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Shadows of Fuzzy Sets and Existence Theorems for Fuzzy Maps

Masato Amemiya

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I dedicate this thesis to my mother Takeko, my sister Eriko, my twin sister Emiko and my lamented father Atsushi who would have been delighted most at the completeness of the thesis and would pray for my success in the grave.

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CHAPTER 1

Introduction

Fuzzy set theory was initiated by Zadeh [**53**] giving the concept of fuzzy sets to aggregations of objects of indistinctness. This was an attempt to provide general frameworks for dealing theoretically with such aggregations or the constituting object itself. Since then, many kinds of fuzzy concepts have been introduced and applied in diverse areas as logical foundations for treating intrinsic indefiniteness or impreciseness of our judgment, ways of behavior, linguistic information, measurement of uncertainty, etc. Regarding mathematics, for instance, the theory has presented numerous topics of concern to such fields as algebra, topology and analysis in connection with specific motivations behind them.

On the other hand, as is well-known, nonlinear functional analysis has suddenly progressed over the past few decades. It has provided versatile and crucial methods to various mathematical fields for investigating each of the characteristic (and inherent) nonlinear problems. As things stand today, it is no longer a subsidiary of linear cases but it is independently and firmly established as a discipline in the broad field of mathematics, both pure and applied. The aim of this thesis is to study some problems, which have arisen in the applications of the theory of fuzzy sets, in the framework of nonlinear functional analysis.

Let X be a nonempty set. A fuzzy set in X is a function of X into $[0,1]$. Let A and B be fuzzy sets in X. We shall write $A = B$ if $A(x) = B(x)$ for all $x \in X$. Let X be a topological space and let A be a fuzzy set in X. The r-level set of A is a subset of X, which is denoted by A_r and is defined by $A_r = \{x \in X : A(x) \geq r\}$ for every $r \in (0,1]$ and by $A_0 = \mathrm{cl}\{x \in X : A(x) > 0\}$ for $r = 0$, respectively, where $\mathrm{cl}K$ denotes the closure of an arbitrary subset K of X. Then A is said to be closed if for each $r \in (0,1]$, A_r is a closed subset of X. Let X be a linear topological space and let A be a fuzzy set in X. Then A is said to be convex if for each $r \in (0,1]$, A_r is a convex subset of X.

In Chapter 2 of this thesis, we study the concept of shadows of fuzzy sets in a real normed linear space, which was introduced by Zadeh [**53**] in the finite dimensional real Euclidean space and was later, generalized by Takahashi and Takahashi [**47**] to the case of a real Hilbert space. We may employ this concept for obtaining sufficient conditions for the equality of fuzzy sets.

Let X be a real normed linear space. We shall denote by X^* the dual of X, that is, the space of all continuous linear functionals defined on X. Then a closed hyperplane H in X is defined by

$$H = \{x \in X : f(x) = c\}$$

for any nonzero continuous linear functional $f \in X^*$ and any real constant c.

Let A be a fuzzy set in a real Hilbert space X and let H be a closed hyperplane in X. The shadow $S_H[A]$ of A on H is a fuzzy set in H defined by

$$S_H[A](x) = \sup\{A(z) : P_H(z) = x\}$$

for every $x \in H$, where P_H denotes the metric projection from X onto H, namely, the mapping of X onto H defined by the relation

$$\|x - P_H(x)\| = \min_{y \in H} \|x - y\|$$

for every $x \in X$; see, for instance, [**50**].

In [**53**], Zadeh presented the following result, which gave a sufficient condition for the equality of convex fuzzy sets; see also [**54**]:

> Let A and B be two convex fuzzy sets in the finite dimensional real Euclidean space $\mathbb{R}^n$. If $S_H[A] = S_H[B]$ for every hyperplane H in $\mathbb{R}^n$, then $A = B$.

However, it was later shown that this admitted a counterexample (see [**47**] and Section 3 of Chapter 2, p. 21 for instance). In [**47**], Takahashi and Takahashi studied the shadows of fuzzy sets in a real Hilbert space and then, by imposing the closedness on the convex fuzzy sets, deduced the following revised version of Zadeh's above result.

THEOREM 2.1 (Takahashi-Takahashi, [**47**]). *Let A and B be two convex fuzzy sets in a real Hilbert space X. If both A and B are closed and if $S_H[A] = S_H[B]$ for every closed hyperplane H in X, then $A = B$.*

The main purpose of Chapter 2 is to extend the above result of Takahashi and Takahashi to the case of a real normed linear space. So, we first state some results concerning the shadows of fuzzy sets in a real Hilbert space and then, prove a theorem which is logically equivalent to the above theorem. Next, for the completeness of the purpose, we introduce the new concept of fuzzy sets in a real normed linear space corresponding to the shadows of fuzzy sets in a real Hilbert space without using the metric projections. This shall be done as follows.

Let X be a real normed liner space, let A be a fuzzy set in X and let H_0^* be a closed hyperplane in X^* containing $0 \in X^*$. Then we define a fuzzy set $\hat{S}_{H_0^*}[A]$ in X with respect to H_0^* by

$$\hat{S}_{H_0^*}[A](x) = \sup\{A(z) : z \in \hat{M}_{H_0^*}(x)\}$$

for every $x \in X$, where $\hat{M}_{H_0^*}(x)$ is a subset of X defined by

$$\hat{M}_{H_0^*}(x) = \bigcap_{f \in H_0^*} \{z \in X : f(z) = f(x)\}$$

for every $x \in X$.

Using the above concept, we shall prove the following theorem.

THEOREM 2.3. *Let A and B be two convex fuzzy sets in a real normed linear space X. If both A and B are closed and if $\hat{S}_{H_0^*}[A] = \hat{S}_{H_0^*}[B]$ for every closed hyperplane H_0^* in X^* containing $0 \in X^*$, then $A = B$.*

In Chapter 3 of this thesis, we discuss fixed point theory for fuzzy mappings in connection with that for set-valued maps in the context of a generalization.

Let X be a nonempty set. We shall denote by $\mathbf{2}^X$ the class of all subsets of X. Then a mapping from X into $\mathbf{2}^X$ is called a set-valued map on X. Let T be a set-valued map on X. Then an element $x_0 \in X$ is said to be a fixed point of T if $x_0 \in Tx_0$. We shall also denote by $\mathcal{F}(X)$ the family of all fuzzy sets in X. Then a mapping from X into $\mathcal{F}(X)$ is called a fuzzy mapping on X. This enables us to regard an arbitrary fuzzy mapping on X as a two variable function from $X \times X$ into $[0,1]$. Then, for any fuzzy mapping, we shall think of the value at the first variable as the value of the fuzzy mapping itself (for details, see Section 2 of Chapter 3, p. 26). Let F be a fuzzy mapping on X. Then an element $x_0 \in X$ is said to be a fixed point of F if $F(x_0, x_0) = 1$.

From the above, we may consider fixed point theory for fuzzy mappings to be a generalization of that for set-valued maps. Indeed, we can always associate any set-valued map T on X with the fuzzy mapping F on X defined by

$$F(x, y) = \mathbf{1}_{Tx}(y)$$

for every $x, y \in X$, where $\mathbf{1}_K$ denotes the characteristic function for an arbitrary set K, that is, the function of X into $\{0,1\}$ defined by the relation

$$\mathbf{1}_K(x) = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \notin K. \end{cases}$$

Then, it is obvious that $F(x_0, x_0) = 1$ if and only if $x_0 \in Tx_0$, which implies that fixed point theorems for set-valued maps can be derived from those for fuzzy mappings. This is a primary fact that motivates us to study the theory of fixed points for fuzzy mappings in the present thesis. In fact, we prove, in Section 3 of Chapter 3, some generalized fixed point theorems for fuzzy mappings on a complete metric space, which guarantee the existence of the fixed points of (almost) all the mappings of contractive type, both set-valued and fuzzy (for specific descriptions of the mappings of this type, see Section 1, pp. 24, 26 and Section 3, p. 34). For stating one of them, we need some definitions and notations.

Let X be a metric space with metric d, let K be a nonempty subset of X and let $x \in X$. Then, the distance from x to K is defined by

$$d(x, K) = \inf\{d(x, y) : y \in K\}.$$

Let $f : X \to (-\infty, \infty]$ be a function. $\mathrm{dom} f$ denotes the subset $\{x \in X : f(x) < \infty\}$ of X and f is called proper if $\mathrm{dom} f \neq \emptyset$. Let F be a fuzzy mapping on X. Then $[Fx]_1$ denotes the 1-level set of the fuzzy set $F(x, \cdot) \in \mathcal{F}(X)$, that is, the subset $\{y \in X : F(x, y) = 1\}$ of X.

As one of our theorems, the following shall be proved.

THEOREM 3.5. *Let X be a complete metric space with metric d, let $f : X \to (-\infty, \infty]$ be a proper function bounded from below and let F be a fuzzy mapping on X. Assume that for each $x \in X$, there exists $y \in X$ such that $F(x, y) = 1$ and $f(y) + d(x, y) \leq f(x)$. Further, assume that for any $z \in X$ with $F(z, z) \neq 1$, $\inf_{x \in X}\{d(x, z) + d(x, [Fx]_1)\} > 0$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

In the rest of Chapter 3, we prove in Section 4 a fixed point theorem for fuzzy mappings on a complete metric space by employing the concept of w-distances (see Section 2, p. 26) which was introduced by Kada, Suzuki and Takahashi [**27**] as a generalized concept of distances.

THEOREM 3.7. *Let X be a complete metric space, let $f : X \to (-\infty, \infty]$ be a proper, lower semicontinuous function bounded from below and let F be a fuzzy mapping on X. Assume that there exists a w-distance p on X such that for each $x \in X$, there is $y \in X$ with $F(x, y) = 1$ and $f(y) + p(x, y) \leq f(x)$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

Further, by applying well-known Fan's fixed point theorem [**19**] (see Section 2, p. 27), we prove in Section 5 a fixed point theorem for fuzzy mappings on a locally convex linear topological space.

THEOREM 3.8. *Let C be a compact convex subset of a locally convex linear topological space and let F be a fuzzy mapping on C such that for every $x \in C$, $F(x, \cdot)$ is a convex fuzzy set in C. Suppose that for every $x \in C$, there exists $y \in C$ such that $F(x, y) = 1$ and that the function $F : C \times C \to [0, 1]$ is upper semicontinuous on $C \times C$. Then there exists $x_0 \in C$ such that $F(x_0, x_0) = 1$.*

We shall also observe that this is logically equivalent to the above Fan's fixed point theorem.

Let X be a real normed linear space, let C be a convex subset of X and let f be a real-valued function on C. Then f is said to be convex if for any $x, y \in C$ and any $\lambda \in [0, 1]$, $f\big(\lambda x + (1 - \lambda)y\big) \leq \lambda f(x) + (1 - \lambda)f(y)$. f is called concave if $-f$ is convex. Let C, I be nonempty sets and let φ be a real-valued function on $C \times I$. Then φ is called concavelike in its second variable if for any $y_1, y_2 \in I$ and any $\lambda \in (0, 1)$, there exists $y_0 \in I$ such that $\varphi(x, y_0) \geq \lambda\varphi(x, y_1) + (1 - \lambda)\varphi(x, y_2)$ for all $x \in C$.

Chapter 4 of this thesis deals with the concepts of fuzzy numbers and fuzzy-valued maps [**15, 22**] (see Section 2, p. 46 and Section 3, p. 47). These were introduced as fuzzy analogues of real numbers and real-valued functions and have led to fuzzy optimization problems with constraints (for details, see Section 4, pp. 53-55). In relation to ordinary optimization problems, such problems may naturally be formulated as follows:

> C being a nonempty subset of a topological space (describing the constraints) and F being a mapping on C with value of fuzzy numbers, find $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$,

where $\preceq$ stands for a partial order relation on the set of all fuzzy numbers (which shall be defined below). In Chapter 4, we prove some theorems in reference to the above problems in the framework of nonlinear analysis, in particular, convex analysis.

Let us first give some definitions and notations regarding fuzzy numbers. We shall use the letter $\mathbb{R}$ to denote the real line as well as the set of all real numbers.

Let A and B be fuzzy sets in $\mathbb{R}$. By Zadeh's extension principle [**55**], we define the addition $\oplus$ to yield a fuzzy set $A \oplus B$ in $\mathbb{R}$ by

$$(A \oplus B)(z) = \sup_{z=x+y, x,y \in \mathbb{R}} \min\big(A(x), B(y)\big)$$

for all $z \in \mathbb{R}$. Also, we define the multiplication $\odot$ to yield a fuzzy set $A \odot B$ in $\mathbb{R}$ by

$$(A \odot B)(z) = \sup_{z=xy, x,y \in \mathbb{R}} \min\bigl(A(x), B(y)\bigr)$$

for all $z \in \mathbb{R}$.

We know that, if both A and B are convex, then $A \oplus B$ is convex. For related results, we may refer to [**36**].

Let A be a fuzzy set in $\mathbb{R}$. Then A is said to be a fuzzy number [**15**] if it satisfies the following conditions:

(i) A is convex;

(ii) there exists a unique real number $m \in \mathbb{R}$ such that $A(m) = 1$;

(iii) A_0 is a bounded subset of $\mathbb{R}$.

We shall denote by $\mathcal{F}$ the set of all fuzzy numbers. Then, for any $A \in \mathcal{F}$ and any $\lambda \in \mathbb{R}$, we shall write λA (respectively $A\lambda$) instead of $\mathbf{1}_{\{\lambda\}} \odot A$ (respectively $A \odot \mathbf{1}_{\{\lambda\}}$). We also know that for any $A \in \mathcal{F}$ and any $\lambda \in \mathbb{R}$, λA belongs to $\mathcal{F}$ (for details, see Section 2, p. 47).

Let $A, B \in \mathcal{F}$. Then we define an order relation $\preceq$ on $\mathcal{F}$ as follows [**41**]:

$A \preceq B$ if and only if $\sup A_r \leq \sup B_r$ and $\inf A_r \leq \inf B_r$ for all $r \in [0, 1]$.

In [**41**], Ramík and Římánek introduced the concept of L-R fuzzy numbers. This is a convenient representation for fuzzy numbers in the form of three parameters and is known to be useful for practical calculations of fuzzy numbers such as the comparison (the order relation) and the algebraic operations of the addition and the multiplication; see also [**15**].

Let S be a function from $\mathbb{R}$ to $(-\infty, 1]$ satisfying the following conditions:

(i) For every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} : S(x) \geq \alpha\}$ is a convex subset of $\mathbb{R}$;

(ii) $S(x) = 1$ if and only if $x = 0$;

(iii) the set $\{x \in \mathbb{R} : S(x) > 0\}$ is a bounded subset of $\mathbb{R}$;

(iv) $S(x) = S(-x)$ for all $x \in \mathbb{R}$.

Then the function S is said to be a shape function. Let S, T be shape functions, let $m \in \mathbb{R}$ and let $\alpha, \beta \geq 0$. Then the L-R fuzzy number μ is a fuzzy number defined as follows:

$$\mu(x) = \begin{cases} \max(S(\frac{x-m}{\alpha}), 0) & \text{if } x \leq m, \\ \max(T(\frac{x-m}{\beta}), 0) & \text{if } x \geq m. \end{cases}$$

Then, we shall say that μ is generated by the shape functions S and T. Further, we shall denote μ by

$$\mu = (m, \alpha, \beta)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation. In the applications of our results, we shall use the concept of L-R fuzzy numbers.

Let X be a nonempty set. Then a mapping from X into $\mathcal{F}$ is called a fuzzy-valued map on X. Let F be a fuzzy-valued map on X. Then, for any $r \in [0,1]$, f_r^F and g_r^F denote the real-valued functions on X defined by $f_r^F(x) = \sup\bigl[F(x)\bigr]_r$ for every $x \in X$ and by $g_r^F(x) = \inf\bigl[F(x)\bigr]_r$ for every $x \in X$, respectively, where $\bigl[F(x)\bigr]_r$ denotes the r-level set of the fuzzy number $F(x) \in \mathcal{F}$. We shall also denote by $\mathcal{P}^F$ the family $\bigl\{f_r^F, g_r^F : r \in [0,1]\bigr\}$ of real-valued functions on X. Let C be a convex subset of a linear space and let F be a fuzzy-valued map on C. Then F is said to be convex if for every $x, y \in C$ and every $\lambda \in (0,1)$,

$$F\bigl(\lambda x + (1-\lambda)y\bigr) \preceq \lambda F(x) \oplus (1-\lambda)F(y).$$

Let X be a topological space and let F be a fuzzy-valued map on X. Then F is said to be lower semicontinuous on X if for every $r \in [0,1]$, both f_r^F and g_r^F are lower semicontinuous on X.

For our purpose of this chapter, we first study, in Section 3 of Chapter 4, the convexity of fuzzy-valued maps and give the characterization of the convexity of them on a convex subset of a linear space, which shall ensure the standpoint of convex analysis. In the course of discussion, we shall prove the following three theorems.

THEOREM 4.2. *Let C be a convex subset of a linear space and let F be a convex fuzzy-valued map on C. Then, for every $r \in [0,1]$, f_r^F is convex.*

THEOREM 4.3. *Let C be a convex subset of a linear space and let F be a convex fuzzy-valued map on C. Then, for every $r \in [0,1]$, g_r^F is convex.*

THEOREM 4.4. *Let C be a convex subset of a linear space and let F be a fuzzy-valued map on C. Assume that for every $r \in [0,1]$, both f_r^F and g_r^F are convex. Then F is convex.*

We remark that Theorems 4.2, 4.3 and 4.4 give the characterization of the convexity of a fuzzy-valued map F in the sense that F is convex if and only if for every $h \in \mathcal{P}^F$, h is convex.

As an immediate consequence of the above theorems, we shall prove the following theorem on the convexity of fuzzy-valued maps with value of L-R fuzzy numbers generated by the same shape functions.

THEOREM 4.5. *Let C be a convex subset of a linear space, let $m : C \to \mathbb{R}$, let $\alpha, \beta : C \to [0, \infty)$ be functions on C and let $S, T : \mathbb{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an* L-R *fuzzy number denoted by*

$$F(x) = \big(m(x), \alpha(x), \beta(x)\big)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation. Assume that both m and β are convex and that α is concave. Then F is convex.

Concerning the lower semicontinuity of fuzzy-valued maps of this type, we shall prove, in Section 4 of Chapter 4, the following theorem.

THEOREM 4.6. *Let C be a nonempty subset of a topological space, let $m : C \to \mathbb{R}$, let $\alpha, \beta : C \to [0, \infty)$ be functions on C and let $S, T : \mathbb{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an* L-R *fuzzy number denoted by*

$$F(x) = \big(m(x), \alpha(x), \beta(x)\big)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation. Assume that both m and β are lower semicontinuous on C and that α is upper semicontinuous on C. Then F is lower semicontinuous on C.

In the rest of Chapter 4, we deal with minimization problems for fuzzy-valued maps in reference to fuzzy optimization problems with constraints (mentioned on page 5) of the form:

> C being a nonempty compact subset of a topological space and F being a lower semicontinuous fuzzy-valued map on C, find $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$.

Then, by the definition of the order relation $\preceq$, we can immediately resolve the above problem into the following form:

> C being a nonempty compact subset of a topological space and $\mathcal{P}^F$ being the family of lower semicontinuous real-valued functions on C, find $x_0 \in C$ such that $h(x_0) \leq h(x)$ for all $h \in \mathcal{P}^F$ and all $x \in C$.

Based on this (which is a primary motivation for studying the topics of this section), we establish, in Section 4 of this chapter, some minimization theorems for fuzzy-valued maps on a compact topological space.

One of our theorems shall be obtained as follows.

THEOREM 4.8. *Let C be a compact and convex subset of a linear topological space, let F be a lower semicontinuous and convex fuzzy-valued map on C. Let φ be a real-valued function on $C \times \mathcal{P}^F$ defined by $\varphi(x,h) = h(x) - \min_{u \in C} h(u)$ for every $(x,h) \in C \times \mathcal{P}^F$. Suppose that φ is concavelike in its second variable, that is, for every $h_1, h_2 \in \mathcal{P}^F$ and every $\lambda \in (0,1)$, there exists $h_0 \in \mathcal{P}^F$ such that*

$$\varphi(x,h_0) \geq \lambda\varphi(x,h_1) + (1-\lambda)\varphi(x,h_2)$$

for all $x \in C$. Then there exists $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$.

We shall, at the end of this section, provide an example of fuzzy-valued maps satisfying the assumptions in the above theorem.

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CHAPTER 2

Generalization of Shadows

1. Introduction

In [**6, 7**], Zadeh introduced the concept of shadows of fuzzy sets with a view to obtaining sufficient conditions for the equality of fuzzy sets. Indeed, he stated the following result for convex fuzzy sets in the finite dimensional real Euclidean space $\mathbb{R}^n$:

> Let A and B be two convex fuzzy sets in the finite dimensional real Euclidean space $\mathbb{R}^n$. If $S_H[A] = S_H[B]$ for every hyperplane H in $\mathbb{R}^n$, then $A = B$,

where $S_H[A]$ is the shadow of A on H.

However, it was later shown that this admitted a counterexample. In [**3**], Takahashi and Takahashi gave a counterexample to Zadeh's above result (see also Section 3 of the present chapter, p. 21) and then, proved the following revised version of it, which were discussed in a real Hilbert space.

THEOREM 2.1 (Takahashi-Takahashi, [**3**]). *Let A and B be two convex fuzzy sets in a real Hilbert space X. If both A and B are closed and if $S_H[A] = S_H[B]$ for every closed hyperplane H in X, then $A = B$.*

In this chapter, we extend the above result of Takahashi and Takahashi to the case of a normed linear space. So, we first study the shadows of fuzzy sets in a real Hilbert space and prove a theorem which is logically equivalent to Theorem 2.1. Next, we extend our result (and equivalently, Theorem 2.1) to the case of a normed linear space. In the course of our discussion, we shall introduce a fuzzy set in a normed linear space, which is in essence a generalization of the shadows of fuzzy sets.

2. Preliminaries

Throughout this chapter, all linear spaces are real and if X is a normed linear space, then X^* denotes its dual, that is, the space of all continuous linear functionals defined on X. We shall also denote by $\mathbb{R}$ the set of real numbers and use the letter $\mathbb{R}$ to denote the real line as well.

Let X be a nonempty set. A fuzzy set in X is a function of X into $[0,1]$. Let A be a fuzzy set in X. Then the complement of A is denoted by A' and is defined by $A'(x) = 1 - A(x)$ for every $x \in X$. Let A and B be fuzzy sets in X. We write $A = B$ if $A(x) = B(x)$ for all $x \in X$. Let X be a topological space and let A be a fuzzy set in X. The r-level set of A is a subset of X, which is denoted by A_r and is defined by $A_r = \{x \in X : A(x) \geq r\}$ for every $r \in (0,1]$ and by $A_0 = \mathrm{cl}\{x \in X : A(x) > 0\}$ for $r = 0$, respectively, where $\mathrm{cl}K$ denotes the closure of an arbitrary subset K of X. Then A is said to be closed if for each $r \in (0,1]$, A_r is a closed subset of X. Let A be a fuzzy set in a topological linear space X. Then A is said to be convex if for each $r \in (0,1]$, A_r is convex in X.

Let X be a normed linear space. A closed hyperplane H in X is defined by

$$H = \{x \in X : f(x) = c\}$$

for any nonzero continuous linear functional $f \in X^*$ and any real constant $c \in \mathbb{R}$. Let X^{**} denote the dual of X^*. Then, we note that X can be embedded in X^{**} by virtue of the natural embedding, that is, an isometric isomorphism $\hat{x}$ from X into X^{**} defined by

$$\hat{x}(f) = f(x)$$

for every $x \in X$ with any $f \in X^*$; see [**2**] for instance. Therefore, for any nonzero element $x \in X$ and any real constant $c \in \mathbb{R}$, the set

$$H_c^* = \{f \in X^* : f(x) = c\}$$

can be regarded as a closed hyperplane in X^*.

Let C be a closed convex subset of a Hilbert space X. Then, it is well-known that for each $x \in X$, there exists a unique element $P_C(x)$ such that

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

The mapping P_C is called the metric projection of X onto C. For the metric projections, we know the following; see, for instance, [**4, 5**].

PROPOSITION 2.1. *Let C be a convex subset of a Hilbert space X, let $x \in X$ and let $x_0 \in C$. Then the following two conditions are equivalent:*

(i) $P_c(x) = x_0$;
(ii) $(x - x_0, x_0 - y) \geq 0$ *for all* $y \in C$.

We also know the following two separation theorems; see [**2**] for instance.

PROPOSITION 2.2. *In a locally convex linear topological space, an open convex subset and an arbitrary convex subset which are disjoint can be separated by a nonzero continuous linear functional.*

PROPOSITION 2.3. *In a locally convex linear topological space, two disjoint closed convex subsets, one of which is compact, can be strongly separated by a nonzero continuous linear functional.*

3. Generalization of Shadows of Fuzzy Sets

Let A be a fuzzy set in a real Hilbert space X and let H be a closed hyperplane in X. The shadow $S_H[A]$ of A on H is a fuzzy set in H defined by

$$S_H[A](x) = \sup\{A(z) : P_H(z) = x\}$$

for every $x \in H$, where P_H denotes the metric projection of X onto H.

In this section, we first discuss some results concerning the shadows of fuzzy sets in a Hilbert space. Let C be a closed convex subset of a Hilbert space X. Then we shall denote by P_C the metric projection of X onto C. Let us begin with proving the following two lemmas.

LEMMA 2.1. *Let H_0 be a closed hyperplane in a Hilbert space X containing $0 \in X$ and let H be a closed hyperplane in X which is parallel to H_0. Then for every $x \in H$,*

$$\{z \in X : P_{H_0}(z) = P_{H_0}(x)\} = \{z \in X : P_H(z) = x\}.$$

PROOF. We observe that for every $z \in X$ and every $p \in H_0$,

$$(z - P_{H_0}(z), p) = 0 \text{ and } (z - P_H(z), p) = 0.$$

Let $x \in H$ be fixed arbitrarily. Then, put

$$A = \{z \in X : P_{H_0}(z) = P_{H_0}(x)\}$$

and put

$$B = \{z \in X : P_H(z) = x\}.$$

We first claim that $A \subset B$. Let $z \in A$ and let $y \in H$. Then, since $P_{H_0}(z) = P_{H_0}(x)$ and $x - y \in H_0$, we infer by the above observation that

$$\begin{aligned}(z - x, x - y) &= (z - P_{H_0}(z) + P_{H_0}(z) - x, x - y)\\ &= (z - P_{H_0}(z) + P_{H_0}(x) - x, x - y)\\ &= (z - P_{H_0}(z), x - y) + (P_{H_0}(x) - x, x - y)\\ &= 0.\end{aligned}$$

Since $y \in H$ is arbitrary, this implies that $P_H(z) = x$, that is, $z \in B$.

Conversely, let $z \in B$ and let $y \in H_0$. Then, since $P_H(z) = x$ and $P_{H_0}(x) - y \in H_0$, it follows that

$$\begin{aligned}
\big(z - P_{H_0}(x), P_{H_0}(x) - y\big) &= \big(z - P_H(z) + P_H(z) - P_{H_0}(x), P_{H_0}(x) - y\big) \\
&= \big(z - P_H(z) + x - P_{H_0}(x), P_{H_0}(x) - y\big) \\
&= 0,
\end{aligned}$$

which implies that $P_{H_0}(z) = P_{H_0}(x)$, that is, $z \in A$. Therefore, we have that $B \subset A$. $\square$

Let A be a fuzzy set in a Hilbert space X and let H be a closed hyperplane in X. Then, we define a fuzzy set $\tilde{S}_H[A]$ of A in X with respect to H by

$$\tilde{S}_H[A](x) = \sup\{A(z) : P_H(z) = P_H(x)\}$$

for every $x \in X$.

LEMMA 2.2. *Let A, B be fuzzy sets in a Hilbert space X, let $\mathcal{H}$ be the class of all closed hyperplanes in X and let $\mathcal{H}_0$ be the class of all closed hyperplanes in X containing $0 \in X$. Then the following two statements are equivalent:*

(i) $S_H[A] = S_H[B]$ *for all* $H \in \mathcal{H}$;
(ii) $\tilde{S}_{H_0}[A] = \tilde{S}_{H_0}[B]$ *for all* $H_0 \in \mathcal{H}_0$.

PROOF. (i) $\Rightarrow$ (ii): Let $H_0 \in \mathcal{H}_0$ and let $x \in X$. Then, there exists $H \in \mathcal{H}$ such that H contains x and is parallel to H_0. Then it follows from Lemma 2.1 that

$$\begin{aligned}
\tilde{S}_{H_0}[A](x) &= \sup\{A(z) : P_{H_0}(z) = P_{H_0}(x)\} \\
&= \sup\{A(z) : P_H(z) = x\} \\
&= S_H[A](x) \\
&= S_H[B](x) \\
&= \sup\{B(z) : P_H(z) = x\} \\
&= \sup\{B(z) : P_{H_0}(z) = P_{H_0}(x)\} = \tilde{S}_{H_0}[B](x).
\end{aligned}$$

Therefore, we have that $\tilde{S}_{H_0}[A] = \tilde{S}_{H_0}[B]$.

(ii) $\Rightarrow$ (i): Let $H \in \mathcal{H}$ and let $x \in H$. Then, we can take $H_0 \in \mathcal{H}_0$ such that H_0 is parallel to H. Then it follows from Lemma 2.1 that

$$\begin{aligned} S_H[A](x) &= \sup\{A(z) : P_H(z) = x\} \\ &= \sup\{A(z) : P_{H_0}(z) = P_{H_0}(x)\} \\ &= \tilde{S}_{H_0}[A](x) \\ &= \tilde{S}_{H_0}[B](x) \\ &= \sup\{B(z) : P_{H_0}(z) = P_{H_0}(x)\} \\ &= \sup\{B(z) : P_H(z) = x\} = S_H[B](x). \end{aligned}$$

Hence, we have that $S_H[A] = S_H[B]$. □

Using Lemma 2.2 and Theorem 2.1, we obtain the following theorem.

THEOREM 2.2. *Let A and B be two convex fuzzy sets in a Hilbert space X. If both A and B (or A' and B') are closed and if $\tilde{S}_H[A] = \tilde{S}_H[B]$ for each closed hyperplane H_0 in X containing $0 \in X$, then $A = B$.*

We also prove the following lemma.

LEMMA 2.3. *Let E be a closed subspace in a Hilbert space X and let $x \in X$ be fixed arbitrarily. Then*

$$\bigcap_{p \in E} \{z \in X : (p, z) = (p, x)\} = \{z \in X : P_E(z) = P_E(x)\}.$$

PROOF. Put

$$A = \bigcap_{p \in E} \{z \in X : (p, z) = (p, x)\}$$

and put

$$B = \{z \in X : P_E(z) = P_E(x)\}.$$

Then, the relation $B \subset A$ is obvious. So, we prove that $A \subset B$. Let $z \in A$ and let $p \in E$. Since $P_E(x) - p \in E$, we infer that

$$\begin{aligned} \|z - p\|^2 - \|z - P_E(x)\|^2 &\geq 2(P_E(x) - p, z - P_E(x)) \\ &= 2(P_E(x) - p, x - P_E(x)) \\ &\geq 0. \end{aligned}$$

This implies that $P_E(z) = P_E(x)$ and hence, that $z \in B$. □

We now extend Theorem 2.2 to the case of a normed linear space. Before doing it, we give a definition.

Let A be a fuzzy set in a normed linear space X and let H_0^* a closed hyperplane in X^* containing $0 \in X^*$. Then we define a fuzzy set $\hat{S}_{H_0^*}[A]$ of A in X with respect to H_0^* by

$$\hat{S}_{H_0^*}[A](x) = \sup\{A(z) : z \in \hat{M}_{H_0^*}(x)\}$$

for all $x \in X$, where $\hat{M}_{H_0^*}(x)$ is a subset of X defined by

$$\hat{M}_{H_0^*}(x) = \bigcap_{g \in H_0^*} \{z \in X : g(z) = g(x)\}$$

for all $x \in X$.

THEOREM 2.3. *Let A and B be two convex fuzzy sets in a normed linear space X. If both A and B (or A' and B') are closed and if $\hat{S}_{H_0^*}[A] = \hat{S}_{H_0^*}[B]$ for every $H_0^* \in \mathcal{H}_0^*$, then $A = B$.*

PROOF. It is sufficient to show that, if there exists an element $x_0 \in X$ such that $A(x_0) \neq B(x_0)$, then there exists $H_0^* \in \mathcal{H}_0^*$ such that

$$\hat{S}_{H_0^*}[A](x_0) \neq \hat{S}_{H_0^*}[B](x_0).$$

We may assume that

$$\alpha = A(x_0) > B(x_0) = \beta.$$

Then, let

$$G = \{z \in X : B(z) > \beta\}$$

and consider the following two cases : (i) G is empty; (ii) G is nonempty.

(i) In this case, it is obvious that for each $H_0^* \in \mathcal{H}_0^*$, $\hat{S}_{H_0^*}[B](x_0) \leq \beta$. Further, since $x_0 \in \hat{M}_{H_0^*}(x_0)$, we have that $\hat{S}_{H_0^*}[A](x_0) \geq \alpha > \beta$. Hence, we deduce that

$$\hat{S}_{H_0^*}[A](x_0) \neq \hat{S}_{H_0^*}[B](x_0).$$

(ii) First, suppose that both A' and B' are closed. Then G is an open convex subset of X, so that, there exists a nonzero continuous linear functional $f \in X^*$ such that

$$\sup\{f(z) : z \in G\} \leq f(x_0).$$

Put

$$F = \{z \in X : f(z) = f(x_0)\}.$$

Then, since G is open and convex and since f is continuous and linear, we infer that

$$z \in G \Rightarrow f(z) < f(x_0)$$

and hence, that

$$z \in F \Rightarrow z \notin G \Rightarrow B(z) \leq \beta.$$

Define a closed hyperplane $H_0^* \in \mathcal{H}_0^*$ by

$$H_0^* = \{g \in X^* : g(p) = 0\},$$

where p is a nonzero element of X such that $f(p) = 0$. Then, since

$$\begin{aligned}\hat{M}_{H_0^*}(x_0) &= \bigcap_{g \in H_0^*} \{z \in X \ : \ g(z) = g(x_0)\} \\ &\subset \{z \in X : f(z) = f(x_0)\} = F,\end{aligned}$$

we have that

$$\begin{aligned}\hat{S}_{H_0^*}[B](x_0) &= \sup\{B(z) : z \in \hat{M}_{H_0^*}(x_0)\} \\ &\leq \sup\{B(z) : z \in F\} \\ &\leq \beta.\end{aligned}$$

On the other hand, since $x_0 \in \hat{M}_{H_0^*}(x_0)$, it follows that

$$\begin{aligned}\hat{S}_{H_0^*}[A](x_0) &= \sup\{A(z) : z \in \hat{M}_{H_0^*}(x_0)\} \\ &\geq A(x_0) = \alpha > \beta.\end{aligned}$$

Thus, we deduce that

$$\hat{S}_{H_0^*}[A](x_0) \neq \hat{S}_{H_0^*}[B](x_0).$$

Next, suppose that both A and B are closed. Then there exists $\varepsilon > 0$ such that $\alpha > \beta + \varepsilon$. Let

$$G_\varepsilon = \{z \in X : B(z) \geq \beta + \varepsilon\}.$$

Then, since G_ε is a nonempty closed convex subset of X, there exist continuous linear functional $f \in X^*$ and a real constant $c \in \mathbb{R}$ such that

$$\sup\{f(z) : z \in G_\varepsilon\} < c < f(x_0).$$

Put

$$F_\varepsilon = \{z \in X : f(z) = f(x_0)\}$$

and define a closed hyperplane $H_0^* \in \mathcal{H}_0^*$ by

$$H_0^* = \{g \in X^* : g(p) = 0\},$$

where p is a nonzero element of X such that $f(p) = 0$. Then we infer that

$$\begin{aligned}\hat{S}_{H_0^*}[B](x_0) &= \sup\{B(z) : z \in \hat{M}_{H_0^*}(x_0)\} \\ &\leq \sup\{B(z) : z \in F_\varepsilon\} \\ &\leq \beta + \varepsilon.\end{aligned}$$

On the other hand, it follows that

$$\begin{aligned}\hat{S}_{H_0^*}[A](x_0) &= \sup\{A(z) : z \in \hat{M}_{H_0^*}(x_0)\} \\ &\geq A(x_0) = \alpha > \beta + \varepsilon.\end{aligned}$$

Thus, we deduce that

$$\hat{S}_{H_0^*}[A](x_0) \neq \hat{S}_{H_0^*}[B](x_0).$$

This completes the proof. □

We remark that the above proof of Theorem 2.3 implies that Theorem 2.2 is logically equivalent to Theorem 2.1. Therefore, Theorem 2.3 is a generalization of Theorem 2.1 to the case of a linear space.

We conclude this section by giving a counterexample to Zadeh's result, which is different from the one presented in [**3**].

EXAMPLE 2.1. Let $X = \mathbb{R}^2$, let

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 \leq 1\} \subset X,$$

let

$$L = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1, 0 \leq x_1 < 1\} \subset K$$

and let $K_0 = K \backslash L$. Define a fuzzy set A in X by

$$A(x) = \begin{cases} 1, & \text{if } x \in K_0, \\ \frac{1}{2}, & \text{if } x \in L, \\ 0, & \text{if } x \notin K \end{cases}$$

for every $x \in X$ and define a fuzzy set B in X by

$$B(x) = \begin{cases} 1, & \text{if } x \in K_0, \\ \frac{1}{3}, & \text{if } x \in L, \\ 0, & \text{if } x \notin K \end{cases}$$

for every $x \in X$. Then, it is straightforward to see that both A and B are convex fuzzy sets in X. Further, it is obvious that for every hyperplane H_0 in X containing $0 \in X$ and every $x \in H_0$, $S_{H_0}[A](x) = S_{H_0}[B](x)$. Therefore, we evidently deduce that for every hyperplane H in X, $S_H[A] = S_H[B]$. However, $A \neq B$.

4. Summary and Further Problems

In this chapter, we have proved a theorem (Theorem 2.2) which is logically equivalent to the result of Takahashi and Takahashi (Theorem 2.1) and extended it to the case of a normed linear space (Theorem 2.3) without the completeness of the space. In the course of the discussion, we have introduced the new concept of a fuzzy set in a normed linear space corresponding to the shadows of fuzzy sets in a Hilbert space. This fuzzy set is defined on the whole space, whereas the shadows of fuzzy sets are defined on a closed hyperplane. With this in mind, we can present the following problems for topics of our next study.

PROBLEM 1. To define the shadow of fuzzy sets in a normed linear space.

PROBLEM 2. To improve the result of the case of a Hilbert space (Theorem 2.1 or Theorem 2.2) by methods used in the discussion of this chapter. For instance, to prove a theorem of the following form:

Let A and B be two convex fuzzy sets in a Hilbert space X. If both A and B (or A' and B') are closed and if $S_{H_0}[A] = S_{H_0}[B]$ for every closed hyperplane H_0 in X containing $0 \in X$, then $A = B$.

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CHAPTER 3

Fixed Point Theorems for Fuzzy Mappings

1. Introduction

The basic concern of the theory of fixed points for fuzzy mappings is to extend fixed point theorems for set-valued maps to the case of fuzzy mappings. So, let us begin with a brief survey of fixed point theorems for set-valued maps of contractive type on complete metric spaces. We shall denote by $H(K_1, K_2)$ the distance between nonempty bounded closed subsets K_1 and K_2 of a metric space (see Section 2, p. 26).

In [**26**], Nadler introduced the concept of contractive set-valued maps on a complete metric space and showed the following fixed point theorem which was a set-valued analogue of the well-known contraction principle of Banach.

THEOREM 3.1 (Nadler, [**26**]). *Let X be a complete metric space with metric d and let T be a set-valued map on X such that for every $x \in X$, Tx is a nonempty bounded closed subset of X. Suppose that there exists $k < 1$ such that $H(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.*

The above concept has been discussed expansively introducing various set-valued maps of contractive type. Amongst these, Papageorgiou [**27**] obtained the following theorem.

THEOREM 3.2 (Papageorgiou, [**27**]). *Let X be a complete metric space with metric d and let T be a continuous set-valued map on X such that for every $x \in X$, Tx is a nonempty bounded closed subset of X. Let $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$ be a function satisfying the following properties:*

(i) *For any $p, q \in \mathbb{R}_+^3$, $\phi(p) \leq \phi(q)$ if $p \leq q$;*
(ii) *there exists $k \in [0, 1)$ such that $\phi(t, t, t) \leq kt$ for all $t \geq 0$;*
(iii) *ϕ is upper semicontinuous,*

where $\mathbb{R}_+ = [0, \infty)$. Assume that for every $x, y \in X$,

$$H(Tx, Ty) \leq \phi\big(d(x, y), d(x, Tx), d(y, Ty)\big).$$

Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

For further investigation concerning fixed points of set-valued maps of this type, we may refer to [**3, 4, 25**] for instance.

On the other hand, the concept of contractive fuzzy mappings was introduced by Heilpern [**16**] as a fuzzy analogue of contractive set-valued maps. Since then, various fuzzy mappings of contractive type have been introduced and numerous fixed point theorems for them have been obtained by many authors [**6, 9, 10, 12, 21, 22, 23, 28, 29, 31**].

In [**19**], Kada, Suzuki and Takahashi introduced the concept of w-distances (see Section 2, p. 26), which was a generalized concept of distances, and improved Caristi's fixed point theorem [**8**], Ekeland's variational principle [**13**] and the nonconvex minimization theorem according to Takahashi [**34**]; see also [**32, 36**].

The purpose of this chapter is to discuss fixed point theorems for fuzzy mappings in connection with those for set-valued maps in the context of a generalization. In Section 3, we prove some theorems on complete metric spaces, which can be applied to guarantee the existence of fixed points of (almost) all the mappings of contractive type, both set-valued and fuzzy. In Section 4, we prove a fixed point theorem for fuzzy mappings in a complete metric space in terms of the concept of w-distances. In Section 5, we prove a fixed point theorem for fuzzy mappings on a locally convex linear topological space by using Fan's fixed point theorem [**14**] (see Section 2, p. 27).

2. Preliminaries

Throughout this chapter, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers and by $\mathbf{1}_K$ the characteristic function for an arbitrary set K. Let f be a function of a nonempty set X to $(-\infty, \infty\,]$. $\mathrm{dom}f$ denotes the set $\{x \in X : f(x) < \infty\}$ and f is said to be proper if $\mathrm{dom}f \neq \emptyset$.

Let X be a nonempty set. A fuzzy set in X is a function of X into $[0,1]$. Let X be a topological space and let A be a fuzzy set in X. The r-level set of A is a subset of X, which is denoted by A_r and is defined by $A_r = \{x \in X : A(x) \geq r\}$ for every $r \in (0,1]$ and by $A_0 = \mathrm{cl}\{x \in X : A(x) > 0\}$ for $r = 0$, respectively, where $\mathrm{cl}K$ denotes the closure of an arbitrary subset K of X. Then A is said to be convex if for each $r \in (0,1]$, A_r is a convex subset of X.

Let X, Y be nonempty sets and let $K(Y)$ denote a class of subsets of Y. Then, a mapping from X into $K(Y)$ is called a set-valued map from X into $K(Y)$. Let T be a set-valued map from X into $K(X)$. Then an element $x_0 \in X$ is called a fixed point of T if $x_0 \in Tx_0$.

Let $\mathcal{F}(X)$ denote the family of all fuzzy sets in X. Then a mapping from X into $\mathcal{F}(X)$ is called a fuzzy mapping on X. This enables us to consider each fuzzy mapping on X to be a two variable function from $X \times X$ into $[0, 1]$. In fact, let F be a two variable function from $X \times X$ into $[0, 1]$. Then, we can define a fuzzy mapping G on X by

$$Gx = F(x, \cdot)$$

for every $x \in X$. Inversely, if G is a fuzzy mapping on X, then we can define a two variable function from $X \times X$ into $[0, 1]$ by

$$F(x, y) = (Gx)(y)$$

for every $(x, y) \in X \times X$. So, it is natural to regard in the same light with a fuzzy mapping on X a two variable function from $X \times X$ into $[0, 1]$. Then, we shall think of the value at the first variable for x as the value of the fuzzy mapping itself at x. Let F be a fuzzy mapping on X. Then an element $x_0 \in X$ is said to be a fixed point of F if $F(x_0, x_0) = 1$.

Let X be a metric space with metric d. Then, for any $x \in X$ and any subset K of X, we define the distance from x to K by

$$d(x, K) = \inf\{d(x, y) : y \in K\}.$$

We shall denote by $\mathrm{CB}(X)$ the class of all nonempty bounded closed subsets of X. Then, for any $K_1, K_2 \in \mathrm{CB}(X)$, H denotes the Hausdorff distance with respect to d (see [**35, 36**] for instance), that is,

$$H(K_1, K_2) = \max\{\sup_{u \in K_1} d(u, K_2), \sup_{v \in K_2} d(v, K_1)\}.$$

Let $x, y \in X$ and let $K_1, K_2 \in \mathrm{CB}(X)$. Then we see at once that $d(x, K_1) \leq d(x, y) + d(y, K_1)$. Also, it is easy to check that $d(x, K_1) = 0$ if and only if $x \in K_1$ and that, if x belongs to K_1, then $d(x, K_2) \leq H(K_1, K_2)$. Moreover, we note that for every sequence $\{K_n\}_{n \in \mathbb{N}}$ of nonempty subsets of X, $\lim_{n \to \infty} d(x, K_n) = 0$ if and only if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of X such that $x_n \in K_n$ and $\lim_{n \to \infty} d(x_n, x) = 0$.

Let T be a set-valued map from X into $\mathrm{CB}(X)$. Then, T is said to be k-contractive if there exists $k < 1$ such that

$$H(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. If for any $x, y \in X$, $H(Tx, Ty) \leq d(x, y)$, T is called nonexpansive. Then, we know that, if T is nonexpansive or k-contractive, the real-valued function g on X defined by

$$g(x) = d(x, Tx)$$

for every $x \in X$ is continuous; see, for instance, [**35, 36**].

Let $p : X \times X \to [0, \infty)$ be a function. Then p is said to be a w-distance on X [**19**] if it satisfies the following:

(i) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(ii) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Some examples of w-distances were stated in [**19, 36**]. Further, the following result was obtained in [**19**].

PROPOSITION 3.1. *Let X be a metric space with metric d and let T be a continuous mapping from X into itself. Then a function $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \max\{d(Tx, y), d(Tx, Ty)\}$$

for every $x, y \in X$ is a w-distance on X.

Let X, Y be topological spaces and let T be a set-valued map from X into the class of subsets of Y such that for each $x \in X$, Tx is a nonempty closed subset of Y. Then, T is said to be upper semicontinuous on X if for every $x \in X$ and every open subset V of Y containing Tx, there exists a neighborhood U of x such that $Ty \subset V$ whenever $y \in U$.

We know the following; see [**35**] for instance.

PROPOSITION 3.2. *Let X be a topological space, let Y be a compact Hausdorff space and let T be a set-valued map from X into the class of all nonempty closed subsets of Y. Then, T is upper semicontinuous on X if and only if the graph of T, namely, the set*

$$G(T) = \{(x, y) \in X \times Y : y \in Tx\}$$

is a closed subset of $X \times Y$.

We also know the following Fan's fixed point theorem [**14**].

THEOREM 3.3 (Fan, [**14**]). *Let C be a compact convex subset of a locally convex linear topological space and let T be an upper semicontinuous set-valued map from C into the class of all nonempty closed convex subsets of C. Then T has a fixed point.*

3. Fixed Point Theorems for Fuzzy-mappings in Complete Metric spaces

We first prove the following fixed point theorem for fuzzy mappings on a complete metric space.

THEOREM 3.4. *Let X be a complete metric space with metric d and let T be a continuous mapping from X to itself. Let $f : X \to (-\infty, \infty]$ be a proper, lower semicontinuous function bounded from below and let*

F be a fuzzy mapping on X. Suppose that for each $x \in X$, there exists $y \in X$ such that $F(Tx, y) = 1$ or $F(Tx, Ty) = 1$ and $f(y) + \max\{d(Tx, y), d(Tx, Ty)\} \leq f(x)$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.

PROOF. Suppose that $F(x, x) \neq 1$ for all $x \in X$, and let u be an element in X such that $f(u) < \infty$. Then, we shall inductively define a sequence $\{u_n\}$, starting with $u_0 = u$. Suppose that $u_{n-1} \in X$ is known, and let

$$S_n = \Big\{w \in X : f(w) + \max\{d(w, Tu_{n-1}), d(Tw, Tu_{n-1})\} \leq f(u_{n-1})\Big\}.$$

Then, by our assumption, there exists $w \in S_n$ such that

$$w \neq Tu_{n-1} \text{ or } Tw \neq Tu_{n-1}.$$

Therefore, since

$$\begin{aligned} 0 &< \max\{d(w, Tu_{n-1}), d(Tw, Tu_{n-1})\} \\ &\leq f(u_{n-1}) - f(w) \\ &\leq f(u_{n-1}) - \inf_{x \in S_n} f(x), \end{aligned}$$

we can choose $u_n \in S_n$ such that

$$\text{(i)} \qquad f(u_n) \leq \inf_{x \in S_n} f(x) + \frac{1}{2}\{f(u_{n-1}) - \inf_{x \in S_n} f(x)\}.$$

We claim that this is a Cauchy sequence. Indeed, if $m > n$, we obtain the following three inequalities:

$$\begin{aligned} d(Tu_n, Tu_m) &\leq \sum_{k=n}^{m-1} d(Tu_k, Tu_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \{f(u_k) - f(u_{k+1})\} \\ \text{(ii)} \qquad &= f(u_n) - f(u_m), \\ \text{(iii)} \qquad d(u_m, Tu_{m-1}) &\leq f(u_{m-1}) - f(u_m), \end{aligned}$$

and

$$\begin{aligned} d(u_n, Tu_{m-1}) &\leq d(u_n, Tu_{n-1}) + \sum_{k=n}^{m-1} d(Tu_{k-1}, Tu_k) \\ &\leq f(u_{n-1}) - f(u_n) + \sum_{k=n}^{m-1} \{f(u_{k-1}) - f(u_k)\} \\ \text{(iv)} \qquad &= f(u_{n-1}) - f(u_n) + f(u_{n-1}) - f(u_{m-1}). \end{aligned}$$

Therefore, we deduce from (iii) and (iv) that

$$\begin{aligned} d(u_n, u_m) &\le d(u_n, Tu_{m-1}) + d(Tu_{m-1}, u_m) \\ &\le f(u_{n-1}) - f(u_n) + f(u_{n-1}) - f(u_{m-1}) + f(u_{m-1}) - f(u_m) \end{aligned}$$

and hence, that $\{u_n\}$ is a Cauchy sequence. Let $u_n \to v$. Then if $m \to \infty$ in (ii), we have that

$$\begin{aligned} d(Tu_n, Tv) &\le f(u_n) - \lim_{m\to\infty} f(u_m) \\ &\le f(u_n) - f(v). \end{aligned} \tag{v}$$

On the other hand, by our hypothesis, there exists $z \in X$ such that

$$F(Tv, z) = 1 \text{ or } F(Tv, Tz),$$

and

$$f(z) + \max\{d(z, Tv), d(Tz, Tv)\} \le f(v). \tag{vi}$$

We may assume that $z \ne Tv$ or $Tz \ne Tv$. Then, by (v) and (vi), we infer that

$$\begin{aligned} f(z) &\le f(v) - d(z, Tv) \\ &\le f(v) - d(z, Tv) + f(u_n) - f(v) - d(Tu_n, Tv) \\ &\le f(u_n) - f(z, Tu_n), \end{aligned}$$

and that

$$\begin{aligned} f(z) &\le f(v) - d(Tz, Tv) \\ &\le f(v) - d(Tz, Tv) + f(u_n) - f(v) - d(Tu_n, Tv) \\ &\le f(u_n) - f(Tz, Tu_n). \end{aligned}$$

Consequently, it follows from these two inequalities that

$$f(z) + \max\{d(z, Tu_n), d(Tz, Tu_n)\} \le f(u_n),$$

which implies that $z \in S_n$. Using (i), we have that

$$2f(u_n) - f(u_{n-1}) \le \inf_{x\in S_n} f(x) \le f(z)$$

and thus, that

$$f(z) < f(v) \le \lim_{n\to\infty} f(u_n) \le f(z).$$

This is a contradiction. Hence, there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$. $\square$

Putting $T = I$ (an identity operator on X) in Theorem 3.4, we immediately obtain the following corollary.

COROLLARY 3.1. *Let X be a complete metric space with metric d. Let $f : X \to (-\infty, \infty\,]$ a proper, lower semicontinuous function, bounded from below and let F be a fuzzy mapping on X. Suppose that for each $x \in X$, there exists $y \in X$ such that $F(x,y) = 1$ and $f(y)+d(x,y) \leq f(x)$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

Using Corollary 3.1, we can prove the following Nadler's fixed point theorem (Theorem 3.1, p. 24).

COROLLARY 3.2 ([**26**]). *Let X be a complete metric space with metric d and let T be a k-contractive set-valued map from X into* $\mathrm{CB}(X)$. *Then, T has a fixed point.*

PROOF. Let F be a fuzzy mapping on X defined by

$$F(x,y) = \mathbf{1}_{Tx}(y)$$

for every $x, y \in X$. Suppose that $d(x, Tx) > 0$ for all $x \in X$ and choose $\varepsilon > 0$ with $\varepsilon < \frac{1}{k} - 1$. Then for each $x \in X$, there exists $y \in Tx$ such that

$$d(x,y) \leq (1+\varepsilon) d(x, Tx).$$

Then, since

$$\begin{aligned} d(y, Ty) &\leq H(Tx, Ty) \\ &\leq kd(x,y) \\ &\leq k(1+\varepsilon)d(x,Tx), \end{aligned}$$

we infer that

$$\begin{aligned} d(x,Tx) - d(y,Ty) &\geq \frac{1}{1+\varepsilon} d(x,y) - k(x,y) \\ &= \left(\frac{1}{1+\varepsilon} - k\right) d(x,y). \end{aligned}$$

Thus, we deduce that for every $x \in X$, there exists $y \in X$ such that $F(x,y) = 1$ and $f(y) + d(x,y) \leq f(x)$, where f is a continuous real-valued function on X defined by

$$f(x) = \left(\frac{1}{1+\varepsilon} - k\right)^{-1} d(x, Tx)$$

for every $x \in X$. Therefore, by Corollary 3.1, there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$, that is, $d(x_0, Tx_0) = 0$. This is a contradiction. $\square$

For studying further, we next prove a fixed point theorem for fuzzy mappings in a complete metric space without the lower semicontinuity of the function f.

Let X be a metric space and let F be a fuzzy mapping on X. Then, for any $x \in X$, we denote by $[Fx]_1$ the 1-level set of the fuzzy set $F(x, \cdot) \in \mathcal{F}(X)$, that is, the subset $\{y \in X : F(x,y) = 1\}$ of X.

THEOREM 3.5. *Let X be a complete metric space with metric d, let $f : X \to (-\infty, \infty]$ be a proper function bounded from below and let F be a fuzzy mapping on X. Assume that for each $x \in X$, there is $y \in X$ with $F(x,y) = 1$ and $f(y) + d(x,y) \leq f(x)$. Further, assume that for any $z \in X$ with $F(z,z) \neq 1$, $\inf_{x \in X}\{d(x,z) + d(x, [Fx]_1)\} > 0$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

PROOF. Let u be an element of X such that $f(u) < \infty$, and we set $u_0 = u$. Then, we shall inductively define a sequence $\{u_n\}$ of elements of X, starting with u_0. Suppose that $u_{n-1} \in X$ is known. Then there exists $u_n \in X$ such that

$$F(u_{n-1}, u_n) = 1 \text{ and } f(u_n) + d(u_{n-1}, u_n) \leq f(u_{n-1}).$$

The latter inequality implies that $f(u_n) \leq f(u_{n-1})$, so that, the sequence $\{f(u_n)\}$ of elements of $\mathbb{R}$ is convergent. Let $r = \lim_{n\to\infty} f(u_n)$. We claim that the sequence $\{u_n\}$ is a Cauchy sequence. Indeed, if $m > n$, then

$$\begin{aligned} d(u_n, u_m) &\leq \sum_{k=n}^{m-1} d(u_k, u_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \{f(u_k) - f(u_{k+1})\} = f(u_n) - f(u_m) \end{aligned} \tag{vii}$$

and hence, $\{u_n\}$ is a Cauchy sequence. Let $u_n \to x_0$. Then, if $m \to \infty$ in (vii), we have that

$$d(u_n, x_0) \leq f(u_n) - r. \tag{viii}$$

On the other hand, the equality $F(u_{n-1}, u_n) = 1$ obviously implies that $d(u_{n-1}, [Fu_{n-1}]_1) \leq d(u_{n-1}, u_n)$. We shall show that $F(x_0, x_0) = 1$. In order to do it, let us suppose that $F(x_0, x_0) \neq 1$. Then, by hypothesis, (vii) and (viii), it follows that

$$\begin{aligned} 0 &< \inf_{x \in X}\{d(x, x_0) + d(x, [Fx]_1)\} \\ &\leq \inf_{n \in \mathbb{N}}\{d(u_n, x_0) + d(u_n, [Fu_n]_1)\} \\ &\leq \inf_{n \in \mathbb{N}}\{d(u_n, x_0) + d(u_n, u_{n+1})\} \\ &\leq \inf_{n \in \mathbb{N}}\{f(u_n) - r + f(u_n) - r\} = 0. \end{aligned}$$

This is a contradiction. Hence, we have that $F(x_0, x_0) = 1$. $\square$

Using Theorem 3.5, we obtain the following corollary which is a fuzzy analogue of Aubin's result [5].

COROLLARY 3.3 ([5]). *Let X be a complete metric space with metric d, let $f : X \to (-\infty, \infty]$ be a proper function bounded from below and let F be a fuzzy mapping on X such that the set $\{(x, y) \in X \times X : F(x, y) = 1\}$ is a nonempty closed subset of $X \times X$. Assume that for each $x \in X$, there is $y \in X$ with $F(x, y) = 1$ and $f(y) + d(x, y) \leq f(x)$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

PROOF. By Theorem 3.5, it is sufficient to show that for any $z \in X$ with $F(z, z) \neq 1$, $\inf_{x \in X}\{d(x, z) + d(x, [Fx]_1)\} > 0$. For doing it, suppose that there exist $z \in X$ and a sequence $\{x_n\}$ of elements of X such that

$$F(z, z) \neq 1 \text{ and } d(x_n, z) + d(x_n, [Fx_n]_1) \to 0$$

as $n \to \infty$. The latter implies that $d(x_n, z) \to 0$ and $d(x_n, [Fx_n]_1) \to 0$ as $n \to \infty$, and consequently, that $d(z, [Fx_n]_1) \to 0$ as $n \to \infty$. Therefore, we infer that there exists a sequence $\{y_n\}$ of elements of X such that $y_n \in [Fx_n]_1$ and $d(y_n, z) \to 0$ as $n \to \infty$. Hence, by the hypothesis of F, we have that $F(z, z) = 1$. This is a contradiction. □

Also, we prove the following theorem.

THEOREM 3.6. *Let X be a complete metric space with metric d and let F be a fuzzy mapping on X such that for each $x \in X$, $[Fx]_1$ belongs to $\mathrm{CB}(X)$. Assume that there exists $k \in [0, 1)$ such that for any $u_0 \in X$ and any $u_1 \in [Fu_0]_1$, there is $u_2 \in [Fu_1]_1$ with $d(u_1, u_2) \leq kd(u_0, u_1)$. Further, assume that for any $z \in X$ with $F(z, z) \neq 1$, $\inf_{x \in X}\{d(x, z) + d(x, [Fx]_1)\} > 0$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

PROOF. Suppose that $F(x, x) \neq 1$ for all $x \in X$, and choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{k} - 1$. Then, since $d(x, [Fx]_1) > 0$ for any $x \in X$, we infer that for each $x \in X$, there exists $y \in [Fx]_1$ such that $d(x, y) \leq (1 + \varepsilon)d(x, [Fx]_1)$. By hypothesis, we have $y_0 \in [Fy]_1$ with $d(y, y_0) \leq kd(x, y)$, which implies that $d(y, [Fy]_1) \leq kd(x, y)$. Then it follows that

$$d(x, [Fx]_1) - d(y, [Fy]_1) \geq \frac{1}{1+\varepsilon}d(x, y) - kd(x, y) = \left(\frac{1}{1+\varepsilon} - k\right) d(x, y).$$

Thus, we deduce that for each $x \in X$, there exists $y \in X$ such that $F(x, y) = 1$ and $f(y) + d(x, y) \leq f(x)$, where f is a real-valued function on X defined by

$$f(x) = \left(\frac{1}{1+\varepsilon} - k\right)^{-1} d(x, [Fx]_1)$$

for every $x \in X$. Hence, by Theorem 3.5, we have $x_0 \in X$ such that $F(x_0, x_0) = 1$. This is a contradiction. $\square$

By Theorem 3.6, we obtain the following result due to Papageorgiou [**27**] (Theorem 3.2, p. 24) without the continuity of a set-valued map T and the upper semicontinuity of the function ϕ of $\mathbb{R}^3_+$ to $\mathbb{R}_+$.

COROLLARY 3.4 ([**27**]). *Let X be a complete metric space with metric d and let T be a set-valued map from X into* $\mathrm{CB}(X)$. *Assume that for every $x, y \in X$,*

$$H(Tx, Ty) \leq \phi\big(d(x,y), d(x,Tx), d(y,Ty)\big),$$

where $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfies the following properties:

(i) *For any $p, q \in \mathbb{R}^3_+$, $\phi(p) \leq \phi(q)$ if $p \leq q$;*
(ii) *there exists $k \in [0,1)$ such that $\phi(t,t,t) \leq kt$ for all $t \geq 0$.*

Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

PROOF. Suppose that $x \notin Tx$ for every $x \in X$. Define a fuzzy mapping F on X by

$$F(x,y) = \mathbf{1}_{Tx}(y)$$

for every $x, y \in X$. Then we have $F(z,z) \neq 1$ for every $z \in X$. Let $u_0 \in X$ and $u_1 \in [Fu_0]_1$, and take $k_1 \in \mathbb{R}$ such that $k < k_1 < 1$. Then we have $u_0 \neq u_1$, $d(u_0, [Fu_0]_1) > 0$. Moreover, we have that

$$\begin{aligned}
d(u_1, [Fu_1]_1) &\leq H([Fu_0]_1, [Fu_1]_1) \\
&\leq \phi\big(d(u_0,u_1), d(u_0,Tu_0), d(u_1,Tu_1)\big) \\
&\leq \phi\big(d(u_0,u_1), d(u_0,u_1), d(u_1,Tu_1)\big) \\
&\leq k \max\big\{d(u_0,u_1), d(u_1,Tu_1)\big\} \\
&= k \max\big\{d(u_0,u_1), d(u_1,[Fu_1]_1)\big\} \\
&< k_1 \max\big\{d(u_0,u_1), d(u_1,[Fu_1]_1)\big\}
\end{aligned}$$

and hence, that $d(u_1, [Fu_1]_1) < k_1 d(u_0, u_1)$. Therefore, there is $u_2 \in [Fu_1]_1$ with $d(u_1, u_2) \leq k_1 d(u_0, u_1)$. Further, we deduce that for every $z \in X$, $\inf_{x \in X} \big\{d(x,z) + d(x, [Fx]_1)\big\} > 0$. For this, let us suppose that there exist $z \in X$ and a sequence $\{x_n\}$ of elements of X such that

$$d(x_n, z) + d(x_n, [Fx_n]_1) \to 0$$

as $n \to \infty$. Then, we have that $d(x_n, z) \to 0$ and $d(x_n, [Fx_n]_1) \to 0$ as $n \to \infty$ and thus, that $d(z, [Fx_n]_1) \to 0$ as $n \to \infty$. Therefore, we infer that there exists a sequence $\{y_n\}$ of elements of X such that $y_n \in [Fx_n]_1$

and $d(y_n, z) \to 0$ as $n \to \infty$. Since y_n belongs to $[Fx_n]_1$, it follows that

$$\begin{aligned} d(y_n, [Fz]_1) &\leq H([Fx_n]_1, [Fz]_1) \\ &\leq \phi\big(d(x_n, z), d(z, Tz), d(x_n, Tx_n)\big) \\ &\leq \phi\big(d(x_n, z), d(z, Tz), d(x_n, y_n)\big) \\ &\leq k\big(d(x_n, z) + d(z, y_n) + d(z, Tz)\big) \\ &= k\big(d(x_n, z) + d(z, y_n) + d(z, [Fz]_1)\big). \end{aligned}$$

Thus, we have that

$$\begin{aligned} 0 &< d(z, [Fz]_1) \\ &\leq d(z, y_n) + d(y_n, [Fz]_1) \\ &\leq (1+k)d(y_n, z) + kd(x_n, z) + kd(z, [Fz]_1). \end{aligned}$$

This implies that $0 < d(z, [Fz]_1) \leq kd(z, [Fz]_1)$. This is a contradiction. Hence, by Theorem 3.6, we have $x_0 \in X$ such that $F(x_0, x_0) = 1$. This is a contradiction. □

Furthermore, we obtain the following corollary; see **[22]**

COROLLARY 3.5 ([**22**]). *Let X be a complete metric space with metric d and let F be a fuzzy mapping on X such that for each $x \in X$, $[Fx]_1$ belongs to $\mathrm{CB}(X)$. Assume that for any $u_0 \in X$, any $u_1 \in [Fu_0]_1$ and any $v_0 \in X$, there exists $v_1 \in [Fv_0]_1$ such that $d(u_1, v_1) \leq a_1 d(u_0, u_1) + a_2 d(v_0, v_1) + a_3 d(v_0, u_1) + a_4 d(u_0, v_1) + a_5 d(u_0, v_0)$, where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and $a_3 \geq a_4$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

PROOF. Suppose that $F(x, x) \neq 1$ for every $x \in X$. Putting $r = \frac{a_1+a_4+a_5}{1-a_2-a_4}$, it is obvious that $0 \leq r < 1$. Let $u_0 \in X$ and let $u_1 \in [Fu_0]_1$. Then, by hypothesis, there exists $u_2 \in [Fu_1]_1$ such that

$$\begin{aligned} d(u_1, u_2) &\leq a_1 d(u_0, u_1) + a_2 d(u_1, u_2) \\ &\quad + a_3 d(u_1, u_1) + a_4 d(u_0, u_2) + a_5 d(u_0, u_1) \\ &\leq a_1 d(u_0, u_1) + a_2 d(u_1, u_2) \\ &\quad + a_4\big\{d(u_0, u_1) + d(u_1, u_2)\big\} + a_5 d(u_0, u_1). \end{aligned}$$

This implies that $d(u_1, u_2) \leq rd(u_0, u_1)$. Moreover, we show that for any $z \in X$, $\inf_{x \in X}\big\{d(x, z) + d(x, [Fx]_1)\big\} > 0$. If not, there exist $z \in X$ and a sequence $\{x_n\}$ of elements of X such that such that

$$d(x_n, z) + d(x_n, [Fx_n]_1) \to 0$$

as $n \to \infty$. Then we have $d(x_n, z) \to 0$, and $d(z, [Fx_n]_1) \to 0$ as $n \to \infty$. This implies the existence of a sequence $\{y_n\}$ of elements of X such that

$y_n \in [Fx_n]_1$, and $d(y_n, z) \to 0$ as $n \to \infty$. Then it follows from hypothesis that for any $n \in \mathbb{N}$, there exists $z_n \in [Fz]_1$ such that

$$\begin{aligned} d(y_n, z_n) &\leq a_1 d(x_n, y_n) + a_2 d(z, z_n) \\ &\quad + a_3 d(z, y_n) + a_4 d(x_n, z_n) + a_5 d(x_n, z) \\ &\leq a_1 \{d(x_n, z) + d(z, y_n)\} + a_2 \{d(z, y_n) + d(y_n, z_n)\} \\ &\quad + a_3 d(z, y_n) + a_4 \{d(x_n, z) + d(z, y_n) + d(y_n, z_n)\} \\ &\quad + a_5 d(x_n, z) \end{aligned}$$

and thus, that $d(y_n, z_n) \leq k\big(d(z, y_n) + d(x_n, z)\big)$, where $k = \frac{a_1+a_2+a_3+a_4+a_5}{1-a_2-a_4}$. Since $z_n \in [Fz]_1$ for any $n \in \mathbb{N}$, we infer that

$$0 < d(z, [Fz]_1) \leq d(z, z_n) \leq d(z, y_n) + d(y_n, z_n) \leq (k+1) d(z, y_n) + k d(x_n, z),$$

which implies that $0 < d(z, [Fz]_1) \leq 0$. This is a contradiction. Therefore, by Theorem 3.6, we have $x_0 \in X$ such that $F(x_0, x_0) = 1$. This is a contradiction. $\square$

We remark that in the applications of Theorem 3.4, the technical difficulties essentially arise in the proof of the lower semicontinuity of f. In fact, as we have observed above, if we negate each of the conclusions of the above corollaries, we can simply obtain a proper function f satisfying the assumption represented in the form of the inequality in Theorem 3.4. However, the verification of the lower semicontinuity of f is not so straightforward. This is a primary reason why we need to obtain a theorem (Theorem 3.5 or Theorem 3.6) without assuming that f is lower semicontinuous.

We conclude this section by showing an example which implies the generality of our results.

EXAMPLE 3.1. Let $X = [0, \infty)$ and let $A_n = [n, n+1)$ for every $n \in \mathbb{N} \cup \{0\}$. Let $f_0 : A_0 \to A_0$ be a function such that $\{0\} \cup f_0(A_0) \neq A_0$, and for every $n \in \mathbb{N}$, let $f_n : A_n \to A_n$ be a function given by $f_n(x) = f_0(x - n) + n$ for every $x \in X$. Then we define a fuzzy mapping $F : X \times X \to [0, 1]$ as follows:

$$F(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A_n \times A_n \text{ and } y \in \{n\} \cup \{f_n(x)\}, \\ \frac{1}{2}, & \text{if } (x, y) \in A_n \times A_n \text{ and } y \notin \{n\} \cup \{f_n(x)\}, \\ 0, & \text{if } (x, y) \in A_m \times A_n \text{ for } m \neq n. \end{cases}$$

Further, assume that there exists $r \in [0, 1)$ such that $|f_0(x) - f_0^2(x)| \leq r|x - f_0(x)|$ for all $x \in A_0$ and that for any $z \in X$ with $z \neq f_0(z)$, $\inf\limits_{x \in A_0} \{|x - z| + |x - f_0(x)|\} > 0$. Then, F satisfies the following two conditions:

(i) For every $u_0 \in X$ and every $u_1 \in [Fu_0]_1$, there exists $u_2 \in [Fu_1]_1$ such that $|u_1 - u_2| \leq r|u_0 - u_1|$;

(ii) for any $z \in X$ with $F(z,z) \neq 1$, $\inf_{x \in X}\{|x-z| + |x - [Fx]_1|\} > 0$.

PROOF. Note that $X = \bigcup_{n=0}^{\infty} A_n$. We first prove that F satisfies condition (i). Let $u_0 \in X$ and let $u_1 \in [Fu_0]_1$. Then, without loss of generality, we may suppose that $u_0 \in A_0$. We have that if $u_1 = 0$, then $[Fu_1]_1 = \{0\} \cup \{f_0(0)\}$ and that if $u_1 = f_0(u_0)$, then $[Fu_1]_1 = \{0\} \cup \{f_0(u_1)\}$. In the case where $u_1 = 0$, our claim is obvious since $u_1 \in [Fu_1]_1$. So, we may assume that $u_1 = f_0(u_0)$ and then, let $u_2 = f_0(u_1)$. By hypothesis, we infer that $|u_1 - u_2| = |f_0(u_0) - f_0^2(u_0)| \leq r|u_0 - f_0(u_0)| = r|u_0 - u_1|$. This is the desired conclusion.

Next, we show that F satisfies condition (ii). Indeed, if there exists $z \in X$ with $F(z,z) \neq 1$ such that $\inf_{x \in X}\{|x-z| + |x - [Fx]_1|\} = 0$, then we deduce that there exist $x_m \in X$ and $y_m \in [Fx_m]_1$ such that $|x_m - z| \to 0$ and $|x_m - y_m| \to 0$ as $m \to \infty$. By our assumption, we may also suppose that $0 < z < 1$. Then we infer that there exists $m_0 \in \mathbb{N}$ such that $0 < x_m < 1$ and $0 < y_m < 1$ for any $m \geq m_0$ and consequently, that $y_m \in \{0\} \cup \{f_0(x_m)\}$ for any $m \geq m_0$. This implies that $y_m = f_0(x_m)$ for any $m \geq m_0$ and hence, that $f_0(x_m) \to z$ as $m \to \infty$. We remark that $z \neq f_0(z)$. Therefore, by hypothesis, we have that $0 < \inf_{x \in A_0}\{|x-z|+|x-f_0(x)|\} \leq \inf_{m \geq m_0}\{|x_m - z| + |x_m - f_0(x_m)|\} = 0$. This is a contradiction. □

In the above example, let f_0 be given by

$$f_0(x) = \begin{cases} \frac{1}{2}x, & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ \frac{1}{2}x + \frac{1}{2}, & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

Then, by letting $r = \frac{1}{2}$, it is easy to check that F, which is defined as in the above example, satisfies the assumptions and that every element of the subset $\bigcup_{n=1}^{\infty}\{n\} \cup \{n+\frac{1}{2}\} \cup \{n+1\}$ of $\mathbb{R}$ is a fixed point of F. However, F does not satisfy the assumption in Corollary 3.5. In fact, let us suppose that there exist $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that for any $u_0 \in X$, any $u_1 \in [Fu_0]_1$ and any $v_0 \in X$, there exists $v_1 \in [Fv_0]_1$ such that $|u_1 - v_1| \leq a_1|u_0 - u_1| + a_2|v_0 - v_1| + a_3|v_0 - u_1| + a_4|u_0 - v_1| + a_5|u_0 - v_0|$. Letting $u_0 = \frac{1}{2}$ and $v_0 = 1$, we have $[Fu_0]_1 = \{0\} \cup \{\frac{1}{2}\}$ and $[Fv_0]_1 = \{1\}$. By our assumption, for $u_1 = \frac{1}{2} \in [Fu_0]_1$, we infer that there exists $v_1 \in [Fv_0]_1$ such that

$|u_1 - v_1| \leq a_1|u_0 - u_1| + a_2|v_0 - v_1| + a_3|v_0 - u_1| + a_4|u_0 - v_1| + a_5|u_0 - v_0|$ and consequently, that $|\frac{1}{2} - v_1| \leq a_2|1 - v_1| + \frac{1}{2}a_3 + a_4|\frac{1}{2} - v_1| + \frac{1}{2}a_5$. However, since $[Fv_0]_1 = \{1\}$, this implies that $a_2 + a_3 + a_4 + a_5 \geq 1$. This is a contradiction.

By the same way, we also observe that a set-valued map T on X defined by $Tx = [Fx]_1$ for every $x \in X$ does not satisfy the assumption in Corollary 3.4.

4. Fixed Point Theorems for Fuzzy Mappings in Terms of w-distances

In this section, by employing the concept of w-distances, we prove a fixed point theorem for fuzzy mappings, which generalizes Theorem 3.4 and the result of Kada, Suzuki and Takahashi [**19**]. Before doing it, we need to prove a lemma.

LEMMA 3.1. *Let X be a complete metric space with metric d and let $\{K_n\}$ be a sequence of nonempty closed subsets of X such that $K_n \supset K_{n+1}$ for all $n \in \mathbb{N}$. Assume that there exist a sequence $\{u_n\}$ of elements of X and a w-distance p on X such that*

$$\sup_{x,y \in K_n} \max\{p(u_n, x), p(u_n, y)\} \to 0$$

as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} K_n$ consists of one point.

PROOF. Put $\delta_{K_n} = \sup\{d(x,y) : x, y \in K_n\}$ for every $n \in \mathbb{N}$. Then it is sufficient to show that $\delta_{K_n} \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. The definition of a w-distance guarantees the existence of $\delta > 0$ such that $p(u,w) \leq \delta$ and $p(u,z) \leq \delta$ imply $d(w,z) \leq \varepsilon$. Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\sup_{x,y \in K_n} \max\{p(u_n, x), p(u_n, y)\} \leq \delta$. Then, it follows that for any $n \geq n_0$ and any $x, y \in K_n$, $p(u_n, x) \leq \delta$ and $p(u_n, y) \leq \delta$. This implies that for any $n \geq n_0$ and any $x, y \in K_n$, $d(x,y) \leq \varepsilon$. Hence, we have proved that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\delta_{K_n} \leq \varepsilon$. $\square$

THEOREM 3.7. *Let X be a complete metric space with metric d, let $f : X \to (-\infty, \infty]$ be a proper, lower semicontinuous function bounded from below and let F be a fuzzy mapping on X. Assume that there exists a w-distance p on X such that for each $x \in X$, there is $y \in X$ with $F(x,y) = 1$ and $f(y) + p(x,y) \leq f(x)$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$ and $p(x_0, x_0) = 0$.*

PROOF. Let u be an element of X such that $f(u) < \infty$ and put $u_0 = u$. Then we set

$$S_1 = \{w \in X : f(w) + p(u_0, w) \leq f(u_0)\}.$$

We shall construct a sequence $\{S_n\}$ of subsets of X by induction, starting with S_1. Suppose that $S_n \subset X$ is known. Then we take $u_n \in S_n$ such that

$$f(u_n) \leq \inf_{z \in S_n} f(z) + \frac{1}{n}$$

and set

$$S_{n+1} = \{w \in X : f(w) + p(u_n, w) \leq f(u_n)\}.$$

Then, since $u_n \in S_n$, it is clear that

$$f(u_n) \leq f(u_n) + p(u_{n-1}, u_n) \leq f(u_{n-1}),$$

so that, the sequence $\{f(u_n)\}$ of elements of $\mathbb{R}$ is convergent. We claim that $\bigcap_{n=1}^{\infty} S_n$ consists of one point. Indeed, S_n is obviously a nonempty closed subset of X for every $n \in \mathbb{N}$. Further, since

$$\begin{aligned} w \in S_{n+1} \ &\Rightarrow \ f(w) + p(u_n, w) \leq f(u_n) \\ &\Rightarrow \ f(w) + p(u_{n-1}, u_n) + p(u_n, w) \leq f(u_n) + p(u_{n-1}, u_n) \\ &\Rightarrow \ f(w) + p(u_{n-1}, w) \leq f(u_n) + p(u_{n-1}, u_n) \leq f(u_{n-1}) \\ &\Rightarrow \ w \in S_n, \end{aligned}$$

we have that $S_n \supset S_{n+1}$. Moreover, putting $\alpha_n = f(u_{n-1}) - \{f(u_n) - \frac{1}{n}\}$, it follows that for any $n \in \mathbb{N}$ and any $w \in S_n$,

$$p(u_{n-1}, w) \leq f(u_{n-1}) - f(w) \leq f(u_{n-1}) - \inf_{z \in S_n} f(z) \leq \alpha_n,$$

which implies that $\sup_{w,z \in S_n} \max\{p(u_{n-1}, w), p(u_{n-1}, z)\} \leq \alpha_n \to 0$ as $n \to \infty$. Thus, we deduce from Lemma 3.1 that $\bigcap_{n=1}^{\infty} S_n$ consists of one point.

Let $\bigcap_{n=1}^{\infty} S_n = \{x_0\}$ and we set

$$S = \{w \in X : f(w) + p(x_0, w) \leq f(x_0)\}.$$

Then there exists $w \in S$ such that $F(x_0, w) = 1$. Further, since

$$\begin{aligned} w \in S \ &\Rightarrow \ f(w) + p(x_0, w) \leq f(x_0) \\ &\Rightarrow \ f(w) + p(u_{n-1}, x_0) + p(x_0, w) \leq f(x_0) + p(u_{n-1}, x_0) \\ &\Rightarrow \ f(w) + p(u_{n-1}, w) \leq f(x_0) + p(u_{n-1}, x_0) \leq f(u_{n-1}) \\ &\Rightarrow \ w \in S_n, \end{aligned}$$

we have that $S \subset S_n$. This implies that $S = \{x_0\}$ and hence, that $F(x_0, x_0) = 1$ and $p(x_0, x_0) = 0$. This completes the proof. $\square$

Applying Theorem 3.7, we have the following, which was obtained in the previous section (see Theorem 3.4, p. 27).

COROLLARY 3.6. *Let X be a complete metric space with metric d and let T be a continuous mapping from X into itself. Let $f : X \to (-\infty, \infty]$ be a proper, lower semicontinuous function bounded from below and let F be a fuzzy mapping on X. Suppose that for each $x \in X$, there exists $y \in X$ such that $F(Tx, y) = 1$ or $F(Tx, Ty) = 1$ and $f(y) + \max\{d(Tx, y), d(Tx, Ty)\} \leq f(x)$. Then there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$.*

PROOF. Since T is continuous, a function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) = \max\{d(Tx, y), d(Tx, Ty)\}$$

for every $x, y \in X$ is a w-distance on X. Let

$$S(x) = \{w \in X : F(Tx, w) = 1\} \cup \{w \in X : F(Tx, Tw) = 1\}$$

for every $x \in X$, and define a fuzzy mapping G on X by

$$G(x, y) = \mathbf{1}_{S(x)}(y)$$

for every $x, y \in X$. Then it follows that for each $x \in X$, there exists $y \in X$ such that $G(x, y) = 1$ and $f(y) + p(x, y) \leq f(x)$. Therefore, by Theorem 3.7, there exists $x_0 \in X$ such that $G(x_0, x_0) = 1$ and $p(x_0, x_0) = 0$, that is, $F(Tx_0, x_0) = 1$ or $F(Tx_0, Tx_0) = 1$, and $Tx_0 = x_0$. Hence, we have $x_0 \in X$ such that $F(x_0, x_0) = 1$. $\square$

We also have the following corollary proved in [**19**], which was a generalization of the result of Takahashi [**34**].

COROLLARY 3.7 ([**19**]). *Let X be a complete metric space with metric d and let $f : X \to (-\infty, \infty]$ be a proper, lower semicontinuous function bounded from below. Assume that there exists a w-distance p on X such that for each $x \in X$ with $\inf_{u \in X} f(u) < f(x)$, there is $y \in X$ with $y \neq x$ and $f(y) + p(x, y) \leq f(x)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{u \in X} f(u)$.*

PROOF. Suppose that $\inf_{u \in X} f(u) < f(x)$ for all $x \in X$. Let

$$S(x) = \{w \in X : w \neq x\}$$

for every $x \in X$ and define a fuzzy mapping F on X by

$$F(x, y) = \mathbf{1}_{S(x)}(y)$$

for every $x, y \in X$. Then we deduce from hypothesis that for each $x \in X$, there is $y \in X$ with $F(x,y) = 1$ and $f(y) + p(x,y) \leq f(x)$. Therefore, by Theorem 3.7, we have $x_0 \in X$ such that $F(x_0, x_0) = 1$, that is, $x_0 \in S(x_0)$. This is a contradiction. Hence, there exists $x_0 \in X$ such that $f(x_0) = \inf_{u \in X} f(u)$. $\square$

5. Fixed Point Theorems for Fuzzy Mappings in Locally Convex Linear Topological Spaces

In this section, we prove a fixed point theorem for fuzzy mappings on a locally convex linear topological space by using Fan's fixed point theorem [**14**] (Theorem 3.3, p. 27).

THEOREM 3.8. *Let C be a compact convex subset of a locally convex linear topological space and let F be a fuzzy mapping on C such that for every $x \in C$, $F(x, \cdot)$ is a convex fuzzy set in C. Suppose that for each $x \in C$, there exists $y \in C$ such that $F(x,y) = 1$ and that the function $F : C \times C \to [0,1]$ is upper semicontinuous on $C \times C$. Then there exists $x_0 \in C$ such that $F(x_0, x_0) = 1$.*

PROOF. Define a set-valued map T on C by

$$Tx = \{y \in C : F(x,y) = 1\}$$

for every $x \in C$. Then for each $x \in C$, Tx is a nonempty closed convex subset of C. Further, putting

$$G(T) = \{(x,y) \in C \times C : F(x,y) = 1\},$$

$G(T)$ is a closed subset of $C \times C$. In fact, let $(x_\alpha, y_\alpha) \to (x,y)$ and $(x_\alpha, y_\alpha) \in G(T)$. Then, since

$$1 \geq F(x,y) \geq \inf_{\alpha} \sup_{\beta \geq \alpha} F(x_\beta, y_\beta) = 1,$$

we have that $(x,y) \in G(T)$. This implies that $G(T)$ is closed in $C \times C$ and hence, that T is upper semicontinuous on C. Therefore, by Fan's fixed point theorem, there exists $x_0 \in C$ such that $x_0 \in Tx_0$, that is, $F(x_0, x_0) = 1$. This completes the proof. $\square$

Further, we can also show that Theorem 3.8 is a generalization of Fan's fixed point theorem as follows: Let C be a compact convex subset of a locally convex linear topological space and let T be an upper semicontinuous set-valued map from C into the class of all nonempty closed convex subsets of C. Define a fuzzy mapping F on C by

$$F(x,y) = \mathbf{1}_{Tx}(y)$$

for every $x, y \in C$. Then F is upper semicontinuous. In fact, if $\alpha \leq 0$, then the set

$$\{(x,y) \in C \times C : F(x,y) \geq \alpha\}$$

is obviously $C \times C$ and if $\alpha > 1$, then the set is empty. Moreover, if $0 < \alpha \leq 1$, we infer that $\{(x,y) \in C \times C : F(x,y) \geq \alpha\} = G(T)$ is a closed subset of $C \times C$. Therefore F is upper semicontinuous. Hence, by Theorem 3.8, there exists $x_0 \in C$ such that $F(x_0, x_0) = 1$, that is, $x_0 \in Tx_0$.

6. Summary and Further Problems

In this chapter, we have proved some generalized fixed point theorems for fuzzy mappings, both in a complete metric space and in a locally convex linear topological space. In connection with the previous papers on these topics [**6, 9, 10, 11, 12, 20, 21, 22, 23, 28, 29, 31**] and the references therein, we can present the following problem for a topic of our next study.

PROBLEM 1. To prove common fixed point theorems for families of fuzzy mappings of contractive type on complete metric spaces.

Further, we may state the following.

PROBLEM 2. To construct (fuzzy) standard mathematical models to which our theorems can be applied.

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CHAPTER 4

Convexity of Fuzzy-Valued Maps and Minimization Theorems

1. Introduction

The concept of fuzzy numbers was introduced by Dubois and Prade [**4**] as a natural way of dealing mathematically with indefiniteness or impreciseness of our judgment about the objects (data) under consideration. This was a fuzzy analogue of the concept of real numbers and led to optimization problems with fuzzy constraints of the form (see Section 4 of the present chapter, pp. 53-55):

> C being a subset (describing the constraints by fuzzy numbers) of a linear space and F being a mapping defined on C with value of fuzzy numbers, find $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$,

where $\preceq$ denotes the partial order relation on the set of all fuzzy numbers (see Section 2, p. 46).

So far, the problems, which can be considered to be fuzzy analogues of linear programming problems, have been studied by many authors [**3, 5, 11, 12, 13**] to a considerable extent because, by employing certain useful concepts (L-R fuzzy numbers for instance; see Section 3, p. 51), these can be transformed to ordinary (nonfuzzy) linear cases.

However, regarding the problems formulated from a standpoint of convex analysis, much more still remain to be done. For instance, minimization theorems for fuzzy-valued maps, which may imply the existence of solutions to those problems, have yet to be proved as of now; see also [**9**].

With this in mind, we study, in this chapter, the convexity of fuzzy-valued maps and establish some minimization theorems for fuzzy-valued maps. In Section 3, we prove some theorems connected with the convexity of fuzzy-valued maps on a convex subset of a linear space. Then we shall have a necessary and sufficient condition for fuzzy-valued maps to be convex, which shall be the fundamental result throughout the rest of this chapter. In Section 4, after a brief introduction of optimization problems with fuzzy constraints, we define the lower semicontinuity of fuzzy-valued maps on a topological space and obtain a sufficient condition for them to be lower semicontinuous. Next, we establish some minimization theorems for lower semicontinuous fuzzy-valued maps on a compact subset of a topological space in connection with the existence of solutions to the above mentioned problems.

2. Preliminaries

Throughout this chapter, all linear spaces are real, and we denote by $\mathbb{R}$ the set of real numbers and by $\mathbf{1}_K$ the characteristic function for an arbitrary set K. We shall also use the letter $\mathbb{R}$ to denote the real line. Let X be a linear topological space and let C, D be subsets of X. Then $\mathrm{cl}C$ denotes the closure of C, and there is defined $C + D = \{c + d : c \in C,\ d \in D\}$ and $\lambda C = \{\lambda c : c \in C\}$ for any $\lambda \in \mathbb{R}$.

Let A be a fuzzy set in $\mathbb{R}$. We denote by A_r the r-level set of A, which is a subset of $\mathbb{R}$ and is defined by $A_r = \{x \in \mathbb{R} : A(x) \geq r\}$ for every $r \in (0, 1]$ and $A_0 = \mathrm{cl}\{x \in \mathbb{R} : A(x) > 0\}$ for $r = 0$, respectively. Then A is said to be convex (respectively closed) if for every $r \in (0, 1]$, A_r is a convex (respectively closed) subset of $\mathbb{R}$.

Let A and B be fuzzy sets in $\mathbb{R}$. By Zadeh's extension principle [**17**], we define the addition $\oplus$ to yield a fuzzy set $A \oplus B$ in $\mathbb{R}$ by

$$(A \oplus B)(z) = \sup_{z=x+y, x,y\in\mathbb{R}} \min\big(A(x), B(y)\big)$$

for all $z \in \mathbb{R}$. Also, we define the multiplication $\odot$ to yield a fuzzy set $A \odot B$ in $\mathbb{R}$ by

$$(A \odot B)(z) = \sup_{z=xy, x,y\in\mathbb{R}} \min\big(A(x), B(y)\big)$$

for all $z \in \mathbb{R}$. Then $[A \oplus B]_r$ and $[A \odot B]_r$ denote the r-level sets of $A \oplus B$ and $A \odot B$ for every $r \in [0, 1]$, respectively. Note that, if both A and B are convex, then $A \oplus B$ is convex; see [**10**].

Let A be a fuzzy set in $\mathbb{R}$. A is said to be a fuzzy number [**4, 7**] if it satisfies the following conditions:

(i) A is convex;
(ii) there exists a unique real number $m \in \mathbb{R}$ such that $A(m) = 1$;
(iii) A_0 is a bounded subset of $\mathbb{R}$.

Then the symbol $\mathcal{F}$ denotes the set of fuzzy numbers. Note that for any $A \in \mathcal{F}$, A_0 is a compact convex subset of $\mathbb{R}$ and that for every $A, B \in \mathcal{F}$, $A \oplus B$ belongs to $\mathcal{F}$.

Let $A, B \in \mathcal{F}$. We define an order relation $\preceq$ on $\mathcal{F}$ [**11**] as follows:

$A \preceq B$ if and only if $\sup A_r \leq \sup B_r$ and $\inf A_r \leq \inf B_r$ for all $r \in [0, 1]$.

Note that the order relation $\preceq$ satisfies the axioms of a partial order relation. Let $A \in \mathcal{F}$ and let $\lambda \in \mathbb{R}$. For the sake of convenience, we shall write $A \preceq \lambda$ instead of $A \preceq \mathbf{1}_{\{\lambda\}}$. Similarly, we shall write $A \oplus \lambda$ (respectively λA, $A\lambda$) for $A \oplus \mathbf{1}_{\{\lambda\}}$ (respectively $\mathbf{1}_{\{\lambda\}} \odot A$, $A \odot \mathbf{1}_{\{\lambda\}}$). Then we observe that if $\lambda \neq 0$, then $(\lambda A)(z) = A(\frac{z}{\lambda})$ for all $z \in \mathbb{R}$ and that $(0A)(z) = \mathbf{1}_{\{0\}}(z)$ for all $z \in \mathbb{R}$. Consequently, it follows that $[\lambda A]_r = \lambda A_r$ for all $r \in [0, 1]$ and hence, that $\lambda A \in \mathcal{F}$.

Let C be a convex subset of a linear space and let f be a real-valued function on C. f is said to be convex if $f\big(\lambda x + (1 - \lambda)y\big) \leq \lambda f(x) + (1 - \lambda)f(y)$ for any $x, y \in C$ and any $\lambda \in (0, 1)$. f is called concave if $-f$ is convex. Moreover, f is said to be quasi-concave if for every $c \in \mathbb{R}$, $\{x \in C : f(x) \geq c\}$ is a convex subset of C. Let C, I be nonempty sets and let φ be a real-valued function on $C \times I$. Then φ is called concavelike in its second variable if for any $y_1, y_2 \in I$ and any $\lambda \in (0, 1)$, there exists $y_0 \in I$ such that $\varphi(x, y_0) \geq \lambda\varphi(x, y_1) + (1 - \lambda)\varphi(x, y_2)$ for all $x \in C$. We know the following Fan's minimax theorem [**6**].

THEOREM 4.1 (Fan, [**6**]). *Let C be a compact convex subset of a linear topological space, let I be a nonempty set and let φ be a real-valued function on $C \times I$ satisfying the following conditions*:

(i) *For each $y \in I$, the function $x \mapsto \varphi(x, y)$ is lower semicontinuous and convex*;

(ii) *φ is concavelike in its second variable.*

Then the following holds:

$$\sup_{y\in I} \min_{x\in C} \varphi(x, y) = \min_{x\in C} \sup_{y\in I} \varphi(x, y).$$

3. Convexity of Fuzzy-Valued Maps

Let X be a nonempty set. A mapping $F : X \to \mathcal{F}$ is called a fuzzy-valued map if for every $x \in X$, $F(x)$ belongs to $\mathcal{F}$. Let C be a convex subset of a linear space and let F be a fuzzy-valued map on C. F is said to be convex [**9**] if for every $x, y \in C$ and every $\lambda \in (0, 1)$,

$$F\big(\lambda x + (1 - \lambda)y\big) \preceq \lambda F(x) \oplus (1 - \lambda)F(y).$$

In this section, we study the convexity of fuzzy-valued maps on a linear space. We first have the following two lemmas.

LEMMA 4.1. *For every $A \in \mathcal{F}$, we have*

$$\lim_{r\to+0} \sup A_r = \sup A_0 \textit{ and } \lim_{r\to+0} \inf A_r = \inf A_0.$$

PROOF. It is easy to see that for every $r_1, r_2 \in (0, 1]$ with $r_1 > r_2$, $A_{r_1} \subset A_{r_2} \subset A_0$. So, we have that $\sup A_{r_1} \leq \sup A_{r_2} \leq \sup A_0$.

Therefore, since A_0 is a nonempty bounded subset of $\mathbb{R}$, $\lim_{r\to+0}\sup A_r = \sup_{r>0}\sup A_r$ exists. Put $\eta = \lim_{r\to+0}\sup A_r$ and $\xi = \sup A_0$. Then, by virtue of the above, it is clear that $\eta \leq \xi$. Conversely, let us take any $x \in \mathbb{R}$ with $A(x) > 0$. Then, by choosing $r_0 > 0$ such that $A(x) > r_0 > 0$, we observe that x belongs to A_{r_0} and hence, that $x \leq \sup A_{r_0} \leq \sup_{r>0}\sup A_r = \eta$. Consequently, since $x \in \mathbb{R}$ with $A(x) > 0$ is arbitrary, we deduce that for every $y \in A_0$, $y \leq \eta$. This implies that $\xi \leq \eta$. Thus, we have proved that $\xi = \eta$. Similarly, we show that $\lim_{r\to+0}\inf A_r = \inf A_0$. □

LEMMA 4.2. *For every $A \in \mathcal{F}$ and every $r \in (0,1]$, we have*

$$\lim_{\delta\to r-0} A_\delta = \inf_{\delta<r}\sup A_\delta = \sup A_r \text{ and } \lim_{\delta\to r-0} A_\delta = \sup_{\delta<r}\inf A_\delta = \inf A_r.$$

PROOF. Let $r \in (0,1]$ be fixed arbitrarily and take any $\delta > 0$ with $\delta < r$. Since $A_r \subset A_\delta$, we see at once that $\lim_{\delta\to r-0} A_\delta = \inf_{\delta<r}\sup A_\delta$ exists. Set $\xi = \sup A_r$ and $\eta = \inf_{\delta<r}\sup A_\delta$. Then, by the above observation, the inequality $\xi \leq \eta$ evidently follows. In order to prove that the inverse inequality holds, let us assume that $\xi < \eta$. Then there exists $x_0 \in \mathbb{R}$ such that $\xi < x_0 < \eta$. We infer that

$$\inf A_\delta \leq \inf A_r \leq \sup A_r = \xi < x_0 < \eta = \inf_{\delta<r}\sup A_\delta \leq \sup A_\delta$$

and hence, by convexity of a subset A_δ of $\mathbb{R}$, that x_0 belongs to A_δ. Since $\delta < r$ is arbitrary, this implies that $x_0 \in \bigcap_{\delta<r} A_\delta = A_r$. Consequently, we have that $x_0 \leq \sup A_r = \xi < x_0$. This is a contradiction. So, we deduce that $\xi = \eta$. By the same way, we have that $\lim_{\delta\to r-0}\inf A_\delta = \sup_{\delta<r}\inf A_\delta = \inf A_r$. □

By Lemmas 4.1and 4.2, we obtain the following lemma.

LEMMA 4.3. *For every $A, B \in \mathcal{F}$ and every $r \in [0,1]$, we have*

$$\sup[A \oplus B]_r = \sup A_r + \sup B_r \text{ and } \inf[A \oplus B]_r = \inf A_r + \inf B_r.$$

PROOF. Let $A, B \in \mathcal{F}$. We know that $A \oplus B$ belongs to $\mathcal{F}$. Let us fix an arbitrary $r \in (0,1]$. Then the inequality $\sup[A\oplus B]_r \geq \sup A_r + \sup B_r$ is obvious. Indeed, if $z \in A_r + B_r$, then there exist $x_0 \in A_r$ and $y_0 \in B_r$ such that $z = x_0 + y_0$. Since $A(x_0) \geq r$ and $B(y_0) \geq r$, we infer that

$$\begin{aligned}(A \oplus B)(z) &= \sup_{z=x+y,x,y\in\mathbb{R}} \min\big(A(x), B(y)\big)\\ &\geq \min\big(A(x_0), B(y_0)\big) \geq r\end{aligned}$$

and hence, that $z \in [A \oplus B]_r$. This implies that $[A \oplus B]_r \supset A_r + B_r$ and consequently, that $\sup[A \oplus B]_r \geq \sup(A_r + B_r) = \sup A_r + \sup B_r$.

We next claim that the inverse inequality holds. Let $z \in [A \oplus B]_r$ and choose any $\delta > 0$ with $\delta < r$. Since $(A \oplus B)(z) > \delta$, there exist $x_0, y_0 \in \mathbb{R}$ such that $\min\big(A(x_0), B(y_0)\big) \geq \delta$ and $z = x_0 + y_0$. This implies that $z \in A_\delta + B_\delta$ and consequently, that $[A \oplus B]_r \subset A_\delta + B_\delta$. Hence, we have that $\sup[A \oplus B]_r \leq \sup A_\delta + \sup B_\delta$. Therefore, since $\delta > 0$ with $\delta < r$ is arbitrary, it follows from Lemma 4.2, that $\sup[A \oplus B]_r \leq \sup A_r + \sup B_r$.

Thus, we deduce that $\sup[A \oplus B]_r = \sup A_r + \sup B_r$ for all $r \in (0, 1]$. Further, by applying Lemma 4.1 to both sides of the above equality, we have that $\sup[A \oplus B]_0 = \sup A_0 + \sup B_0$. By the same method, we prove that $\inf[A \oplus B]_r = \inf A_r + \inf B_r$ for all $r \in [0, 1]$. Hence, we have proved the lemma. □

Let F be a fuzzy-valued map on a nonempty set X. Then $\big[F(x)\big]_r$ denotes the r-level set of $F(x) \in \mathcal{F}$ for every $x \in X$ and every $r \in [0, 1]$.

We now prove the following theorem.

THEOREM 4.2. *Let C be a convex subset of a linear space and let F be a convex fuzzy-valued map on C. Then, for any $r \in [0, 1]$, the real-valued function f_r on C defined by $f_r(x) = \sup\big[F(x)\big]_r$ for every $x \in C$ is convex.*

PROOF. Let $x, y \in C$, let $\lambda \in (0, 1)$ and let $r \in [0, 1]$. We know that $\big[\lambda F(x)\big]_r = \lambda\big[F(x)\big]_r$ and that $\lambda F(x) \in \mathcal{F}$. Since F is convex, it is obvious that

$$\begin{aligned} f_r\big(\lambda x + (1-\lambda)y\big) &= \sup\Big[F\big(\lambda x + (1-\lambda)y\big)\Big]_r \\ &\leq \sup\big[\lambda F(x) \oplus (1-\lambda)F(y)\big]_r. \end{aligned}$$

Moreover, we infer by Lemma 4.3 that

$$\begin{aligned} \sup\big[\lambda F(x) \oplus (1-\lambda)F(y)\big]_r &= \sup\big[\lambda F(x)\big]_r + \sup\big[(1-\lambda)F(y)\big]_r \\ &= \sup \lambda\big[F(x)\big]_r + \sup(1-\lambda)\big[F(y)\big]_r \\ &= \lambda \sup\big[F(x)\big]_r + (1-\lambda)\sup\big[F(y)\big]_r. \end{aligned}$$

Hence, we deduce that for any $r \in [0, 1]$,

$$f_r\big(\lambda x + (1-\lambda)y\big) \leq \lambda f_r(x) + (1-\lambda) f_r(y).$$

This completes the proof. □

By the same way, we prove the following theorem.

THEOREM 4.3. *Let C be a convex subset of a linear space and let F be a convex fuzzy-valued map on C. Then, for any $r \in [0, 1]$, the real-valued function g_r on C defined by $g_r(x) = \inf\big[F(x)\big]_r$ for every $x \in C$ is convex.*

Let $A, B \in \mathcal{F}$ and let us assume that both A and B are closed, that is, for every $r \in [0, 1]$, both A_r and B_r is a closed subset of $\mathbb{R}$. Then we know that $[A \oplus B]_r = A_r + B_r$ for all $r \in [0,1]$, so that, the proof of Theorems 4.2 and 4.3 is straightforward; see [**10**]. However, if both A and B are not closed, we can not use this assertion. Indeed, the following example implies that if A is not closed, then there exists $r \in [0,1]$ such that $[A \oplus B]_r \neq A_r + B_r$.

EXAMPLE 4.1. Define $A \in \mathcal{F}$ by

$$A(x) = \begin{cases} 0, & \text{if } x < -1, \\ 1 - |x|, & \text{if } -1 \leq x < \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \leq x, \end{cases}$$

and define $B \in \mathcal{F}$ by

$$B(x) = \begin{cases} 0, & \text{if } x < -\frac{3}{2}, \\ 1 - |x+1|, & \text{if } -\frac{3}{2} \leq x \leq 0, \\ 0, & \text{if } 0 < x. \end{cases}$$

Then we observe that $(A \oplus B)(0) = \sup_{x \in \mathbb{R}} \min\big(A(x), B(-x)\big) = \frac{1}{2}$ and consequently, $0 \in [A \oplus B]_{\frac{1}{2}}$. On the other hand, since $A_{\frac{1}{2}} = [-\frac{1}{2}, \frac{1}{2})$ and $B_{\frac{1}{2}} = [-\frac{3}{2}, -\frac{1}{2}]$, it follows that $0 \notin [-2, 0) = A_{\frac{1}{2}} + B_{\frac{1}{2}}$. Therefore, we have that $[A \oplus B]_{\frac{1}{2}} \neq A_{\frac{1}{2}} + B_{\frac{1}{2}}$.

Let F be a fuzzy-valued map on a nonempty set X and let $r \in [0,1]$. Throughout the rest of this chapter, f_r^F and g_r^F denote the real-valued functions on X defined by $f_r^F(x) = \sup\big[F(x)\big]_r$ for every $x \in X$ and by $g_r^F(x) = \inf\big[F(x)\big]_r$ for every $x \in X$, respectively.

Using Lemma 4.3, we also have the following theorem.

THEOREM 4.4. *Let C be a convex subset of a linear space and let F be a fuzzy-valued map on C. Assume that for every $r \in [0,1]$, both f_r^F and g_r^F are convex. Then F is convex.*

PROOF. Let $x, y \in C$, let $\lambda \in (0,1)$ and let $r \in [0,1]$. By Lemma 4.3, we infer that

$$\begin{aligned} \sup\big[\lambda F(x) \oplus (1-\lambda)F(y)\big]_r &= \sup\big[\lambda F(x)\big]_r + \sup\big[(1-\lambda)F(y)\big]_r \\ &= \lambda \sup\big[F(x)\big]_r + (1-\lambda)\sup\big[F(y)\big]_r \\ &= \lambda f_r^F(x) + (1-\lambda) f_r^F(y). \end{aligned}$$

Therefore, it follows from hypothesis that

$$\begin{aligned} \sup\Big[F\big(\lambda x + (1-\lambda)y\big)\Big]_r &= f_r^F\big(\lambda x + (1-\lambda)y\big) \\ &\leq \lambda f_r^F(x) + (1-\lambda) f_r^F(y) \\ &= \sup\big[\lambda F(x) \oplus (1-\lambda)F(y)\big]_r. \end{aligned}$$

Similarly, we deduce that

$$\inf\Big[F\big(\lambda x + (1-\lambda)y\big)\Big]_r \le \inf\big[\lambda F(x) \oplus (1-\lambda)F(y)\big]_r.$$

Hence, we have proved the theorem. □

We remark that Theorems 4.2, 4.3 and 4.4 give the characterization of the convexity of a fuzzy-valued map F in the sense that F is convex if and only if for every $r \in [0,1]$, both f_r^F and g_r^F are convex.

For further comprehension, let us employ the concept of L-R fuzzy numbers [**11**].

Let S be a function from $\mathbb{R}$ to $(-\infty, 1]$. S is called a shape function [**7**] if it satisfies the following conditions:

(i) For every $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R} : S(x) \ge \alpha\}$ is a convex subset of $\mathbb{R}$;
(ii) $S(x) = 1$ if and only if $x = 0$;
(iii) the set $\{x \in \mathbb{R} : S(x) > 0\}$ is a bounded subset of $\mathbb{R}$;
(iv) $S(x) = S(-x)$ for all $x \in \mathbb{R}$.

Then, for any shape function S and any $r \in (0,1]$, S_r and S_0 denote the subsets $\{x \in \mathbb{R} : S(x) \ge r\}$ and $\mathrm{cl}\{x \in \mathbb{R} : S(x) > 0\}$ of $\mathbb{R}$, respectively. Moreover, for every $r \in [0,1]$, we put $k_r^S = \sup S_r \in [0, \infty)$.

Let S, T be shape functions, let $m \in \mathbb{R}$ and let $\alpha, \beta \ge 0$. Then the L-R fuzzy number μ is a fuzzy number defined by

$$\mu(x) = \begin{cases} \max\big(S(\frac{x-m}{\alpha}), 0\big), & \text{if } x \le m, \\ \max\big(T(\frac{x-m}{\beta}), 0\big), & \text{if } x \ge m. \end{cases}$$

Then we shall say that μ is generated by the shape functions S and T. Further, we shall denote the L-R fuzzy number μ by

$$\mu = (m, \alpha, \beta)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation.

We should mention that the above definition of μ includes the cases where $\alpha = 0$ and $\beta > 0$ for instance. In these cases, we define μ by

$$\mu(x) = \begin{cases} \mathbf{1}_{\{m\}}(x), & \text{if } x \le m, \\ \max\big(T(\frac{x-m}{\beta}), 0\big), & \text{if } x \ge m. \end{cases}$$

Further, in the case where $\alpha, \beta = 0$, μ is defined by $\mu = \mathbf{1}_{\{m\}}$.

As an immediate consequence of Theorem 4.4, we obtain the following theorem regarding the convexity of fuzzy-valued maps whose range consist of L-R fuzzy numbers generated by the same shape functions.

THEOREM 4.5. *Let C be a convex subset of a linear space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0, \infty)$ be functions and let*

$S, T : \mathbb{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an L-R *fuzzy number denoted by*

$$F(x) = \big(m(x), \alpha(x), \beta(x)\big)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation. Assume that both m and β are convex and that α is concave. Then F is convex.

PROOF. Without loss of generality, we may assume that $S(x) \geq 0$ and $T(x) \geq 0$ for all $x \in \mathbb{R}$. Let $x \in C$ and let $r \in [0, 1]$. We observe that

$$\big[F(x)\big]_r = \big(m(x) + \alpha(x)S_r\big) \cup \big(m(x) + \beta(x)T_r\big)$$

and therefore, that

$$f_r^F(x) = \sup\big[F(x)\big]_r = m(x) + \sup T_r\beta(x) = m(x) + k_r^T\beta(x).$$

Similarly, we have that $g_r^F(x) = \inf\big[F(x)\big]_r = m(x) + \inf S_r\alpha(x)$. Further, we infer by the definition of a shape function that $k_r^S + \inf S_r = \sup S_r + \inf S_r = 0$ and consequently, that $g_r^F(x) = m(x) - k_r^S\alpha(x)$. Hence, since $k_r^S, k_r^T \geq 0$, we deduce from the assumptions of m, α and β that both f_r^F and g_r^F are convex. Thus, by Theorem 4.4, we have proved that F is convex. □

We next present the following result relating to the comparison of two L-R fuzzy numbers of the same type, which was essentially stated in [**7**, **11**].

PROPOSITION 4.1 ([**7**, **11**]). *Let $m, n \in \mathbb{R}$, let $\alpha, \beta, \gamma, \delta \geq 0$ and let S, T be shape functions. Then, for two* L-R *fuzzy numbers A and B denoted respectively by $A = (m, \alpha, \beta)_{\mathrm{L}_S\mathrm{R}_T}$ and $B = (n, \gamma, \delta)_{\mathrm{L}_S\mathrm{R}_T}$ in the form of a parametric representation, $A \preceq B$ if and only if $\sup A_1 \leq \sup B_1$, $\sup A_0 \leq \sup B_0$ and $\inf A_0 \leq \inf B_0$.*

PROOF. For the sake of completeness, we give a proof. Since the "only if part" is obvious by the definition of the order relation $\preceq$, it suffices to prove the "if part". Let $r \in [0, 1]$. We observe that $\sup A_r = m + k_r^T\beta$ and $\sup B_r = n + k_r^T\delta$. Therefore, since $\sup A_1 = m$ and $\sup B_1 = n$, we infer that

$$\begin{aligned} k_0^T(\sup A_r - \sup B_r) &= k_0^T\big(m + k_r^T\beta - (n + k_r^T\delta)\big) \\ &= k_0^T(m - n) + k_r^T k_0^T(\beta - \delta) \\ &= (k_0^T - k_r^T)(\sup A_1 - \sup B_1) \\ &\quad + k_r^T(\sup A_0 - \sup B_0). \end{aligned}$$

It is obvious that $k_0^T \geq k_r^T \geq 0$, so that, we have that $\sup A_r \leq \sup B_r$. By the same way, we deduce that $\inf A_r \leq \inf B_r$. This completes the proof. □

Applying Lemma 4.3 and Proposition 4.1, we simply obtain the following result, which was essentially presented in [**7**].

PROPOSITION 4.2 ([**7**]). *Let C be a convex subset of a linear space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0, \infty)$ be functions and let $S, T : \mathbb{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an* L-R *fuzzy number denoted by*

$$F(x) = \big(m(x), \alpha(x), \beta(x)\big)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation. Then, F is convex if and only if f_1^F, f_0^F and g_0^F are convex.

PROOF. Let $x, y \in C$ and let $\lambda \in (0, 1)$. By Lemma 4.3 and Proposition 4.1, we infer that $F\big(\lambda x + (1 - \lambda)y\big) \preceq \lambda F(x) \oplus (1 - \lambda)F(y)$ if and only if $f_1^F\big(\lambda x + (1 - \lambda)y\big) \le \lambda f_1^F(x) + (1 - \lambda)f_1^F(y)$, $f_0^F\big(\lambda x + (1 - \lambda)y\big) \le \lambda f_0^F(x) + (1 - \lambda)f_0^F(y)$ and $g_0^F\big(\lambda x + (1 - \lambda)y\big) \le \lambda g_0^F(x) + (1 - \lambda)g_0^F(y)$. This completes the proof. $\square$

4. Minimization Theorems for Fuzzy-Valued Maps

In this section, we establish some minimization theorems for fuzzy-valued maps on a compact topological space. First, let us begin with a brief introduction of optimization problems with fuzzy constraints, which were discussed in the previous papers (see [**3, 5, 11**] for instance).

Let $X = \mathbb{R}^n$, let $E = \mathbb{R}^n_+ = [0, \infty)^n \subset X$, let $c_1, c_2, \ldots, c_n \in \mathbb{R}$ and let f be a real-valued function on E defined by

$$f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$. Let $a_{ij} \in \mathbb{R}$ for every $i = 1, 2, \ldots, m$ and every $j = 1, 2, \ldots, n$, let $g_1, g_2, \ldots, g_m$ be real-valued functions on E defined respectively by

$$g_i(x) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$ and every $i = 1, 2, \ldots, m$ and let

$$E_0 = \big\{x \in E : g_i(x) \le b_i, i = 1, 2, \ldots, m\big\},$$

where $b_i \in \mathbb{R}$. Then, as a usual linear programming problem, we know the following:

Find $x_0 \in E_0$ such that $f(x_0) \le f(x)$ for all $x \in E_0$.

However, in actual problems, the coefficients of the constraint functions g_i are often fuzzy because of, for instance, vagueness or impreciseness of our judgment (estimation, evaluation or measurement) about the data. The concept of fuzzy numbers might be applied to these cases.

Let S, T be shape functions and let A_{ij} be L-R fuzzy numbers denoted by $A_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{\mathrm{L}_S\mathrm{R}_T}$ in the form of a parametric representation for every $i = 1, 2, \ldots, m$ and every $j = 1, 2, \ldots, n$, where $\alpha_{ij}, \beta_{ij} \geq 0$. Then we define fuzzy-valued maps $G_1, G_2, \ldots, G_m$ on E respectively by

$$G_i(x) = A_{i1}x_1 \oplus A_{i2}x_2 \oplus \cdots \oplus A_{in}x_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$ and every $i = 1, 2, \ldots, m$ and let

$$E_1 = \{x \in E : G_i(x) \preceq b_i, i = 1, 2, \ldots, m\}.$$

Then, as a contingent plan, the following fuzzy optimization problem can be stated:

Find $x_0 \in E_1$ such that $f(x_0) \leq f(x)$ for all $x \in E_1$.

In the above, the objective function f is nonfuzzy, whereas the constraint functions are fuzzy. So, it could be appropriate to consider that the objective function is also fuzzy, namely, a fuzzy-valued map on E. In that situation, we might for instance employ a fuzzy-valued map F on E instead of f defined by

$$F(x) = C_1x_1 \oplus C_2x_2 \oplus \cdots \oplus C_nx_n$$

for every $x = (x_1, x_2, \ldots, x_n) \in E$, where $C_1, C_2, \ldots, C_n$ are L-R fuzzy numbers denoted respectively by

$$C_i = (c_i, \gamma_i, \delta_i)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation for every $i = 1, 2, \ldots, n$, where $\gamma_i, \delta_i \geq 0$. Consequently, the following minimization problem has been presented:

Find $x_0 \in E_1$ such that $F(x_0) \preceq F(x)$ for all $x \in E_1$.

It may be said from the above discussion that these problems are fuzzy analogues of linear programming problems. Naturally, the following optimization problem in terms of fuzzy-valued maps arises as a fuzzy analogue of convex problems of the generalized form; see also [**9, 14**]:

> C being a convex subset of a linear space and F being a convex fuzzy-valued map on C, find $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$.

In the rest of this chapter, we shall prove some theorems in reference to the above problems. First of all, let us give a definition of the lower semicontinuity of fuzzy-valued maps on a topological space.

Let F be a fuzzy-valued map on a topological space X. F is said to be lower semicontinuous on X if for every $r \in [0,1]$, both f_r^F and g_r^F are lower semicontinuous on X.

Next, concerning the concept of L-R fuzzy numbers, we prove the following theorem, which gives a sufficient condition for fuzzy-valued maps on a topological space to be lower semicontinuous.

THEOREM 4.6. *Let C be a nonempty subset of a topological space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0,\infty)$ be functions and let $S, T : \mathbb{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an* L-R *fuzzy number denoted by*

$$F(x) = \big(m(x), \alpha(x), \beta(x)\big)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation. Assume that both m and β are lower semicontinuous on C and that α is upper semicontinuous on C. Then F is lower semicontinuous on C.

PROOF. Let $x \in C$ and let $r \in [0,1]$. Since $f_r^F(x) = m(x) + k_r^T \beta(x)$, we infer by the assumptions of m, β that f_r is lower semicontinuous on C. Similarly, by the equation $g_r^F(x) = m(x) - k_r^S \alpha(x)$, it follows from the assumption of α that g_r is lower semicontinuous on C. This completes the proof. ☐

Moreover, we need the following definition and notation. Let X, I be a nonempty set and let $\{h_i : i \in I\}$ be a family of real-valued functions X. $x_0 \in X$ is said to be a common minimizer of $\{h_i : i \in I\}$ in X if $h_i(x_0) \le h_i(x)$ for every $x \in X$ and every $i \in I$. Let F be a fuzzy-valued map on X. We denote by $\mathcal{P}^F$ the family $\{f_r^F, g_r^F : r \in [0,1]\}$ of real-valued functions on X. Then, for any $x_0 \in X$, it is easy to check that $F(x_0) \preceq F(x)$ for every $x \in X$ if and only if x_0 is a common minimizer of $\mathcal{P}^F$ in X.

Now, we establish the following minimization theorem.

THEOREM 4.7. *Let F be a lower semicontinuous fuzzy-valued map on a compact topological space X. Suppose that every finite subfamily of $\mathcal{P}^F$ has a common minimizer in X. Then there exists $x_0 \in X$ such that $F(x_0) \preceq F(x)$ for every $x \in X$.*

PROOF. For every $r \in [0,1]$, let $C_r = \{x \in X : f_r^F(x) = \inf_{y \in X} f_r^F(y)\}$ and let $D_r = \{x \in X : g_r^F(x) = \inf_{y \in X} g_r^F(y)\}$. We observe that both C_r and D_r are nonempty closed subsets of X. Further, we infer by hypothesis that the family $\{C_r \cap D_r : r \in [0,1]\}$ has finite intersection property. Therefore, we deduce that $\bigcap_{r\in[0,1]} (C_r \cap D_r) \neq \emptyset$. This implies that there exists $x_0 \in X$ such that for every $x \in X$ and every $r \in [0,1]$, $f_r^F(x_0) \leq f_r^F(x)$ and $g_r^F(x_0) \leq g_r^F(x)$, that is, x_0 is a common minimizer of $\mathcal{P}^F$ in X. Hence, we have $x_0 \in X$ such that $F(x_0) \preceq F(x)$ for every $x \in X$. This completes the proof. □

Further, we obtain the following minimization theorem for lower semicontinuous and convex fuzzy-valued maps on a compact convex subset of a linear topological space.

THEOREM 4.8. *Let C be a compact convex subset of a linear topological space and let F be a lower semicontinuous and convex fuzzy-valued map on C. Let φ be a real-valued function on $C \times \mathcal{P}^F$ defined by $\varphi(x,h) = h(x) - \min_{u\in C} h(u)$ for every $(x,h) \in C \times \mathcal{P}^F$. Suppose that φ is concavelike in its second variable. Then there exists $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$.*

PROOF. By the hypothesis of F, we deduce from Theorems 4.2 and 4.3 that for every $h \in \mathcal{P}^F$, the function $x \mapsto \varphi(x,h)$ is convex. Therefore, we infer that φ satisfies the assumptions of minimax theorem. Hence, we have that

$$\sup_{h\in\mathcal{P}^F} \min_{x\in C} \varphi(x,h) = \min_{x\in C} \sup_{h\in\mathcal{P}^F} \varphi(x,h)$$

and consequently, that $\min_{x\in C} \sup_{h\in\mathcal{P}^F} \varphi(x,h) = 0$. This implies that there exists $x_0 \in C$ such that $\varphi(x_0,h) \leq 0$ for all $h \in \mathcal{P}^F$, that is, $h(x_0) \leq \min_{u\in C} h(u)$ for all $h \in \mathcal{P}^F$. This completes the proof. □

Moreover, we prove the following two theorems relating to the concept of L-R fuzzy numbers.

Let F be a fuzzy-valued map on a nonempty set X. The symbol $\mathcal{P}_0^F$ denotes the family $\{f_1^F, f_0^F, g_0^F\}$ of real-valued functions on X.

THEOREM 4.9. *Let C be a nonempty subset of a topological space, let m be a real-valued function on C, let $\alpha, \beta : C \to [0,\infty)$ be functions and let $S, T : \mathbb{R} \to (-\infty, 1]$ be shape functions. Let F be a fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an L-R fuzzy number denoted by*

$$F(x) = \big(m(x), \alpha(x), \beta(x)\big)_{\mathrm{L}_S\mathrm{R}_T}$$

in the form of a parametric representation. Then there exists $x_0 \in X$ such that $F(x_0) \preceq F(x)$ for all $x \in X$ if and only if there exists $x_0 \in X$ such that x_0 is a common minimizer of $\mathcal{P}_0^F$ in X.

PROOF. Let $x_0 \in X$. By Proposition 4.1, it is easy to verify that $F(x_0) \preceq F(x)$ for all $x \in X$ if and only if for every $x \in X$, $f_1^F(x_0) \leq f_1^F(x)$, $f_0^F(x_0) \leq f_0^F(x)$ and $g_0^F(x_0) \leq g_0^F(x)$, that is, x_0 is a common minimizer of $\mathcal{P}_0^F$ in X. This completes the proof. □

THEOREM 4.10. *Let C be a compact convex subset of a linear topological space, let F be a convex fuzzy-valued map on C such that for every $x \in C$, $F(x)$ is an* L-R *fuzzy number generated respectively by the same shape functions. Suppose that for every $h \in \mathcal{P}_0^F$, h is lower semicontinuous on C and that the real-valued function φ_0 on $C \times \mathcal{P}_0^F$ defined by $\varphi_0(x, h) = h(x) - \min\limits_{u \in C} h(u)$ for every $(x, h) \in C \times \mathcal{P}_0^F$ is concavelike in its second variable. Then there exists $x_0 \in C$ such that $F(x_0) \preceq F(x)$ for all $x \in C$.*

PROOF. Applying minimax theorem, we have that

$$\sup_{h \in \mathcal{P}_0^F} \min_{x \in C} \varphi_0(x, h) = \min_{x \in C} \sup_{h \in \mathcal{P}_0^F} \varphi_0(x, h).$$

Consequently, we deduce that there exists $x_0 \in C$ such that $\varphi_0(x_0, h) \leq 0$ for all $h \in \mathcal{P}_0^F$, that is, $h(x_0) \leq \min\limits_{u \in C} h(u)$ for all $h \in \mathcal{P}_0^F$. Hence, by Theorem 4.9, the statement ensues. □

By the above discussion, we remark that, when we impose on a fuzzy-valued map F being "L-R fuzzy-valued", we can confine ourselves to the family $\mathcal{P}_0^F$.

We conclude this section by providing an example of fuzzy-valued maps satisfying the assumptions in Theorem 4.10.

EXAMPLE 4.2. Let C be a compact convex subset of a linear topological space, let $m : C \to \mathbb{R}$ be a lower semicontinuous and convex function, let $\alpha, \beta \geq 0$ and let $S, T : \mathbb{R} \to (-\infty, 1]$ be shape functions. By virtue of Theorems 4.5 and 4.6, let us define a lower semicontinuous and convex fuzzy-valued map F on C by the relation

$$F(x) = \big(m(x), \alpha, \beta\big)_{\mathrm{L}_S\mathrm{R}_T}$$

for every $x \in C$. Then we infer that for every $r \in [0, 1]$, $f_r^F(x) = m(x) + k_r^T \beta$ and $g_r^F(x) = m(x) - k_r^S \alpha$ and therefore, that φ_0 is concavelike in its second variable.

5. Summary and Further Problems

In this chapter, we have dealt with the concepts of fuzzy numbers and fuzzy-valued maps in connection with fuzzy optimization problems and proved some minimization theorems for fuzzy-valued maps in the framework of nonlinear analysis, in particular, convex analysis. So, in connection with ordinary convex problems, we can present the following problem as a topic of our next study.

PROBLEM 1. To introduce the concept of derivatives of fuzzy-valued maps.

Moreover, we have observed that problems on the solutions of fuzzy optimization problems are logically equivalent to those on the common minimum solutions of a family of real-valued functions (see Section 4, p. 55). Therefore, we can also state the following.

PROBLEM 2. To study mathematical areas to which our theorems or our methods used in the proofs may be applied.

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