

論文 / 著書情報
Article / Book Information

題目(和文)	高精度統計計算による惑星集積過程の研究
Title(English)	Modeling of the foramtion of protoplanets without nebular gas by solving the statistical accumulation equation
著者(和文)	稲葉知士
Author(English)	
出典(和文)	学位:博士(理学), 学位授与機関:東京工業大学, 報告番号:甲第3978号, 授与年月日:1999年3月26日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor (Science), Conferring organization: Tokyo Institute of Technology, Report number:甲第3978号, Conferred date:1999/3/26, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

Abstract

We have investigated the early stage of the planetary accumulation which is one of the main themes of the study on the origin of the solar system. Since the number of planetesimals is quite large in the early stage, a direct N -body simulation is too hard to be performed. For the description of the planetary accumulation in that stage, the statistical method is widely used. However, in previous works authors did not check carefully the validity and accuracy of the used methods. In the present study, incorporating the latest results on the planetesimal dynamics, we develop the statistical method by which the planetary accumulation can be simulated accurately. To test our method, we compared our numerical results with those of the N -body simulation and found that these results coincide well each other within a surprising high accuracy.

Using our numerical method, we have performed the simulations of the planetary accumulation and found that the growth of protoplanets is divided into two stages. At first, protoplanets increase their masses exponentially, which is named the exponential growth. After the mass of a protoplanet reaches a certain critical mass, the growth of the protoplanet slows down. We have also showed that the runaway growth starts at the beginning of the accumulation and the orderly growth does not occur although it is believed that planetesimals start their growth through the orderly growth. Furthermore, we have derived the non-dimensional basic equations describing the planetary accumulation, which contain only one non-dimensional parameter. Combining the scaling law derived from the non-dimensional basic equations with the results of the numerical simulations, we have found that, irrelevant to the adopted physical parameters of the planetary accumulation (i.e., the surface density of solid materials, the initial average mass of planetesimals, and the semimajor axis), planetesimals grow through the two modes of the planetary accumulation.

Contents

1. Introduction	1
2. Basic equations describing the planetary accumulation process	9
2.1. Statistical coagulation equation	9
2.2. Collision rate between planetesimals	11
2.3. Velocity evolution of planetesimals	15
2.3.1. Velocity evolution due to gravitational interaction	16
2.3.2. Velocity evolution due to gas drag	19
2.3.3. Velocity evolution due to collisional dissipation	22
3. Methods of the simulation	24
3.1. Evolution of the mass distribution	25
3.2. Evolution of eccentricity and inclination	28
4. Test of accuracy of the statistical simulation code	32
4.1. Comparisons with the N -body simulation	32
4.1.1. Case of $\Delta a = 0.092$ AU	34
4.1.2. Case of $\Delta a = 0.021$ AU	37
4.2. Comparison with previous statistical simulations	40
4.3. Summary of the accuracy test	42
5. Accumulation process without nebular gas	45
5.1. The scaling law in the planetary accumulation process	46
5.2. The modes of the planetary growth	49
5.3. Check of the scaling law	57

6. Conclusions and concluding remarks	65
Acknowledgments	67
References	68

1. Introduction

The Kyoto model (Hayashi et al. 1985) is nowadays widely accepted as the standard model of the origin of the planetary system. According to the Kyoto model, the planetary accumulation proceeds as follows. After a disk-like gaseous nebula is formed around the central star (i.e., the protosun), dust grains sink toward the equatorial plane of the gaseous nebula and form a dust layer. When the density of the dust layer becomes high enough, the dust layer becomes gravitationally unstable owing to the self-gravity (Safronov 1969, Hayashi 1972, Goldreich and Ward, 1973) and fragments into a number of small bodies (which are called planetesimals). For the minimum mass solar nebula (Hayashi 1981), the typical initial mass of planetesimals m_0 is evaluated as (Hayashi et al. 1985)

$$m_0 = \begin{cases} 1 \times 10^{18} \left(\frac{a}{1\text{AU}} \right)^{3/2} \text{ g} & (a < 2.7\text{AU}), \\ 1 \times 10^{21} \left(\frac{a}{5\text{AU}} \right)^{3/2} \text{ g} & (a > 2.7\text{AU}), \end{cases} \quad (1.1.1)$$

where a is a distance from the protosun to a region considered. Planetesimals go around the protosun in the nebular gas and occasionally collide mutually to coalesce. As a result, they grow gradually to a small number of protoplanets. Assuming that a planetesimal grows through mutual collisions with other planetesimals with comparable masses, we obtain the growth time T_g of the planetesimal as (Hayashi et al. 1985)

$$T_g = \frac{m}{dm/dt} = \frac{(4\pi\rho_m/3)^{2/3}m^{1/3}}{8\pi^2(1+(v_{\text{esc}}/v)^2)\sigma_{\text{solid}}}T_K, \quad (1.1.2)$$

where σ_{solid} is the surface density of solid component, m and ρ_m are, respectively, the mass and the material density of a planetesimal, v is the relative velocity between planetesimals, and T_K is the Keplerian time at the orbit considered. Furthermore, v_{esc} is the surface escape velocity given by

$$v_{\text{esc}} = \sqrt{\frac{2Gm}{r}}, \quad (1.1.3)$$

where r is the radius of the planetesimal with mass m . Hayashi et al. (1985) and Ohtsuki et al. (1993) show that v is approximately given by the escape velocity v_{esc} . Putting $v = v_{\text{esc}}$, we obtain T_g as

$$T_g = \begin{cases} 8.8 \times 10^6 \left(\frac{m}{M_{\oplus}}\right)^{1/3} \left(\frac{a}{1 \text{ AU}}\right)^3 \text{ years} & (a < 2.7 \text{ AU}), \\ 2.7 \times 10^8 \left(\frac{m}{10M_{\oplus}}\right)^{1/3} \left(\frac{a}{5 \text{ AU}}\right)^3 \text{ years} & (a > 2.7 \text{ AU}). \end{cases} \quad (1.1.4)$$

In the above, we adopt the minimum mass nebular model (Hayashi 1981, Hayashi et al. 1985) in which σ_{solid} is given by

$$\sigma_{\text{solid}} = \begin{cases} 7.1 \left(\frac{a}{1 \text{ AU}}\right)^{-3/2} \text{ g/cm}^2 & (a < 2.7 \text{ AU}), \\ 30 \left(\frac{a}{5 \text{ AU}}\right)^{-3/2} \text{ g/cm}^2 & (a > 2.7 \text{ AU}). \end{cases} \quad (1.1.5)$$

Note that the discontinuity at $a = 2.7 \text{ AU}$ relates to the condensation of icy materials. After the core mass of the proto-Jovian planets becomes about 10 Earth's mass, they capture the nebula gas as their envelop to form the present Jovian planets as we observe.

Although the Kyoto model seems to give a satisfactory scenario on the formation of the planets of our solar system, two important problems are left unsolved. First, we cannot succeed in explaining of the "wide spacing" between the planets (difficulty of planetary spacing). According to the Kyoto model, too many tiny planets (say, 20 planets) are formed in the region of the terrestrial type planets. Secondly, we are faced with difficulty of formation time of the Jovian planets. As readily known from Eq. (1.1.4), the formation time becomes long with an increase in the distance from the protosun to a planet considered. For example, the formation time of the proto-Neptune with $10M_{\oplus}$ is 5.8×10^{10} years which is about 10 times as long as

the age of the solar system. Furthermore, observations of the T Tauri stars presume that the dissipation time of the solar nebula is within 1×10^7 years (Adams et al. 1987, 1988). The growth of the proto-Jovian core is too slow to capture a large amount of the nebular gas before the escape of the gaseous nebula.

In order to overcome these serious difficulties, i.e., difficulty of planetary spacing and difficulty of formation time, many detailed investigations are made. Stewart and Wetherill (1988) and Ida (1990) make clear that the velocity of a large body is suppressed through the gravitational scatterings with small bodies (this effect is called the dynamical friction) and suggest the occurrence of the runaway growth of the planets. The runaway growth means that the growth of only some bodies are much faster than that of other bodies and is expected as a mechanism to solve the problem of formation time of the Jovian planets. Wetherill and Stewart (1989, 1993) perform the numerical simulation of the planetary accumulation on the basis of the statistical method and confirm the occurrence of the runaway growth. On the other hand, Ohtuki et al. (1993) investigate the growth of planetesimals and give the following picture as the planetary accumulation process. Initially orderly growth, which means that the growth rates of all planetesimals are not so different from each other, is observed. After relatively large planetesimals grow to a certain critical mass and the effect of the gravitational focusing is important, the runaway growth starts and only some planetesimals (protoplanets) grow rapidly. However, by the use of the N -body simulation, Kokubo and Ida (1996) reconfirm that the runaway growth starts in the early stage of the planetary growth. Furthermore, it is shown that a runaway body grow at almost the same growth rate as that of next runaway bodies keeping the distance from them at about 10 Hill radii (Kokubo and Ida 1998). This growth mode is called oligarchic growth.

Clearly, there exists a conflict among the previous works. Because of a simple estimation, the explanation of Ohtsuki et al. may not be realistic, or otherwise the numerical procedure

developed by Wetherill and Stewart may not be suitable to describe the planetary accumulation process. Anyway, for overcoming the serious two difficulties mentioned earlier it is very important to make clear whether planetesimals start to grow through the stage of the orderly growth or not and when the runaway growth starts. In the runaway growth, the growth equation of a protoplanet is given by (Kokubo and Ida 1996)

$$\frac{dM}{dt} \propto M^{4/3} e_m^{-2}, \quad (1.1.6)$$

where M is the mass of a protoplanet and e_m is a typical eccentricity of planetesimals. If the runaway growth starts when the mean mass of planetesimals is still small, the growth rates of protoplanets becomes quite fast because small planetesimals suffer strongly from the gas drag due to the solar nebula and have very small e_m . For example, Wetherill and Stewart (1993) show that the formation time of a protoplanet with lunar mass is about 3×10^5 years at 1 AU and that, if we take the effect of collisional fragmentation of planetesimals into account, the formation time is shorten to 1×10^5 years. After the Kyoto model, the growth time of a protoplanet with lunar mass is about 2×10^6 years (see Eq.(1.1.4)) and is about 10 times as long as that obtained by Wetherill and Stewart. Therefore, in order to understand the growth of the protoplanets precisely, it is necessary to fit the starting time of the runaway growth correctly, as pointed out by Wetherill (1990).

Recent observations of extra-solar planets suggest that there is a wide variety of planetary systems (Mayor and Queloz 1995, Butler et al. 1997). It is difficult for the Kyoto model to bring about such a variety; the Kyoto model stands on the less massive nebular model, say, 0.01 to $0.02 M_\odot$. Observations of the T Tauri stars also show that a typical dust disk mass is between 1×10^{-4} and $1 \times 10^{-3} M_\odot$ (Beckwith and Sargent 1996). This tells us that a typical gas disk mass is of 1×10^{-2} or $1 \times 10^{-1} M_\odot$ since the total mass of gas is 100 times larger than that of solids. On the other hand, Cuzzi et al. (1993) assert that dust grains cannot sink within a

narrow-plane of the protoplanetary nebula owing to the turbulent flow in the nebular gas and planetesimals cannot be formed by the gravitational instability. Instead of the gravitational instability, Stepinski and Valageas (1996, 1997) consider the formation of planetesimals by the direct collision of dust grains though it is uncertain if their theory is correct. Thus, there is no established theory about the formation of planetesimals yet and, hence, the initial mass of a planetesimal is not definite, either. The physical parameters such as the surface density of solid component, the gas density of the solar nebula, and the initial average mass of planetesimals must influence essentially the final masses, positions, and the formation time of the planets. Hence, in order to understand a wide variety of planetary systems, the formation process of the planets should be studied by changing extensively the values of the physical parameters. Despite of the importance, nobody studies the formation process over the wide ranges of the parameters, probably because of the hardness of the numerical calculations.

As is well known, the accumulation process of the planets has so far been investigated by alternative methods: the N -body simulation and the statistical method. In the N -body simulation (e.g., Lecar and Aarseth 1986, Beaugé and Aarseth 1990, Aarseth et al. 1993, Kokubo and Ida 1996), they integrate numerically the orbits of all bodies (planetesimals) rotating around the protosun taking account of the mutual gravitational force precisely and describe the planetary growth assuming that the contacting two bodies merge to one with mass of their total. In the N -body simulation, they do not use any approximation other than the perfect accretion (i.e., contacting two bodies merge always to one), so that the N -body simulation can describe the accumulation process precisely. However, for the sake of the computational ability, it takes a long computing time even for only one calculation and the number of bodies treated is limited to 1×10^4 at most. In the early stage of the planetary growth, on the other hand, the number of planetesimals is inferred to be 1×10^{10} (Greenberg et al. 1978, Hayashi et al. 1985). Therefore, it is impossible to describe the early stage of the

accumulation process by the N -body simulation.

To investigate the planetesimal accumulation in the early stage, instead of the N -body simulation, many researchers adopt the statistical method (e.g., Greenberg et al. 1978, Nakagawa et al. 1983, Ohtsuki et al. 1988, Barge and Pellat 1991, Spaute et al. 1991, Wetherill and Stewart 1989, Weidenshilling et al. 1997). In the statistical method, the time evolution of the distribution function of planetesimals is calculated. This method has two merits: First, it is possible to calculate the evolution of a number of bodies because it does not treat the individual physical quantities of all bodies but only the average quantities. Secondly, the computing time of the accumulation process is very short compared with the N -body simulation. At the same time, the statistical method accompanies some risks. In this method, physical quantities, which characterize the mass distribution and the kinetic behaviors of planetesimals, are all replaced by their suitable averages. If we make averages carelessly, the obtained solution is disturbed seriously and fails to predict, even qualitatively, the planetary accumulation process since it is one of the strongly non-linear phenomenon. Furthermore, as mentioned earlier we face, sooner or later, the stage of the runaway growth which cannot be described by the statistical manner without a special device. Thus, it is very important to establish the numerical procedures of the statistical code which can pursue precisely and efficiently the accumulation process in order to obtain a correct picture of the planetary growth.

The aims of our present study are

- (1) to develop and establish the statistical numerical code which can pursue the accumulation process of the planets with sufficiently high accuracy

and

- (2) to answer, by the use of the established numerical simulator, the following key problems in relation to the early stage accumulation of planetesimals:

- (a) Do planetesimals experience orderly growth? Or, does the runaway growth start at the early stage of the accumulation process?
- (b) How do the parameters (such as the surface density of solid materials, the initial mass of a planetesimal, and the distance from the protosun) influence the growth time of the protoplanets in the case without nebular gas?

On the basis of Wetherill and Stewart (1989), we have already developed the statistical numerical code which can describe the accumulation process quantitatively (Inaba et al. 1999). The statistical code contains some parameters controlling the accuracy and the efficiency of the practical numerical computation. Among these, two parameters are essentially important: the mass division parameter which determines the mass ratio of the adjacent mass batches and the time division parameter which controls a time step of numerical integration. Through a number of numerical simulations for the simple cases where the collision rate depends only on masses of colliding bodies, Inaba et al. (1999) check the validity of their statistical code by comparing the analytical solutions to the stochastic coagulation equation and find a suitable set of the two parameters for the simulations of the accumulation process. The real planetary accumulation process is more complex since the collision rate depends strongly on the kinetic behavior of planetesimals as well as the masses of colliding bodies. Recently, Stewart and Ida (1999) derive new expressions describing the velocity evolution of planetesimals and find, by comparing with the results of the N -body simulations, that their method gives reliable results. Adopting the expressions of the velocity evolution by Stewart and Ida and the collision rate by Greenzweig and Lissauer (1992) who take account of the effect of the central star (i.e., the protosun), we complete the statistical code which calculates the velocity evolution of planetesimals simultaneously. In our present study, we compare the numerical results obtained by our statistical code with those of the N -body simulation and check the reliability of our numerical code for a realistic system of planetesimals.

In section 2, we will review the statistical coagulation equation as well as the basic equations relevant to the processes of the planetary accumulation, that is, (1) coalescence between planetesimals, (2) enhancement of the random velocity due to the gravitational stirring, and (3) its dissipation due to the gas drag as well as collisions. The numerical method which is developed by Inaba et al.(1999) will be explained in section 3. In section 4, we will compare the numerical results obtained by our statistical method with those of the N -body simulation to test the validity of our statistical method. We will see that our statistical code gives the same results as the N -body simulations and can describe the accumulation process with sufficiently high accuracy.

If we can simulate the planetary accumulation processes with various physical parameters (e.g., the surface density of solid, the gas density of the solar nebula, and the initial average mass of planetesimals), we seem to explain the variety of the planetary system observed in extrasolar objects. However, it is difficult practically to perform the simulations because of many parameters. In section 5, we will first rewrite the basic equations into non-dimensional forms and find a scaling law describing the growth of planetesimals in an early stage. Furthermore, by using our code, we will investigate the planetesimal accumulation for the case without nebular gas. It will be shown that the runaway growth starts irrelevant to the adopted any initial conditions. If we consider the effect of nebular gas on the planetary growth, the nebular gas decreases the random velocities of planetesimals and the collisional probability between planetesimals increases. As a result, the growth rates of planetesimals become large. This suggests that the runaway growth occurs in general. We will also justify in this section the validity of the scaling law, found above, by comparing with the results of the numerical simulations with various physical parameters.

The results of our present study will be summarized briefly in section 6.

2. Basic equations describing the planetary accumulation process

In the statistical method, the accumulation process is described as a time evolution of the mass distribution function of planetesimals based on the statistical coagulation equation. In subsection 2.1, we describe the coagulation equation for the planetary accumulation. In subsection 2.2, we present the collision rate (which we mean the rate of coalescence) between two planetesimals. The collision rate depends on dispersions of eccentricities and inclinations of planetesimals' orbits, which are the measure of deviations from a circular and planar orbit. In subsection 2.3, we introduce equations describing the time evolution of dispersions of planetesimals' eccentricities and inclinations due to their mutual gravitational interaction and the gas drag force.

2.1. Statistical coagulation equation

We consider a large number of bodies (i.e., planetesimals) rotating around a central star with mass M_c . We assume that the masses of bodies, m , are much smaller than M_c . We also assume that the eccentricity e and the inclination i of each body are much smaller than unity. In such a case, the bodies form a disk-like system around the central star because $i \ll 1$.

As a result of coalescences between planetesimals, planetesimals with various kinds of mass are formed. We denote the surface number density of planetesimals with masses between m and $m + dm$ by $n_s(m)dm$. Generally, the function $n_s(m)$ depends on a distance from the central star. Henceforth we are concerned with the accumulation process in a ring-like region with a major radius a and a narrow width Δa and assume that $n_s(m)$ is uniform throughout the ring-like region.

Let us consider collisions between two populations of planetesimals which belong to mass ranges $(m_1, m_1 + dm_1)$ and $(m_2, m_2 + dm_2)$. The collision rate depends on the distributions

of eccentricity, e , and inclination, i , in each population. As shown by Ida and Makino (1992), the distribution function of e and i is approximately given by the Rayleigh-type distribution function $f_R(e, i; e^*, i^*)$:

$$f_R(e, i; e^*, i^*)dedi = \frac{4ei}{e^{*2}i^{*2}} \exp\left(-\frac{e^2}{e^{*2}} - \frac{i^2}{i^{*2}}\right)dedi, \quad (2.1.1)$$

where e^{*2} and i^{*2} are the dispersions of e and i , respectively. From now on, we denote e^* and i^* of planetesimals with mass m_j by e_j^* and i_j^* , respectively. The averaged number of collisions between two populations of planetesimals per unit area of the disk and per unit time, $\langle \Gamma_{\text{col}} \rangle_{12}$, is given by (Nakazawa et al. 1989, Ohtsuki 1999)

$$\langle \Gamma_{\text{col}} \rangle_{12} = h_{12}^2 a^2 \Omega_K n_s(m_1) n_s(m_2) dm_1 dm_2 \langle P_{\text{col}} \rangle_{12}, \quad (2.1.2)$$

where $\langle P_{\text{col}} \rangle_{12}$ is the mean collision rate between planetesimals with masses m_1 and m_2 (i.e., the collision rate, P_{col} , averaged by the distribution function $f_R(e, i; e^*, i^*)$ in each population), a is the semimajor axis, Ω_K is the Keplerian angular velocity, and h_{12} is the reduced Hill radius of two planetesimals with masses m_1 and m_2 given by

$$h_{12} = \left(\frac{m_1 + m_2}{3M_c} \right)^{1/3}. \quad (2.1.3)$$

Formally speaking, the mean collision rate $\langle P_{\text{col}} \rangle_{12}$ is obtained by averaging P_{col} by two distribution functions (i.e., those in two populations). However, as shown by Ohtsuki (1999), if the distribution functions of e and i are of the Rayleigh-type, the mean collision rate $\langle P_{\text{col}} \rangle_{12}$ is simply obtained by averaging over a single Rayleigh-type distribution function:

$$\langle P_{\text{col}} \rangle_{12} = \int P_{\text{col}}(e, i) f_R(e, i; e_{12}^*, i_{12}^*) dedi, \quad (2.1.4)$$

where the dispersions e_{12}^{*2} and i_{12}^{*2} are given by

$$\begin{cases} e_{12}^{*2} = e_1^{*2} + e_2^{*2}, \\ i_{12}^{*2} = i_1^{*2} + i_2^{*2}. \end{cases} \quad (2.1.5)$$

Greenzweig and Lissauer (1992) investigate numerically the collision rate between planetesimals with the Rayleigh-type distribution of eccentricities and inclinations in detail and obtain an approximate formula of $\langle P_{\text{col}} \rangle_{12}$. In our present study we use their results which are presented in the next subsection.

Using Eq. (2.1.2), we can readily derive the coagulation equation for the disk system:

$$\begin{aligned} \frac{dn_s(m_1)}{dt} = & \frac{1}{2} \Omega_K \int_0^{m_1} dm_2 \int_0^\infty dm_3 (h_{23}a)^2 n_s(m_2) n_s(m_3) \langle P_{\text{col}} \rangle_{23} \delta(m_1 - m_2 - m_3) \\ & - \Omega_K n_s(m_1) \int_0^\infty dm_2 (h_{12}a)^2 n_s(m_2) \langle P_{\text{col}} \rangle_{12}. \end{aligned} \quad (2.1.6)$$

The coagulation equation is the basic equation describing the evolution of the mass distribution function $n_s(m)$.

2.2. Collision rate between planetesimals

Prior to the description of the mean collision rate, we introduce the reduced eccentricity, $\tilde{e}$, and the reduced inclination, $\tilde{i}$, for the later convenience, which are defined as (Nakazawa et al. 1989)¹

$$\tilde{e} = e/h_{12} \quad \text{and} \quad \tilde{i} = i/h_{12}, \quad (2.2.1)$$

respectively, where h_{12} is the reduced Hill radius given by Eq. (2.1.3). The reduced dispersions $\tilde{e}^{*2}$ and $\tilde{i}^{*2}$ are also defined by the same manner. When $\tilde{e}, \tilde{i} \gg 1$, such a body (case) is called

¹For the sake of simplicity we frequently abbreviate “reduced” from the naming of $\tilde{e}$ or $\tilde{i}$ as long as there is no possibility of confusion.

that of the high-velocity regime whereas it is called that of the low-velocity regime for the case of $\tilde{e}, \tilde{i} \ll 1$.

Now, the collision rate between planetesimals is studied analytically or numerically by many researchers (e.g., Ida and Nakazawa 1989, Greenzweig and Lissauer 1990,1992, Dones and Tremaine 1993). Almost all of these works give not analytical expressions of the collision rate but numerical results. Greenzweig and Lissauer (1990, 1992) investigate numerically the (mean) collision rate among planetesimals and, at the same time, give an approximate analytic formula of the mean collision rate in the limit of high-eccentricity. According to Greenzweig and Lissauer (1990), in the high-velocity regime ($\tilde{e}, \tilde{i} \gg 1$) the collision rate P_{col} is approximately given by

$$P_{\text{col}}(\tilde{e}, \tilde{i})_{\text{high}} = \frac{\tilde{r}_p^2}{2\pi} \left(F(I) + \frac{6}{\tilde{r}_p} \frac{G(I)}{\tilde{e}^2} \right), \quad (2.2.2)$$

where

$$\tilde{r}_p = \frac{r_1 + r_2}{h_{12}a}, \quad (2.2.3)$$

and r_1 and r_2 are the radii of colliding planetesimals and $I = \tilde{i}/\tilde{e}$. Functions $F(I)$ and $G(I)$ are defined, respectively, by

$$F(I) = \frac{4\sqrt{1+I^2}}{I} E(k) \quad \text{and} \quad G(I) = \frac{4}{I\sqrt{1+I^2}} K(k), \quad (2.2.4)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and the second kind, respectively, and $k^2 = 3/(4+4I^2)$. Averaging over the distribution function, we have the mean collision rate $\langle P_{\text{col}} \rangle$ given by Eq. (2.1.4) (Greenzweig and Lissauer 1992):

$$\langle P_{\text{col}} \rangle_{\text{high}} = \frac{\tilde{r}_p^2}{2\pi} \left(\mathcal{F}(I^*) + \frac{6}{\tilde{r}_p} \frac{\mathcal{G}(I^*)}{\tilde{e}^{*2}} \right), \quad (2.2.5)$$

where $I^* = \tilde{i}^*/\tilde{e}^*$ and functions $\mathcal{F}(I^*)$ and $\mathcal{G}(I^*)$ are given, respectively, by

$$\mathcal{F}(I^*) = 8 \int_0^1 d\lambda \frac{I^{*2} E(\sqrt{3(1-\lambda^2)}/2)}{(I^{*2} + (1-I^{*2})\lambda^2)^2} \quad \text{and} \quad \mathcal{G}(I^*) = 8 \int_0^1 d\lambda \frac{K(\sqrt{3(1-\lambda^2)}/2)}{(I^{*2} + (1-I^{*2})\lambda^2)}. \quad (2.2.6)$$

In Eq. (2.2.5), the second term denotes the enhancement of the collision rate due to the gravitational focusing. This term becomes larger than the first term whenever we observe the runaway growth (Wetherill 1990, Ohtsuki et al. 1993).

On the other hand, Ida and Nakazawa (1989) show that when $\tilde{e}, \tilde{i} \ll 1$, the collision rate P_{col} is approximately given by

$$P_{\text{col}}(\tilde{e} = \tilde{i} = 0) = 11.3\sqrt{\tilde{r}_p}. \quad (2.2.7)$$

Thus, for the low-velocity regime (i.e., $\tilde{e}^*, \tilde{i}^* \ll 1$), the mean collision rate is simply given by

$$\langle P_{\text{col}} \rangle_{\text{low}} = 11.3\sqrt{\tilde{r}_p}. \quad (2.2.8)$$

In a system of planetesimals, it is known that $\tilde{i}^*$ is of the same order of magnitude as $\tilde{e}^*$. So, it is sufficient for us to cover only three cases: (i) $\tilde{e}^*, \tilde{i}^* \gg 1$, (ii) $\tilde{e}^*, \tilde{i}^* \ll 1$, and (iii) the intermediate case. After some trials we find that the mean collision rate is well approximated by the following expression over the three cases:

$$\langle P_{\text{col}} \rangle = (\langle P_{\text{col}} \rangle_{\text{high}}^{-2} + \langle P_{\text{col}} \rangle_{\text{low}}^{-2})^{-1/2}. \quad (2.2.9)$$

In Fig. 2.1, we compare the mean collision rate evaluated by the above expression with the results obtained by the 3 body numerical calculations (Greenzweig and Lissauer 1992), for the cases of $\tilde{e}^* = \tilde{i}^*$, $2\tilde{i}^*$, and $4\tilde{i}^*$. Clearly, approximate formula (2.2.9) can almost reproduce the numerical results within an accuracy of factor 2. In our present study, we use this expression for the mean collision rate between planetesimals.

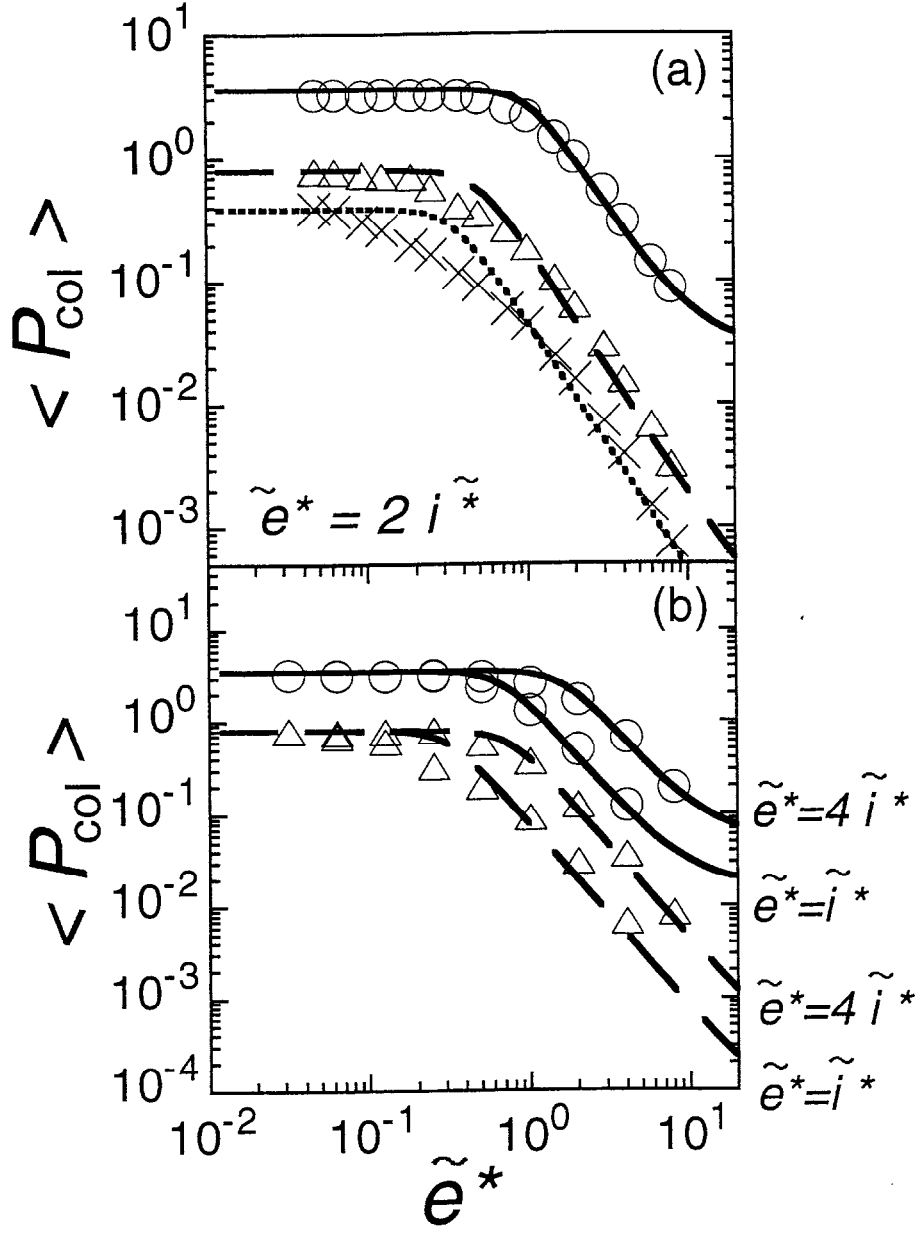


Fig. 2.1. Comparison of the mean collision rate $\langle P_{\text{col}} \rangle$ evaluated from the approximate formula (curves) with those of the corresponding numerical solutions (circles, triangles, and crosses). The mean collision rate is shown as a function of $\tilde{e}^*$ for the cases of $\tilde{e}^* = 2\tilde{i}^*$ (panel (a)) and $\tilde{e}^* = \tilde{i}^*$ and $4\tilde{i}^*$ (panel (b)). Solid, dashed, and dotted curves indicate the mean collision rate for $\tilde{r}_p = 0.1, 0.005$, and 0.0012 , respectively.

2.3. Velocity evolution of planetesimals

Since the mean collision rate between planetesimals depends on the dispersions of eccentricities and inclinations (i.e., $\tilde{e}^{*2}$ and $\tilde{i}^{*2}$) of a swarm of planetesimals, we have to calculate them at every moment of the growth of planetesimals. In this subsection, we summarize the equations describing the evolution of $\tilde{e}^{*2}$ and $\tilde{i}^{*2}$ of planetesimals. As readily conjectured, the dispersions depend on the mass of planetesimals and, hence, we have to evaluate the dispersions $\tilde{e}_j^{*2}$ and $\tilde{i}_j^{*2}$ for each mass m_j . They are enhanced by the gravitational perturbations due to other planetesimals and decreased by the drag due to the nebular gas and collisions. The time variation of the dispersions is given, as the sum of these three effects, by

$$\begin{cases} \frac{d\tilde{e}_j^{*2}}{dt} = \left(\frac{d\tilde{e}_j^{*2}}{dt}\right)_{\text{grav}} + \left(\frac{d\tilde{e}_j^{*2}}{dt}\right)_{\text{gas}} + \left(\frac{d\tilde{e}_j^{*2}}{dt}\right)_{\text{col}}, \\ \frac{d\tilde{i}_j^{*2}}{dt} = \left(\frac{d\tilde{i}_j^{*2}}{dt}\right)_{\text{grav}} + \left(\frac{d\tilde{i}_j^{*2}}{dt}\right)_{\text{gas}} + \left(\frac{d\tilde{i}_j^{*2}}{dt}\right)_{\text{col}}, \end{cases} \quad (2.3.1)$$

where terms with subscripts 'grav', 'gas', and 'col' express the time variations due to the gravitational perturbation, the gas drag, and the collisions, respectively. In our present work, we concentrate ourselves to the study of the planetary accumulation for the case without the solar nebula. In this sense, we need not include the effect of the gaseous drag in Eq. (2.3.1). However, as will be presented in section 4, we make a comparison with previous works quantitatively to confirm the validity of our statistical method. Since the gas drag due to the solar nebula is taken into account in most previous works, we also include the effect of the gas drag in our statistical code.

Note that, hereafter, we call simply $\tilde{e}^*$ (or $\tilde{i}^*$) “the random velocity” in place of “the root mean square of eccentricities (or inclinations)”. Such a naming is due to the facts that $(\tilde{e}^{*2} + \tilde{i}^{*2})^{1/2}$ is almost equal to the mean (non-dimensional) random velocity of a planetesimal circulating around the protosun and that in most cases we have $\tilde{e}^* \simeq \tilde{i}^*$ in a swarm of planetesimals.

2.3.1. Velocity evolution due to gravitational interaction

Ida (1990), Ohtsuki (1999), and Stewart and Ida (1999) investigate the evolution of the dispersions of eccentricity and inclination due to the mutual gravitational interaction. According to them, time variations of $\tilde{e}_1^{*2}$ and $\tilde{i}_1^{*2}$ of planetesimals with mass m_1 are described, respectively, as

$$\left(\frac{d\tilde{e}_1^{*2}}{dt}\right)_{\text{grav}} = \Omega_K a^2 \int dm_2 n_s(m_2) \frac{h_{12}^2 m_2}{(m_1 + m_2)^2} \left(m_2 \langle P_{VS} \rangle + \langle P_{DF} \rangle \frac{m_2 \tilde{e}_2^{*2} - m_1 \tilde{e}_1^{*2}}{\tilde{e}_{12}^{*2}} \right) \quad (2.3.2)$$

and

$$\left(\frac{d\tilde{i}_1^{*2}}{dt}\right)_{\text{grav}} = \Omega_K a^2 \int dm_2 n_s(m_2) \frac{h_{12}^2 m_2}{(m_1 + m_2)^2} \left(m_2 \langle Q_{VS} \rangle + \langle Q_{DF} \rangle \frac{m_2 \tilde{i}_2^{*2} - m_1 \tilde{i}_1^{*2}}{\tilde{i}_{12}^{*2}} \right). \quad (2.3.3)$$

In the above, $n_s(m_2)dm_2$ is the surface number density of planetesimals with masses between m_2 and $m_2 + dm_2$ and h_{12} is the reduced Hill radius given by Eq. (2.1.3). The first terms in the brackets represent the changing rate by the viscous stirring due to encounter with other planetesimals. The second terms correspond to the changing rate by the dynamical friction which leads to the energy equipartition among planetesimals with different masses. Note that the definitions of $\langle P_{VS} \rangle$, $\langle P_{DF} \rangle$, $\langle Q_{VS} \rangle$, and $\langle Q_{DF} \rangle$ are of Ohtsuki (1999) and are different from those of Stewart and Ida (1999) by a factor of h_{12}^4 .

Tanaka and Ida (1996) and Stewart and Ida (1999) derive analytic expressions of the mean viscous stirring rates, $\langle P_{VS} \rangle$ and $\langle Q_{VS} \rangle$, and the mean dynamical friction rates, $\langle P_{DF} \rangle$ and $\langle Q_{DF} \rangle$. In the present study, we use the expressions of Stewart and Ida. According to them, the mean viscous stirring rate, $\langle P_{VS} \rangle$, consists of two parts:

$$\langle P_{VS} \rangle = \langle P_{VS, \text{close}} \rangle + \langle P_{VS, \text{dist}} \rangle, \quad (2.3.4)$$

where $\langle P_{VS, \text{close}} \rangle$ is the viscous stirring rate brought by close encounter with other planetesimals and $\langle P_{VS, \text{dist}} \rangle$ is that by distant encounter. They are given by

$$\begin{cases} \langle P_{\text{VS, close}} \rangle = \frac{9}{\sqrt{\pi} I^* \tilde{e}^{*2}} \left(\ln(\Lambda^2 + 1) - \frac{\Lambda^2}{\Lambda^2 + 1} \right) J_e, \\ \langle P_{\text{VS, dist}} \rangle = 137 \frac{\exp(\frac{2}{\tilde{e}^{*2}}) E_1(\frac{2}{\tilde{e}^{*2}}) - \exp(\frac{2}{\tilde{i}^{*2}}) E_1(\frac{2}{\tilde{i}^{*2}})}{\tilde{e}^{*2} - \tilde{i}^{*2}}, \end{cases} \quad (2.3.5)$$

where

$$J_e = \frac{8I^*}{\sqrt{\pi}} \int_0^1 d\lambda \frac{5K(\sqrt{3(1-\lambda^2)}/2) - \frac{12(1-\lambda^2)E(\sqrt{3(1-\lambda^2)}/2)}{(1+3\lambda^2)}}{(I^{*2} + (1-I^{*2})\lambda^2)}, \quad (2.3.6)$$

$$\Lambda = (\tilde{i}^* + 1)(\tilde{e}^{*2} + \tilde{i}^{*2})/3, \quad (2.3.7)$$

and $E(k)$ and $K(k)$ are the complete elliptic integrals. Furthermore, $E_1(x)$ is the exponential integral function given by

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt \quad (2.3.8)$$

and is expressed approximately as (Abramowitz and Stegun 1972):

$$E_1(x) = -\ln x + a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5, \quad (2.3.9)$$

where

$$\begin{cases} a_0 = -0.57721566, & a_1 = 0.99999193, & a_2 = -0.24991055, \\ a_3 = 0.05519968, & a_4 = -0.00976004, & \text{and } a_5 = 0.00107857 \end{cases} \quad (2.3.10)$$

The mean viscous stirring rate for $\tilde{i}^{*2}$, $\langle Q_{\text{VS}} \rangle$, is given by the similar functional form, that is,

$$\langle Q_{\text{VS}} \rangle = \frac{9}{\sqrt{\pi} I^* \tilde{e}^{*2}} \left(\ln(\Lambda^2 + 1) - \frac{\Lambda^2}{\Lambda^2 + 1} \right) J_i, \quad (2.3.11)$$

where

$$J_i = \frac{8I^*}{\sqrt{\pi}} \int_0^1 d\lambda \frac{K(\sqrt{3(1-\lambda^2)}/2) - \frac{12(1-\lambda^2)E(\sqrt{3(1-\lambda^2)}/2)}{(1+3\lambda^2)}}{(I^{*2} + (1-I^{*2})\lambda^2)}. \quad (2.3.12)$$

Note that we neglect a term due to the distant encounter in Eq. (2.3.11) since the effect of the distant encounter to the viscous stirring rate for $\tilde{i}^{*2}$ is quite small (Stewart and Ida 1999).

The mean dynamical friction rates $\langle P_{\text{DF}} \rangle$ and $\langle Q_{\text{DF}} \rangle$ are given, respectively, by (Stewart and Ida 1999)

$$\langle P_{\text{DF}} \rangle = 0.7 \frac{18}{\sqrt{\pi} I^* \tilde{e}^{*2}} \ln(\Lambda^2 + 1) H_e \quad (2.3.13)$$

and

$$\langle Q_{\text{DF}} \rangle = 0.7 \frac{18}{\sqrt{\pi} I^* \tilde{e}^{*2}} \ln(\Lambda^2 + 1) H_i, \quad (2.3.14)$$

where

$$H_e = \frac{32I^*}{\sqrt{\pi}} \int_0^1 d\lambda \frac{(1-\lambda^2)E(\sqrt{3(1-\lambda^2)}/2)}{(1+3\lambda^2)(I^{*2} + (1-I^{*2})\lambda^2)} \quad (2.3.15)$$

and

$$H_i = \frac{32I^*}{\sqrt{\pi}} \int_0^1 d\lambda \frac{\lambda^2 E(\sqrt{3(1-\lambda^2)}/2)}{(1+3\lambda^2)(I^{*2} + (1-I^{*2})\lambda^2)}. \quad (2.3.16)$$

In the above, J_e , J_i , H_e , and H_i are all functions of I^* . Prior to the simulation of the planetary growth process, we calculate numerically these functions for the typical values of I^* and prepare the numerical tables. In making a simulation we use the values interpolated from these tables.

2.3.2. Velocity evolution due to gas drag

Adachi et al. (1976) investigate extensively the gas drag effect on the motion of a planetesimal rotating around the protosun. They obtain the approximate formulae of the time evolution of the semimajor axis, a , the eccentricity, e , and the inclination, i , averaging over one Keplerian period. Assuming that one of e , i , and η is much larger than others and retaining the leading terms, they obtain the approximate equations: ²

$$\frac{\tau}{a} \left(\frac{da}{dt} \right)_{\text{gas}} = -2\eta \left(\frac{2E+K}{3\pi} e + \frac{2}{\pi} i + \eta \right), \quad (2.3.17)$$

$$\frac{\tau}{e} \left(\frac{de}{dt} \right)_{\text{gas}} = - \left(\frac{2E}{\pi} e + \frac{2}{\pi} i + \frac{3}{2}\eta \right), \quad (2.3.18)$$

and

$$\frac{\tau}{i} \left(\frac{di}{dt} \right)_{\text{gas}} = -\frac{1}{2} \left(\frac{2E}{\pi} e + \frac{8}{3\pi} i + \eta \right), \quad (2.3.19)$$

where $E = E(3/4)$ and $K = K(3/4)$. In the above, η is a parameter denoting a measure of importance of the radial pressure gradient in the gaseous disk and is defined as

$$\eta = \frac{v_K - v_{\text{gas}}}{v_K}, \quad (2.3.20)$$

where v_K and v_{gas} are the Keplerian velocity and the circular velocity of gas, respectively. The characteristic time for the drag dissipation, τ , is given by

$$\tau = \left(\frac{1}{2} C_D \pi r^2 \rho_{\text{gas}} a \Omega_K / m \right)^{-1}, \quad (2.3.21)$$

where C_D is the drag coefficient, r and m are the radius and mass of a planetesimal, respectively, and ρ_{gas} is the gas density of the solar nebula.

²As indicated by Kary et al. (1993), there is an error in the expression of de/dt (Eq. (4.12)) of Adachi et al. (1976).

As conjectured from the way of derivation of Eqs. (2.3.17) to (2.3.19), errors of these equations become large when e , i , and η are comparable with one another. Hence, instead of them, we propose the following new equations:

$$\frac{\tau}{a} \left(\frac{da}{dt} \right)_{\text{gas}} = -2\eta \left(\left(\frac{2E+K}{3\pi} e \right)^2 + \left(\frac{2}{\pi} i \right)^2 + \eta^2 \right)^{1/2}, \quad (2.3.22)$$

$$\frac{\tau}{e} \left(\frac{de}{dt} \right)_{\text{gas}} = - \left(\left(\frac{2E}{\pi} e \right)^2 + \left(\frac{2}{\pi} i \right)^2 + \left(\frac{3}{2} \eta \right)^2 \right)^{1/2}, \quad (2.3.23)$$

and

$$\frac{\tau}{i} \left(\frac{di}{dt} \right)_{\text{gas}} = -\frac{1}{2} \left(\left(\frac{2E}{\pi} e \right)^2 + \left(\frac{8}{3\pi} i \right)^2 + \eta^2 \right)^{1/2}. \quad (2.3.24)$$

Although we cannot show the validity of the new equations mathematically, we can confirm numerically that the errors of the new equations are small compared with those of Adachi et al. In Fig. 2.2, we compare the time evolution of a , e , and i calculated from the equations of Adachi et al. (1976) and our new equations with the results of the orbital calculation (the test calculations are made under the condition that $\tau = 200T_K$ and $\eta = 0.01$ and start from the initial condition where $e = i = 0.01$ and $a = 1\text{AU}$). As seen from this figure, our equations reproduce the results of the orbital calculation much better than those of Adachi et al.

Averaging each term separately by the distribution function of e and i (i.e., the Rayleigh-type distribution function) and summing up them in the similar way to Eqs. (2.3.22) to (2.3.24), we obtain the following equations (note that the dispersions are not of the “reduced” type):

$$\frac{\tau}{e^{*2}} \left(\frac{de^{*2}}{dt} \right)_{\text{gas}} = -2 \left(\frac{9}{4\pi} E^2 e^{*2} + \frac{1}{\pi} i^{*2} + \frac{9}{4} \eta^2 \right)^{1/2} \quad (2.3.25)$$

and

$$\frac{\tau}{i^{*2}} \left(\frac{di^{*2}}{dt} \right)_{\text{gas}} = - \left(\frac{1}{\pi} E^2 e^{*2} + \frac{4}{\pi} i^{*2} + \eta^2 \right)^{1/2}. \quad (2.3.26)$$

The above equations will be referred to in section 4.

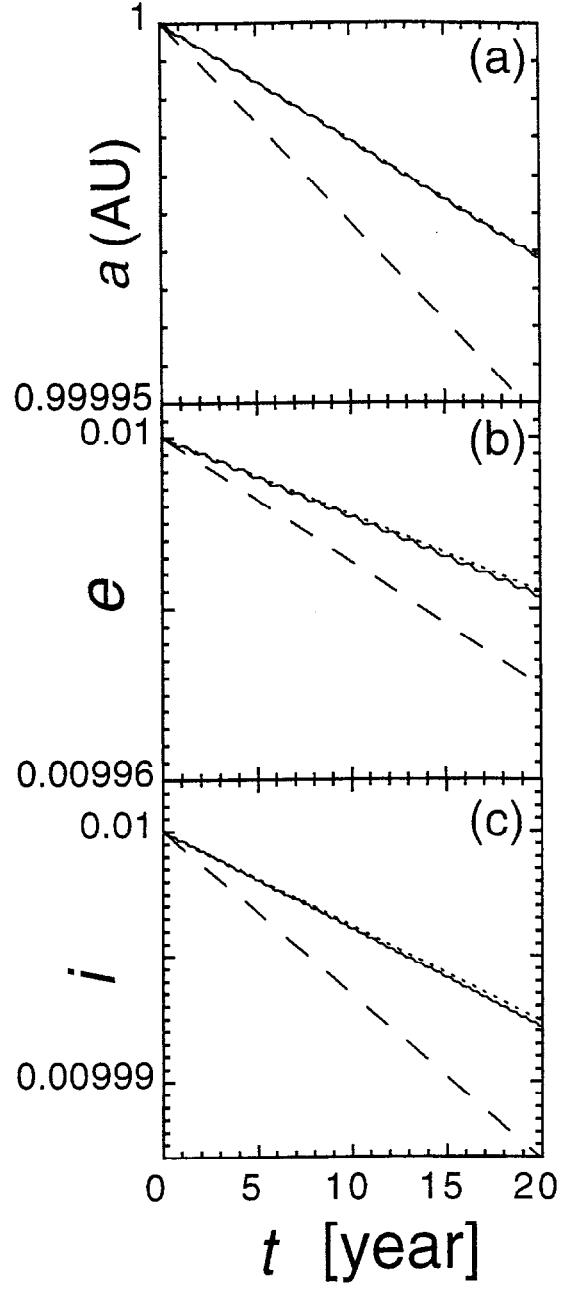


Fig. 2.2. Time evolution of the semimajor axis, a , the eccentricity, e , and the inclination, i , due to gaseous drag. These are calculated from the equations derived by Adachi et al. (1976) (dashed curves) and the new equations (dotted curves) as well as the orbital calculation (solid curves). For the initial condition we choose $e = i = 0.01$ and $a = 1\text{AU}$ and for τ and η we put to be $\tau = 200T_K$ and $\eta = 0.01$.

2.3.3. Velocity evolution due to collisional dissipation

We have also to consider the effect of the collisional dissipation on the kinetic behavior of planetesimals. When two planetesimals (with masses m_j and m_k) coalesce, the orbit of a merged planetesimal is given by that of the center of mass. The eccentricity E_{jk} and inclination I_{jk} of the barycenter's orbit are given by

$$\begin{cases} E_{jk}^2 = \left(\frac{m_j}{m_j + m_k}\right)^2 e_j^2 + \left(\frac{m_k}{m_j + m_k}\right)^2 e_k^2 + \frac{2m_j m_k}{(m_j + m_k)^2} e_j e_k \cos(\tau_j - \tau_k), \\ I_{jk}^2 = \left(\frac{m_j}{m_j + m_k}\right)^2 i_j^2 + \left(\frac{m_k}{m_j + m_k}\right)^2 i_k^2 + \frac{2m_j m_k}{(m_j + m_k)^2} i_j i_k \cos(\omega_j - \omega_k), \end{cases} \quad (2.3.27)$$

where τ_j and ω_j are the longitudes of the perihelion and the ascending nodes, respectively. Assuming the random distribution of the phase angles τ_j and ω_j and averaging by the Rayleigh-type distribution (2.1.1), we obtain the dispersions of E_{jk} and I_{jk} as

$$\begin{cases} E_{jk}^{*2} = \left(\frac{m_j}{m_j + m_k}\right)^2 e_j^{*2} + \left(\frac{m_k}{m_j + m_k}\right)^2 e_k^{*2}, \\ I_{jk}^{*2} = \left(\frac{m_j}{m_j + m_k}\right)^2 i_j^{*2} + \left(\frac{m_k}{m_j + m_k}\right)^2 i_k^{*2}. \end{cases} \quad (2.3.28)$$

Remember that the averaged number of collisions between two populations of planetesimals per unit area and per unit time is given by Eq. (2.1.2). Noting that coalescent bodies have the dispersions given by Eq. (2.3.28), we have for the variation rate of $\tilde{e}_l^{*2}$ and $\tilde{i}_l^{*2}$ of planetesimals with mass m_l due to coalescence

$$\begin{aligned} \left(\frac{de_l^{*2}}{dt}\right)_{\text{col}} = & - \frac{\Omega_K}{2n_s(m_l)} \int dm_j \int dm_k (h_{jk} a)^2 n_s(m_j) n_s(m_k) \\ & \times (e_l^{*2} - E_{jk}^{*2}) \langle P_{\text{col}} \rangle_{jk} \delta(m_l - m_j - m_k) \end{aligned} \quad (2.3.29)$$

and

$$\begin{aligned}
\left(\frac{di_l^{\star 2}}{dt}\right)_{\text{col}} = & - \frac{\Omega_K}{2n_s(m_l)} \int dm_j \int dm_k (h_{jk}a)^2 n_s(m_j) n_s(m_k) \\
& \times (i_l^{\star 2} - I_{jk}^{\star 2}) \langle P_{\text{col}} \rangle_{jk} \delta(m_l - m_j - m_k). \tag{2.3.30}
\end{aligned}$$

3. Methods of the simulation

In this section, we explain the numerical procedures of the statistical simulation code for the planetary accumulation. The statistical simulation code for describing the accumulation process, especially, describing the runaway growth of the planets, is first developed by Wetherill and Stewart (1989) and presented in more detail in Wetherill (1990). The statistical code includes some technical parameters which determine the accuracy and the efficiency of the numerical simulation. Though they treat the technical parameters carefully in their works, they do not check how the statistical code can reproduce the exact solutions to the stochastic coagulation equation (Marcus 1968, Tanaka and Nakazawa 1993) and how the accuracy depends on the technical parameters. Inaba et al. (1999) reconstruct the statistical simulation code according to Wetherill and Stewart and test the accuracy of the code very carefully by comparing the numerical results with the analytical solutions to the stochastic coagulation equation. They make clear the validity and usefulness of the code and find the suitable range of the technical parameters to describe the accumulation process precisely. In subsection 3.1, we present briefly the numerical procedures of our code for simulating the accumulation process.

In order to simulate the planetary growth, we have to pursue the time evolution of the random velocity, $\tilde{e}^*$ and $\tilde{i}^*$, simultaneously since the collision rate between planetesimals depends essentially on their random velocities as mentioned in section 2. Though, in most previous works, the time evolution of the random velocity is treated by a simple (and, hence, an inaccurate) manner, we describe it as precisely as possible taking account of the recent results found in the N -body simulations. The numerical method for calculation of the time evolution of the random velocity is explained in subsection 3.2.

3.1. Evolution of the mass distribution

To describe the mass distribution of planetesimals, we use a set of discrete mass batches instead of the (continuous) mass coordinate. The mass m_j of the j -th batch is given by a geometric series:

$$m_j = m_0 \delta^j, \quad j = 0, 1, \dots, n_b - 2 \quad (3.1.1)$$

where n_b is the total number of batches and δ (>1) is a certain constant (one of the technical parameters) of the order of unity. When the value of δ is too large, physically realistic description of the accumulation process might fail. When it is very close to 1, on the other hand, we are forced to prepare a large number of numerical batches and spend a long computing time though the simulation could describe the accumulation process precisely. Thus, we have to choose the value of δ appropriately in the numerical simulations. Inaba et al. (1999) show that the simulation with $\delta \leq 1.3$ can describe the runaway growth precisely. In our present study, we choose the value of δ to be 1.1

Let $N_{s,j}$ be the surface number density of planetesimals belonging to batch j (i.e., with the mass m_j). A set of the surface number densities $\{N_{s,j}\}$ describes the mass distribution function $n_s(m)$ mentioned in the previous section. From Eq. (2.1.6), we can readily write the equation describing the time evolution of the surface number density in each batch:

$$\frac{dN_{s,l}}{dt} = \frac{1}{2} \Omega_K \sum_{j,k}' (h_{jk}a)^2 N_{s,j} N_{s,k} \langle P_{\text{col}} \rangle_{jk} - \Omega_K N_{s,l} \sum_j (h_{jl}a)^2 N_{s,j} \langle P_{\text{col}} \rangle_{jl}, \quad (3.1.2)$$

where $\sum_{j,k}'$ expresses the sum over the pair of batches j and k which satisfy $(m_l + m_{l-1})/2 < m_j + m_k < (m_l + m_{l+1})/2$. In the simulation, instead of the surface number density $N_{s,j}$, we calculate the number of planetesimals in the ring-like region we consider, N_j , which is given by

$$N_j = 2\pi a \Delta a N_{s,j}, \quad (3.1.3)$$

where a and Δa is the major radius and the width of the ring-like region, respectively.

We calculate the evolution of $\{N_j\}$ as follows. The number of collisions during a (numerical) time interval δt between bodies belonging to a target batch (j) and a projectile batch ($k; k \leq j$), $\delta N_{\text{col},jk}$, is given by

$$\delta N_{\text{col},jk} = h_{jk}^2 a^2 \Omega_K N_{s,j} N_k \langle P_{\text{col}} \rangle_{jk} \delta t. \quad (3.1.4)$$

The growth of bodies belonging to batch j in a time interval δt is treated in two ways. When the mass of merged bodies, $m_j + m_k$, is smaller than $(m_j + m_{j+1})/2$, the bodies still belongs to batch j ; the number of bodies in batch k is decreased by $\delta N_{\text{col},jk}$ and the number of bodies in batch j remains unchanged whereas the total mass in batch j is increased by $m_k \delta N_{\text{col},jk}$ and the total mass in batch k is decreased by $m_k \delta N_{\text{col},jk}$. On the other hand, when the combined mass, $m_j + m_k$, is greater than $(m_j + m_{j+1})/2$, the merged bodies are placed to a higher batch (say, batch l); the numbers of bodies in both batches j and k are decreased by $\delta N_{\text{col},jk}$ and the total masses in batches j and k are decreased by $m_j \delta N_{\text{col},jk}$ and $m_k \delta N_{\text{col},jk}$, respectively. Furthermore, the number of bodies in batch l is increased by $\delta N_{\text{col},jk}$ and its mass is increased by $(m_j + m_k) \delta N_{\text{col},jk}$.

We make the above procedures for all of pairs of batches j and k using N_j^n at time $t = t_n$, through which we can calculate the net changes in the number δN_j and the mass δM_j for each batch. The number of bodies of batch j at time $t_{n+1} = t_n + \delta t$ is then given by

$$N_j^{n+1} = N_j^n + \delta N_j. \quad (3.1.5)$$

Furthermore, a new average mass of batch j is evaluated by dividing the newly obtained total mass by the newly obtained number of bodies contained in the batch. That is,

$$m_j^{n+1} = (M_j^n + \delta M_j) / N_j^{n+1}. \quad (3.1.6)$$

In order to keep the number of planetesimals N_j to be an integer, we make $\delta N_{\text{col},jk}$ an integer

according to Wetherill and Stewart (1993). When $\delta N_{\text{col},jk} < 2 \times 10^9$, $\delta N_{\text{col},jk}$ is replaced by

$$\delta N'_{\text{col},jk} = [\delta N_{\text{col},jk}] + \epsilon, \quad (3.1.7)$$

where $[\delta N_{\text{col},jk}]$ is integer part of $\delta N_{\text{col},jk}$ and ϵ is given by

$$\epsilon = \begin{cases} 1 & : \delta N_{\text{col},jk} - [\delta N_{\text{col},jk}] \geq \zeta, \\ 0 & : \delta N_{\text{col},jk} - [\delta N_{\text{col},jk}] < \zeta. \end{cases} \quad (3.1.8)$$

In the above, ζ is a random number between 0 and 1 created by the random number generator if necessary. On the other hand, when $\delta N_{\text{col},jk}$ is equal to or greater than 2×10^9 , we permit $\delta N_{\text{col},jk}$ to be a fractional value.

Furthermore, another device is introduced to describe properly the growth of the runaway bodies. As seen in the results of the N -body simulation (Kokubo and Ida 1998), some runaway bodies are formed in a swarm of planetesimals and the radial separation of their orbits is about 10 times as large as the Hill radius of bodies. Their orbits are almost circular and coplanar because of the dynamical friction due to small bodies. As a result, orbits of the runaway bodies never cross and they cannot collide each other. According to Wetherill and Stewart(1993), we take account of this isolation of the bodies by the following way. Let N_{run} be the number of the largest runaway bodies which cannot collide each other. Then N_{run} should be so defined that the sum of their orbital separation δa_i is equal to the width of the computation zone, Δa . In the simulation, the bodies contained in all batches heavier than batch j' is assumed to be isolated; the batch number j' is defined by the condition

$$\sum_{i=j'+1}^{n_b} N_i \delta a_i < \Delta a < \sum_{i=j'}^{n_b} N_i \delta a_i, \quad (3.1.9)$$

where n_b is the total number of batches. By the use of j' N_{run} is given by

$$N_{\text{run}} = \sum_{i=j'}^{n_b} N_i. \quad (3.1.10)$$

Furthermore, the orbital separation, δa_i , is defined by

$$\delta a_i = 10 \left(\frac{m_i + \bar{m}_{\text{run}}}{3M_{\odot}} \right)^{1/3} a, \quad (3.1.11)$$

where numerical factor, 10, comes from the results of Kokubo and Ida (1998) and $\bar{m}_{\text{run}}$ is the mean mass of the runaway bodies:

$$\bar{m}_{\text{run}} = \sum_{i=j'}^{n_b} m_i N_i / N_{\text{run}}. \quad (3.1.12)$$

In the runaway growth as well as the oligarchic growth, masses of the runaway bodies do not differ so much that it is adequate to adopt the mean mass of runaway bodies in Eq. (3.1.11). In section 4 we will justify such a choice of the separation δa_i .

3.2. Evolution of eccentricity and inclination

From Eqs. (2.3.2) and (2.3.3), we can readily write the equations describing the time evolution of $\tilde{e}_l^{*2}$ and $\tilde{i}_l^{*2}$ of planetesimals belonging to batch l due to the gravitational interaction: they are

$$\left(\frac{d\tilde{e}_l^{*2}}{dt} \right)_{\text{grav}} = \Omega_K a^2 \sum_{j=1}^{n_b} \left(\frac{N_j - \delta_{jl}}{2\pi a \Delta a} \right) \frac{h_{jl}^2 m_j}{(m_j + m_l)^2} \left(m_j \langle P_{\text{VS}} \rangle + \langle P_{\text{DF}} \rangle \frac{m_j \tilde{e}_j^{*2} - m_l \tilde{e}_l^{*2}}{\tilde{e}_{jl}^{*2}} \right) \quad (3.2.1)$$

and

$$\left(\frac{d\tilde{i}_l^{*2}}{dt} \right)_{\text{grav}} = \Omega_K a^2 \sum_{j=1}^{n_b} \left(\frac{N_j - \delta_{jl}}{2\pi a \Delta a} \right) \frac{h_{jl}^2 m_j}{(m_j + m_l)^2} \left(m_j \langle Q_{\text{VS}} \rangle + \langle Q_{\text{DF}} \rangle \frac{m_j \tilde{i}_j^{*2} - m_l \tilde{i}_l^{*2}}{\tilde{i}_{jl}^{*2}} \right). \quad (3.2.2)$$

Except the runaway bodies, we use Eqs. (2.3.4), (2.3.11), (2.3.13), and (2.3.14) as $\langle P_{\text{VS}} \rangle$, $\langle Q_{\text{VS}} \rangle$, $\langle P_{\text{DF}} \rangle$, and $\langle Q_{\text{DF}} \rangle$ of the above equations, respectively. We have to pay special attention to the gravitational interaction between the runaway bodies since they exist away from each other.

In such a case, for the expressions of the stirring rate and the dynamical friction rate we use those of the distant encounter given by (Hasegawa and Nakazawa 1990)

$$\langle P_{\text{VS}} \rangle = \frac{21a}{\delta a_i^3}, \quad (3.2.3)$$

and

$$\langle P_{\text{DF}} \rangle = \langle Q_{\text{VS}} \rangle = \langle Q_{\text{DF}} \rangle = 0, \quad (3.2.4)$$

where δa_i is the orbital separation given by Eq. (3.1.11).

In order to check the validity of the above-mentioned procedures for calculating e^* and i^* , we compare the results with those of the N -body simulation by Stewart and Ida (1999). They postulate two components of planetesimals. The mass and the number of planetesimals of component 1 (which is hereafter called planetesimal 1) are $1 \times 10^{24}\text{g}$ and 1000, respectively, whereas those of component 2 are $1 \times 10^{25}\text{g}$ and 100. We consider the same situation as theirs and pursue the time evolution of e^* and i^* by our code. The results are shown in Fig. 3.1 together with those of the N -body simulation by Stewart and Ida. From this figure we see that e^* and i^* of planetesimals 1 and 2 increase with time owing to the gravitational interaction and that both e^* and i^* of planetesimals 1 are larger than those of planetesimals 2 owing to the dynamical friction. Such results are quite reasonable for a planetesimal system composed of two populations. Anyway, Fig. 3.1 shows that our code can reproduce well the results of the N -body simulation.

As described in 2.3.2, $\bar{e}^*$ and $\bar{i}^*$ decrease by the gas drag as well as the collisional dissipation. Their time evolution due to the gas drag is calculated according to Eqs. (2.3.25) and (2.3.26). Furthermore, evolution of e_l^* and i_l^* due to the collisional effect are described, respectively, by Eqs. (2.3.29) and (2.3.30). In our numerical simulation, the new values of $\bar{e}_l^{*2}$ and $\bar{i}_l^{*2}$ of batch l are given, respectively, by

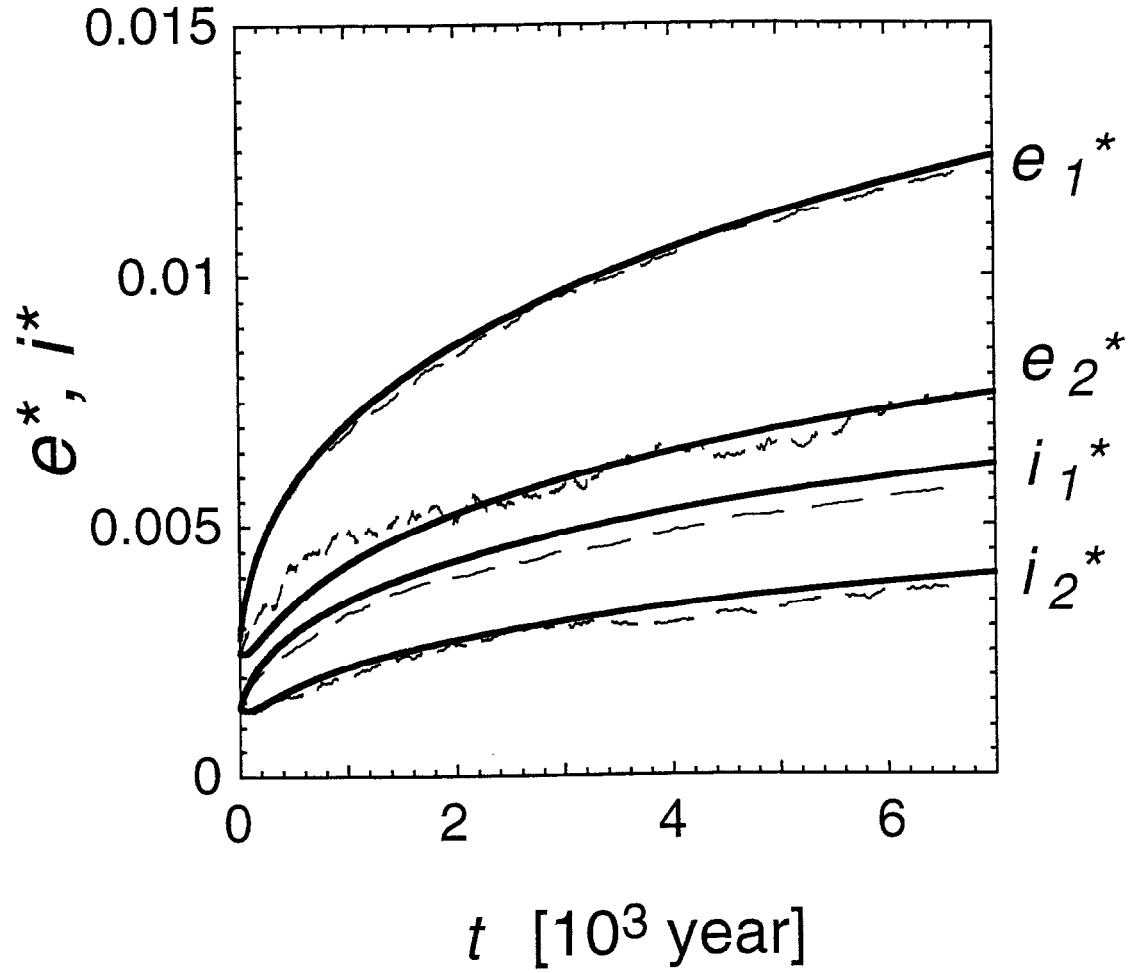


Fig. 3.1. Time evolution of the random velocity obtained by our method (solid curves). The corresponding time evolution found from the N -body simulation (Stewart and Ida 1999) are also shown by dashed curves. Subscripts 1 and 2 show values of planetesimals 1 and 2, respectively.

$$e_{l,\text{new}}^{*2} = \frac{e_l^{*2}(N_l - \sum_{j,k}' \delta N_{\text{col},jk}) + \sum_{j,k}' E_{jk}^{*2} \delta N_{\text{col},jk}}{N_l} \quad (3.2.5)$$

and

$$i_{l,\text{new}}^{*2} = \frac{i_l^{*2}(N_l - \sum_{j,k}' \delta N_{\text{col},jk}) + \sum_{j,k}' I_{jk}^{*2} \delta N_{\text{col},jk}}{N_l}, \quad (3.2.6)$$

where $\sum_{j,k}'$ has the same meaning as that in Eq. (3.1.2). Furthermore, $\delta N_{\text{col},jk}$ is the number of collisions defined by Eq. (3.1.4) and both E_{jk}^{*2} and I_{jk}^{*2} are already given by Eq. (2.3.28).

4. Test of accuracy of the statistical simulation code

In order to test the accuracy of our statistical simulation code, we perform numerical simulations and compare the results with those of the N -body simulations in subsection 4.1. We will see that our statistical simulation code can describe the accumulation process with high accuracy. We also compare our numerical results with those of previous studies based on the statistical simulations and discuss a difference between ours and others in subsection 4.2. Important results are summarized in short in subsection 4.3.

4.1. Comparisons with the N -body simulation

Kokubo and Ida (1999) perform the N -body simulations of the planetary accumulation in a ring-like region around $a = 1$ AU. As the width of the ring-like region they consider two cases, i.e., cases of $\Delta a = 0.092$ AU and 0.021 AU, and examine the effect of the boundary condition on the accumulation process by simulating for such two cases. In order to test quantitatively the accuracy of the numerical results due to the statistical method by comparing with those of the N -body simulation, we perform the numerical simulation with the same physical parameters as theirs. The adapted parameters are given in Table I.

As pointed out by Inaba et al. (1999), in a single simulation the mass distribution changes discontinuously owing to the stochastic behavior of coalescence. In order to avoid this trouble, it is necessary to take an ensemble mean obtained from a large number of independent numerical simulations. The ensemble mean of a quantity X at time t , $\bar{X}$, is given by

$$\bar{X} = \frac{1}{N_s} \sum_{m=1}^{N_s} X^m, \quad (4.1.1)$$

where N_s is the number of the independent simulations (in our present study, we adopt $N_s = 100$) and X^m is the quantity at time t , obtained by the m -th numerical simulation. Furthermore,

in order to compare the results obtained by the different methods in a quantitative manner, we introduce a standard deviation, σ_X , defined as

$$\sigma_X = \sqrt{\frac{\sum_{m=1}^{N_s} (X^m - \bar{X})^2}{N_s}}. \quad (4.1.2)$$

Table I. Adopted parameters for the test simulation. In the table, η is a non-dimensional parameter defined by Eq. (2.3.20).

Quantity	Symbol	Value
Distance from the protosun	a	1AU
Ring width	Δa	0.092AU (Case 1) 0.021AU (Case 2)
Material density of a planetesimal	ρ_{mat}	2g/cm ³
Gas drag coefficient	C_D	2
Nebular gas density	ρ_{gas}	2×10^{-9} g/cm ³
	η	1.85×10^{-3}

As for X , we consider four physical quantities: the cumulative numbers of the mass distribution, the root mean squares of the eccentricity and the inclination of a swarm of planetesimals, the root mean square of masses of planetesimals, and the mass of the largest planetesimal. The cumulative number $N_{c,j}$ is defined as the sum of the number of planetesimals with mass larger than m_j ;

$$N_{c,j} = \sum_{k=j}^{n_b} N_k, \quad (4.1.3)$$

where n_b is the total number of batches. The root mean squares of eccentricities and inclinations of planetesimals are given by

$$\langle e^{*2} \rangle^{1/2} = \sqrt{\frac{\sum_{j=1}^{n_b} N_j e_j^{*2}}{N_{c,0}}} \quad (4.1.4)$$

and

$$\langle i^{*2} \rangle^{1/2} = \sqrt{\frac{\sum_{j=1}^{n_b} N_j i_j^{*2}}{N_{c,0}}}, \quad (4.1.5)$$

respectively. The root mean square of mass of all planetesimals is defined by

$$\langle m^2 \rangle^{1/2} = \sqrt{\frac{\sum_{j=1}^{n_b} N_j m_j^2}{N_{c,0}}}. \quad (4.1.6)$$

4.1.1. Case of $\Delta a = 0.092$ AU

Kokubo and Ida (1996) show that the mass distribution of planetesimals is described by a single power law as $n_s(m)dm \propto m^{-2.5}dm$ just after the runaway growth starts. According to Kokubo and Ida (1999), we calculate the accumulation process from the power-law initial mass distribution; The minimum mass of the distribution, m_0 , is put to be 2×10^{23} g and the mass of the largest body is $20m_0$. The total mass of planetesimals is set to be $6507m_0$. The dispersions of the eccentricities and the inclinations of planetesimals are assumed to be

$$e_m^* = 2 \times i_m^* = 0.002. \quad (4.1.7)$$

We show the snapshots of the cumulative number as well as the relative velocity e^* and i^* at $t = 1 \times 10^5$, 2×10^5 , and 4×10^5 years in Figs. 4.1a and 4.1b. The initial mass distribution ($t = 0$ year) is shown in the snapshots at $t = 1 \times 10^5$ years by the dashed curve. At $t = 1 \times 10^5$ years, bodies with mass larger than $100m_0$ begin to be formed; e^* and i^* are suppressed by the dynamical friction due to small bodies. Small bodies with mass between $1 m_0$ and 10

m_0 have almost constant e^* and i^* independently of mass. The mass range of planetesimals with constant e^* and i^* spread with time and the eccentricity and the inclination increase monotonously owing to the formation of the larger bodies. At $t = 4 \times 10^5$ years, planetesimals with masses smaller than $200m_0$ have the almost same e^* and i^* which are 1.6 times larger than those at $t = 1 \times 10^5$ years.

From Fig. 4.1a, we see that in the mass range of 1 to $10 m_0$ the cumulative numbers calculated from the N -body simulation and the statistical method coincide each other almost completely at any time. For the relatively large bodies, there is a minor difference in the cumulative numbers. The cumulative number of planetesimals with $30m_0$ given by the N -body simulation is 1.7 times smaller than that of the statistical method. The differences may come from the discontinuous change of the numerical results obtained by the N -body simulation due to the statistical fluctuation. As seen from Fig. 4.1b, e^* and i^* of small bodies in the N -body simulation agree with those of the statistical method. For large bodies, it is difficult to compare them because they scatter for the case of the N -body simulation. However, the trends of the two results resemble each other.

In Fig. 4.2a, we show the time evolution of the mass of the largest body, $M_{\max}$ and the root mean square of mass of all planetesimals, $\langle m^2 \rangle^{1/2}$. Initially the mass of the largest body is $20m_0$ and becomes about $1000m_0$ at $t = 5 \times 10^5$ years. In Fig. 4.2a, together with the ensemble mean $\bar{M}_{\max}$, curves of $\bar{M}_{\max} + \sigma_M$ and $\bar{M}_{\max} - \sigma_M$ are also shown where σ_M is the standard deviation of the mass of the largest body. Despite of the discontinuous change of the results of the N -body simulation, the mass of the largest body of the N body simulation almost exist within the region bounded by $\bar{M}_{\max} + \sigma_M$ and $\bar{M}_{\max} - \sigma_M$. Furthermore, Fig. 4.2b shows the time evolution of the root mean squares of eccentricities, $\langle e^{*2} \rangle^{1/2}$, and inclinations, $\langle i^{*2} \rangle^{1/2}$, of planetesimals. They increase monotonously with time. As in the case of the mass of the largest body, we show the ensemble mean $\overline{\langle e^{*2} \rangle^{1/2}}$ and $\overline{\langle i^{*2} \rangle^{1/2}}$ as well as $\overline{\langle e^{*2} \rangle^{1/2}} \pm \sigma_e$ and

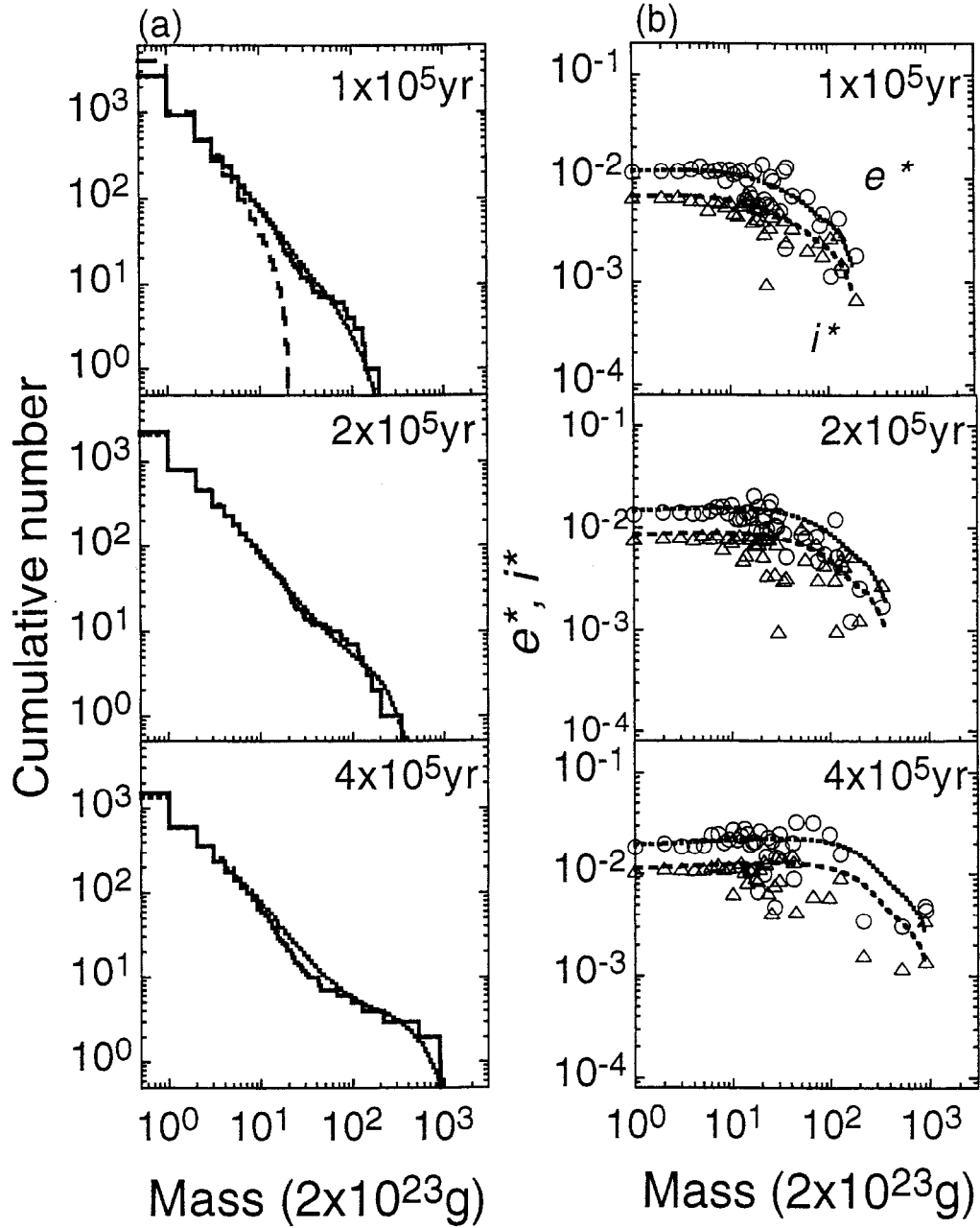


Fig. 4.1. Comparison between the numerical results obtained from the statistical method and the N -body simulation at $t = 1 \times 10^5$, 2×10^5 , and 4×10^5 years. The cumulative numbers are shown in panels (a) for the cases of the statistical method (dotted curve) and the N -body simulation (solid curve). Initial mass distribution is also shown by the dashed curve in the top of panels (a). Furthermore, e^* and i^* calculated by the statistical method (solid and dashed curves) and the N -body simulation (circles and triangles) are illustrated in panels (b).

$\overline{\langle i^{*2} \rangle^{1/2}} \pm \sigma_i$. The root mean squares obtained by the N -body simulation fall on the regions bounded by $\overline{\langle e^{*2} \rangle^{1/2}} \pm \sigma_e$ and $\overline{\langle i^{*2} \rangle^{1/2}} \pm \sigma_i$. Thus, we can say that the statistical method succeeds in reproducing the results of the N -body simulation.

In the above calculation, the orbital separation of the runaway bodies δa_i , which is one of the technical parameters of our numerical code, is put to be 10 Hill radii of them as mentioned in subsection 3.1 (see Eq. (3.1.11)). We perform two additional simulations with different orbital separations, $\delta a_i = 5$ and 15 Hill radii, to see the dependence on the adopted orbital separation. Figure 4.3 shows the expectation values of the mass of the largest bodies obtained by the statistical simulations with different orbital separations and that of the N -body simulation. In the case of $\delta a_i = 15$ Hill radii the mass of the runaway body becomes too large since the simulation does not prevent large bodies from coagulating compared with those with other orbital separations, while the growth of the runaway bodies is decelerated in the case of 5 Hill radii. Only for the case of 10 Hill radii we can simulate properly the growth of the runaway bodies in accordance with the result of the N -body simulation.

4.1.2. Case of $\Delta a = 0.021$ AU

We make briefly a comparison with the N -body simulation for the case of a narrow region with $\Delta a = 0.021$ AU (see Table I). In accordance with Kokubo and Ida (1999), the initial mass, m_0 , and the total number of planetesimals are put, respectively, to be 1×10^{23} g and 3000. The initial dispersions of eccentricities and inclinations are given by $e_0^* = 2 \times i_0^* = 0.0043$.¹

The time evolution of the mass of the largest body, $M_{\max}$, and the root mean squares of

¹As described in section 2, e^* and i^* are enhanced by the gravitational scattering with other planetesimals and are damped by the gaseous drag in the solar nebula. As a result, e^* and i^* approach their equilibrium values. The above initial values of e^* and i^* are of the equilibrium ones.

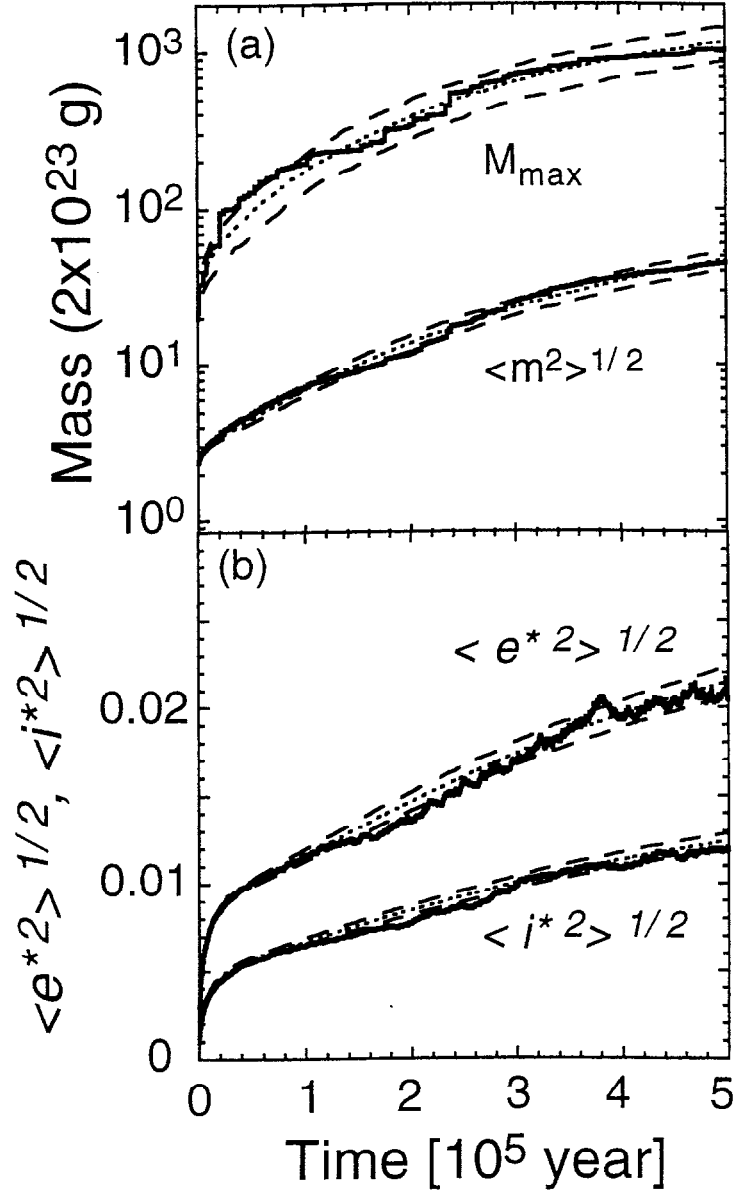


Fig. 4.2. Time evolution of the mass of the largest body, $\bar{M}_{\max}$ and the root mean square of mass, $\langle m^2 \rangle^{1/2}$ (panel (a)). The dotted and dashed curves correspond to $\bar{M}_{\max}$ and $\bar{M}_{\max} \pm \sigma_M$ obtained by the statistical method, respectively where $\bar{M}_{\max}$ and σ_M are the ensemble mean and its standard deviation deduced from the statistical simulations. The solid curves are of the N -body simulation. Time evolution of the root mean squares of eccentricities and inclinations of planetesimals is shown in panel (b).

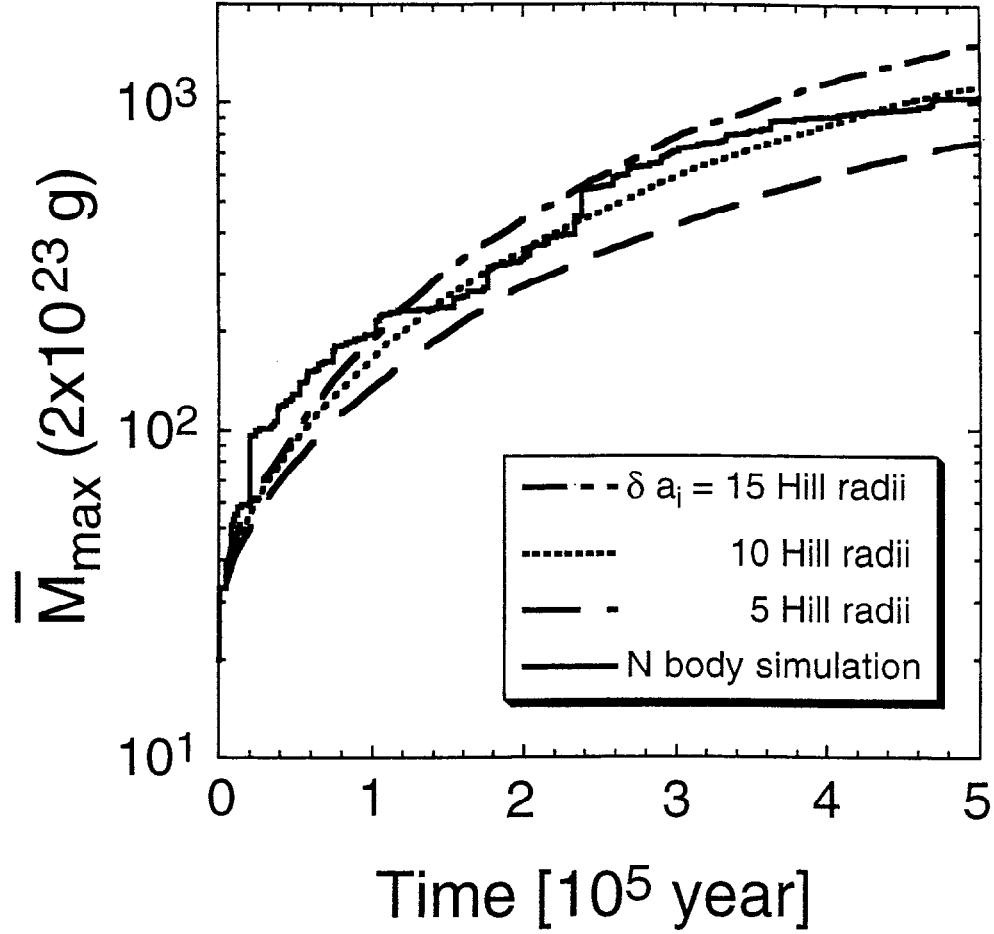


Fig. 4.3. Evolution of the expectation value of the mass of the largest body obtained by the statistical method with different three orbital separations. The result of the N -body simulation (Kokubo and Ida 1999) is also shown by the solid curve.

eccentricities, $\langle e^{*2} \rangle^{1/2}$, and inclinations, $\langle i^{*2} \rangle^{1/2}$, are shown in Figs. 4.4a and 4.4b, respectively. In these figures, $\bar{M}_{\max}$ and $\bar{M}_{\max} \pm \sigma_{M_{\max}}$ obtained by the statistical method are shown by the same manner as Fig. 4.2. At the end of the calculation ($t = 3 \times 10^5$ years), $M_{\max}$ attains about $400 m_0$ and $\langle e^{*2} \rangle^{1/2}$ becomes 3 times as large as the initial value. For the mass of the largest body, the statistical method give the same evolution curve as that of the N -body simulation. On the other hand, the root mean squares of eccentricities and inclinations obtained by the N -body simulation are smaller than $\overline{\langle e^{*2} \rangle^{1/2}} - \sigma_e$ and $\overline{\langle i^{*2} \rangle^{1/2}} - \sigma_i$, respectively (σ_e and σ_i are the standard deviations of the root mean squares of e^* and i^* defined by Eq. (4.1.2)). But the difference is quite small (it is confined within a factor of 1.2). Thus, regardless of the width of the computation region where planetesimals distribute, our statistical method can describe the growth process of planetesimals as well as their kinetic behaviors.

4.2. Comparison with previous statistical simulations

As described in Introduction, a number of simulations of the planetary accumulation are carried out by many researchers on the basis of the statistical method. Recently, Wetherill and Stewart (1993) study the accumulation process of planetesimals using their own velocity evolution equation, in which the effect of the solar gravity is taken into account, and including the effect of the collisional fragmentation. They show that the collisional fragmentation of planetesimals accelerates the runaway growth of protoplanets.

Since our code is based on their method, if we adopt the same collision rate and velocity evolution equation, we must have the same results as theirs. Unfortunately, Wetherill and Stewart (1993) give little information on the velocity evolution and the collision rate formula so that we cannot reproduce the same situation as theirs. Nevertheless, it is meaningful to make a comparison with the result of Wetherill and Stewart (1993) for a case without the

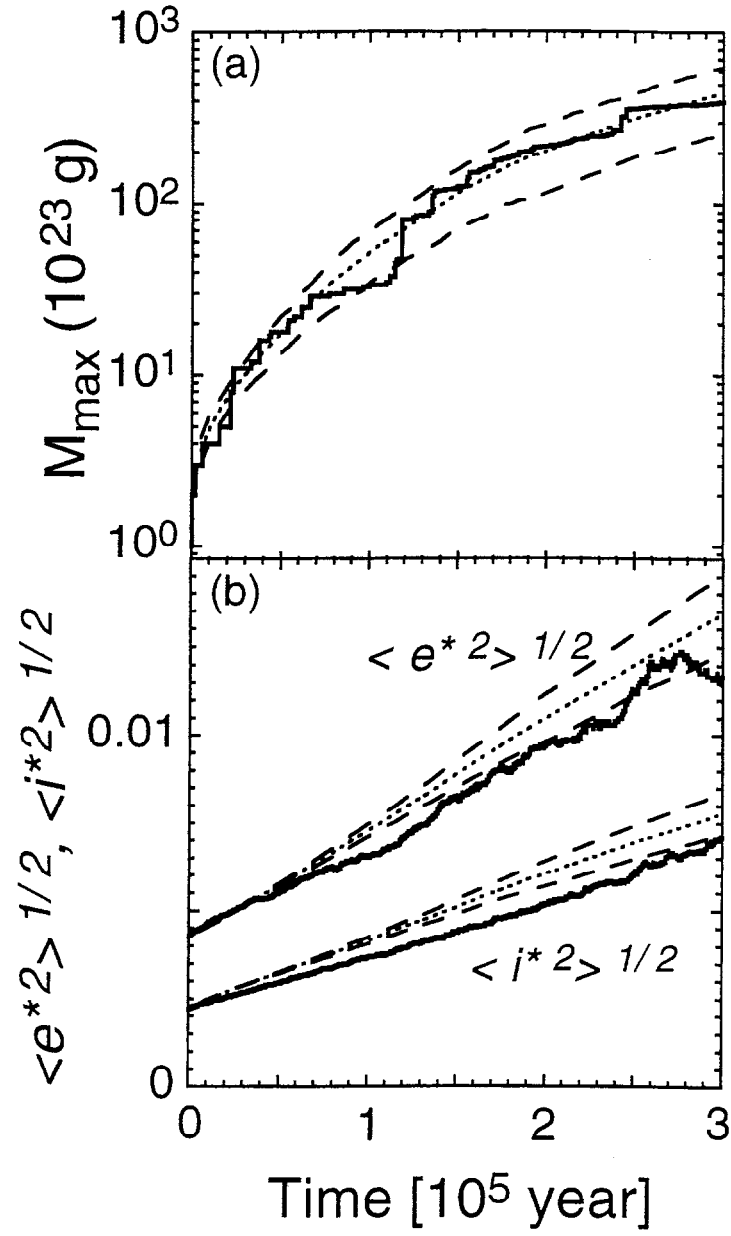


Fig. 4.4. The same as Fig. 4.2, but for the case of $\Delta a = 0.021$ AU.

collisional fragmentation because we can see the effect of our improvement where we use the new velocity evolution equation developed by Stewart and Ida (1999) and the formula for the collision rate due to Greenzweig and Lissauer (1992). In this simulation we reset the orbital separation, δa_i , by the same way as Wetherill and Stewart (1993) in place of Eq. (3.1.11), i.e., $\delta a_i = 2\sqrt{3}h_{m_i, \bar{m}_r}a + 2ae_i$ where e_i is the eccentricity of runaway body i . The initial mass, m_0 , is put to be 4.8×10^{18} g.

Figures 4.5a and 4.5b show the snapshots of the relative velocity e^* , and the cumulative number, respectively. At $t = 5 \times 10^2$ years, both results coincide well each other. At $t = 1 \times 10^4$ years, however, larger planetesimals are formed in Wetherill and Stewart (1993); the mass of the largest body is about 1.8×10^{24} g in their case whereas 6×10^{22} g in our case. In reflection of this difference of the planetary growth, the relative velocity e^* differs greatly between two studies. Furthermore, at $t = 1.5 \times 10^5$ years, the runaway bodies are larger in their case and, as a result, the number of the runaway bodies is small compared with ours. At the same time, the cumulative number for $m \lesssim 1 \times 10^{21}$ g deviates quite largely between two results though the distribution of e^* is very similar. It is important to seek the reason(s) of these differences since the planetary accumulation is the strongly non-linear phenomena; a numerical error, even a bit, leads inevitably to the qualitatively different picture of the planetary growth. Unfortunately, to find the reason of the differences is difficult because Wetherill and Stewart (1993) give little information on their numerical procedures. Anyway we can say, at least, that the accuracy of Wetherill and Stewart (1993) is not enough.

4.3. Summary of the accuracy test

In this section, we test our statistical method by comparing with the N -body simulations, performed by Kokubo and Ida (1999) for the two cases where planetesimals distribute within a ring-like region with relatively wide width ($\Delta a = 0.092$ AU) and with narrow width (0.021

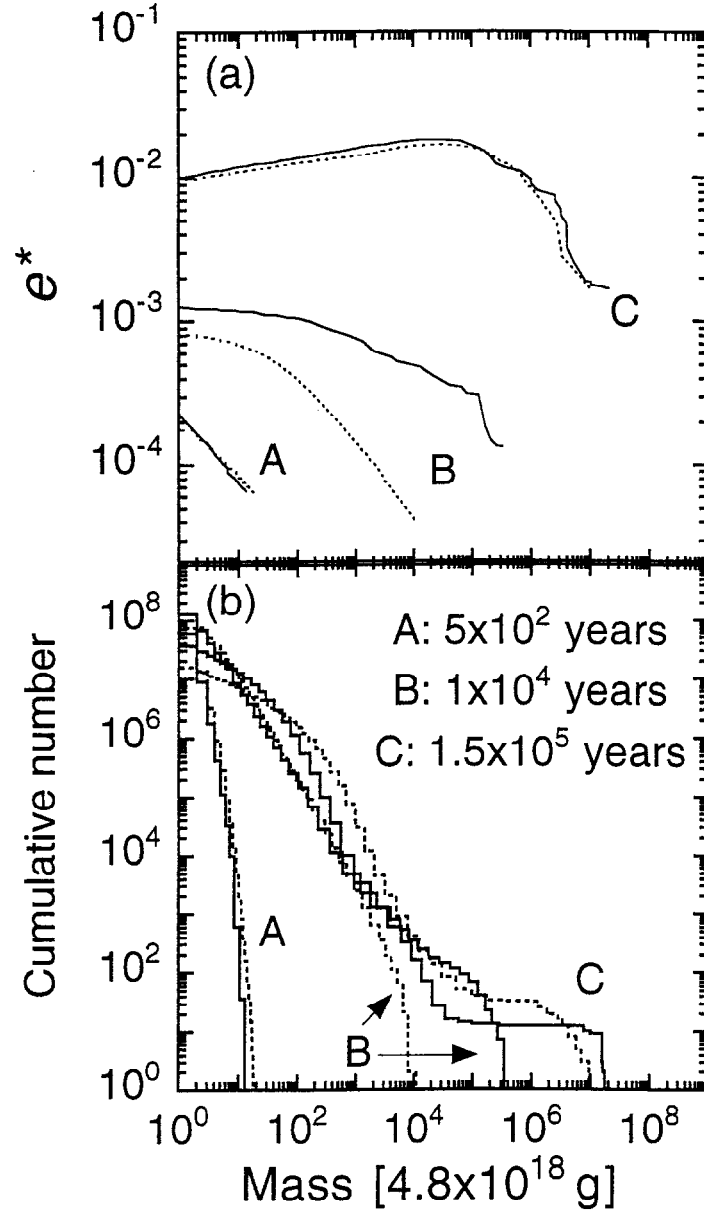


Fig. 4.5. Snapshots of the cumulative number and the relative velocity e^* at 5×10^2 , 1×10^4 , and 1.5×10^5 years. The numerical results obtained by our code are shown by the dotted curves and those of Wetherill and Stewart (1993) by the solid curves.

AU). From the comparison in the case with a wide width, we find that the results of the statistical method agree well with that of the N -body simulation. For the case of a narrow width, the root mean squares of eccentricities and inclinations in the statistical simulation are slight larger than those in the N -body simulation but the deviation is small. Consequently, we conclude that the statistical method can describe the accumulation process with a sufficiently high accuracy. Furthermore, we make a comparison with the previous statistical simulation given by Wetherill and Stewart (1993). From the comparison, we find that the accuracy of their numerical procedures is not enough and their results contains some errors.

5. Accumulation process without nebular gas

Recent observations of the extra solar planets reveal that there exist various kinds of planetary systems (Mayor and Queloz 1995, Butler et al. 1997). The variety of the planetary systems may come from the differences in the physical parameters of the protoplanetary system (e.g., the surface density of solid materials, gas density of the solar nebula and others). Therefore, it is necessary to examine the accumulation processes with different parameters of the gaseous nebula and the planetesimal disk. Since there are many parameters which affect the accumulation process of the planets, it is hard to perform the numerical simulations of all sets of the parameters. Rather, it is fruitful to find an appropriate scaling law in the accumulation process.

As the first step of our investigation, we examine the accumulation process without the nebular gas. By neglecting the effect of the nebular gas, the number of parameters to be considered decreases and the problem is simplified. Despite of the simplification, since nobody knows if the accumulation of the terrestrial planets proceeds in the nebular gas, this calculation itself may be of great value.

In subsection 5.1, we will first derive a scaling law which describes the early stage accumulation process of planetesimals and show that there is a non-dimensional parameter which governs the growth process of planetesimals. In subsection 5.2, making the simulation of the accumulation process by our statistical simulation code, we will show that the accumulation process is divided into two stages, i.e., the first stage where runaway bodies increase their mass exponentially and the second stage where the mass of the runaway body exceeds a certain critical mass and the growth rate decreases. Furthermore, it will be shown that a small number of planetesimals grow rapidly, not through the orderly growth stage, from the beginning of the accumulation process. In subsection 5.3 we will show that the scaling law is well satisfied until the mass of the runaway body approaches the critical mass and that the growth of runaway bodies does not depend appreciably on the non-dimensional parameter.

5.1. The scaling law in the planetary accumulation process

We rewrite the basic equations into the non-dimensional forms, introducing non-dimensional variables. As a unit of mass, we adopt the initial mean mass of planetesimals, m_0 . The accumulation process in a disk system is characterized by the surface number density of planetesimals. The unit of the surface number density, n_0 , is suitably given by

$$n_0 = \sigma_s / m_0, \quad (5.1.1)$$

where σ_s is the surface density of the planetesimal disk. Furthermore, the time is normalized by the characteristic collision time of planetesimals, t_c , which is defined by

$$t_c = (n_0 \pi (2r_0)^2 \Omega_K)^{-1}, \quad (5.1.2)$$

where r_0 is the radius of a planetesimal with mass m_0 and Ω_K is the Keplerian angular velocity. The eccentricity and the inclination of planetesimals are properly scaled by the ratio, $v_{\text{esc},0}/v_K$, where v_K is the Keplerian velocity and $v_{\text{esc},0}$ is the (surface) escape velocity from a planetesimal with mass m_0 , given by

$$v_{\text{esc},0} = \sqrt{2Gm_0/r_0}. \quad (5.1.3)$$

Using these typical values, we normalize the mass, the surface number density, the time, and the random velocity as follows:

$$\left\{ \begin{array}{l} \hat{m}_k = \frac{m_k}{m_0}, \\ \hat{n}_s(m) = \frac{n_s(m)}{n_0}, \\ \hat{t} = n_0 \pi (2r_0)^2 \Omega_K t, \\ \hat{e}_k^* = \frac{e_k^* v_K}{v_{\text{esc},0}}, \quad \text{and} \quad \hat{i}_k^* = \frac{i_k^* v_K}{v_{\text{esc},0}}. \end{array} \right. \quad (5.1.4)$$

Using the non-dimensional variables introduced above and assuming that the eccentricity and the inclination are much larger than the reduced Hill radius of planetesimals (i.e., $\tilde{e}^*, \tilde{i}^* \gg 1$), we rewrite the coagulation equation (2.1.6) as

$$\begin{aligned} \frac{d\hat{n}_s(\hat{m}_1)}{d\hat{t}} &= \frac{1}{16\pi^2} \int_0^{\hat{m}_1} d\hat{m}_2 \int_0^\infty d\hat{m}_3 \hat{n}_s(\hat{m}_2) \hat{n}_s(\hat{m}_3) (\hat{m}_2^{1/3} + \hat{m}_3^{1/3})^2 \\ &\quad \times \left(\mathcal{F}(I_{23}^*) + \frac{\mathcal{G}(I_{23}^*)}{\hat{e}_2^{*2} + \hat{e}_3^{*2}} \frac{\hat{m}_2 + \hat{m}_3}{\hat{m}_2^{1/3} + \hat{m}_3^{1/3}} \right) \delta(\hat{m}_1 - \hat{m}_2 - \hat{m}_3) \\ &- \frac{1}{8\pi^2} \hat{n}_s(\hat{m}_1) \int_0^\infty d\hat{m}_2 \hat{n}_s(\hat{m}_2) (\hat{m}_1^{1/3} + \hat{m}_2^{1/3})^2 \\ &\quad \times \left(\mathcal{F}(I_{12}^*) + \frac{\mathcal{G}(I_{12}^*)}{\hat{e}_1^{*2} + \hat{e}_2^{*2}} \frac{\hat{m}_1 + \hat{m}_2}{\hat{m}_1^{1/3} + \hat{m}_2^{1/3}} \right), \end{aligned} \quad (5.1.5)$$

where

$$I_{jk}^* = \sqrt{\frac{\hat{i}_j^{*2} + \hat{i}_k^{*2}}{\hat{e}_j^{*2} + \hat{e}_k^{*2}}}, \quad (5.1.6)$$

and $\mathcal{F}(I_{jk}^*)$ and $\mathcal{G}(I_{jk}^*)$ are given by Eq. (2.2.6). In the above, the term $\hat{m}_1^{1/3} + \hat{m}_2^{1/3}$ expresses the sum of the radii of the colliding bodies normalized by $2r_0$ and, hence, the dependence of Eq. (5.1.5) on the material density of planetesimals disappear. In Eq. (5.1.5), for the mean collision rate $\langle P_{\text{col}} \rangle$, we adopt the expression in high-velocity regime (Eq. (2.2.5)). Note that no non-dimensional parameter is included in the normalized coagulation equation (5.1.5). In high-velocity case, we rewrite Eqs. (2.3.2) and (2.3.3) as

$$\frac{de_1^{\star 2}}{d\hat{t}} = \frac{1}{2\pi^{3/2}} \int d\hat{m}_2 \hat{n}_s(\hat{m}_2) \hat{m}_2^2 \left(\frac{J_e}{e_1^{\star 2} + e_2^{\star 2}} + \frac{1.4H_e}{e_1^{\star 2} + e_2^{\star 2}} \frac{e_2^{\star 2} - e_1^{\star 2} \hat{m}_1/\hat{m}_2}{e_1^{\star 2} + e_2^{\star 2}} \right) \frac{\ln(R i_{12}^{\star}(e_{12}^{\star 2} + i_{12}^{\star 2}))}{I^*} \quad (5.1.7)$$

and

$$\frac{di_1^{\star 2}}{d\hat{t}} = \frac{1}{2\pi^{3/2}} \int d\hat{m}_2 \hat{n}_s(\hat{m}_2) \hat{m}_2^2 \left(\frac{J_i}{i_1^{\star 2} + i_2^{\star 2}} + \frac{1.4H_i}{i_1^{\star 2} + i_2^{\star 2}} \frac{i_2^{\star 2} - i_1^{\star 2} \hat{m}_1/\hat{m}_2}{i_1^{\star 2} + i_2^{\star 2}} \right) \frac{\ln(R i_{12}^{\star}(e_{12}^{\star 2} + i_{12}^{\star 2}))}{I^*}, \quad (5.1.8)$$

where

$$R = \sqrt{\frac{8\pi\rho_{\text{mat}}a^3}{3M_c}}, \quad (5.1.9)$$

and J_e, H_e, J_i , and H_i are given by Eqs. (2.3.6), (2.3.15), (2.3.12), and (2.3.16), respectively. Equations (5.1.7) and (5.1.8) include only one non-dimensional parameter R . When $M_c = 1M_\odot$, R is given numerically by

$$R = 6.5 \times 10^3 \left(\frac{\rho_{\text{mat}}}{3\text{g/cm}^3} \right)^{1/2} \left(\frac{a}{1\text{AU}} \right)^{3/2}. \quad (5.1.10)$$

If we perform a simulation of the accumulation process with a value of R , we can tell the other accumulation processes with the same value of R as long as these basic equations are valid for the description of the accumulation process.

It should be noted that R will not affect seriously the results of the simulation of the planetary accumulation since R is included only in the logarithmic form. In this sense, we can presume that the mode of the planetary accumulation in the early stage hardly depends on the adopted physical parameters (i.e., the surface density, the initial mean mass, the material density, and the semimajor axis). In fact, these will be justified in subsection 5.3 by performing simulations with various values of R .

5.2. The modes of the planetary growth

In order to see in detail the planetary growth in the early stage, we perform the simulation of the accumulation process of planetesimals which distribute randomly in a ring-like region. Initially all planetesimals have the same mass, m_0 , and the initial dispersions of the random velocities are given by

$$e_0^* = 2 \times i_0^* = \sqrt{\frac{2Gm_0}{r_0}}/v_K, \quad (5.2.1)$$

where r_0 is the radius of a planetesimal with mass m_0 . Other conditions are shown in Table II.

Table II. Adopted parameters for simulation

Quantity	Symbol	Value
Distance from the protosun	a	1AU
Material density of a planetesimal	ρ_{mat}	3g/cm ³
Surface density of solid	σ_s	7.1g/cm ²
Initial mass	m_0	$1 \times 10^{17}\text{g}$
Ring width	Δa	0.20 AU

Figure 5.1 shows the cumulative numbers and the dispersions of eccentricities at $t = 1$ and 5×10^4 years. At $t = 1 \times 10^4$ years, the mass distribution is approximately given by the power law, $N_c \propto m^{-\alpha}$, where the exponent α is about 1.5. At $t = 5 \times 10^4$ years, a bulge, which consists of about 30 runaway bodies, is formed at the high mass end of the distribution. Formation of a bulge corresponds to the start of the oligarchic growth proposed by Kokubo and Ida (1998) where some large bodies grow at the similar growth rate keeping their orbital separation of 10 Hill radii. The eccentricities of relatively large planetesimals are almost determined by

the energy equipartition and small planetesimals have almost constant eccentricity since the equilibrium eccentricity weakly depends on mass of a planetesimal (Ida and Makino 1993).

In Fig. 5.2, we show the time evolution of the mass of the largest body, the average mass, and the root mean squares of the eccentricities and inclinations of planetesimals. Until the mass of the largest body, $M_{\max}$, attains to about $2 \times 10^5 m_0$, it increases exponentially. Such a growth is called the exponential growth from now on. By applying the least squares fit to the growth of $M_{\max}$, we find that $M_{\max}$ is approximated by

$$M_{\max} = 1.7 \times 10 m_0 \exp(8.9 \times 10^{-4} t). \quad (5.2.2)$$

The increase of the average mass is much slower than the growth of the largest bodies. This means that the runaway growth occurs from the beginning of the planetary accumulation. The mass of the largest body at the stage where the exponential growth stops is estimated as $2 \times 10^5 m_0$ (see the next paragraph). The exponential growth stage is followed by the stage of the oligarchic growth as we observe in Fig. 5.1. The growth rate of the largest body becomes small and is expressed approximately as

$$M_{\max} = 2.3 \times 10^3 m_0 t^{0.93}, \quad (5.2.3)$$

which we find again by the use of the least squares fit. As mentioned in the previous subsection, the non-dimensional coagulation equation (5.1.5) is based on the regular functional forms of the mean collision rate $\langle P_{\text{col}} \rangle$ and the distribution function $n_s(m)$ and, hence, cannot describe the stage of the oligarchic growth of the planetary accumulation where the runaway bodies do not collide each other. This suggests that, if we rewrite Eq. (5.2.2) in non-dimensional form, the numerical coefficients appearing in the equation are universal as long as we are concerned with the cases with same value of the non-dimensional parameter, R . On the other hand, the

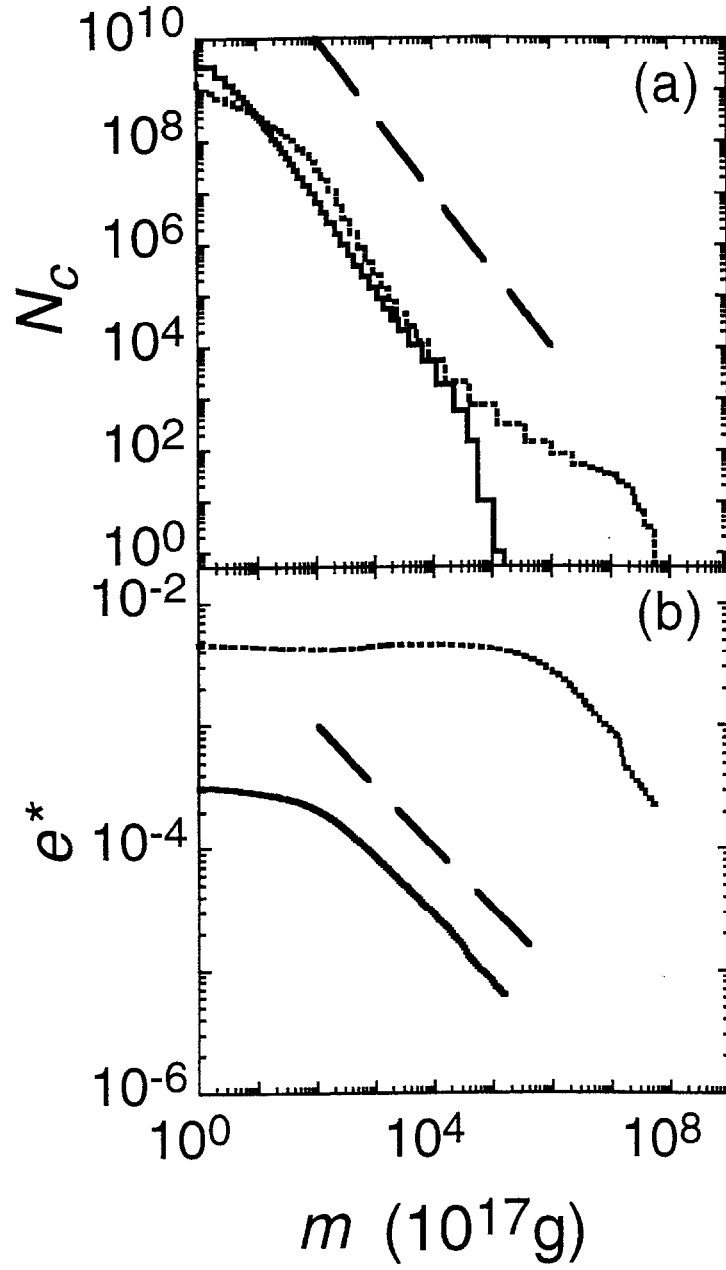


Fig. 5.1. The cumulative numbers (panel (a)) and the dispersions of the eccentricities (panel (b)) at $t = 1 \times 10^4$ years (solid lines) and 5×10^4 years (dotted lines). The dashed lines denote $N_c \propto m^{-3/2}$ in panel (a) and $e^* \propto m^{-1/2}$ in panel (b).

coefficients appearing in Eq. (5.2.3) will depend on the physical parameters such as the surface density of solid components and the initial (mean) mass of planetesimals.

The root mean square of the eccentricities increases monotonically and, at the end of the simulation ($t = 5000$ years), it becomes 50 times as large as the escape velocity of the planetesimal with mass m_0 . Because of the large eccentricity (i.e., the large relative velocity), colliding bodies does not always coalesce but probably fragment. Such a problem will be discussed briefly in the last of this section.

The transition mass from the exponential growth to the oligarchic growth is readily estimated as follows. As the runaway growth starts, the surface number density relaxes and approaches the distribution described by the single power law (Kokubo and Ida 1996):

$$n_s(m)dm = 0.5 \frac{\sigma_s}{m_0^2} \left(\frac{m}{m_0} \right)^{-5/2} dm \quad (m_0 \leq m \leq M_{\max}). \quad (5.2.4)$$

As stated in subsection 3.1, in the stage of the oligarchic growth, one runaway body occupies a ring-like region with the width, δa , which is approximately given by (see Eq. (3.1.11))

$$\delta a = 10 \left(\frac{2M_{\max}}{3M_{\odot}} \right)^{1/3} a. \quad (5.2.5)$$

Thus, we have

$$\int_{M_{\max}/2}^{M_{\max}} 2\pi a \delta a n_s(m) dm = 1. \quad (5.2.6)$$

From Eqs. (5.2.4), (5.2.5), and (5.2.6), we can readily derive

$$\frac{M_{\max}}{m_0} = 2 \times 10^5 \left(\frac{\sigma_s(1 \text{ AU})}{7.1 \text{ g/cm}^2} \right)^{6/7} \left(\frac{a}{1 \text{ AU}} \right)^{3/7} \left(\frac{m_0}{10^{17} \text{ g}} \right)^{-4/7}, \quad (5.2.7)$$

where, according to Hayashi (1981), σ_s is put to be

$$\sigma_s = \sigma_s(1 \text{ AU}) \left(\frac{a}{1 \text{ AU}} \right)^{-3/2}. \quad (5.2.8)$$

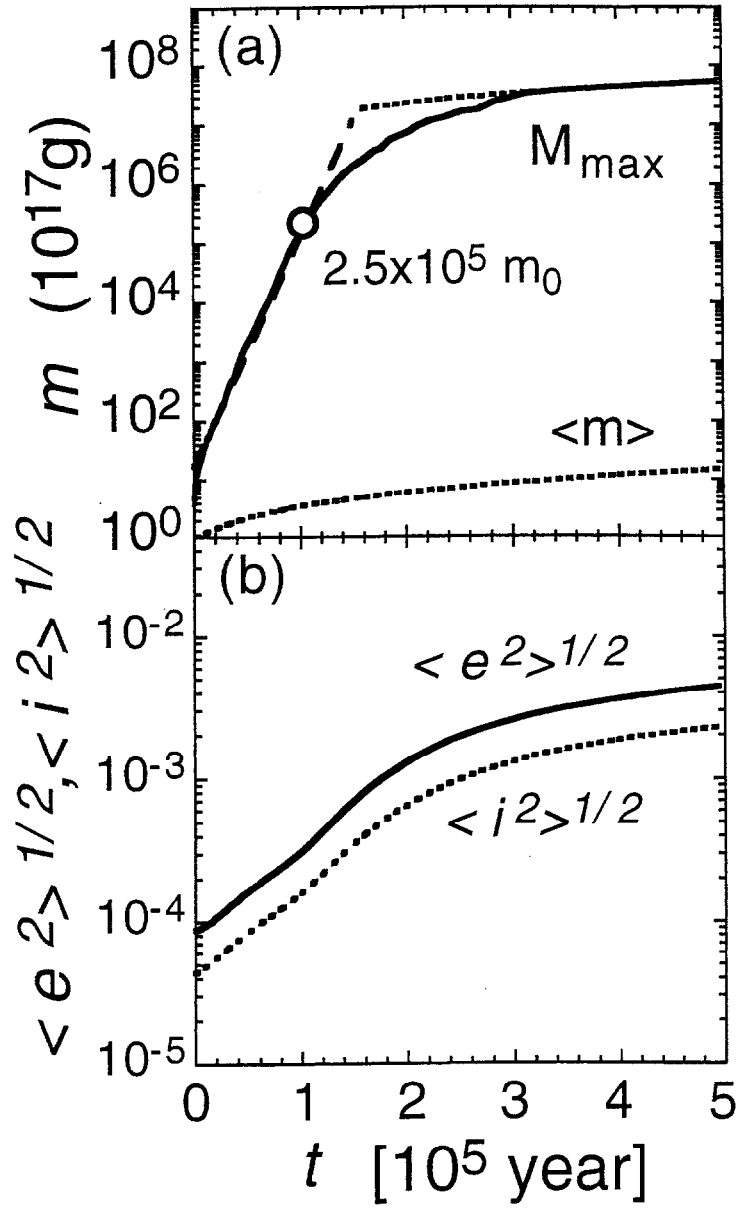


Fig. 5.2. Time evolution of the mass of the largest body, $M_{\max}$, and the average mass $\langle m \rangle$ (panel (a)) and the root mean squares of the eccentricities and inclinations, $\langle e^2 \rangle^{1/2}$ and $\langle i^2 \rangle^{1/2}$. (panel (b)). The approximate growth curves of $M_{\max}$ evaluated from Eqs. (5.2.2) and (5.2.3) are also shown by the dashed lines in panel (a). The critical mass which determines the end of the exponential growth is shown by a circle in panel (a).

Hereafter, the transition mass given by Eq. (5.2.7) is called the critical mass and denoted by m_{crit} . Note that the lower limit of integral (5.2.6) is put to be $M_{\text{max}}/2$ but m_{crit} does not change so much (within a factor of about 2) by another choice (say, M_{max}/e).

As the growth mode of planetesimals, there is another mode called the orderly growth (Ohtsuki et al. 1993). In the orderly growth, the growth rates of all planetesimals are almost the same. As stated in subsection 2.2, when the gravitational focusing is effective, the runaway growth occurs; otherwise the growth mode is orderly. The term of the gravitational focusing becomes effective when the random velocity ev_K is smaller than the escape velocity (see Eq.(2.2.2)). Figure 5.3 shows the root mean square of the eccentricities normalized by the escape velocity of the largest body. Immediately the value decreases owing to the rapid growth of the largest body and becomes smaller than the unity. Hence, it is quite natural to say that the mode of the accumulation is the runaway growth. Even if we consider the effect of the gas drag of the solar nebula on planetesimals, the growth mode will certainly be a runaway-type since the gas drag decreases the eccentricities and the gravitational focusing becomes effective. Hence, we conclude that planetesimals always start their growth through the runaway growth from the beginning and orderly growth does not occur in the planetary accumulation.

Finally, we will add a short comment on the collisional fragmentation. In this study, we assume the perfect accretion (i.e., the colliding bodies become one body with the sum of their masses) and do not consider the effect of the collisional fragmentation of planetesimals. However, as stated in the above, the relative velocity between small planetesimals becomes much larger than their escape velocity (see Fig. 5.3). In such a case collisions between them lead to fragmentation and the assumption of the perfect accretion breaks down. We perform the more realistic simulation, adopting the following “imperfect accretion model”: if the relative eccentricity e_{12}^* is larger than $2 \times v_{\text{esc}}/v_K$, we make the colliding bodies not to coalesce. In Fig. 5.4, the time evolution of the mass of the largest body and the average mass in that simu-

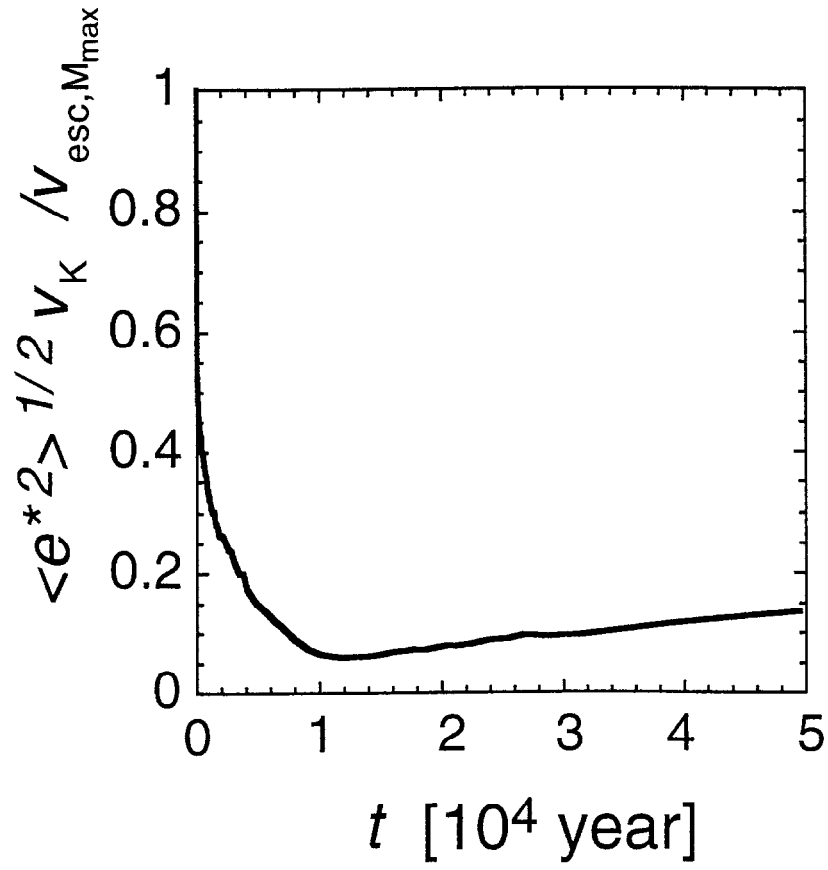


Fig. 5.3. Time evolution of the root mean square of the eccentricities of planetesimals normalized by the escape velocity from the largest body.

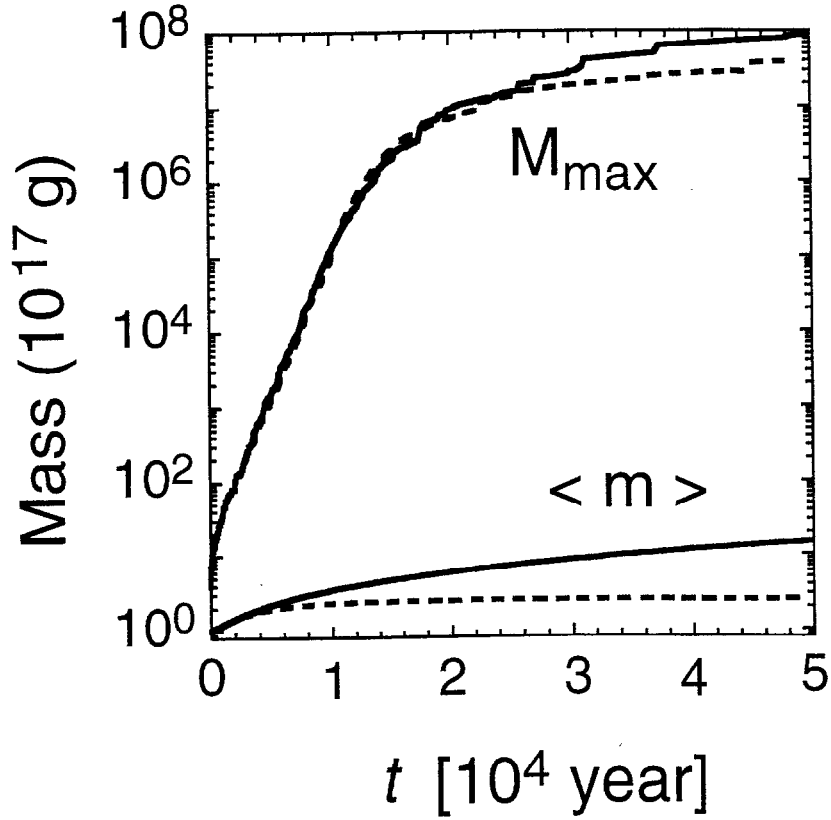


Fig. 5.4. Time evolution of the mass of the largest body, $M_{\max}$, and the average mass, $\langle m \rangle$, for the cases of the imperfect accretion (dotted line) and of the perfect accretion (solid line)

lation are shown. Since small bodies cannot coalesce owing to their large relative velocities, the average mass is small compared with the case of the perfect accretion after $t \gtrsim 5 \times 10^3$ years. In the stage of the exponential growth, we cannot find the effect of the imperfect accretion on the growth of protoplanets. But after that, the growth rate of the largest body becomes small since it captures smaller bodies. Detailed study of the effect of the collisional fragmentation should be made.

5.3. Check of the scaling law

As shown in subsection 5.1, the basic equations (i.e., Eqs. (5.1.5), (5.1.7), and (5.1.8)) are derived on the assumption that the mean collision rate is well described by that of the high-velocity limit. Before we confirm the scaling law mentioned in the previous subsection, we will first investigate how the numerical results depend on the initial dispersions of eccentricities, $\hat{e}_0^*$, and inclinations, $\hat{i}_0^*$. As $\hat{e}_0^*$ and $\hat{i}_0^*$, the following three cases are considered:

$$\hat{e}_0^* = 2 \times \hat{i}_0^* = 1, 0.8, \text{ and } 0.5. \quad (5.3.1)$$

Other parameters are fixed to be the same as in the case of subsection 5.2. Figure 5.5 shows the time evolution of the mass of the largest body, the average mass, and the root mean square of eccentricities of planetesimals. In spite of the difference of $\hat{e}_0^*$, the numerical results are very similar. The memory of the different choice of $\hat{e}_0^*$ seems to disappear at $\hat{t} = 1$, which corresponds to the mean collision time of planetesimals with m_0 .

As the planetary accumulation proceeds and enter the phase of the oligarchic growth, a small number of large planetesimals are formed. Then the coagulation equation (5.1.5) cannot describe the accumulation process since it stands on the assumption that there exist a sufficiently large number of planetesimals in each relevant mass range. Inversely, the scaling law should be satisfied until a small number of large planetesimals appear. In order to verify this statement we make the simulations of the planetary accumulation with various physical parameters. The adopted parameters for the simulations are presented in Table III. Initially all planetesimals have the same mass which is denoted by m_0 . Cases 1 to 5 have the same value of R ($R = 6.50 \times 10^3$) and cases 6 and 7 have the different values of R . In the above we show that the choice of $\hat{e}_0^*$ does not affect the results of the simulation. So $\hat{e}_0^*$ is put to be 1 for the all cases.

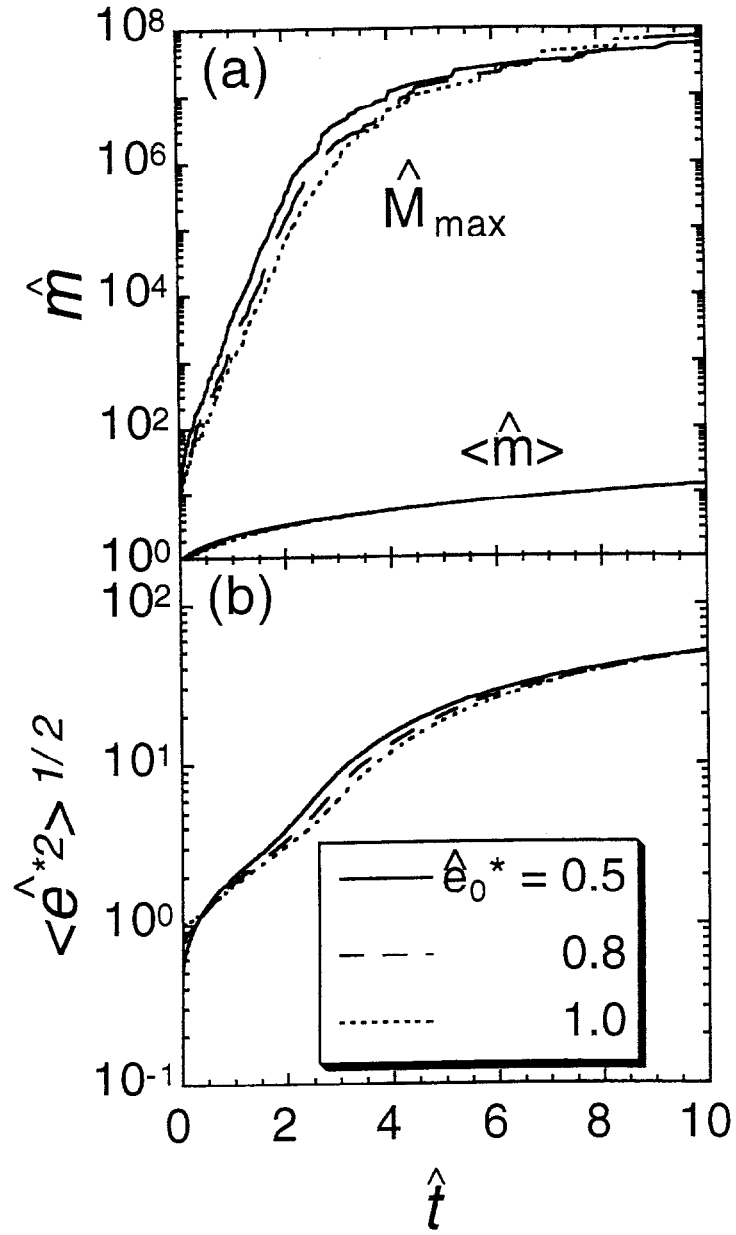


Fig. 5.5. Time evolution of the mass of the largest body, $\hat{M}_{\max}$, and the average mass, $\langle \hat{m} \rangle$ (panel (a)) and the root mean square of eccentricities, $\langle \hat{e}^{*2} \rangle^{1/2}$ (panel (b)) obtained from the simulations with three different values of $\hat{e}_0^*$.

Table III. The adopted parameters for the simulations

Case	m_0 [g]	σ_s [g/cm ²]	a [AU]	ρ_{mat} [g/cm ³]	R
1	1×10^{17}	7.1	1	3	6.5×10^3
2	1×10^{16}	7.1	1	3	6.5×10^3
3	1×10^{14}	7.1	1	3	6.5×10^3
4	1×10^{17}	71	1	3	6.5×10^3
5	1×10^{17}	0.71	1	3	6.5×10^3
6	1×10^{17}	2.7	5	1	4.2×10^4
7	1×10^{17}	0.18	30	1	6.2×10^5

First, we present the results of the simulations for cases 1, 2, and 3 and examine how the different initial mass m_0 of planetesimals affects the numerical results. In Fig. 5.6, the time evolution of the mass of the largest body, the average mass, and the root mean square of eccentricities are shown. Clearly, the time evolution of the average mass is scaled completely and no difference is observed among the three cases. Until the mass of the largest body reaches the critical mass which determines the end of the exponential growth, the growth of the largest body is well scaled. The critical masses evaluated from Eq. (5.2.7) are

$$\frac{m_{\text{crit}}}{m_0} = \begin{cases} 1.3 \times 10^7 & (m_0 = 10^{14} \text{g}), \\ 9.2 \times 10^5 & (m_0 = 10^{16} \text{g}), \\ 2.5 \times 10^5 & (m_0 = 10^{17} \text{g}). \end{cases} \quad (5.3.2)$$

After the mass of the largest body reaches each critical mass, the growth curve of the largest body diverges each other and the growth rate becomes small. The root mean squares of the eccentricities are almost the same for the three cases and are scaled almost completely. The

average quantities (e.g., the average mass and the root mean square of eccentricities) are well scaled although the growth of a small number of large planetesimals cannot be described by the scaling law.

As alternative test to confirm the scaling law, the simulations with different surface densities of solid, σ_s , are carried out (cases 1, 4, and 5). As same as in the case of Fig. 5.6, the time evolution of the average mass is completely scaled (see Fig. 5.7). As to the largest body, the scaling law is satisfied until the mass of the largest body approaches the critical mass given by

$$\frac{m_{\text{crit}}}{m_0} = \begin{cases} 1.8 \times 10^6 & (\sigma_s = 71\text{g/cm}^2), \\ 2.5 \times 10^5 & (\sigma_s = 7.1\text{g/cm}^2), \\ 3.5 \times 10^4 & (\sigma_s = 0.71\text{g/cm}^2). \end{cases} \quad (5.3.3)$$

In these cases, the difference of the root mean squares of eccentricities seems to be large after $\hat{M}_{\text{max}} > m_{\text{crit}}/m_0$ in comparison with the cases of Fig. 5.6. This is due to the fact that the difference of the masses of the largest planetesimal becomes large compared with that in Fig. 5.6. From Figs. 5.6 and 5.7, we can readily confirm that the scaling law is realized until a small number of large planetesimals are formed.

The above simulations have the same value of R ($R = 6.50 \times 10^3$). To examine the effect of the adopted value of R on the accumulation process, we perform the numerical simulation with different values of R (i.e., $R = 4.2 \times 10^4$ and 6.2×10^5 , which correspond to the regions of orbits of Jupiter and Neptune, respectively). The value of R at the orbit of Neptune is about 100 times as large as that at Earth's orbit. Figure 5.8 shows the time evolution of the mass of the largest body, the average mass, and the root mean square of the eccentricities. Despite of the different values of R , the average mass is well scaled. On the other hand, the growth rate of the largest body becomes small with an increase of R . In other words, when the growth of the

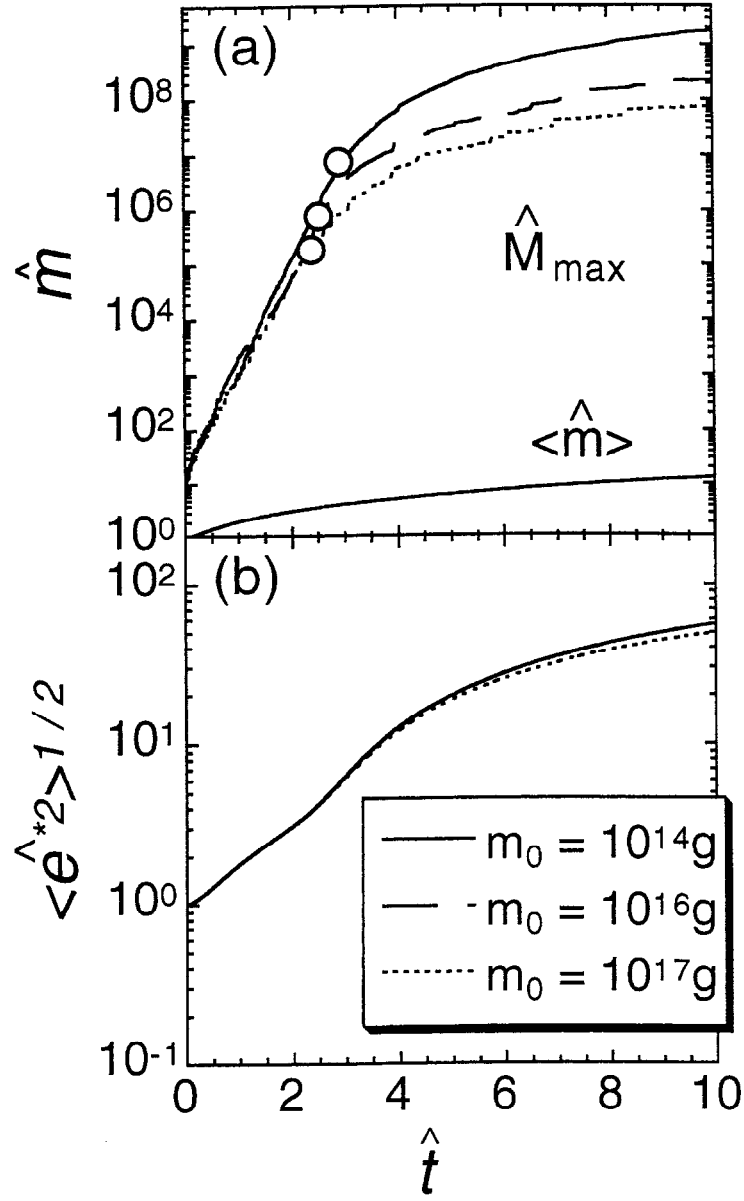


Fig. 5.6. Time evolution of the mass of the largest body, $\hat{M}_{\max}$, and the average mass, $\langle \hat{m} \rangle$ (panel (a)) and the root mean square of eccentricities, $\langle e^{*2} \rangle^{1/2}$ (panel (b)) obtained from the simulations with different initial mass of planetesimals, m_0 . The critical masses which determine the end of the exponential growth are shown by circles in panel (a).

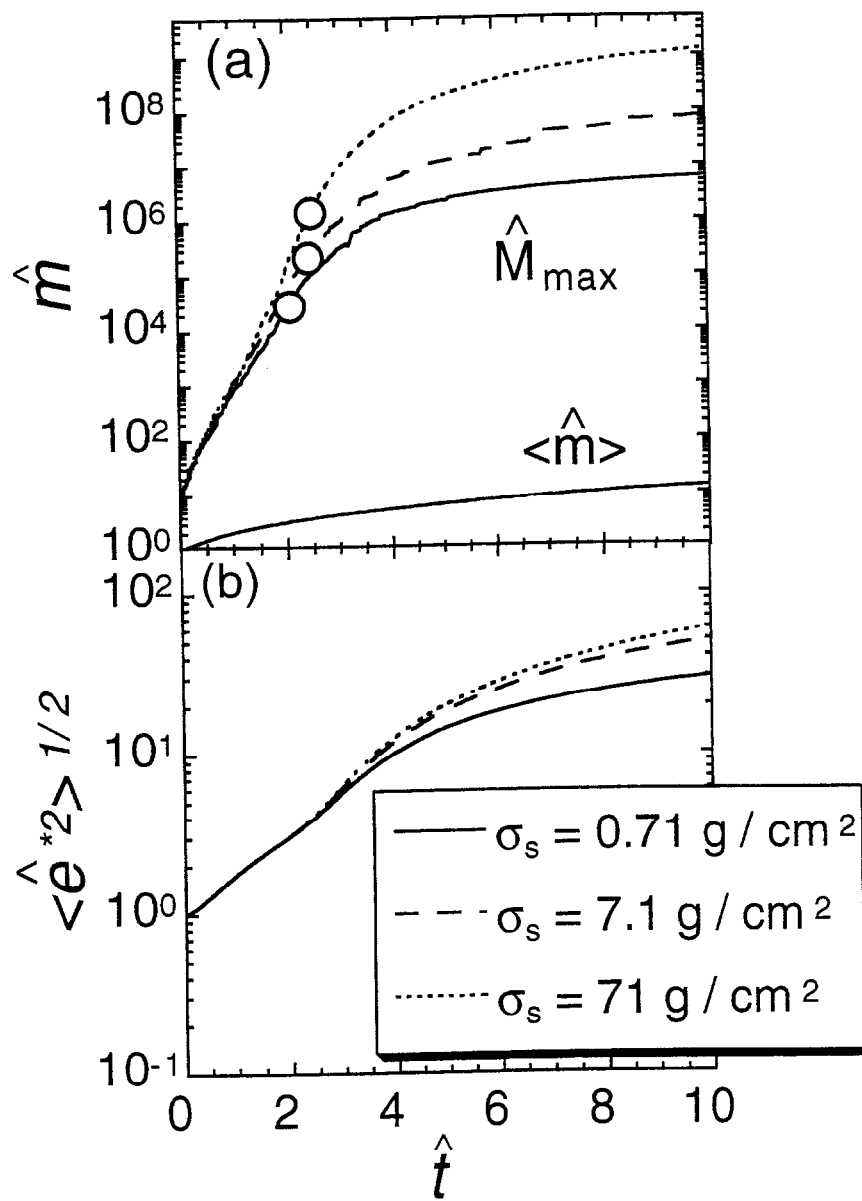


Fig. 5.7. The same as Fig. 5.6, but for the cases with different surface density of solid, σ_s .

largest body is expressed as $\hat{M}_{\max} \propto \exp(\alpha \hat{t})$, the value of α decreases as R increases. Using the least squares fit, we find

$$\alpha = 6.5 - 0.66 \times \log R. \quad (5.3.4)$$

This shows that α depends on R very weakly.

Circles attached to Figs. 5.6 to 5.8 indicate the critical mass, m_{crit} , introduced in subsection 5.2, across which the mode of the planetary growth changes from the runaway growth to the oligarchic growth. As seen from the figures, the growth rates of the largest body become small beyond the circles. This means that the critical mass given by Eq. (5.2.7) is properly evaluated and that the scaling law, which is assumed in deriving the expression of m_{crit} , is valid as long as the mass of the largest body is smaller than the critical mass.

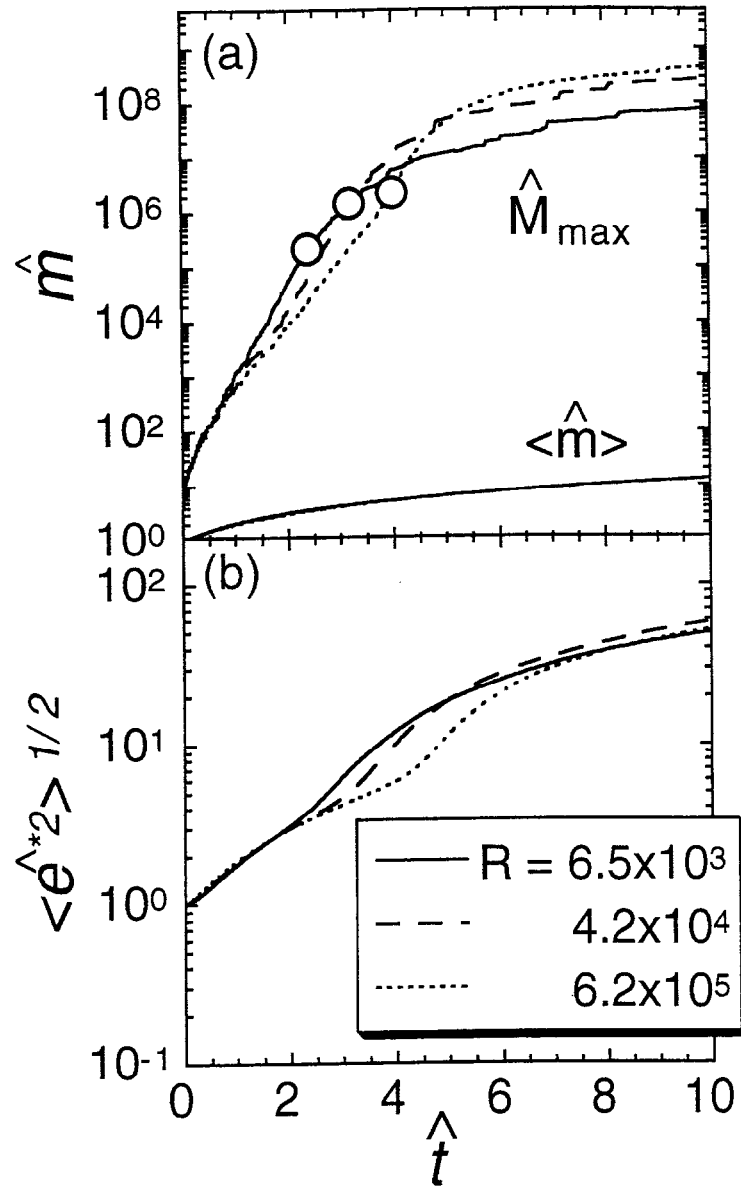


Fig. 5.8. The same as Fig. 5.6, but for the cases with different values of R .

6. Conclusions and concluding remarks

In the present study, we have checked the validity of the statistical code, developed by us, for describing the planetary accumulation process by comparing with the numerical solutions obtained by the N -body simulations. As shown in section 4, the statistical code developed by us can reproduce the numerical results of the N -body simulation with sufficiently high accuracy. Furthermore, we have made a simulation of the planetary accumulation under the same situation as Wetherill and Stewart (1993) and obtained the different results from theirs. Although we cannot make clear the reason of the differences because of little information of their numerical method, it seems to be difficult to describe precisely the planetary accumulation process by their method.

In order to understand the early stage of the planetary accumulation process, we have studied the planetesimal growth under the limited condition that there exists no nebular gas. As the preparation we have first rewritten the basic equations describing the time evolution of the mass and velocity distributions into non-dimensional forms and shown that the non-dimensional basic equations contain only one non-dimensional parameter, R . Furthermore, it has been shown that the planetesimal growth depends weakly on R .

Next, we have simulated the planetary accumulation process of the early stage for the various cases with different physical parameters such as the surface density of solid of materials, the initial mean mass of planetesimals, and the semimajor axis. The obtained results are summarized as follows:

- (1) The planetary growth is divided into two modes: From the beginning of the accumulation of planetesimals the runaway growth starts and the runaway bodies increase their mass exponentially. When the mass of the runaway body exceeds a certain critical mass, the growth mode changes to the oligarchic growth with a small growth rate.

- (2) By the different choice of the physical parameters we have different growth time. However, the mode of the planetary accumulation, mentioned above, is not affected by the adopted parameters.
- (3) In the stage of the runaway growth, the growth rates of runaway bodies are completely scaled by the initial (mean) mass, m_0 , and the mean collision time, t_c , of a planetesimal with mass m_0 as long as we consider the planetesimal system.
- (4) The critical mass, m_{crit} , which divides the mode of the planetary accumulation can be written in an analytical form as a function of the physical parameters. This functional form is consistent with the numerical result.
- (5) The previous works reported that the planetary accumulation starts as, so called, the orderly growth in the early stage. However, as mentioned above, the planetary accumulation starts as the runaway growth.

The above results which we have obtained in this study throw new light on the study of the planetary formation. Especially, conclusions (1) and (5) suggest strongly that we have to investigate extensively and carefully the various process in the very early stage of the planetary formation, i.e., formation mechanism of planetesimals, initial mass distribution of planetesimals, material properties of planetesimals in relation to their collision, spatial variation in mass and number of planetesimal, and other important problems. Until now, it is believed that we are not so interested in the detailed picture of the planetesimal formation because the planetary growth starts as the orderly growth and, hence, does not depend on the adopted initial stage of the planetesimal system. But our results break such an easy conjecture: the planetary growth starts as the runaway growth from the beginning and depends on the initial condition. The problems mentioned above are hoped to be solved in near future.

Acknowledgments

The author expresses his sincere gratitude to H. Tanaka for valuable advice. He is grateful to K. Nakazawa for critical reading of this paper and indebted to S. Ida and H. Emori for continuous encouragement. He also thanks E. Kokubo for giving him the data of the N -body simulation and G. W. Wetherill for giving him the digital data of the statistical simulation. The computation has been made by Cray C916 at the Computer Center of Tokyo Institute of Technology.

References

- AARSETH, S. J., D. N. C. LIN, AND P. L. PALMER 1993. Evolution of planetesimals. II. Numerical simulations. *Astrophys. J.* **403**, 351-376.
- ABRAMOWITZ, M. AND I. A. STEGUN 1992. *Handbook of Mathematical Functions (New York: Dover)*
- ADACHI, I., C. HAYASHI, AND K. NAKAZAWA 1976. The gas drag on the elliptic motion of a solid body in the primordial solar nebula. *Prog. Theor. Phys.* **56**, 1756-1771.
- ADAMS, C. F., C. J. LADA, AND F. H. SHU 1987. Spectral evolution of young stellar objects. *Astrophys. J.* **312**, 788-806.
- ADAMS, C. F., C. J. LADA, AND F. H. SHU 1988. The disk of T Tauri stars with flat infrared spectra. *Astrophys. J.* **326**, 865-883.
- BARGE, P. AND R. PELLAT 1991. Mass spectrum and velocity dispersions during planetesimal accumulation. I. Accretion. *Icarus* **93**, 270-287.
- BEAUGÉ, C. AND S. J. AARSETH 1990. *N*-body simulation of planetary formation. *Mon. Not. R. Astron. Soc.* **245**, 30-39.
- BECKWITH, S. V. W. AND A. I. SARGENT 1996. Circumstellar disks and the search for neighboring planetary systems. *Nature* **383**, 139-144.
- BUTLER, R. P., G. W. MARCY, E. WILLIAMS, H. HAUSER, AND P. SHIRTS 1997. Three new "51 Pegasi-type" planets. *Astrophys. J. Letter* **474**, L115-L118.
- CUZZI, J. N., A. R. DOBROVOLSKIS, AND J. M. CHAMPNEY 1993. Particle-gas dynamics in the midplane of a protoplanetary nebula. *Icarus* **106**, 102-134.
- DONES, L. AND S. TREMAINE 1993. On the origin of planetary spins. *Icarus* **103**, 67-92.
- GOLDREICH, P. AND W. R. WARD 1973. The formation of planetesimals, *Astrophys. J.* **183**, 1051-1061.

- GREENBERG, R., J. WACKER, C. R. CHAPMAN, AND W. K. HARTMANN 1978. Planetesimals to planets: Numerical simulation of collisional evolution. *Icarus* **35**, 1-26.
- GREENZWEIG, Y. AND J. J. LISSAUER 1990. Accretion rates of protoplanets. *Icarus* **87**, 40-77.
- GREENZWEIG, Y. AND J. J. LISSAUER 1992. Accretion rates of protoplanets II. Gaussian distributions of planetesimal velocities. *Icarus* **100**, 440-463.
- HASEGAWA, M. AND K. NAKAZAWA 1990. Distant encounter between Keplerian particles. *Astron. Astrophys.* **227**, 619-627.
- HAYASHI, C. 1972. Origin of the solar system. *Proc. 5th Lunar Planet. Symp. ISAS, Tokyo*, 13-18.
- HAYASHI, C. 1981. Structure of the solar nebula, growth and decay of magnetic fields and effects of magnetic and turbulent viscosities on the nebula, *Prog. Theor. Phys. Suppl.* **70**, 163-196.
- HAYASHI, C., K. NAKAZAWA, AND Y. NAKAGAWA 1985. Formation of the Solar System. In *Protostars and Planets II* (Eds. D. C. Black and M. S. Matthews, Univ. Arizona Press, Tucson), pp. 1100-1153.
- IDA, S. 1990. Stirring and dynamical friction of planetesimals in the solar gravitational field. *Icarus* **88**, 129-145.
- IDA, S. AND K. NAKAZAWA 1989. Collisional probability of planetesimals revolving in the solar gravitational field. III. *Astron. Astrophys.* **224**, 303-315.
- IDA, S. AND J. MAKINO 1992. *N*-body simulation of gravitational interaction between planetesimals and a protoplanet. I. Velocity distribution of planetesimals. *Icarus* **96**, 107-120.
- INABA, S., H. TANAKA, K. OHTSUKI, AND K. NAKAZAWA 1999. High-Accuracy Statistical Simulation of Planetary Accretion: I. Test of the accuracy by comparison with the solution

- to the stochastic coagulation equation. *Earth, Planets and Space* [to be published]
- KARY, D. M., J. J. LISSAUER, AND Y. GREENZWEIG 1993. Nebular Gas Drag and Planetary Accretion. *Icarus* **106**, 288-307.
- KOKUBO, E. AND S. IDA 1996. On runaway growth of planetesimals. *Icarus* **123**, 180-191.
- KOKUBO, E. AND S. IDA 1998. Oligarchic Growth of Protoplanets. *Icarus* **131**, 171-178.
- KOKUBO, E. AND S. IDA 1999. Formation of Protoplanets from Planetesimals in the Solar Nebula. *Icarus* [Submitted].
- LECAR, M. AND S. J. AARSETH 1986. A numerical simulation of the formation of the terrestrial planets. *Astrophys. J.* **305**, 564-579.
- MARCUS, A. H. 1968. Stochastic coalescence. *Technometrics* **10**, 133-143.
- MAYOR, M. AND D. QUELOZ 1995. A jupiter-mass companion to a solar-type star. *Nature* **378**, 355.
- NAKAGAWA, Y., C. HAYASHI, AND K. NAKAZAWA 1983. Accumulation of planetesimals in the solar nebula. *Icarus* **54**, 361-376.
- NAKAZAWA, K., S. IDA, AND Y. NAKAGAWA 1989. Collisional probability of planetesimals revolving in the solar gravitational field I. Basic formulation. *Astron. Astrophys.* **220**, 293-300.
- OHTSUKI K. 1999. Evolution of particle velocity dispersion in a circumplanetary disk due to inelastic collisions and gravitational interactions. *Icarus* **137**, 152-177.
- OHTSUKI, K., Y. NAKAGAWA, AND K. NAKAZAWA 1988. Growth of the earth in nebula gas. *Icarus* **75**, 552-565.
- OHTSUKI K., S. IDA, Y. NAKAGAWA, AND K. NAKAZAWA 1993. Planetary accretion in the solar gravitational field. In *Protostars and Planets III* (Eds. E. H. Levy and J. I. Lunine, Univ. Arizona Press, Tucson), pp.1061-1088.
- SAFRONOV, V. S. 1969. In *Evolution of the protoplanetary cloud and formation of the Earth*

- and planets* (Nauka, Moscow, [Transl. 1972 NASA TT F-677]).
- SPAUTE, D., S. J. WEIDENSCHILLING, D. R. DAVIS, AND F. MARZARI 1991. Accretional evolution of a planetesimal swarm: 1. A new simulation. *Icarus* **92**, 147-164.
- STEPINSKI, T. F. AND P. VALAGEAS 1996. Global evolution of solid matter in turbulent protoplanetary disks. I. Aerodynamics of sold particles. *Astron. Astrophys.* **309**, 301-312.
- STEPINSKI, T. F. AND P. VALAGEAS 1997. Global evolution of solid matter in turbulent protoplanetary disks. II. Development of icy planetesimals. *Astron. Astrophys.* **319**, 1007-1019.
- STEWART, G. R. AND G. W. WETHERILL 1988. Evolution of planetesimal velocities. *Icarus* **74**, 542-553.
- STEWART, G. R. AND S. IDA 1999. Velocity evolution of planetesimals: Unified analytical formulae and comparison with N -body simulations. *Icarus* [Submitted].
- TANAKA, H. AND K. NAKAZAWA 1993. Stochastic coagulation equation and validity of the statistical coagulation equation. *J. Geomag. Geoelectr.* **45**, 361-381.
- TANAKA, H. AND S. IDA 1996. Distribution of planetesimals around a protoplanet in the nebula gas I. The semi-analytic calculation of the gravitational scattering by a protoplanet. *Icarus* **120**, 371-386.
- WEIDENSCHILLING, S. J., SPAUTE, D., DAVIS D. R., MARZARI F., AND K. OHTSUKI 1997. Accretional evolution of a planetesimal swarm 2. the terrestrial zone. *Icarus* **128**, 429-455.
- WETHERILL, G. W. 1990. Comparison of analytical and physical modeling of planetesimal accumulation. *Icarus* **88**, 336-354.
- WETHERILL, G. W. AND G. R. STEWART 1989. Accumulation of a swarm of small planetesimals. *Icarus* **77**, 330-357.
- WETHERILL, G. W. AND G. R. STEWART 1993. Formation of planetary embryos: Ef-

fects of fragmentation, low relative velocity, and independent variation of eccentricity and inclination. *Icarus* **106**, 190-209.