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ON A SIMPLIFIED ESTIMATE OF CORRELOGRAM

BY MITUAKI HUZII

CONTENTS

CHAPTER 1

INTRODUCTION

Let X(t), $t \in (-\infty, \infty)$, be a real-valued weakly stationary process and we assume EX(t) = 0 for simplicity. We denote

$$EX(t)^2 = \sigma^2 \quad , \quad EX(t)X(t+h) = \sigma^2 \rho_h \; .$$

This paper is a report of the study on a simplified estimate of correlogram ρ_h.

Let us assume that the process X(t) is observed at t = 1, 2, ..., N + h. Then we usually estimate ρ_h by the value which consists of the statistic

$$\frac{1}{N} \sum_{n=1}^{N} X(n) X(n+h) \; .$$

But in this paper, we consider another estimate for ρ_h. Let us consider this attempt from another point of view. In many fields of applied statistics, ρ_h is evaluated not only by the usual method, but also by the simplified methods. The latter can be obtained by means of the modifications of the former on the physical considerations : These modifications give many conveniences to use them. It seems that these simplified methods contain many interesting problems in mathematical and statistical point of view, but such investigation has scarcely been carried out. We aim at the study of a simplified method from the mathematical and statistical standpoint.

The main point of the construction of the simplified estimate treated in this paper is that the term $X(n)X(n+h)$ in the estimate is replaced by the term $X(n)\,\mathrm{sgn}(X(n+h))$, where sgn(y) means 1, 0, -1 correspondingly as $y > 0$, $y = 0$, $y < 0$. We shall discuss the statistical properties of the estimate constructed as the above. The contents of this paper are as follows.

In Chapters 2 and 3, we deal with the case that σ^2 is known.

It is shown that

$$\gamma_k = \sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \frac{1}{N} \sum_{n=1}^{N} X(n)\, sgn\,(X(n+k))$$

is an unbiased estimate of ρ_k, when X(n) is a Gaussian process. In Chapter 2, we discuss the statistical properties of γ_k for a simple Markov process and, in Chapter 3, we discuss them for a general process satisfying some conditions. The crux of this discussion is how to evaluate the variance of γ_k. In Chapter 2, we evaluate it directly by the Markov property and, in Chapter 3, we evaluate it by the projection based on the linear regression. It is also shown that γ_k is a consistent estimate, when $\lim_{k\to\infty} \rho_k = 0$. Further the variance of γ_k is compared, for the typical cases, with that of the usual estimate

$$\tilde{\gamma}_k = \frac{1}{\sigma^2} \frac{1}{N} \sum_{n=1}^{N} X(n)X(n+k) .$$

Chapter 4 deals with the simplified estimation of ρ_k when σ^2 is unknown. We shall discuss it about the case that N is sufficiently large. Here, the statistic

$$\Gamma_k = \frac{\sum_{n=1}^{N} X(n)\, sgn\,(X(n+k))}{\sum_{n=1}^{N} |X(n)|}$$

comes into problem. We show that the distibution function of $\sqrt{N}\,(\Gamma_k - \rho_k)$ tends to the Gaussian distribution function with mean zero as $N \longrightarrow \infty$, if X(n) is a Gaussian process having a finite moving average representation and satisfying some other conditions. And we evaluate the asymptotic variance of $\sqrt{N}\,(\Gamma_k - \rho_k)$. These results are shown by using mainly a kind of extended central limit theorem and the similar method to the one in Chapter 3. Concerning the usual estimate

$$\tilde{\Gamma}_k = \frac{\sum_{n=1}^{N} X(n)X(n+k)}{\sum_{n=1}^{N} |X(n)|^2} ,$$

we can find analogous results to those of Γ_k. For typical cases, we shall compare the asymptotic variance of $\sqrt{N}\,(\Gamma_k - \rho_k)$ with that of $\sqrt{N}(\tilde{\Gamma}_k - \rho_k)$.

In Chapter 5, we shall discuss the bias of γ_k when the condition that X(n) is a Gaussian process is not satisfied. We consider this problem in the meaning that X(n) departs slightly from a Gaussian process. First of all, a question arises as to how we define a process which departs slightly from a Gaussian process. We define this process by putting a few conditions on the simultaneous distribution of X(n) and $X(n+k)$ for any n. This definition aims at the orthogonal development, using the Hermite polynomials, of the density of the simultaneous distribution function. The bias of γ_k is expressed by the coefficients, which appear in the orthogonal development and indicate the degree of the departure from the Gaussian distribution. These coefficients are also expressed by moments.

Non-Gaussian cases are systematically considered in Chapter 6. Here, we mainly discuss the variance of γ_k for the processes which depart from a Gaussian process. The definition of non-Gaussian processes in Chapter 5 is not sufficient for this purpose. We define them by the same idea as in Chapter 5, but more strictly and more systematically. This definition is related to the orthogonal development, using the Hermite polynomials in $L_2(R^k)$, of the density of the k-dimensional simultaneous distribution function of $(X(n_1), X(n_2), \cdots, X(n_k))$. The influence of the departure on the variance is expressed by the coefficients in this orthogonal development, which indicate the degree of the departure from Gaussian property and can be represented by moments. And the variance is shown in the forms connected with the results in Chapters 2 and 3. In this chapter, we make discussions generally as much as possible so that we may use these methods in case of discussions about the properties of the other statistic related to a stationary process departing from a Gaussian process.

The method of constructing the estimate of correlogram in this paper is originally found in Takahasi and Husimi's paper [13]. Takahasi and Husimi have used this method to determine the period and the decrement of a vibrating system exposed to irregular forces.

They have treated the process X(n) satisfying the relation

$$\frac{d^2 x(t)}{dt^2} + 2\lambda \frac{dx(t)}{dt} + \omega^2 x(t) = \frac{dB(t)}{dt},$$

where B(t) is a Brownian motion process. They say that this method saves greatly the labour of the calculator. They have applied this method, for example, to the analysis of the atmospheric pressure and have shown the comparison between this method and the usual method for practical data. This result agrees with ours.

Recently, the analogous attempts have been proposed in the engineering field. Correlators have been made on the similar principle to the above (see Imai [2]). It is shown that the circuits for these correlators become very simple. Also in the field of fishery science, the statistic

$$\frac{1}{N} \sum_{t=1}^{N} x(t)\, \mathrm{sgn}\, (x(t+k))$$

is employed to simplify calculations of the covariance in analysis of ship's rolling records by Kawashima [5] .

Some other simplified methods of the estimation of correlogram are as follows.

(i) A correlator based on the principle slightly different from ours is presented in Imai [2] . We assume that X(t) is a strongly stationary process with

$$E X(t) = 0 \quad \text{and} \quad E X(t)^2 < +\infty .$$

For a sufficiently small positive number δ , A is a positive number such that

$$E X(t)^2 - E X(t)^2 \chi_A (X(t)) < \delta ,$$

where

$$\chi_A(x) = \begin{cases} 1, & \text{if } |x| < A, \\ 0, & \text{if } |x| \geqq A. \end{cases}$$

Let $U_1, U_2, \cdots, U_N$ be independent random variables, each having a rectangular distribution on the interval $(-A, A)$. Furthermore, U_j $(j = 1, 2, \cdots, N)$ is assumed to be independent of X(t) for any t. Now let us consider the statistic

$$\gamma_k^{(1)} = \frac{1}{N}\frac{A}{\sigma^2}\sum_{t=1}^{N} X(t)\, sgn\left(X(t+k) + U_t\right),$$

when $EX(t)^2 = \sigma^2$ is known.

Then we have

$$E(\gamma_k^{(1)}) \doteqdot \rho_k. \qquad \text{------} (1)$$

$\gamma_k^{(1)}$ is not an unbiased estimate of ρ_k.

A merit of this estimate is that the relation (1) holds for any strictly stationary process satisfying some conditions. We shall compare the variance of $\gamma_k^{(1)}$ with that of our estimate γ_k for the cases treated in Chapter 3.

It holds

$$E(\gamma_k^{(1)2}) = \frac{1}{N^2}\frac{A^2}{\sigma^4}\left\{\sum_{t=1}^{N} EX(t)^2 \right.$$

$$\left. + \sum_{\substack{t=1 \\ (t \neq s)}}^{N}\sum_{s=1}^{N} EX(t)X(s)\, sgn\left(X(t+h)+U_t\right) sgn\left(X(s+h)+U_s\right)\right\}$$

$$\doteqdot \frac{1}{N^2}\frac{A^2}{\sigma^4}\left\{N\sigma^2 + \frac{1}{A^2}\sum_{\substack{t=1 \\ (t \neq s)}}^{N}\sum_{s=1}^{N} EX(t)X(s)X(t+k)X(s+h)\right\}.$$

Therefore, we find that the variance of $\gamma_k^{(1)}$ is generally greater than that of γ_k (see Chapter 3).

(ii) Let X(t), $EX(t) = 0$, be a stationary process such that

$$|X(t)| < A,$$

where A is a constant. When $EX(t)^2 = \sigma^2$ is known, we shall construct the statistic

$$\gamma_k^{(2)} = \frac{1}{N} \frac{A^2}{\sigma^2} \cdot \sum_{t=1}^{N} Sgn\left(X(t) + Y_t\right) Sgn\left(X(t+h) + Z_t\right),$$

where Y_1, Y_2 -----, Y_N and Z_1, Z_2, ----, Z_N are mutually independent random variables, each of which is uniformly distributed over (-A, A) and independent of X(t) for any t.

Then it is shown that $\gamma_k^{(2)}$ is an unbiased estimate of ρ_k. (see Jespers. Chu and Fettweis [3], Stato and Kawarada [12] and Veltman and Kwakernaak [14]). A strong point of this estimate is that the unbiasedness holds for any weakly stationary process, provided that

$$|X(t)| < A .$$

The processes treated in this discussion are not same as ours, so we can not compare two estimates from the viewpoint of estimation theory.

The similar method is proposed in Sato [11] . Let X(t) be a weakly stationary process such that

$$0 < X(t) < A .$$

We assume that Y_1, Y_2, ----, Y_N and Z_1, Z_2, ----, Z_N are mutually independent random variables, each of which has a rectangular distribution on (0, A) and is independent of X(t) for any t. Let us consider the random variables

$$X_t = X(t) - Y_t \quad \text{and} \quad X_{t+h} = X(t+h) - Z_t, \quad t = 1, 2, ----, N.$$

And we define the random variable Qt as follows :

$$Q_t = \begin{cases} 1, & \text{if } X_t > 0 \text{ and } X_{t+k} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then the statistic

$$\gamma_k^{(3)} = \frac{A^2}{N} \sum_{t=1}^{N} Q_t$$

is an unbiased estimate of $E\,X(t)\,X(t+k)$.

(iii) Let X(t) be a stationary Gaussian process with mean zero. We shall calculate the value

$$\gamma_k^{(4)} = \frac{1}{N}\sum_{t=1}^{N} sgn\left(X(t)\right) sgn\left(X(t+k)\right) .$$

Then we have

$$E(\gamma_k^{(4)}) = \frac{2}{\pi} \sin^{-1} \rho_k$$

(see Ekre [1], Kashiwagi and Kataoka [4], McFadden [6] and Veltman and Kwakernaak [14]).
This is called the polarity correlation.

The similar method is also employed in Ruchkin [10]. Let X(t) be a stationary ergodic Gaussian process with mean zero.
In this case, the sampling intervals are assumed to be such that the samples are independent. Then the random variable C_t is defined as follows,

$$C_t = \begin{cases} 1, & \text{if } X(t) \text{ and } X(t+k) \text{ have the same polarity,} \\ 0, & \text{if } X(t) \text{ and } X(t+k) \text{ have opposite polarities.} \end{cases}$$

Now we shall construct the statistic

$$P_N = \frac{1}{N}\sum_{t=1}^{N} C_t .$$

Then we have

$$E(P_N) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1}(\rho_k) = \frac{1}{\pi}\cos^{-1}(-\rho_k) .$$

So let us consider the statistic

$$\gamma_k^{(5)} = -\cos(\pi P_N)$$

for the estimation of ρ_k. It is shown that

$$E\gamma_k^{(5)} \cong \rho_k \exp\left[-\pi^2(EP_N)(1-(EP_N))/2N\right]$$

$$Var.(\gamma_k^{(5)}) \cong \frac{\pi^2(EP_N)(1-(EP_N))}{2N}\left(1-\cos 2\pi(EP_N)\right)$$

for large N. The standard deviation of $r_k^{(5)}$ is always greater than that of usual estimate. The ratio of the former to the latter increases without limit as $|\rho_k|$ approaches unity. In our treatment, the sampling intervals are not assumed to be such that the samples are independent. So, we can not compare these results with ours (cf. Chapter 4).

(iv) We shall quote Morishita [9]. Let X(t) be a stationary Gaussian process with mean zero. It is shown that we can obtain correlogram by calculating the mean value of sampled data $x(s+k)$, where S is given by

$$x(s) = \xi$$

for an arbitrarily preset non-zero level ξ.

For the related topics, we can find the studies which deal with the axis crossings of a stationary process (see McFadden [7], [8]).

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CHAPTER 2

THE CASE OF A SIMPLE MARKOV GAUSSIAN PROCESS WITH KNOWN VARIANCE

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ON A SIMPLIFIED METHOD OF THE ESTIMATION OF THE CORRELOGRAM FOR A STATIONARY GAUSSIAN PROCESS

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1. Introduction

Let $x(t)$ be a real valued weakly stationary process with discrete time parameter t, such that $Ex(t)=0$, $Ex(t)^2=\sigma^2$, $Ex(t)x(t+h)=\sigma^2\rho_h$. The problem considered in this paper is concerned with the estimation of the correlogram ρ_h. For this, we assume the variance σ^2 to be known. Further, since the correlogram ρ_h is symmetric about $h=0$, we take h to be nonnegative. For the correlogram ρ_h, the unbiased estimate

$$\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{t=1}^{N}x(t)x(t+h)$$

is usually considered. However, we do not take this estimate, as it is, but make modification. The essential part of our modification is to replace $x(t)x(t+h)$ by $x(t)sgn(x(t+h))$ in the estimate $\tilde{\gamma}_h$, where $sgn(y)$ means 1, 0, -1 correspondingly as $y>0$, $y=0$, $y<0$. The statistic thus obtained originates in Takahasi and Husimi's method of determining the period and decrement of a vibrating system exposed to irregular statistical forces [4]. This statistic has been used as a simplified estimate of the correlogram in many practical fields. However, its validity has not been assertained. This problem was first presented to the author by Prof. R. Kawashima, the Faculty of Fisheries, Hokkaido University. He used $(1/N)\sum_{t=1}^{N}x(t)sgn(x(t+h))$ to simplify calculations of the covariance in analysis of ship's rolling records [3].

In the present paper, we shall investigate mathematical and statistical properties of the statistic obtained by that replacement from $\tilde{\gamma}_h$. In section 2, it will be shown that, when $x(t)$ is a stationary Gaussian process, we can give an unbiased estimate of the correlogram, in terms of $x(t)sgn(x(t+h))$. The estimate is

$$\gamma_h=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\frac{1}{N}\sum_{t=1}^{N}x(t)sgn(x(t+h)).$$

The variance of this estimate will also be evaluated. In general, it is not easy to evaluate the variance, and, our discussion is mainly restricted to the case where $x(t)$ is a Markov process. Further, in section 3, we shall give numerical comparisons of the variances of our estimate γ_h with those of the ordinary estimate $\tilde{\gamma}_h$. From this comparison, it will be seen, at least in that Markov case, that our estimate has smaller variance than the ordinary one for small lag h.

2. Mean and variance of estimate γ_h

First, we shall show that, if $x(t)$ is a stationary Gaussian process, γ_h is an unbiased estimate of ρ_h.

In fact, for $h\neq 0$, putting simply $x(t)=x$ and $x(t+h)=y$, we have

$$
\begin{aligned}
&Ex(t)sgn(x(t+h))\\
&=\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} x sgn(y)\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy\\
&=\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=0}^{\infty}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy\\
&-\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=-\infty}^{0}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy
\end{aligned}
$$

and

$$\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=0}^{\infty}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy=\frac{\sigma\rho_h}{\sqrt{2\pi}}$$

$$\frac{1}{2\pi\sigma^2\sqrt{1-\rho_h^2}}\int_{y=-\infty}^{0}\int_{x=-\infty}^{\infty} x\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x^2-2\rho_h xy+y^2)\right)dxdy=-\frac{\sigma\rho_h}{\sqrt{2\pi}}\ .$$

Consequently,

$$Ex(t)sgn(x(t+h))=\sqrt{\frac{2}{\pi}}\,\sigma\rho_h\,,$$

and finally

$$E(\gamma_h)=\rho_h\ .$$

For $h=0$, we can also show

$$E(\gamma_0)=1.$$

In the next place, we consider the variance $V(\gamma_h)$ of estimate γ_h. Evaluation of the variance of γ_0 proceeds as follows. We have

$$V(\gamma_0)=E(\gamma_0-1)^2=E\gamma_0^2-1\,,$$

$$E\gamma_0^2=E\left(\sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma}\,\frac{1}{N}\sum_{t=1}^{N}x(t)sgn(x(t))\right)^2$$

$$=\pi\,\frac{1}{\sigma^2}\,\frac{1}{N^2}\sum_{t=1}^{N}\sum_{\substack{s=1\\ s>t}}^{N}Ex(s)sgn(x(s))x(t)sgn(x(t))$$

$$+\frac{\pi}{2}\,\frac{1}{\sigma^2}\,\frac{1}{N^2}\sum_{t=1}^{N}Ex(t)^2sgn^2(x(t)),$$

and, for simplicity, putting $x(s)=x$ and $x(t)=y$,

$$Ex(s)sgn(x(s))x(t)sgn(x(t))=E|x(s)||x(t)|$$

$$=\frac{1}{2\pi\sigma^2\sqrt{1-\rho_{s-t}^2}}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}|x||y|\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(x^2-2\rho_{s-t}xy+y^2)\right)dxdy.$$

Using the expansion (S. O. Rice [2], section 3.5)

$$\int_0^\infty\int_0^\infty u^l v^m\exp(-u^2-v^2-2auv)dudv$$

$$=\frac{1}{4}\sum_{k=0}^{\infty}\frac{(-2a)^k}{k\,!}\,\Gamma\left(\frac{l+k+1}{2}\right)\Gamma\left(\frac{m+k+1}{2}\right),$$

we get, for example,

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty}xy\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t}^2)}(x^2-2\rho_{s-t}xy+y^2)\right)dxdy$$

$$=\sigma^4(1-\rho_{s-t}^2)^2\left(\sum_{k=0}^{\infty}\frac{(2\rho_{s-t})^k}{k\,!}\,\Gamma\left(\frac{k+2}{2}\right)^2\right).$$

Thus, we have

$$Ex(s)sgn(x(s))x(t)sgn(x(t))$$

$$=\frac{2\sigma^2(1-\rho_{s-t}^2)^{3/2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2\,m)\,!}\,\Gamma(m+1)^2\right),$$

and finally,

$$V(\gamma_0)=\frac{2}{N^2}\sum_{t=1}^{N}\sum_{\substack{s=1\\ s>t}}^{N}(1-\rho_{s-t}^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2\,m)\,!}\Gamma(m+1)^2\right)+\frac{\pi}{2}\,\frac{1}{N}-1$$

$$=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)+\frac{\pi}{2}\,\frac{1}{N}-1.$$

For $h>0$, the variance of γ_h is given as follows. As was stated in section 1, we hereafter restrict our attention to the case where $x(t)$ is a stationary Gaussian Markov process. A process $x(t)$ is called a Markov process in the sense of J. L. Doob [1] when $x(t)$ satisfies the following condition:

for any integer $n \geqq 1$ and any parameter values $t_1 < t_2 < \cdots < t_n$, the conditional probabilities of $x(t_n)$, relative to $x(t_1), x(t_2), \cdots, x(t_{n-1})$, are the same as those relative to $x(t_{n-1})$ in the sense that for each λ

$$P\{x(t_n) \leqq \lambda | x(t_1), x(t_2), \cdots, x(t_{n-1})\} = P\{x(t_n) \leqq \lambda | x(t_{n-1})\}$$

with probability 1.

In this case, the correlogram is expressed as

$$\rho_h = a^{|h|} \qquad (|a| \leqq 1)$$

(see J. L. Doob [1]). Let

$$f_n(x_1, x_2, \cdots, x_n)$$

be the probability density function of the n-dimensional Gaussian distribution. Then, for a Gaussian Markov process $x(t)$, and for any $t_1 < t_2 < \cdots < t_{n-2} < t_{n-1} < t_n$, we have

$$\begin{aligned} &f_n(x(t_1), x(t_2), \cdots, x(t_{n-2}), x(t_{n-1}), x(t_n)) \\ &= \frac{f_2(x(t_1), x(t_2))}{f_1(x(t_2))} \cdots \frac{f_1(x(t_{n-2}), x(t_{n-1}))}{f_1(x(t_{n-1}))} f_2(x(t_{n-1}), x(t_n)). \end{aligned}$$

We use this fact for calculation of the variance of γ_h. For simplicity, we assume that N is sufficiently large and $N > h$. Then we obviously, have

$$\text{variance of } \gamma_h = V(\gamma_h) = E(\gamma_h - \rho_h)^2 = E\gamma_h^2 - \rho_h^2,$$

and

$$\begin{aligned} E\gamma_h^2 &= E\left(\sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \frac{1}{N} \sum_{t=1}^{N} x(t) sgn(x(t+h))\right)^2 \\ &= \frac{\pi}{2} \frac{1}{\sigma^2} \frac{1}{N^2} E\left(\sum_{t=1}^{N} \sum_{s=1}^{N} x(t) sgn(x(t+h)) x(s) sgn(x(s+h))\right) \\ &= \frac{\pi}{2} \frac{1}{\sigma^2} \frac{1}{N^2} \Big[2 \sum_{t=1}^{N} \sum_{\substack{s=1 \\ s>t \\ t+h>s}}^{N} Ex(t) sgn(x(t+h)) x(s) sgn(x(s+h)) \\ &\qquad + 2 \sum_{t=1}^{N} \sum_{\substack{s=1 \\ s>t+h}}^{N} Ex(t) sgn(x(t+h)) x(s) sgn(x(s+h)) \\ &\qquad + 2 \sum_{t=1}^{N-h} Ex(t) sgn(x(t+h)) x(t+h) sgn(x(t+2h)) \\ &\qquad + \sum_{t=1}^{N} Ex(t)^2 sgn^2(x(t+h)) \Big]. \end{aligned}$$

In the following we evaluate each part of summation.

i) When $s > t$ and $t + h > s$, we have

$$f_4(x(t), x(s), x(t+h), x(s+h))$$
$$=\frac{f_2(x(t), x(s))}{f_1(x(s))}\frac{f_2(x(s), x(t+h))}{f_1(x(t+h))}f_2(x(t+h), x(s+h)).$$

For simplicity, we put

$$x(t)=x,\ x(s)=y,\ x(t+h)=\tilde{x},\ x(s+h)=\tilde{y}.$$

Then we have

$$f_4(x, y, \tilde{x}, \tilde{y})$$
$$=\frac{1}{(2\pi)^2\sigma^4(1-\rho^2_{s-t})\sqrt{1-\rho^2_{t+h-s}}}\exp\left(-\frac{1}{2\sigma^2(1-\rho^2_{s-t})}(x-\rho_{s-t}y)^2\right)$$
$$\times\exp\left(-\frac{1}{2\sigma^2(1-\rho^2_{t+h-s})}(y-\rho_{t+h-s}\tilde{x})^2-\frac{1}{2\sigma^2(1-\rho^2_{s-t})}(\tilde{x}^2-2\rho_{s-t}\tilde{x}\tilde{y}+\tilde{y}^2)\right).$$

Now

$$Ex(t)x(s)sgn(x(t+h))sgn(x(s+h))$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xysgn(\tilde{x})sgn(\tilde{y})f_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$=\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$-\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$-\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$+\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y},$$

and

$$\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$=\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$
$$=\frac{\rho_{s-t}(1-\rho^2_{t+h-s})}{2\pi\sqrt{1-\rho^2_{s-t}}}\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\exp\left(-\frac{1}{2\sigma^2(1-\rho^2_{s-t})}(\tilde{x}^2-2\rho_{s-t}\tilde{x}\tilde{y}+\tilde{y}^2)\right)d\tilde{x}d\tilde{y}$$
$$+\frac{\rho_{s-t}\rho^2_{t+h-s}}{2\pi\sigma^2\sqrt{1-\rho^2_{s-t}}}\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\tilde{x}^2\exp\left(-\frac{1}{2\sigma^2(1-\rho^2_{s-t})}(\tilde{x}^2-2\rho_{s-t}\tilde{x}\tilde{y}+\tilde{y}^2)\right)d\tilde{x}d\tilde{y}.$$

Using the expansion formula used in the evaluation of $V(\gamma_0)$, we can get

$$\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\frac{\sigma^2\rho_{s-t}(1-\rho_{t+h-s}^2)\sqrt{1-\rho_{s-t}^2}}{4\pi}\left(\sum_{k=0}^{\infty}\frac{(2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+1}{2}\right)^2\right)$$

$$+\frac{\sigma^2\rho_{s-t}\rho_{t+h-s}^2(1-\rho_{s-t}^2)^{3/2}}{2\pi}\left(\sum_{k=0}^{\infty}\frac{(2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{k+1}{2}\right)\right).$$

Similarly, we get

$$\int_{\tilde{y}=0}^{\infty}\int_{\tilde{x}=-\infty}^{0}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\int_{\tilde{y}=-\infty}^{0}\int_{\tilde{x}=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} xyf_4(x, y, \tilde{x}, \tilde{y})dxdyd\tilde{x}d\tilde{y}$$

$$=\frac{\sigma^2\rho_{s-t}(1-\rho_{t+h-s}^2)\sqrt{1-\rho_{s-t}^2}}{4\pi}\left(\sum_{k=0}^{\infty}\frac{(-2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+1}{2}\right)^2\right)$$

$$+\frac{\sigma^2\rho_{s-t}\rho_{t+h-s}^2(1-\rho_{s-t}^2)^{3/2}}{2\pi}\left(\sum_{k=0}^{\infty}\frac{(-2\rho_{s-t})^k}{k\,!}\Gamma\left(\frac{k+3}{2}\right)\Gamma\left(\frac{k+1}{2}\right)\right).$$

Therefore we obtain

$$Ex(t)x(s)sgn(x(t+h)sgn(x(s+h))$$

$$=\frac{\sigma^2\rho_{s-t}(1-\rho_{t+h-s}^2)\sqrt{1-\rho_{s-t}^2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\Gamma(m+1)^2\right)$$

$$+\frac{2\sigma^2\rho_{s-t}\rho_{t+h-s}^2(1-\rho_{s-t}^2)^{3/2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\Gamma(m+2)\Gamma(m+1)\right).$$

ii) When $s>t+h$, we have, using the same notation as in i),

$$f_4(x(t), x(t+h), x(s), x(s+h))$$

$$=\frac{1}{(2\pi)^2\sigma^4(1-\rho_h^2)\sqrt{1-\rho_{s-t-h}^2}}\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x-\rho_h\tilde{x})^2\right)$$

$$\times\exp\left(-\frac{1}{2\sigma^2(1-\rho_{s-t-h}^2)}(\tilde{x}-\rho_{s-t-h}y)^2-\frac{1}{2\sigma^2(1-\rho_h^2)}(y^2-2\rho_h y\tilde{y}+\tilde{y}^2)\right).$$

In this case, we have

$$E(x/\tilde{x}, y, \tilde{y})=\rho_h\tilde{x}$$

$$E(xy/\tilde{x}, \tilde{y})=E(yE(x/\tilde{x}, y, \tilde{y})/\tilde{x}, \tilde{y})$$

$$=\frac{\rho_{s-t}(1-\rho_h^2)}{1-\rho_{s-t}^2}\tilde{x}^2+\frac{\rho_h^2(1-\rho_{s-t-h}^2)}{1-\rho_{s-t}^2}\tilde{x}\tilde{y}.$$

Consequently, we get

$$Ex(t)sgn(x(t+h))x(s)sgn(x(s+h))$$
$$=\frac{2\sigma^2\rho_{s-t}(1-\rho_h^2)\sqrt{1-\rho_{s-t}^2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)$$
$$+\frac{2\sigma^2\rho_h^2(1-\rho_{s-t-h}^2)\sqrt{1-\rho_{s-t}^2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2m)!}\Gamma(m+1)^2\right).$$

iii) When $s=t+h$, the necessary joint probability density function is $f_3(x(t), x(t+h), x(t+2h))$. With the same notation as in i), we have

$$f_3(x(t), x(t+h), x(t+2h))$$
$$=\frac{f_2(x(t), x(t+h))}{f_1(x(t+h))}f_2(x(t+h), x(t+2h))$$
$$=\frac{1}{(2\pi)^{3/2}\sigma^3(1-\rho_h^2)}\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(x-\rho_h y)^2\right)$$
$$\times\exp\left(-\frac{1}{2\sigma^2(1-\rho_h^2)}(y^2-2\rho_h y\tilde{y}+\tilde{y}^2)\right).$$

Using this expression, we get

$$Ex(t)sgn(x(t+h))x(t+h)sgn(x(t+2h))$$
$$=\frac{2\sigma^2\rho_h(1-\rho_h^2)^{3/2}}{\pi}\left(\sum_{m=0}^{\infty}\frac{(2\rho_h)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right).$$

iv) When $t=s$, we have

$$Ex(t)^2(sgn(x(t+h)))^2=\sigma^2.$$

Using these results, we finally get

$$V(\gamma_h)=E\gamma_h^2-\rho_h^2$$
$$=\frac{1}{N^2}\Bigg[\sum_{k=1}^{h-1}(N-k)\Bigg\{\rho_k(1-\rho_{h-k}^2)\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+1)^2\right)$$
$$+2\rho_k\rho_{h-k}^2(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)\Bigg\}$$
$$+2\sum_{k=h+1}^{N-1}(N-k)\Bigg\{\rho_k(1-\rho_h^2)\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)$$
$$+\rho_h^2(1-\rho_{k-h}^2)\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)\Bigg\}$$
$$+2(N-h)\rho_h(1-\rho_h^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_h)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right)+\frac{\pi}{2}N\Bigg]$$
$$-\rho_h^2.$$

3. Comparison of γ_h with $\tilde{\gamma}_h$

The variance of $\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{t=1}^{N}x(t)x(t+h)$ for a stationary Gaussian process is as follows:

i) When $h=0$, we have

$$\text{variance of } \tilde{\gamma}_0=V(\tilde{\gamma}_0)=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1+2\rho_k^2)+\frac{3}{N}-1.$$

ii) When $h\neq 0$, assuming Markov property, we have

$$\begin{aligned}\text{variance of } \tilde{\gamma}_h=V(\tilde{\gamma}_h)&=E\tilde{\gamma}_h^2-\rho_h^2\\ &=\frac{1}{N^2}\Big[2\sum_{k=1}^{h-1}(N-k)\rho_k^2(1+2\rho_{h-k}^2)\\ &\quad+2\sum_{k=h+1}^{N-1}(N-k)\rho_h^2(1+2\rho_{k-h}^2)\\ &\quad+6(N-h)\rho_h^2+N(1+2\rho_h^2)\Big]\\ &\quad-\rho_h^2.\end{aligned}$$

In this section, we compare, numerically, the variance of γ_h with that of $\tilde{\gamma}_h$.

We are considering a Markov process, so we have

$$\rho_k=a^{|k|} \qquad (|a|\leqq 1).$$

Numerical comparisons are made for the following cases:

$$a=(0.8)^5, \qquad 0.8,$$
$$N=50, \qquad 500.$$

The results are shown in Table 1 and Figure 1.

Taking into account the present numerical results and the ease of computation of γ_h, we can say that the estimate γ_h is a fairly good estimate of the correlogram for a stationary Gaussian Markov process. This will also be referred to in future by M. Sibuya from the point of view of estimation theory.

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TABLE 1*)

$a=0.8$

h	ρ_h	$N=500$ $V(\gamma_h)$	$N=500$ $V(\tilde{\gamma}_h)$	$N=50$ $V(\gamma_h)$	$N=50$ $V(\tilde{\gamma}_h)$
0	1.0000	0.0048	0.0181	0.0464	0.1743
1	0.8000	0.0070	0.0174	0.0630	0.1671
2	0.6400	0.0084	0.0160	0.0808	0.1535
3	0.5120	0.0093	0.0146	0.0894	0.1391
4	0.4096	0.0098	0.0133	0.0949	0.1263
5	0.3277	0.0102	0.0122	0.0984	0.1159
6	0.2621	0.0104	0.0113	0.1007	0.1079
7	0.2097	0.0106	0.0107	0.1021	0.1019
8	0.1678	0.0107	0.0102	0.1031	0.0975
9	0.1342	0.0107	0.0099	0.1037	0.0943
10	0.1074	0.0108	0.0096	0.1040	0.0921
11	0.0859	0.0108	0.0095	0.1043	0.0916
12	0.0687	0.0108	0.0093	0.1044	0.0901
13	0.0550	0.0108	0.0093	0.1045	0.0892
14	0.0440	0.0108	0.0092	0.1046	0.0885
15	0.0352	0.0108	0.0092	0.1046	0.0879
20	0.0115	0.0108	0.0091	0.1047	0.0873
25	0.0038	0.0108	0.0091	0.1047	0.0872
30	0.0012	0.0108	0.0091	0.1047	0.0872

$a=(0.8)^5=0.32768$

h	ρ_h	$N=500$ $V(\gamma_h)$	$N=500$ $V(\tilde{\gamma}_h)$	$N=50$ $V(\gamma_h)$	$N=50$ $V(\tilde{\gamma}_h)$
0	1.00000	0.0014	0.0050	0.0138	0.0494
1	0.32768	0.0034	0.0032	0.0338	0.0316
2	0.10737	0.0036	0.0026	0.0359	0.0259
3	0.03518	0.0036	0.0025	0.0362	0.0249
4	0.01153	0.0036	0.0025	0.0362	0.0247
5	0.00378	0.0036	0.0025	0.0362	0.0247
10	0.00001	0.0036	0.0025	0.0362	0.0247

comments and advices. Thanks are also due to Mr. T. Komazawa and Miss E. Ozaki for performing all the necessary programmings and operations of the FACOM-128 computer to prepare the numerical results.

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*) As was stated in H. Akaike [5], the variance of $\tilde{\gamma}_h$ is asymptotically of order $1/N$. The present results are in accordance with this fact.

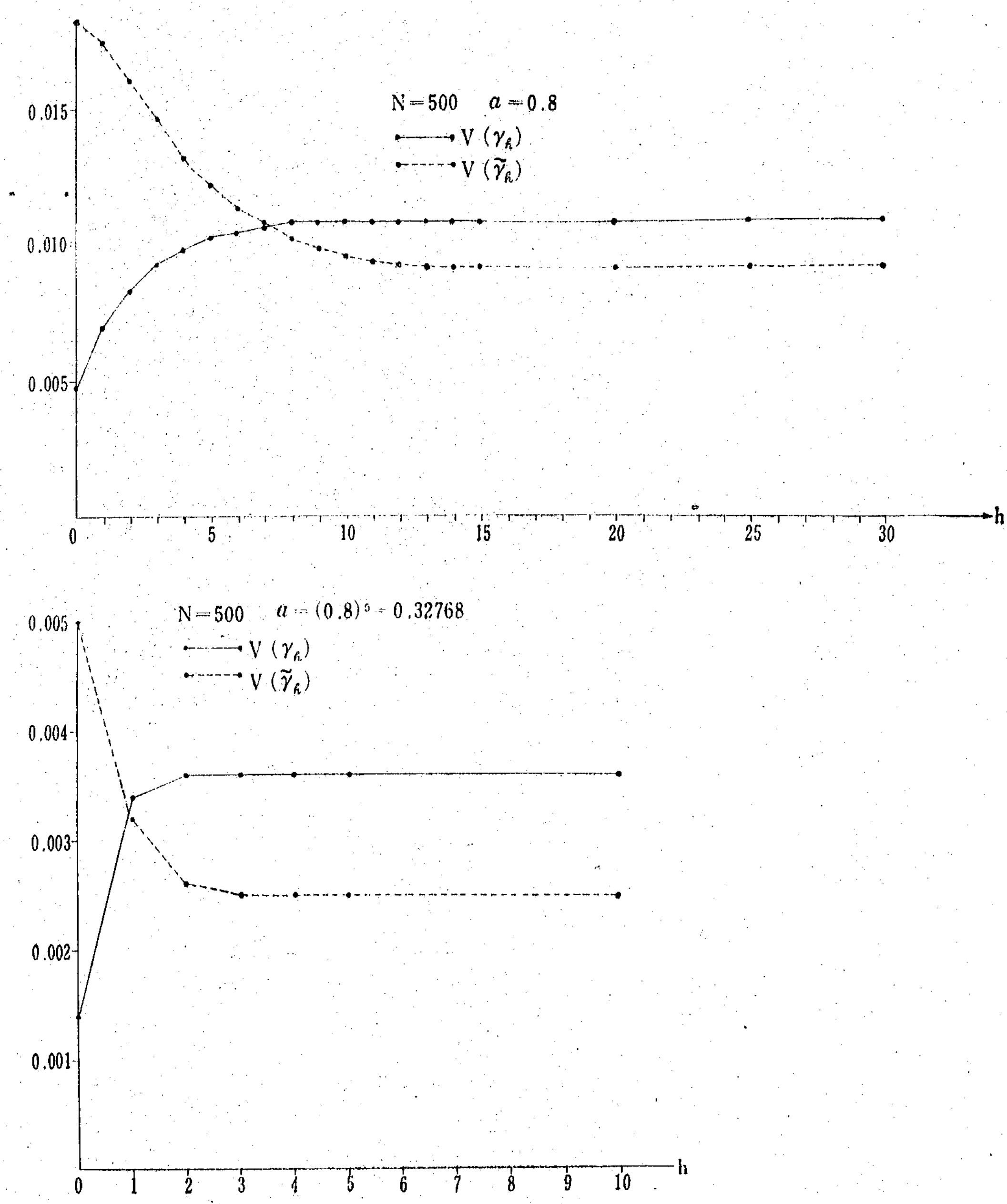

Fig. 1.

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CHAPTER 3

THE CASE OF A GAUSSIAN PROCESS

WITH KNOWN VARIANCE

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ON A SIMPLIFIED METHOD OF THE ESTIMATION OF THE CORRELOGRAM FOR A STATIONARY GAUSSIAN PROCESS, II

By Mituaki Huzii

§1. Summary.

For the estimation of the correlogram of a real valued weakly stationary process $x(t)$, we usually use the estimate using the term $x(t)x(t+h)$. We try to replace the term $x(t)x(t+h)$ by the term $x(t)\,\mathrm{sgn}(x(t+h))$. In the previous paper [2], we showed that, when the variance is known, we can get an unbiased estimate by this replacement for a Gaussian process, and also showed its variance for a simple markov Gaussian process. In this paper, we shall evaluate its variance for a general Gaussian process, and show that this estimate is a consistent estimate under a [redacted] condition. And especially, we compare, numerically, its variance with that of usual estimate, for the second-order process.

§2. The estimate and its variance.

Let $x(t)$ be a real valued weakly stationary process with continuous time parameter t, such that $Ex(t)=0$, $Ex(t)^2=\sigma^2$, $Ex(t)x(t+h)=\sigma^2\rho_h$. We assume the variance σ^2 to be known. And, given observations $\{x(t),\ t=1, 2, \cdots, N, \cdots, N+h\}$, we consider to estimate the correlogram ρ_h, where N and h are positive integers. We shall try to replace the term $x(t)x(t+h)$ of the usual estimate

$$\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{t=1}^{N}x(t)x(t+h)$$

by the term $x(t)\ \mathrm{sgn}(x(t+h))$, where $\mathrm{sgn}(y)$ means 1, 0 and -1, correspondingly as $y>0$, $y=0$ and $y<0$.

For a Gaussian process, the estimate

$$\gamma_h=\sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma}\,\frac{1}{N}\sum_{t=1}^{N}x(t)\ \mathrm{sgn}(x(t+h))$$

is an unbiased estimate [2]. We shall determine the variance of this estimate. Now,

$$\mathrm{Var.}(\gamma_h)=E\gamma_h^2-\rho_h^2,$$

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$$E\gamma_h^2 = E\left(\sqrt{\frac{\pi}{2}}\ \frac{1}{\sigma}\ \frac{1}{N}\sum_{t=1}^{N} x(t)\ \mathrm{sgn}(x(t+h))\right)^2$$

$$= \frac{\pi}{2}\ \frac{1}{\sigma^2}\cdot\frac{1}{N^2} E\left(\sum_{t=1}^{N}\sum_{s=1}^{N} x(t)\ \mathrm{sgn}(x(t+h))x(s)\ \mathrm{sgn}(x(s+h))\right)$$

and, we shall evaluate the value of

$$E(x(t)\ \mathrm{sgn}(x(t+h))x(s)\ \mathrm{sgn}(x(s+h))).$$

i) When t, s, $t+h$ and $s+h$ are all distinct, we put

$$x(t) = Ax(s) + Bx(t+h) + Cx(s+h) + \varepsilon(t),$$

where A, B and C are constants and $\varepsilon(t)$ is a stochastic process such that

a) $E\varepsilon(t)=0$,

b) $\varepsilon(t)$ has no correlation with $x(s)$, $x(t+h)$ and $x(s+h)$.

So, A, B and C are all determined by the relation

$$\begin{aligned} &E(x(t)-Ax(s)-Bx(t+h)-Cx(s+h))x(s)=0,\\ &E(x(t)-Ax(s)-Bx(t+h)-Cx(s+h))x(t+h)=0,\\ &E(x(t)-Ax(s)-Bx(t+h)-Cx(s+h))x(s+h)=0.\end{aligned} \tag{1}$$

As $x(t)$ is a real-valued process, we have the equivalence

$$\rho_l = \rho_{-l}.$$

Using this, we can rewrite the relation (1) as follows:

$$\begin{aligned} &A \qquad + B\rho_{s-t-h} + C\rho_h \quad = \rho_{s-t},\\ &A\rho_{s-t-h} + B \qquad + C\rho_{s-t} = \rho_h,\\ &A\rho_h \quad + B\rho_{s-t} \quad + C \qquad = \rho_{s-t+h}.\end{aligned} \tag{2}$$

From the equation (2), we have

$$A = \frac{1}{\Delta}\begin{vmatrix} \rho_{s-t} & \rho_{s-t-h} & \rho_h \\ \rho_h & 1 & \rho_{s-t} \\ \rho_{s-t+h} & \rho_{s-t} & 1 \end{vmatrix},\quad B = \frac{1}{\Delta}\begin{vmatrix} 1 & \rho_{s-t} & \rho_h \\ \rho_{s-t-h} & \rho_h & \rho_{s-t} \\ \rho_h & \rho_{s-t+h} & 1 \end{vmatrix},$$

and $$C=\frac{1}{\Delta}\begin{vmatrix} 1 & \rho_{s-t-h} & \rho_{s-t} \\ \rho_{s-t-h} & 1 & \rho_h \\ \rho_h & \rho_{s-t} & \rho_{s-t+h} \end{vmatrix},$$

where

$$\Delta=\begin{vmatrix} 1 & \rho_{s-t-h} & \rho_h \\ \rho_{s-t-h} & 1 & \rho_{s-t} \\ \rho_h & \rho_{s-t} & 1 \end{vmatrix}.$$

Therefore, we have

$$\begin{aligned} & E(x(t)/x(s),\ x(t+h),\ x(s+h)) \\ &= E(Ax(s)+Bx(t+h)+Cx(s+h)+\varepsilon(t)/x(s),\ x(t+h),\ x(s+h)) \\ &= Ax(s)+Bx(t+h)+Cx(s+h). \end{aligned}$$

And, so it holds

$$\begin{aligned} & E(x(t)x(s)/x(t+h),\ x(s+h)) \\ &= E[x(s)(E(x(t)/x(s),\ x(t+h),\ x(s+h)))/x(t+h),\ x(s+h)] \\ &= E[x(s)(Ax(s)+Bx(t+h)+Cx(s+h))/x(t+h),\ x(s+h)] \\ &= E[Ax(s)^2+Bx(s)x(t+h)+Cx(s)x(s+h)/x(t+h),\ x(s+h)]. \end{aligned}$$

In the next place, let us put

$$x(s)=Fx(t+h)+Gx(s+h)+\eta(s),$$

where $\eta(s)$ is a stochastic process such as

a′) $E\eta(s)=0$,

b′) $\eta(s)$ has no correlation with $x(t+h)$ and $x(s+h)$.

From this condition, we can express as

$$\begin{aligned} & E(x(s)-Fx(t+h)-Gx(s+h))x(t+h)=0, \\ & E(x(s)-Fx(t+h)-Gx(s+h))x(s+h)=0. \end{aligned} \tag{3}$$

Writing (3) in the correlogram's terms, we have

$$F \qquad +G\rho_{s-t}=\rho_{s-t-h},$$
$$F\rho_{s-t}+G \qquad =\rho_h. \tag{4}$$

By solving the equation (4), we have

$$F=\frac{1}{D}\begin{vmatrix}\rho_{s-t-h} & \rho_{s-t}\\ \rho_h & 1\end{vmatrix} \text{ and } G=\frac{1}{D}\begin{vmatrix}1 & \rho_{s-t-h}\\ \rho_{s-t} & \rho_h\end{vmatrix},$$

where

$$D=\begin{vmatrix}1 & \rho_{s-t}\\ \rho_{s-t} & 1\end{vmatrix}.$$

Substituting the above expression, we get

$$\begin{aligned}
&E(x(t)x(s)/x(t+h),\ x(s+h))\\
&=E[A(Fx(t+h)+Gx(s+h)+\eta(s))^2+B(Fx(t+h)+Gx(s+h)+\eta(s))x(t+h)\\
&\quad+C(Fx(t+h)+Gx(s+h)+\eta(s))x(s+h)/x(s+h),\ x(t+h)]\\
&=(AF^2+BF)x(t+h)^2+(2AFG+BG+CF)x(t+h)x(s+h)\\
&\quad+(AG^2+CG)x(s+h)^2+AE(\eta(s)^2/x(t+h),\ x(s+h)).
\end{aligned}$$

And, as $\eta(s)$ is independent of $x(t+h)$ and $x(s+h)$, we have

$$\begin{aligned}
&E(\eta(s)^2/x(t+h),\ x(s+h))\\
&=E[(x(s)-Fx(t+h)-Gx(s+h))^2/x(t+h),\ x(s+h)]\\
&=E[(x(s)-Fx(t+h)-Gx(s+h))^2/x(t+h)=0,\ x(s+h)=0]\\
&=E[x(s)^2/x(t+h)=0,\ x(s+h)=0]=\frac{\sigma^2\Delta}{D}.
\end{aligned}$$

Consequently, it follows

$$\begin{aligned}
&E(x(t)\operatorname{sgn}(x(t+h))x(s)\cdot\operatorname{sgn}(x(s+h)))\\
&=E[\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h))(E(x(t)x(s)/x(t+h),\ x(s+h)))]\\
&=(AF^2+BF)E(x(t+h)^2\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h)))\\
&\quad+(2AFG+BG+CF)E(|x(t+h)||x(s+h)|)\\
&\quad+(AG^2+CG)E(x(s+h)^2\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h)))\\
&\quad+A\frac{\sigma^2\Delta}{D}E(\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h))).
\end{aligned}$$

Now, we shall put, for simplicity, $x(t+h)=x$ and $x(s+h)=y$ and further put

$$f(x, y)=\frac{1}{2\pi\sigma^2\sqrt{D}}e^{-(x^2-2\rho_{s-t}xy+y^2)/2\sigma^2 D}$$

Then we have

$$E(x(t+h)^2\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h)))$$

$$=\int_{y=0}^{\infty}\int_{x=0}^{\infty}x^2f(x, y)dxdy-\int_{y=0}^{\infty}\int_{x=-\infty}^{0}x^2f(x, y)dxdy-\int_{y=-\infty}^{0}\int_{x=0}^{\infty}x^2f(x, y)dxdy$$

$$+\int_{y=-\infty}^{0}\int_{x=-\infty}^{0}x^2f(x, y)dxdy.$$

Being

$$\int_{y=0}^{\infty}\int_{x=-\infty}^{0}x^2f(x, y)dxdy=\int_{y=0}^{\infty}\int_{x=0}^{\infty}x^2f(-x, y)dxdy=\int_{y=0}^{\infty}\int_{x=0}^{\infty}x^2f(x, -y)dxdy$$

$$=\int_{y=-\infty}^{0}\int_{x=0}^{\infty}x^2f(x, y)dxdy$$

and

$$\int_{y=-\infty}^{0}\int_{x=-\infty}^{0}x^2f(x, y)dxdy=\int_{y=0}^{\infty}\int_{x=0}^{\infty}x^2f(-x, -y)dxdy=\int_{y=0}^{\infty}\int_{x=0}^{\infty}x^2f(x, y)dxdy,$$

so it holds

$$E(x(t+h)^2\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h)))$$

$$=2\left(\int_{y=0}^{\infty}\int_{x=0}^{\infty}x^2f(x, y)dxdy-\int_{y=0}^{\infty}\int_{x=0}^{\infty}x^2f(x, -y)dxdy\right).$$

Let us use the expansion (Rice [4], section 3. 5)

$$\int_0^{\infty}\int_0^{\infty}u^lv^m\exp(-u^2-v^2-2auv)dudv=\frac{1}{4}\sum_{k=0}^{\infty}\frac{(-2a)^k}{k!}\Gamma\left(\frac{l+k+1}{2}\right)\Gamma\left(\frac{m+k+1}{2}\right)$$

and put

$$I(-2a, l, m)=\sum_{k=0}^{\infty}\frac{(-2a)^k}{k!}\Gamma\left(\frac{l+k+1}{2}\right)\Gamma\left(\frac{m+k+1}{2}\right).$$

Consequently, we get

$$E(x(t+h)^2\operatorname{sgn}(x(t+h))\operatorname{sgn}(x(s+h)))$$

$$=2\times\frac{\sigma^2D^{3/2}}{2\pi}(I(2\rho_{s-t}, 2, 0)-I(-2\rho_{s-t}, 2, 0))=\frac{\sigma^2D^{3/2}}{\pi}S_1(\rho_{s-t})$$

where

$$S_1(\rho_{s-t})=2\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\,\Gamma(m+2)\Gamma(m+1)\right).$$

Similarly,

$$E(|x(t+h)||x(s+h)|)$$

$$=2\times\frac{\sigma^2D^{3/2}}{2\pi}(I(2\rho_{s-t},1,1)+I(-2\rho_{s-t},1,1))=\frac{\sigma^2D^{3/2}}{\pi}S_2(\rho_{s-t}),$$

$$E(x(s+h)^2\,\mathrm{sgn}(x(t+h))\,\mathrm{sgn}(x(s+h)))$$

$$=2\times\frac{\sigma^2D^{3/2}}{2\pi}(I(2\rho_{s-t},0,2)-I(-2\rho_{s-t},0,2))=\frac{\sigma^2D^{3/2}}{\pi}S_1(\rho_{s-t})$$

and

$$E(\mathrm{sgn}(x(t+h))\,\mathrm{sgn}(x(s+h)))$$

$$=2\times\frac{\sqrt{D}}{4\pi}(I(2\rho_{s-t},0,0)-I(-2\rho_{s-t},0,0))=\frac{\sqrt{D}}{2\pi}S_3(\rho_{s-t}),$$

where

$$S_2(\rho_{s-t})=2\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m}}{(2m)\,!}\,\Gamma(m+1)^2\right)$$

and

$$S_3(\rho_{s-t})=2\left(\sum_{m=0}^{\infty}\frac{(2\rho_{s-t})^{2m+1}}{(2m+1)\,!}\,\Gamma(m+1)^2\right).$$

As the result, we obtain

$$E[x(t)\,\mathrm{sgn}(x(t+h))x(s)\,\mathrm{sgn}(x(s+h))]$$

$$=(AF^2+BF)\frac{\sigma^2D^{3/2}}{\pi}S_1(\rho_{s-t})+(2AFG+BG+CF)\frac{\sigma^2D^{3/2}}{\pi}S_2(\rho_{s-t})$$

$$+(AG^2+CG)\frac{\sigma^2D^{3/2}}{\pi}S_1(\rho_{s-t})+A\frac{\sigma^2\Delta}{2\pi\sqrt{D}}S_3(\rho_{s-t}).$$

ii) When $s=t+h$, $s+h=t+2h$. The situation is the same when $t=s+h$. In this case, we have

$$E(x(t)x(s)\,\mathrm{sgn}(x(s+h))\,\mathrm{sgn}(x(t+h)))$$

$$=E(x(t)x(t+h)\,\mathrm{sgn}(x(t+h))\,\mathrm{sgn}(x(t+2h)))$$

$$=E[x(t+h)\,\mathrm{sgn}(x(t+h))\,\mathrm{sgn}(x(t+2h))E(x(t)/x(t+h),x(t+2h))].$$

As the above, we shall put

$$x(t)=Hx(t+h)+Kx(t+2h)+\delta(t)$$

where H and K are constants and $\delta(t)$ is a stochastic process such as

a″) $E\delta(t)=0$,

b″) $\delta(t)$ has no correlation with $x(t+h)$ and $x(t+2h)$.

H and K are determined by the conditions

$$\begin{aligned} E(x(t)-Hx(t+h)-Kx(t+2h))x(t+h)&=0, \\ E(x(t)-Hx(t+h)-Kx(t+2h))x(t+2h)&=0. \end{aligned} \tag{5}$$

These [redacted] conditions are equivalent to

$$\begin{aligned} H+K\rho_h&=\rho_h, \\ H\rho_h+K&=\rho_{2h}. \end{aligned} \tag{6}$$

By solving (6), we get

$$H=\frac{1}{D_h}\begin{vmatrix} \rho_h & \rho_h \\ \rho_{2h} & 1 \end{vmatrix} \quad \text{and} \quad K=\frac{1}{D_h}\begin{vmatrix} 1 & \rho_h \\ \rho_h & \rho_{2h} \end{vmatrix},$$

where

$$D_h=\begin{vmatrix} 1 & \rho_h \\ \rho_h & 1 \end{vmatrix}.$$

We have

$$E(x(t)/x(t+h),\ x(t+2h))=Hx(t+h)+Kx(t+2h)$$

and

$$\begin{aligned} &E[x(t)x(t+h)\,\mathrm{sgn}(x(t+h))\,\mathrm{sgn}(x(t+2h))] \\ &=HE(x(t+h)^2\,\mathrm{sgn}(x(t+h))\,\mathrm{sgn}(x(t+2h)))+KE(|x(t+h)||x(t+2h)|). \end{aligned}$$

Using the same method as in i), we get

$$E(x(t)x(t+h)\,\mathrm{sgn}(x(t+h))\,\mathrm{sgn}(x(t+2h)))=H\frac{\sigma^2 D_h^{3/2}}{\pi}S_1(\rho_h)+K\frac{\sigma^2 D_h^{3/2}}{\pi}S_2(\rho_h).$$

iii) When $t=s$, it holds that

$$E(x(t)^2\,\mathrm{sgn}^2(x(t+h)))=Ex(t)^2=\sigma^2.$$

Therefore, using the above results, we obtain, by putting $s-t=k$,

$$\mathrm{Var.}(\gamma_h)=E^2\gamma_h-\rho_h^2$$

$$=\frac{1}{N^2}\left\{\sum_{k=1}^{h-1}+\sum_{k=h+1}^{N-1}\right\}(N-k)\Big[(AF^2+BF)D^{3/2}S_1(\rho_k)+(2AFG+BG+CF)D^{3/2}S_2(\rho_k)$$

$$+(AG^2+CG)D^{3/2}S_1(\rho_k)+\frac{A\Delta}{2D^{1/2}}S_3(\rho_k)\Big]$$

$$+\frac{1}{N^2}(N-h)[HD_h^{3/2}S_1(\rho_h)+KD_h^{3/2}S_2(\rho_h)]+\frac{\pi}{2}\cdot\frac{1}{N}-\rho_h^2.$$

3. Comparison of γ_h with $\tilde{\gamma}_h$.

Now we shall compare the estimate γ_h with the estimate $\tilde{\gamma}_h$, which is usually used. The estimate γ_h and the estimate $\tilde{\gamma}_h$ are both unbiased estimates. Here the comparison is made on the point of variance.

It holds, for a stationary Gaussian process with mean 0, that

$$E(x(t)x(t+h)x(s)x(s+h))$$
$$=(Ex(t)x(t+h))(Ex(s)x(s+h))+(Ex(t)x(s))(Ex(t+h)x(s+h))$$
$$+(Ex(t)x(s+h))(Ex(s)x(t+h)),$$

when $t<s$ and $t+h\neq s$. Using the above relation, we obtain

$$\mathrm{Var.}(\tilde{\gamma}_h)=E(\tilde{\gamma}_h^2)-\rho_h^2$$
$$=E\left(\frac{1}{\sigma^2}\,\frac{1}{N}\sum_{t=1}^{N}x(t)x(t+h)\right)^2-\rho_h^2$$
$$=\frac{1}{\sigma^4}\,\frac{1}{N^2}\sum_{t=1}^{N}\sum_{s=1}^{N}Ex(t)x(t+h)x(s)x(s+h)-\rho_h^2$$
$$\fallingdotseq\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(\rho_k^2+\rho_h^2+\rho_{h+k}\rho_{h-k})+\frac{1}{N}(1+2\rho_h^2)-\rho_h^2.$$

Let us compare the variance of γ_h with that of $\tilde{\gamma}_h$, numerically. For this, we shall consider the second-order process in the sense of Bartlett [1]. That is, $x(t)$ is subjected to the equation

$$d\dot{x}(t)+\alpha\dot{x}(t)dt+\beta x(t)dt=dy(t), \tag{7}$$

where $\dot{x}(t)$ is a mean square differential coefficient of $x(t)$, $d\dot{x}(t)$ is the change in $\dot{x}(t)$ in dt and $y(t)$ is the orthogonal process of the accumulated impulse effects.

Then we find that correlogram ρ_τ satisfies the equation

$$\rho_\tau''+\alpha\rho_\tau'+\beta\rho_\tau=0 \qquad (\tau>0),$$

where $\rho_\tau' = d\rho_\tau/d\tau$, etc., whence we have

$$\rho_\tau = Ae^{\lambda_1\tau} + Be^{\lambda_2\tau} \qquad (\tau > 0),$$

where λ_1 and λ_2 are the roots of $\lambda^2 + \alpha\lambda + \beta = 0$. Furthermore ρ_τ must satisfy the condition

$$\rho_0 = 1 \quad \text{and} \quad \rho_0' = 0.$$

This leads finally to

$$\rho_\tau = \frac{\lambda_1}{\lambda_1 - \lambda_2} e_2^{\lambda_2\tau} - \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1\tau} \qquad (\tau > 0).$$

Table 1

h	ρ_h	$N=50$		$N=250$	
		Var. (γ_h)	Var. $(\tilde{\gamma}_h)$	Var. (γ_h)	Var. $(\tilde{\gamma}_h)$
1	0.9572	0.0720	0.2487	0.0149	0.0516
2	0.8536	0.0772	0.2269	0.0159	0.0471
3	0.7192	0.0850	0.1985	0.0173	0.0411
4	0.5759	0.0954	0.1706	0.0194	0.0353
5	0.4384	0.1074	0.1480	0.0218	0.0306
6	0.3154	0.1199	0.1327	0.0244	0.0273
7	0.2116	0.1315	0.1243	0.0268	0.0256
8	0.1282	0.1412	0.1212	0.0289	0.0250
9	0.0647	0.1487	0.1214	0.0305	0.0250
10	0.0189	0.1539	0.1230	0.0317	0.0254
11	−0.0119	0.1573	0.1251	0.0324	0.0259
12	−0.0307	0.1592	0.1268	0.0329	0.0263
13	−0.0403	0.1602	0.1280	0.0331	0.0265
14	−0.0432	0.1606	0.1286	0.0332	0.0267
15	−0.0416	0.1608	0.1289	0.0332	0.0267
16	−0.0373	0.1608	0.1289	0.0332	0.0268
17	−0.0316	0.1607	0.1288	0.0332	0.0267
18	−0.0254	0.1607	0.1287	0.0332	0.0267
19	−0.0194	0.1607	0.1286	0.0332	0.0267
20	−0.0140	0.1607	0.1285	0.0332	0.0267
21	−0.0095	0.1608	0.1285	0.0332	0.0267
22	−0.0058	0.1608	0.1285	0.0332	0.0266
23	−0.0030	0.1608	0.1285	0.0332	0.0266
24	−0.0009	0.1608	0.1285	0.0332	0.0266
25	0.0004	0.1608	0.1285	0.0332	0.0266
30	0.0016	0.1608	0.1285	0.0332	0.0267

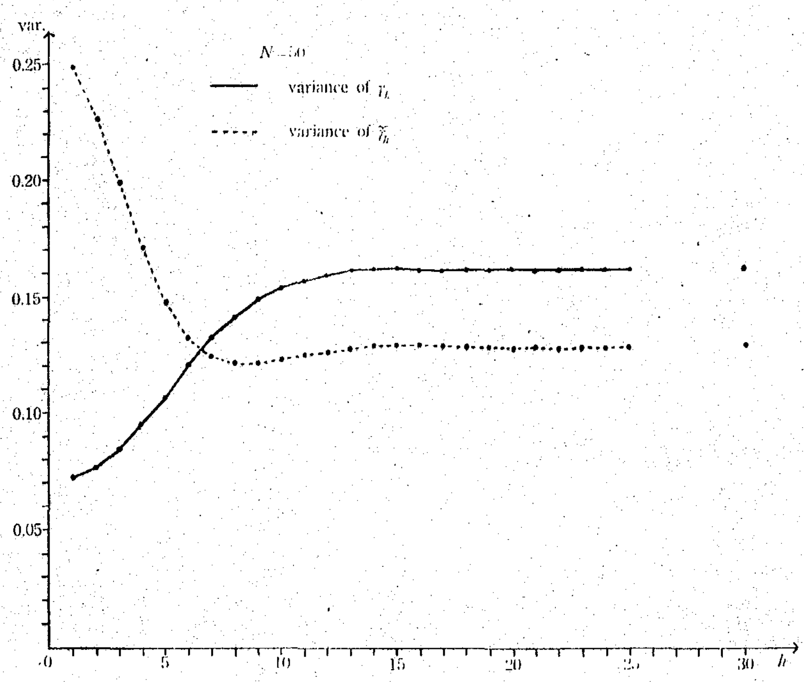

Fig. 1

For numerical computation, we shall take

$$\alpha=-2\log 0.8 \quad \text{and} \quad \beta=2(\log 0.8)^2.$$

In this case, the correlogram is

$$\rho_\tau=\sqrt{2}\,(0.8)^{|\tau|}\cos(|\tau|\log 0.8+\pi/4).$$

By taking $N=50$ and 250, the numerical results are shown in Table 1 and Figure 1.

This results are like as in the case of a simple Markov process [2]. For a small value of lag h, the variances of the estimate γ_h are smaller than that of the estimate $\tilde{\gamma}_h$.

Originally, the model (7) is discussed in Takahasi and Husimi [5]. So we have taken this model in this time and discussed from the statistical point of view. As we have shown, γ_h is a fairly good estimate for a Stationary Gaussian process.

4. Consistency of the estimate γ_h.

Let us further assume that the correlogram ρ_τ has the following property:

for any positive number ε, there exists a number τ_ε such that $|\rho_\tau|<\varepsilon$ is statisfied for any number τ such as $\tau>\tau_\varepsilon$, i.e. $\lim_{\tau\to\infty}\rho_\tau=0$.

In this case, we can prove that γ_h is a consistent estimate. The proof is as follows.

In the expression of the variance of γ_h, we shall put, for simplicity,

$$U_1(k)=AF^2+BF, \qquad U_2(k)=2AFG+BG+CF, \qquad U_3(k)=AG^2+CG.$$

Then, it holds

$$\frac{1}{N^2}\sum_{k=1}^{h-1}(N-k)\Big[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)$$
$$+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\Delta}{2D^{1/2}}S_3(\rho_k)\Big]=O\left(\frac{1}{N}\right)$$

and

$$\frac{1}{N^2}(N-h)[HD_h{}^{3/2}S_1(\rho_h)+KD_h{}^{3/2}S_2(\rho_h)]=O\left(\frac{1}{N}\right).$$

Now we shall evaluate the value of

$$\frac{1}{N^2}\sum_{k=h+1}^{N-1}(N-k)\Big[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\Delta}{2D^{1/2}}S_3(\rho_k)\Big].$$

For any positive number ε, there exist a positive number $K=K(\varepsilon)$ such that

$$|\rho_k|<\varepsilon \quad \text{and} \quad |\rho_{k-h}|<\varepsilon$$

are satisfied for any k being larger than K.

It holds

$$|A|=\frac{1}{\Delta}\begin{vmatrix}\rho_k & \rho_{k-h} & \rho_h\\ \rho_h & 1 & \rho_k\\ \rho_{k+h} & \rho_k & 1\end{vmatrix}=\left|\frac{\rho_k+\rho_k\rho_{k+h}\rho_{k-h}+\rho_h^2\rho_k-\rho_h\rho_{k+h}-\rho_h\rho_{k-h}-\rho_k^3}{1+2\rho_k\rho_h\rho_{k-h}-\rho_h^2-\rho_{k-h}^2-\rho_k^2}\right|$$

and, for any $k>K$,

$$|\rho_k+\rho_k\rho_{k+h}\rho_{k-h}+\rho_h^2\rho_k-\rho_h\rho_{k+h}-\rho_h\rho_{k-h}-\rho_k^3|$$
$$\leqq|\rho_k|+|\rho_k\rho_{k+h}\rho_{k-h}|+|\rho_h^2\rho_k|+|\rho_h\rho_{k+h}|+|\rho_h\rho_{k-h}|+|\rho_k^3|$$
$$<6\varepsilon,$$
$$|1+2\rho_k\rho_h\rho_{k-h}-\rho_h^2-\rho_{k-h}^2-\rho_k^2|$$
$$\geqq 1-\rho_h^2-2|\rho_k\rho_h\rho_{k-h}|-\rho_{k-h}^2-\rho_k^2$$
$$\geqq 1-\rho_h^2-4\varepsilon^2=(1-\rho_h^2)\left(1-\frac{4\varepsilon^2}{1-\rho_h^2}\right).$$

Now we can say

$$1-\frac{4\varepsilon^2}{1-\rho_h^2}\geqq\frac{1}{2}.$$

So we have

$$|A|\leqq\frac{12}{1-\rho_h^2}\varepsilon=a\varepsilon,\qquad a=\frac{12}{1-\rho_h^2}.$$

In the next place,

$$|B|=\left|\frac{\rho_h+\rho_h\rho_{k+h}\rho_{k-h}+\rho_h\rho_k^2-\rho_k\rho_{k-h}-\rho_k\rho_{k+h}-\rho_h^3}{1+2\rho_k\rho_h\rho_{k-h}-\rho_h^2-\rho_{k-h}^2-\rho_k^2}\right|$$

$$\leqq\frac{|\rho_h-\rho_h^3|+4\varepsilon^2}{(1-\rho_h^2)\left(1-\dfrac{4\varepsilon^2}{1-\rho_h^2}\right)}$$

$$\doteqdot|\rho_h|+\frac{4\varepsilon^2|\rho_h|+4\varepsilon^2}{(1-\rho_h^2)\left(1-\dfrac{4\varepsilon^2}{1-\rho_h^2}\right)}$$

$$\leqq|\rho_h|+b\varepsilon^2.$$

Similary we obtain

$$|C|\leqq\varepsilon c,\ |F|\leqq\varepsilon f\ \text{ and }\ |G|\leqq|\rho_h|+\varepsilon^2 g.$$

In the above expression, a, b, c, f and g are constants which are independent of k and N.

Let us evaluate the value of $S_1(\rho_k)$, $S_2(\rho_k)$ and $S_3(\rho_k)$.

$$|S_1(\rho_k)|=2\left|\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\right|$$

$$\leqq4|\rho_k|\sum_{m=0}^{\infty}(2\rho_k)^{2m}\leqq4\varepsilon\sum_{m=0}^{\infty}(2\varepsilon)^{2m}=\frac{4\varepsilon}{1-(2\varepsilon)^2}\leqq l_1\varepsilon,$$

$$|S_2(\rho_k)|=2\left|\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right|$$

$$=2\left(1+(2\rho_k)^2\left(\sum_{m=1}^{\infty}\frac{(2\rho_k)^{2(m-1)}}{(2m)!}\Gamma(m+1)^2\right)\right)$$

$$\leqq2\left(1+(2\varepsilon)^2\frac{1}{1-(2\varepsilon)^2}\right)\leqq2(1+l_2\varepsilon^2)$$

and

$$|S_3(\rho_k)|=2\left|\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+1)^2\right|$$

$$\leqq 4|\rho_k|\left(\sum_{m=1}^{\infty}(2\rho_k)^{2m}\right)\leqq \frac{4\varepsilon}{1-(2\varepsilon)^2}\leqq l_3\varepsilon,$$

where l_1, l_2 and l_3 are constants which are independent of k and N. From the above results, it holds

$$|U_1(k)D^{3/2}S_1(\rho_k)|\leqq(af^2\varepsilon^3+f|\rho_h|\varepsilon+bf\varepsilon^3)\cdot 1\cdot l_1\varepsilon\leqq\varepsilon^2 d_1,$$

$$|U_2(k)D^{3/2}S_2(\rho_k)|\leqq(2af|\rho_h|\varepsilon^2+2afg\varepsilon^4+|\rho_h|^2+b|\rho_h|\varepsilon^2+g|\rho_h|\varepsilon^2$$
$$+bg\varepsilon^4+fc\varepsilon^2)\cdot 1\cdot 2(1+l_2\varepsilon^2)\leqq 2(\rho_h^2+\varepsilon^2 d_2),$$

$$|U_3(k)D^{3/2}S_1(\rho_k)|\leqq(a(|\rho_h|+\varepsilon^2 g)^2\varepsilon+c(|\rho_h|+\varepsilon^2 g)\varepsilon)\cdot 1\cdot l_1\varepsilon\leqq d_3\varepsilon^2$$

and

$$\left|\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\right|\leqq|A||\varDelta||S_3(\rho_k)|\leqq(a\varepsilon)(1-\rho_h^2+4\varepsilon^2)(l_3\varepsilon)\leqq d_4\varepsilon^2,$$

where d_1, d_2, d_3 and d_4 are constants which are independent of k and N. Accordingly, we get

$$\frac{1}{N^2}\sum_{k=K+1}^{N-1}(N-k)\left[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\right]$$

$$\leqq\frac{1}{N^2}\sum_{k=K+1}^{N-1}(N-k)[\varepsilon^2 d_1+2\rho_h^2+2\varepsilon^2 d_2+\varepsilon^2 d_3+\varepsilon^2 d_4]$$

$$=(\varepsilon^2 d_1+2\rho_h^2+2\varepsilon^2 d_2+\varepsilon^2 d_3+\varepsilon^2 d_4)\left(\frac{1}{N^2}\sum_{k=K+1}^{N-1}(N-k)\right)$$

$$=(\varepsilon^2 d_1+2\rho_h^2+2\varepsilon^2 d_2+\varepsilon^2 d_3+\varepsilon^2 d_4)\frac{1}{2}\left(1-\frac{K+1}{N}\right)\left(1-\frac{K}{N}\right)$$

$$=\rho_h^2-\frac{(2K+1)}{N}\rho_h^2+\frac{K(K+1)}{N^2}\rho_h^2+\frac{\varepsilon^2}{2}(d_1+2d_2+d_3+d_4)\left(1-\frac{K+1}{N}\right)\left(1-\frac{K}{N}\right)$$

$$=\rho_h^2+O\left(\frac{1}{N}\right)+O(\varepsilon^2).$$

And it holds that

$$\frac{1}{N^2}\sum_{k=h+1}^{K}(N-k)\left[U_1(k)D^{3/2}S_1(\rho_k)+U_2(k)D^{3/2}S_2(\rho_k)+U_3(k)D^{3/2}S_1(\rho_k)+\frac{A\varDelta}{2D^{1/2}}S_3(\rho_k)\right]$$

$$=O\left(\frac{1}{N}\right).$$

Finally, we obtain

$$\text{Var.}(\gamma_h)=O\left(\frac{1}{N}\right)+O(\varepsilon^2),$$

so

$$P(|\gamma_h-\rho_h|>\theta)\leqq\frac{\text{Var.}(\gamma_h)}{\theta^2}=\frac{1}{\theta^2}\left(O\left(\frac{1}{N}\right)+O(\varepsilon^2)\right).$$

This shows that the estimate γ_h is a consistent estimate.

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CHAPTER 4

THE CASE OF A GAUSSIAN PROCESS

WITH UNKNOWN VARIANCE

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ON A SIMPLIFIED METHOD OF THE ESTIMATION OF THE CORRELOGRAM FOR A STATIONARY GAUSSIAN PROCESS, III

By Mituaki Huzii

§ 1. Introduction.

In this paper we shall deal with a simplified method for the estimation of the correlogram for a stationary process.

Let $X(n)$ be a real-valued stationary process with discrete time parameter n. We assume $EX(n)=0$. We put

$$EX(n)^2=\sigma^2, \qquad EX(n)X(n+h)=\sigma^2\rho_h,$$

and we consider to estimate the correlogram ρ_h.

In the previous papers [4], [5], we discussed a simplified method for the estimation of the correlogram when σ^2 is known. But in the present paper, we discuss the case when σ^2 is unknown. For simplicity, let us assume the process $X(n)$ to be observed at $n=1, 2, \cdots, N, \cdots, N+h$.

Usually, in order to estimate the correlogram ρ_h, we use the estimate

$$\tilde{\Gamma}_h=\frac{\sum_{n=1}^{N} X(n)X(n+h)}{\sum_{n=1}^{N} X(n)^2}.$$

Now we shall modify the estimate $\tilde{\Gamma}_h$. The essential part of our modification is to replace $X(n)X(n+h)$ by $X(n)\,\mathrm{sgn}\,(X(n+h))$, where $\mathrm{sgn}\,(y)$ means $1, 0, -1$ correspondingly as $y>0$, $y=0$, $y<0$. The new estimate is

$$\Gamma_h=\frac{\sum_{n=1}^{N} X(n)\,\mathrm{sgn}\,(X(n+h))}{\sum_{n=1}^{N} |X(n)|}.$$

This new estimate Γ_h may be considered as follows. We make a nonlinear operation on the input $X(n)$ and assume that the output is $Y(n)=\mathrm{sgn}\,(X(n))$. Then, the estimate Γ_h consists of the cross-correlation of the input $X(n)$ and the output $Y(n)$.

We shall show below that when $X(n)$ is a Gaussian process satisfying some conditions, the estimate Γ_h is an asymptotically unbiased estimate of the correlogram

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ρ_h as $N\to\infty$. We evaluate the asymptotic variance of Γ_h. The estimate $\tilde{\Gamma}_h$ is also an asymptotically unbiased estimate of ρ_h. Further, Γ_h and $\tilde{\Gamma}_h$ are both consistent estimates of ρ_h. We compare, for the typical cases, the asymptotic variance of Γ_h with that of $\tilde{\Gamma}_h$.

§ 2. The estimate Γ_h.

Let $X(n)$ be a stationary Gaussian process having a finite moving average repre sentation

$$X(n)=G_0\xi(n)+G_1\xi(n-1)+\cdots+G_M\xi(n-M), \tag{1}$$

where $\xi(n)$ is the white noise with

$$E\xi(n)=0, \quad E\xi(n)^2=1,$$
$$E\xi(n_1)\xi(n_2)=0 \quad \text{when } n_1\neq n_2,$$

M is some positive number and $\{G_k\text{'s}\}$ are constants.

Let $L_2(X;n)$ denote the closed linear manifold generated by $\{X(j);j\leqq n\}$ and $L_2(\xi;n)$ denote the closed linear manifold generated by $\{\xi(j);j\leqq n\}$.

LEMMA 1. *If $X(n)$ is a stationary Gaussian process which has the moving average representation* (1) *and if the condition*

$$L_2(X;n)=L_2(\xi;n) \tag{2}$$

holds for an arbitrary integer n, $\xi(n)$ is a stationary Gaussion process.

In fact, we consider the joint distribution of $\xi(n_1),\cdots,\xi(n_k)$. As $\xi(n_\nu)\in L_2(X;n_\nu)$, there are constants $\{a_l;l=0,1,2,\cdots\}$ such that

$$\xi(n_\nu)=\underset{N\to\infty}{\text{l. i. m.}}\sum_{l=0}^{N}a_lX(n_\nu-l).$$

Therefore for any real numbers $A_1,A_2,\cdots,A_k$,

$$\begin{aligned}&A_1\xi(n_1)+A_2\xi(n_2)+\cdots+A_k\xi(n_k)\\&=\underset{N\to\infty}{\text{l. i. m}}\left\{A_1\left(\sum_{l=0}^{N}a_lX(n_1-l)\right)+A_2\left(\sum_{l=0}^{N}a_lX(n_2-l)\right)+\cdots+A_k\left(\sum_{l=0}^{N}a_lX(n_k-l)\right)\right\}.\end{aligned}$$

The distribution of

$$A_1\left(\sum_{l=0}^{N}a_lX(n_1-l)\right)+A_2\left(\sum_{l=0}^{N}a_lX(n_2-l)\right)+\cdots+A_k\left(\sum_{l=0}^{N}a_lX(n_k-l)\right)$$

is Guassian, so the distribution function of

$$A_1\xi(n_1)+A_2\xi(n_2)+\cdots+A_k\xi(n_k)$$

is Gaussian. This shows $\xi(n)$ is a Gaussian process.

As $\xi(n)$ is a white noise, $\xi(n_1)$ and $\xi(n_2)$ are orthogonal, for any $n_1\neq n_2$, so that, by the above lemma, $\xi(n_1)$ and $\xi(n_2)$ are mutually independent.

Now we determine the asymptotic distribution of the estimate Γ_h. Without loss of generality, we can assume that $h>0$. We have

$$\sqrt{N}(\Gamma_h-\rho_h)=\sqrt{N}\left(\frac{\sum_{n=1}^{N} X(n)\operatorname{sgn}(X(n+h))}{\sum_{n=1}^{N}|X(n)|}-\rho_h\right)$$

$$=\sqrt{N}\left(\frac{\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N} X(n)\operatorname{sgn}(X(n+h))}{\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}|X(n)|}-\rho_h\right)$$

$$=\frac{\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}}{\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}|X(n)|}.$$

In the first place, we consider the statistic

$$\gamma_0=\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}|X(n)|.$$

Using the results in Huzii [4], we have

$$E(\gamma_0)=1$$

and

$$V(\gamma_0)=\text{the variance of } \gamma_0$$

$$=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)+\frac{\pi}{2}\frac{1}{N}-1.$$

LEMMA 2. *If $X(n)$ is a process having the representation* (1), *then $V(\gamma_0)\to 0$ as $N\to\infty$.*

Proof. For our process $X(n)$, $\rho_k=0$ when $|k|>M$. So we have

$$V(\gamma_0)=\frac{2}{N^2}\sum_{k=1}^{M}(N-k)(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)$$
$$+\frac{2}{N^2}\sum_{k=M+1}^{N}(N-k)+\frac{\pi}{2}\frac{1}{N}-1.$$

Now,

$$\frac{2}{N^2}\sum_{k=M+1}^{N}(N-k)=\frac{2}{N^2}\cdot\frac{(N-M-1)(N-M)}{2}=1-\frac{(2M+1)}{N}+\frac{M(M+1)}{N^2}.$$

Therefore we get

$$V(\gamma_0)=\frac{2}{N}\sum_{k=1}^{M}\left(1-\frac{k}{N}\right)(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)$$
$$-\frac{(2M+1)}{N}+\frac{M(M+1)}{N^2}+\frac{\pi}{2}\frac{1}{N}.$$

This shows $V(\gamma_0)\to 0$ as $N\to\infty$.

From this Lemma 2, we can find the following result:

THEOREM 1. *γ_0 converges in probability to 1 as $N\to\infty$.*

In the next place, we consider the numerator of $\sqrt{N}(\Gamma_h-\rho_h)$, that is,

$$\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}.$$

Let us denote

$$Y(n)=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}.$$

Since the process $X(n)$ has the representation (1) and the $\xi(n)$'s are mutually independent, $Y(n_1)$ and $Y(n_2)$ are mutually independent if $|n_1-n_2|>M+h$.

Here, we quote the result in Diananda [2].

DEFINITION 1 (Diananda). Let d_n be a function of n. Suppose $\{X_i\}$ $(i=1, 2, \cdots)$ is a sequence of random variables such that the two sets of variables $(X_1, X_2, \cdots, X_r)$ and $(X_s, X_{s+1}, \cdots, X_n)$ are independent whenever $s-r>d_n$. Then we say that $\{X_i\}$ $(i=1, 2, \cdots)$ is a sequence of *d_n-dependent variables* or is a *d_n-dependent process.*

LEMMA 3 (Diananda). *Let $\{X_i\}$ $(i=1, 2, \cdots)$ be a sequence of stationary m-dependent scalar variables with the mean zero and $E(X_iX_j)=C_{i-j}$. Then the distribution function of the random variable $(X_1+X_2+\cdots+X_n)/\sqrt{n}\to$ the normal distribution function with the mean zero and the variance $\sum_{-m}^{m}C_p$ as $n\to\infty$.*

In our case, $Y(n)$ is a sequence of $(M+h)$-dependent variables and since $X(n)$ is a stationary Gaussian process, $Y(n)$ is a stationary process. It is clear that $EY(n)=0$. Let us denote $EY(n)Y(m)=C(n-m)$. From the above Lemma 3, the distribution function of the random variable

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}Y(n)=\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}$$

tends to the normal distribution function with the mean zero and the variance $\sum_{k=-(M+h)}^{M+h}C(k)$ *as* $N\to\infty$.

Now, we shall evaluate the value of $C(k)=EY(n)Y(n+k)$.

$$C(k)=EY(n)Y(n+k)$$
$$=\frac{\pi}{2}\frac{1}{\sigma^2}E\{(X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|)$$
$$\cdot(X(n+k)\operatorname{sgn}(X(n+k+h))-\rho_h|X(n+k)|)\}$$
$$=\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)\operatorname{sgn}(X(n+h))X(n+k)\operatorname{sgn}(X(n+k+h))$$
$$-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h EX(n)\operatorname{sgn}(X(n+h))|X(n+k)|$$
$$-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h E|X(n)|X(n+k)\operatorname{sgn}(X(n+k+h))+\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2 E|X(n)||X(n+k)|.$$

(i) When k is neither zero nor $\pm h$, we have, by using the results in the previous paper [5],

$$\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)\operatorname{sgn}(X(n+h))X(n+k)\operatorname{sgn}(X(n+k+h))$$
$$=\frac{1}{2}\Big\{(AF^2+BF)D^{3/2}S_1(\rho_k)+(2AFG+BG+CF)D^{3/2}S_2(\rho_k)$$
$$+(AG^2+CG)D^{3/2}S_1(\rho_k)+A\cdot\frac{\Delta}{2\sqrt{D}}S_3(\rho_k)\Big\}$$

and

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2E|X(n)|\,|X(n+k)|=\frac{1}{2}\rho_h^2D^{3/2}S_2(\rho_k),$$

where

$$\Delta=\begin{vmatrix}1 & \rho_{k-h} & \rho_h\\ \rho_{k-h} & 1 & \rho_k\\ \rho_h & \rho_k & 1\end{vmatrix},\qquad A=\frac{1}{\Delta}\begin{vmatrix}\rho_k & \rho_{k-h} & \rho_h\\ \rho_h & 1 & \rho_k\\ \rho_{k+h} & \rho_k & 1\end{vmatrix},$$
$$B=\frac{1}{\Delta}\begin{vmatrix}1 & \rho_k & \rho_h\\ \rho_{k-h} & \rho_h & \rho_k\\ \rho_h & \rho_{k+h} & 1\end{vmatrix},\qquad C=\frac{1}{\Delta}\begin{vmatrix}1 & \rho_{k-h} & \rho_k\\ \rho_{k-h} & 1 & \rho_h\\ \rho_h & \rho_k & \rho_{k+h}\end{vmatrix},$$
$$D=\begin{vmatrix}1 & \rho_k\\ \rho_k & 1\end{vmatrix},\qquad F=\frac{1}{D}\begin{vmatrix}\rho_{k-h} & \rho_k\\ \rho_h & 1\end{vmatrix},\qquad G=\frac{1}{D}\begin{vmatrix}1 & \rho_{k-h}\\ \rho_k & \rho_h\end{vmatrix},$$
$$S_1(\rho_k)=2\Big(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\Big),$$
$$S_2(\rho_k)=2\Big(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\Big),$$
$$S_3(\rho_k)=2\Big(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+1)^2\Big).$$

Now, the value of

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h EX(n)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|$$

is as follows. Suppose that

$$X(n)=U_1X(n+k)+V_1X(n+h)+\nu_1(n),$$

where $\nu_1(n)$ is a Gaussian process with the mean zero and satisfies

$$E\nu_1(n)X(n+k)=0, \qquad E\nu_1(n)X(n+h)=0.$$

Then, U_1 and V_1 are determined by the following conditions:

$$E(X(n)-U_1X(n+k)-V_1X(n+h))X(n+k)=0,$$
$$E(X(n)-U_1X(n+k)-V_1X(n+h))X(n+h)=0.$$

From these, we get

$$U_1=\frac{1}{D_1}\begin{vmatrix}\rho_k & \rho_{h-k}\\ \rho_h & 1\end{vmatrix} \quad \text{and} \quad V_1=\frac{1}{D_1}\begin{vmatrix}1 & \rho_k\\ \rho_{h-k} & \rho_h\end{vmatrix},$$

where

$$D_1=\begin{vmatrix}1 & \rho_{h-k}\\ \rho_{h-k} & 1\end{vmatrix}.$$

The new random variable $\nu_1(n)$, determined in the above, is independent of $X(n+k)$, $X(n+h)$ and $(X(n+k), X(n+h))$. Using these results, we have

$$\begin{aligned}&EX(n)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|\\ &=E(U_1X(n+k)+V_1X(n+h)+\nu_1(n))\,\mathrm{sgn}\,(X(n+h))|X(n+k)|\\ &=U_1EX(n+k)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|+V_1E|X(n+h)||X(n+k)|\\ &=U_1\frac{\sigma^2}{\pi}D_1^{3/2}S_1(\rho_{h-k})+V_1\frac{\sigma^2}{\pi}D_1^{3/2}S_2(\rho_{h-k}).\end{aligned}$$

So we have

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h EX(n)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|=\frac{\rho_h}{2}\{U_1D_1^{3/2}S_1(\rho_{h-k})+V_1D_1^{3/2}S_2(\rho_{h-k})\}.$$

Similarly, we get

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h E|X(n)|X(n+k)\,\mathrm{sgn}\,(X(n+k+h))=\frac{\rho_h}{2}\{U_2D_2^{3/2}S_1(\rho_{k+h})+V_2D_2^{3/2}S_2(\rho_{k+h})\},$$

where

$$D_2=\begin{vmatrix}1 & \rho_{h+k}\\ \rho_{h+k} & 1\end{vmatrix}, \qquad U_2=\frac{1}{D_2}\begin{vmatrix}\rho_k & \rho_{k+h}\\ \rho_h & 1\end{vmatrix} \quad \text{and} \quad V_2=\frac{1}{D_2}\begin{vmatrix}1 & \rho_k\\ \rho_{k+h} & \rho_h\end{vmatrix}.$$

Consequently, using the above results, we obtain

$$C(k)=EY(n)Y(n+k)$$

$$=\frac{1}{2}\Bigg[\Big\{(AF^2+BF)D^{3/2}S_1(\rho_k)+(2AFG+BG+CF)D^{3/2}S_2(\rho_k)$$

$$+(AG^2+CG)D^{3/2}S_1(\rho_k)+A\cdot\frac{4}{2\sqrt{D}}S_3(\rho_k)\Big\}$$

$$-\rho_h D_1^{3/2}\{U_1S_1(\rho_{h-k})+V_1S_2(\rho_{h-k})\}$$

$$-\rho_h D_2^{3/2}\{U_2S_1(\rho_{h+k})+V_2S_2(\rho_{h+k})\}+\rho_h^2D^{3/2}S_2(\rho_k)\Bigg].$$

(ii) Here we shall treat the case $|k|=h$. In the first place, let us consider the case $k=h$.

$$C(h)=\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)|X(n+h)|\operatorname{sgn}(X(n+2h))$$

$$-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hEX(n)X(n+h)-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hE|X(n)|X(n+h)\operatorname{sgn}(X(n+2h))$$

$$+\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2E|X(n)||X(n+h)|.$$

In this expression,

$$\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)|X(n+h)|\operatorname{sgn}(X(n+2h))=\frac{1}{2}D_h^{3/2}(H_1S_1(\rho_h)+K_1S_2(\rho_h)),$$

where

$$D_h=\begin{vmatrix}1 & \rho_h\\ \rho_h & 1\end{vmatrix},\qquad H_1=\frac{1}{D_h}\begin{vmatrix}\rho_h & \rho_h\\ \rho_{2h} & 1\end{vmatrix}\qquad\text{and}\qquad K_1=\frac{1}{D_h}\begin{vmatrix}1 & \rho_h\\ \rho_h & \rho_{2h}\end{vmatrix}.$$

And

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hEX(n)X(n+h)=\frac{\pi}{2}\rho_h^2.$$

We treat the term

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hE|X(n)|X(n+h)\operatorname{sgn}(X(n+2h))$$

as the following. Let us put

$$X(n+h)=H_2X(n)+K_2X(n+2h)+\delta_2(n),$$

where $\delta_2(n)$ is independent of $X(n)$, $X(n+2h)$ and $(X(n), X(n+2h))$. The above condition is satisfied by determining the constants H_2 and K_2 from the following relations:

$$E\delta_2(n)X(n)=0\qquad\text{and}\qquad E\delta_2(n)X(n+2h)=0.$$

Then H_2 and K_2 are

$$H_2=\frac{1}{D_{2h}}\begin{vmatrix}\rho_h & \rho_{2h}\\ \rho_h & 1\end{vmatrix}\quad\text{and}\quad K_2=\frac{1}{D_{2h}}\begin{vmatrix}1 & \rho_h\\ \rho_{2h} & \rho_h\end{vmatrix},$$

where

$$D_{2h}=\begin{vmatrix}1 & \rho_{2h}\\ \rho_{2h} & 1\end{vmatrix}.$$

Hence we have

$$\begin{aligned}&\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h E|X(n)|X(n+h)\operatorname{sgn}(X(n+2h))\\ &=\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h H_2 EX(n)^2\operatorname{sgn}(X(n))\operatorname{sgn}(X(n+2h))\\ &\quad+\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h K_2 E|X(n)||X(n+2h)|\\ &=\frac{1}{2}\rho_h D_{2h}^{3/2}(H_2S_1(\rho_{2h})+K_2S_2(\rho_{2h})).\end{aligned}$$

Lastly, it is shown

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2 E|X(n)||X(n+h)|=\frac{1}{2}\rho_h^2 D_h^{3/2}S_2(\rho_h).$$

Consequently, we obtain

$$\begin{aligned}C(h)=\frac{1}{2}[&D_h^{3/2}(H_1S_1(\rho_h)+K_1S_2(\rho_h))\\ &-\pi\rho_h^2-\rho_h D_{2h}^{3/2}(H_2S_1(\rho_{2h})+K_2S_2(\rho_{2h}))+\rho_h^2D_h^{3/2}S_2(\rho_h)].\end{aligned}$$

In the next place, when $k=-h$, we can consider

$$C(-h)=C(h).$$

(iii) When $k=0$,

$$\begin{aligned}C(0)&=\frac{\pi}{2}\frac{1}{\sigma^2}E(X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|)^2\\ &=\frac{\pi}{2}\frac{1}{\sigma^2}(EX(n)^2-2\rho_h EX(n)^2\operatorname{sgn}(X(n))\operatorname{sgn}(X(n+h))+\rho_h^2EX(n)^2)\\ &=\frac{\pi}{2}\frac{1}{\sigma^2}\left(\sigma^2-2\rho_h\frac{\sigma^2}{\pi}D_h^{3/2}S_1(\rho_h)+\rho_h^2\sigma^2\right)\\ &=\frac{\pi}{2}-\rho_h D_h^{3/2}S_1(\rho_h)+\frac{\pi}{2}\rho_h^2.\end{aligned}$$

From the above results, we have

$$
\begin{aligned}
C_h &= \sum_{k=-(M+h)}^{M+h} C(k) = C(0) + 2\sum_{k=1}^{M+h} C(k) \\
&= \frac{\pi}{2} - \rho_h D_h^{3/2} S_1(\rho_h) - \frac{\pi}{2}\rho_h^2 + D_h^{3/2}(H_1 S_1(\rho_h) + K_1 S_2(\rho_h)) \\
&\quad - \rho_h D_{2h}^{3/2}(H_2 S_1(\rho_{2h}) + K_2 S_2(\rho_{2h})) + \rho_h^2 D_h^{3/2} S_2(\rho_h) \\
&\quad + \sum_{\substack{k=1 \\ (k \neq h)}}^{M+h} \Big[(AF^2 + BF) D^{3/2} S_1(\rho_k) + (2AFG + BG + CF) D^{3/2} S_2(\rho_k) \\
&\quad + (AG^2 + CG) D^{3/2} S_1(\rho_k) + A \cdot \frac{\Delta}{2\sqrt{D}} S_3(\rho_k) - \rho_h D_1^{3/2}(U_1 S_1(\rho_{h-k}) + V_1 S_2(\rho_{h-k})) \\
&\quad - \rho_h D_2^{3/2}(U_2 S_1(\rho_{h+k}) + V_2 S_2(\rho_{h+k})) + \rho_h^2 D^{3/2} S_2(\rho_k) \Big].
\end{aligned} \tag{3}
$$

Now we shall make the following assumptions:

(A, 1) The determinants Δ, D, D_1 and D_2 are not zero when $k \geqq 1$ and $k \neq h$.

(A, 2) $D_h \neq 0$ and $D_{2h} \neq 0$.

Here we rearrange the above results.

THEOREM 2. *If $X(n)$ is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2) *and if the correlogram has the properties* (A, 1) *and* (A, 2), *the distribution function of $\sum_{n=1}^{N} Y(n)/\sqrt{N}$ tends to the normal distribution function with the mean zero and the variance C_h as $N \to \infty$.*

Now, we shall consider the distribution function of $\sqrt{N}(\Gamma_h - \rho_h)$. By Theorem 1,

$$\gamma_0 = \frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}|X(n)|$$

converges in probability to 1 as $N \to \infty$. And by Theorem 2, the distribution function of

$$\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\,\mathrm{sgn}\,(X(n+h)) - \rho_h|X(n)|\}$$

tends to the normal distribution function with the mean zero and the variance C_h as $N \to \infty$. Therefore we have the following theorem.

THEOREM 3. *If $X(n)$ is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2) *and if the correlogram has the properties* (A, 1) *and* (A, 2), *the distribution function of $\sqrt{N}(\Gamma_h - \rho_h)$ tends to the normal distribution function with the mean zero and the variance C_h as $N \to \infty$.*

§ 3. The estimate $\tilde{\Gamma}_h$.

In this section, we shall consider, with respect to the estimate $\tilde{\Gamma}_h$, the same as we did in § 2. Let the process $X(n)$ have the same properties as § 2.

Now we have

$$\sqrt{N}(\tilde{\Gamma}_h-\rho_h)=\sqrt{N}\left(\frac{\sum_{n=1}^{N}X(n)X(n+h)}{\sum_{n=1}^{N}X(n)^2}-\rho_h\right)=\sqrt{N}\left(\frac{\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)X(n+h)}{\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)^2}-\rho_h\right)$$

$$=\frac{\frac{1}{\sqrt{N}}\frac{1}{\sigma^2}\sum_{n=1}^{N}(X(n)X(n+h)-\rho_h X(n)^2)}{\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)^2}.$$

We shall denote

$$\tilde{\gamma}_0=\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)^2.$$

Then from the results in Huzii [4],

$$E(\tilde{\gamma}_0)=1$$

and

$$V(\tilde{\gamma}_0)=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1+2\rho_k^2)+\frac{3}{N}-1$$

$$=\frac{1}{N}\left\{2+4\sum_{k=1}^{M}\left(1-\frac{k}{N}\right)\rho_k^2\right\}.$$

Hence, we have following lemma and theorem.

LEMMA 4. *If $X(n)$ is a stationary Gaussian process which has a finite moving average representation* (1), *$V(\tilde{\gamma}_0)$ tends to zero as $N\to\infty$.*

THEOREM 4. *If $X(n)$ is a stationary Gaussian process having a finite moving average representation* (1), *$\tilde{\gamma}_0$ converges in probability to* 1.

Now, we shall consider the statistic

$$\frac{1}{\sqrt{N}}\frac{1}{\sigma^2}\sum_{n=1}^{N}(X(n)X(n+h)-\rho_h X(n)^2).$$

Let us put

$$\tilde{Y}(n)=\frac{1}{\sigma^2}(X(n)X(n+h)-\rho_h X(n)^2).$$

As $X(n)$ is a stationary Gaussian process having a finite moving average representation (1) which satisfies the condition (2), $\tilde{Y}(n)$ is a $(M+h)$-dependent variable and $E\tilde{Y}(n)=0$. Clearly, $\tilde{Y}(n)$ is a stationary process. We shall denote

$$E\tilde{Y}(n)\tilde{Y}(m)=\tilde{C}(n-m).$$

By using the result of Lemma 3, the distribution function of the random variable

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\tilde{Y}(n)=\frac{1}{\sqrt{N}}\frac{1}{\sigma^2}\sum_{n=1}^{N}(X(n)X(n+h)-\rho_h X(n)^2)$$

tends to the normal distribution function with the mean zero and the variance $\sum_{k=-(M+h)}^{M+h}\tilde{C}(k)$ as $N\to\infty$.

Combining the above result with Theorem 4, we can say that the distribution function of $\sqrt{N}(\tilde{\Gamma}_k-\rho_h)$ tends to the normal distriburion function with the mean zero and the variance $\sum_{k=-(M+h)}^{M+h}\tilde{C}(k)$ as $N\to\infty$.

Let us now compute the value of $\sum_{k=-(M+h)}^{M+h}\tilde{C}(k)$.

$$\begin{aligned}\tilde{C}(k)&=E\tilde{Y}(n)\tilde{Y}(n+k)\\&=\frac{1}{\sigma^4}E(X(n)X(n+h)-\rho_h X(n)^2)(X(n+k)X(n+k+h)-\rho_h X(n+k)^2)\\&=\frac{1}{\sigma^4}\{EX(n)X(n+k)X(n+h)X(n+k+h)-\rho_h EX(n)^2X(n+k)X(n+k+h)\\&\qquad-\rho_h EX(n)X(n+k)^2X(n+h)+\rho_h^2EX(n)^2X(n+k)^2\}.\end{aligned}$$

(i) When k is neither zero nor $\pm h$,

$$\begin{aligned}\tilde{C}(k)&=(\rho_k^2+\rho_h^2+\rho_{h-k}\rho_{k+h})-\rho_h(\rho_h+2\rho_k\rho_{k+h})-\rho_h(\rho_h+2\rho_k\rho_{h-k})+\rho_h^2(1+2\rho_k^2)\\&=\rho_k^2+\rho_{h-k}\rho_{h+k}-2\rho_h\rho_k\rho_{k+h}-2\rho_h\rho_k\rho_{h-k}+2\rho_h^2\rho_k^2.\end{aligned}$$

(ii) When $k=h$,

$$\tilde{C}(h)=\rho_{2h}+2\rho_h^4-\rho_h^2-2\rho_h^2\rho_{2h}$$

and when $k=-h$,

$$\tilde{C}(-h)=\tilde{C}(h).$$

(iii) When $k=0$,

$$\tilde{C}(0)=1-\rho_h^2.$$

Putting

$$\tilde{C}_h=\sum_{k=-(M+h)}^{M+h}\tilde{C}(k),$$

we obtain, from the above results,

$$\begin{aligned}\tilde{C}_h=&1-\rho_h^2+2(\rho_{2h}+2\rho_h^4-\rho_h^2-2\rho_h^2\rho_{2h})\\&+2\sum_{\substack{k=1\\(k\neq h)}}^{M+h}(\rho_k^2+\rho_{h-k}\rho_{h+k}-2\rho_h\rho_k\rho_{h+k}-2\rho_h\rho_k\rho_{h-k}+2\rho_h^2\rho_k^2).\end{aligned}$$

Hence we have the following theorems:

THEOREM 5. *If $X(n)$ is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2), *the distribution function* *of* $\sum_{n=1}^{N}\tilde{Y}(n)/\sqrt{N}$ *tends to the normal distribution function with the mean zero and the variance* $\tilde{C}_h$ *as* $N\to\infty$.

THEOREM 6. *If* $X(n)$ *is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2), *the distribution function of* $\sqrt{N}(\tilde{\Gamma}_h-\rho_h)$ *tends to the normal distribution function with the mean zero and the variance* $\tilde{C}_h$ *as* $N\to\infty$.

§ 4. Comparison of the estimate Γ_h with the estimate $\tilde{\Gamma}_h$.

We shall compare the estimate Γ_h with the estimate $\tilde{\Gamma}_h$ on the viewpoint of the variance. Without loss of generality, we can assume $h>0$.

a) When $X(n)$ is a white noise, we have $\rho_k=0$ for any $k\neq 0$. So we have

$$C_h=\frac{\pi}{2} \quad \text{and} \quad \tilde{C}_h=1$$

for any $h\geqq 1$.

b) Let us assume

$$\rho_k=\begin{cases}\dfrac{1-\rho^{2(M-|k|+1)}}{1-\rho^{2(M+1)}}\cdot\rho^{|k|}\cos k\theta; & 0\leqq|k|\leqq M,\\ 0; & |k|\geqq M+1,\end{cases} \tag{4}$$

where ρ and θ are constants and $0\leqq\rho<1$. For simplicity, we write

$$\alpha_k=\frac{1-\rho^{2(M-|k|+1)}}{1-\rho^{2(M+1)}}\cos k\theta.$$

Then we have $|\alpha_k|<1$ and $\rho_k=\alpha_k\rho^{|k|}$.

In this case, we can say as follows:

THEOREM 7. *If* $|\rho_{h_0}|<\rho^{h_0}<\varepsilon$ *holds for sufficiently small positive number* ε, C_h *and* $\tilde{C}_h$ *are given approximately for any* $h\geqq h_0$ *as follows;*

$$C_h\sim\frac{\pi}{2}+2\sum_{k\geqq 1}\rho_k^2\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m+1)!}\cdot(m!)^2\right)$$

and

$$\tilde{C}_h\sim 1+2\sum_{k\geqq 1}\rho_k^2,$$

where the sign $\sim$ *is used to indicate that the left side and the right side are coinside by ignoring the magnitude of the order* ε.

Proof. Here, we shall prove this theorem only when $M\geqq h$. The stituation is the same when $h\geqq M+1$.

As $\rho_k=\alpha_k\rho^{|k|}$, we have for $h>k>0$, in the expression (3),

$$\Delta=1-\alpha_k^2\rho^{2k}-\alpha_{h-k}^2\rho^{2(h-k)}+O(\varepsilon),\qquad D=1-\alpha_k^2\rho^{2k},$$
$$D_1=1-\alpha_{h-k}^2\rho^{2(h-k)},\qquad D_2=1+O(\varepsilon^2),$$
$$D_h=1+O(\varepsilon^2),\qquad D_{2h}=1+O(\varepsilon^4),$$
$$A=\alpha_k\rho^k+O(\varepsilon),\qquad F=\alpha_{h-k}\rho^{h-k}/(1-\alpha_k^2\rho^{2k})+O(\varepsilon).$$

And each of B, C, G, H_1, K_1 is $O(\varepsilon)$. Further $AF^2D^{3/2}S_1(\rho_k)$ is $O(\varepsilon^2)$. Now we have

$$A\cdot\frac{\Delta}{2\sqrt{D}}\cdot S_3(\rho_k)$$
$$=(\alpha_k\rho^k+O(\varepsilon))\cdot\frac{(1-\alpha_k^2\rho^{2k}-\alpha_{h-k}^2\rho^{2(h-k)}+O(\varepsilon))}{2\sqrt{1-\alpha_k^2\rho^{2k}}}\cdot 2\left(\sum_{m=0}^{\infty}\frac{(2\alpha_k\rho^k)^{2m+1}}{(2m+1)!}(m!)^2\right)$$
$$=2\alpha_k^2\rho^{2k}\sqrt{1-\alpha_k^2\rho^{2k}}\left(\sum_{m=0}^{\infty}\frac{(2\alpha_k\rho^k)^{2m}}{(2m+1)!}(m!)^2\right)+O(\varepsilon).$$

Using the above results, we obtain

$$C_h=\frac{\pi}{2}+\sum_{k\geqq 1}A\cdot\frac{\Delta}{2\sqrt{D}}S_3(\rho_k)+O(\varepsilon)$$
$$=\frac{\pi}{2}+2\sum_{k\geqq 1}\alpha_k^2\rho^{2k}\sqrt{1-\alpha_k^2\rho^{2k}}\left(\sum_{m=0}^{\infty}\frac{(2\alpha_k\rho^k)^{2m}}{(2m+1)!}(m!)^2\right)+O(\varepsilon)$$
$$=\frac{\pi}{2}+2\sum_{k\geqq 1}\rho_k^2\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m+1)!}(m!)^2\right)+O(\varepsilon).$$

Similarly we have

$$\tilde{C}_h=1+2\sum_{k\geqq 1}\rho_k^2+O(\varepsilon).$$

Concerning the relation between C_h and $\tilde{C}_h$, we can obtain the following theorem:

THEOREM 8. *If the value of $|\rho_{h_0}|$ is sufficiently small, that is, $|\rho_{h_0}|<\rho^{h_0}<\varepsilon$ holds for sufficiently small positive number ε, it holds*

$$\frac{\pi}{2}\tilde{C}_h\geqq C_h>\tilde{C}_h$$

for any $h\geqq h_0$.

Proof. In the first place, we shall prove that $C_h>\tilde{C}_h$. By Theorem 7,

$$C_h\sim\frac{\pi}{2}+2\sum_{k\geqq 1}\rho_k^2\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m+1)!}(m!)^2\right)$$

and

$$\tilde{C}_h \sim 1+2\sum_{k\geqq 1} \rho_k^2.$$

We shall show

$$\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty} \frac{(2\rho_k)^{2m}}{(2m+1)!}(m!)^2\right) \geqq 1$$

for each k. For simplicity we put $\rho_k^2 = X$, then the above relation is

$$\sqrt{1-X}\left(\sum_{m=0}^{\infty} \frac{2^{2m}(m!)^2}{(2m+1)!} X^m\right) \geqq 1.$$

We consider the function

$$f(X) = \left(\sum_{m=0}^{\infty} \frac{2^{2m}(m!)^2}{(2m+1)!} X^m\right) - \frac{1}{\sqrt{1-X}}$$

for $0 \leqq X < 1$. We have $f(0)=0$. Further

$$\begin{aligned} f'(X) &= \sum_{m=1}^{\infty} \frac{2^{2m}(m!)^2 m}{(2m+1)!} X^{m-1} - \frac{1}{2}(1-X)^{-3/2} \\ &= \sum_{m=0}^{\infty} \frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!} X^m - \sum_{m=0}^{\infty} \frac{(2m+1)(2m-1)\cdots 5\cdot 3\cdot 1}{m!\,2^{m+1}} X^m \\ &= \sum_{m=0}^{\infty} \left(\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!} - \frac{(2m+1)!!}{m!\,2^{m+1}}\right) X^m. \end{aligned}$$ [1]

Now we write

$$b_m = \frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!}, \qquad c_m = \frac{(2m+1)!!}{m!\,2^{m+1}}$$

and

$$a_m = b_m - c_m.$$

Then

$$f'(X) = \sum_{m=1}^{\infty} a_m X^m.$$

We have

$$b_0 = \frac{2}{3} > c_0 = \frac{1}{2} \quad \text{and} \quad a_0 = \frac{2}{3} - \frac{1}{2} > 0.$$

If $b_m > c_m$ holds, we find

$$b_{m+1} = b_m \cdot \frac{2^2(m+2)^3}{(2m+4)(2m+5)(m+1)} > c_{m+1} = c_m \cdot \frac{(2m+3)}{2(m+1)},$$

because

$$\frac{2^2(m+2)^3}{(2m+4)(2m+5)(m+1)} > \frac{(2m+3)}{2(m+1)}. \tag{5}$$

1) $(2m+1)!! = 1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1)\cdot(2m+1)$.

So we have $a_m>0$ for any positive integer m and this shows $f'(X)>0$ for $X\geqq 0$. This result shows $f(X)\geqq 0$ for $0\leqq X<1$ and we obtain

$$\sqrt{1-X}\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right)\geqq 1.$$

Consequently we have $C_h>\tilde{C}_h$.

In the next place we shall prove that $(\pi/2)\tilde{C}_h\geqq C_h$. For this purpose, we show

$$\frac{\pi}{2}>\sqrt{1-X}\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right),$$

by writing $\rho^{2k}=X$ as the above. Let us consider the function

$$g(X)=\frac{\pi}{2}\frac{1}{\sqrt{1-X}}-\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m$$

for $0\leqq X<1$. We have $g(0)=\pi/2-1>0$ and

$$g'(X)=\frac{\pi}{4}(1-X)^{-3/2}-\sum_{m=1}^{\infty}\frac{2^{2m}(m!)^2 m}{(2m+1)!}X^{m-1}$$
$$=\sum_{m=0}^{\infty}\left(\frac{\pi}{4}\frac{(2m+1)!!}{m!\,2^m}-\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!}\right)X^m.$$

We shall write

$$e_m=\frac{\pi}{4}\frac{(2m+1)!!}{m!\,2^m},\qquad f_m=\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!}$$

and

$$g_m=e_m-f_m.$$

Then we have

$$e_0=\frac{\pi}{4}>f_0=\frac{2}{3}\qquad\text{and}\qquad g_0>0.$$

We show $g_m\geqq 0$ for any positive integer m. Let us assume that, for a certain integer m, $g_m<0$, that is, $e_m<f_m$. Then we find

$$e_{m+1}=e_m\cdot\frac{(2m+3)}{2(m+1)}<f_{m+1}=f_m\cdot\frac{2^2(m+2)^3}{(2m+4)(2m+5)(m+1)},$$

by using the relation (5). This shows $g_{m'}<0$ for any $m'\geqq m$ and we have

$$1>\frac{e_m}{f_m}>\frac{e_{m+1}}{f_{m+1}}>\frac{e_{m+2}}{f_{m+2}}>\cdots.$$

On the other hand,

$$\lim_{m\to\infty}\frac{e_m}{f_m}=\lim_{m\to\infty}\frac{\pi}{4}\frac{(2m+1)!!}{m!\,2^m}\cdot\frac{(2m+3)!}{2^{2(m+1)}((m+1)!)^2(m+1)}$$
$$=\lim_{m\to\infty}\frac{\pi}{4}\frac{(2m+1)!}{2^{2m}(m!)^2}\frac{(2m+3)!}{2^{2(m+1)}((m+1)!)^2(m+1)}.$$

Using Stirling's formula

$$n! \sim (2\pi)^{1/2} n^{n+1/2} e^{-n},$$

we have

$$\lim_{m\to\infty}\frac{e_m}{f_m}=\lim_{m\to\infty}\frac{\pi}{4}\cdot\frac{(2\pi)^{1/2}(2m+1)^{2m+3/2}e^{-(2m+1)}}{2^{2m}(2\pi)m^{2m+1}e^{-2m}}\cdot\frac{(2\pi)^{1/2}(2m+3)^{2m+7/2}e^{-(2m+3)}}{2^{2(m+1)}(m+1)(2\pi)(m+1)^{2m+3}e^{-2(m+1)}}$$

$$=\frac{1}{e^2}\lim_{m\to\infty}\left(1+\frac{1}{2m}\right)^{2m}\left(1+\frac{1}{2m+2}\right)^{2m+2}\left(1+\frac{1}{2m}\right)^{3/2}\left(1+\frac{3}{2m}\right)^{3/2}\cdot\frac{1}{\left(1+\frac{1}{m}\right)^2}=1.$$

This is a contradiction. Consequently we have $g_m \geqq 0$ for all positive integer m.

From this result, we obtain $g'(X)>0$ for $0 \leqq X<1$ and $g(X)>0$ for $0 \leqq X<1$. This implies

Table 1.

h	ρ_h	C_h	$\tilde{C}_h$
1	0.4322	0.484	0.279
2	−0.2663	1.244	0.829
3	−0.5069	2.000	1.423
4	−0.2677	2.630	1.948
5	0.0929	3.101	2.360
6	0.2517	3.430	2.661
7	0.1581	3.650	2.870
8	−0.0244	3.793	3.010
9	−0.1223	3.887	3.103
10	−0.0901	3.950	3.165
11	0.0004	3.992	3.207
12	0.0580	4.022	3.236
13	0.0499	4.044	3.256
14	0.0060	4.060	3.271
15	−0.0267	4.073	3.284
16	−0.0269	4.084	3.293
17	−0.0062	4.091	3.300
18	0.0119	4.096	3.305
19	0.0142	4.098	3.307
20	0.0047	4.098	3.307
21	−0.0050	4.099	3.308
22	−0.0072	4.099	3.308
23	−0.0031	4.099	3.308
24	0.0019	4.099	3.308
25	0.0035	4.099	3.308
30	0.0001	4.100	3.309

$$\frac{\pi}{2} > \sqrt{1-X}\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right)$$

and we obtain $(\pi/2)\tilde{C}_h \geqq C_h$.

c) As it is difficult to compare C_h with $\tilde{C}_h$ generally, we make a comparison numerically.

For this purpose, we treat the case when the correlogram ρ_k is defined by (4).

Considering the case

$$\rho=0.8, \quad \theta=0.25 \quad \text{and} \quad M=30,$$

we obtain the result of numerical comparison as Table 1. This result is also shown as Figure 1.

The situation of the other cases, assuming each of the parameters ρ, θ and M to have various values, will be similar to that of the above case. Generally, C_h will be greater than $\tilde{C}_h$.

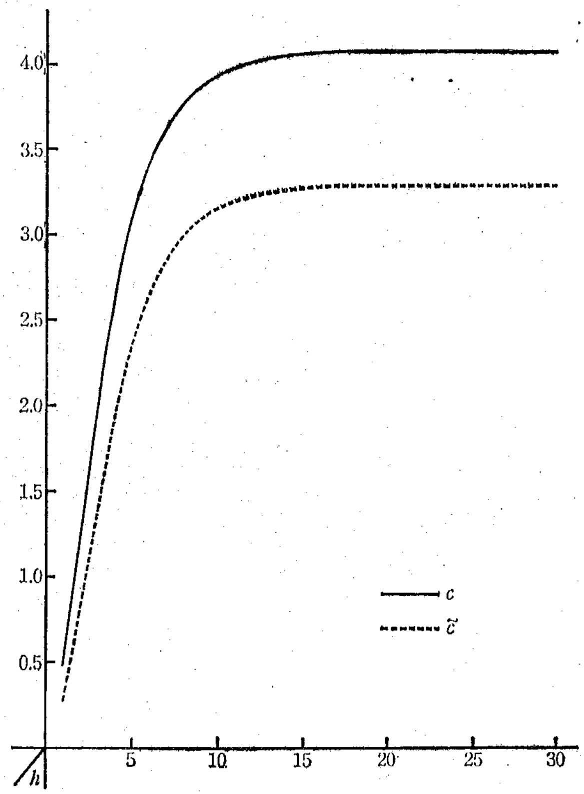

Figure 1.

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CHAPTER 5

THE BIAS OF THE ESTIMATE FOR A NON-GAUSSIAN PROCESS

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ON THE BIAS OF A SIMPLIFIED ESTIMATE OF CORRELOGRAM

BY MITUAKI HUZII

§ 1. Introduction.

Let $X(n)$ be a real-valued weakly stationary process with discrete time parameter n. For simplicity, we assume $EX(n)=0$.

We shall denote

$$EX(n)^2=\sigma^2 \quad \text{and} \quad EX(n)X(n+h)=\sigma^2\rho_h$$

and consider to estimate the correlogram ρ_h when σ^2 is known. We assume $X(n)$ to be observed at $n=1, 2, 3, \cdots, N, \cdots, N+h$. Usually, we use the estimate

$$\tilde{\gamma}_h=\frac{1}{\sigma^2}\frac{1}{N}\sum_{n=1}^{N}X(n)X(n+h)$$

for the estimation of ρ_h. $\tilde{\gamma}_h$ is an unbiased estimate of ρ_h.

We have shown that when $X(n)$ is a Gaussian process,

$$\gamma_h=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\frac{1}{N}\sum_{n=1}^{N}X(n)\,\mathrm{sgn}\,(X(n+h))$$

is also an unbiased estimate of ρ_h, where $\mathrm{sgn}(y)$ means $1, 0, -1$ correspondingly as $y>0$, $y=0$, $y<0$, and we have evaluated the variance of γ_h ([3], [4]).

In this paper, we discuss the bias of the estimate γ_h when the assumption that $X(n)$ is a Gaussian process is not satisfied. For a class of stationary processes, which are not Gaussian, we shall show the bias of γ_h and its properties.

§ 2. Stationary processes which deviate from a Gaussian process.

In this paper, we shall assume a stationary process $X(n)$ which deviates from a Gaussian process to be as follows.

Let $X(n)$ be, furthermore, a strictly stationary process and $\tilde{f}(x, y)$ denote the probability density of the joint distribution of the variables $X(n)$ and $X(n+h)$. Clearly, $\tilde{f}(x, y)$ does not depend on n. We have

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$$EX(n)=EX(n+h)=0, \qquad EX(n)^2=EX(n+h)^2=\sigma^2$$

and

$$EX(n)X(n+h)=\sigma^2\rho_h.$$

Let $\Phi_2(x, y; \sigma^2, \sigma^2\rho_h)$ denote the probability density function of the two-dimensional Gaussian distribution function with the mean vector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the variance-covariance matrix

$$\begin{pmatrix} \sigma^2 & \sigma^2\rho_h \\ \sigma^2\rho_h & \sigma^2 \end{pmatrix}.$$

Now, we shall assume that $\tilde{f}(x, y)$ satisfies

$$(1) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\tilde{f}^2(x, y)}{\Phi_2(x, y; \sigma^2, \sigma^2\rho_h)} dxdy<+\infty.$$

Let us use the notations

$$L_2(R)=\left\{g(x); \int_{-\infty}^{\infty} g^2(x)dx<+\infty\right\}$$

and

$$L_2(R^2)=\left\{h(x, y); \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^2(x, y)dxdy<+\infty\right\}.$$

Then the condition (1) can be written as

$$\frac{\tilde{f}(x, y)}{\sqrt{\Phi_2(x, y; \sigma^2, \sigma^2\rho_h)}} \in L_2(R^2).$$

Now we shall make two random variables

$$U(n)=X(n)-\rho_h X(n+h),$$

$$V(n+h)=X(n+h)$$

and treat these random variables $U(n)$ and $V(n+h)$ instead of $X(n)$ and $X(n+h)$. Clearly we have

$$EU(n)V(n+h)=0.$$

Corresponding to the above transformation, we change the variables as follows:

$$u=x-\rho_h y, \qquad v=y.$$

By this transformation, we assume $\tilde{f}(x, y)$ is transformed into $f(u, v)$.

Let us denote

$$\Phi_1(x, \sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}.$$

Then we find

$$\Phi_2(x, y;\ \sigma^2, \sigma^2\rho_h)=\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)$$

and the condition (1) can be written as

$$(2)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f^2(u, v)}{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}dudv<+\infty,$$

that is

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))}\sqrt{\Phi_1(v, \sigma^2)}}\in L_2(R^2).$$

§3. A complete orthonormal system of $L_2(R^2)$.

Here we shall prepare for an orthogonal development of the function which belongs to $L_2(R^2)$.

We assume that $H_n(x)$ represents the Hermite polynomial defined by the relation

$$\left(\frac{d}{dx}\right)^n e^{-x^2/2}=(-1)^n H_n(x)e^{-x^2/2}\qquad (n=0, 1, 2, \cdots).$$

$H_n(x)$ is a polynomial of degree n, and we have

$$H_0(x)=1,\quad H_1(x)=x,\quad H_2(x)=x^2-1,$$

$$H_3(x)=x^3-3x,\qquad H_4(x)=x^4-6x^2+3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

Then, as is generally known, the system

$$\left\{\frac{1}{\sqrt{n!}}\frac{1}{(2\pi)^{1/4}}H_n(x)e^{-x^2/4}\right\}$$

is a complete orthonormal system on $(-\infty, \infty)$:

$$\frac{1}{\sqrt{m!}\sqrt{n!}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H_n(x)H_m(x)e^{-x^2/2}\,dx=\begin{cases}1 \text{ for } m=n,\\ 0 \text{ for } m\neq n\end{cases}\qquad (m, n=0, 1, 2, \cdots).$$

We write

$$\varphi_n(x, 1)=\frac{1}{\sqrt{n!}}H_n(x)\sqrt{\Phi_1(x, 1)}\qquad (n=0, 1, 2, \cdots).$$

Some properties of the Hermite polynomials are as follows:

(a) $H_{2k}(x)$ is an even function of x for $k=0, 1, 2, \cdots$,

(b) $H_{2k+1}(x)$ is an odd function of x for $k=0, 1, 2, \cdots$,

(c) $H_{k+1}(x)-xH_k(x)+kH_{k-1}(x)=0$,

Now let us define $\psi_{m,n}(x, 1; y, 1)$ by

$$\psi_{m,n}(x, 1; y, 1)=\varphi_m(x, 1)\varphi_n(y; 1) \qquad (m, n=0, 1, 2, \cdots).$$

Then the system

$$\{\psi_{m,n}(x, 1; y, 1)\}$$

is a complete orthonormal system of $L_2(R^2)$.

§4. An orthogonal expansion of $f(u, v)$ derived from the two-dimensional Gaussian distribution.

In this section, we shall discuss an expansion of $f(u, v)$ by orthogonal functions which are induced in §3. The two-dimensional Gaussian distribution plays a leading part in this expansion. We consider $f(u, v)$ to be slightly different from the two-dimensional Gaussian distribution function, that is, $\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)$.

In accordance with the section 3, we define $\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)$ by

$$\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)=\frac{1}{\sqrt{p!}}H_p\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\frac{1}{\sqrt{q!}}H_q\left(\frac{v}{\sigma}\right)\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}.$$

Then $\{\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)\}$ is a complete orthonormal system of $L_2(R^2)$.

Now, by the condition (2), we have

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}\in L_2(R^2),$$

so we can find the expansion such that

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}=\underset{P,Q\to\infty}{\mathrm{l.i.m.}}\sum_{p,q=0}^{P,Q} a_{p,q}\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma),$$

where

$$a_{p,q}=\iint\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}\psi_{p,q}(u, \sigma\sqrt{1-\rho_h^2}; v, \sigma)dudv$$

$$=\frac{1}{\sqrt{p!}\sqrt{q!}}\iint H_p\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_q\left(\frac{v}{\sigma}\right)f(u, v)dudv.$$

In the above expression, we find

$$a_{0,0}=\iint f(u, v)dudv=1,$$

$$a_{1,0}=\iint H_1\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)f(u, v)dudv=\frac{EU(n)}{\sigma\sqrt{1-\rho_h^2}}=0,$$

$$a_{0,1}=\iint H_1\left(\frac{v}{\sigma}\right)f(u, v)dudv=\frac{EV(n+h)}{\sigma}=0,$$

$$a_{2,0}=\frac{1}{\sqrt{2}}\iint H_2\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)f(u, v)dudv=\frac{1}{\sqrt{2}}\left(\frac{EU(n)^2}{\sigma^2(1-\rho_h^2)}-1\right)=0,$$

$$a_{1,1}=\iint H_1\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_1\left(\frac{v}{\sigma}\right)f(u, v)dudv=\frac{EU(n)V(n+h)}{\sigma^2\sqrt{1-\rho_h^2}}=0,$$

$$a_{0,2}=\frac{1}{\sqrt{2}}\iint H_2\left(\frac{v}{\sigma}\right)f(u, v)dudv=\frac{1}{\sqrt{2}}\left(\frac{EV(n+h)^2}{\sigma^2}-1\right)=0.$$

So we have

$$\frac{f(u, v)}{\sqrt{\Phi(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}}$$

$$=\underset{P,Q\to\infty}{\text{l.i.m.}}\left[\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))\Phi_1(v, \sigma^2)}+\sum_{\substack{p,q=0\\p+q\geqq 3}}^{P,Q}a_{p,q}\phi_{p,q}(u, \sigma\sqrt{1-\rho_h^2};\ v, \sigma)\right].$$

§5. An orthogonal expansion of $(u+\rho_h v)\,\mathrm{sgn}\,(v)$.

At the beginning, let us arrange our discussion. The essential point of our discussion is to evaluate the value of $EX(n)\,\mathrm{sgn}\,(X(n+h))$. Now, the value of $EX(n)\mathrm{sgn}(X(n+h))$ is as follows:

$$EX(n)\,\mathrm{sgn}\,(X(n+h))=\iint x\,\mathrm{sgn}\,(y)\tilde{f}(x, y)dxdy$$

$$=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)f(u, v)dudv.$$

The function $(u+\rho_h v)\,\mathrm{sgn}\,(v)$ does not belong to $L_2(R^2)$. But by the condition (2),

$$\frac{f(u, v)}{\sqrt{\Phi_1(u, \sigma^2(1-\rho_h^2))}\sqrt{\Phi_1(v, \sigma^2)}}$$

belongs to $L_2(R^2)$. So, let us express the above value as follows:

$$EX(n)\,\mathrm{sgn}\,(X(n+h))=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)f(u,\,v)dudv$$

$$=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)}\cdot\frac{f(u,\,v)}{\sqrt{\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)}}dudv.$$

Then both

$$(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)}\ \text{ and }\ \frac{f(u,\,v)}{\sqrt{\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)}}$$

belong to $L_2(R^2)$.

Here we shall discuss an orthogonal expansion of the function

$$(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)}.$$

As this function belongs to $L_2(R^2)$, we can expand this function by the orthogonal system

$$\{\phi_{k,\,l}(u,\,\sigma\sqrt{1-\rho_h{}^2};\,v,\,\sigma)\}.$$

We consider that this expansion is

$$(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)}=\underset{K,L\to\infty}{\mathrm{l.i.m.}}\sum_{k=0}^{K}\sum_{l=0}^{L}c_{k,\,l}\phi_{k,\,l}(u,\,\sigma\sqrt{1-\rho_h{}^2};\,v,\,\sigma).$$

Now we have

$$c_{k,\,l}=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)}\,\phi_{k,\,l}(u,\,\sigma\sqrt{1-\rho_h{}^2};\,v,\,\sigma)dudv$$

$$=\frac{1}{\sqrt{k!}\sqrt{l!}}\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)dudv$$

$$=\frac{1}{\sqrt{k!}\sqrt{l!}}\iint u\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)dudv$$

$$+\frac{\rho_h}{\sqrt{k!}\sqrt{l!}}\iint v\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)dudv.$$

The first term of the above expression is

$$\frac{1}{\sqrt{k!}\sqrt{l!}}\iint u\,\mathrm{sgn}\,(v)H_k\Big(\frac{u}{\sigma\sqrt{1-\rho_h{}^2}}\Big)H_l\Big(\frac{v}{\sigma}\Big)\Phi_1(u,\,\sigma^2(1-\rho_h{}^2))\Phi_1(v,\,\sigma^2)dudv$$

$$=\left\{\frac{1}{\sqrt{k!}}\int uH_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\}\left\{\frac{1}{\sqrt{l!}}\int \mathrm{sgn}\,(v)H_l\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\}.$$

$$=\begin{cases}\sigma\sqrt{1-\rho_h^2}\cdot\dfrac{1}{\sqrt{(2i+1)\,!}}\displaystyle\int \mathrm{sgn}\,(v)H_{2i+1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv, & k=1,\ l=2i+1\ \ (i=0,1,2,\cdots),\\ 0, & \text{otherwise.}\end{cases}$$

The second term is as follows. As stated in § 3, it holds

$$\frac{v}{\sigma}H_l\left(\frac{v}{\sigma}\right)=H_{l+1}\left(\frac{v}{\sigma}\right)+lH_{l-1}\left(\frac{u}{\sigma}\right).$$

Using this relation, we have

$$\frac{\rho_h}{\sqrt{k!}\sqrt{l!}}\iint v\,\mathrm{sgn}\,(v)H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_l\left(\frac{v}{\sigma}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))\Phi_1(v,\ \sigma^2)dudv$$

$$=\rho_h\left\{\frac{1}{\sqrt{k!}}\int H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\}\left\{\frac{1}{\sqrt{l!}}\int v\,\mathrm{sgn}\,(v)H_l\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\}$$

$$=\begin{cases}\rho_h\left\{\dfrac{1}{\sqrt{k!}}\displaystyle\int H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\}\\ \quad\times\left\{\dfrac{\sigma}{\sqrt{l!}}\displaystyle\int \mathrm{sgn}\,(v)H_{l+1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv+\frac{l\sigma}{\sqrt{l!}}\int \mathrm{sgn}\,(v)H_{l-1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\}, & l\geqq 1,\\ \rho_h\left\{\dfrac{1}{\sqrt{k!}}\displaystyle\int H_k\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)\Phi_1(u,\ \sigma^2(1-\rho_h^2))du\right\}\left\{\int|v|\Phi_1(v,\ \sigma^2)dv\right\}, & l=0,\end{cases}$$

$$=\begin{cases}\rho_h\displaystyle\int|v|\Phi_1(v,\ \sigma^2)dv, & k=0,\ \ l=0,\\ \rho_h\sigma\dfrac{1}{\sqrt{(2j)\,!}}\left\{\displaystyle\int \mathrm{sgn}(v)H_{2j+1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right.\\ \quad\left.+(2j)\displaystyle\int \mathrm{sgn}(v)H_{2j-1}\left(\frac{v}{\sigma}\right)\Phi_1(v,\ \sigma^2)dv\right\} & k=0,\ \ l=2j\ \ \ (j\geqq 1),\\ 0, & \text{otherwise.}\end{cases}$$

Therefore we find

$$c_{k,l}=\begin{cases}\rho_h\int|v|\Phi_1(v,\sigma^2)dv, & k=0,\quad l=0,\\ \rho_h\sigma\dfrac{1}{\sqrt{(2j)!}}\Big\{\int\mathrm{sgn}(v)H_{2j+1}\Big(\dfrac{v}{\sigma}\Big)\Phi_1(v,\sigma^2)dv \\ \qquad +(2j)\int\mathrm{sgn}(v)H_{2j-1}\Big(\dfrac{v}{\sigma}\Big)\Phi_1(v,\sigma^2)dv\Big\}, & k=0,\quad l=2j\quad(j\geqq1),\\ \sigma\sqrt{1-\rho_h^2}\dfrac{1}{\sqrt{(2i+1)!}}\int\mathrm{sgn}(v)H_{2i+1}\Big(\dfrac{v}{\sigma}\Big)\Phi_1(v,\sigma^2)dv, & k=1,\quad l=2i+1\ (i\geqq0),\\ 0, & \text{ortherwise.}\end{cases}$$

Consequently we have

$$\begin{aligned}&(u+\rho_h v)\,\mathrm{sgn}(v)\sqrt{\Phi_1(u,\sigma^2(1-\rho_h^2))\Phi_1(v,\sigma^2)}\\ &=\underset{K,L\to\infty}{\mathrm{l.i.m.}}\Big\{\sqrt{\frac{2}{\pi}}\sigma\rho_h\phi_{0,0}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\\ &\qquad+\sum_{i=1}^{K}c_{0,2i}\phi_{0,2i}(u,\sigma\sqrt{1-\sigma_h^2};v,\sigma)+\sum_{i=0}^{L}c_{1,2i+1}\phi_{1,2i+1}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\Big\}.\end{aligned}$$

§6. Evaluation of the bias of the estimate γ_h.

Using the results in §4 and §5, we shall, in the first place, evaluate the value of $EX(n)\,\mathrm{sgn}\,(X(n+h))$.

$$\begin{aligned}&EX(n)\,\mathrm{sgn}\,(X(n+h))=\iint x\,\mathrm{sgn}\,(y)\tilde{f}(x,y)dxdy\\ &=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)f(u,v)dudv\\ &=\iint(u+\rho_h v)\,\mathrm{sgn}\,(v)\sqrt{\Phi_1(u,\sigma^2(1-\rho_h^2))\Phi_1(v,\sigma^2)}\frac{f(u,v)}{\sqrt{\Phi_1(u,\sigma^2(1-\rho_h^2))\Phi_1(v,\sigma^2)}}dudv\\ &=\lim_{\substack{K,L\\P,Q}\to\infty}\iint\Big\{\sqrt{\frac{2}{\pi}}\sigma\rho_h\phi_{0,0}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\\ &\qquad+\sum_{i=1}^{K}c_{0,2i}\phi_{0,2i}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)+\sum_{i=0}^{L}c_{1,2i+1}\phi_{1,2i+1}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\Big\}\\ &\qquad\times\Big\{\phi_{0,0}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)+\sum_{\substack{p,q=0\\p+q\geqq3}}^{P,Q}a_{p,q}\phi_{p,q}(u,\sigma\sqrt{1-\rho_h^2};v,\sigma)\Big\}dudv\end{aligned}$$

$$=\sqrt{\frac{2}{\pi}}\sigma\rho_h+\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}.$$

So we have

$$\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}EX(n)\operatorname{sgn}(X(n+h))=\rho_h+\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

This means

$$E(\gamma_h)=\frac{1}{N}\sum_{n=1}^{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}EX(n)\operatorname{sgn}(X(n+h))$$

$$=\rho_h+\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

Therefore the estimate γ_h has the bias

$$\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

THEOREM 1. *When a strictly stationary process $X(n)$ satisfies the condition* (1), *the estimate γ_h of ρ_h has the property:*

$$E(\gamma_h)=\rho_h+b_h,$$

where b_h is the bias and

$$b_h=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,2i}a_{0,2i}+\sum_{i=1}^{\infty}c_{1,2i+1}a_{1,2i+1}\right\}.$$

§7. Some properties of $a_{p,q}$ and the relations between $a_{p,q}$ and moments.

In this section, we shall consider the relation between $a_{p,q}$ and moments, and also the relation between $a_{p,q}$ and Gaussian properties.

Now,

$$a_{p,q}=\frac{1}{\sqrt{p!}\sqrt{q!}}\iint H_p\left(\frac{u}{\sigma\sqrt{1-\rho_h^2}}\right)H_q\left(\frac{v}{\sigma}\right)f(u,v)dudv.$$

If $f(u, v)$ is the probability density of two-dimensional Gaussian distribution function, $U(n)$ is independent of $V(h+h)$. So we have clearly the following facts:

LEMMA 1. *When the joint distribution of $U(n)$ and $V(n+h)$ is two-dimensional Gaussian distribution, we have*

$$a_{p,q}=\begin{cases}1 & for \quad p=q=0,\\ 0 & for \quad p\neq 0 \quad or \quad q\neq 0.\end{cases}$$

LEMMA 2. *If the joint distribution of $U(n)$ and $V(n+h)$ is Gaussian, the joint distribution of $X(n)$ and $X(h+h)$ is also Gaussian. And the converse is also true.*

LEMMA 3. *When $X(n)$ is a Gaussian process, we have*

$$a_{p,q}=\begin{cases}1 & for \quad p=0 \quad and \quad q=0,\\ 0 & for \quad p\neq 0 \quad or \quad q\neq 0,\end{cases}$$

and γ_h is an unbiased estimate of ρ_h.

LEMMA 4. *When $X(n)$ is a strictly stationary process, $a_{p,q}$ depends only on h.*

Now let us put

$$M_{k,l}=EU(n)^k V(n+h)^l=\iint u^k v^l f(u, v)dudv$$

and

$$m_{k,l}=EX(n)^k X(n+h)^l=\iint x^k y^l \tilde{f}(x, y)dxdy.$$

Clearly we have

$$M_{0,l}=m_{0,l}.$$

Let

$$\beta_j^i(\omega_1, \omega_2, \cdots, \omega_k)$$

denote a linear combination of $\omega_1, \omega_2, \cdots, \omega_{k-1}$ and ω_k with constant coefficients. Then we have the following result.

LEMMA 5. *It holds*

$$a_{2k,2l}=\alpha_{2l}^{2k}(M_{0,0}, M_{0,2}, \cdots, M_{0,2l}, M_{2,0}, M_{2,2}, \cdots, M_{2,2l}, \cdots, M_{2k,0}, M_{2k,2}, \cdots, M_{2k,2l}),$$

$$a_{2k,2l+1}=\alpha_{2l+1}^{2k}(M_{0,1}, M_{0,3}, \cdots, M_{0,2l+1}, M_{2,1}, M_{2,3}, \cdots, M_{2,2l+1}, \cdots, M_{2k,1}, M_{2k,3}, \cdots, M_{2k,2l+1}),$$

$$a_{2k+1,2l}=\alpha_{2l}^{2k+1}(M_{1,0}, M_{1,2}, \cdots, M_{1,2l}, M_{3,0}, M_{3,2}, \cdots, M_{3,2l}, \cdots, M_{2k+1,0}, M_{2k+1,2}, \cdots, M_{2k+1,2l}),$$

and

$$a_{2k+1,2l+1}=\alpha_{2l+1}^{2k+1}(M_{1,1}, M_{1,3}, \cdots, M_{1,2l+1}, M_{3,1}, M_{3,3}, \cdots, M_{3,2l+1}, \cdots, M_{2k+1,1}, M_{2k+1,3}, \cdots, M_{2k+1,2l+1}) \qquad (k, l=0, 1, 2, \cdots).$$

As we have seen in the above, the bias of the estimate ρ_h is

$$b_h=\sqrt{\frac{\pi}{2}}\ \frac{1}{\sigma}\left\{\sum_{i=2}^{\infty}c_{0,\,2i}a_{0,\,2i}+\sum_{i=1}^{\infty}c_{1,\,2i+1}a_{1,\,2i+1}\right\}$$

and this shows that the bias is affected only by $\{a_{0,\,2i}\}$ and $\{a_{1,\,2i+1}\}$.

Now we have

$$M_{0,\,2l}=m_{0,\,2l}=m_{2l,\,0}$$

and

$$M_{1,\,2l+1}=EU(n)V(n+h)^{2l+1}=E(X(n)-\rho_hX(n+h))X(n+h)^{2l+1}$$

$$=EX(n)X(n+h)^{2l+1}-\rho_hEX(n+h)^{2l+2}=m_{1,\,2l+1}-\rho_hm_{0,\,2l+2}.$$

So we have

(3)
$$a_{0,\,2i}=a_{2i}^0(M_{0,\,0},\ M_{0,\,2},\ \cdots,\ M_{0,\,2i})$$
$$=a_{2i}^0(m_{0,\,0},\ m_{0,\,2},\ \cdots,\ m_{0,\,2i})$$

and

(4)
$$a_{1,\,2i+1}=a_{2i+1}^1(M_{1,\,1},\ M_{1,\,3},\ \cdots,\ M_{1,\,2i+1})$$
$$=a_{2i+1}^1(m_{1,\,1},\ m_{1,\,3},\ \cdots,\ m_{1,\,2i+1},\ m_{0,\,2},\ m_{0,\,4},\ \cdots,\ m_{0,\,2i+2}).$$

Examples.

$$a_{0,\,4}=\frac{1}{\sqrt{4!}}\left(\frac{1}{\sigma^4}M_{0,\,4}-\frac{6}{\sigma^2}M_{0,\,2}+3\right)$$

$$=\frac{1}{\sqrt{4!}}\left(\frac{1}{\sigma^4}m_{0,\,4}-\frac{6}{\sigma^2}m_{0,\,2}+3\right)=\frac{1}{\sqrt{4!}}\left(\frac{1}{\sigma^4}m_{0,\,4}-3\right),$$

$$a_{0,\,6}=\frac{1}{\sqrt{6!}}\left(\frac{1}{\sigma^6}M_{0,\,6}-\frac{15}{\sigma^4}M_{0,\,4}+\frac{45}{\sigma^2}M_{0,\,2}-15\right)$$

$$=\frac{1}{\sqrt{6!}}\left(\frac{1}{\sigma^6}m_{0,\,6}-\frac{15}{\sigma^4}m_{0,\,4}+\frac{45}{\sigma^2}m_{0,\,2}-15\right)$$

$$=\frac{1}{\sqrt{6!}}\left(\frac{1}{\sigma^6}m_{0,\,6}-\frac{15}{\sigma^4}m_{0,\,4}+30\right),$$

$$a_{1,\,3}=\frac{1}{\sqrt{3!}}\left(\frac{1}{\sigma^4\sqrt{1-\rho_h{}^2}}M_{1,\,3}-\frac{3}{\sigma^2\sqrt{1-\rho_h{}^2}}M_{1,\,1}\right)$$

$$=\frac{1}{\sqrt{3!}}\frac{1}{\sigma^4\sqrt{1-\rho_h{}^2}}M_{1,\,3}$$

$$=\frac{1}{\sqrt{3\,!}}\frac{1}{\sigma^4\sqrt{1-\rho_h{}^2}}(-\rho_h m_{0,4}+m_{1,3}),$$

$$a_{1,5}=\frac{1}{\sqrt{5\,!}}\left(\frac{1}{\sigma^6\sqrt{1-\rho_h{}^2}}M_{1,5}-\frac{10}{\sigma^4\sqrt{1-\rho_h{}^2}}M_{1,3}+\frac{15}{\sigma^2\sqrt{1-\rho_h{}^2}}M_{1,1}\right)$$

$$=\frac{1}{\sqrt{5\,!}}\left(\frac{1}{\sigma^6\sqrt{1-\rho_h{}^2}}M_{1,5}-\frac{10}{\sigma^4\sqrt{1-\rho_h{}^2}}M_{1,3}\right)$$

$$=\left(-\frac{\rho_h}{\sigma^6\sqrt{1-\rho_h{}^2}}m_{0,6}+\frac{10\rho_h}{\sigma^4\sqrt{1-\rho_h{}^2}}m_{0,4}+\frac{1}{\sigma^6\sqrt{1-\rho_h{}^2}}m_{1,5}-\frac{10}{\sigma^4\sqrt{1-\rho_h{}^2}}m_{1,3}\right).$$

When $X(n)$ is a Gaussian process, it holds

$$M_{0,2k}=(2k-1)\,!!\ M_{0,2}^k=(2k-1)\,!!\ m_{0,2}^k$$

and

$$M_{1,2k+1}=0,\quad \text{that is,}\quad m_{1,2k+1}=\rho_h m_{0,2k+2}=(2k+1)\,!!\ \rho_h m_{0,2}^{k+1}.$$

Then, we have

$$a_{2i}^0(1, M_{0,2}, \cdots, (2i-1)\,!!\ M_{0,2}^i)=a_{2i}^0(1, m_{0,2}, \cdots, (2i-1)\,!!\ m_{0,2}^i)=0$$

and

$$a_{2i+1}^1(0, 0, \cdots, 0)$$

$$=a_{2i+1}^1(\rho_h m_{0,2}, 3\,!!\ \rho_h m_{0,2}^2, \cdots, (2i+1)\,!!\ \rho_h m_{0,2}^{i+1}, m_{0,2}, 3\,!!\ m_{0,2}^2, \cdots, (2i+1)\,!!\ m_{0,2}^{i+1})$$

$$=0$$

By the above results, we can say as follows:

THEOREM 2. *If $X(n)$ is a strictly stationary process satisfying the condition* (1) *and if $a_{0,2i}=0$ for $i\geqq 2$ and $a_{1,2i+1}=0$ for $i\geqq 1$, γ_h is an unbiased estimate of ρ_h. $a_{0,2i}$ and $a_{1,2i+1}$ can be expressed in the form of* (3) *and* (4) *respectively.*

If $\sum_{i=2}^{\infty}a_{0,2i}^2$ and $\sum_{i=1}^{\infty}a_{1,2i+1}^2$ are sufficiently small in comparison with $|\rho_h|$, $E\gamma_h$ is approximately equal to ρ_h. As we have stated in the above, $a_{0,4}$ is related to the coefficient of excess. Let us consider the situation in (u, v, z)-space. The value of $a_{0,4}$ gives a measure of flattening of the frequency curve on a section paprallel to the (v, z)-plane. $a_{0,2i}$ will have a meaning similar to $a_{0,4}$. On the other hand, $a_{1,2i+1}$ gives a measure of the two-dimensional asymmetry.

The other features of the frequency surface, e.g. the one-sided asymmetry, etc., do not influence the bias of the estimate γ_h.

Like the bias, will be a problem the effect on the variance of γ_h, when $X(n)$ deviates from the Gaussian process. This problem will be treated by the method similar to the above. We shall treat this subject in the future.

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CHAPTER 6

THE VARIANCE OF THE ESTIMATE FOR A NON-GAUSSIAN PROCESS

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ON A SIMPLIFIED ESTIMATE OF CORRELOGRAM FOR A STATIONARY NON-GAUSSIAN PROCESS

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1. Introduction.

Let X(n) (n = 0, ±1, ±2, ...) be a real-valued weakly stationary process with EX(n) = 0 and we denote

$$E X(n)^2 = \sigma^2 \quad \text{and} \quad E X(n) X(n+k) = \sigma^2 \rho_k .$$

We shall discuss the properties of a simplified estimate of the correlogram ρ_k, assuming that σ^2 is known.

In the previous papers [3] [4], we have shown that when X(n) is a stationary Gaussian process,

$$\gamma_k = \sqrt{\frac{\pi}{2}} \frac{1}{\sigma} \frac{1}{N} \sum_{n=1}^{N} X(n) \operatorname{Sgn}(X(n+k)) ,$$

$$\left(\operatorname{sgn}(y) = \begin{cases} 1 & ; \; y > 0 \\ 0 & ; \; y = 0 \\ -1 & ; \; y < 0 \end{cases} \right)$$

is an unbiased estimate of ρ_k and we have obtained its variance. Also we have evaluated the variance of the usual estimate

$$\tilde{\gamma}_k = \frac{1}{\sigma^2} \frac{1}{N} \sum_{n=1}^{N} X(n) X(n+k)$$

and compared the variance of γ_k with that of $\tilde{\gamma}_k$. In [5], we have discussed the bias of γ_k when X(n) departs from a Gaussian process. In this paper, using the same idea, we define more generally the processes which depart from a Gaussian process and discuss the influence on the variances of γ_k and $\tilde{\gamma}_k$.

2. Stationary processes which depart from a Gaussian process.

In [5] , we have defined the stationary processes which depart from a Gaussian process and used this definition to discuss the bias of the estimate γ_k . In this paper, in order to discuss the variances of the estimates γ_k and $\tilde{\gamma}_k$, we define, by more strict conditions, the class of the stationary processes which depart from a Gaussian process. We shall define it generally.

Let $f^{(n)}(x_1, x_2, \text{-----}, x_n)$ be the probability density function of a n-dimensional distribution function and $\varphi_f^{(n)}(x_1, x_2, \text{-----}, x_n)$ be the probability density of the n-dimensional Gaussian distribution which has the same man vector and the same variance-covariance matrix as $f^{(n)}(x_1, x_2, \text{----}, x_n)$. For example, we sometimes write

$$f^{(n)}(\mathfrak{x}) = f^{(n)}(x_1, x_2, \text{-----}, x_n)$$
$$\varphi^{(n)}(\mathfrak{x}) = \varphi^{(n)}(x_1, x_2, \text{----}, x_n) = \varphi_f^{(n)}(x_1, x_2, \text{---}, x_n)$$

for simplicity. Let $\mathfrak{F}_n$ be the class of n-dimensional probability density functions, of which each function $f^{(n)}(x_1, x_2, \text{----}, x_n)$ satisfies the condition

$$\int_{R^n}\!\!\cdots\!\int \frac{f^{(n)}(x_1, x_2, \text{----}, x_n)^2}{\varphi^{(n)}(x_1, x_2, \text{---}, x_n)}\, dx_1\, dx_2 \cdots dx_n < +\infty$$

Then we have the following lemma.

<u>Lemma 1.</u> If $f^{(n)}(x_1, x_2, \text{---}, x_n) \in \mathfrak{F}_n$ holds, we have

$$\int_s f^{(n)}(x_1, x_2, \text{---}, s, \text{---} x_n)\, ds \in \mathfrak{F}_{n-1}.$$

Proof. Let $\varphi^{(n-1)}(x_1, x_2, \text{---} x_{k-1}, x_{k+1}, \text{---} x_n)$ be the (n-1) - dimensional Gaussian distribution function which has the same mean vector and the same variance-covariance matrix as

$$\int_s f^{(n)}(x_1, x_2, \text{----}, x_{k-1}, s, x_{k+1}, \text{----}, x_n)\, ds .$$

Then we have

$$\varphi^{(n-1)}(x_1, x_2, \text{---}, x_{k-1}, x_{k+1}, \text{---}, x_n) = \int_s \varphi^{(n)}(x_1, x_2, \text{---}, x_{k-1}, s, x_{k+1}, \text{---}, x_n)\, ds.$$

We have

$$\int \cdots \int_{R^{n-1}} \frac{\left(\int_s f^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)\, ds \right)^2}{\left(\int_s \varphi^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)\, ds \right)} dx_1 \cdots dx_{k-1}\, dx_{k+1} \cdots dx_n$$

$$= \int \cdots \int_{R^{n-1}} \frac{\left(\int_s f^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)\, \varphi^{(n)}(x_1, \cdots x_{k-1}, s, x_{k+1} \cdots, x_n)^{-\frac{1}{2}} \varphi^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)^{\frac{1}{2}} ds \right)^2 dx_1 \cdots dx_n}{\left(\int_s \varphi^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)\, ds \right)}$$

$$\leq \int \cdots \int_{R^{n-1}} \frac{\left(\int_s f^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)^2 \varphi^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots x_n)^{-1} ds \right) \times \left(\int_s \varphi^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)\, ds \right) dx_1 \cdots dx_n}{\left(\int_s \varphi^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)\, ds \right)}$$

$$= \int \cdots \int_{R^{n-1}} \int_s f^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)^2 \varphi^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)^{-1} ds\, dx_1 \cdots dx_n$$

$$< +\infty .$$

This means

$$\int_s f^{(n)}(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n)\, ds \in \mathfrak{F}_{n-1} .$$

Definition 1. Let $X_0(n)$ be a stationary Gaussian process. A process $X(n)$ is said to belong to the class $U^{(k)}(X_0)$ if it has the following properties :

(i) $X(n)$ is a strictly stationary process with

$$E\,X(n) = 0 \quad , \quad E\,X(n)^2 = \sigma^2 < +\infty .$$

(ii) It holds

$$EX(n_1)X(n_2) = EX_0(n_1)X_0(n_2)$$

for any n_1 and n_2.

(iii) If $n_1 < n_2 < \cdots < n_k$ are parameter values, the joint distribution function of $(X(n_1), X(n_2), \cdots, X(n_k))$ is non-degenerate and absolutely continuous, and its density function belongs to the class $\mathfrak{F}_k$.

From Lemma 1, we find the following facts.

Theorem 1. If $X(n) \in U^{(k)}(X_0)$, we have $X(n) \in U^{(k-1)}(X_0)$. This means

$$U^{(1)}(X_0) \geqq U^{(2)}(X_0) \geqq U^{(3)}(X_0) \geqq \cdots\cdots$$

Theorem 2. If $X(n) \in U^{(k)}(X_0)$, we have

$$E\,|X(n_1)^{l_1} X(n_2)^{l_2} \cdots X(n_k)^{l_k}| < +\infty$$

for any integers $n_1 < n_2 < \cdots < n_k$ and any non-negative integers $l_1, l_2, \cdots, l_k$.

In fact,

$$E\,|X(n_1)^{l_1} X(n_2)^{l_2} \cdots X(n_k)^{l_k}| = \int\cdots\int_{R^k} |x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}|\, f^{(k)}(x_1, \cdots, x_k)\, dx_1 \cdots dx_k$$

$$= \int\cdots\int |x_1^{l_1} \cdot x_2^{l_2} \cdots\cdots x_k^{l_k}|\, \frac{f^{(k)}(x_1, \cdots, x_k)}{\varphi^{(k)}(x_1, \cdots, x_k)^{\frac{1}{2}}}\, \varphi^{(k)}(x_1, x_2, \cdots, x_k)^{\frac{1}{2}} dx_1 \cdots dx_k$$

$$\leqq \sqrt{\int\cdots\int_{R^k} \frac{f^{(k)}(x_1, \cdots, x_k)^2}{\varphi^{(k)}(x_1, \cdots, x_k)}\, dx_1 \cdots dx_k}\ \sqrt{\int\cdots\int_{R^k} |x_1^{l_1} \cdot x_2^{l_2} \cdots\cdots x_k^{l_k}|^2\, \varphi^{(k)}(x_1, \cdots, x_k)\, dx_1 \cdots dx_k}$$

$$< +\infty$$

In the following, we consider to expand a k-dimensional distribution function, using the k-dimensional Gaussian distribution function. This is the reason why we consider the class $\mathfrak{F}_k$. Using this expansion, we shall discuss the bias and the variance of γ_k and also the variance of $\tilde{\gamma}_k$.

§ 3. The variance of γ_h for a process belonging to $U^{(4)}(X_0)$.

Let $X_0(n)$ be a stationary Gaussian process. To discuss the variance of γ_k for a stationary process which departs from X_0 (n), we shall mainly consider the process belonging to the class $U^{(4)}(X_0)$. In this section, we shall show the variance of γ_k for a process X(n) belonging to $U^{(4)}(X_0)$, in the form of the variation from the variance of $X_0(n)$. For this purpose we shall assume that the correlogram ρ_k of the process X(n), which is equal to that of the process $X_0(n)$, satisfies the following conditions :

(P, 1) The determinant

$$\begin{vmatrix} 1 & \rho_{k-h} & \rho_h \\ \rho_{k-h} & 1 & \rho_k \\ \rho_h & \rho_k & 1 \end{vmatrix}$$

is not zero when $k \geqq 1$ and $k \neq h$.

(P, 2) The determinant

$$\begin{vmatrix} 1 & \rho_k \\ \rho_k & 1 \end{vmatrix}$$

is not zero when $k \geqq 1$.

Let $L_2(R^k)$ denote the totality of measurable function $h(x)$ such that

$$\int \cdots \int_{R^k} |h(x)|^2 dx < +\infty .$$

And let $H_n(x)$ be the Hermite polynomials defined by the relations

$$\left(\frac{d}{dx}\right)^n e^{-\frac{x^2}{2}} = (-1)^n H_n(x) e^{-\frac{x^2}{2}} \quad (n = 0, 1, 2, ---) .$$

We shall denote

$$\phi_n^{(1)}(x, \sigma) = \frac{1}{(n!)^{\frac{1}{2}}(2\pi)^{\frac{1}{4}}\sigma^{\frac{1}{2}}} H_n\left(\frac{x}{\sigma}\right) e^{-\frac{x^2}{4\sigma^2}} ,$$

then $\{\phi_n^{(1)}(x,\sigma)\}$ is a complete orthonormal system of $L_2(R)$. And also let us denote

$$\phi^{(k)}_{n_1,n_2,\cdots,n_k}(x_1,\sigma_1;\, x_2,\sigma_2;\cdots;\, x_k,\sigma_k)$$

$$=\phi^{(1)}_{n_1}(x_1,\sigma_1)\,\phi^{(1)}_{n_2}(x_2,\sigma_2)\cdots\cdots\phi^{(1)}_{n_k}(x_k,\sigma_k)$$

$$(n_1,n_2,\cdots,n_k=0,1,2,\cdots).$$

Then the system

$$\{\phi^{(k)}_{n_1,n_2,\cdots,n_k}(x_1,\sigma_1;\, x_2,\sigma_2;\cdots;\, x_k,\sigma_k)\}$$

is a complete orthonormal system of $L_2(R^k)$.

Let $V_X(\gamma_k)$ denote the variance of γ_k for a process X(n). Then we have

$$V_X(\gamma_k)=E\gamma_k^2-(E\gamma_k)^2$$

$$=\frac{\pi}{2}\frac{1}{\sigma^2}\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}EX(n)X(m)\,sgn(X(n+k))\,sgn(X(m+k))$$

$$-\left(\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}EX(n)\,sgn(X(n+k))\right)^2$$

$$=\frac{\pi}{\sigma^2}\frac{1}{N^2}\sum_{\substack{h=1\\(h\neq k)}}^{N-1}(N-h)\,EX(n)X(n+h)\,sgn(X(n+k))\,sgn(X(n+h+k))$$

$$+\frac{\pi}{\sigma^2}\frac{(N-k)}{N^2}EX(n)\,|X(n+k)|\,sgn(X(n+2k))$$

$$+\frac{\pi}{2}\frac{1}{N}-(\rho_k+b_k)^2,$$

where $b_k=E\gamma_k-\rho_k$ and the symbol $\sum_{\substack{h=1\\(h\neq k)}}^{N-1}$ means

$$\sum_{\substack{h=1\\(h\neq k)}}^{N-1}d_h=\sum_{h=1}^{N-1}d_h-d_k.$$

We shall evaluate the each term of $V_X(\gamma_k)$.

(i) $E X(n) X(n+k) \operatorname{sgn}(X(n+h)) \operatorname{sgn}(X(n+k+h))$.

Now

$$E X(n) X(n+k) \operatorname{sgn}(X(n+h)) \operatorname{sgn}(X(n+k+h))$$

$$= \int \cdots \int_{R^4} x_n x_{n+k} \operatorname{sgn}(x_{n+h}) \operatorname{sgn}(x_{n+k+h}) \tilde{f}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})\, dx_n\, dx_{n+k}\, dx_{n+h}\, dx_{n+k+h},$$

where $\tilde{f}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})$ indicates the density function of the joint probability distribution of $(X(n), X(n+k), X(n+h), X(n+k+h))$.

And let $\tilde{\varphi}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})$ be the Gaussian density function of the joint distribution of

$$(X_0(n), X_0(n+k), X_0(n+h), X_0(n+k+h)).$$

We use the notation, such as

$$\mathbb{x} = (x_n, x_{n+k}, x_{n+h}, x_{n+k+h}),$$

$$\tilde{f}^{(4)}(\mathbb{x}) = \tilde{f}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})$$

and

$$\tilde{\varphi}^{(4)}(\mathbb{x}) = \tilde{\varphi}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h}),$$

when it is not confused. So by the assumption,

$$\frac{\tilde{f}^{(4)}(\mathbb{x})}{\sqrt{\tilde{\varphi}^{(4)}(\mathbb{x})}} \in L_2(R^4) \qquad \text{------ (1)}$$

Now we shall put the random vectors

$$\mathbb{X} = (X(n), X(n+k), X(n+h), X(n+k+h))$$

Then by the assumptions (P, 1) and (P, 2), we can find a linear transformation T_4 which satisfies the following conditions :

(a) $T_4 \mathbb{X}^t = \mathbb{U}^t$, where $\mathbb{U} = (U_n, U_{n+k}, U_{n+h}, U_{n+k+h})$ is the random vector whose variance-covariance matrix is a diagonal matrix.

(b) T_4 is the matrix such as

$$T_4 = \begin{pmatrix} 1 & -A & -B & -C \\ 0 & 1 & -F & -G \\ 0 & 0 & 1 & -\rho_k \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where A, B, C, F and G are constants (see [4]).

Corresponding to this linear transformation T_4, let us assume $\tilde{f}^{(4)}(\mathbb{X})$ is transformed to

$$f^{(4)}(\mathbb{U}) = f^{(4)}(u_n, u_{n+k}, u_{n+h}, u_{n+k+h}).$$

Here, we shall use the notation

$$\psi^{(1)}(x, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

Writing $E\, U_e^2 = \sigma_e^2$, we can express

$$\tilde{\varphi}^{(4)}(X) = \psi^{(1)}(u_n, \sigma_n)\, \psi^{(1)}(u_{n+k}, \sigma_{n+k})\, \psi^{(1)}(u_{n+h}, \sigma_{n+h})$$

$$\times\, \psi^{(1)}(u_{n+k+h}, \sigma_{u+k+h}).$$

Putting

$$\varphi^{(4)}(\mathbb{U}) = \psi^{(1)}(u_n, \sigma_n)\, \psi^{(1)}(u_{n+k}, \sigma_{n+k})\, \psi^{(1)}(u_{n+h}, \sigma_{n+h})\, \psi^{(1)}(u_{n+k+h}, \sigma_{n+k+h}),$$

we have

$$\frac{f^{(4)}(\mathfrak{u})}{\sqrt{\varphi^{(4)}(\mathfrak{u})}} \in L_2(R^4)$$

by the relation (1). So we can expand the function

$$\frac{f^{(4)}(\mathfrak{u})}{\sqrt{\varphi^{(4)}(\mathfrak{u})}}$$

by the complete orthonormal system

$$\left\{ \phi^{(4)}_{i,j,p,q} \left(u_n, \sigma_n ; u_{n+k}, \sigma_{n+k} ; u_{n+h}, \sigma_{n+h} ; u_{n+k+h}, \sigma_{n+k+h} \right) \right\}.$$

We shall write simply

$$\phi^{(4)}_{i,j,p,q}(\mathfrak{u}) = \phi^{(4)}_{i,j,p,q} \left(u_n, \sigma_n ; u_{n+k}, \sigma_{n+k} ; u_{n+h}, \sigma_{n+h} ; u_{n+k+h}, \sigma_{n+k+h} \right).$$

Let the orthogonal expansion be

$$\frac{f^{(4)}(\mathfrak{u})}{\sqrt{\varphi^{(4)}(\mathfrak{u})}} = \sum_{i,j,p,q \geq 0} a^{k}_{i,j,p,q} \; \phi^{(4)}_{i,j,p,q}(\mathfrak{u}),$$

where the sign of equality means that the left side and the right side coincide in the mean and

$$a^{k}_{i,j,p,q} = \iiiint_{R^4} \frac{f^{(4)}(\mathfrak{u})}{\sqrt{\varphi^{(4)}(\mathfrak{u})}} \; \phi^{(4)}_{i,j,p,q}(\mathfrak{u}) \, d\mathfrak{u}$$

$$= \frac{1}{\sqrt{i!}\sqrt{j!}\sqrt{p!}\sqrt{q!}} \iiiint_{R^4} H_i\left(\frac{u_n}{\sigma_n}\right) H_j\left(\frac{u_{n+k}}{\sigma_{n+k}}\right) H_p\left(\frac{u_{n+h}}{\sigma_{n+h}}\right) H_q\left(\frac{u_{n+k+h}}{\sigma_{n+k+h}}\right)$$

$$\times f^{(4)}(u_n, u_{n+k}, u_{n+h}, u_{n+k+h}) \, du_n \, du_{n+k} \, du_{n+h} \, du_{n+k+h}.$$

In the above expansion, we have

$$a^{k}_{0,0,0,0} = 1$$

and

$$a^{k}_{i,j,p,q} = 0 \qquad \text{for } 0 < i+j+p+q \leqq 2.$$

So we have

$$\frac{f^{(4)}(\mathfrak{U})}{\sqrt{\varphi^{(4)}(\mathfrak{U})}} = \phi^{(4)}_{0,0,0,0}(\mathfrak{U}) + \sum_{\substack{i,j,p,q \geqq 0 \\ i+j+p+q \geqq 3}} a^{k}_{i,j,p,q}\ \phi^{(4)}_{i,j,p,q}(\mathfrak{U}).$$

Here we shall rearrange our discussion. It holds

$$E\, X(n)X(n+k)\,sgn\,(X(n+h))\,sgn\,(X(n+k+h))$$

$$= \int\!\cdots\!\int_{R^4} x_n\, x_{n+k}\, sgn(x_{n+h})\, sgn\,(x_{n+k+h})\ \tilde{f}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})$$
$$dx_n\, dx_{n+k}\, dx_{n+h}\, dx_{n+k+h}$$

$$= \int\!\cdots\!\int_{R^4} x_n\, x_{n+k}\, sgn\,(x_{n+h})\, sgn\,(x_{n+k+h})\ \sqrt{\tilde{\varphi}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})}$$
$$\times\ \frac{\tilde{f}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})}{\sqrt{\tilde{\varphi}^{(4)}(x_n, x_{+k}, x_{n+h}, x_{n+k+h})}}\ dx_n dx_{n+k}\, dx_{n+h}\, dx_{n+k+h}$$

$$= \int\!\cdots\!\int_{R^4} (u_n + \alpha u_{n+k} + \beta u_{n+h} + \gamma u_{n+k+h})(u_{n+k} + \alpha' u_{n+h} + \beta' u_{n+k+h})$$
$$\times\ sgn\,(u_{n+h} + \alpha'' u_{n+k+h})\, sgn\,(u_{n+k+h})\sqrt{\varphi^{(4)}(u_n, u_{n+k}, u_{n+h}\ u_{n+k+h})}$$
$$\times\ \frac{f^{(4)}(u_n, u_{n+k}, u_{n+h}, u_{n+k+h})}{\sqrt{\varphi^{(4)}(u_n, u_{n+k}, u_{n+h}, u_{n+k+h})}}\ du_n\, du_{n+k}\, du_{n+h}\, du_{n+k+h}\ ,$$

where $\alpha, \beta, \gamma, \alpha', \beta'$, and α'' are constants such that

$$T_4^{-1} = \begin{pmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & \alpha' & \beta' \\ 0 & 0 & 1 & \alpha'' \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, we shall consider to expand the function

$$h_4(\mathbb{U}) \sqrt{\varphi^{(4)}(\mathbb{U})}$$

$$\equiv (U_n + \alpha U_{n+k} + \beta U_{n+h} + \gamma U_{n+k+h})(U_{n+k} + \alpha' U_{n+h} + \beta' U_{n+k+h})$$

$$\times \operatorname{sgn}(U_{n+h} + \alpha'' U_{n+k+h}) \operatorname{sgn}(U_{n+k+h}) \sqrt{\varphi^{(4)}(U_n, U_{n+k}, U_{n+h}, U_{n+k+h})}.$$

As

$$h_4(\mathbb{U}) \sqrt{\varphi^{(4)}(\mathbb{U})} \in L_2(R^4),$$

we can expand this function by the system $\left\{ \phi^{(4)}_{i,j,p,q}(\mathbb{U}) \right\}$ and assume this expansion is

$$h_4(\mathbb{U}) \sqrt{\varphi^{(4)}(\mathbb{U})} = \sum_{i,j,p,q \geqq 0} C^{k}_{i,j,p,q} \, \phi^{(4)}_{i,j,p,q}(\mathbb{U}),$$

where

$$C^{k}_{i,j,p,q} = \iiiint_{R^4} h_4(\mathbb{U}) \sqrt{\varphi^{(4)}(\mathbb{U})} \, \phi^{(4)}_{i,j,p,q}(\mathbb{U}) \, d\mathbb{U}$$

$$= \frac{1}{\sqrt{i!}\sqrt{j!}\sqrt{p!}\sqrt{q!}} \iiiint_{R^4} h_4(U_n, U_{n+k}, U_{n+h}, U_{n+k+h}) H_i\left(\frac{U_n}{\sigma_n}\right) H_j\left(\frac{U_{n+k}}{\sigma_{n+k}}\right)$$

$$\times H_p\left(\frac{U_{n+h}}{\sigma_{n+h}}\right) H_q\left(\frac{U_{n+k+h}}{\sigma_{n+k+h}}\right) \varphi^{(4)}(U_n, U_{n+k}, U_{n+h}, U_{n+k+h}) \, dU_n \, dU_{n+k} \, dU_{n+h} \, dU_{n+k+h}.$$

Now we have

$C^{k}_{i,j,p,q} = 0$ for any (i, j) except for the following cases :

i = 0, 1

j = 0, 1, 2

and

$$C^{k}_{0,0,0,0} = E X_0(n) X_0(n+k) sgn(X_0(n+h)) sgn(X_0(n+k+h)).$$

Using the above results, we have

$$E X(n) X(n+k) sgn(X(n+h)) sgn(X(n+k+h))$$

$$= \int \cdots \int_{R^4} h_4(w) \sqrt{\varphi^{(4)}(w)} \frac{f(w)}{\sqrt{\varphi^{(4)}(w)}} dw$$

$$= \int \cdots \int_{R^4} \Big(\sum_{\substack{0 \leq i \leq 1 \\ 0 \leq j \leq 2 \\ p, q \geqq 0}} C^{k}_{i,j,p,q} \phi^{(4)}_{i,j,p,q}(w) \Big)$$

$$\times \Big(\phi^{(4)}_{0,0,0,0}(w) + \sum_{\substack{i,j,p,q \geqq 0 \\ i+j+p+q \geqq 3}} a^{k}_{i,j,p,q} \phi^{(4)}_{i,j,p,q}(w) \Big) dw$$

$$= C^{k}_{0,0,0,0} + \sum_{\substack{0 \leq i \leq 1 \\ 0 \leq j \leq 2 \\ p, q \geqq 0 \\ i+j+p+q \geqq 3}} C^{k}_{i,j,p,q} a^{k}_{i,j,p,q}.$$

We have the following result :

<u>Lemma 3.</u> $E X(n) X(n+k) sgn(X(n+h)) sgn(X(n+k+h))$

$$= E X_0(n) X_0(n+k) sgn(X_0(n+h)) sgn(X_0(n+k+h)) + \sum_{\substack{0 \leq i \leq 1 \\ 0 \leq j \leq 2 \\ p, q \geqq 0 \\ i+j+p+q \geqq 3}} C^{k}_{i,j,p,q} a^{k}_{i,j,p,q}.$$

(ii) $E X(n) \, |X(n+h)| \, sgn\,(X(n+2h))$.

Let $\tilde{f}^{(3)}(x_n, x_{n+h}, x_{n+2h})$ (or simply $\tilde{f}^{(3)}(\mathbf{x})$) denote the probability density function of the joint distribution of $(X(n), X(n+h), X(n+2h))$. Then we have

$$E X(n) \, |X(n+h)| \, sgn\,(X(n+2h))$$

$$= \iiint_{R^3} x_n \, |x_{n+h}| \, sgn\,(x_{n+2h}) \, \tilde{f}^{(3)}(x_n, x_{n+h}, x_{n+2h}) \, dx_n \, dx_{n+h} \, dx_{n+2h}.$$

Let $\tilde{\varphi}^{(3)}(x_n, x_{n+h}, x_{n+2h})$ be the Gaussian density function with the same mean vector and the same variance-covariance matrix as $\tilde{f}^{(3)}(x_n, x_{n+h}, x_{n+2h})$.

As $X(n) \in U^{(4)}(X_0)$, we have

$$\frac{\tilde{f}^{(3)}(\mathbf{x})}{\sqrt{\tilde{\varphi}^{(3)}(\mathbf{x})}} \in L_2(R^3).$$

Let us put

$$\mathbb{X} = (X(n), X(n+h), X(n+2h)).$$

Then, by the assumptions (P, 1) and (P, 2), we can find a linear transformation T_3 which satisfies the following conditions :

(a') $T_3 \mathbb{X}^t = \mathbb{V}^t$;
where $\mathbb{V} = (V_n, V_{n+h}, V_{n+2h})$ and the variance-covariance matrix of V is diagonal.

(b') T_3 is the matrix such as

$$T_3 = \begin{pmatrix} 1 & -H & -K \\ 0 & 1 & -\rho_h \\ 0 & 0 & 1 \end{pmatrix},$$

where H and K are constants.

We shall put

$$T_3^{-1} = \begin{pmatrix} 1 & \theta_\alpha & \theta_\beta \\ 0 & 1 & \theta_\gamma \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$E\, X(n)\,|X(n+h)|\, sgn\,(X(n+2h))$$

$$= \iiint_{R^3} x_n\ |x_{n+h}|\ sgn\,(x_{n+2h})\ \tilde{f}^{(3)}(x_n, x_{n+h}, x_{n+2h})\ dx_n\ dx_{n+h}\ dx_{n+2h}$$

$$= \iiint_{R^3} (v_n + \theta_\alpha v_{n+h} + \theta_\beta v_{n+2h})\ |v_{n+h} + \theta_\gamma v_{n+2h}|\ sgn\,(v_{n+2h})$$

$$\times f^{(3)}(v_n, v_{n+h}, v_{n+2h})\ dv_n\, dv_{n+h}\, dv_{n+2h}.$$

Let us assume that $\tilde{\varphi}^{(3)}(x_n, x_{n+h}, x_{n+2h})$ is transformed to $\varphi^{(3)}(v_n, v_{n+h}, v_{n+2h})$ by T_3 and denote simply

$$h_3(v) = (v_n + \theta_\alpha v_{n+h} + \theta_\beta v_{n+2h})\ |v_{n+h} + \theta_\gamma v_{n+2h}|\ sgn\,(v_{n+2h})$$

$$f^{(3)}(v) = f^{(3)}(v_n, v_{n+h}, v_{n+2h})$$

and

$$\varphi^{(3)}(v) = \varphi^{(3)}(v_n, v_{n+h}, v_{n+2h}).$$

Then we have

$$E X(n)\,|X(n+h)|\, sgn\,(X(n+2h))$$

$$= \iiint_{R^3} h_3(v)\cdot f^{(3)}(v)\, dv$$

$$= \iiint_{R^3} h_3(v)\sqrt{\varphi^{(3)}(v)}\cdot\frac{f^{(3)}(v)}{\sqrt{\varphi^{(3)}(v)}}\, dv$$

We shall consider the functions

$$\frac{f^{(3)}(\mathbf{v})}{\sqrt{\varphi^{(3)}(\mathbf{v})}} \quad \text{and} \quad h_3(\mathbf{v})\sqrt{\varphi^{(3)}(\mathbf{v})} .$$

As

$$\frac{f^{(3)}(\mathbf{v})}{\sqrt{\varphi^{(3)}(\mathbf{v})}} \in L_2(R^3) \quad \text{and} \quad h_3(\mathbf{v})\sqrt{\varphi^{(3)}(\mathbf{v})} \in L_2(R^3),$$

we can expand these functions by the complete orthonormal system $\{ \phi^{(3)}_{i,j,p}(\mathbf{v}) \}$, where

$$\phi^{(3)}_{i,j,p}(\mathbf{v}) = \phi^{(3)}_{i,j,p}(V_n, \tau_n; V_{n+h}, \tau_{n+h}; V_{n+2h}, \tau_{n+2h})$$

and

$$\tau_\ell^2 = E V_\ell^2 \quad \text{for} \quad \ell = n,\ n+h,\ n+2h .$$

Let these expansions be

$$\frac{f^{(3)}(\mathbf{v})}{\sqrt{\varphi^{(3)}(\mathbf{v})}} = \sum_{i,j,p \geq 0} a^{h}_{i,j,p}\ \phi^{(3)}_{i,j,p}(\mathbf{v})$$

and

$$h_3(\mathbf{v})\sqrt{\varphi^{(3)}(\mathbf{v})} = \sum_{i,j,p \geq 0} C^{h}_{i,j,p}\ \phi^{(3)}_{i,j,p}(\mathbf{v}),$$

where

$$a^{h}_{i,j,p} = \iiint_{R^3} \frac{f^{(3)}(\mathbf{v})}{\sqrt{\varphi^{(3)}(\mathbf{v})}}\ \phi^{(3)}_{i,j,p}(\mathbf{v})\, d\mathbf{v}$$

$$= \frac{1}{\sqrt{i!}\sqrt{j!}\sqrt{p!}} \iiint_{R^3} H_i\left(\frac{V_n}{\tau_n}\right) H_j\left(\frac{V_{n+h}}{\tau_{n+h}}\right) H_p\left(\frac{V_{n+2h}}{\tau_{n+2h}}\right) f^{(3)}(V_n, V_{n+h}, V_{n+2h})\, dV_n\, dV_{n+h}\, dV_{n+2h}$$

and

$$C_{i,j,p}^{h} = \iiint_{R^3} h_3(v)\sqrt{\varphi^{(3)}(v)}\; \phi_{i,j,p}^{(3)}(v)\, dv$$

$$= \frac{1}{\sqrt{i!}\sqrt{j!}\sqrt{p!}} \iiint_{R^3} h_3(v_n, v_{n+h}, v_{n+2h}) H_i\left(\frac{v_n}{\sigma_n}\right) H_j\left(\frac{v_{n+h}}{\sigma_{n+h}}\right) H_p\left(\frac{v_{n+2h}}{\sigma_{n+2h}}\right) \varphi^{(3)}(v_n, v_{n+h}, v_{n+2h})\, dv_n\, dv_{n+h}\, dv_{n+2h}.$$

In the above expressions, we have

$$a_{0,0,0}^{h} = 1$$

$$a_{i,j,p}^{h} = 0 \quad \text{for} \quad 0 < i+j+p \leqq 2 .$$

Also we can see easily the following facts :

$$C_{i,j,p}^{h} = 0 \quad \text{for any} \quad i \geqq 2 ,$$

$$C_{0,0,0}^{h} = EX_0(n)\,|X_0(n+h)|\,sgn\left(X_0(n+2h)\right).$$

Consequently we obtain

$$EX(n)\,|X(n+h)|\,sgn\left(X(n+2h)\right)$$

$$= \iiint_{R^3} h_3(v)\sqrt{\varphi^{(3)}(v)} \cdot \frac{f^{(3)}(v)}{\sqrt{\varphi^{(3)}(v)}}\, dv$$

$$= \iiint_{R^3} \Big(\sum_{\substack{0\le i\le 1\\ j,p\ge 0}} C_{i,j,p}^{h}\, \phi_{i,j,p}^{(3)}(v)\Big)\Big(\phi_{0,0,0}^{(3)}(v) + \sum_{\substack{i,j,p\ge 0\\ i+j+p\ge 3}} a_{i,j,p}^{h}\, \phi_{i,j,p}^{(3)}(v)\Big) dv$$

$$= C_{0,0,0}^{h} + \sum_{\substack{0\le i\le 1\\ j,p\ge 0\\ i+j+p\ge 3}} C_{i,j,p}^{h}\, a_{i,j,p}^{h} .$$

From the above results, we have the following fact :

Lemma 4. $E X(n) |X(n+h)| sgn(X(n+2h))$

$$= E X_0(n) |X_0(n+h)| sgn(X_0(n+2h)) + \sum_{\substack{0 \le i \le 1 \\ j, p \ge 0 \\ i+j+p \ge 3}} C^h_{i,j,p}\, a^h_{i,j,p} .$$

In the previous paper [5] , we have obtained

$$E\gamma_h = \rho_h + \sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma}\left\{\sum_{i=2}^{\infty} C_{0,2i}\, a_{0,2i} + \sum_{i=1}^{\infty} C_{1,2i+1}\, a_{1,2i+1}\right\},$$

where

$$a_{i,j} = \frac{1}{\sqrt{i!}\sqrt{j!}} \iint H_i\left(\frac{x_n - \rho_h x_{n+h}}{\sigma\sqrt{1-\rho_h^2}}\right) H_j\left(\frac{x_{n+h}}{\sigma}\right) \tilde{f}^{(2)}(x_n, x_{n+h})\, dx_n\, dx_{n+h}$$

and

$$C_{i,j} = \frac{1}{\sqrt{i!}\sqrt{j!}} \iint x_n\, sgn(x_{n+h})\, H_i\left(\frac{x_n - \rho_h x_{n+h}}{\sigma\sqrt{1-\rho_h^2}}\right) H_j\left(\frac{x_{n+h}}{\sigma}\right)$$

$$\times\ \varphi^{(1)}(x_n - \rho_h x_{n+h}, \sigma\sqrt{1-\rho_h^2})\, \varphi^{(1)}(x_{n+h}, \sigma)\, dx_n\, dx_{n+h} .$$

Combining the above results, we have

$$V_x(\gamma_h) = \frac{\pi}{2}\frac{1}{\sigma^2}\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} E X(n)\, sgn\big(X(n+h)\big) X(m)\, sgn\big(X(m+h)\big)$$

$$-\left(\sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma} E X(n)\, sgn\big(X(n+h)\big)\right)^2$$

$$= \frac{\pi}{2}\frac{1}{\sigma^2}\frac{2}{N^2}\sum_{\substack{k=1\\k\neq h}}^{N-1}(N-k)\, E X(n)\, sgn\big(X(n+h)\big) X(n+k)\, sgn\big(X(n+k+h)\big) .$$

$$+\frac{\pi}{2}\frac{1}{\sigma^2}\frac{2}{N^2}(N-h)\, E X(n) \,|X(n+h)|\, sgn\big(X(n+2h)\big)$$

$$+\frac{\pi}{2}\frac{1}{N} - \left(\sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma} E X(n)\, sgn\big(X(n+h)\big)\right)^2$$

$$= V_{x0}(\gamma_h) + \frac{\pi}{\sigma^2}\frac{1}{N^2}\sum_{\substack{k=1\\k\neq h}}^{N-1}(N-k)\left(\sum_{\substack{0\le i\le 1\\0\le j\le 2\\p,q\ge 0\\i+j+p+q\ge 3}} C^{k}_{i,j,p,q}\; a^{k}_{i,j,p,q}\right)$$

$$+\frac{\pi}{\sigma^2}\frac{1}{N}\left(1-\frac{h}{N}\right)\left(\sum_{\substack{0\le i\le 1\\j,p\ge 0\\i+j+p\ge 3}} C^{h}_{i,j,p}\; a^{h}_{i,j,p}\right)$$

$$-\left\{\frac{\pi}{2}\frac{1}{\sigma^2}\left(\sum_{i=2}^{\infty} C_{0,2i}\, a_{0,2i} + \sum_{i=1}^{\infty} C_{1,2i+1}\, a_{1,2i+1}\right)^2\right.$$

$$\left.+2\rho_h\sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma}\left(\sum_{i=2}^{\infty} C_{0,2i}\, a_{0,2i} + \sum_{i=1}^{\infty} C_{1,2i+1}\, a_{1,2i+1}\right)\right\} \text{-----}(2)$$

Therefore, we have the following result.

Theorem 3. Let $X_0(n)$ be a stationary Gaussian process. If $X(n)$ is a stationary process belonging to the class $U^{(4)}(X_0(n))$, the variance of the estimate

$$\gamma_h = \sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma}\,\frac{1}{N}\sum_{n=1}^{N} X(n)\,\mathrm{sgn}(X(n+h))$$

is given by (2).

4. Comparison of the variance of γ_h with that of $\tilde{\gamma}_h$.

The variance of $\tilde{\gamma}_h$ is as follows.

$$V_x(\tilde{\gamma}_h) = \frac{1}{N^2}\frac{1}{\sigma^2}\sum_{n=1}^{N}\sum_{m=1}^{N} EX(n)\,x(n+h)\,x(m)\,x(m+h) - \rho_h^{\,2}$$

$$= \frac{1}{N^2}\frac{2}{\sigma^2}\sum_{\substack{k=1\\ k\neq h}}^{N-1}(N-k)\,EX(n)\,X(n+h)\,x(n+k)\,x(n+k+h)$$

$$+ \frac{1}{N^2}\frac{2}{\sigma^2}(N-h)\,EX(n)\,x(n+h)^2\,x(n+2h)$$

$$+ \frac{1}{N}\frac{1}{\sigma^2}\,EX(n)^2\,X(n+h)^2 - \rho_h^{\,2}$$

Now we shall consider the same transformation T_4 or T_3 as in §3. And we shall denote

$$h_4^*(u) = (u_n + \alpha u_{n+k} + \beta u_{n+h} + \gamma u_{n+k+h})$$

$$\times\,(u_{n+k} + \alpha' u_{n+h} + \beta' u_{n+k+h})(u_{n+h} + \alpha'' u_{n+k+h})\,u_{n+k+h}.$$

Then we have

$$h_4^*(u)\sqrt{\varphi^{(4)}(u)} \in L_2(R^4),$$

and so we can expand this function by the system $\{\phi^{(4)}_{i,j,p,q}(u)\}$ as the following :

$$h_4^*(u)\sqrt{\varphi^{(4)}(u)} = \sum_{i,j,p,q\geq 0} C^*_{i,j,p,q}\,\phi^{(4)}_{i,j,p,q}(u),$$

where

$$C^{*\,k}_{i,j,p,q} = \iiiint_{R^4} h_4^{*}(\mathbb{u}) \sqrt{\varphi^{(4)}(\mathbb{u})}\ \phi^{(4)}_{i,j,p,q}(\mathbb{u})\, d\mathbb{u}$$

$$= \frac{1}{\sqrt{i!}\sqrt{j!}\sqrt{p!}\sqrt{q!}} \iiiint h_4^{*}(u_n, u_{n+k}, u_{n+h}, u_{n+k+h}) H_i\left(\frac{u_n}{\sigma_n}\right) H_j\left(\frac{u_{n+k}}{\sigma_{n+k}}\right) H_p\left(\frac{u_{n+h}}{\sigma_{n+h}}\right) H_q\left(\frac{u_{n+k+h}}{\sigma_{n+k+h}}\right)$$

$$\times \varphi^{(4)}(u_n, u_{n+k}, u_{n+h}, u_{n+k+h})\, du_n\, du_{n+k}\, du_{n+h}\, du_{n+k+h}.$$

As is easily seen, it holds

$$C^{*}_{i,j,p,q} = 0 \quad \text{except for } 0 \le i \le 1,\ 0 \le j \le 2,\ 0 \le p \le 3,\ 0 \le q \le 4$$

and

$$C^{*\,k}_{0,0,0,0} = E X_0(n) X_0(n+k) X_0(n+h) X_0(n+k+h).$$

Therefore we get the following fact :

Lemma 5.

$$E X(n) X(n+k) X(n+h) X(n+k+h)$$

$$= \iiiint_{R^4} h_4^{*}(\mathbb{u}) \sqrt{\varphi^{(4)}(\mathbb{u})} \cdot \frac{f^{(4)}(\mathbb{u})}{\sqrt{\varphi^{(4)}(\mathbb{u})}}\, d\mathbb{u}$$

$$= E X_0(n) X_0(n+k) X_0(n+h) X_0(n+k+h)$$

$$+ \sum_{\substack{0 \le i \le 1 \\ 0 \le j \le 2 \\ 0 \le p \le 3 \\ 0 \le q \le 4 \\ i+j+p+q \ge 3}} C^{*\,k}_{i,j,p,q}\ a^{k}_{i,j,p,q}.$$

In the next place, we shall put

$$h_3^*(\mathfrak{v}) = (v_n + \theta_\alpha v_{n+k} + \theta_\beta v_{n+2k})(v_{n+k} + \theta_\gamma v_{n+2k})^2 v_{n+2k}.$$

As we have

$$h_3^*(\mathfrak{v})\sqrt{\varphi^{(3)}(\mathfrak{v})} \in L_2(R^3),$$

we can expand this function by the system $\{ \phi_{i,j,p}^{(3)}(\mathfrak{v}) \}$ as the following :

$$h_3^*(\mathfrak{v})\sqrt{\varphi^{(3)}(\mathfrak{v})} = \sum_{i,j,p \geqq 0} C_{i,j,p}^{*\,k} \phi_{i,j,p}^{(3)}(\mathfrak{v}),$$

where

$$C_{i,j,p}^{*\,k} = \iiint_{R^3} h_3^*(\mathfrak{v})\sqrt{\varphi^{(3)}(\mathfrak{v})}\, \phi_{i,j,p}^{(3)}(\mathfrak{v})\, d\mathfrak{v}$$

$$= \frac{1}{\sqrt{i!}\sqrt{j!}\sqrt{p!}} \iiint_{R^3} h_3^*(v_n, v_{n+k}, v_{n+2k}) H_i\left(\frac{v_n}{\tau_n}\right) H_j\left(\frac{v_{n+k}}{\tau_{n+k}}\right) H_p\left(\frac{v_{n+2k}}{\tau_{n+2k}}\right) \varphi^{(3)}(v_n, v_{n+k}, v_{n+2k})\, dv_n\, dv_{n+k}\, dv_{n+2k}.$$

It holds

$$C_{i,j,p}^{*\,k} = 0 \qquad \text{except for } 0 \leqq i \leqq 1,\ 0 \leqq j \leqq 3,\ 1 \leqq p \leqq 4$$

and

$$C_{0,0,0}^{*\,k} = E X_0(n) X_0(n+k)^2 X_0(n+2k).$$

Consequently we obtain the following fact :

<u>Lemma 6.</u> $E X(n) X(n+k)^2 X(n+2k)$

$$= E X_0(n) X_0(n+k)^2 X_0(n+2k) + \sum_{\substack{0 \leqq i \leqq 1 \\ 0 \leqq j \leqq 3 \\ 1 \leqq p \leqq 4 \\ i+j+p \geqq 3}} C_{i,j,p}^{*\,k}\, a_{i,j,p}^{k}.$$

Lastly we have

$$E\,X(n)^2 X(n+k)^2 = E\,X_0(n)^2 X_0(n+k)^2 + \sum_{\substack{0 \leqq i \leqq 2 \\ 2 \leqq j \leqq 4 \\ i+j \geqq 3}} C_{i,j}\, a_{ij},$$

where

$$C_{i,j} = \frac{1}{\sqrt{i!}\sqrt{j!}} \iint_{R^2} x_n^2\, x_{n+k}^2\, H_i\left(\frac{x_n - \rho_k x_{n+k}}{\sigma\sqrt{1-\rho_k^2}}\right) H_j\left(\frac{x_{n+k}}{\sigma}\right)$$

$$\times \frac{1}{2\pi\sigma^2\sqrt{1-\rho_k^2}}\, e^{-\frac{1}{2\sigma^2(1-\rho_k^2)}(x_n^2 - 2\rho_k x_n x_{n+k} + x_{n+k}^2)}\, dx_n\, dx_{n+k}$$

Combining the above results, we obtain

$$V_X(\tilde{\gamma}_k) = V_{X_0}(\tilde{\gamma}_k) + \frac{1}{N^2}\frac{2}{\sigma^2}\sum_{\substack{k=1 \\ k \neq k}}^{N-1} (N-k)\left(\sum_{\substack{0 \leqq i \leqq 1 \\ 0 \leqq j \leqq 2 \\ 0 \leqq p \leqq 3 \\ 0 \leqq q \leqq 4 \\ i+j+p+q \geqq 3}} C^{*\,k}_{i,j,p,q}\, a^{k}_{i,j,p,q}\right)$$

$$+ \frac{1}{N^2}\frac{2}{\sigma^2}(N-k)\left(\sum_{\substack{0 \leqq i \leqq 1 \\ 0 \leqq j \leqq 3 \\ 1 \leqq p \leqq 4 \\ i+j+p \geqq 3}} C^{*\,k}_{i,j,p}\, a^{k}_{i,j,p}\right)$$

$$+ \frac{1}{N}\frac{1}{\sigma^2}\left(\sum_{\substack{0 \leqq i \leqq 2 \\ 2 \leqq j \leqq 4 \\ i+j \geqq 3}} C^{*}_{i,j}\, a_{i,j}\right) \qquad \text{-----}\ (3)$$

Therefore we have the following theorem.

Theorem 4. Let $X_0(n)$ be a stationary Gaussion process. If X(n) is a stationary process belonging to the class $U^{(4)}(X_0(n))$, the variance of the estimate

$$\tilde{\gamma}_h = \frac{1}{\sigma^2} \frac{1}{N} \sum_{n=1}^{N} x(n)x(n+h)$$

is given by (3).

§ 5. On the variation of the variance.

In §3 and §4, we have evaluated the variances of the estimates γ_k and $\tilde{\gamma}_k$. Using this evaluation, we shall discuss the variation of the variances.

At first, we shall discuss this problem in a general way. Let $X_o(n)$ be a stationary Gaussian process. We assume that X(n) and Y(n) are stationary processes belonging to the class $U^{(k)}(X_o)$.
Let $\tilde{f}^{(k)}(x_{n_1}, x_{n_2}, ----, x_{n_k})$ and $\tilde{g}^{(k)}(x_{n_1}, x_{n_2}, ----, x_{n_k})$ be the probability density functions of the k-dimensional joint distribution functions of $(X(n_1), X(n_2), ----, X(n_k))$ and $(Y(n_1), Y(n_2), ---, Y(n_k))$ respectively. Without loss of generality, we can assume

$$n_1 < n_2 < ----- < n_k.$$

Now we shall define the norm

$$\| \tilde{f}^{(k)}(x_{n_1}, x_{n_2}, \cdots, x_{n_k}) - \tilde{g}^{(k)}(x_{n_1}, x_{n_2}, ---, x_{n_k}) \|_{\mathfrak{F}_k}$$

as the following :

$$\| \tilde{f}^{(k)}(x_{n_1}, x_{n_2}, ---, x_{n_k}) - \tilde{g}^{(k)}(x_{n_1}, x_{n_2}, ---, x_{n_k}) \|^2_{\mathfrak{F}_k}$$

$$= \int_{R^k}\!\!\cdots\!\!\int \left| \tilde{f}^{(k)}(x_{n_1}, x_{n_2}, \cdots, x_{n_k}) - \tilde{g}^{(k)}(x_{n_1}, x_{n_2}, \cdots, x_{n_k}) \right|^2 \frac{dx_{n_1}\, dx_{n_2} --- dx_{n_k}}{\tilde{\varphi}^{(k)}(x_{n_1}, x_{n_2}, ---, x_{n_k})} .$$

Then we have the following relation.

<u>Theorem 5.</u> It holds

$$\| \tilde{f}^{(\ell-1)}(x_{n_1}, ---, x_{n_{i-1}}, x_{n_{i+1}}, ---, x_{n_\ell}) - \tilde{g}^{(\ell-1)}(x_{n_1}, ---, x_{n_{i-1}}, x_{n_{i+1}}, \cdots, x_{n_\ell}) \|_{\mathfrak{F}_{\ell-1}}$$

$$\leqq \| \tilde{f}^{(\ell)}(x_{n_1}, ---, x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1}, ----, x_{n_\ell}) \|_{\mathfrak{F}_\ell}$$

for any ℓ and i which satisfy the conditions

$$0 < \ell \leqq k$$

$$1 \leqq i \leqq \ell .$$

Proof. As

$$\tilde{f}^{(\ell-1)}(x_{n_1},\cdots,x_{n_{i-1}},\ x_{n_{i+1}},\cdots,x_{n_\ell}) = \int \tilde{f}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})\,dx_{n_i}$$

and

$$\tilde{g}^{(\ell-1)}(x_{n_1},\cdots,x_{n_{i-1}},x_{n_{i+1}},\cdots,x_{n_\ell}) = \int \tilde{g}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})\,dx_{n_i}\ ,$$

we have

$$\|\tilde{f}^{(\ell-1)}(x_{n_1},\cdots,x_{n_{i-1}},x_{n_{i+1}},\cdots,x_{n_\ell}) - \tilde{g}^{(\ell-1)}(x_{n_1},\cdots,x_{n_{i-1}},x_{n_{i+1}},\cdots,x_{n_\ell})\|^2_{\mathfrak{F}_{\ell-1}}$$

$$\leqq \underset{R^{\ell-1}}{\int\cdots\int}\left(\int|\tilde{f}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})|\,dx_{n_i}\right)^2 \frac{dx_{n_1}\cdots dx_{n_{i-1}}\,dx_{n_{i+1}}\cdots dx_{n_\ell}}{\tilde{\varphi}^{(\ell-1)}(x_{n_1},\cdots,x_{n_{i-1}},x_{n_{i+1}},\cdots,x_{n_\ell})}$$

$$= \underset{R^{\ell-1}}{\int\cdots\int}\left(\int \frac{|\tilde{f}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})|}{\tilde{\varphi}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})^{\frac{1}{2}}}\,\tilde{\varphi}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})^{\frac{1}{2}}\,dx_{n_i}\right)^2$$

$$\times\ \frac{dx_{n_1}\cdots dx_{n_{i-1}}\,dx_{n_{i+1}}\cdots dx_{n_\ell}}{\tilde{\varphi}^{(\ell-1)}(x_{n_1},\cdots,x_{n_{i-1}},x_{n_{i+1}},\cdots,x_{n_\ell})}$$

$$\leqq \underset{R^{\ell-1}}{\int\cdots\int}\left(\int \frac{|\tilde{f}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})|^2}{\tilde{\varphi}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})}\,dx_{n_i}\right)\left(\int \tilde{\varphi}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})\,dx_{n_i}\right)$$

$$\times\ \frac{dx_{n_1}\cdots dx_{n_{i-1}}\,dx_{n_{i+1}}\cdots dx_{n_\ell}}{\tilde{\varphi}^{(\ell-1)}(x_{n_i}\cdots,x_{n_{i-1}},x_{n_{i+1}},\cdots x_{n_\ell})}$$

$$= \underset{R^{\ell}}{\int\cdots\int} \frac{|\tilde{f}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})|^2}{\tilde{\varphi}^{(\ell)}(x_{n_1},\cdots,x_{n_\ell})}\,dx_{n_1}\cdots dx_{n_\ell}$$

$$= \| \tilde{f}^{(\ell)}(x_{n_1}, \cdots, x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1}, \cdots, x_{n_\ell}) \|^2_{\mathcal{F}_\ell} .$$

Now, we shall restrict our attention to the process belonging to the class $U^{(4)}(X_0)$ and consider the variance of the estimate γ_k. Let X(n) and Y(n) be two stationary processes belonging to the class $U^{(4)}(X_0)$. And, as in §3, let $\tilde{f}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})$ denote the joint probability density function of $(X(n), X(n+k), X(n+h), X(n+k+h))$ and $\tilde{g}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h})$ denote that of $(Y(n), Y(n+k), Y(n+h), Y(n+k+h))$.

Also, we assume $\tilde{f}^{(3)}(x_n, x_{n+h}, x_{n+2h})$ and $\tilde{g}^{(3)}(x_n, x_{n+h}, x_{n+2h})$ to denote the joint probability density functions of $(X(n), X(n+h), X(n+2h))$ and $(Y(n), Y(n+h), Y(n+2h))$ respectively.

Definition 2. Let X(n) be a stationary process belonging to the class $U^{(4)}(X_0)$ and ε be a positive number. A stationary process Y(n) is said to belong to the class $V_\varepsilon(X)$ if it has the following properties :

(i) $Y(n) \in U^{(4)}(X_0)$,

(ii) $$\max\left[\max_{\substack{1 \leq k \leq N-1 \\ k \neq h}} \left(\| \tilde{g}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h}) - \tilde{f}^{(4)}(x_n, x_{n+k}, x_{n+h}, x_{n+k+h}) \|_{\mathcal{F}_4}, \right.\right.$$

$$\left.\| \tilde{g}^{(3)}(x_n, x_{n+h}, x_{n+2h}) - \tilde{f}^{(3)}(x_n, x_{n+h}, x_{n+2h}) \|_{\mathcal{F}_3} \right] < \varepsilon.$$

Then we have the following theorem.

Theorem 6. It holds

$$| V_Y(\gamma_k) - V_X(\gamma_k) | < C\,\varepsilon \quad \text{and} \quad | V_Y(\tilde{\gamma}_k) - V_X(\tilde{\gamma}_k) | < \tilde{C}\,\varepsilon .$$

for any $Y(n) \in V_\varepsilon(X)$, where C and $\tilde{C}$ are constants.

The above result can be easily obtained by the use of the following lemma and Theorem 5.

Lemma 7. Let $X(n), Y(n) \in U^{(\ell)}(X_0)$ We shall assume $h(x_1, x_2, \cdots, x_\ell)$ to be a measurable function of ℓ variables. Then we have

$$|E h(X(n_1), X(n_2), \cdots, X(n_\ell)) - E h(Y(n_1), Y(n_2), \cdots, Y(n_\ell))|$$

$$\leqq K^{\frac{1}{2}} \| \tilde{f}^{(\ell)}(x_{n_1}, \cdots, x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1}, \cdots, x_{n_\ell}) \| \tilde{f}_\ell ,$$

if

$$K = \int_{R^\ell}\cdots\int h^2(x_{n_1}, x_{n_2}, \cdots, x_{n_\ell}) \tilde{\varphi}^{(\ell)}(x_{n_1}, x_{n_2}, \cdots, x_{n_\ell}) \, dx_{n_1} dx_{n_2} \cdots dx_{n_\ell} < +\infty .$$

In fact, we have

$$|E h(X(n_1), X(n_2), \cdots, X(n_\ell)) - E h(Y(n_1), Y(n_2), \cdots, Y(n_\ell))|$$

$$= \Big| \int\cdots\int h(x_{n_1}, \cdots, x_{n_\ell}) \tilde{f}^{(\ell)}(x_{n_1}, \cdots, x_{n_\ell}) \, dx_{n_1} \cdots dx_{n_\ell}$$

$$- \int_{R^\ell}\cdots\int h(y_{n_1}, \cdots, y_{n_\ell}) \tilde{g}^{(\ell)}(y_{n_1}, \cdots, y_{n_\ell}) \, dy_{n_1} \cdots dy_{n_\ell} \Big|$$

$$\leqq \int_{R^\ell}\cdots\int |h(x_{n_1}, \cdots, x_{n_\ell})| \, |\tilde{f}^{(\ell)}(x_{n_1}, \cdots, x_{n_\ell}) - \tilde{g}^{(\ell)}(x_{n_1}, \cdots, x_{n_\ell})| \, dx_{n_1} \cdots dx_{n_\ell}$$

$$= \int \cdots \int |h(x)| \frac{|\tilde{f}^{(\ell)}(x) - \tilde{g}^{(\ell)}(x)|}{\tilde{\varphi}^{(\ell)}(x)^{\frac{1}{2}}} \tilde{\varphi}^{(\ell)}(x)^{\frac{1}{2}} dx$$

$$\leqq \left(\int \cdots \int_{R^\ell} h^2(x) \tilde{\varphi}^{(\ell)}(x) \, dx \right)^{\frac{1}{2}} \left(\int \cdots \int_{R^\ell} \frac{|\tilde{f}^{(\ell)}(x) - \tilde{g}^{(\ell)}(x)|^2}{\tilde{\varphi}^{(\ell)}(x)} dx \right)^{\frac{1}{2}} .$$

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