcal{O}_E(1),$$

a contradiction. Since $\mathrm{Ext}^1([\pi^*\mathcal{O}(3) - E], [\pi^*\mathcal{O}(2) + E]) \simeq H^1(\Sigma_1, \pi^*\mathcal{O}(-1) + 2E) \simeq \mathbb{C}^1$, we see that $\mathcal{E}$ is unique up to isomorphism if it exists. On the other hand, $\mathcal{T}_{\mathbb{P}^2}(1)$ satisfies the condition of $\mathcal{E}$. Thus we conclude that $\mathcal{E} \simeq \mathcal{T}_{\mathbb{P}^2}(1)$. This is the case (III-iii).

If $c_1(\mathcal{E}) = 4$, then we get $4 \leq c_2(\mathcal{E}) \leq 6$. In case $(c_1(\mathcal{E}), c_2(\mathcal{E})) = (4,4)$, we have $c_2(\mathcal{E}) = 2c_1(\mathcal{E}) - 4$, and hence $\mathcal{E} \simeq \mathcal{O}(2)^{\oplus 2}$ by (4.4). This is the case (I-i). In case $(c_1(\mathcal{E}), c_2(\mathcal{E})) = (4,5)$, we find that $\chi(\mathcal{E}(-2)) = 1$, $h^2(\mathcal{E}(-2)) = 0$, and $h^0(\mathcal{E}(-3)) = 0$. Hence there exists a non-zero section $s \in H^0(\mathcal{E}(-2))$ such that $\dim(s)_0 \leq 0$. Since $c_1(\mathcal{E}(-2)) = 0$ and $c_2(\mathcal{E}(-2)) = 1$, the section s induces an exact sequence

$$0 \to \mathcal{O} \xrightarrow{\cdot s} \mathcal{E}(-2) \to \mathcal{I}_x \to 0,$$

where x is a point of $\mathbb{P}^2$. Then tensoring with $\mathcal{O}(2)$ gives an exact sequence

$$0 \to \mathcal{O}(2) \to \mathcal{E} \to \mathcal{I}_x(2) \to 0.$$

Note that $\mathcal{E}$ is semistable since $h^0(\mathcal{E}(-3)) = 0$. This is the case (II-ii). In case $(c_1(\mathcal{E}), c_2(\mathcal{E})) = (4,6)$, we find that $\chi(\mathcal{E}(-1)) = 4$, $h^2(\mathcal{E}(-1)) = 0$, and $h^0(\mathcal{E}(-3)) = 0$. If $h^0(\mathcal{E}(-2)) > 0$, then a non-zero section $s \in H^0(\mathcal{E}(-2))$ induces an exact sequence

$$0 \to \mathcal{O} \xrightarrow{\cdot s} \mathcal{E}(-2) \to \mathcal{I}_Z \to 0,$$

where Z is a zero-dimensional subscheme of $\mathbb{P}^2$ with $\deg Z = c_2(\mathcal{E}(-2)) = 2$. Then we have $Z = x + x'$, where x and x' are two points of $\mathbb{P}^2$ (not necessarily distinct). Let L be the line passing through x and x'. Since $0 \neq s|_L \in H^0(\mathcal{E}(-2)|_L)$, we see that $\mathcal{E}(-2)|_L \simeq \mathcal{O}_L(t) \oplus \mathcal{O}_L(-t)$ for some integer $t \geq 2$. It follows that $\mathcal{E}|_L \simeq \mathcal{O}_L(2+t) \oplus \mathcal{O}_L(2-t)$, which is a contradiction since $\mathcal{E}|_L$ is ample. Thus we get $h^0(\mathcal{E}(-2)) = 0$. Then there exists a non-zero section $s \in H^0(\mathcal{E}(-1))$ such that $\dim(s)_0 \leq 0$. The section s induces an exact sequence

$$0 \to \mathcal{O} \xrightarrow{\cdot s} \mathcal{E}(-1) \to \mathcal{I}_Z(2) \to 0,$$

where Z is a zero-dimensional subscheme of $\mathbb{P}^2$ with $\deg Z = c_2(\mathcal{E}(-1)) = 3$. Then tensoring with $\mathcal{O}(1)$ gives an exact sequence

$$0 \to \mathcal{O}(1) \to \mathcal{E} \to \mathcal{I}_Z(3) \to 0.$$

Note that $\mathcal{E}$ is stable since $h^0(\mathcal{E}(-2)) = 0$. This is the case (III-iv). □

Remark 4.6. We make some comments on (4.5).

The existence of $\mathcal{E}$ in the cases (II-ii) and (III-ii) is shown by Fujisawa [5, Example (3.9)]; furthermore, the existence of $\mathcal{E}$ in the case (III-iv) is shown by Fujisawa [5, Example (3.11)].

In the case (III-iv), by Szurek-Wiśniewski [18, p. 298, REMARK], $\mathcal{E}(-1)$ is spanned and the evaluation $\mathcal{O}^{\oplus 4} \to \mathcal{E}(-1)$ induces an exact sequence $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{E}(-1) \to 0$.

Using the theorems above, we can classify rank two ample vector bundles with small c_2 on $\mathbb{P}^2$.

Corollary 4.7. *Rank two ample vector bundles $\mathcal{E}$ with $c_2(\mathcal{E}) \leq 6$ on $\mathbb{P}^2$ are the following:*

(1) $c_2(\mathcal{E}) = 1$ *and* $\mathcal{E} \simeq \mathcal{O}(1)^{\oplus 2}$;
(2) $c_2(\mathcal{E}) = 2$ *and* $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(2)$;
(3) $c_2(\mathcal{E}) = 3$ *and* $\mathcal{E} \simeq \mathcal{T}_{\mathbb{P}^2}$;
(4) $c_2(\mathcal{E}) = 3$ *and* $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(3)$;
(5) $c_2(\mathcal{E}) = 4$ *and* $\mathcal{E} \simeq \mathcal{O}(2)^{\oplus 2}$;
(6) $c_2(\mathcal{E}) = 4$ *and* $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(4)$;
(7) $c_2(\mathcal{E}) = 5$, *$\mathcal{E}$ is semistable, but not stable, and there exists an exact sequence* $0 \to \mathcal{O}(2) \to \mathcal{E} \to \mathcal{I}_x(2) \to 0$, *where $\mathcal{I}_x$ is the ideal sheaf of a point $x \in \mathbb{P}^2$;*

(8) $c_2(\mathcal{E}) = 5$ *and* $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(5)$;
(9) $c_2(\mathcal{E}) = 6$, $\mathcal{E}$ *is stable, and there exists an exact sequence* $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{E}(-1) \to 0$;
(10) $c_2(\mathcal{E}) = 6$ *and* $\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(3)$;
(11) $c_2(\mathcal{E}) = 6$ *and* $\mathcal{E} \simeq \mathcal{O}(1) \oplus \mathcal{O}(6)$.

Proof. Suppose that $c_2(\mathcal{E}) \leq 6$. In case $c_1(\mathcal{E}) \leq 3$, $\mathcal{E}$ is a vector bundle of the type (1), (2), or (3) by (4.1). In case $c_1(\mathcal{E}) \geq 4$, we have $4 \leq c_1(\mathcal{E}) \leq c_2(\mathcal{E}) + 1 \leq 7$ by (4.3). Then we see that $c_2(\mathcal{E}) \leq c_1(\mathcal{E}) + 2$ and hence $\mathcal{E}$ is of the type (4), (5), (6), (7), (8), (9), (10), or (11) by (4.3), (4.5), and (4.6). □

Remark 4.8. We conclude with a remark that (4.3) gives the equality condition in (1.5) on $\mathbb{P}^2$.

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ON POLARIZED MANIFOLDS OF SECTIONAL GENUS THREE

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§ 1. Introduction

Let L be an ample line bundle on a complex projective manifold M of dimension $n\geqq 2$. The sectional genus $g=g(M, L)$ of a polarized manifold (M, L) is defined by the formula $2g(M, L)-2=(K+(n-1)L)L^{n-1}$, where K is the canonical bundle of M. For polarized manifolds over $\boldsymbol{C}$, it is known that g takes non-negative integers ([F1; Corollary 1] or [I2; Lemma 7]).

In many papers the structure of (M, L) with low g has been studied: see [F1] or [I2] for $g\leqq 1$; [BeLP] for $g=n=2$; [F2] for $g=2$; [Ma] for $g=3$ and $n=2$. As for the case $g=3$ and $n\geqq 3$, we see from the results of [F1] or [I2] that (M, L) is one of the following types.

(1.1) There is an effective divisor E on M such that $(E, L_E)\cong(\boldsymbol{P}^{n-1}, \mathcal{O}(1))$ and $[E]_E=\mathcal{O}(-1)$.

(1.2) There is a fibration $\Phi: M\to C$ over a smooth curve C such that every fiber F of Φ is a hyperquadric in $\boldsymbol{P}^n$ and $L_F=\mathcal{O}(1)$.

(1.3) There is a fibration $\Phi: M\to C$ over a smooth curve C such that $(F, L_F)\cong(\boldsymbol{P}^2, \mathcal{O}(2))$ for every fiber F of Φ.

(1.4) (M, L) is a scroll over a smooth surface S.

(1.5) $K+(n-2)L$ is nef.

(1.6) (M, L) is a scroll over a smooth curve of genus three.

In the case (1.6), we have nothing more to say.

In the case (1.1), using the theory of minimal reduction (e.g. [I2; (0.11)], [F2; (1.9)], or [F; (11.11)]), we see (M, L) is obtained by a finite number of simple blow-ups of a polarized manifold (M', L') which is of type (1.3) or (1.5).

The cases (1.2) and (1.3) are further studied in § 2 and § 3, which is the main part of this paper. We shall see our classification results are similar to those in case $g=2$, but the computations are more complicated.

In the case (1.4), $(M, L)\cong(\boldsymbol{P}_S(\mathcal{E}), H(\mathcal{E}))$ and $g(S, \det\mathcal{E})=3$ for some vector bundle $\mathcal{E}$ on S, thus the classification of (M, L) is reduced to the classification of ample vector bundles $\mathcal{E}$ for each polarized surface with $g=3$. Under the

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additional condition that L is spanned, the classification was obtained in [BiLL]. Without this condition, however, we have only some partial results and our classification is not yet complete. The author hopes this case will be treated in a future paper.

The case (1.5) is a kind of "general type". For any fixed n, there are only finitely many deformation types of (M, L). (See [F; (13.1)].) But it seems to be difficult to enumerate all such deformation types.

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Notation and Terminology

Basically we use the customary notation in algebraic geometry as in [H2]. All varieties are defined over $\boldsymbol{C}$ and assumed to be complete. Vector bundles are often identified with locally free sheaves of their sections and these words are used interchangeably. Line bundles are identified with linear equivalence classes of Cartier divisors, and their tensor products are denoted additively, while we use multiplicative notation for intersection products in Chow rings. The linear equivalence class is denoted by [ ], and its corresponding invertible sheaf is denoted by $\mathcal{O}[\ ]$. We use { } for the homology class of an algebraic cycle.

Given a morphism $f: X \to Y$ and a line bundle A on Y, we denote f^*A by A_X, or sometimes by A for short when there is no danger of confusion. The canonical bundle of a manifold M is denoted by K^M, unlike the customary notation K_M. The $\mathcal{O}(1)$'s of projective spaces $\boldsymbol{P}_\alpha$, $\boldsymbol{P}_\beta$, $\cdots$ will be denoted by H_α, H_β, $\cdots$. Given a vector bundle $\mathcal{E}$ on X, we denote by $\boldsymbol{P}_X(\mathcal{E})$ (or $\boldsymbol{P}(\mathcal{E})$) the associated projective space bundle, and denote by $H(\mathcal{E})$ the tautological line bundle on $\boldsymbol{P}(\mathcal{E})$ in the sense of [H2]. The pair $(\boldsymbol{P}(\mathcal{E}), H(\mathcal{E}))$ is called the scroll of $\mathcal{E}$.

§ 2. The case of a hyperquadric fibration over a curve

In this section, we study the case (1.2), following the idea in [F2; § 3].

(2.1) Since $h^0(F, L_F)=n+1$, $\mathcal{E}:=\Phi_*\mathcal{O}_M[L]$ is a locally free sheaf of rank $n+1$ on C and a natural map $\Phi^*\mathcal{E}\to L$ is surjective. This yields a C-morphism $\rho: M\to \boldsymbol{P}_C(\mathcal{E})$ and for every point x on C the restriction of ρ to $F_x:=\Phi^{-1}(x)$ is an embedding of F_x into $\boldsymbol{P}^n$. Hence ρ itself is an embedding and M is a member of $|2H(\mathcal{E})+B_{P(\mathcal{E})}|$ for some line bundle B on C. We put $d=L^n$, $e=c_1(\mathcal{E})$, $b=\deg B$ and denote by $g(C)$ the genus of C. After simple computation, we get $d=2e+b$, $2g(C)+e+b=4$, and $s:=2e+(n+1)b\geqq 0$. Furthermore in the last inequality, equality holds if and only if every fiber of Φ is smooth by [F2; (3.3)]. From these results, we have $(n+1)d+s+4ng(C)=8n$, hence $g(C)$

$=0$ or 1.

(2.2) We first study the case $g(C)=1$. In this case, C is an elliptic curve and we have $e=d-2$ and $b=4-d$ from the equality above. Hence we obtain $d\leqq 6$, since $s\geqq 0$ and $n\geqq 3$.

(2.3) We consider the ampleness of $\mathcal{E}$. If $\mathcal{E}$ is ample, then $\det\mathcal{E}$ is ample and $e=c_1(\mathcal{E})>0$. It follows that $d>2$, hence $\mathcal{E}$ is not ample when $d\leqq 2$. On the other hand, $\mathcal{E}$ is ample when $d\geqq 5$ by the argument in [F2; (3.13)]. In general, for any indecomposable vector bundle $\mathcal{F}$ on an elliptic curve, $\mathcal{F}$ is ample if and only if $c_1(\mathcal{F})>0$ (for a proof, see e.g. [H1]). Thus when $d=3$ or 4, $\mathcal{E}$ is ample if it is indecomposable.

(2.4) When $d=3$ or 4, we can find an example of (M, L) similarly as in [F2; (3.12)]. We can also find an example of (M, L) with $d=6$ as follows. Let C be a smooth elliptic curve and take a line bundle $\mathcal{L}$ on C with $\deg\mathcal{L}=1$. We put $\mathcal{E}=\mathcal{L}^{\oplus 4}$, then $\mathcal{E}$ is ample, $c_1(\mathcal{E})=4$, $\boldsymbol{P}_C(\mathcal{E})\cong C\times\boldsymbol{P}^3_\sigma$, and $H(\mathcal{E})=H_\sigma+\mathcal{L}_{P(\mathcal{E})}$, where H_σ is the pullback of $\mathcal{O}(1)$ on $\boldsymbol{P}^3_\sigma$. Putting $B=-2\mathcal{L}$, we have $\deg B=-2$ and $2H(\mathcal{E})+B_{P(\mathcal{E})}=2H_\sigma$. Then a general member M of $|2H(\mathcal{E})+B_{P(\mathcal{E})}|$ is smooth and, putting $L=[H(\mathcal{E})]_M$, we obtain an expected example of (M, L) with $d=6$.

(2.5) From now on, we study the case $g(C)=0$. In this case, $C\cong\boldsymbol{P}^1_\xi$ and we have $e=d-4$ and $b=8-d$ from the equality in (2.1). Hence we obtain $d\leqq 12$, since $s\geqq 0$ and $n\geqq 3$. Furthermore when $d=11$ or 12, we have $n=3$; when $d=12$, we have $s=0$ and Φ is a $\boldsymbol{P}^1\times\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1_\xi$.

(2.6) We put $P=\boldsymbol{P}_C(\mathcal{E})$, $H=H(\mathcal{E})$, and denote by H_ξ the pullback of $\mathcal{O}(1)$ on $\boldsymbol{P}^1_\xi$. Since $C\cong\boldsymbol{P}^1_\xi$, we can describe $\mathcal{E}\cong\mathcal{O}(e_0)\oplus\cdots\oplus\mathcal{O}(e_n)$, where $e_0, \cdots, e_n\in\boldsymbol{Z}$, $e_0\leqq\cdots\leqq e_n$, and $\sum_{i=0}^n e_i=e$. We denote $\mathcal{O}(e_0)\oplus\cdots\oplus\mathcal{O}(e_n)$ by $\mathcal{O}(e_0, \cdots, e_n)$ for simplicity. We shall classify $\mathcal{E}\cong\mathcal{O}(e_0, \cdots, e_n)$ for each $d=1, 2, \cdots, 12$.

(2.7)

LEMMA. $2(e_{n-1}+e_n)<d$ *when* $e_0\leqq 0$.

Proof. (cf. [F2; (3.24)]). A natural surjection $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-1})$ gives a prime divisor $D_1:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-1}))$ on P. Similarly $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-2}, e_n)$ gives a prime divisor $D_2:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-2}, e_n))$ on P and $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-2})$ gives a subvariety $W:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-2}))$ of P. We have $D_1\in|H-e_nH_\xi|$, $D_2\in|H-e_{n-1}H_\xi|$, and $W=D_1\cap D_2$ as schemes. When $e_0\leqq 0$, we have $W\not\subset M$ since H_W is not ample. Hence $\dim(M\cap W)=n-2$ and $0<L^{n-2}\{M\cap W\}=H^{n-2}(2H+bH_\xi)(H-e_nH_\xi)(H-e_{n-1}H_\xi)=d-2(e_{n-1}+e_n)$. □

(2.8) Suppose that $d=1$. We have $e=-3$, $b=7$, and $M\in|2H+7H_\xi|$. By (2.7), $\mathcal{E}\cong\mathcal{O}(-3, 0, \cdots, 0)$, $\mathcal{O}(-2, -1, 0, \cdots, 0)$, or $\mathcal{O}(-1, -1, -1, 0, \cdots, 0)$.

(2.8.1) When $\mathcal{E}\cong\mathcal{O}(-1, -1, -1, 0, \cdots, 0)$, we have $n\leqq 4$ by the argument in [F2; (3.21)]. Indeed, we have

$$P \cong \left\{ \begin{array}{c} (\xi_0 : \xi_1) \times (\sigma_0 : \sigma_1 : \sigma_2 : \sigma_{30} : \sigma_{31} : \cdots : \sigma_{n0} : \sigma_{n1}) \in \boldsymbol{P}^1_\xi \times \boldsymbol{P}^{2n-2}_\sigma \\ | \xi_0 : \xi_1 = \sigma_{30} : \sigma_{31} = \cdots = \sigma_{n0} : \sigma_{n1} \end{array} \right\},$$

$H = H_\sigma - H_\xi$, and $M \in |2H_\sigma + 5H_\xi|$. Thus we can describe

$$M = \{q_0(\sigma)\xi_0^5 + q_1(\sigma)\xi_0^4\xi_1 + \cdots + q_6(\sigma)\xi_1^5 = 0 \text{ in } P\},$$

where $q_0, \cdots, q_6$ are homogeneous polynomials of degree two in $\sigma_0, \sigma_1, \cdots, \sigma_{n1}$. In this defining equation of M, we put

$$\sigma_0 = a_{00}\xi_0 + a_{01}\xi_1,\ \sigma_1 = a_{10}\xi_0 + a_{11}\xi_1,\ \sigma_2 = a_{20}\xi_0 + a_{21}\xi_1,$$

$$\sigma_{30} = a_3\xi_0,\ \sigma_{31} = a_3\xi_1,\ \cdots,\ \sigma_{n0} = a_n\xi_0,\ \sigma_{n1} = a_n\xi_1,$$

where $a_{00}, a_{01}, \cdots, a_n$ are constants. Then we obtain an equation

$$Q_0(a)\xi_0^7 + Q_1(a)\xi_0^6\xi_1 + \cdots + Q_7(a)\xi_1^7 = 0,$$

where $Q_0, \cdots, Q_7$ are homogeneous polynomials of degree two in $(a) := (a_{00}, a_{01}, \cdots, a_n)$. If $n \geqq 5$, then $Q_0(a) = \cdots = Q_7(a) = 0$ has a non-trivial solution. We fix such a solution (a) and define a rational map $\alpha : \boldsymbol{P}^1_\xi \to \boldsymbol{P}^{2n-2}_\sigma$ by

$$\alpha(\xi_0 : \xi_1) := (a_{00}\xi_0 + a_{01}\xi_1 : a_{10}\xi_0 + a_{11}\xi_1 : a_{20}\xi_0 + a_{21}\xi_1)$$
$$: a_3\xi_0 : a_3\xi_1 : \cdots : a_n\xi_0 : a_n\xi_1).$$

If α is not a morphism, then $a_{00} : a_{10} : a_{20} = a_{01} : a_{11} : a_{21}$ and $a_3 = \cdots = a_n = 0$. Since (a) is non-trivial, the equations

$$\sigma_0 : \sigma_1 : \sigma_2 = a_{00} : a_{10} : a_{20} = a_{01} : a_{11} : a_{21},\ \sigma_{30} = \sigma_{31} = \cdots = \sigma_{n0} = \sigma_{n1} = 0$$

determine a point z on $\boldsymbol{P}^{2n-2}_\sigma$. Let Z be the fiber of $\boldsymbol{P}^1_\xi \times \boldsymbol{P}^{2n-2}_\sigma \to \boldsymbol{P}^{2n-2}_\sigma$ over z. Then we have $Z \subset M$ by the definition of Z, hence $0 < LZ = HZ = (H_\sigma - H_\xi)Z = -1$. This is a contradiction, thus α is a morphism. Let Γ be the graph of α. Then $\Gamma \subset M$ by the definition of α, hence $0 < L\Gamma = H\Gamma = (H_\sigma - H_\xi)\Gamma$. However, since $H_\sigma\Gamma = H_\xi\Gamma = 1$, this is a contradiction too. Hence we have proved that $n \leqq 4$, thus $\mathcal{E} \cong \mathcal{O}(-1, -1, -1, 0)$ or $\mathcal{O}(-1, -1, -1, 0, 0)$. If $\mathcal{E} \cong \mathcal{O}(-1, -1, -1, 0)$, then $P \cong \{(\xi_0 : \xi_1) \times (\sigma_0 : \sigma_1 : \sigma_2 : \sigma_{30} : \sigma_{31}) \in \boldsymbol{P}^1_\xi \times \boldsymbol{P}^4_\sigma | \xi_0 : \xi_1 = \sigma_{30} : \sigma_{31}\}$. Thus the projection $\mu : P \to \boldsymbol{P}^4_\sigma$ is the blowing-up of $\boldsymbol{P}^4_\sigma$ with center $W := \{\sigma_{30} = \sigma_{31} = 0$ in $\boldsymbol{P}^4_\sigma\} \cong \boldsymbol{P}^2$. Since the exceptional divisor E of μ is a member of $|H_\sigma - H_\xi|$, we have $M \in |7H_\sigma - 5E|$. Hence M is the strict transform of a hypersurface of degree seven in $\boldsymbol{P}^4_\sigma$, which has singularities with multiplicity five along W.

(2.8.2) When $\mathcal{E} \cong \mathcal{O}(-2, -1, 0, \cdots, 0)$, we claim that $n \leqq 4$. The following argument is similar to (2.8.1). We have

$$P \cong \left\{ \begin{array}{c} (\xi_0 : \xi_1) \times (\sigma_0 : \sigma_{10} : \sigma_{11} : \sigma_{20} : \sigma_{21} : \sigma_{22} : \cdots : \sigma_{n0} : \sigma_{n1} : \sigma_{n2}) \in \boldsymbol{P}^1_\xi \times \boldsymbol{P}^{3n-1}_\sigma \\ |\xi_0 : \xi_1 = \sigma_{10} : \sigma_{11} = \sigma_{20} : \sigma_{21} = \sigma_{21} : \sigma_{22} = \cdots = \sigma_{n0} : \sigma_{n1} = \sigma_{n1} : \sigma_{n2} \end{array} \right\},$$

$H = H_\sigma - 2H_\xi$, and $M \in |2H_\sigma + 3H_\xi|$. Thus $M = \{q_0(\sigma)\xi_0^3 + q_1(\sigma)\xi_0^2\xi_1 + q_2(\sigma)\xi_0\xi_1^2 + q_3(\sigma)\xi_1^3 = 0 \text{ in } P\}$, where $q_0, \cdots, q_3$ are quadric polynomials in (σ). We put

$$\sigma_0 = a_{00}\xi_0^2 + a_{01}\xi_0\xi_1 + a_{02}\xi_1^2,\ \sigma_{10} = \xi_0(a_{10}\xi_0 + a_{11}\xi_1),\ \sigma_{11} = \xi_1(a_{10}\xi_0 + a_{11}\xi_1),$$

$$\sigma_{20} = a_2\xi_0^2,\ \sigma_{21} = a_2\xi_0\xi_1,\ \sigma_{22} = a_2\xi_1^2,\ \cdots,\ \sigma_{n0} = a_n\xi_0^2,\ \sigma_{n1} = a_n\xi_0\xi_1,\ \sigma_{n2} = a_n\xi_1^2.$$

Then from the defining equation of M above, we obtain an equation

$$Q_0(a)\xi_0^7 + Q_1(a)\xi_0^6\xi_1 + \cdots + Q_7(a)\xi_1^7 = 0,$$

where $Q_0, \cdots, Q_7$ are quadric polynomials in $(a) := (a_{00}, a_{01}, \cdots, a_n)$. If $n \geqq 5$, then $Q_0(a) = \cdots = Q_7(a) = 0$ has a non-trivial solution (a). We fix it and define a rational map $\alpha : \boldsymbol{P}^1_\xi \to \boldsymbol{P}^{3n-1}_\sigma$ by

$$\alpha(\xi_0 : \xi_1) := (a_{00}\xi_0^2 + a_{01}\xi_0\xi_1 + a_{02}\xi_1^2 : \xi_0(a_{10}\xi_0 + a_{11}\xi_1) : \xi_1(a_{10}\xi_0 + a_{11}\xi_1) :$$
$$a_2\xi_0^2 : a_2\xi_0\xi_1 : a_2\xi_1^2 : \cdots : a_n\xi_0^2 : a_n\xi_0\xi_1 : a_n\xi_1^2).$$

If α is not a morphism, then $a_2 = \cdots = a_n = 0$ and for some $(c_0 : c_1) \in \boldsymbol{P}^1_\xi$, we have $a_{10}c_0 + a_{11}c_1 = 0$ and $a_{00}c_0^2 + a_{01}c_0c_1 + a_{02}c_1^2 = 0$. In the case $a_{10} = a_{11} = 0$, let Z be the fiber of $\boldsymbol{P}^1_\xi \times \boldsymbol{P}^{3n-1}_\sigma \to \boldsymbol{P}^{3n-1}_\sigma$ over $z := (1 : 0 : \cdots : 0)$. Then we have $Z \subset M$, hence $0 < LZ = HZ = (H_\sigma - 2H_\xi)Z = -2$. This is a contradiction, thus $a_{10} \neq 0$ or $a_{11} \neq 0$.

In this case, $a_{00}\xi_0^2 + a_{01}\xi_0\xi_1 + a_{02}\xi_1^2$ is devided by $a_{10}\xi_0 + a_{11}\xi_1$ in $\boldsymbol{C}[\xi_0, \xi_1]$; we denote by $b_0\xi_0 + b_1\xi_1$ its quotient. We put

$$Z = \{\sigma_0 = b_0\sigma_{10} + b_1\sigma_{11},\ \sigma_{20} = \cdots = \sigma_{n2} = 0 \text{ in } P\}.$$

Then $\dim Z = 1$ and $Z \subset M$ by the definition of Z, hence $0 < LZ = HZ = (H_\sigma - 2H_\xi)Z$. However, since $H_\sigma Z = 1$ and $H_\xi Z = 1$, this is a contradiction too. Thus α is a morphism.

Let Γ be the graph of α. We have $\Gamma \subset M$ and then $0 < L\Gamma = H\Gamma = (H_\sigma - 2H_\xi)\Gamma$. However, since $H_\sigma\Gamma = 2$ and $H_\xi\Gamma = 1$, this is also a contradiction. Hence we have proved that $n \leqq 4$, thus $\mathcal{E} \cong \mathcal{O}(-2, -1, 0, 0)$ or $\mathcal{O}(-2, -1, 0, 0, 0)$.

(2.8.3) When $\mathcal{E} \cong \mathcal{O}(-3, 0, \cdots, 0)$, we claim that $n \leqq 4$ as before. P is isomorphic to

$$\left\{ \begin{array}{l} (\xi_0 : \xi_1) \times (\sigma_0 : \sigma_{10} : \sigma_{11} : \sigma_{12} : \sigma_{13} : \cdots : \sigma_{n0} : \sigma_{n1} : \sigma_{n2} : \sigma_{n3}) \in \boldsymbol{P}^1_\xi \times \boldsymbol{P}^{4n}_\sigma | \\ \xi_0 : \xi_1 = \sigma_{10} : \sigma_{11} = \sigma_{11} : \sigma_{12} = \sigma_{12} : \sigma_{13} = \cdots = \sigma_{n0} : \sigma_{n1} = \sigma_{n1} : \sigma_{n2} = \sigma_{n2} : \sigma_{n3} \end{array} \right\},$$

$H = H_\sigma - 3H_\xi$, and $M \in |2H_\sigma + H_\xi|$. Thus $M = \{q_0(\sigma)\xi_0 + q_1(\sigma)\xi_1 = 0 \text{ in } P\}$, where q_0 and q_1 are quadric polynomials in (σ). We put

$$\sigma_0 = a_{00}\xi_0^3 + a_{01}\xi_0^2\xi_1 + a_{02}\xi_0\xi_1^2 + a_{03}\xi_1^3,$$

$$\sigma_{10} = a_1\xi_0^3,\ \sigma_{11} = a_1\xi_0^2\xi_1,\ \sigma_{12} = a_1\xi_0\xi_1^2,\ \sigma_{13} = a_1\xi_1^3,\ \cdots,$$

$$\sigma_{n0} = a_n\xi_0^3,\ \sigma_{n1} = a_n\xi_0^2\xi_1,\ \sigma_{n2} = a_n\xi_0\xi_1^2,\ \sigma_{n3} = a_n\xi_1^3.$$

Then from the defining equation of M above, we obtain an equation

$$Q_0(a)\xi_0^7 + Q_1(a)\xi_0^6\xi_1 + \cdots + Q_7(a)\xi_1^7 = 0,$$

where $Q_0, \cdots, Q_7$ are quadric polynomials in $(a) := (a_{00}, a_{01}, \cdots, a_n)$. If $n \geqq 5$, then $Q_0(a) = \cdots = Q_7(a) = 0$ has a non-trivial solution (a). We fix it and define a rational map $\alpha: \boldsymbol{P}_\xi^1 \to \boldsymbol{P}_\sigma^{4n}$ by

$$\alpha(\xi_0 : \xi_1) := (a_{00}\xi_0^3 + a_{01}\xi_0^2\xi_1 + a_{02}\xi_0\xi_1^2 + a_{03}\xi_1^3 : a_1\xi_0^3 : a_1\xi_0^2\xi_1 : a_1\xi_0\xi_1^2 : a_1\xi_1^3 :$$

$$\cdots : a_n\xi_0^3 : a_n\xi_0^2\xi_1 : a_n\xi_0\xi_1^2 : a_n\xi_1^3).$$

If α is not a morphism, then $a_1 = \cdots = a_n = 0$. Let Z be the fiber of $\boldsymbol{P}_\xi^1 \times \boldsymbol{P}_\sigma^{4n} \to \boldsymbol{P}_\sigma^{4n}$ over $z := (1 : 0 : \cdots : 0)$. We have $Z \subset M$ and then $0 < LZ = HZ = (H_\sigma - 3H_\xi)Z = -3$. This is a contradiction, hence α is a morphism. Let Γ be the graph of α. We have $\Gamma \subset M$ and then $0 < L\Gamma = H\Gamma = (H_\sigma - 3H_\xi)\Gamma$. However, since $H_\sigma\Gamma = 3$ and $H_\xi\Gamma = 1$, this is a contradiction too. Hence we have proved that $n \leqq 4$, thus $\mathcal{E} \cong \mathcal{O}(-3, 0, 0, 0)$ or $\mathcal{O}(-3, 0, 0, 0, 0)$.

(2.9) Now we study the case $d=2$. We have $e=-2$, $b=6$, and $M \in |2H + 6H_\xi|$. By (2.7), $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0)$ or $\mathcal{O}(-1, -1, 0, \cdots, 0)$.

(2.9.1) When $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, \cdots, 0)$, we have $n \leqq 4$ as in (2.8.1). Hence $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, 0)$ or $\mathcal{O}(-1, -1, 0, 0, 0)$.

(2.9.2) When $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0)$, we have $n \leqq 4$ as in (2.8.2). Hence $\mathcal{E} \cong \mathcal{O}(-2, 0, 0, 0)$ or $\mathcal{O}(-2, 0, 0, 0, 0)$.

(2.10) Suppose that $d=3$. Then $e=-1$, $b=5$, and $M \in |2H + 5H_\xi|$. From (2.7), we have $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0, 1)$, $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, \cdots, 0, 1)$, or $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0)$.

(2.10.1) When $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0)$, we have $n \leqq 4$ as in (2.8.1). Hence $\mathcal{E} \cong \mathcal{O}(-1, 0, 0, 0)$ or $\mathcal{O}(-1, 0, 0, 0, 0)$.

(2.10.2) When $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, \cdots, 0, 1)$, we have $n \leqq 4$ by the argument in [F2; (3.23.2)] which is similar to (2.8.1). Hence $\mathcal{E} \cong \mathcal{O}(-1, -1, 0, 1)$ or $\mathcal{O}(-1, -1, 0, 0, 1)$.

(2.10.3) When $\mathcal{E} \cong \mathcal{O}(-2, 0, \cdots, 0, 1)$, we have $n \leqq 4$ as in (2.8.2) and (2.10.2). Hence $\mathcal{E} \cong \mathcal{O}(-2, 0, 0, 1)$ or $\mathcal{O}(-2, 0, 0, 0, 1)$.

The next lemma is useful for $d \geqq 4$.

(2.11)
LEMMA. *When* $d \geqq 4$, -1 *does not appear twice in* $\{e_0, \cdots, e_n\}$.

We can prove this lemma by the argument in [F2; (3.18)].

(2.12) Now we study the case $d=4$. We have $e=0$, $b=4$, and $M \in |2H+4H_\xi|$. By (2.7) and (2.11), $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0, 1)$ or $\mathcal{O}(0, \cdots, 0)$.

(2.12.1) When $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0, 1)$, we have $n \leqq 4$ as in (2.10.2). Hence $\mathcal{E} \cong \mathcal{O}(-1, 0, 0, 1)$ or $\mathcal{O}(-1, 0, 0, 0, 1)$.

(2.12.2) When $\mathcal{E} \cong \mathcal{O}(0, \cdots, 0)$, by the argument in [F2; (3.23.1)], we have $n \leqq 4$, $P \cong \boldsymbol{P}_\xi^1 \times \boldsymbol{P}_\sigma^n$, $\mathrm{Bs}|L|=\phi$, and the morphism $\varphi: M \to \boldsymbol{P}_\sigma^n$ defined by $|L|$ is a finite morphism of degree four. Conversely, a general member M of $|2H_\sigma+4H_\xi|$ on P does not contain any fiber of the projection $P \to \boldsymbol{P}_\sigma^n$, thus $L := H_M$ is ample and (M, L) is a polarized manifold of the above type.
The next lemma is useful for $d \geqq 5$.

(2.13)
LEMMA. $e_0 \geqq -1$ *when* $d \geqq 5$.

We can prove this lemma by the argument in [F2; (3.19)].
Similarly we obtain the following two lemmas.

(2.14)
LEMMA. $e_0 \geqq 0$ *when* $d \geqq 7$.

(2.15)
LEMMA. $e_0 \geqq 1$ *when* $d \geqq 9$.

(2.16) Now we study the case $d=5$. We have $e=1$, $b=3$, and $M \in |2H+3H_\xi|$. By (2.11) and (2.13), $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0, 2)$, $\mathcal{O}(-1, 0, \cdots, 0, 1, 1)$, or $\mathcal{O}(0, \cdots, 0, 1)$.

(2.16.1) When $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0, 2)$, we have $n \leqq 3$ similarly as in (2.10.2), hence $\mathcal{E} \cong \mathcal{O}(-1, 0, 0, 2)$. Furthermore $\mathrm{Bs}|L|$ is one point as in [F2; (3.23.2)].

(2.16.2) When $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0, 1, 1)$, we have $n \leqq 4$ and $\mathrm{Bs}|L|$ is one point as in (2.16.1). Thus $\mathcal{E} \cong \mathcal{O}(-1, 0, 1, 1)$ or $\mathcal{O}(-1, 0, 0, 1, 1)$.

(2.16.3) When $\mathcal{E} \cong \mathcal{O}(0, \cdots, 0, 1)$, by the argument in [F2; (3.24)], we have $n \leqq 4$ and $|L|$ makes M the normalization of a hypersuface of degree five in $\boldsymbol{P}^{n+1}$, which has triple points along a $\boldsymbol{P}^2$ in $\boldsymbol{P}^{n+1}$.

(2.17) Suppose that $d=6$. We have $e=2$, $b=2$, and $M \in |2H+2H_\xi|$. From (2.7), (2.11), and (2.13), we have $\mathcal{E} \cong \mathcal{O}(-1, 0, \cdots, 0, 1, 1, 1)$, $\mathcal{O}(0, \cdots, 0, 1, 1)$, $\mathcal{O}(0, \cdots, 0, 2)$.

(2.17.1) When $\mathcal{O}\cong\mathcal{E}(-1, 0, \cdots, 0, 1, 1, 1)$, we show that $n=3$ similarly as in (2.7). Natural surjections $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-1})$, $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-2}, e_n)$, and $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-3}, e_{n-1}, e_n)$ give prime divisors $D_1:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-1}))$, $D_2:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-2}, e_n))$, and $D_3:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-3}, e_{n-1}, e_n))$ respectively. A natural surjection $\mathcal{E}\to\mathcal{O}(e_0, \cdots, e_{n-3})$ gives a subvariety $W:=\boldsymbol{P}(\mathcal{O}(e_0, \cdots, e_{n-3}))$ of $P=\boldsymbol{P}(\mathcal{E})$. We have $D_1\in|H-e_nH_\xi|$, $D_2\in|H-e_{n-1}H_\xi|$, $D_3\in|H-e_{n-2}H_\xi|$, and $W=D_1\cap D_2\cap D_3$ as schemes. Since H_W is not ample, we have $W\not\subset M$, hence $\dim(M\cap W)=n-3$ and $0<L^{n-3}\{M\cap W\}=H^{n-3}(2H+2H_\xi)(H-H_\xi)^3=2e-4=0$ if $n\geqq 4$. This is a contradiction, thus we have $n=3$ and $\mathcal{E}\cong\mathcal{O}(-1, 1, 1, 1)$. By the argument in [F2; (3.26)], M is a double covering of $\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^2$ and its branch locus is a smooth member of $|4H_\xi+2H_\sigma|$. We also have $L=[H_\xi+H_\sigma]_M$.

(2.17.2) When $\mathcal{E}\cong\mathcal{O}(0, \cdots, 0, 1, 1)$, we have $n\leqq 4$ as in (2.16.3), hence $\mathcal{E}\cong\mathcal{O}(0, 0, 1, 1)$ or $\mathcal{O}(0, 0, 0, 1, 1)$. We show the existence of (M, L). When $\mathcal{E}\cong\mathcal{O}(0, 0, 1, 1)$, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^5|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}\}$ and $H=H_\sigma$. Let M be a general member of $|2H_\sigma+2H_\xi|$ and put $L=[H_\sigma]_M$. Then $\mathrm{Bs}\,|L|=\phi$ and the restriction of $P\to\boldsymbol{P}_\sigma^5$ to M is the morphism φ defined by $|L|$. If $\varphi: M\to\varphi(M)$ is not finite, M contains a fiber Z of $P\to\boldsymbol{P}_\sigma^5$ over one point z on the line $l:=\{\sigma_{20}=\sigma_{21}=\sigma_{30}=\sigma_{31}=0$ in $\boldsymbol{P}_\sigma^5\}$. Using homogeneous polynomials q_0, q_1, and q_2 of degree two in (σ), we can describe that $M=\{q_0(\sigma)\xi_0^2+q_1(\sigma)\xi_0\xi_1+q_2(\sigma)\xi_1^2=0$ in $P\}$. Then $Z\subset M$ if and only if $q_0(z)=q_1(z)=q_2(z)=0$. Thus if we choose q_0, q_1, and q_2 generally to satisfy that $l\cap\{q_0(\sigma)=q_1(\sigma)=q_2(\sigma)=0$ in $\boldsymbol{P}_\sigma^5\}=\phi$, then φ becomes finite and L is ample. Similarly we can find an example of (M, L) when $\mathcal{E}\cong\mathcal{O}(0, 0, 0, 1, 1)$.

(2.17.3) When $\mathcal{E}\cong\mathcal{O}(0, \cdots, 0, 2)$, we have $n\leqq 3$ as in (2.16.3), hence $\mathcal{E}\cong\mathcal{O}(0, 0, 0, 2)$. We can show the existence of (M, L) similarly as above.

When $d\geqq 7$, the situation is much simpler.

(2.18)

LEMMA. *$Bs|L|=\phi$ and L is very ample when $d\geqq 7$.*

We can prove this lemma similarly as in [F2; (3.31)]. This lemma tells us that our results overlap [I1; Theorem 4.3], but our method is different from his.

(2.19) Now we study the case $d=7$. We have $e=3$, $b=1$, and $M\in|2H+H_\xi|$. Furthermore $e_0\geqq 0$ by (2.14), and $e_2\geqq 1$ by the argument in [F2; (3.25)]. Hence $\mathcal{E}\cong\mathcal{O}(0, 0, 1, 2)$, $\mathcal{O}(0, 1, 1, 1)$, or $\mathcal{O}(0, 0, 1, 1, 1)$. In each case, (M, L) exists similarly as in (2.17.2). By the morphism defined by $|L|$, M is isomorphic to a manifold of degree seven in $\boldsymbol{P}^{n+3}$.

(2.20) Suppose that $d=8$. We have $e=4$, $b=0$, and $M\in|2H|$. Furthermore $e_0\geqq 0$ by (2.14), and $e_1\geqq 1$ by the argument in [F2; (3.26)]. Hence $\mathcal{E}\cong\mathcal{O}(0, 1, 1, 2)$, $\mathcal{O}(0, 1, 1, 1, 1)$, or $\mathcal{O}(1, 1, 1, 1)$.

(2.20.1) When $\mathcal{E}\cong\mathcal{O}(1,1,1,1)$, we have $P\cong \boldsymbol{P}^1_\xi\times\boldsymbol{P}^3_\sigma$, $H=H_\xi+H_\sigma$, and $M\in|2H_\sigma+2H_\xi|$. Hence M is a smooth divisor of bidegree (2, 2) on P. Conversely, let M be a general member of $|2H_\xi+2H_\sigma|$ and put $L=[H_\xi+H_\sigma]_M$. Since $\mathcal{E}$ is ample, L is ample and (M, L) is a polarized manifold of the above type.

(2.20.2) When $\mathcal{E}\cong\mathcal{O}(0,1,1,1,1)$, by the argument in [F2; (3.26)], M is a double covering of $\boldsymbol{P}^1_\xi\times\boldsymbol{P}^3_\sigma$ and its branch locus is a smooth member of $|2H_\xi+2H_\sigma|$. We have also $L=[H_\xi+H_\sigma]_M$.

(2.20.3) Even when $\mathcal{E}\cong\mathcal{O}(0,1,1,2)$, by the argument in [F2; (3.26)], we have a morphism $h: M\to\boldsymbol{P}^1_\xi\times\boldsymbol{P}^3_\sigma$ and $L=h^*(H_\xi+H_\sigma)$. Since L is ample, $h: M\to h(M)$ is finite and $h(M)\in|a_1H_\xi+a_2H_\sigma|$ for some non-negative integers a_1 and a_2. Then $8=L^3=(\deg h)\cdot[H_\xi+H_\sigma]^3_{h(M)}=(\deg h)(a_1+3a_2)$. From the construction of h, we get $\deg h=2$ and $a_1=a_2=1$. Hence $h(M)\in|H_\xi+H_\sigma|$ and $M\to h(M)$ is a double covering.

(2.21) Suppose that $d=9$. We have $e=5$, $b=-1$, and $M\in|2H-H_\xi|$. Since $e_0\geq 1$ by (2.15), $\mathcal{E}\cong\mathcal{O}(1,1,1,2)$ or $\mathcal{O}(1,1,1,1,1)$.

(2.21.1) When $\mathcal{E}\cong\mathcal{O}(1,1,1,1,1)$, similarly as in [F2; (3.27)], the restriction of the projection $P\cong\boldsymbol{P}^1_\xi\times\boldsymbol{P}^4_\sigma\to\boldsymbol{P}^4_\sigma$ to M is a blowing-up of $\boldsymbol{P}^4_\sigma$ and its center is a complete intersection of two hyperquadrics in $\boldsymbol{P}^4_\sigma$.

(2.21.2) When $\mathcal{E}\cong\mathcal{O}(1,1,1,2)$, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_2:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}^1_\xi\times\boldsymbol{P}^4_\sigma|\xi_0:\xi_1=\sigma_{30}:\sigma_{31}\}$, hence P is the blowing-up of $\boldsymbol{P}^4_\sigma$ with center $\{\sigma_{30}=\sigma_{31}=0 \text{ in } \boldsymbol{P}^4_\sigma\}$. The exceptional divisor E is $\{\sigma_{30}=\sigma_{31}=0 \text{ in } P\}\in|H_\sigma-H_\xi|$, thus $M\in|3H_\sigma-E|$ and M is the strict transform of a smooth hypercubic in $\boldsymbol{P}^4_\sigma$.

(2.22) Suppose that $d=10$. We have $e=6$, $b=-2$, and $M\in|2H-H_\xi|$. Since $e_0\geq 1$ by (2.15), $\mathcal{E}\cong\mathcal{O}(1,1,1,3)$, $\mathcal{O}(1,1,2,2)$, $\mathcal{O}(1,1,1,1,2)$, or $\mathcal{O}(1,1,1,1,1,1)$.

(2.22.1) When $\mathcal{E}\cong\mathcal{O}(1,1,1,1,1,1)$, we have $P\cong\boldsymbol{P}^1_\xi\times\boldsymbol{P}^5_\sigma$, $H=H_\xi+H_\sigma$, $M\in|2H_\sigma|$, and $L=[H_\xi+H_\sigma]_M$. Hence $M\cong\boldsymbol{P}^1_\xi\times Q$, where Q is a smooth hyperquadric in $\boldsymbol{P}^5_\sigma$.

(2.22.2) When $\mathcal{E}\cong\mathcal{O}(1,1,1,1,2)$, by the argument in [F2; (3.28)], we have M is the blowing-up of a hyperquadric in $\boldsymbol{P}^5_\sigma$ and its center is a smooth quadric surface.

(2.22.3) When $\mathcal{E}\cong\mathcal{O}(1,1,2,2)$, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}^1_\xi\times\boldsymbol{P}^5_\sigma|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}\}$, $H=H_\xi+H_\sigma$, $M\in|2H_\sigma|$, and $L=[H_\xi+H_\sigma]_M$. Since $\mathcal{E}$ is ample, H is ample and then L is ample for any general member M of $|2H_\sigma|$. Because of (2.18), M is embedded in $\boldsymbol{P}^9$ as a manifold of degree nine by the morphism defined by $|L|$. On the other hand, the restriction of the projection $\mu: P\to\boldsymbol{P}^5_\sigma$ to M is the morphism defined by $|L-$

$H_\xi|$, and M is birationally mapped onto $\mu(M)$. We have $10=L^3=3[H_\xi]_M[H_\sigma]_M^2+[H_\sigma]_M^3$ and $[H_\xi]_M[H_\sigma]_M^2=2$ since $M\to\boldsymbol{P}_\xi^1$ is a hyperquadric fibration. Thus the degree of $\mu(M)$ is four. Furthermore, since $\mu(P)=\{\sigma_{20}\sigma_{31}-\sigma_{30}\sigma_{21}=0 \text{ in } \boldsymbol{P}_\sigma^5\}$ and $M\in|2H_\sigma|$, $\mu(M)$ is a complete intersection of two hyperquadrics in $\boldsymbol{P}_\sigma^5$. Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 3)$, we have the same result as above.

(2.23) Suppose that $d=11$. We have $e=7$, $b=-3$, and $M\in|2H-3H_\xi|$. Since $e_0\geqq 1$ by (2.15), and since $n=3$ by (2.5), $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 4)$, $\mathcal{O}(1, 1, 2, 3)$, or $\mathcal{O}(1, 2, 2, 2)$.

(2.23.1) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 4)$, we claim that (M, L) does not exist. Assume that (M, L) exists. A natural surjection $\mathcal{E}\to\mathcal{O}(1, 1, 1)$ gives a prime divisor $W:=\boldsymbol{P}(\mathcal{O}(1, 1, 1))$ on P. We have $W\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^2$, $H_W=H_\xi+H_\sigma$, and $W\not\subset M$, hence $[M]_W=M\cap W\in|2H_W-3H_\xi|=|2H_\sigma-H_\xi|$. This is a contradiction, thus we have proved the claim.

(2.23.2) Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 2, 3)$, we can show that (M, L) does not exist. We have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31}:\sigma_{32})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^6|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}=\sigma_{31}:\sigma_{32}\}$ and $H=H_\sigma+H_\xi$. Assume that there exists a smooth member M of $|2H_\sigma-H_\xi|$. Then there is an exact sequence of normal bundles

$$0\longrightarrow\mathcal{N}_{B/M}\longrightarrow\mathcal{N}_{B/P}\longrightarrow[\mathcal{N}_{M/P}]_B\longrightarrow 0,$$

where $B:=\mathrm{Bs}|2H_\sigma-H_\xi|=\{\sigma_{20}=\sigma_{21}=\sigma_{30}=\sigma_{31}=\sigma_{32}=0 \text{ in } P\}\cong\boldsymbol{P}(\mathcal{O}(1, 1))$. Since B is the complete intersection of $D_1:=\{\sigma_{20}=\sigma_{21}=0 \text{ in } P\}\cong\boldsymbol{P}(\mathcal{O}(1, 1, 3))$ and $D_2:=\{\sigma_{30}=\sigma_{31}=\sigma_{32}=0 \text{ in } P\}\cong\boldsymbol{P}(\mathcal{O}(1, 1, 2))$, we have $\mathcal{N}_{B/P}\cong[\mathcal{N}_{D_1/P}]_B\oplus[\mathcal{N}_{D_2/P}]_B\cong[H_\sigma-H_\xi]_B\oplus[H_\sigma-2H_\xi]_B$. Also we have $\mathcal{N}_{M/P}\cong[2H_\sigma-H_\xi]_B$. Then the morphism $\varphi:[H_\sigma-H_\xi]_B\oplus[H_\sigma-2H_\xi]_B\to[2H_\sigma-H_\xi]_B$ corresponding to $\mathcal{N}_{B/P}\to[\mathcal{N}_{M/P}]_B$ is given by some $\varphi_1\in H^0(B, [H_\sigma]_B)$ and $\varphi_2\in H^0(B, [H_\sigma+H_\xi]_B)$. Since $[H_\sigma]_B[H_\sigma+H_\xi]_B=1$, φ_1 and φ_2 have a common zero point, at which φ is not surjective. This is a contradiction and (M, L) does not exist.

(2.23.3) When $\mathcal{E}\cong\mathcal{O}(1, 2, 2, 2)$, we can show the existence of (M, L). We have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_{10}:\sigma_{11}:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^6|\xi_0:\xi_1=\sigma_{10}:\sigma_{11}=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}\}$ and $H=H_\sigma+H_\xi$. Putting $U_i=\{\xi_i\neq 0 \text{ in } P\}$ and $V_j=\{\sigma_j\neq 0 \text{ in } P\}$, we take a rational section $s_1:=\{(U_i\cap V_j, \sigma_0^2/\xi_0\cdot\xi_i/\sigma_j^2)\}_{i,j}$ of $2H_\sigma-H_\xi$. Note that $h^0(P, 2H-3H_\xi)=h^0(\boldsymbol{P}_\xi^1, S^2(\mathcal{E})\otimes[-3H_\xi])=15$. Let $f_1, \cdots, f_{15}$ be rational functions on P such that

$$f_1=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{10}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{11}}{\xi_1},\qquad f_2=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{20}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{21}}{\xi_1},$$

$$f_3=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{30}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{31}}{\xi_1},\qquad f_4=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}^2}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{11}}{\xi_1},$$

$$f_5=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{11}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{11}^2}{\xi_1},\qquad f_6=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{20}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{21}}{\xi_1},$$

$$f_7=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{21}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{11}\sigma_{21}}{\xi_1},\qquad f_8=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{30}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{31}}{\xi_1},$$

$$f_9=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{31}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{11}\sigma_{31}}{\xi_1},\qquad f_{10}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}^2}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{21}}{\xi_1},$$

$$f_{11}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{21}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{21}^2}{\xi_1},\qquad f_{12}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{30}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_1},$$

$$f_{13}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{21}\sigma_{31}}{\xi_1},\qquad f_{14}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{30}^2}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_1},$$

$$f_{15}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_0}=\frac{\xi_0}{\sigma_0^2}\cdot\frac{\sigma_{31}^2}{\xi_1}.$$

Then $\boldsymbol{C}\langle f_1, \cdots, f_{15}\rangle$, the vector space spanned by $f_1, \cdots, f_{15}$ over $\boldsymbol{C}$, is isomorphic to $H^0(P, 2H_\sigma-H_\xi)$ by mapping each f_i to $f_i\cdot s_1$. Thus we can describe

$$|2H_\sigma-H_\xi|=\{\mathrm{div}(f\cdot s_1)\,|\,f\in\boldsymbol{C}\langle f_1, \cdots, f_{15}\rangle-0\},$$

where $\mathrm{div}(f\cdot s_1)$ is an effective divisor defined by a regular section $f\cdot s_1$ of $2H_\sigma-H_\xi$. Since $\mathrm{Bs}|2H_\sigma-H_\xi|=\{\sigma_{10}=\sigma_{11}=\cdots=\sigma_{31}=0 \text{ in } P\}\cong\boldsymbol{P}_\xi^1\times\{(1:0:\cdots:0)\}$, if we take $f=\sum_{i=1}^{15}c_if_i\in\boldsymbol{C}\langle f_1, \cdots, f_{15}\rangle$ with $(c_1, c_2, c_3)\neq(0, 0, 0)$, $\mathrm{div}(f\cdot s_1)$ is nonsingular along $\mathrm{Bs}|2H_\sigma-H_\xi|$. Thus a general member M of $|2H_\sigma-H_\xi|$ is smooth by Bertini's theorem. For such M, $L:=H_M$ is ample since $\mathcal{E}$ is ample, hence (M, L) is a polarized manifold as desired. Furthermore, similarly as in (2.16.3), $|L-H_\xi|$ makes M a desingularization of a variety of degree five in $\boldsymbol{P}_\sigma^6$.

(2.24) Suppose that $d=12$. We have $e=8$, $b=-4$, and $M\in|2H-4H_\xi|$. Since $e_0\geqq1$ by (2.15), and since $n=3$ by (2.5), $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 5)$, $\mathcal{O}(1, 1, 2, 4)$, $\mathcal{O}(1, 1, 3, 3)$, $\mathcal{O}(1, 2, 2, 3)$, or $\mathcal{O}(2, 2, 2, 2)$.

(2.24.1) When $\mathcal{E}\cong\mathcal{O}(2, 2, 2, 2)$, we have $P\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^3$, $H=H_\sigma+2H_\xi$, $M\in|2H_\sigma|$, and $L=[H_\sigma+H_\xi]_M$. Hence $M\cong\boldsymbol{P}_\xi^1\times Q$, where Q is a smooth quadric surface in $\boldsymbol{P}_\sigma^3$. Since $Q\cong\boldsymbol{P}_\mu^1\times\boldsymbol{P}_\lambda^1$, we have $M\cong\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\mu^1\times\boldsymbol{P}_\lambda^1$ and $L=2H_\xi+H_\mu+H_\lambda$.

(2.24.2) When $\mathcal{E}\cong\mathcal{O}(1, 1, 1, 5)$, (M, L) does not exist by the argument in (2.23.1).

(2.24.3) Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 2, 4)$, we can show that (M, L) does not exist similarly as in (2.23.2).

(2.24.4) When $\mathcal{E}\cong\mathcal{O}(1, 2, 2, 3)$, we can show the existence of (M, L) similarly as in (2.23.3). In fact, we have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_{10}:\sigma_{11}:\sigma_{20}:\sigma_{21}:\sigma_{30}:\sigma_{31}:\sigma_{32})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^7\,|\,\xi_0:\xi_1=\sigma_{10}:\sigma_{11}=\sigma_{20}:\sigma_{21}=\sigma_{30}:\sigma_{31}=\sigma_{31}:\sigma_{32}\}$, $H=H_\sigma+H_\xi$, and $h^0(P, 2H-4H_\xi)=h^0(\boldsymbol{P}_\xi^1, S^2(\mathcal{E})\otimes[-4H_\xi])=11$. We take a rational section $s_2:=\{(U_i\cap V_j, \sigma_0^2/\xi_0^2\cdot\xi_i^2/\sigma_j^2)\}_{i,j}$ of $2H_\sigma-2H_\xi$, where U_i and V_j are the same as in (2.23.3). Let $f_1, \cdots, f_{11}$ be rational functions on P such that

$$f_1=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{30}}{\xi_0^2},\quad f_2=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}^2}{\xi_0^2},\quad f_3=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{20}}{\xi_0^2},\quad f_4=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{30}}{\xi_0^2},$$

$$f_5=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{10}\sigma_{31}}{\xi_0^2},\quad f_6=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}^2}{\xi_0^2},\quad f_7=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{30}}{\xi_0^2},\quad f_8=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_0^2},$$

$$f_9=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}^2}{\xi_0^2},\quad f_{10}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_0^2},\quad f_{11}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{32}}{\xi_0^2}.$$

Then $H^0(P, 2H_\sigma-2H_\xi)\cong \boldsymbol{C}\langle f_1, \cdots, f_{11}\rangle$ and $\mathrm{Bs}|2H_\sigma-2H_\xi|=\boldsymbol{P}_\xi^1\times\{(1:0:\cdots:0)\}$. For any $f=\sum_{i=1}^{11}c_if_i$ with $c_1\neq 0$, $\mathrm{div}(f\cdot s_2)$ is nonsingular along $\mathrm{Bs}|2H_\sigma-2H_\xi|$, thus a general member M of $|2H_\sigma-2H_\xi|$ is smooth. Putting $L=H_M$, we obtain a polarized manifold (M, L) as desired. In this case, $|L-H_\xi|$ makes M a desingularization of a variety of degree six in $\boldsymbol{P}^7$.

(2.24.5) Even when $\mathcal{E}\cong\mathcal{O}(1, 1, 3, 3)$, we can show the existence of (M, L) similarly. We have $P\cong\{(\xi_0:\xi_1)\times(\sigma_0:\sigma_1:\sigma_{20}:\sigma_{21}:\sigma_{22}:\sigma_{30}:\sigma_{31}:\sigma_{32})\in\boldsymbol{P}_\xi^1\times\boldsymbol{P}_\sigma^7|\xi_0:\xi_1=\sigma_{20}:\sigma_{21}=\sigma_{21}:\sigma_{22}=\sigma_{30}:\sigma_{31}=\sigma_{31}:\sigma_{32}\}$ and $H^0(P, 2H_\sigma-2H_\xi)\cong \boldsymbol{C}\langle f_1, \cdots, f_{13}\rangle$, where

$$f_1=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{20}}{\xi_0^2},\quad f_2=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_0\sigma_{30}}{\xi_0^2},\quad f_3=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_1\sigma_{20}}{\xi_0^2},\quad f_4=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_1\sigma_{30}}{\xi_0^2},$$

$$f_5=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}^2}{\xi_0^2},\quad f_6=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{21}}{\xi_0^2},\quad f_7=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{21}^2}{\xi_0^2},\quad f_8=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{30}}{\xi_0^2},$$

$$f_9=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{20}\sigma_{31}}{\xi_0^2},\quad f_{10}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{21}\sigma_{31}}{\xi_0^2},\quad f_{11}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}^2}{\xi_0^2},\quad f_{12}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{30}\sigma_{31}}{\xi_0^2},$$

$$f_{13}=\frac{\xi_0^2}{\sigma_0^2}\cdot\frac{\sigma_{31}^2}{\xi_0^2}.$$

Since $\mathrm{Bs}|2H_\sigma-2H_\xi|=\{\sigma_{20}=\sigma_{21}=\cdots=\sigma_{32}=0 \text{ in } P\}$, if we take $f=\sum_{i=1}^{13}c_if_i$ with $c_1c_4-c_2c_3\neq 0$, then $\mathrm{div}(f\cdot s_2)$ is nonsingular along $\mathrm{Bs}|2H_\sigma-2H_\xi|$. Thus a general member M of $|2H_\sigma-2H_\xi|$ is smooth. Putting $L=H_M$, we obtain a polarized manifold (M, L) as desired, and $|L-H_\xi|$ makes M a desingularization of a variety of degree six in $\boldsymbol{P}^7$.

(2.25) Summarizing the results above, we obtain the following.

THEOREM. *Let (M, L) be a polarized manifold of the type* (1.2). *Then $g(C)$, the genus of C, is 0 or 1, $\mathcal{E}:=\Phi_*\mathcal{O}_M[L]$ is a locally free sheaf on C, $M\in|2H(\mathcal{E})+B_{P(\mathcal{E})}|$ for some line bundle B on C, and $L=[H(\mathcal{E})]_M$. Putting $d=L^n$, $e=c_1(\mathcal{E})$, and $b=\deg B$, we have the following results.*

When $g(C)=1$, we have $1\leq d\leq 6$, $e=d-2$, $b=4-d$, and

(i) *if $d=1$ or 2, then $\mathcal{E}$ is not ample;*

(ii) *if $d=3$ or 4, then $\mathcal{E}$ is ample as long as it is indecomposable;*

(iii) *if $d=5$ or 6, then $\mathcal{E}$ is ample.*

When g(C) we have $C \cong \boldsymbol{P}^1_\xi$, $1 \leq d \leq 12$, $e=d-4$, $b=8-d$, $M \in |2H(\mathcal{E})+bH_\xi|$, *and their lists are in the table below.*

d	$\mathcal{E}$	(M, L)
1	$\mathcal{O}(-3, 0, 0, 0)$ $\mathcal{O}(-3, 0, 0, 0, 0)$ $\mathcal{O}(-2, -1, 0, 0)$ $\mathcal{O}(-2, -1, 0, 0, 0)$ $\mathcal{O}(-1, -1, -1, 0)$ $\mathcal{O}(-1, -1, -1, 0, 0)$	The existence is uncertain.
2	$\mathcal{O}(-2, 0, 0, 0)$ $\mathcal{O}(-2, 0, 0, 0, 0)$ $\mathcal{O}(-1, -1, 0, 0)$ $\mathcal{O}(-1, -1, 0, 0, 0)$	The existence is uncertain.
3	$\mathcal{O}(-2, 0, 0, 1)$ $\mathcal{O}(-2, 0, 0, 0, 1)$ $\mathcal{O}(-1, -1, 0, 1)$ $\mathcal{O}(-1, -1, 0, 0, 1)$ $\mathcal{O}(-1, 0, 0, 0)$ $\mathcal{O}(-1, 0, 0, 0, 0)$	The existence is uncertain.
4	$\mathcal{O}(-1, 0, 0, 1)$ $\mathcal{O}(-1, 0, 0, 0, 1)$ $\mathcal{O}(0, 0, 0, 0)$ $\mathcal{O}(0, 0, 0, 0, 0)$	The existence is uncertain. The existence is uncertain. $\|L\|$ makes M a quadruple covering of $\boldsymbol{P}^3$. $\|L\|$ makes M a quadruple covering of $\boldsymbol{P}^4$.
5	$\mathcal{O}(-1, 0, 0, 2)$ $\mathcal{O}(-1, 0, 1, 1)$ $\mathcal{O}(-1, 0, 0, 1, 1)$ $\mathcal{O}(0, 0, 0, 1)$ $\mathcal{O}(0, 0, 0, 0, 1)$	$\mathrm{Bs}\|L\|$ is a point. $\mathrm{Bs}\|L\|$ is a point. $\mathrm{Bs}\|L\|$ is a point. $\|L\|$ makes M the normalization of a hypersurface of degree five in $\boldsymbol{P}^4$. $\|L\|$ makes M the normalization of a hypersurface of degree five in $\boldsymbol{P}^5$.

d	$\mathcal{E}$	(M, L)
6	$\mathcal{O}(-1, 1, 1, 1)$	M is a double covering of $\boldsymbol{P}_{\xi}^{1}\times\boldsymbol{P}_{\sigma}^{2}$ with branch locus being a smooth divisor of bidegree (4,2). $L=[H_{\xi}+H_{\sigma}]_{M}$.
	$\mathcal{O}(0, 0, 1, 1)$	Exist.
	$\mathcal{O}(0, 0, 0, 1, 1)$	Exist.
	$\mathcal{O}(0, 0, 0, 2)$	Exist.
7	$\mathcal{O}(0, 0, 1, 2)$	Exist.
	$\mathcal{O}(0, 1, 1, 1)$	Exist.
	$\mathcal{O}(0, 0, 1, 1, 1)$	Exist.
8	$\mathcal{O}(0, 1, 1, 1, 1)$	M is a double covering of $\boldsymbol{P}_{\xi}^{1}\times\boldsymbol{P}_{\sigma}^{3}$ with branch locus being a smooth divisor of bidegree (2,2). $L=[H_{\xi}+H_{\sigma}]_{M}$.
	$\mathcal{O}(0, 1, 1, 2)$	M is a double covering of a divisor of bidegree (1, 1) on $\boldsymbol{P}_{\xi}^{1}\times\boldsymbol{P}_{\sigma}^{3}$. $L=[H_{\xi}+H_{\sigma}]_{M}$.
	$\mathcal{O}(1, 1, 1, 1)$	M is a smooth divisor of bidegree (2,2) on $\boldsymbol{P}_{\xi}^{1}\times\boldsymbol{P}_{\sigma}^{3}$. $L=[H_{\xi}+H_{\sigma}]_{M}$.
9	$\mathcal{O}(1, 1, 1, 1, 1)$	M is the blowing-up of $\boldsymbol{P}_{\sigma}^{4}$ with center being a complete intersection of two hyperquadrics. $L=[H_{\xi}+H_{\sigma}]_{M}$.
	$\mathcal{O}(1, 1, 1, 2)$	M is the strict transform of a smooth hypercubic in $\boldsymbol{P}_{\sigma}^{4}$ by the blowing-up of $\boldsymbol{P}_{\sigma}^{4}$ with center being a $\boldsymbol{P}^{2}$. $L=[H_{\xi}+H_{\sigma}]_{M}$.
10	$\mathcal{O}(1, 1, 1, 1, 1, 1)$	$M\cong\boldsymbol{P}_{\xi}^{1}\times Q$, where Q is a smooth hyperquadric in $\boldsymbol{P}_{\sigma}^{5}$. $L=[H_{\xi}+H_{\sigma}]_{M}$.
	$\mathcal{O}(1, 1, 1, 1, 2)$	M is the blowing-up of a hyperquadric in $\boldsymbol{P}_{\sigma}^{5}$ with center being a smooth quadric surface. $L=[H_{\xi}+H_{\sigma}]_{M}$.
	$\mathcal{O}(1, 1, 2, 2)$	M is a desingularization of a complete intersection of two hyperquadrics in $\boldsymbol{P}_{\sigma}^{5}$. $L=[H_{\xi}+H_{\sigma}]_{M}$.
	$\mathcal{O}(1, 1, 1, 3)$	M is a desingularization of a complete intersection of two hyperquadrics in $\boldsymbol{P}_{\sigma}^{5}$. $L=[H_{\xi}+H_{\sigma}]_{M}$.
11	$\mathcal{O}(1, 2, 2, 2)$	$\|L-H_{\xi}\|$ makes M a desingularization of a three-dimensional variety of degree five in $\boldsymbol{P}^{6}$.
12	$\mathcal{O}(1, 1, 3, 3)$	$\|L-H_{\xi}\|$ makes M a desingularization of a three-dimensional variety of degree six in P^{7}.
	$\mathcal{O}(1, 2, 2, 3)$	$\|L-H_{\xi}\|$ makes M a desingularization of a three-dimensional variety of degree six in P^{7}.
	$\mathcal{O}(2, 2, 2, 2)$	$M\cong\boldsymbol{P}_{\xi}^{1}\times\boldsymbol{P}_{\mu}^{1}\times\boldsymbol{P}_{\lambda}^{1}$ and $L=2H_{\xi}+H_{\mu}+H_{\lambda}$.

§ 3. The case of a Veronese fibration over a curve

In this section we study the case (1.3), using the argument in [F; (13.10)].

(3.1) Put $H=K+2L$, then $\mathcal{E}:=\Phi_*\mathcal{O}_M[H]$ is a locally free sheaf of rank three on C and (M, H) is the scroll of $\mathcal{E}$. We have $L=2H+\Phi^*B$ for some $B\in\mathrm{Pic}(C)$. Similarly as before, we put $d=L^3$, $e=c_1(\mathcal{E})$, $b=\deg B$ and denote by $g(C)$ the genus of C. Then $e\geqq 0$, $e+b=1$, and $d=8e+12b$. By the canonical bundle formula, we obtain that $K^C+\det\mathcal{E}+2B=0$, hence $2g(C)-2+e+2b=0$. From these results, $(e, d)=(0, 12)$ or $(2, 4)$.

(3.2) When $(e, d)=(0, 12)$, we have $b=1$ and $g(C)=0$, hence $C\cong \boldsymbol{P}^1$, $B=\mathcal{O}(1)$, and $\mathcal{E}\cong\mathcal{O}(e_1)\oplus\mathcal{O}(e_2)\oplus\mathcal{O}(e_3)$ for $e_1, e_2, e_3\in\boldsymbol{Z}$. For each $1\leqq i\leqq 3$, a natural surjection $\mathcal{E}\to\mathcal{O}(e_i)$ gives a section Z_i of Φ and $H_{Z_i}=\mathcal{O}(e_i)$. Since $e_1+e_2+e_3=e=0$ and $L_{Z_i}=\mathcal{O}(2e_i+1)$ is ample, we have $e_1=e_2=e_3=0$ and $\mathcal{E}\cong\mathcal{O}_C^{\oplus 3}$, thus $M\cong\boldsymbol{P}^1_\xi\times\boldsymbol{P}^2_\sigma$ and $L=H_\xi+2H_\sigma$.

(3.3) When $(e, d)=(2, 4)$, we have $b=-1$ and $g(C)=1$. Hence C is an elliptic curve and $\det\mathcal{E}+2B=0$ since $K^C=\mathcal{O}_C$. Let Q be any quotient bundle of $\mathcal{E}$. If rank $Q=1$, then $Z:=\boldsymbol{P}_C(Q)$ is a section of Φ and $HZ=c_1(Q)$. Then $c_1(Q)\geqq 1$ since $0<LZ=2c_1(Q)-1$. If rank $Q=2$, then $D:=\boldsymbol{P}_C(Q)\in|H-\Phi^*\mathcal{F}|$, where $\mathcal{F}$ is the kernel of $\mathcal{E}\to Q$. Since $0<L^2D=4(1-c_1(\mathcal{F}))$, we have $c_1(Q)=e-c_1(\mathcal{F})\geqq 2$. In both cases we have (rank $Q)\cdot c_1(\mathcal{E})<(\mathrm{rank}\mathcal{E})\cdot c_1(Q)$, hence $\mathcal{E}$ is stable. Conversely, let $\mathcal{E}$ be a semistable vector bundle on C with rank $\mathcal{E}=3$ and $c_1(\mathcal{E})=2$. We put $M=\boldsymbol{P}_C(\mathcal{E})$, $H=H(\mathcal{E})$ and let $\Phi: M\to C$ be the bundle map. By the semistability criterion in [Mi; (3.1)], $3H-\Phi^*(\det\mathcal{E})$ is nef. Since C is an elliptic curve, we can find some $B\in\mathrm{Pic}(C)$ satisfying $\det\mathcal{E}+2B=0$. Then $3(2H+\Phi^*B)=2(3H+\Phi^*(2B))-\Phi^*B$ is ample. Hence $L:=2H+\Phi^*B$ is ample and (M, L) is a polarized manifold of the type (1.3).

(3.4) Summing up, we obtain the following theorem.

THEOREM. *Let (M, L) be a polarized of the type* (1.3). *We put $d=L^3$ and denote by $g(C)$ the genus of C. Then (M, L) is one of the following two types.*

(I) $g(C)=0$, *heuce* $C\cong\boldsymbol{P}^1_\xi$; $d=12$, $M\cong\boldsymbol{P}^1_\xi\times\boldsymbol{P}^2_\sigma$, *and* $L=H_\xi+2H_\sigma$.

(II) $g(C)=1$ *and* $M\cong\boldsymbol{P}_C(\mathcal{E})$, *where* $\mathcal{E}:=\Phi_*\mathcal{O}_M[K+2L]$ *is a stable vector bundle of rank three on C with* $c_1(\mathcal{E})=2$; $d=4$ *and* $L=2H(\mathcal{E})+\Phi^*B$, *where* $B\in$ *Pic*(C) *with* $\det\mathcal{E}+2B=0$.

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COMPLEX MANIFOLDS POLARIZED BY AN AMPLE AND SPANNED LINE BUNDLE OF SECTIONAL GENUS THREE

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ABSTRACT. Let X be a complex projective manifold with $\dim X \geq 3$ and let L be an ample and spanned line bundle on X. We classify polarized manifolds (X, L) of sectional genus three mainly by adjunction theory and the classification theory of Δ-genus.

Introduction.

A pair (X, L) is said to be a polarized manifold if X is a nonsingular projective variety and L is an ample line bundle on X. The sectional genus $g(L)$ of a polarized manifold (X, L) is an important invariant and is defined by $g(L) := 1 + (1/2)(K_X + (n-1)L)L^{n-1}$, where $n = \dim X$ and K_X is the canonical bundle of X. Complex polarized manifolds (X, L) with $g(L) \leq 2$ have been classified by [F4], [Io2], [BeLP], and [F5]. As for the case $g(L) = 3$, both [M] for $n = 2$ and [Is] for $n \geq 3$ gave partial classification; it seems that complete classification is very difficult. Spannedness of line bundles generally makes the situation clearer and more geometric. Indeed, under the assumption that L is very ample, classification of (X, L) with $g(L) = 3$ was obtained by [Io1]. Under the assumption that L is spanned by global sections, classification of (X, L) with $g(L) = 3$ and $n = 2$ was obtained by [LL1]; that with $g(L) = 3$, $n = 3$, $L^3 = 3$, and $h^0(L) = 4$ was obtained by [LL2].

The purpose of this note is a generalization of these classifications above, that is, to classify complex polarized manifolds (X, L) with $g(L) = 3$ and $n \geq 3$ under the assumption that L is spanned. To do this, we mainly depend on the results of [Is] (and also [BiLL]) and then we use the classification theory of Δ-genus, which is defined by $\Delta(L) := n + L^n - h^0(L)$ for (X, L). In Appendix, we give the list of complex polarized manifolds (X, L) such that $g(L) = 2$, $n \geq 3$, and L is spanned. Although these polarized manifolds have been already classified in [F5] without the spannedness of L, we believe that our list is non-trivial and useful for applications.

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§ 1. Preliminaries.

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Definition 1.1. Let X be a smooth projective variety with $\dim X = n$ and let L be a line bundle on X. Then we say that (X, L) is a *scroll over* Y if there exists a fiber space $\pi : X \to Y$ such that every fiber F of π is isomorphic to $\mathbb{P}^{n-m}$ and $L_F := L|_F = \mathcal{O}_{\mathbb{P}^{n-m}}(1)$, where $1 \leq m = \dim Y < \dim X$.

Definition 1.2. Let (X, L) and (X', L') be polarized manifolds. Then (X, L) is called a *simple blowing up* of (X', L') if X is a blowing up at one point on X' and $L = \mu^* L' - E$, where $\mu : X \to X'$ is the blowing up and E is the μ-exceptional effective reduced divisor.

Lemma 1.3. *Let (X, L) be a polarized manifold with $\dim X = n$. If L is spanned and $g(L) \geq 1$, then $d := L^n \geq 2$.*

Proof. If L is spanned and $d = 1$, then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$, which shows $g(L) = 0$. □

Lemma 1.4. *Let (X, L) be a polarized manifold with $\Delta(L) = 1$.*

(1) *If $d = 2$, then there is a double covering $\pi : X \to \mathbb{P}^n$ and $L = \pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$.*
(2) *If L is spanned and $g(L) \geq 2$, then $d = 2$.*

Proof. See [F1]. □

Lemma 1.5. *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that $|L|$ has no fixed component, $\Delta(L) = 2$, $d = 4$ and $g(L) = 3$. Then (X, L) is either*

(1) *X is a hyperquartic in $\mathbb{P}^{n+1}$ and $L = \mathcal{O}_X(1)$, or*
(2) *$|L|$ makes X a double covering of a smooth hyperquadric $\mathbb{Q}^n$ with the branch locus $B \in |\mathcal{O}_{\mathbb{Q}^n}(4)|$.*

Proof. See [F3;(0.6)]. □

Lemma 1.6. *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that L is spanned, $d = 3$ and $g(L) = 3$. Then $n = 3$ and $q(X) := h^1(\mathcal{O}_X) = 0$.*

Proof. By assumption, we get $h^0(L) = n + 1$ because of Lemma 1.4. Let $\rho : X \to \mathbb{P}^n$ be the morphism defined by $|L|$. Then ρ is a triple covering such that $L = \rho^*(\mathcal{O}_{\mathbb{P}^n}(1))$.

If $n \geq 4$, then by Proposition 3.2 in [La], $K_X = \rho^*(\mathcal{O}_{\mathbb{P}^n}(k))$ for some $k \in \mathbb{Z}$. It follows that $K_X + (n-1)L = \rho^*(\mathcal{O}_{\mathbb{P}^n}(k + n - 1))$, hence $2g(L) - 2 = 3(k + n - 1)$, a contradiction. So we have $n = 3$. Then by Theorem 1 in [La], we obtain $q(X) = 0$. This completes the proof of Lemma 1.6. □

Lemma 1.7. *Let $\mathcal{F}$ be an indecomposable vector bundle on an elliptic curve. Then:*

(1) *$h^0(\mathcal{F}) = c_1(\mathcal{F})$ if $c_1(\mathcal{F}) > 0$;*
(2) *$h^0(\mathcal{F}) \leq 1$ if $c_1(\mathcal{F}) = 0$.*

Proof. See Lemma 15 in [A].

§ 2. The case in which g(L) = 3.

Theorem 2.1. *Let (X, L) be a polarized manifold over $\mathbb{C}$ with $\dim X = n \geq 3$. Assume that $g(L) = 3$ and L is spanned. Then (X, L) is one of the following.*

(I) *There is a fibration $f : X \to C$ over a smooth curve C with $g(C) \leq 1$ such that every fiber F of f is a hyperquadric in $\mathbb{P}^n$ and $L_F = \mathcal{O}(1)$. Then $\mathcal{E} := f_*\mathcal{O}(L)$ is a locally free sheaf of rank $n+1$ on C, $X \in |2H(\mathcal{E}) + \pi^*B|$ on $\mathbb{P}(\mathcal{E})$ for some line bundle B on C, and $L = H(\mathcal{E})|_X$, where π is the projection $\mathbb{P}(\mathcal{E}) \to C$, and $H(\mathcal{E})$ is the tautological line bundle on $\mathbb{P}(\mathcal{E})$. We put $d = L^n$, $e = c_1(\mathcal{E})$ and $b = \deg B$.*

- (I-1) *When $g(C) = 1$, we have $n = 3$, $d = 6$, $e = 4$, $b = -2$ and $\mathcal{E}$ is ample.*
- (I-2) *When $g(C) = 0$, we have $C \cong \mathbb{P}^1_\xi$, $4 \leq d \leq 12$, $e = d-4$, $b = 8-d$, $X \in |2H(\mathcal{E}) + bH_\xi|$ (H_ξ is the pull back of $\mathcal{O}_{\mathbb{P}^1_\xi}$), $\mathcal{E} \cong \mathcal{O}(e_0) \oplus \cdots \oplus \mathcal{O}(e_n) =: \mathcal{O}(e_0, \cdots, e_n)$, and their lists are in Table 1.*

(II) *(X, L) is a scroll over a smooth curve of genus three.*

(III) *(X, L) is a scroll over a smooth surface S. Let $\mathcal{E}$ be a locally free sheaf of rank $n-1$ on S such that $(\mathbb{P}(\mathcal{E}), H(\mathcal{E})) \cong (X, L)$. Let F_e be the rational $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ with $e \geq 0$, σ the minimal section, and f a fiber of $F_e \to \mathbb{P}^1$. Then $(S, \mathcal{E})$ is as follows:*

- (III-1a) $S = \mathbb{P}^2$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4}$;
- (III-1b) $S = \mathbb{P}^2$, *and either* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ or $\mathcal{E} = T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$;
- (III-1c) $S = \mathbb{P}^2$, $\operatorname{rank} \mathcal{E} = 2$ and $\det \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(4)$;
- (III-2a) $S = F_0$, *and either* $\mathcal{E} = [\sigma + f] \oplus [\sigma + 3f]$ *or* $\mathcal{E} = [\sigma + 2f]^{\oplus 2}$;
- (III-2b) $S = F_1$, $\mathcal{E} = [\sigma + 2f] \oplus [\sigma + 3f]$;
- (III-2c) $S = F_2$, $\mathcal{E} = [\sigma + 3f]^{\oplus 2}$;
- (III-3) *S is a Del Pezzo surface with $K_S^2 = 2$ and either $\mathcal{E} = [-K_S]^{\oplus 2}$, or $\mathcal{E} = \psi^*(\mathcal{Q}|_Y)$, where ψ is a birational morphism from S to a congruence Y of bidegree (4,4) in the Grassmannian of lines of $\mathbb{P}^3$, and $\mathcal{Q}$ is the universal rank two quotient bundle;*
- (III-4) *$S = \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a rank two vector bundle over an elliptic curve C with $c_1(\mathcal{F}) = 1$ and $\mathcal{E} = H(\mathcal{F}) \otimes \rho^*\mathcal{G}$, where $\rho : S \to C$ is the bundle projection and $\mathcal{G}$ is any rank two vector bundle on C defined by a non splitting exact sequence*

$$0 \to \mathcal{O}_C \to \mathcal{G} \to \mathcal{O}_C(x) \to 0,$$

where $x \in C$.

(IV) *X is a smooth hyperquartic in $\mathbb{P}^{n+1}$ and $L = \mathcal{O}_X(1)$.*

(IV′) *(X, L) is a simple blowing up of another polarized threefold of the type (IV).*

(V) *$|L|$ makes X a double covering of a variety W. Let B be the branch locus. Then*

- (V-1) *$W = \mathbb{P}^n$ and $B \in |\mathcal{O}_{\mathbb{P}^n}(8)|$, or*
- (V-2) *$W = \mathbb{Q}^n$ and $B \in |\mathcal{O}_{\mathbb{Q}^n}(4)|$.*

Table 1

d	$\mathcal{E}$	(X, L)
4	$\mathcal{O}(0,0,0,0)$	$\lvert L\rvert$ makes X a quadruple covering of $\mathbb{P}^3$.
	$\mathcal{O}(0,0,0,0,0)$	$\lvert L\rvert$ makes X a quadruple covering of $\mathbb{P}^4$.
5	$\mathcal{O}(0,0,0,1)$	$\lvert L\rvert$ makes X the normalization of a hypersurface of degree five in $\mathbb{P}^4$.
	$\mathcal{O}(0,0,0,0,1)$	$\lvert L\rvert$ makes X the normalization of a hypersurface of degree five in $\mathbb{P}^5$.
6	$\mathcal{O}(-1,1,1,1)$	X is a double covering of $\mathbb{P}^1_\xi \times \mathbb{P}^2_\sigma$ with branch locus being a smooth divisor of bidegree (4,2). $L = [H_\xi + H_\sigma]_X$.
	$\mathcal{O}(0,0,1,1)$	Exist.
	$\mathcal{O}(0,0,0,1,1)$	Exist.
	$\mathcal{O}(0,0,0,2)$	Exist.
7	$\mathcal{O}(0,0,1,2)$	Exist.
	$\mathcal{O}(0,1,1,1)$	Exist.
	$\mathcal{O}(0,0,1,1,1)$	Exist.
8	$\mathcal{O}(0,1,1,1,1)$	X is a double covering of $\mathbb{P}^1_\xi \times \mathbb{P}^3_\sigma$ with branch locus being a smooth divisor of bidegree (2,2). $L = [H_\xi + H_\sigma]_X$.
	$\mathcal{O}(0,1,1,2)$	X is a double covering of a divisor of bidegree (1,1) on $\mathbb{P}^1_\xi \times \mathbb{P}^3_\sigma$. $L = [H_\xi + H_\sigma]_X$.
	$\mathcal{O}(1,1,1,1)$	X is a smooth divisor of bidegree (2,2) on $\mathbb{P}^1_\xi \times \mathbb{P}^3_\sigma$. $L = [H_\xi + H_\sigma]_X$.
9	$\mathcal{O}(1,1,1,1,1)$	X is the blowing-up of $\mathbb{P}^4_\sigma$ with center being a complete intersection of two hyperquadrics. $L = [H_\xi + H_\sigma]_X$.
	$\mathcal{O}(1,1,1,2)$	X is the strict transform of a smooth hyperqubic in $\mathbb{P}^4_\sigma$ by the blowing-up of $\mathbb{P}^4_\sigma$ with center being a $\mathbb{P}^2$. $L = [H_\xi + H_\sigma]_X$.
10	$\mathcal{O}(1,1,1,1,1,1)$	$X \cong \mathbb{P}^1_\xi \times \mathbb{Q}^4$, where $\mathbb{Q}^4$ is a smooth hyperquadric in $\mathbb{P}^5_\sigma$. $L = [H_\xi + H_\sigma]_X$.
	$\mathcal{O}(1,1,1,1,2)$	X is the blowing-up of a hyperquadric in $\mathbb{P}^5_\sigma$ with center being a smooth quadric surface. $L = [H_\xi + H_\sigma]_X$.
	$\mathcal{O}(1,1,2,2)$	X is a desingularization of a complete intersection of two hyperquadrics in $\mathbb{P}^5_\sigma$. $L = [H_\xi + H_\sigma]_X$.
	$\mathcal{O}(1,1,1,3)$	X is a desingularization of a complete intersection of two hyperquadrics in $\mathbb{P}^5_\sigma$. $L = [H_\xi + H_\sigma]_X$.
11	$\mathcal{O}(1,2,2,2)$	$\lvert L - H_\xi\rvert$ makes X a desingularization of a three-dimensional variety of degree five in $\mathbb{P}^6$.
12	$\mathcal{O}(1,1,3,3)$	$\lvert L - H_\xi\rvert$ makes X a desingularization of a three-dimensional variety of degree six in $\mathbb{P}^7$.
	$\mathcal{O}(1,2,2,3)$	$\lvert L - H_\xi\rvert$ makes X a desingularization of a three-dimensional variety of degree six in $\mathbb{P}^7$.
	$\mathcal{O}(2,2,2,2)$	$X \cong \mathbb{P}^1_\xi \times \mathbb{P}^1_\mu \times \mathbb{P}^1_\lambda$ and $L = 2H_\xi + H_\mu + H_\lambda$.

Proof. Let (X, L) be a polarized manifold over $\mathbb{C}$ with $\dim X = n \geq 3$. We see from the results [F4] or [Io2], (X, L) is one of the following types:

(A) There is an effective divisor E on X such that $(E, L_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and $\mathcal{O}_E(E) = \mathcal{O}(-1)$.
(B) There is a fibration $f : X \to C$ over a smooth curve C such that every fiber F of f is a hyperquadric in $\mathbb{P}^n$ and $L_F = \mathcal{O}(1)$.
(C) There is a fibration $f : X \to C$ over a smooth curve C such that $(F, L_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for every fiber F of f.
(D) (X, L) is a scroll over a smooth surface S.
(E) (X, L) is a scroll over a smooth curve C.
(F) $K_X + (n-2)L$ is nef.

We consider the above cases separately. Assume that $g(L) = 3$ and L is spanned.

(1) Case (B).

We obtain the former part of (I) by the arguments in [Is;(2.1)].

(1-1) The case in which $g(C) = 1$.

Then $1 \leq d \leq 6$, $e = d - 2$, and $b = 4 - d$ by [Is;(2.2)]. The case $d = 1$ is ruled out by Lemma 1.3.

(1-1-1) The case in which $g(C) = 1$ and $d = 2$.

Then $\Delta(L) = 1$ since L is ample and spanned. But by Lemma 1.4, $K_X + (n-2)L$ is ample and this is impossible.

(1-1-2) The case in which $g(C) = 1$ and $d = 3$ or 4.

Then $\mathcal{E}$ is ample as long as $\mathcal{E}$ is indecomposable. Hence if $\mathcal{E}$ is indecomposable, then $h^0(L) = h^0(\mathcal{E}) = c_1(\mathcal{E}) \leq 2$. But this is impossible because L is ample and spanned. Hence $\mathcal{E}$ is decomposable.

(1-1-2-1) The case in which $g(C) = 1$ and $d = 3$.

By Lemma 1.6, we get $n = 3$ and $q(X) = 0$. But this case cannot occur since $g(C) = 1$.

(1-1-2-2) The case in which $g(C) = 1$ and $d = 4$.

Then $c_1(\mathcal{E}) = 2$, and by an argument similar to that in [F5] we can prove that $\mathcal{E}$ is semipositive. If $h^0(L) \geq n + 2$, then $\Delta(L) = 2$ by Lemma 1.4 (2). By Lemma 1.5, we obtain that $K_X + (n-2)L$ is nef. But this is impossible. Hence $h^0(L) = n + 1$.

Let $\mathcal{E} = \oplus_{i=1}^m \mathcal{E}_i$, where each $\mathcal{E}_i$ is an indecomposable vector bundle on C. We have $e_i := \deg \mathcal{E}_i \geq 0$ since $\mathcal{E}$ is semipositive. Then we may assume that $(e_1, \ldots, e_m) = (0, \ldots, 0, 1, 1)$ or $(0, \ldots, 0, 2)$. Since $\operatorname{rank} \mathcal{E} = n + 1$ and $h^0(\mathcal{E}) = h^0(L) = n + 1$, we find that $m = n$ or $n + 1$ by Lemma 1.7. In case $m = n + 1$, each $\mathcal{E}_i$ is a line bundle and we get

(i) $\mathcal{E} \cong \mathcal{O}_C^{\oplus n-1} \oplus \mathcal{F}$, where $\mathcal{F}$ is a vector bundle of rank two on C.

In case $m = n$, we see that $(e_1, \ldots, e_n) = (0, \ldots, 0, 2)$ and $\operatorname{rank} \mathcal{E}_n \leq 2$. If $\operatorname{rank} \mathcal{E}_n = 2$, then we get (i). If $\operatorname{rank} \mathcal{E}_n = 1$, then we get

(ii) $\mathcal{E} \cong \mathcal{O}_C^{\oplus n-2} \oplus \mathcal{E}_n \oplus \mathcal{G}$, where $\mathcal{G}$ is an indecomposable vector bundle of rank two on

C with $\deg \mathcal{G} = 0$ and $h^0(\mathcal{G}) = 1$.

Note that we have a non-trivial extension $0 \to \mathcal{O}_C \to \mathcal{G} \to \mathcal{O}_C \to 0$. Hence in both cases (i) and (ii), we have a natural surjection $\mathcal{E} \to \mathcal{O}_C^{\oplus 2}$. This surjection determines a subvariety $W := \mathbb{P}(\mathcal{O}_C^{\oplus 2})$ of $\mathbb{P}(\mathcal{E})$. We see that $W \cap X \neq \emptyset$ since $f : X \to C$ is a hyperquadric fibration. Take a point $p \in W \cap X$ and let C' be a section of $W \to C$ in $|H(\mathcal{E})|_W|$ passing through p. Then $C' \subset X$ since $X \cdot C' = (2H(\mathcal{E}) + \pi^* B)C' = 0$. It follows that $LC' = H(\mathcal{E}) \cdot C' = 0$, a contradiction to the ampleness of L. Thus these cases cannot occur.

(1-1-3) The case in which $g(C) = 1$ and $d = 5$ or 6.

Then $\mathcal{E}$ is ample. So we obtain $h^0(L) = h^0(\mathcal{E}) = c_1(\mathcal{E}) = d - 2 \leq 4$. On the other hand $h^0(L) \geq n+1$. Hence the $d = 5$ case is impossible, and if $d = 6$, then $n = 3$ and $h^0(L) = 4$. An example of (X, L) for $d = 6$ will be given in Example 2.4.

(1-2) The case in which $g(C) = 0$.

Then by [Is; §2], we get $1 \leq d \leq 12$, $e = d - 4$, $b = 8 - b$, and (X, L) is one of the types in the table of [Is;(2.25)]. Since L is spanned, we obtain Table 1 in Theorem 2.1 by the following lemma.

Lemma 2.2. *Let (X, L) be a polarized manifold as in* (B). *Then* $\mathrm{Bs}\,|L| \neq \emptyset$ *if $g(L) = 3$, $C \cong \mathbb{P}^1$, $1 \leq d \leq 5$, and $e_0 < 0$.*

Proof of the Lemma. Assume that $g(L) = 3$, $C \cong \mathbb{P}^1$, $1 \leq d \leq 5$, and $e_0 < 0$. The projection $\mathcal{E} \to \mathcal{O}(e_0)$ determines a section Z of $\mathbb{P}(\mathcal{E}) \to C$. Then $H^0(Z, H(\mathcal{E})|_Z) \cong H^0(\mathbb{P}^1, \mathcal{O}(e_0)) = 0$, hence we have $Z \subset \mathrm{Bs}\,|H(\mathcal{E})|$. Since $H^0(\mathbb{P}(\mathcal{E}), H(\mathcal{E})) \to H^0(M, L)$ is surjective, we see that $M \cap Z \subset M \cap \mathrm{Bs}\,|H(\mathcal{E})| \subset \mathrm{Bs}\,|L|$. By assumption,

$$MZ = (2H(\mathcal{E}) + bH_\xi)Z = 2e_0 + b = 2e_0 + (8 - d) > 0.$$

Hence $M \cap Z \neq \emptyset$ and then $\mathrm{Bs}\,|L| \neq \emptyset$. □

(2) Case (D).

Then by the classification results of [BiLL], we obtain the case (III) in Theorem 2.1.

(3) Case (C).

Then by [Is;(3.4)], we obtain (X, L) is one of the following two types.

(3-1) $g(C) = 0$ and $(X, L) \cong (\mathbb{P}^1_\xi \times \mathbb{P}^2_\sigma, H_\xi + 2H_\sigma)$;

(3-2) $g(C) = 1$ and $(X, L) \cong (\mathbb{P}_C(\mathcal{E}), 2H(\mathcal{E}) + f^*B)$, where $\mathcal{E}$ is a stable vector bundle of rank three on C with $c_1(\mathcal{E}) = 2$, and $B \in \mathrm{Pic}(C)$ with $\det \mathcal{E} + 2B = 0$.

The case (3-1) is a special case of (III-1c) in Theorem 2.1.

In case (3-2), let S be a general member of $|L|$. We have $g(L_S) = 3$, $q(S) = 1$, $L_S^2 = 4$, and $K_S^2 = -1$. Then by p.279 Case 3 in [LL1], S is an elliptic $\mathbb{P}^1$-bundle S' blown-up at a single point p, and $L_S = \eta^*(4\sigma' + F') - 2E$, where $\eta : S \to S'$ is the blowing up, σ' is a minimal section with $\sigma'^2 = 0$, F' is a fiber of the $\mathbb{P}^1$-bundle, and $E = \eta^{-1}(p)$. Let σ be the strict transform of σ' under η. Since

$$0 < L_S \cdot \sigma = (4\sigma' + F')\sigma' - 2E\sigma = 1 - 2E\sigma,$$

we see that $E\sigma = 0$ and $L_S \cdot \sigma = 1$. It follows that $\sigma' \cong \sigma \cong \mathbb{P}^1$ since L_S is spanned. This is a contradiction, thus the case (3-2) is ruled out.

(4) Case (F).

Since $g(L) = 3$ and $K_X + (n-2)L$ is nef, we have $2 \leq d \leq 4$.

(4-1) The case in which $d = 2$.

Then $\Delta(L) = 1$ since $g(L) = 3$. By Lemma 1.4 (1), we obtain (V-1).

(4-2) The case in which $d = 3$.

By Lemma 1.6, $n = 3$ and $K_X + L$ is nef. Then there is a fibration $f : X \to W$ onto a normal variety W such that $K_X + L = f^*A$ for some ample line bundle A on W (see [F0;(0.4.15)]). We get $L^3 \cdot (L(f^*A)^2) \leq (L^2(f^*A))^2$ by Index Theorem (see [F0;(0.4.6)]). Since

$$L^2(f^*A) = (K_X + L)L^2 = 2g(L) - 2 - L^3 = 1,$$

we find $L(f^*A)^2 = 0$, which implies that W is a smooth curve. Then we get $\deg A = 1$ and $L^2F = 1$ for any general fiber F of f. Since L is spanned, it follows that $F \cong \mathbb{P}^2$, $L_F = \mathcal{O}(1)$, and $K_F = \mathcal{O}(-3)$. On the other hand, we have $K_F + L_F = [K_X + L]_F = [f^*A]_F = \mathcal{O}_F$. This is a contradiction, thus this case cannot occur.

(4-3) The case in which $d = 4$.

Then $(K_X + (n-2)L)L^{n-1} = 2g(L) - 2 - d = 0$. Using Fibration Theorem as in (4-2), we get a fibration $f : X \to W$ such that $K_X + (n-2)L = f^*A$ for some ample $A \in \mathrm{Pic}(W)$. Then we have $K_X + (n-2)L = \mathcal{O}_X$ since $(f^*A)L^{n-1} = 0$. By Riemann-Roch Theorem and Vanishing Theorem, we get $h^0(L) = g(L) + n - 1 = n + 2$. Hence $\Delta(L) = n + L^n - h^0(L) = 2$. By Lemma 1.5, we obtain (IV) and (V-2) in Theorem 2.1.

(5) Case (A).

In this case, by using the theory of minimal reduction (e.g. [Io2;(0.11)], [F5;(1.9)], or [F0;(11.11)]), we see (X, L) is obtained by a finite number of simple blowing ups of a polarized manifold (Y, A) which is of the type (C) or (F). Let $\pi : X \to Y$ be its birational morphism.

(5-1) The case in which (Y, A) is of the type (C).

Then $n = 3$ and (Y, A) is of the type (3-1) or (3-2).

(5-1-1) The case in which (Y, A) is of the type (3-1).

Then we can find a curve Z in Y such that $AZ = 1$ and Z passes through a point which is blown up by π. This is a contradiction since $LZ' \leq 0$ for the strict transform Z' of Z under π.

(5-1-2) The case in which (Y, A) is of the type (3-2).

We have $L^3 \leq 3$. If $L^3 \leq 2$, then $\Delta(L) = 1$ and $L^3 = 2$. By Lemma 1.4, we obtain $\kappa(X) = 0$, a contradiction. Hence $L^3 = 3$, but this case cannot occur as in the case (1-1-2-1).

(5-2) The case in which (Y, A) is of the type (F).

Then $L^n = 2$ or 3 since $A^n \leq 4$ and L is spanned.

If $L^n = 2$, then $\Delta(L) = 1$ and X is a double cover of $\mathbb{P}^n$ whose branch locus B is a hypersurface of degree 8 by Lemma 1.4. But in this case, $K_X + (n-2)L$ is ample and this is impossible.

If $L^n = 3$, then $A^n = 4$ and $\pi : X \to Y$ is the blowing up at one point $y \in Y$. Note that $n = 3$ by Lemma 1.6. Then we find that $(K_Y + A)A^2 = (K_X + L)L^2 - 1 = 0$. Since $K_Y + A$ is nef, as in the case (4-3), we infer that $K_Y + A = \mathcal{O}_Y$ and $\Delta(A) = 2$. Since L is spanned, $|A|$ has no fixed component. Hence (Y, A) is of the type (IV) or (V-2) in Theorem 2.1 by Lemma 1.5. If (Y, A) is of the type (V-2), then we have a double covering $\rho : Y \to \mathbb{Q}^3$ such that $A = \rho^*\mathcal{O}_{\mathbb{Q}^3}(1)$. Set $\rho^{-1}(\rho(y)) = \{y, y'\}$; then $|L|$ has a base point on $\pi^{-1}(y')$ (possibly $y = y'$). This is a contradiction, thus (X, L) is of the type (IV′).

This completes the proof of Theorem 2.1. □

Remark 2.3. The existence of the case (IV′) in Theorem 2.1 can be shown as follows.

Let X' be a general hyperquartic in $\mathbb{P}^4$. Then X' is irreducible and smooth; moreover, X' cannot be a union of lines in $\mathbb{P}^4$ (see, e.g., Theorem 8 in [BaV]). Take a point $p \in X'$ which is not on any line contained in X' and let $\pi : M \to \mathbb{P}^4$ be the blowing up of $\mathbb{P}^4$ at p. Then $H := \pi^*\mathcal{O}_{\mathbb{P}^4}(1) - E$ is spanned, where $E = \pi^{-1}(p)$. We have $h^0(M, H) = 4$ and we denote by $\rho : M \to \mathbb{P}^3$ the morphism defined by $|H|$. Note that ρ is surjective and the fibers of ρ consist of the strict transforms of lines in $\mathbb{P}^4$ passing through p. Let X be the strict transform of X' under π. Then $\pi|_X : X \to X'$ is the blowing up at p and $H_X = (\pi|_X)^*(\mathcal{O}_{X'}(1)) - E_X$, where $E_X = (\pi|_X)^{-1}(p)$. Since X' does not contain any line passing through p, we infer that X does not contain any fiber of ρ and $\rho|_X : X \to \mathbb{P}^3$ is a finite surjective morphism. Hence $L := H_X = (\rho|_X)^*\mathcal{O}_{\mathbb{P}^3}(1)$ is ample and spanned. Thus (X, L) is a simple blowing up of $(X', \mathcal{O}_{X'}(1))$ and we easily see $g(L) = 3$.

Example 2.4. We give an example of the case (I-1) in Theorem 2.1 (informed by T. Fujita).

Let C be any smooth elliptic curve and let $\{p_i\}_{i=0}^3$ be four points of order two in the group structure on C. We set $\mathcal{E} = \oplus_{i=0}^3 \mathcal{O}_C(p_i)$. Then $\mathcal{E}$ is ample and so is the tautological line bundle $H(\mathcal{E})$ on $\mathbb{P}(\mathcal{E})$. Let $\pi : \mathbb{P}(\mathcal{E}) \to C$ be the projection. For each i, a natural injection $\mathcal{O}_C(p_i) \to \mathcal{E}$ determines a prime divisor $D_i \in |H(\mathcal{E}) - \pi^*\mathcal{O}_C(p_i)|$ on $\mathbb{P}(\mathcal{E})$; a natural surjection $\mathcal{E} \to \mathcal{O}_C(p_i)$ determines a section $C_i := \mathbb{P}(\mathcal{O}_C(p_i))$ of π. Then $C_i = D_j \cap D_k \cap D_l$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$ as schemes. Let q_i be the point $C_i \cap F_i$, where $F_i := \pi^{-1}(p_i)$. Since $D_i + F_i \in |H(\mathcal{E})|$, we have $\operatorname{Bs}|H(\mathcal{E})| \subset \cap_{i=0}^3 (D_i + F_i) = \{q_0, \dots, q_3\}$. Set $B = \mathcal{O}_C(-2p_0)$; then $B \cong \mathcal{O}_C(-2p_i)$ for each i. Note that $h^0(2H(\mathcal{E}) + \pi^*B) = h^0(S^2(\mathcal{E}) \otimes B) = 4$. Since $2D_i \in |2H(\mathcal{E}) + \pi^*B|$, we have $\operatorname{Bs}|2H(\mathcal{E}) + \pi^*B| \subset \cap_{i=0}^3 D_i = \emptyset$. Hence a general member X of $|2H(\mathcal{E}) + \pi^*B|$ is irreducible and smooth by Bertini's Theorem, and $q_i \notin X$ $(i = 0, 1, 2, 3)$. Then $L := H(\mathcal{E})|_X$ is ample and spanned, $g(L) = 3$, and $d = 6$. Thus (X, L) gives an expected example.

Appendix. The case in which $\mathbf{g(L) = 2}$.

Theorem. *Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. If $g(L) = 2$ and L is spanned, then (X, L) is one of the following:*

(I) X is a double covering of $\mathbb{P}^n$ with branch locus being a smooth hypersurface of degree 6, and L is the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

(II) (X, L) is a scroll over a smooth surface S. Let $\mathcal{E}$ be a locally free sheaf of rank two on S such that $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E})) \cong (X, L)$. Then $(S, \mathcal{E})$ is either

(II-1) $S \cong \mathbb{P}^1_\alpha \times \mathbb{P}^1_\beta$ and $\mathcal{E} \cong [H_\alpha + 2H_\beta] \oplus [H_\alpha + H_\beta]$, or

(II-2) S is the blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} \cong [2H - E]^{\oplus 2}$, where H is the pull back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and E is the exceptional divisor.

(III) (X, L) is a scroll over a smooth curve of genus two.

(IV) (X, L) is a hyperquadric fibration as in the case (I) of Theorem 2.1. We have $C \cong \mathbb{P}^1_\xi$, $3 \leq d \leq 9$, $e = d - 3$, $b = 6 - d$, $X \in |2H(\mathcal{E}) + bH_\xi|$, and their lists are in Table 2.

Table 2

d	$\mathcal{E}$	(X, L)
3	$\mathcal{O}(0,0,0,0)$	*$\|L\|$ makes X a triple covering of $\mathbb{P}^3$.*
4	$\mathcal{O}(0,0,0,1)$	*$\|L\|$ makes X the normalization of a hypersurface of degree four in $\mathbb{P}^4$, which has double points along a line.*
5	$\mathcal{O}(0,0,1,1)$	*Exist.*
6	$\mathcal{O}(0,1,1,1)$	*X is a double covering of $\mathbb{P}^1_\xi \times \mathbb{P}^2_\sigma$ with branch locus being a smooth divisor of bidegree (2,2). $L = [H_\xi + H_\sigma]_X$.*
7	$\mathcal{O}(1,1,1,1)$	*X is the blowing-up of $\mathbb{P}^3_\sigma$ with center being a complete intersection of two hyperquadrics. $L = [H_\xi + H_\sigma]_X$.*
8	$\mathcal{O}(1,1,1,2)$	*X is the blowing-up of hyperquadric in $\mathbb{P}^4_\sigma$ with center being a conic in $\mathbb{P}^2$. $L = [H_\xi + H_\sigma]_X$.*
	$\mathcal{O}(1,1,1,1,1)$	*$X \cong \mathbb{P}^1_\xi \times \mathbb{Q}$, where $\mathbb{Q}$ is a smooth hyperquadric in $\mathbb{P}^4_\sigma$. $L = [H_\xi + H_\sigma]_X$.*
9	$\mathcal{O}(1,1,2,2)$	*$X \cong \mathbb{P}^1_\lambda \times \Sigma_1$, where Σ_1 is the blowing-up of $\mathbb{P}^2_\sigma$ at a point. $L = [H_\xi + H_\lambda + H_\sigma]_X$.*

Proof. By [F5], [F6], [BiLL], and by an argument similar to that in Theorem 2.1, we can prove this Theorem. □

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