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題目(和文)	
Title(English)	A study on control system design from input-output data
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出典(和文)	学位:博士(工学), 学位授与機関:東京工業大学, 報告番号:甲第4665号, 授与年月日:2001年3月26日, 学位の種別:課程博士, 審査員:
Citation(English)	Degree:Doctor (Engineering), Conferring organization: Tokyo Institute of Technology, Report number:甲第4665号, Conferred date:2001/3/26, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

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A Study on Control System Design from Input-Output Data

A Dissertation submitted to

Department of Control Engineering
the Graduate School of Science and Engineering

Tokyo Institute of Technology

by

Masami Iwase

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Engineering

supervised by

Associate Professor Masaki Yamakita
Professor Katsuhisa Furuta

January 2001

To all with whom I have shared precious time

Acknowledgments

I would like to express my best appreciation, first of all, to Professor Katsuhisa Furuta and Professor Masaki Yamakita. Properly speaking, I must write both professor's name on the same position, but it is impossible. So I'd like to beg pardon for my rudeness. I owe Professor Furuta my thanks for his continuous supports, suitable guidances and useful suggestions during the past four years of my study. Professor Yamakita made great efforts to help me complete this dissertation in the last year which was a most serious and important. To study with both marvelous professors was a happy and valuable experience for me.

Examiners, Professor S. Hara, Professor T. Mita, Professor M. Sampei and Professor Y. Matsuo have given me a lot of excellent and timely advices constantly at various occasions including 'ATACS'. I am deeply indebted to them for my research and impressed with their earnest attitude toward researches.

I have much to be thankful to all mates, colleagues and fellows in Furuta Laboratory, Yamakita Laboratory and Sampei Laboratory in my public and my private life. Nothing can take the place of the five years which I have shared with them. In particular, I am grateful from the bottom of my heart to Dr. T. Hoshino, an assistant in former Furuta Laboratory, and Professor M. Koga, a developer of MATX, and Dr. S. Nakaura, an assistant in Sampei Laboratory, and Mr. H. Yamazoe, a manager in SWCC Co., Ltd.

Lastly, my special thanks go to Mr. Satoru Tanaka and my all friends, parents and Ms. Kinuyo Uehara; they have constantly encouraged and supported me with warm friendship and affection during my satisfactory student life at Tokyo Institute of Technology.

I wish all success and good luck.

Masami Iwase

Oh-Okayama, Tokyo
January, 2001

Abstract

This dissertation is concerned with the outcome of a study on control system designs from input-output data.

Recently, because of a suggestion that system identification and controller design should be performed simultaneously and a theoretical interest that some control systems can be designed with the operation closed in input-output data, control system designs based on input-output data are recognized again. However, the majority of former works take interests in LQG optimal feedback control of linear time invariant discrete-time systems. To provide new control system designs for other systems and control problems is a future work in this field. On this subject, three new control system designs are proposed in this dissertation.

First, a "Laguerre system" is introduced by expanding a linear continuous-time system into a series expansion with the Laguerre orthonormal basis function, and a new LQG control system design from input-output data for the system is proposed using the Laguerre system. The actual algorithm, that the controller is composed by merging the optimal feedback law and the observer and is realized in the form of the state space representation, is given.

Second, for linear time invariant discrete-time systems, an adaptive-learning control design based on input-output response is proposed. In this method, control objects are represented with Markov parameters, an impulse response of system, and then the adaptive-learning control law is designed so that a quadratic cost function is optimized and so that control sequences and Markov parameters are updated simultaneously. To verify the effectiveness, it is applied to an experimental device whose modeling is difficult.

Last, it is shown that some optimal control systems, e.g. time optimal control, for linear time invariant discrete-time system are designed by formulating the design problems in the generalized numerical optimizations using input-output data. By introducing an auxiliary problem of the original one, control design can be transformed into a nonlinear optimization problem in the case of nonlinear input-affine systems or into a linear programming in the case of linear systems. The efficiency is investigated through numerical simulations.

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Chapter 1

Introduction

1.1 Background and Motivation

When one designs a control system, information on a control object is considered as most important factor. Since Kalman has proposed the state space representation [44], the modern control system design theory has evolved by penetrating system behaviour deeply with a control object model. That is, mathematical models such as transfer function and state-space equation given by modeling are utilized to obtain system information. Therefore, in present standard design procedures, the mathematical models allow us to design control system rationally.

However, recently, if we go back to the starting point of control system design, i.e., "Find a input by which an output behaves in desirable way.", then a question whether mathematical models are needed really arises [56].

Before the state space representation is proposed, control system is designed directly based on system responses like Bode and Nyquist plots and Nichols charts etc. PID controller, which is regarded as an extremely useful controller used historically long time and even in the present, is tuned on the basis of system responses caused by test signals like the methods proposed by Ziegler and Nichols etc., namely, in classical control system, not models but some system responses have been used directly in order to design.

Now considering the modern control case, system responses are reflected in mathematical models by considering an identification problem to decide model parameters so as to approximate behaviour of control objects exceedingly. In other words, the modern control theory has proposed design algorithms for a class considered as a model set such as linear/nonlinear systems, time-invariant/time-variant systems, mechanical systems and plant systems, therefore, practical control system designs need the identification step in order to obtain the required model out of a class.

These model-based control designs, LQG, H_∞ and etc., promise good performances if accurate models are given. They are especially effective in such systems, for example mechanical systems, that accurate structure and parameters can be obtained relatively easily. Otherwise, as long as modeling error between actual systems and models can be estimated sufficiently like flexible structure systems, robust control designs such as H_∞ promise good performance even if accurate models cannot be constructed perfectly.

However, there are not always these cases generally. There are many cases that even model structures cannot be given. In such case, it is necessary that under the assumption of a class including the real object a model is chosen from the class and is constructed so as to approximate the system behaviour sufficiently. This identification problem is called non-parametric identification while the problem to identify parameters of a model given its structure information fully is called parametric identification [30]. In non-parametric identification, to obtain an accurate model is more difficult than in the parametric case, and the obtained model has physical structure information no longer. This indicates that non-parametric models give no other information except approximation of the real system input-output relation.

Considering the mentioned question again, it can be answered from above observations as follows, namely, if models can be obtained exactly, nothing can be better than the way using the model in order to design control system. However, let us consider the more general case where available information is only an input-output response. Considering a class assumed to include real control objects, if there exists such a reasonable method that is closed on operation of input-output data and that can design the desirable control from direct operation of input-output data, the method is better than the model-based method in the non-parametric case where the identified parameters have no physical sense. Furthermore to design control system from input-output data directly reduces the possibility of occurrence of artificial mistakes and meets an argument that modeling and design should be integrated because they depend on each other [46].

Since these concepts have appeared, the data-based control system design comes to be paid many attention, recently. Many works have been reported, but the majority of these works assumed only linear time invariant discrete-time system for a class. Skelton and others proposed a LQG controller design method from the Markov parameters which is the impulse response of discrete-time systems [45]. Furuta and others showed that not only LQG controller but also H_∞ controller and disturbance attenuation controller can be obtained by the same Markov parameters [22][42]. Ikeda and others gave basis vectors forming the input-output data space and considered a tracking problem and LQ problem regarding the basis vectors as a new system representation [56] [57]. Chan obtained an equivalent output feedback controller to the optimal state feedback law by deciding controller parameters in a sense of the least square without separating LQG control system design problem into the optimal feedback gain and observer designs [17] [18] [19]. Hjalmarsson and Gevers introduced a gradient method called Iterative Feedback Tuning for the minimum variance control problem and applied to LQG controller design also [12] [13] [14]. Kawamura proposed an iterative LQG controller tuning [58]. Sugie and others approached a new model-free control input design problem from a convex programming viewpoint [53]. Favoreel and others proposed the LQG control system design method for linear and bilinear systems from arbitrary input-output data utilizing the subspace identification method [54]. Moreover, in a broad sense, the direct adaptive control design, learning control design and behaviour approach may be considered as the control design based on input-output data. Therefore it is possible that connecting these methods to the response-based method yields a new control system design method.

On the mentioned background, this dissertation also shall take the same position similar to the previous works and shall consider a control system design based on input-output data. The mentioned references handled only discrete-time systems, and no data-based control design for other classes has been proposed, so this dissertation shall propose a new LQG control design for linear time invariant continuous-time systems to extend the class with which data-based control can deal. In this method, a Laguerre series expansion is introduced to deal continuous time signals with finite data, then a optimal controller can be represented with the expansion coefficients obtained from responses generated by injecting the time response of the Laguerre function to the system as input. Furthermore a state space realization method of a continuous-time controller is given.

Next, to illustrate the efficiency of control system design based on input-output data for such system that its structure and dynamics cannot be given explicitly, one of applications shall be given as an example.

In this example, an active vibration isolation system, which provides low-vibration environment on the table by moving the table to suppress the vibration from the out environment by some actuators, is considered. The table carries an external machine which needs low-vibration environment, for example a stepper, an electron microscope and etc.

The external machine moves repeatedly in the same pattern and causes a periodic disturbance. Therefore these disturbances must be suppressed, but modeling of this system is difficult due to complexity of the coupling dynamics between the table and the external machine. Because of the micro-vibration, linear time invariant discrete-time systems are assumed to be appropriate class in this case. Under the situation a new adaptive learning control design based on input-output data is proposed and is applied to the system having the periodic disturbance and the complex modeling.

Moreover, for examples of the control system design closed on the operation of input-output data, new design methods of a time-optimal control, fuel-optimal control and etc. are proposed for linear time invariant discrete-time systems. In this methods, each design problem can be formulated in the linear programming, and its constraints can be represented with some system responses. As a result, time optimal control and other controls, which are difficult to be obtained by analytical methods in general, can be easily given by any usual solver of the linear programming.

Organization of dissertation and outlines of each chapter are given as shown in the next section.

1.2 Outline

This dissertation consists of six chapters and is organized as follows.

Chapter 1: Introduction

In this chapter background and motivation of this dissertation are mentioned, and the outline of each chapter is given respectively. Notation used in each chapter is also collected and shown, here.

Chapter 2: A LQG Controller Design for Linear Continuous-Time Systems based on Laguerre Series Expansion

In this chapter, a new control system design shall be proposed for linear time invariant continuous-time systems from input-output data. This method is different from the previous works with respect to considering continuous-time systems as a class.

How continuous signals are treated is one of crucial topics of the proposed method and to solve this problem a series expansion representation based on the Laguerre fuction is introduced. Although the Laguerre series expansion is widely used in the fields of identification and system representation, the control system design using the Laguerre series expansion representation has never studied yet. We shall first equip the system representation based on the Laguerre series expansion, which we shall call Laguerre system. After that, for the Laguerre systems of considered continuous-time systems, we give design methods of an optimal feedback gain and full order observer and then show that merging them leads to a LQG dynamic controller.

Furthermore, we shall show that the LQG controller is described by expansion coefficients of a response yielded by injecting the time response of the Laguerre basis function to the control object as input, and give a realization method which gives the state space representation of the obtained LQG controller. Theoretically, this method provides almost same controller with one designed based on the state space representation.

Finally, a numerical example is given to investigate the proposed method. The method is applied to LQG control design for a 2nd-order system and compared with the state space case.

Chapter 3: An Adaptive Learning Control Using Markov Parameters

A learning control is an effective approach which makes the system output track a desired trajectory perfectly. If the system is completely known, there is no need for the learning control because the input that yields the desired trajectory may be derived by the inverse system. However, due to the inaccuracy of the parameters, such an inverse system cannot be obtained.

In this chapter, two simple adaptive iterative learning control method for linear time invariant discrete time systems, which are effective in the case where system parameters are unknown and/or there are modeling errors, are proposed. The proposed methods give the learning control laws based on a quadratic criterion and both input update and parameter estimation are obtained at each iteration simultaneously.

Method 1 is based on an algorithm in which a dual operator of the system is used for its update law explicitly in the same way as [39] [40]. However, if some disturbance is injected to the system, the convergence cannot be guaranteed. To overcome this problem, in method 2, the disturbance is explicitly taken account of using the inverse system. In these method the Markov parameters are used for the unknown system representation, therefore the methods can cope with all class of linear discrete-time systems.

To illustrate the efficiency, the proposed method 2 is applied to a micro tremor isolation vibration system practically.

Chapter 4: Time Optimal Control Design with Nonlinear Programming

This chapter studies a time optimal control system design using bounded input through application to swing-up control of the pendulum. Time optimal control of a nonlinear system can be formulated by Pontryagin's Maximum Principle, which is, however, hard to compute practically. In this chapter, a new computational approach is presented to attain a numerical solution of the time optimal problem.

Time optimal control problems is described as minimization of the achievable time to attain the terminal state under the bounded input amplitude, although algorithms to solve this problem are known complicated. Therefore, in this paper, it is shown how the optimal time swing-up control is formulated as an auxiliary problem that the minimal input amplitude is searched so that the terminal state satisfies a specification at a given time. Through the proposed approach, time optimal control can be solved by nonlinear optimization.

Its approach is evaluated by numerical simulations of a simplified pendulum model, is checked satisfying the necessary condition of Maximum Principle, and is experimentally verified via application to swing-up of a rotating type pendulum from the pendant to the inverted state which is known one of most difficult control problems, since the system is nonlinear, underactuated, and has uncontrollable states.

The result of this chapter leads to some extensions as shown in next chapter. Because of the extensions, time optimal, fuel optimal and some optimal controllers for linear time invariant discrete-time systems can be designed by operation closed in input-output data.

Chapter 5: Time and Fuel Optimal Control Design by Linear Programming Using Input-Output Data

In this chapter, considering a finite dimensional, linear time invariant, discrete-time system, we show first that the time optimal control, the fuel optimal control and the mixed time-fuel optimal control problems can be formulated as the linear programming problems.

However, the time optimal and fuel optimal control problems cannot be formulated as the linear programming in a straightforward way. Hence, we need to consider an auxiliary problem formulated in the linear programming for its design problem, and solve the original problem utilizing the auxiliary problem. Especially about the time optimal control, the auxiliary problem is practically set on the analogy of such an auxiliary problem, which is proposed in the previous chapter, that we considered how the optimal control problem for nonlinear control systems is formulated as a nonlinear optimization problem.

Moreover, in this approach, if we were confronted with the case that a model of an object for design isn't given explicitly, the optimal control could be designed from some system responses as long as the responses can be measured. As a result, we can design these optimal control laws easily from some system responses by any usual solvers for the linear programming.

To verify the effectiveness, we apply the methods to the design problem of a positioning control for track seek motion of a hard disk drive. On each evaluation of the time optimal, fuel optimal and mixed time-fuel optimal, the optimal input is designed under the input and output restrictions.

Chapter 6: Conclustions

This chapter summarizes the results and contributions of each chapters and suggests future research directions.

1.3 Notation

The notation used for simplicity throughout each chapter is presented here briefly. Note that other used notation is defined in each chapter suitably.

Chapter 2:

$R,\ R_+$	reals and nonnegative reals
$\mathcal{L}_p(R_+)$	p-norm Lebesue space
$\|\cdot\|_p$	p-norm in $\mathcal{L}_p$ space
$\{\cdot\}_{n\leq k}$	sequence
$\phi_i(t)$	Laguerre function
$\mathcal{L}_a[\cdot]$	Laguerre transform
$(A,\ B,\ C)$	continuous-time system
$(A_N,\ B_N,\ C_N)$	block-diagonal state space representation
D_N	differential of Laguerre system
$x(t)$	continuous time signal
X_N	coefficient vector of Laguerre series expansion
X_0	initial condition of Laguerre system

Chapter 3:

$(A,\ B,\ C\ ,D)$	discrete-time system
x_i:(subscript)	i-th step signal sequence
X:(capital character)	Toeplitz form of signal sequence
x:(small character)	vector form of signal sequence
$\tilde{x}$:(tilde)	error siginal
$\hat{x}$:(hat)	estimation of signal
$<,>$	inner product of vector
$\|\cdot\|_Q$	2-norm with weight matrix Q

Chapter 4:

x	continuous signal
x_k:(with subscript)	discretized signal
x_0:(with subscript 0)	initial contidion
x_f:(with subscript f)	terminal condition
$\tilde{x}$:(tilde)	variable of nonlinear optimization
$(A_{eq},\ b_{eq})$	linear equality constraint
$(A,\ b)$	linear inequality constraint
$c_{eq}()$	nonlinear equality constraint
$c()$	nonlinear inequality constraint
$l_b,\ u_b$	lower bound and upper bound

Chapter 5:

$(\Phi,\ \Gamma,\ C,\ D)$	discrete-time system
Γ_N	Observability matrix
Φ_N	Toeplitz matrix of Markov parametes
Y_N:(with subscript)	vector form of signal sequence
ξ	variable of Linear Programming

Chapter 2

A LQG Controller Design for Linear Continuous Time Systems based on Laguerre Series Expansion

2.1 Introduction

Recently, various control system designs are considered for systems giving less prior information about themselves. The direct adaptive method, learning control and data-based control are taken as examples. In such a case, given information on systems is only input-output data and the structure of its model is not given mostly.

Since the state space representation has been proposed by R. E. Kalman [44], the approach, in which some mathematical models are derived and then control laws are designed based on the models, has been established in the modern control theory, and a lot of works on LQG, H_∞ and other control designs have been studied. Therefore, we may say that the modern control theory has been evolved by deep insight into models [58].

In the modern control system theory the assumption that models are given is a starting-point to discuss the design algorithm. Hence, if accurate models can be obtained, the model-based controller promises good performance. However, this assumption causes another problem of how to construct a model to reflect the considered system sufficiently, i.e., identification problem. Therefore, these control system designs need two steps, identification and control law design.

If control objects are systems giving structure information of its model easily and clearly such as mechanical systems, an relatively accurate model can be obtained by modeling and identification of its parameters. But a prior knowledge on the system cannot be given sometimes. In this case, we should consider a class, i.e., a model set such as linear discrete time system, bilinear system, mechanical system and etc. Then, we choose an appropriate class to which the considered system can belong, and identify the model reflecting the system behaviour out of the class. This procedure is called non-parametric identification. A model given by the non-parametric identification does not have the physical sense about its structure and parameters any longer except reflecting the input-output behaviour of systems.

If we go back to the starting-point of control system designs, "Find a input which lets the output of a control object behave in desirable way", a question whether models are indispensable to design control systems is caused. In the case that information obtained from a control object is only input-output response, to construct the model is not always needed. That is, if control systems are designed reasonably by close operations using only input-output data, it is enough.

Such control system designs directly using input-output data meet a claim that the modeling and control design problems are not independent problems and they should be integrated into a continuous operation [46].

Before, the response-based control is regarded as merely another design method of an usual one. However, since the concept of integrated design has appeared, the response-based control comes to be paid many attention. In recent works, supposing that a control object belongs to the class of linear time invariant discrete-time systems, the control system has been designed from input-output data. Skelton [45], Furuta and others [22] [42] have proposed LQG, H_2 and H_∞ optimal controller design methods. Ikeda and others [56][57] have considered vector bases forming the input-output data space and have given a new system representation using these bases, and tracking and LQ controllers have been designed based on the system representation. Chan [17][18][19] has led a output feedback control equivalent to an optimal state feedback control, deciding controller's parameters in a sense of least square and not separating the design problem into the optimal regulator and observer design problems. Hjalmarsson, Gevers and others [12][13][14] have introduced a gradient method called Iterative Feedback Tuning for the minimum variance control problem and have applied it to the LQG controller design. Kawamura [58] also has proposed an iterative LQG control tuning. Sugie and others [53] have approached a new model-free control input design method from a convex programming viewpoint. Favoreel [54] has designed LQG controllers for linear or bilinear systems from arbitrary input-output data utilizing the subspace identification method.

In this chapter, we also take the standpoint of the control system design based on input-output data and propose a new data-based control design for linear time invariant continuous-time systems. Our method is different from the above mentioned methods with respect to considering continuous-time systems as a class.

How continuous signals are treated is one of crucial topics of the proposed method and to solve this problem a series expansion representation based on the Laguerre function is introduced. Although the Laguerre series expansion is widely used in the fields of identification and system representation, control system designs using the Laguerre series expansion representation have never been studied yet. We shall first provide a system representation based on the Laguerre series expansion, which we shall call Laguerre system. After that, considering the Laguerre system representation of a continuous-time system, we give design methods of an optimal feedback gain and full order observer, and show that to integrate them leads to a LQG dynamic controller. Furthermore, we shall show that the LQG controller is obtained from expansion coefficients of a response yielded by injecting the time response of the Laguerre basis function to the control object as input, and then propose a realization method which gives the state space representation of the obtained LQG controller. Basically, our method provides almost same controller as one based on the state space representation.

Finally, the proposed method is investigated via a numerical example. The method is applied to a LQG control design for a 2nd-order system, and is compared with the state space case.

This chapter is organized as follows. In the next section, notation and preliminary results on the Laguerre series expansion are summarized. In Section 3, main results of a new LQG controller design for Laguerre systems are given. In Section 4, we shall show the designed controller is represented with system responses and a controller realization method is given in Section 5. Finally, we give a illustrative numerical example and some concluding remarks in Section 6 and the last section, respectively.

2.2 Preliminalies

Let us start by defining notation and terminology used throughout this chapter and summarizing some results on the Laguerre series expansion. R and R_+ mean the reals and the nonnegative reals respectively. $\mathcal{L}_p(R_+)$ consists of real-valued vector Lebesgue measurable functions f on R_+ such that $||x(t)||_p = \left[\int_{R_+} |f(t)|^p\right]^{1/p} < \infty$. The notation $||\cdot||_p$ means p-norm in $\mathcal{L}_p$ space. So long as there is no notice especially, we use $\mathcal{L}_2$ and $||\cdot||$ for the shorthand of $\mathcal{L}_2(R_+)$ and $||\cdot||_2$, respectively. Furthermore, let $\{f_n\}_{n\geq k}$ denote a sequence.

We shall consider only a continuous-time, finite dimensional, linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

$$y(t) = Cx(t) \tag{2.2}$$

where $x(t) \in R^{n\times 1}$ is the state, $u(t) \in R^{m\times 1}$ is the input, $y(t) \in R^{p\times 1}$ is the output and (A, B, C) have appropriate dimension respectively. It is assumed that (A, B) is controllable and (C, A) is observable.

In our approach, a dynamical controller can be obtained by collective matrix and vector operations of data which are gotten observing responses when test signals are injected to the system. The practical operations are made by computers, therefore they have the restriction that computer's memory is finite. That is, the continuous-time signals cannot be handled as they are as long as they aren't changed into some discrete form. Here, the laguerre function, which is widely used in the fields of identification, is introduced. We consider the laguerre function as a basis of the $\mathcal{L}_2$ space and expand continuous signals into expansion coefficients of the basis. The reason why the laguerre function is introduced is described afterward.

Laguerre functions are obtained by normalization and orthogonalization of the series $\left\{(pt)^i e^{-pt}\right\}_{i\geq 0}$. The generalized form of the sequence $\{\phi_i(t)\}_{i\geq 0}$ of Laguerre functions is given as following polynomials.

$$\phi_i(t) = \sqrt{2p}\sum_{k=0}^{i}(-1)^{i-k}(2p)^k \frac{i!}{k!k!(i-k)!}t^k e^{-pt} \tag{2.3}$$

As (2.3) shows, the Laguerre function has a parameter p. The Laguerre function is transformed into such a rational polynomial in s domain,

$$\phi_i(s) = \frac{\sqrt{2p}}{p+s}\left(\frac{p-s}{p+s}\right)^i \tag{2.4}$$

by the Laplace transform. It can be easily seen that the Laguerre function consists of a first-order lowpass filter and n all-pass functions, which have poles at $-p$. Therefore we can also consider that the parameter p determines the mode of time response of the Laguerre function.

Expansion coefficients of Laguerre functions are obtained by the inner product operation. Let's consider that a continuous-time signal $v(t) \in R^{j\times 1}$ is expanded to expansion coefficients $\{v_i\}_{i\geq 0}$. $v_i \in R^{j\times 1}$ is calculated as follows.

$$v_i = \int_0^{\infty} v(t)\phi_i(t)\,dt \tag{2.5}$$

Proposition 1 *[11] The Laguerre functions $\{\phi_n(t)\}_{n\geq 0}$ form an orthonormal basis of $\mathcal{L}_2$. Furthermore, the system of the laguerre functions is complete.*

This proposition shows some useful fact about the system of Laguerre functions, which are led from its complete orthonomal property.

Proposition 2 *Let $\{\phi_n(t)\}_{n\geq 0}$ be the complete orthonormal basis of $\mathcal{L}_2$, which consists of the Laguerre functions, and let $\{v_i\}_{i\geq 0}$ be obtained by the inner product (2.5). Then, for any $v(t) \in \mathcal{L}_2$ following equations hold.*

1.

$$v(t) = \sum_{k=0}^{\infty} v_k \phi_k(t) \tag{2.6}$$

2.

$$\|v(t)\|^2 = \sum_{k=0}^{\infty} |v_k|^2 \ . \tag{2.7}$$

The expansion coefficients $\{v_i\}_{i\geq 0}$ are given to approximate the $\mathcal{L}_2$ space in a sense of 2-norm exceedingly. An element of the $\mathcal{L}_2$ space is represented by the infinite linear combination of the Laguerre basis and coefficients, and can be approximated with an arbitrary accuracy. (2.7) is called *Parseval's identity* which shows a relation between 2-norm in $\mathcal{L}_2$ space and the expansion coefficients.

These properties are useful for application to the LQG control design problem of linear continuous-time systems. The LQG problem is defined as minimization of a cost function in the form of 2-norm, so the cost function can be represented with the expansion coefficients of the Laguerre series by the Parseval's identity. If the continuous signals of systems and the cost function are approximated by the Laguerre series expansion, it is possible that the LQG problem is formulated using only the coefficients. It is the just reason why the Laguerre function is introduced.

Considering the approximation of a continuous signal $v(t)$ with N terms of the Laguerre series:

$$v(t) \approx \sum_{k=0}^{N-1} v_k \phi_k(t) \tag{2.8}$$

, we shall let V_N denote a vector form of the coefficients $\{v_k\}$, and define $\mathcal{L}_a[\cdot]$, which we shall call *Laguerre transform*, as an operation that V_N is formed out of $v(t)$ as follows.

$$\mathcal{L}_a[v(t)] := V_N \tag{2.9}$$

$$V_N := \begin{bmatrix} v_0^T & v_1^T & \cdots & v_{N-1}^T \end{bmatrix}^T \tag{2.10}$$

This operation has next properties.

Proposition 3 *Let $\mathcal{L}_a[\cdot]$ be the operation that expansion coefficients of the Laguerre basis are formed out of a continuous signal. Then, the operation holds the linearity.*

1.

$$\mathcal{L}_a[\alpha v(t)] = \alpha \mathcal{L}_a[v(t)] \tag{2.11}$$

2.

$$\mathcal{L}_a\left[v(t)+w(t)\right]=\mathcal{L}_a\left[v(t)\right]+\mathcal{L}_a\left[w(t)\right] \tag{2.12}$$

Proof. It is obvious that these equations are satisfied from the fact that the coefficients are calculated by the inner product (2.5). □

In next section, we shall consider how the continuous-time systems are represented with the Laguerre series expansion and how the LQG problem is formulated and solved under this expression.

2.3 Main Results

2.3.1 Representation of systems with the Laguerre series expansion

First we shall describe a representation of the system (2.1)-(2.2) using the expansion coefficients of the Laguerre series. Before dealing with (2.1)-(2.2), we consider the Laguerre transform of the initial state response of the following autonomous system with an initial condition $x(0)$.

Lemma 1 *Consider an autonomous system (2.13) with an initial condition $x(0)$.*

$$\dot{x}=Ax, \quad \text{with } \; x(0) \tag{2.13}$$

If the initial state response of (2.13) with the initial condition $x(0)$ is transformed into expansion coefficients by the Laguerre transform, the coefficients can be represented as follows:

$$x_k=\sqrt{2p}\,(pI-A)^{-(k+1)}\,(pI+A)^k\,x(0), \qquad k\geq 0 \tag{2.14}$$

where p is the parameter of the Laguerre function and A is the system matrix of (2.13).

Proof. We consider how the initial state response of (2.13) with $x(0)$,

$$x(t)=e^{At}x(0) \tag{2.15}$$

is expanded into the Laguerre series. From the definition, we can obtain the coefficients by calculating each inner product:

$$x_k=\int_0^\infty e^{At}x(0)\cdot\phi_k(t)dt \tag{2.16}$$

In practice, the result of the above integration can be constructed from

$$c_k=\int_0^\infty e^{At}x(0)\cdot t^k e^{-pt}dt=k!\,(pI-A)^{(k+1)}\,x(0) \tag{2.17}$$

, because $\phi_k(t)$ consists of $t^k e^{-pt}$. (See Details 2.8.1). We shall use this relation and prove that the solution of (2.16) is in the form of (2.14) with mathematical induction. In $k=0$ case, the inner product,

$$x_0=\sqrt{2p}\;c_0=\sqrt{2p}\,(pI-A)^{-1}\,x(0) \tag{2.18}$$

satisfies (2.14). Next, it is supposed that

$$x_k = \int_0^\infty e^{At}x(0)\cdot\phi_k(t)dt = \int_0^\infty e^{At}x(0)\left\{\sqrt{2p}\sum_{i=0}^{k}(-1)^{k-i}(2p)^i\frac{k!}{i!i!\,(k-i)!}t^ie^{-pt}\right\}dt$$
$$= \sqrt{2p}\sum_{i=0}^{k}(-1)^{k-i}(2p)^i\frac{k!}{i!(k-i)!}(pI-A)^{-(i+1)}x(0) \tag{2.19}$$

is equivalent to $\sqrt{2p}\,(pI-A)^{-(k+1)}(pI+A)^k x(0)$.

Multiplying (2.19) by $(pI-A)^{-1}(pI+A)$ from the left-hand side and using the following relations:

$$(pI-A)^{-1}(pI+A) = -I + 2p\,(pI-A)^{-1} \tag{2.20}$$
$$(pI-A)^{-1}(pI+A) = (pI+A)(pI-A)^{-1} \tag{2.21}$$

, then we get

$$(pI-A)^{-1}(pI+A)\,x_k = \sqrt{2p}\sum_{i=0}^{k+1}(-1)^{k+1-i}(2p)^i\frac{(k+1)!}{i!\,(k+1-i)!}(pI-A)^{-(i+1)}x(0) \tag{2.22}$$

(See Details 2.8.2). Because (2.22) is equivalent to $\sqrt{2p}\,(pI-A)^{-(k+2)}(pI+A)^{k+1}x(0)$, i.e., (2.14) holds in the $k+1$ case, the proof is completed. □

Furthermore, using the expansion (see Details 2.8.3)

$$(pI-A)^{-i}(pI+A)^i = \sum_{k=0}^{i-1}(-1)^{i-k-1}2p\,(pI-A)^{-(k+1)}(pI+A)^k + (-1)^i I \tag{2.23}$$

, we shall show an algebraic equation which the expansion coefficients should satisfy.

Theorem 1 *Consider an autonomous system (2.13) with an initial condition $x(0)$.*

$$\dot{x} = Ax, \quad \text{with } x(0)$$

If the initial state response of (2.13) with the initial condition $x(0)$ is represented with the Laguerre series expansion, the expansion coefficients obtained by the Laguerre transform have to satisfy the following algebraic equation:

$$D_N X_N - X_0 = A_N X_N \tag{2.24}$$

where

$$D_N = \begin{bmatrix} pI & & & & \\ -2pI & pI & & & \\ 2pI & -2pI & pI & & \\ \vdots & & & \ddots & \\ (-1)^{N-1}2pI & (-1)^{N-2}2pI & \cdots & -2pI & pI \end{bmatrix}$$
$$X_N = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}, \quad X_0 = \sqrt{2p}\begin{bmatrix} I \\ -I \\ \vdots \\ (-1)^{N-1}I \end{bmatrix}x(0)$$
$$A_N = \mathrm{diag}\underbrace{(A,\ A,\ \cdots,\ A)}_{N} \tag{2.25}$$

Proof. Apply (2.23) to the result of *Lemma 1*:

$$x_k = \sqrt{2p}\,(pI - A)^{-1} \cdot (pI - A)^{-k}\,(pI + A)^k\,x(0)$$

$$x_k = \sqrt{2p}\,(pI - A)^{-1} \left[\sum_{i=0}^{k-1} (-1)^{k-i-1}\,2p\,(pI - A)^{-(i+1)}\,(pI + A)^i + (-1)^k\,I\right] x(0)$$

$$(pI - A)\,x_k = \sum_{i=0}^{k-1} (-1)^{k-i-1}\,2px_k + \sqrt{2p}\,(-1)^k\,Ix(0) \tag{2.26}$$

Stacking up (2.26) from $k = 1$ to $N - 1$ leads to its vector form (2.24). □

An algebraic equation of the continuous-time system (2.1)-(2.2), which the Laguerre series expansion coefficients of the system should satisfy, is obtained by the result of *Theorem 1* and the linearity of Laguerre transform.

Corollary 1 *Consider a continuous-time, finite dimensional, linear time invariant system (2.1)-(2.2):*

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

If we shall transform signals $x(t)$, $u(t)$ and $y(t)$ into expansion coefficients by the Laguerre transform, then the coefficients of each signal have to satisfy the following algebraic equation which is constructed by these coefficients, the system matrix and an initial state $x(0)$:

$$D_N X_N - X_0 = A_N X_N + B_N U_N \tag{2.27}$$

$$Y_N = C_N X_N \tag{2.28}$$

where B_N and C_N are defined similarly to A_N, and Y_N and U_N are defined similarly to X_N.

We shall call the representation (2.27)-(2.28) of the continuous-system (2.1)-(2.2) *Laguerre System* throughout the chapter.

2.3.2 LQ problem

We have already had the Laguerre system which the coefficients obtained by expanding the continuous system to the Laguerre series should satisfy. Additionally we consider here how the standard LQ problem for the continuous system is formulated and solved with the Laguerre series expansion. First let us start by considering the Parsevel identity about the cost function of the LQ problem.

The LQ problem is to find the control input $u(t)$ minimizing a quadratic cost function

$$J = \int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt, \qquad Q \geq 0, \quad R > 0 \tag{2.29}$$

where Q is a semi-positive definite matrix and R is a positive definite matrix under the constraint $\dot{x}(t) = Ax(t) + Bu(t)$. (2.29) can be rewritten in the norm form:

$$J = ||x(t)||_Q^2 + ||u(t)||_R^2 \tag{2.30}$$

, and it can be represented by the expansion coefficients of $x(t)$ and $u(t)$ as

$$J = \sum_{k=0}^{\infty} \left(x_k^T Q x_k + u_k^T R u_k\right) \tag{2.31}$$

because of the Parseval identity. If (2.31) can be approximated using finite number of expansion coefficients, it is possible to state the LQ problem by the Laguerre expansion coefficients as follows:

LQ problem: *Find the control input U_N minimizing the cost function*

$$J = X_N^T Q_N X_N + U_N^T R_N U_N \tag{2.32}$$

, where Q_N and U_N are defined similarly to A_N, under the constraint (2.27):

$$D_N X_N - X_0 = A_N X_N + B_N U_N \ .$$

The optimal control input U_N and the minimum value of the cost function are given by the next theorem.

Theorem 2 *The optimal control U_N minizing the cost function (2.32) under the constraint (2.27) is given as*

$$U_N = -\left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q \bar{A} X_0 \tag{2.33}$$

where $\bar{A}_N = (D_N - A_N)^{-1}$, and the mininum value of the cost function is given as

$$\min_{U_N} J = X_0^T P_L X_0 \tag{2.34}$$

$$P_L = \bar{A}_N^T Q_N \bar{A}_N - \bar{A}_N^T Q_N \bar{A}_N B_N \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q_N \bar{A}_N \tag{2.35}$$

Proof. From (2.27) X_N is

$$X_N = (D_N - A_N)^{-1} X_0 + (D_N - A_N)^{-1} B_N U_N = \bar{A}_N X_0 + \bar{A}_N B_N U_N \tag{2.36}$$

, and substituting (2.36) into the cost function (2.32) leads to the modified:

$$\begin{aligned} J &= \begin{bmatrix} X_0 \\ U_N \end{bmatrix}^T \begin{bmatrix} \bar{A}_N^T Q_N \bar{A}_N & \bar{A}_N^T Q_N \bar{A}_N B_N \\ B_N^T \bar{A}_N^T Q_N \bar{A}_N & R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N \end{bmatrix} \begin{bmatrix} X_0 \\ U_N \end{bmatrix} \\ &= \left[U_N^T + X_0^T \bar{A}_N^T Q_N \bar{A}_N B_N \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1}\right] \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right) \\ &\times \left[U_N + \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q_N \bar{A}_N X_0\right] + X_0^T P_L X_0 \\ P_L &:= \bar{A}_N^T Q_N \bar{A}_N - \bar{A}_N^T Q_N \bar{A}_N B_N \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q_N \bar{A}_N \end{aligned} \tag{2.37}$$

Therefore, from (2.37) the optimal input is given as

$$U_N = -\left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q_N \bar{A}_N X_0$$

and the minimum value is $X_0^T P_L X_0$. □

Note that, from (2.33), the optimal control is given as not the feedback law but the feedforward law using the initial condition X_0. If we consider the case that some disturbance signals are injected to the system, there is difference between the feedback and the feedforward, that is, the former type control input reflects the influence of the disturbance but the latter type does not reflect. However, owing to the Parseval identity, the relation with respect to the minimum of the cost function

$$\min_{u(t)} \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt = \min_{U_N} \lim_{N\to\infty} \left(X_N^T Q_N X_N + U_N^T R_N U_N\right) \quad (2.38)$$

is held, and the feedback type control law using the Laguerre series expansion coefficients is given in the next corollary.

Corollary 2 *Let P be a solution of the riccati equation:*

$$A^T P + PA + Q - RBR^{-1}B^T P = O \quad (2.39)$$

, and the optimal feedback gain F of (2.1) is given as $F = -R^{-1}B^T P$. Then, because of the Parseval identity P is equivalent to

$$P = I_N^T P_L I_N, \qquad I_N = \begin{bmatrix} I & -I & \cdots & (-1)^{N-1} I \end{bmatrix}^T \quad (2.40)$$

Furthermore, F can be represented as

$$F = -R^{-1}B^T I_N^T P_L I_N \quad (2.41)$$

Proof. The statement is obvious from (2.38). □

2.3.3 Full order observer

In the case that the state $x(t)$ cannot be measured directly from the output for the system (2.1)-(2.2), the optimal state feedback control law cannot be applied in practice, so we estimate the state utilizing the observer generally. The final aim of this chapter is to design a dynamical output feedback controller by integrating the state feedback law and observer represented in the form of the Laguerre expansion coefficients, and the state feedback law has been obtained. Then, we shall consider how a full order observer is denoted as a Laguerre system.

First the full order observer is given as

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + K\left[y(t) - C\tilde{x}(t)\right] \quad (2.42)$$

For a technical reason, we shall divide (2.42) into two equations as follows:

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= A\tilde{x}_1(t) + Bu(t) \\ \dot{\tilde{x}}_2(t) &= A\tilde{x}_2(t) + K\left[y(t) - C\tilde{x}(t)\right] \end{aligned}$$

, and let these equations be rewritten as the Laguerre systems:

$$\begin{aligned} D_N\tilde{X}_{N1} - \tilde{X}_{10} &= A_N\tilde{X}_{N1} + B_N\tilde{U}_N \\ D_N\tilde{X}_{N2} - \tilde{X}_{20} &= A_N\tilde{X}_{N2} + K_N\left[Y_N - C_N\tilde{X}_N\right] \end{aligned}$$

Throughout the chapter, the initial state of the observer is assumed to be zero, $\tilde{x}(0) = 0$, i.e., we assume that $\tilde{X}_0 = \tilde{X}_{10} = \tilde{X}_{20} = O$. Then, the Laguerre system of the observer (2.42) is denoted as

$$\begin{aligned}\tilde{X} &= \tilde{X}_{10} + \tilde{X}_{20} \\ &= \bar{A}_N B_N U_N - [D_N - (A_N - K_N C_N)]^{-1} K_N C_N \bar{A}_N B_N U_N + [D_N - (A_N - K_N C_N)]^{-1} K_N Y_N\end{aligned} \tag{2.43}$$

In addition, we show that $[D_N - (A_N - K_N C_N)]^{-1} K_N$ in (2.43) can be obtained by considering the LQ problem for a dual system of (2.1)-(2.2).

Using the dual system of (2.1)-(2.2)

$$\dot{\tilde{x}}(t) = A^T \tilde{x}(t) + C^T \tilde{u}(t) \tag{2.44}$$

$$\tilde{y}(t) = B^T \tilde{x}(t) \tag{2.45}$$

, the LQ problem for this system is formulated in the next lemma.

Lemma 2 *For the Laguerre system of the dual system (2.44)*

$$\tilde{D}_N \tilde{X}_N - \tilde{X}_0 = A_N^T \tilde{X}_N + C_N^T \tilde{U}_N \tag{2.46}$$

, the optimal input $\tilde{U}_N$ which minimizes the quadratic cost function

$$\tilde{J} = \tilde{X}_N^T \tilde{Q}_N \tilde{X}_N + \tilde{U}_N^T \tilde{R}_N \tilde{U}_N \tag{2.47}$$

is given as

$$\tilde{U}_N = -\left(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T \tilde{X}_0, \qquad \tilde{A}_N = \left(D_N - A_N^T\right)^{-T} \tag{2.48}$$

, and the minimum value of the cost function is given as

$$\begin{aligned}&\min_{\tilde{U}_N} \tilde{J} = \tilde{X}_0^T \tilde{P}_L \tilde{X}_0 \\ &\tilde{P}_L = \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T - \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T \left(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T\end{aligned} \tag{2.49}$$

Proof. This statement is led by replacing the coefficients according to $A_N \to \tilde{A}_N$, $B_N \to \tilde{C}_N^T$, $Q_N \to \tilde{Q}_N$ and $R_N \to \tilde{R}_N$ in *Theorem 2*. □

We make $\tilde{U}_N = -K^T \tilde{X}$ denote the optimal feedback law of the dual system and substitute it into $D_N \tilde{X}_N - \tilde{X}_0 = A_N^T \tilde{X}_N - C_N^T K_N^T \tilde{X}_N$, then

$$\tilde{U}_N = -K_N^T \left[D_N - (A_N^T - C_N^T K_N^T)\right]^{-1} \tilde{X}_0 \tag{2.50}$$

is obtained. Comparing (2.50) with (2.48)

$$K_N^T \left[D_N - (A_N^T - C_N^T K_N^T)\right]^{-1} = \left(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T \tag{2.51}$$

is held. Furthermore, because the dual system has the special structure with respect to $\tilde{A}_N$, there is the relation between the original system and the dual system as follows. (See appendix

2.8.4).

$$\begin{aligned} [D_N - (A_N - K_N C_N)]^{-1} K_N &= \begin{bmatrix} \delta_1 K & & & \\ \delta_2 K & \delta_1 K & & \\ \vdots & & \ddots & \\ \delta_N K & \delta_{N-1} K & \cdots & \delta_1 K \end{bmatrix} \\ K_N^T \left[D_N - \left(A_N^T - C_N^T K_N^T\right)\right]^{-1} &= \begin{bmatrix} K^T \delta_1^T & & & \\ K^T \delta_2^T & K^T \delta_1^T & & \\ \vdots & & \ddots & \\ K^T \delta_N^T & K^T \delta_{N-1}^T & \cdots & K^T \delta_1^T \end{bmatrix} \end{aligned} \tag{2.52}$$

Therefore, we calculate (2.48) and rearrange the result according to (2.52).
So $[D_N - (A_N - K_N C_N)]^{-1} K_N$ can be represented by using the expansion coefficients of the Laguerre series.

Theorem 3 *The full order observer in the form of the Laguerre system is constructed as follows:*

$$\tilde{X} = \bar{A}_N B_N U_N - [D_N - (A_N - K_N C_N)]^{-1} K_N C_N \bar{A}_N B_N U_N + [D_N - (A_N - K_N C_N)]^{-1} K_N Y_N$$

where $[D_N - (A_N - K_N C_N)]^{-1} K_N$ *can be obtained by the following procedure. For the dual system*

$$\tilde{D}_N \tilde{X}_N - \tilde{X}_0 = A_N^T \tilde{X}_N + C_N^T \tilde{U}_N$$

, we shall find the optimal feedback law which minimize the cost function

$$\tilde{J} = \tilde{X}_N^T \tilde{Q}_N \tilde{X}_N + \tilde{U}_N^T \tilde{R}_N \tilde{U}_N$$

Then, we get

$$K_N^T \left[D_N - \left(A_N^T - C_N^T K_N^T\right)\right]^{-1} = \left(\tilde{R}_N + C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N \tilde{Q}_N \tilde{A}_N^T$$

Finally, it is rearranged according to (2.52).

2.3.4 Dynamic output feedback controller

From *Theorem 2*, *Corollary 2* and *Theorem 3*, integrating the state feedback law and the full order observer leads to design a dynamic output feedback controller which is represented by the expansion coefficients of the Laguerre series.

Theorem 4 *For the system (2.1)-(2.2), an optimal dynamic output feedback controller is given as follows.*

$$\begin{aligned} \tilde{U}_N = {} & F_N \bar{A}_N B_N U_N - F_N [D_N - (A_N - K_N C_N)]^{-1} K_N C_N \bar{A}_N B_N U_N \\ & + F_N [D_N - (A_N - K_N C_N)]^{-1} K_N Y_N \end{aligned} \tag{2.53}$$

$$F = -R^{-1} B^T I_N^T \left[\bar{A}_N^T Q_N \bar{A}_N - \bar{A}_N^T Q_N \bar{A}_N B_N \left(R_N + B_N^T \bar{A}_N^T Q_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T Q_N \bar{A}_N\right] \tag{2.54}$$

where F_N *is defined similarly to* A_N.

Remark 1 *The proposed dynamic controller is formed by the combination of the optimal state feedback law and the full order observer. This controller is the same as a LQG controller formed by the combination of an optimal state feedback law and a Kalman filter. The optimal state feedback law is equal to one designed with the same weight matrices Q and R to minimize the following cost function:*

$$J = \lim_{t_f \to \infty} \mathcal{E}\left\{ \int_0^{t_f} \left(x^T Q x + u^T R u\right) dt \right\} \tag{2.55}$$

, where $\mathcal{E}\{\cdot\}$ means a expectation operator, for a statistical system

$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ y &= Cx + v \end{aligned} \tag{2.56}$$

. Furthermore, the full order observer comes to be same with the Kalman filter if disturbances w, v and the initial state x_0 satisfy the following statistical condition:

$$\mathcal{E}\{w(t)\} = O, \quad \mathcal{E}\{v(t)\} = O \tag{2.57}$$

$$\mathcal{E}\left\{ \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} w^T(\tau) & v^T(\tau) \end{bmatrix} \right\} = \begin{bmatrix} \tilde{Q} & O \\ O & \tilde{R} \end{bmatrix} \delta(t-\tau) \tag{2.58}$$

$$\mathcal{E}\{x(0)\} = O \tag{2.59}$$

Therefore, we can say that the proposed dynamical controller is a LQG controller in a sense that it coincides with the LQG controller designed under the conditions (2.57)-(2.59) for the system (2.56).

2.4 LQG controller representation by system responses

As shown in the previous section, the LQG dynamic output feedback controller has been represented with the expansion coefficients of the Laguerre series. Furthermore we shall show the controller can be represented with some system responses, here. This is one of the main purposes in this chapter. First, we shall consider these system responses and their representation, and next, show the optimal feedback gain (2.33) and the observer (2.43) are represented by them. Finally combining these results yields the Laguerre series representation of the LQG controller.

2.4.1 System response to Laguerre input

An input-output relation of the Laguerre system (2.27)-(2.28) with the zero initial state $x(0) = 0$ is described as

$$Y_N = C_N \bar{A}_N B_N U_N = \begin{bmatrix} l_1 & & & \\ l_2 & l_1 & & \\ \vdots & & \ddots & \\ l_N & l_{N-1} & \cdots & l_1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \tag{2.60}$$

$$l_i = \begin{cases} C\left(pI - A\right)^{-1} B, & i = 1 \\ 2p \cdot C\left(pI + A\right)^{i-2} \left(pI - A\right)^{-i} B, & i > 1 \end{cases} \tag{2.61}$$

(See Details 2.8.5).

Let us consider the following system instead of (2.1)-(2.2), here.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Ew(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.62}$$

The Laguerre system of (2.62) is given as

$$\begin{aligned} D_N X_N - X_0 &= A_N X_N + E_N W_N + B_N U_N \\ Y_N &= C_N X_N \end{aligned} \tag{2.63}$$

(2.63) has two kinds of input, W_N and U_N. We shall consider two responses from W_N to Y_N and from U_N to Y_N. For each path, its input-output relation with the zero initial state is described similarly to (2.60)-(2.61) as follows.

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} g_1 & & & \\ g_2 & g_1 & & \\ \vdots & & \ddots & \\ g_N & g_{N-1} & \cdots & g_1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix} \tag{2.64}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} h_1 & & & \\ h_2 & h_1 & & \\ \vdots & & \ddots & \\ h_{N-1} & h_{N-2} & \cdots & h_1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \tag{2.65}$$

where

$$g_i = \begin{cases} C\,(pI - A)^{-1} E, & i = 1 \\ 2p \cdot C\,(pI + A)^{i-2}\,(pI - A)^{-i} E, & i > 1 \end{cases}$$

$$h_i = \begin{cases} C\,(pI - A)^{-1} B, & i = 1 \\ 2p \cdot C\,(pI + A)^{i-2}\,(pI - A)^{-i} B, & i > 1 \end{cases}$$

Since (2.64)-(2.65) have the same structures with linear discrete time systems, a special response such as the Markov parameters of linear discrete time systems can be considered. Namely it indicates a response to unit pulse input. For example the response from u to y, $\left[h_1^T,\ h_2^T,\ \cdots,\ h_N^T\right]^T$, is given by injecting the input sequence

$$u_i = \begin{cases} I & i = 0 \\ O & i \neq 0 \end{cases} \tag{2.66}$$

into the Laguerre system (2.63) and the response from w to y, $\left[g_1^T,\ g_2^T,\ \cdots,\ g_N^T\right]^T$, is given in the same way.

The unit pulse input (2.66) can be generated practically by injecting the continuous signal of the Laguerre basis $\phi_0(t)$ in (2.3) to the original system (2.1)-(2.2) because the Laguerre basis is orthogonal to other bases, therefore the expansion coefficient of $\phi_i(t)$ is given as $[O,\ \cdots,\ O,\ I,\ O,\ \cdots,\ O]^T$ where I is at the i-th block row. Owing to the toeplitz structure of (2.64)-(2.65), the special response can be obtained by not only $\{u_k : u_0 = I,\ \ u_k = O\ (k \neq 0)\}$ but also $\{u_k : u_i = I,\ \ u_k = O\ (k \neq i)\}$. However, note that the Laguerre system case is different from the linear discrete time system case, that is, the input signal $\phi_i(t)$ injected to the original system in order to generate the unit pulse input $\{u_k : u_i = I,\ \ u_k = O\ (k \neq i)\}$ is different from other signals $\phi_j(t)$ $(i \neq j)$ although the unit pulse input is only delayed in discrete time system case.

2.4.2 Optimal feedback gain case

Let us consider the LQ problem again to find a control input $u(t)$ minimizing a cost function:

$$\begin{aligned} J &= \int_0^\infty \left(y^T(t)y(t) + u^T(t)Ru(t)\right) dt \\ &= Y_N^T Y_N + U_N^T R_N U_N = X_N^T C_N^T C_N X_N + U_N^T R_N U_N \end{aligned} \tag{2.67}$$

for the Laguerre system (2.63).

Substituting $Q_N = C_N^T C_N$ in *Corollary 2* leads to the optimal control:

$$\begin{aligned} U_N = &-R^{-1}B^T I_N \left[\bar{A}_N^T C_N^T C_N \bar{A}_N - \bar{A}_N^T C_N^T C_N \bar{A}_N B_N \right. \\ &\left. \times \left(R_N + B_N^T \bar{A}_N^T C_N^T C_N \bar{A}_N B_N\right)^{-1} B_N^T \bar{A}_N^T C_N^T C_N \bar{A}_N \right] I_N \end{aligned} \tag{2.68}$$

It can be easily seen that (2.68) consists of R, $C_N \bar{A}_N B_N$, $B^T I_N \bar{A}_N^T C_N$ and $C_N \bar{A}_N I_N$. R is assumed to be given and $C_N \bar{A}_N B_N$ is obtained by the unit pulse response (2.65) of the Laguerre system. Similarly $B^T I_N \bar{A}_N^T C_N$ can be also represented with (2.65). (See Details 2.8.6). But $C_N \bar{A}_N I_N$ alone cannot be represented with (2.64)-(2.65), therefore we shall show that it can be represented with the responses combining it with coefficients of the observer, at later.

2.4.3 Full order observer case

Similarly to (2.43), the observer for the Laguerre system (2.63) is given such as

$$\begin{aligned} \tilde{X} = &\bar{A}_N B_N U_N - \left[D_N - (A_N - K_N C_N)\right]^{-1} K_N C_N \bar{A}_N B_N U_N \\ &+ \left[D_N - (A_N - K_N C_N)\right]^{-1} K_N Y_N \end{aligned} \tag{2.69}$$

In (2.69), it is obvious that $C_N \bar{A}_N B_N$ can be represented with (2.65) or $\bar{A}_N B_N$ cannot, and it is not obvious with respect to $\left[D_N - (A_N - K_N C_N)\right]^{-1} K_N$. Here, the part of observer gain, $\left[D_N - (A_N - K_N C_N)\right]^{-1} K_N$, is designed again with considering an optimal control of a dual system:

$$\begin{aligned} D_N \tilde{X}_N - \tilde{X}_0 &= A_N^T \tilde{X}_0 + C_N^T \tilde{U}_N \\ \tilde{Z}_N &= E_N^T \tilde{X}_N \end{aligned} \tag{2.70}$$

The cost function is given as

$$\tilde{J} = \tilde{Z}_N^T \tilde{Z}_N = \tilde{X}_N^T E_N E_N^T \tilde{X}_N + \tilde{U}_N^T \tilde{R}_N \tilde{U}_N \tag{2.71}$$

Substituting $\tilde{Q}_N = E_N E_N^T$ in *Lemma 2* yields

$$\begin{aligned} &K_N^T \left[D_N - \left(A_N^T - C_N^T K_N^T\right)\right]^{-1} \\ &= \left(\tilde{R}_N + C_N \tilde{A}_N E_N E_N^T \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N E_N E_N^T \tilde{A}_N^T \end{aligned} \tag{2.72}$$

(2.72) cosists of $\tilde{R}_N$, $C_N \tilde{A}_N E_N$ and $E_N^T \tilde{A}_N^T$. $\tilde{R}_N$ is assumed to be given and $C_N \tilde{A}_N E_N$ are represented with (2.64), but $E_N^T \tilde{A}_N^T$ alone cannot. (See Details 2.8.7). After all, $\bar{A}_N B_N$ in (2.69) and $E_N^T \tilde{A}_N^T$ in (2.72) need to be combined with $C_N \bar{A}_N I_N$ of the optimal feedback gain to describe the whole controller with (2.64)-(2.65).

2.4.4 Representation of dynamic controller with the unit pulse response

To represent the whole controller using (2.64)-(2.65), we shall integrate the optimal feedback gain and the full order observer considered previously. Especially the parts which cannot be represented alone with the unit pulse response are focused on.

From *Theorem 4*, the dynamic controller is described briefly as

$$U_N = \mathrm{diag}(\underbrace{F,\ F,\ \cdots,\ F}_{N})\tilde{X}$$

Then, we shall extract the previous mentioned parts from this controller and show them:

$$\begin{bmatrix} C_N\bar{A}_N I_N & & & \\ & C_N\bar{A}_N I_N & & \\ & & \ddots & \\ & & & C_N\bar{A}_N I_N \end{bmatrix} \bar{A}_N B_N \tag{2.73}$$

$$\begin{bmatrix} C_N\bar{A}_N I_N & & & \\ & C_N\bar{A}_N I_N & & \\ & & \ddots & \\ & & & C_N\bar{A}_N I_N \end{bmatrix} [D_N - (A_N - K_N C_N)]^{-1} K_N \tag{2.74}$$

As a matter of fact, (2.73)-(2.74) can be described with (2.64)-(2.65) (See Details 2.8.8), and this fact leads to the following theorem.

Theorem 5 *The dynamic output feedback controller (2.53)-(2.54) can be represented with only the unit pulse response (2.64)-(2.65) and weight matrices $R > 0,\ \tilde{R} > 0$ as follows.*

$$U_N = \left[I - \mathcal{F}_N + \mathcal{K}_N \bar{H}_N\right]^{-1} \mathcal{K}_N Y_N \tag{2.75}$$

$$\mathcal{F}_N = \begin{bmatrix} F_G\gamma_1 & & \\ \vdots & \ddots & \\ F_G\gamma_N & \cdots & F_G\gamma_1 \end{bmatrix}, \quad \mathcal{K}_N = \begin{bmatrix} F_G\tilde{\gamma}_1 & & \\ \vdots & \ddots & \\ F_G\tilde{\gamma}_N & \cdots & F_G\tilde{\gamma}_1 \end{bmatrix} K_G \tag{2.76}$$

$$F_G = -R^{-1}\left[\ \bar{h}_1^T\ \cdots\ \bar{h}_N^T\ \right]\left[I - \bar{H}_N\left(R_N + \bar{H}_N^T\bar{H}_N\right)^{-1}\bar{H}_N^T\right] \tag{2.77}$$

$$K_G^T = \left[\tilde{R}_N + \bar{G}_N\bar{G}_N^T\right]^{-1}\bar{G}_N \tag{2.78}$$

$$\bar{H}_N = \begin{bmatrix} h_1 & & \\ \vdots & \ddots & \\ h_N & \cdots & h_1 \end{bmatrix}, \quad \bar{G}_N = \begin{bmatrix} g_1 & \cdots & g_N \\ & \ddots & \vdots \\ & & g_1 \end{bmatrix} \tag{2.79}$$

$$\begin{bmatrix} \bar{h}_1 \\ \vdots \\ \bar{h}_N \end{bmatrix} = \begin{bmatrix} h_1 \\ \vdots \\ \sum_{i=1}^{N}(-1)^{i-1}h_{N-i+1} \end{bmatrix} \tag{2.80}$$

$$\gamma_i = \frac{\sqrt{2p}}{2p}\begin{bmatrix} h_{i+1} + h_i \\ \vdots \\ h_{i+N} + h_{i+N-1} \end{bmatrix}, \quad \tilde{\gamma}_i = \frac{\sqrt{2p}}{2p}\begin{bmatrix} g_{i+1} + g_i \\ \vdots \\ g_{i+N} + g_{i+N-1} \end{bmatrix} \tag{2.81}$$

Remark 2 *Note that the representation of the LQG dynamic controller (2.75) has the toeplitz structure because $\mathcal{F}_N$, $\mathcal{K}_N$ and $\bar{H}_N$ have the same structure also.*

2.5 Realization of a continuous time dynamic controller

In previous section, we have shown that the LQG dynamical controller is represented with the expansion coefficients of the responses generated by injecting the Laguerre basis signal $\phi_i(t)$ to the system (2.62). As a result, the LQG controller is given such as (2.75)-(2.81) which are the relations that expansion coefficients of the optimal input and output should satisfy. In this section, we consider how this relation is realized as the continuous time dynamical system.

From *Remark 2*, (2.75) has the toeplitz structure, which is shown as

$$U_N = \begin{bmatrix} \Phi_1 & & & \\ \Phi_2 & \Phi_1 & & \\ \vdots & & \ddots & \\ \Phi_N & \cdots & \Phi_2 & \Phi_1 \end{bmatrix} Y_N \tag{2.82}$$

On the other hand, let $(A_c,\ B_c,\ C_c)$ denote a LQG dynamic controller realized as a continuous system from (2.75), and then its Laguerre system is given as

$$U_N = C_{c,N}\left(D_N - A_{c,N}\right) B_{c,N} Y_N \tag{2.83}$$

Comparing (2.82) and (2.83) yields the following relation.

$$\Phi_i = \begin{cases} C_c\left(pI - A_c\right)^{-1} B_c, & i = 1 \\ 2p \cdot C_c\left(pI + A_c\right)^{i-2}\left(pI - A_c\right)^{-i} B_c, & i > 1 \end{cases} \tag{2.84}$$

In (2.84) we shall define

$$\begin{aligned} \bar{A}_c &= \left(pI + A_c\right)\left(pI - A_c\right)^{-1}, \quad & \bar{B}_c &= \sqrt{2p}\left(pI - A_c\right)^{-1} B_c \\ \bar{C}_c &= \sqrt{2p} C_c \left(pI - A_c\right)^{-1}, \quad & \bar{D}_c &= C_c\left(pI - A_c\right)^{-1} B_c \end{aligned} \tag{2.85}$$

and then modifying (2.85) with respect to $(A_c,\ B_c,\ C_c)$ leads to

$$\begin{aligned} & A_c = p \cdot \left(\bar{A}_c + I\right)^{-1}\left(A_c - I\right), \quad B_c = \frac{1}{\sqrt{2p}}\left(pI - A_c\right)\bar{B}_c \\ & C_c = \frac{1}{\sqrt{2p}} C_c \left(pI - A_c\right) \end{aligned} \tag{2.86}$$

(2.84) is rewritten as

$$\Phi_i = \begin{cases} \bar{D}_c, & i = 1 \\ \bar{C}_c \bar{A}_c^{i-1} \bar{B}_c, & i > 1 \end{cases} \tag{2.87}$$

$\left(\bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{D}_c\right)$ can be realized from Φ_i according to Ho and Kalman's minimum realization method, for example.

Therefore we shall propose a realization algorithm for the continuous time dynamic controller.

Algorithm

Step 1: Design the LQG dynamic controller according to (2.75)-(2.81).
Step 2: Extract the controller parameters Φ_i from the result of Step 1.
Step 3: Realize $\left(\bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{D}_c\right)$ from Φ_i according to Ho and Kalman's minimum realization method.
Step 4: Transform the result of Step 3 into the state space representation of the continuous time dynamic controller, $(A_c,\ B_c\ ,C_c)$

2.6 Numerical examples

To illustrate efficiency of the proposed method, we shall give numerical examples. For the comparison to the traditional LQG control design based on the state space representation, the following 2nd order system is considered.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -100 & -10 \end{bmatrix} x + \begin{bmatrix} 30 & 0 \\ 0 & 3 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned} \tag{2.88}$$

Moreover, the weight matrix of input R and $\tilde{R}$ is set as $R = [0.01]$ and $\tilde{Q} = I$, respectively. This case is equivalent to choosing $Q = \text{diag}(1,\ 0)$, $R = [0.01]$ as weight matrices for the optimal feedback gain and $\tilde{Q} = \text{diag}(900,\ 9)$, $\tilde{R} = I$ as weight matrices for the observer gain. With these weights, the optimal feedback gain F and observer gain K were calculated using the model (2.88) as

$$\begin{aligned} F &= \begin{bmatrix} -4.9876 \times 10^{-01} & -4.9752 \times 10^{-02} \end{bmatrix} \\ K &= \begin{bmatrix} -2.7848 \times 10^{+01} & 6.224 \times 10^{+01} \end{bmatrix} \end{aligned} \tag{2.89}$$

where the solution of the Riccati equation corresponding to each gain is given as

$$\begin{aligned} P_F &= \begin{bmatrix} 9.9876 \times 10^{+00} & 4.9876 \times 10^{-01} \\ 4.9876 \times 10^{-01} & 4.9752 \times 10^{-02} \end{bmatrix} \\ P_K &= \begin{bmatrix} 2.7848 \times 10^{+01} & -6.2240 \times 10^{+01} \\ -6.2240 \times 10^{+01} & 4.2916 \times +02 \end{bmatrix} \end{aligned} \tag{2.90}$$

Basically, the optimal controller for the Laguerre system should be equal to the one designed based on the state space representation in the sense of the limit. However, since the controller must be approximated with finite one, designers should choose parameters, i.e. the pole of the Laguerre function, p and the length of the expansion coefficient, N appropriately. This chapter cannot suggest the optimal choice in presence stage. This problem is our future work. Here, comparing with the state space case, results with $p = 5,\ N = 5$ and $p = 5,\ N = 12$ are shown in Figure 2.1-2.6.

From Figures, it can be seen that the proposed method gives the almost same LQG dynamic controller with one designed based on the state space representation as long as N is enough large. The required N might depend on the choice of p, that is, it is possible that the good choice of p provides less necessity of long length N. Of course, since continuous signals are approximated with the continuous signal bases in the proposed method, we will not need the large data relatively to represent original signals even if we compare other system representations using data set with our method. This is one of our method's merits.

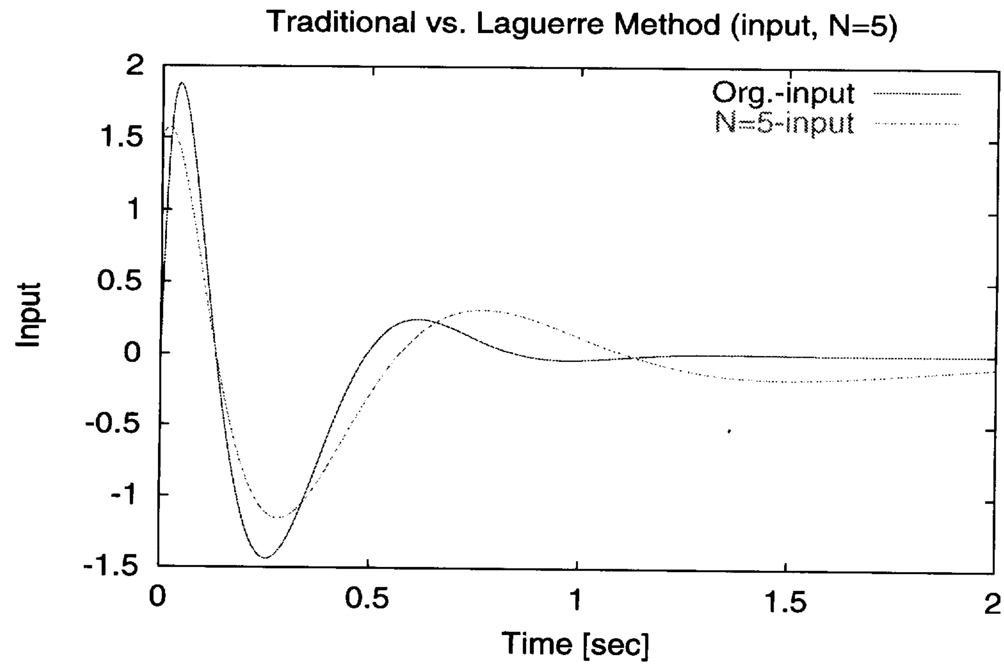


Figure 2.1: time response of optimal input ($p = 5,\ N = 5$)

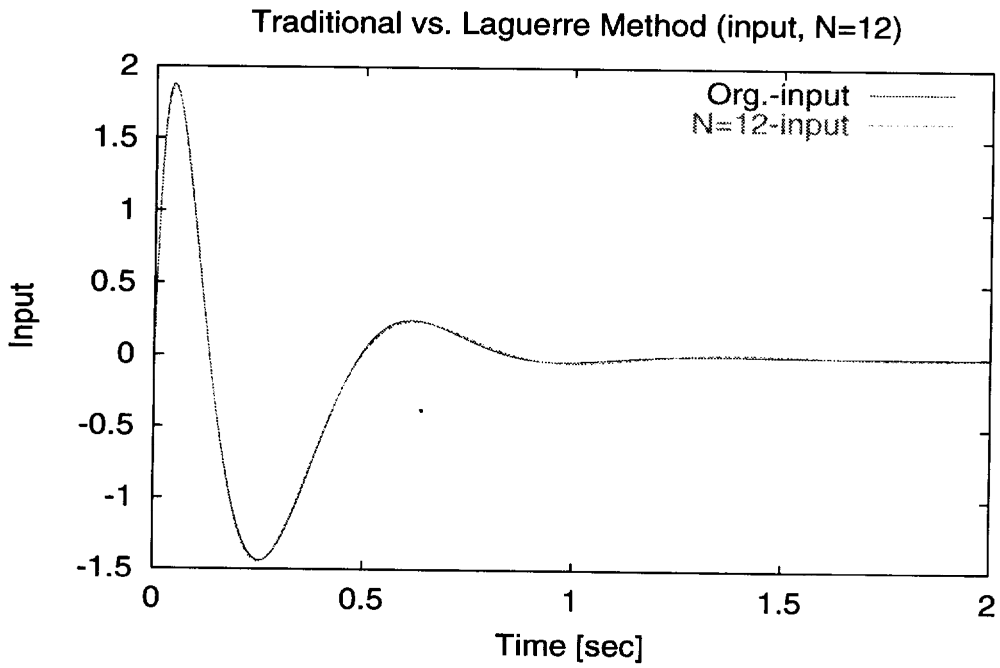


Figure 2.2: time response of optimal input ($p = 5,\ N = 12$)

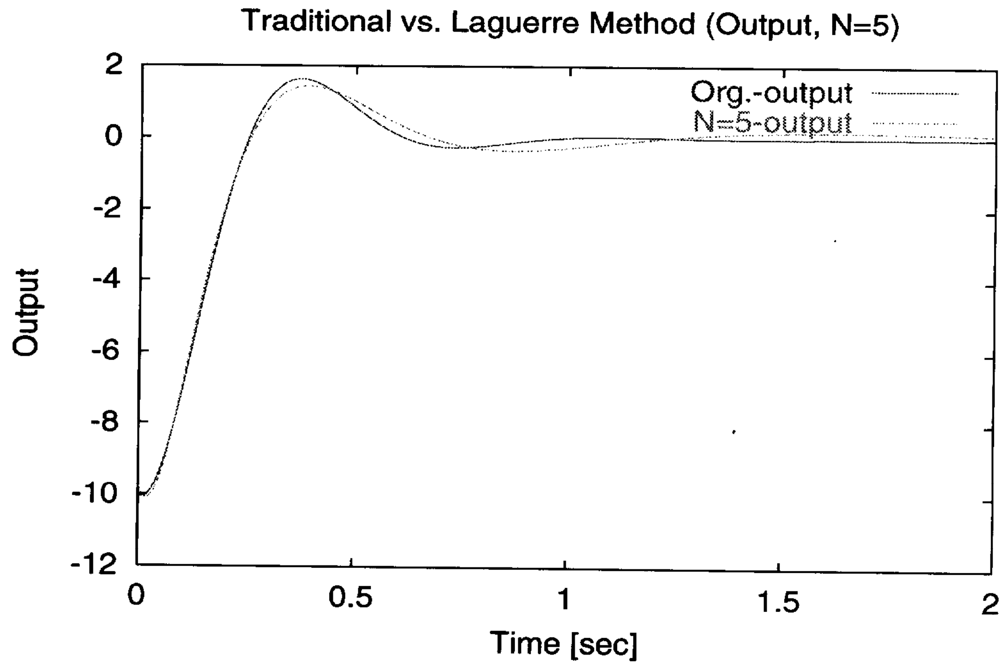


Figure 2.3: time response of output to optimal input ($p = 5,\ N = 5$)

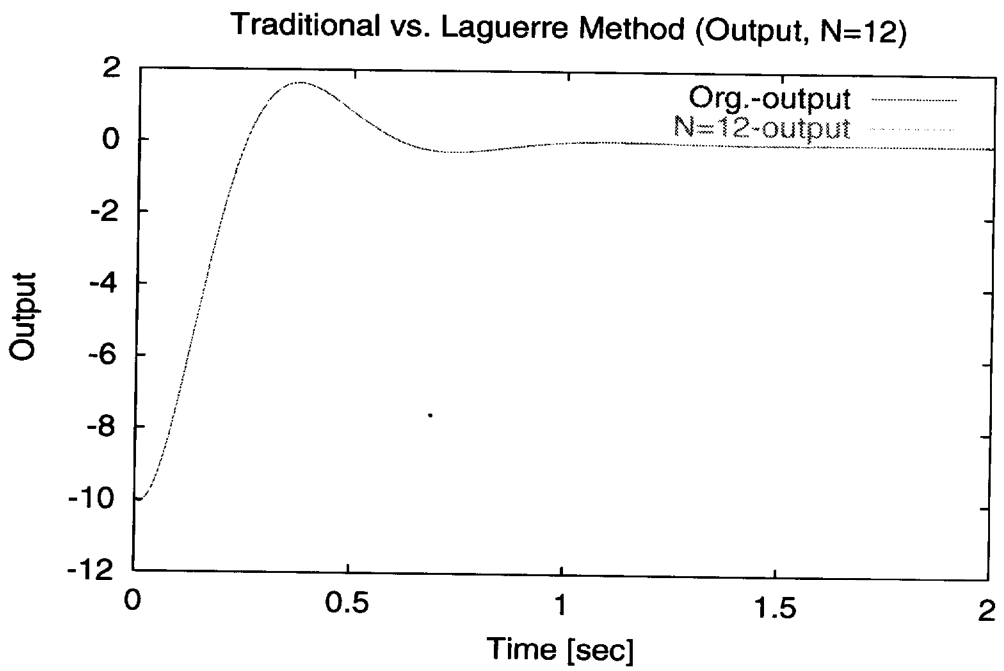


Figure 2.4: time response of output to optimal input ($p = 5,\ N = 12$)

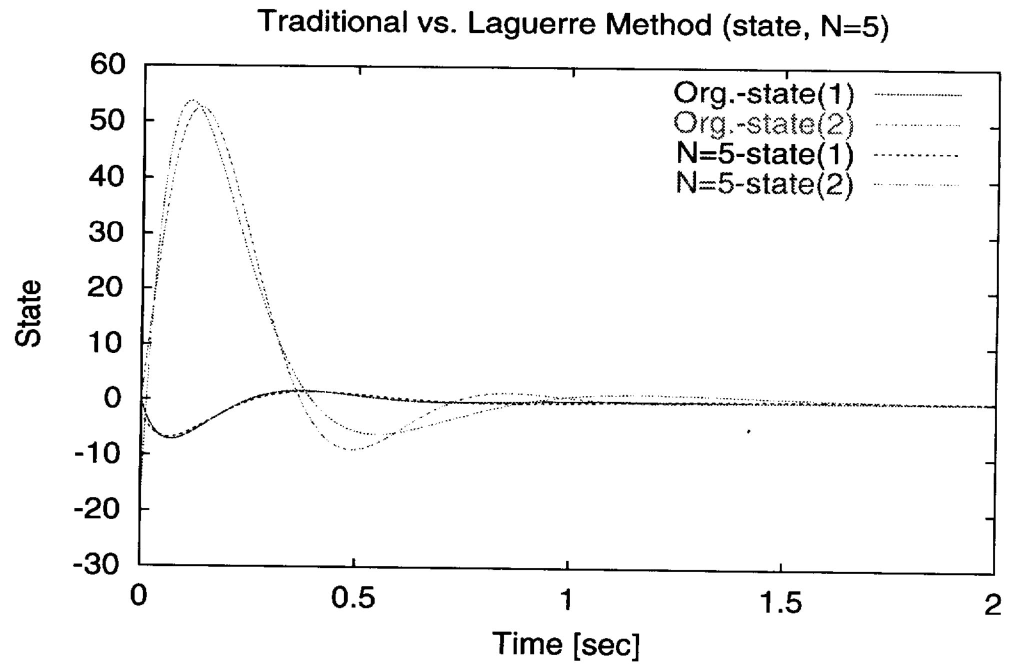


Figure 2.5: time response of state to optimal input ($p = 5,\ N = 5$)

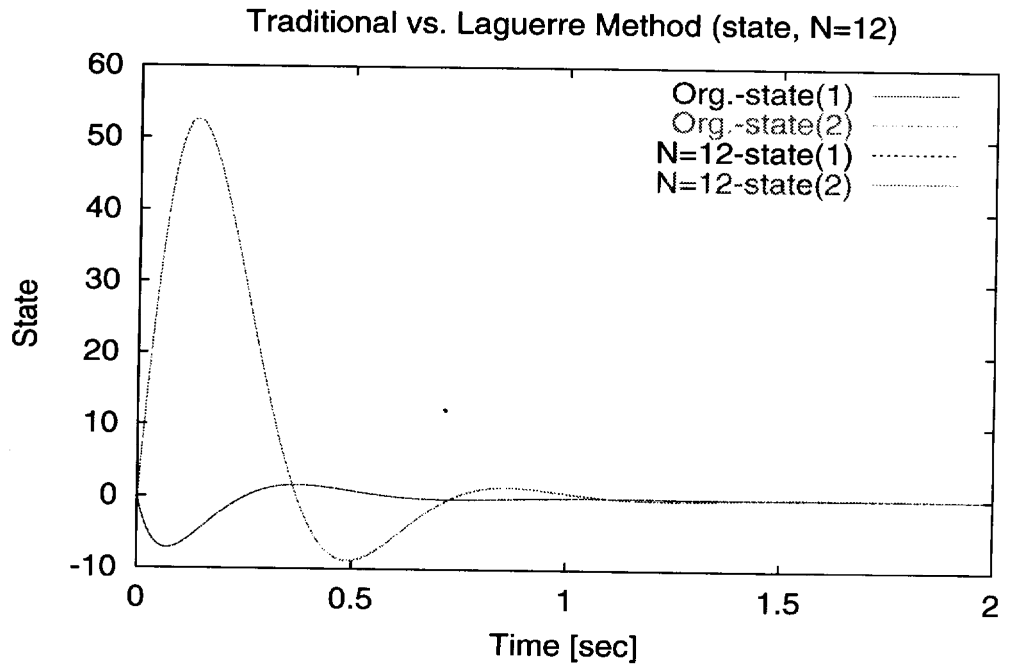


Figure 2.6: time response of state to optimal input ($p = 5,\ N = 12$)

2.7 Concluding Remarks

This chapter has proposed a new design method of a LQG dynamic controller for linear continuous-time systems by using input-output data set. In this method, the Laguerre basis is introduced in order to discretize continuous signals and to extend the system to coefficients of the Laguerre series expansion. Then a new system representation called "Laguerre system" has been proposed. Using this Laguerre system representation, the design methods of the optimal state feedback and full order observer have been given, and then integrating them has led to a LQG controller consisting of the expansion coefficients. Furthermore, considering the representation of the designed controller with input-output responses, it has been shown that the controller can be described by the expansion coefficients of responses generated by injecting the time response of the Laguerre basis to the original system. Finally this chapter has provided a realization method of the continuous-time state space model of the designed LQG controller in the form of the Laguerre expansion coefficients. According to the proposed algorithm, the continuous-time LQG controller can be designed from only system responses. To verify the proposed method, it has been applied to a 2nd order system and has been compared with the controller designed based on the state space representation. Theoretically the proposed method shall give an almost same controller to the state-space-based controller. Simulation results have shown that this statement holds as long as the coefficient's length is long enough.

2.8 Details

2.8.1 Proof of (2.17)

In $k = 1$ case,

$$c_0 = \int_0^{\infty} e^{At}x_0 \cdot e^{-pt}dt = (pI - A)^{-1}\, x_0 \tag{2.91}$$

We shall partially differentiate (2.91) with respect to p,

$$\frac{\partial^k}{\partial p^k} c_0 = \int_0^{\infty} e^{At}x_0 \cdot (-t)^k\, e^{-pt}dt = (-1)^k\, k!\, (pI - A)^{-1}\, x_0 \tag{2.92}$$

□

2.8.2 Proof of (2.20)-(2.22)

Equation (2.20):

$$(pI - A)^{-1}(pI + A) = (pI - A)^{-1}\left[-(pI - A) + 2pI\right] = -I + 2p\,(pI - A)^{-1} \tag{2.93}$$

Equation (2.21):

$$\begin{aligned}(pI + A)(pI - A) &= p^2 I - A^2 = (pI - A)(pI + A)\\ (pI - A)^{-1}(pI + A) &= (pI + A)(pI - A)^{-1}\end{aligned} \tag{2.94}$$

Equation (2.22):

$$\begin{aligned}
&(pI-A)^{-1}(pI+A)\,x_k \\
&= -\sqrt{2p}\sum_{i=0}^{k}(-1)^{k-i}(2p)^i\frac{k!}{i!\,(k-i)!}(pI-A)^{-(i+1)}x_0 \\
&\quad+\sqrt{2p}\sum_{i=0}^{k}(-1)^{k-i}(2p)^{i+1}\frac{k!}{i!\,(k-i)!}(pI-A)^{-(i+2)}x_0 \\
&= -\sqrt{2p}\sum_{i=0}^{k}(-1)^{k-i}(2p)^i\frac{k!}{i!\,(k-i)!}(pI-A)^{-(i+1)}x_0 \\
&\quad+\sqrt{2p}\sum_{i=1}^{k+1}\frac{i}{k+1}(-1)^{k+1-i}(2p)^i\frac{(k+1)!}{i!\,(k+1-i)!}(pI-A)^{-(i+1)}x_0 \\
&= \sqrt{2p}(-1)^{k+1}(pI-A)^{-1}x_0-\sqrt{2p}\sum_{i=1}^{k}(-1)^{k-i}(2p)^i\frac{k!}{i!\,(k-i)!}(pI-A)^{-(i+1)}x_0 \\
&\quad+\sqrt{2p}\sum_{i=1}^{k}\frac{i}{k+1}(-1)^{k+1-i}(2p)^i\frac{(k+1)!}{i!\,(k+1-i)!}(pI-A)^{-(i+1)}x_0 \\
&\quad+\sqrt{2p}(2p)^{k+1}(pI-A)^{-(k+2)}x_0 \\
&= \sqrt{2p}(-1)^{k+1}(pI-A)^{-1}x_0 \\
&\quad+\sqrt{2p}\sum_{i=1}^{k}\left(\frac{k}{i+1}+\frac{i-k+1}{i+1}\right)(-1)^{k+1-i}(2p)^i\frac{(k+1)!}{i!\,(k+1-i)!}(pI-A)^{-(i+1)}x_0 \\
&\quad+\sqrt{2p}(2p)^{k+1}(pI-A)^{-(k+2)}x_0 \\
&= \sqrt{2p}\sum_{i=0}^{k+1}(-1)^{k+1-i}(2p)^i\frac{(k+1)!}{i!\,(k+1-i)!}(pI-A)^{-(i+1)}x_0
\end{aligned}\tag{2.95}$$

□

2.8.3 Proof of (2.23)

From (2.93) and (2.94)

$$\begin{aligned}
(pI-A)^{-i}(pI+A)^i &= (pI-A)(pI+A)(pI-A)^{-(i-1)}(pI+A)^{i-1} \\
&= -(pI-A)^{-(i-1)}(pI+A)^{i-1}+2p(pI-A)^{-(i-1)}(pI+A)^{i-1}
\end{aligned}\tag{2.96}$$

and repeating the expansion (2.96), finally (2.23) can be obtained. □

2.8.4 Structure of A matrix in dual system

From (2.24)-(2.25)

$$D_N-A_N=\begin{bmatrix} pI-A & & & \\ -2pI & pI-A & & \\ \vdots & & \ddots & \\ (-1)^{N-1}2pI & \cdots & -2pI & pI-A \end{bmatrix}\tag{2.97}$$

As long as A has no pole at p, the inverse matrix of (2.97) exists and has the toeplitz structure because of the same toeplitz structure of $D_N - A_N$. Let us represent the inverse matrix as

$$(D_N - A_N)^{-1} = \begin{bmatrix} \alpha_1 & & & \\ \alpha_2 & \alpha_1 & & \\ \vdots & & \ddots & \\ \alpha_N & \cdots & \alpha_2 & \alpha_1 \end{bmatrix} \tag{2.98}$$

On the other hand, for the dual Laguerre system

$$D_N - A_N^T = \begin{bmatrix} pI - A^T & & & \\ -2pI & pI - A^T & & \\ \vdots & & \ddots & \\ (-1)^{N-1} 2pI & \cdots & -2pI & pI - A^T \end{bmatrix} \tag{2.99}$$

is transposed to

$$\left(D_N - A_N^T\right)^T = \begin{bmatrix} pI - A & -2pI & \cdots & (-1)^{N-1} 2pI \\ & \ddots & & \vdots \\ & & pI - A & -2pI \\ & & & pI - A \end{bmatrix} \tag{2.100}$$

From (2.98) the inverse matrix of (2.100) is given as

$$\left(D_N - A_N^T\right)^{-T} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ & \ddots & & \vdots \\ & & \alpha_1 & \alpha_2 \\ & & & \alpha_1 \end{bmatrix} \tag{2.101}$$

Then, it shows the special structure of the dual system. □

2.8.5 Components of $(D_N - A_N)^{-1}$

We shall give practical components of (2.98). From *Theorem 1* the autonomous Laguerre system must satisfy

$$\begin{gathered} X_N = (D_N - A_N)^{-1} X_0 \\ \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \sqrt{2p} \begin{bmatrix} \alpha_1 & & & \\ \alpha_2 & \alpha_1 & & \\ \vdots & & \ddots & \\ \alpha_N & \cdots & \alpha_2 & \alpha_1 \end{bmatrix} \begin{bmatrix} I \\ -I \\ \vdots \\ (-1)^{N-1} I \end{bmatrix} x(0) \\ x_k = \sqrt{2p}\,(pI - A)^{-(k+1)} (pI + A)^k x(0) \end{gathered} \tag{2.102}$$

From the structure of (2.102), the next relation is held.

$$\begin{aligned} x_0 &= \sqrt{2p}\;\alpha_1 x(0) = \sqrt{2p}\,(pI - A)^{-1} x(0) \\ x_i &= \sqrt{2p}\;\alpha_{i+1} x(0) - x_{i-1} = \sqrt{2p}\,(pI - A)^{-(i+1)} (pI + A)^i x(0) \end{aligned} \tag{2.103}$$

Using and substituting $(pI + A) = [2pI - (pI - A)]$, (2.103) is arranged as

$$x_i = \sqrt{2p}2p\,(pI - A)^{-(i+1)}\,(pI + A)^{i-1}\,x(0) - x_{i-1} \tag{2.104}$$

Comparing (2.103) and (2.104) results in

$$\alpha_i = \begin{cases} (pI - A)^{-1} & i = 1 \\ 2p\,(pI + A)^{i-2}\,(pI - A)^{-i} & i > 1 \end{cases} \tag{2.105}$$

□

2.8.6 Representation of the optimal feedback gain with the unit pulse response

From (2.64),

$$C_N\bar{A}_NB_N = \begin{bmatrix} h_1 & & & \\ h_2 & h_1 & & \\ \vdots & & \ddots & \\ h_N & \cdots & h_2 & h_1 \end{bmatrix} \tag{2.106}$$

and from (2.102)

$$\bar{A}I_N = \begin{bmatrix} \alpha_1 & & & \\ \alpha_2 & \alpha_1 & & \\ \vdots & & \ddots & \\ \alpha_N & \cdots & \alpha_2 & \alpha_1 \end{bmatrix} \begin{bmatrix} I \\ -I \\ \vdots \\ (-1)^{N-1}I \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_1 - \alpha_2 \\ \vdots \\ \sum_{i=1}^{N}(-1)^{i-1}\alpha_{N-i+1} \end{bmatrix} \tag{2.107}$$

Then, $C_N\bar{A}_NI_NB$ is obtained as

$$C_N\bar{A}_NI_NB = \begin{bmatrix} h_1 \\ h_2 - h_1 \\ \vdots \\ \sum_{i=1}^{N}(-1)^{i-1}h_{N-i+1} \end{bmatrix} \tag{2.108}$$

□

2.8.7 Representation of the full order observer with the unit pulse response

From (2.48) and (2.65),

$$C_N\bar{A}_NB_N = \begin{bmatrix} h_1 & & & \\ h_2 & h_1 & & \\ \vdots & & \ddots & \\ h_N & \cdots & h_2 & h_1 \end{bmatrix}, \quad C_N\tilde{A}_NE_N = \begin{bmatrix} g_1 & g_2 & \cdots & g_N \\ & \ddots & & \vdots \\ & & g_2 & g_1 \\ & & & g_1 \end{bmatrix} \tag{2.109}$$

□

2.8.8 Combination of the optimal gain and the observer

Let us consider (2.73) first. Extracting the part regarding the A matrix from (2.73) leads to

$$\begin{bmatrix} \bar{A}_N I_N & & \\ & \ddots & \\ & & \bar{A}_N I_N \end{bmatrix} \bar{A}_N = \begin{bmatrix} \bar{A}_N I_N & & \\ & \ddots & \\ & & \bar{A}_N I_N \end{bmatrix} \begin{bmatrix} \alpha_1 & & \\ \vdots & \ddots & \\ \alpha_N & \cdots & \alpha_1 \end{bmatrix} \tag{2.110}$$

$\bar{A}_N I_N \alpha_i$ denotes

$$\bar{A}_N I_N \alpha_i = \sqrt{2p} \begin{bmatrix} \alpha_1 & & & \\ \alpha_2 & \alpha_1 & & \\ \vdots & & \ddots & \\ \alpha_N & \cdots & \alpha_2 & \alpha_1 \end{bmatrix} \begin{bmatrix} I \\ -I \\ \vdots \\ (-1)^{N-1} \end{bmatrix} \alpha_i = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix} \alpha_i \tag{2.111}$$

where $\beta_k = \sum_{i=1}^{k} (-1)^{i-1} \alpha_{k-i+1}$.

From (2.102) β_k is also described as

$$\beta_k = \sqrt{2p}\,(pI + A)^{k-1} (pI - A)^{-k} \tag{2.112}$$

Then,

$$\begin{aligned} \beta_k \alpha_i &= \sqrt{2p}\,(pI + A)^{k-1} (pI - A)^{-k} \cdot 2p\,(pI + A)^{i-2} (pI - A)^{-i} \\ &= 2p\sqrt{2p}\,(pI + A)^{k+i-3} (pI - A)^{-k-i} \end{aligned} \tag{2.113}$$

On the other hand,

$$\begin{aligned} \alpha_{k+i} + \alpha_{k+i-1} &= 2p\,(pI + A)^{k+i-2} (pI - A)^{-k-i} + 2p\,(pI + A)^{k+i-3} (pI - A)^{-k-i+1} \\ &= 2p\left[(pI + A) + (pI - A)\right](pI + A)^{k+i-3} (pI - A)^{-k-i} \\ &= 2p\left[2p\,(pI + A)^{k+i-3} (pI - A)^{-k-i}\right] \end{aligned} \tag{2.114}$$

Comparing (2.113) and (2.114) gives

$$\beta_k \alpha_i = \frac{\sqrt{2p}}{2p} (\alpha_{k+i} + \alpha_{k+i-1}) \tag{2.115}$$

As a result of integrating them,

$$\begin{bmatrix} C_N \bar{A}_N I_N & & \\ & \ddots & \\ & & C_N \bar{A}_N I_N \end{bmatrix} \bar{A} B_N = \begin{bmatrix} \gamma_1 & & \\ \vdots & \ddots & \\ \gamma_N & \cdots & \gamma_1 \end{bmatrix}, \quad \gamma_i = \frac{\sqrt{2p}}{2p} \begin{bmatrix} h_{i+1} + h_i \\ \vdots \\ h_{i+N} + h_{i+N-1} \end{bmatrix} \tag{2.116}$$

Next, let us consider the combination F_N and $[D_N - (A_N - K_N C_N)] K_N$. At first, F is separated into F_G and $C_N \bar{A} I_N$:

$$F = F_G C_N \bar{A}_N I_N \tag{2.117}$$

where F_G can be represented with (2.64)-(2.65). The problem part is discribed using (2.52) and (2.117) as

$$\begin{bmatrix} C_N\bar{A}_N I_N & & \\ & \ddots & \\ & & C_N\bar{A}_N I_N \end{bmatrix} \begin{bmatrix} \delta_1 K & & \\ \vdots & \ddots & \\ \delta_N K & \cdots & \delta_1 K \end{bmatrix}$$
$$= \begin{bmatrix} C_N\bar{A}_N I_N \delta_1 K & & \\ \vdots & \ddots & \\ C_N\bar{A}_N I_N \delta_N K & \cdots & C_N\bar{A}_N I_N \delta_1 K \end{bmatrix} \tag{2.118}$$

Transposing each components of (2.118) gives

$$\begin{bmatrix} K^T\delta_1^T I_N^T \bar{A}_N^T C_N^T & & \\ \vdots & \ddots & \\ K^T\delta_N^T I_N^T \bar{A}_N^T C_N^T & \cdots & K^T\delta_1^T I_N^T \bar{A}_N^T C_N^T \end{bmatrix}$$
$$= K_N^T \left[D_N - (A_N^T - C_N^T K_N^T)\right]^{-1} \begin{bmatrix} I_N^T \bar{A}_N^T C_N^T & & \\ & \ddots & \\ & & I_N^T \bar{A}_N^T C_N^T \end{bmatrix}$$
$$= \left(\tilde{R}_N + C_N\tilde{A}_N E_N E_N^T \tilde{A}_N^T C_N^T\right)^{-1} C_N \tilde{A}_N E_N E_N^T \tilde{A}_N^T \begin{bmatrix} I_N^T \bar{A}_N^T C_N^T & & \\ & \ddots & \\ & & I_N^T \bar{A}_N^T C_N^T \end{bmatrix} \tag{2.119}$$

where

$$\left\{ E_N^T \tilde{A}_N^T \begin{bmatrix} I_N^T \bar{A}_N^T C_N^T & & \\ & \ddots & \\ & & I_N^T \bar{A}_N^T C_N^T \end{bmatrix} \right\}^T = \begin{bmatrix} C_N\bar{A}_N I_N & & \\ & \ddots & \\ & & C_N\bar{A}_N I_N \end{bmatrix} \tilde{A}_N E_N$$
$$= \begin{bmatrix} C_N\bar{A}_N I_N & & \\ & \ddots & \\ & & C_N\bar{A}_N I_N \end{bmatrix} \begin{bmatrix} \alpha_1 & \cdots & \alpha_N \\ & \ddots & \vdots \\ & & \alpha_1 \end{bmatrix} \begin{bmatrix} E & & \\ & \ddots & \\ & & E \end{bmatrix} \tag{2.120}$$

The part on the A matrix of (2.120) is given in the same way of the optimal feedback gain case. Then, (2.120) is represented as follows.

$$\begin{bmatrix} C_N\bar{A}_N I_N & & \\ & \ddots & \\ & & C_N\bar{A}_N I_N \end{bmatrix} \tilde{A}_N E_N = \begin{bmatrix} \tilde{\gamma}_1 & \cdots & \tilde{\gamma}_N \\ & \ddots & \vdots \\ & & \tilde{\gamma}_1 \end{bmatrix}, \quad \tilde{\gamma}_i = \frac{1}{2p} \begin{bmatrix} g_{i+1} + g_i \\ \vdots \\ g_{i+N} + g_{i+N-1} \end{bmatrix} \tag{2.121}$$

Using (2.121), (2.119) is rewritten as

$$K_N^T \left[D_N - (A_N^T - C_N^T K_N^T)\right]^{-1} \begin{bmatrix} I_N^T \bar{A}_N^T C_N^T & & \\ & \ddots & \\ & & I_N^T \bar{A}_N^T C_N^T \end{bmatrix}$$
$$= K_G^T \begin{bmatrix} \tilde{\gamma}_1^T & & \\ \vdots & \ddots & \\ \tilde{\gamma}_N^T & \cdots & \tilde{\gamma}_1^T \end{bmatrix} \tag{2.122}$$

where $K_G^T = \left(\tilde{R}_N + C_N\tilde{A}_N E_N E_N^T \tilde{A}_N^T C_N^T\right)^{-1} C_N\tilde{A}_N E_N$.

Transposing each components of (2.122) again gives the following final form.

$$F_N\left[D_N - (A_N - K_N C_N)\right] K_N = \begin{bmatrix} F_G\tilde{\gamma}_1 & & & \\ F_G\tilde{\gamma}_2 & F_G\tilde{\gamma}_1 & & \\ \vdots & & \ddots & \\ F_G\tilde{\gamma}_N & F_G\tilde{\gamma}_{N-1} & \cdots & F_G\tilde{\gamma}_1 \end{bmatrix} K_G \tag{2.123}$$

Chapter 3

An Adaptive Learning Control Using Markov Parameters

3.1 Introduction

A learning control is an effective approach which makes the system output track a desired trajectory perfectly. If the system is completely known, there is no need for the learning control because the input that yields the desired trajectory may be derived by the inverse system. However, due to the inaccuracy of the parameters, such an inverse system cannot be obtained.

Several iterative learning control methods surveyed in [27] for example have been proposed, and if some given conditions are satisfied, they showed advantages of the iterative learning control as a priori knowledge of the system information is not needed too much for perfect tracking during a fixed period of time. There are, however, many systems which cannot meet these conditions and which are completely unknown. Therefore, in this chapter, we propose simple adaptive iterative learning control methods for linear time invariant discrete time systems, which are effective in the case where system parameters are unknown and/or there are modeling errors.

In our method, the Markov parameters are used to represent the target system, and the control law and each signals are defined in the form of sequences. Because of this approach, the adaptive learning control can be designed on the basis of system responses measured at each steps and can cope with systems which have unknown dynamics.

Meanwhile, in the literatures of iterative learning control methods, several iterative learning control algorithm with adaptation has been proposed. B. H. Park used the parameterization of unknown mechanical systems, and studied an adaptive iterative learning control for robotic systems with some uncertainties in [3]. Recently, M. French has reported a non-linear adaptive learning control [34], in which some adaptive controller design methods based on Lyapunov functions are used in order to design the learning control law for continuous time systems. In his method the 'learning' is achieved during each iteration, i.e., on-line, so the implementation of its controller became slightly complex.

Our methods, which are different from former works, are based on an algorithm in which a dual operator of the system is used for its update law explicitly in the same way as [39] [40]. The proposed methods give the learning control laws based on a quadratic criterion and both input update and parameter estimation are obtained at each iteration simultaneously. Although we use the Markov parameters for the unknown system representation similarly to

M. Q. Phan who used the same parameters in his work [37], our approach is different from his work with respect to the way of the parameter estimation, i.e., his approach is indirect estimation, however, our methods are direct ones. Furthermore, an effect of a disturbance signal is explicitly taken account of in the algorithm.

In this chapter some numerical simulations are given to show the effectiveness of the proposed methods, and the method is also applied for an active vibration isolation system. In the considered system a repeated disturbance is generated by a motion of a part of the system. The iterative learning control method is very efficient for the application. The results will show the validity of the proposed methods and the practical efficiency for the suppression of the repeated vibration generated by machines installed on the stage.

3.2 Problem Formulation

In this chapter we consider the following discrete linear time invariant system:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0, \tag{3.1}$$
$$y(k) = Cx(k) + Du(k), \tag{3.2}$$

where $u(k), x(k), y(k)$ are input, state and output respectively, and in this chapter the plant is assumed to be SISO system, though it is possible to extend the proposed methods for MIMO systems. Since one iterative step has N samples, then the input sequence of the i-th step is defined by

$$u_i := [u_i(1), u_i(2), \dots, u_i(N)]^T,$$

and y_i is also defined similarly.

Then the relationship of the input/output can be described by :

$$y_i = Hu_i \tag{3.3}$$

where

$$H = \begin{bmatrix} D & & O \\ CB & D & \\ \vdots & & \ddots \\ CA^{N-2}B & \dots & D \end{bmatrix}.$$

$$h := [D, CB, \dots, CA^{N-2}B]^T.$$

The inner product and norm with an arbitrary weight matrix $Q > 0$ are defined as follows:

$$\langle x, y \rangle := x^T y$$
$$\|x\|_Q^2 := \langle x, Qx \rangle.$$

The purpose of learning control algorithms is, under a given desired trajectory y_d and a defined error $e_i := y_d - y_i$, to find the input u_i which achieves $\|e_i\| \to 0$ $(i \to \infty)$ by iterations.

3.3 Adaptive Learning Control Algorithm

The i-th estimated model of the system (A, B, C, D) is represented by $(\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i)$, and $\hat{H}_i$ is also defined like H. The estimation error is defined as $\tilde{a}_i := a - \hat{a}_i$.

The control input and estimated model parameters are updated according to the following laws:

$$u_{i+1} = u_i + \gamma_i z_i \tag{3.4}$$
$$\hat{h}_{i+1} = \hat{h}_i + \gamma_i w_i. \tag{3.5}$$

z_i, w_i and γ_i are determined for each control method which is defined later.

(3.4) is the general form of learning control [39] [40], and (3.5) is given corresponding to the form of (3.4). Using these control laws we present two adaptive learning control methods in this chapter.

The proposed methods are given by **Method 1** and **Method 2** as follows.

From the definition, the error between the system and the model for a same input u_i is described by

$$\tilde{e}_i = Hu_i - \hat{H}_i u_i = \tilde{H}_i u_i = U_i \tilde{h}_i, \tag{3.6}$$

where

$$U_i = \begin{bmatrix} u_i(1) & & O \\ u_i(2) & u_i(1) & \\ \vdots & & \ddots \\ u_i(N) & \cdots & u_i(1) \end{bmatrix}. \tag{3.7}$$

In MIMO case, input and output sequences are described by

$$u_i := \begin{bmatrix} u_i^T(1) & u_i^T(2) & \cdots & u_i^T(N) \end{bmatrix}^T \tag{3.8}$$
$$y_i := \begin{bmatrix} y_i^T(1) & y_i^T(2) & \cdots & y_i^T(N) \end{bmatrix}^T \tag{3.9}$$

where $u_i(k) \in R^{m\times 1}$ and $y_i(k) \in R^{p\times 1}$. h_i and U_i need to be redefined such as

$$h_i = \left[Vec(D)^T, \cdots, Vec(CA^{N-2}B)^T\right]^T \tag{3.10}$$
$$U_i = \begin{bmatrix} u_i^T(1)\otimes I_m & & \\ u_i^T(2)\otimes I_m & & \\ \vdots & \ddots & \\ u_i^T(N)\otimes I_m & \cdots & u_i^T(1)\otimes I_m \end{bmatrix} \tag{3.11}$$

where $Vec(X)$ denotes the vector formed by stacking each column of X into one long vector:

$$Vec(X) := \begin{bmatrix} x_1^T & x_2^T & \cdots & x_n^T \end{bmatrix}^T \tag{3.12}$$
$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \tag{3.13}$$

and $\otimes$ operator is Kronecker product. With these notations, we can consider MIMO systems as well as SISO, so only the SISO case is handled in the following methods.

[Method 1] Consider a criterion

$$J_i = \frac{1}{2}\|e_i\|_Q^2 + \frac{1}{2}\|\tilde{h}_i\|^2 \tag{3.14}$$

with any $Q > 0$ and chose each signal as follows:

$$z_i = \hat{H}_i^T Q e_i, \tag{3.15}$$

$$w_i = -Z_i^T Q e_i + U_i^T \tilde{e}_i, \tag{3.16}$$

$$\gamma_i = \frac{\|z_i\|^2 + \|\tilde{e}_i\|^2}{\|Hz_i\|_Q^2 + \|w_i\|^2}, \tag{3.17}$$

where Z_i is defined like U_i.

Then J_i will decrease in the steepest way at each iteration, and we have $\|z_i\| \to 0$ and $\|\tilde{e}_i\| \to 0$.

Proof. From $e_{i+1} = y_d - y_{i+1}$, $y_{i+1} = Hu_{i+1}$, $\hat{h}_{i+1} = h - \tilde{h}_{i+1}$ and (3.4)-(3.5),

$$\begin{aligned} J_{i+1} &= \frac{1}{2}\|y_d - H(u_i + \gamma_i z_i)\|_Q^2 + \frac{1}{2}\|\tilde{h}_i - \gamma_i w_i\|^2 \\ &= \frac{1}{2}\|e_i - \gamma_i H z_i\|_Q^2 + \frac{1}{2}\|\tilde{h}_i - \gamma_i w_i\|^2 \end{aligned} \tag{3.18}$$

then

$$\begin{aligned} J_{i+1} - J_i &= -\gamma_i e_i^T Q H z_i - \gamma_i \tilde{h}_i^T w_i + \frac{\gamma_i^2}{2}(\|Hz_i\|_Q^2 + \|w_i\|^2) \\ &= -\gamma_i e_i^T Q \hat{H}_i z_i - \gamma_i e_i^T Q \tilde{H}_i z_i + \gamma_i \tilde{h}_i^T Z_i^T Q e_i - \gamma_i \tilde{h}_i^T U_i^T \tilde{e}_i + \frac{\gamma_i^2}{2}(\|Hz_i\|_Q^2 + \|w_i\|^2) \end{aligned}$$

Using (3.15), $\tilde{H}_i z_i = Z_i \tilde{h}_i$ and $\tilde{e} = \tilde{H}_i u_i = U_i \tilde{h}_i$,

$$\begin{aligned} J_{i+1} - J_i &= -\gamma_i(\|z_i\|^2 + \|\tilde{e}_i\|^2) + \frac{\gamma_i^2}{2}(\|Hz_i\|_Q^2 + \|w_i\|^2) \\ &= \frac{-(\|z_i\|^2 + \|\tilde{e}_i\|^2)^2}{2(\|Hz_i\|_Q^2 + \|w_i\|^2)} \le 0 \end{aligned} \tag{3.19}$$

□

Remark

- If u_i is persistently exciting, the parameters of the model will converge to the true values according to $\tilde{e}_i \to 0$, which means $\|e_i\| \to 0$.
- It should be noted that if a disturbance is imposed on the system, the decreasing is not ensured.
- Hz_i in γ_i can be obtained from a response of the system with the input sequence z_i. Concretely, if $u_i + \epsilon z_i$, where ϵ is a small number, is injected to the system in an additional step, we observe the output as

$$y_i' := H(u_i + \epsilon z_i) = y_i + \epsilon H z_i \tag{3.20}$$

, then Hz_i can be calculated as

$$Hz_i = (y_i' - y_i)/\epsilon \tag{3.21}$$

If the ϵ is selected small enough, the injection of the input will be allowed, however, the step is still impractical. In that case we can modify the gain as follows. If a lower bound of the estimate of $||H||_\infty$ is given previously, we can use the relation

$$||Hz_i|| \leq ||H||_\infty ||z_i|| \tag{3.22}$$

and replace $||Hz_i||$ to $||H||_\infty ||z_i||$ in γ_i, and the convergence can be also guaranteed. In practice, Hz_i is usually replaced by $\hat{H}z_i$, but the convergence cannot be guaranteed theoretically.

Notes

- We can also consider a weighted norm of the parameters estimation error as $\|\tilde{h}_i\|_R^2$ with any matrix $R > 0$. Notes that, in this case, the Method 1 handles the problem how the controller is designed in order to reduces the tracking error e_i^R of the weighted system, which is redefined as

$$e_i^R = y_d - U_i R h.$$

To overcome the problem in the remark, we present a second method using the inverse system, and consider the following relationship.

$$y_i = H(v_i + d) \Leftrightarrow Ly_i = v_i + d, \tag{3.23}$$
$$v_i = \hat{L}_i y_d + u_i, \tag{3.24}$$

where u_i is the input sequence, d is the disturbance which is assumed to be repeated at each iteration and L is the inverse system of H.

From (3.23), (3.24), and $\tilde{u}_i := u_i + d$, we obtain

$$\begin{aligned}\tilde{u}_i &= Ly_i - \hat{L}_i y_d \\ &= \tilde{L}_i y_i - \hat{L}_i e_i \\ &= Y_i \tilde{l}_i - \hat{L}_i e_i,\end{aligned} \tag{3.25}$$

where Y_i is defined like U_i, and $\tilde{l}_i$ like $\tilde{h}_i$. Parameter estimation laws are defined by the following

$$u_{i+1} = u_i + \gamma_i z_i, \tag{3.26}$$
$$\hat{l}_{i+1} = \hat{l}_i + \gamma_i w_i. \tag{3.27}$$

[Method2] Consider a criterion

$$J_i = \frac{1}{2}\|\tilde{u}_i\|_Q^2 + \frac{1}{2}\|\tilde{l}_i\|_R^2 \tag{3.28}$$

where any $Q > 0,\ R > 0$ and chose each signal as follows:

$$z_i = \hat{L}_i e_i, \tag{3.29}$$
$$w_i = R^{-1} Y_i^T Q z_i, \tag{3.30}$$
$$\gamma_i = \frac{\|z_i\|_Q^2}{\|z_i\|_Q^2 + \|w_i\|_R^2}, \tag{3.31}$$

where Y_i is defined like U_i. Then J_i will decrease in the steepest way at each iteration and $\|z_i\| \to 0$.

Proof. Because $\tilde{u}_{i+1} = \tilde{u}_i + \gamma_i z_i$ and $\tilde{l}_{i+1} = \tilde{l}_i - \gamma_i w_i$ from update laws,

$$J_{i+1} - J_i = \gamma_i \tilde{u}_i^T Q z_i - \gamma_i \tilde{l}_i^T R w_i + \frac{\gamma_i^2}{2}(\|z_i\|_Q^2 + \|w_i\|_R^2)$$

and from (3.25)

$$\begin{aligned} J_{i+1} - J_i &= \gamma_i (Y_i \tilde{l}_i - \hat{L}_i e_i)^T Q z_i - \gamma_i \tilde{l}_i^T R w_i + \frac{\gamma_i^2}{2}(\|z_i\|_Q^2 + \|w_i\|_R^2) \\ &= -\gamma_i (\hat{L}_i e_i)^T Q z_i + \gamma_i \tilde{l}_i^T (Y_i^T Q z_i - R w_i) + \frac{\gamma_i^2}{2}(\|z_i\|_Q^2 + \|w_i\|_R^2) \\ &= -\gamma_i \|z_i\|_Q^2 + \frac{\gamma_i^2}{2}(\|z_i\|_Q^2 + \|w_i\|_R^2) \\ &= \frac{-(\|z_i\|_Q^2)^2}{2(\|z_i\|_Q^2 + \|w_i\|_R^2)} \leq 0 \end{aligned} \quad (3.32)$$

□

Remark

- If the system is strictly proper, we should consider the input/output relationship shifted by a delay.

3.4 Numerical Simulation

We present results of some computer simulations to demonstrate the effectiveness of the proposed methods. Let the plant be a system whose transfer function is described by

$$G(z) = \frac{5.0(z+0.1)(z-0.9)}{(z-0.1)(z-0.8)(z-0.9)}, \quad (3.33)$$

and a nominal model of this plant to be updated is chosen by

$$\hat{G}_0(z) = \frac{6.0(z+0.3)(z-0.9)}{(z-0.1)(z-0.7)(z-0.8)}. \quad (3.34)$$

We set $N = 100$, chose $Q = I$ and $R = I$ as weight matrices in the criterions, and performed 20 iterations.

3.4.1 Tracking Problem

A tracking control is performed using **Method 1**. Figure 3.2, Figure 3.3 and Figure 3.4 show J_i, the tracking error $\frac{1}{2}\|e_i\|^2$ and the parameter estimation error $\frac{1}{2}\|\tilde{h}_i\|^2$ respectively. These figures show that J_i decreases at each step and $\|e_i\| \to 0$. Figure 3.1 shows the desired trajectory and the output of the system at 5 and 20 iterations.

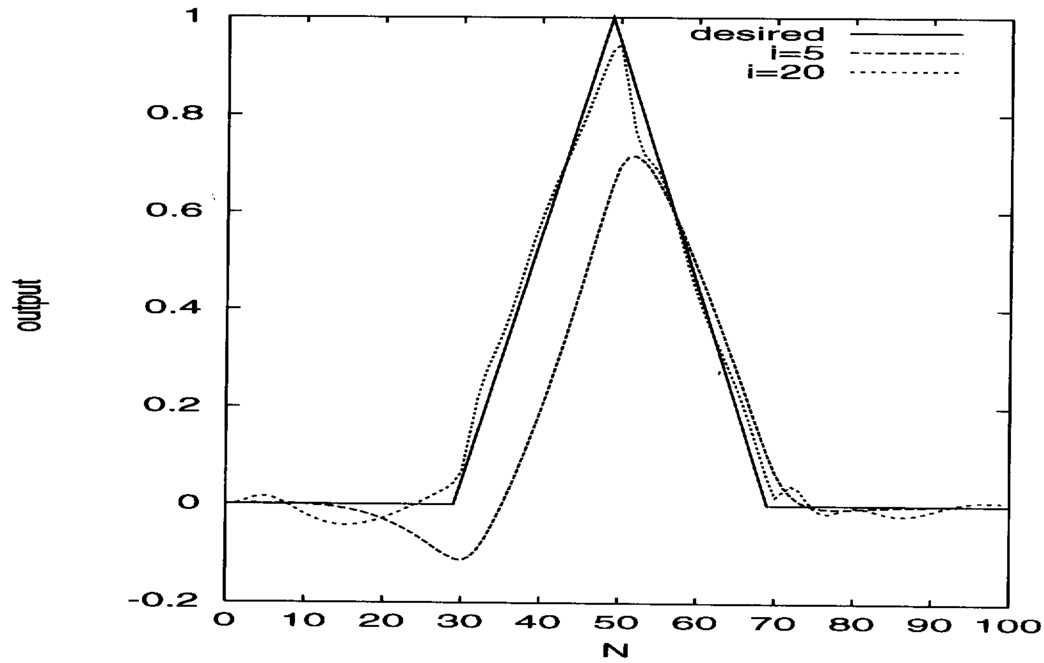


Figure 3.1: Output of the system by tracking control (step i=5 and 20)

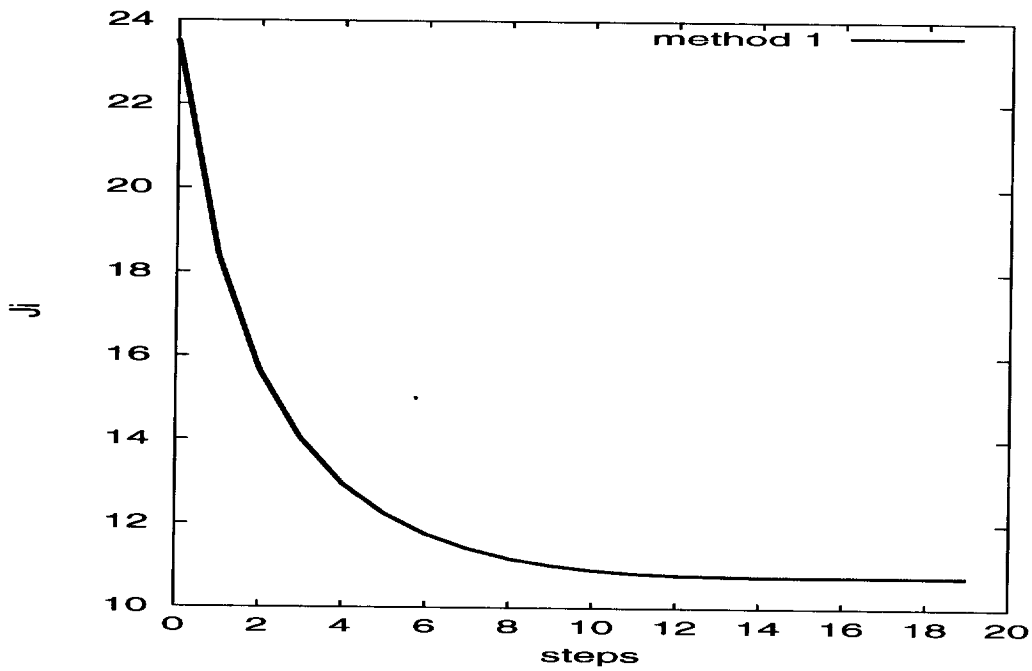


Figure 3.2: Cost function : J_i

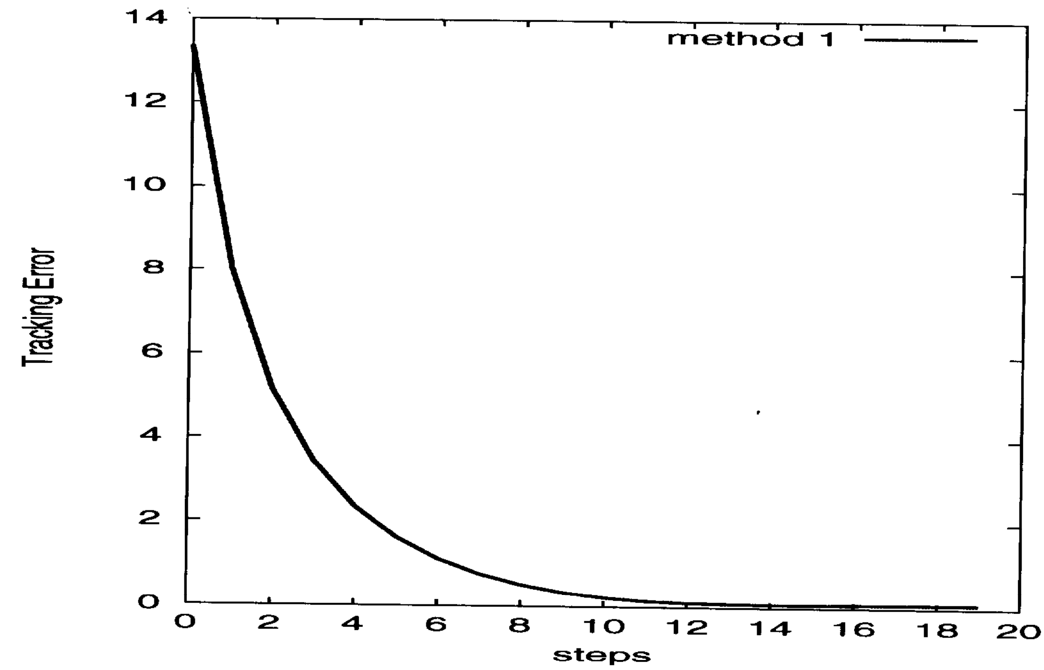


Figure 3.3: Tracking error : $||e_i||^2/2$

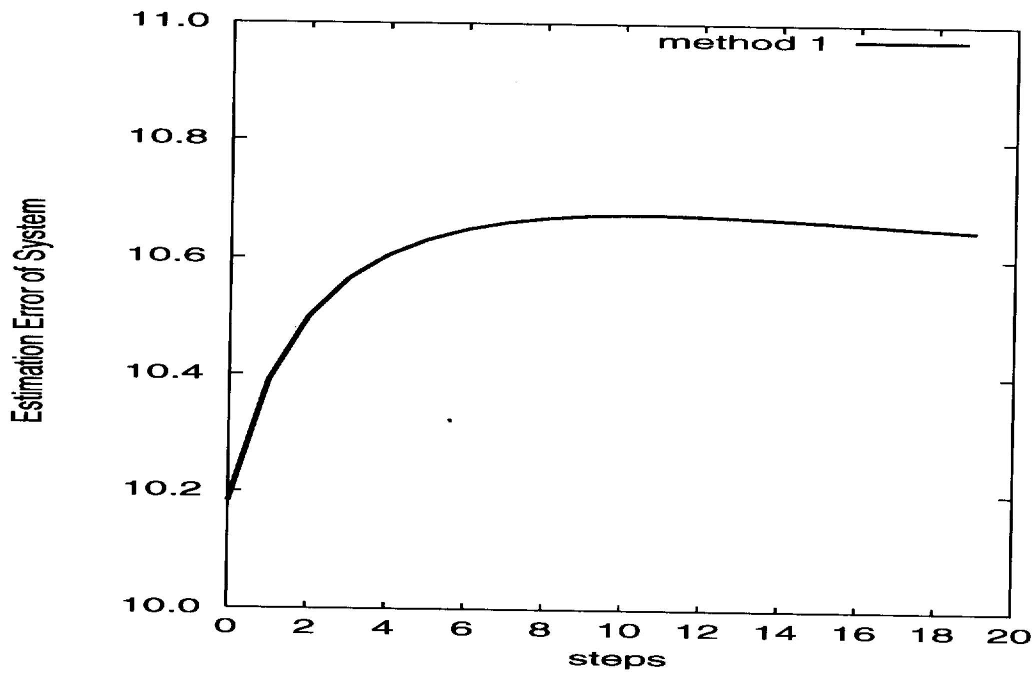


Figure 3.4: Estimation error of the Markov parameters : $||\tilde{h}_i||^2/2$

3.4.2 Existence of repeated disturbance

We assume a disturbance in (3.23) which is specified as

$$d(k) = \begin{cases} 0 & (1 \le k < 20) \\ 0.01\sin(2\pi\frac{k-20}{4}) & (20 \le k < 60) \\ 0 & (60 \le k \le 100) \end{cases}, \tag{3.35}$$

and the desired trajectory $y_d = O$. **Method 2** is applied to this case. Figure 3.6, Figure3.7, Figure 3.8 and Figure 3.9 show J_i, the input criterion $\frac{1}{2}||\tilde{u}_i||^2$, the parameter estimation error $\frac{1}{2}||\tilde{l}_i||^2$ and the tracking error, respectively. These figures show that J_i decreases at each step and $||e_i|| \to 0$. Figure 3.5 shows the desired trajectory and the output of the system at 1, 10 and 20 iteration.

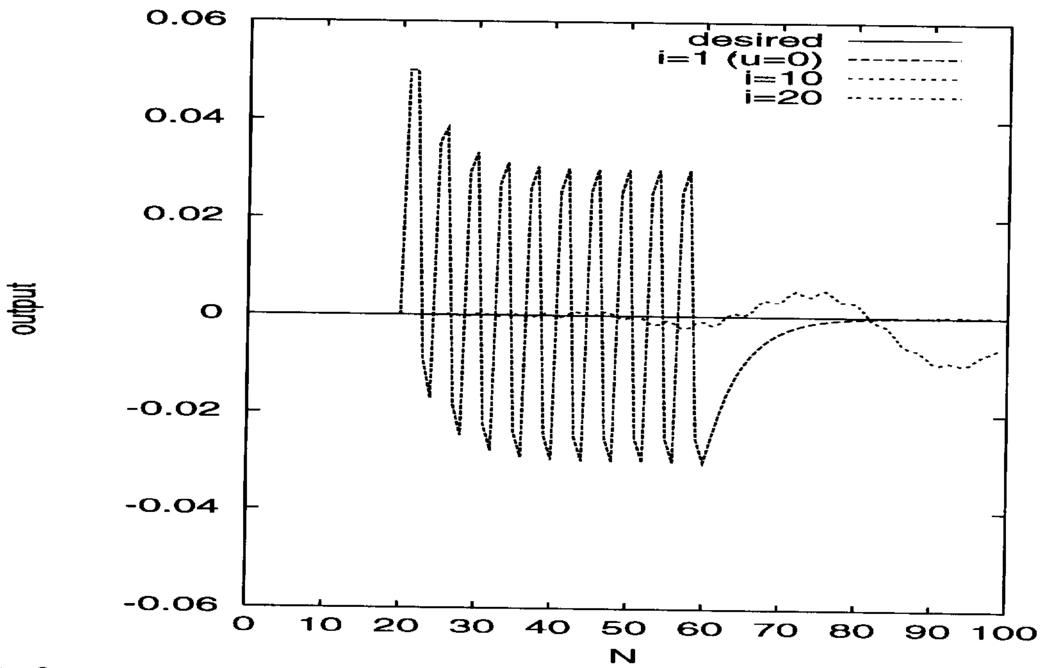


Figure 3.5: Output of the system by tracking control with a repeated disturbance (step i=1, 10 and 20)

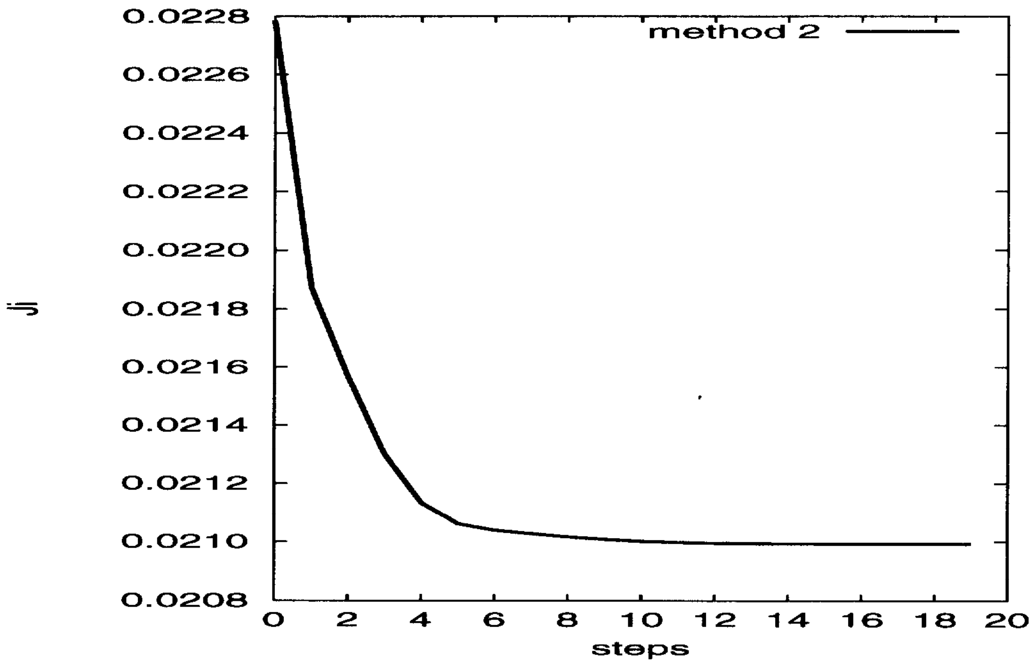


Figure 3.6: Cost function : J_i

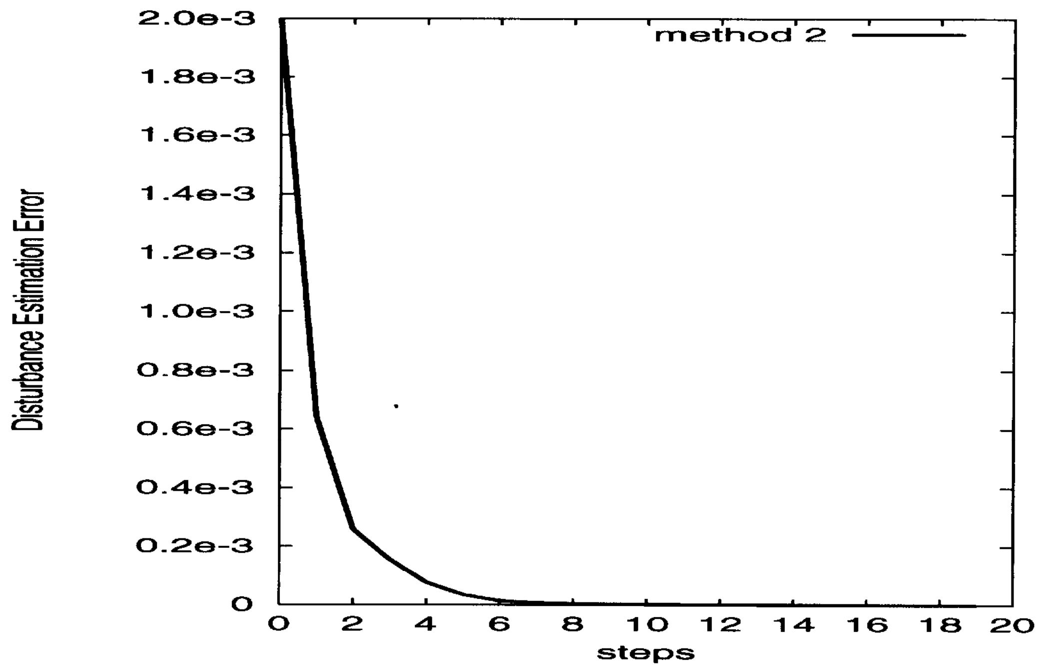


Figure 3.7: Estimation error of disturbance : $\|\tilde{u}_i\|^2/2$

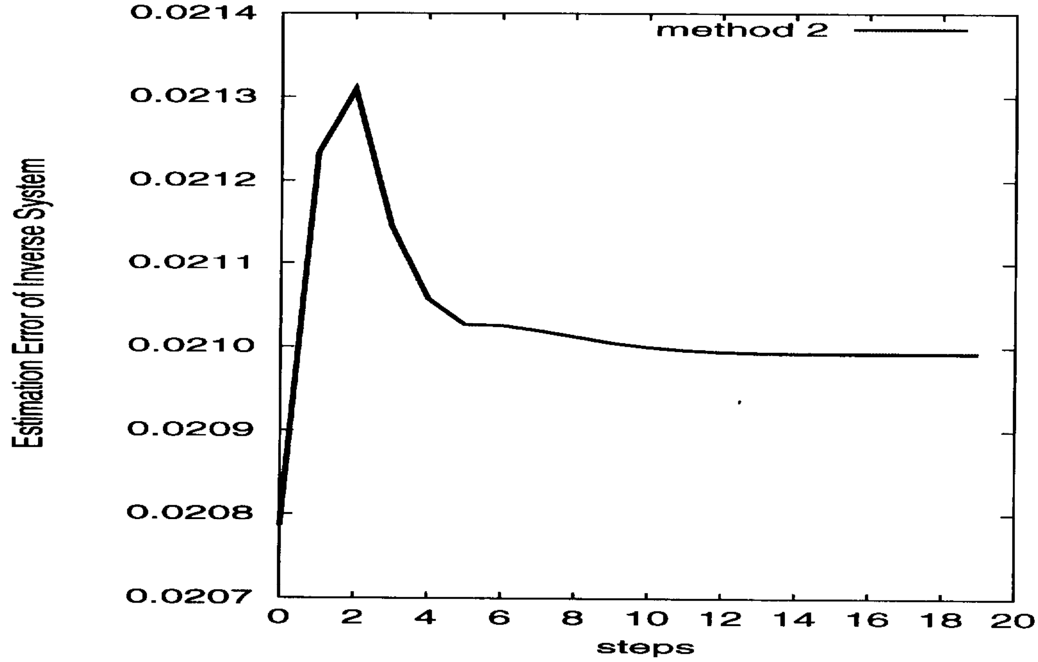


Figure 3.8: Estimation error of the inverse system : $||\tilde{l}_i||^2/2$

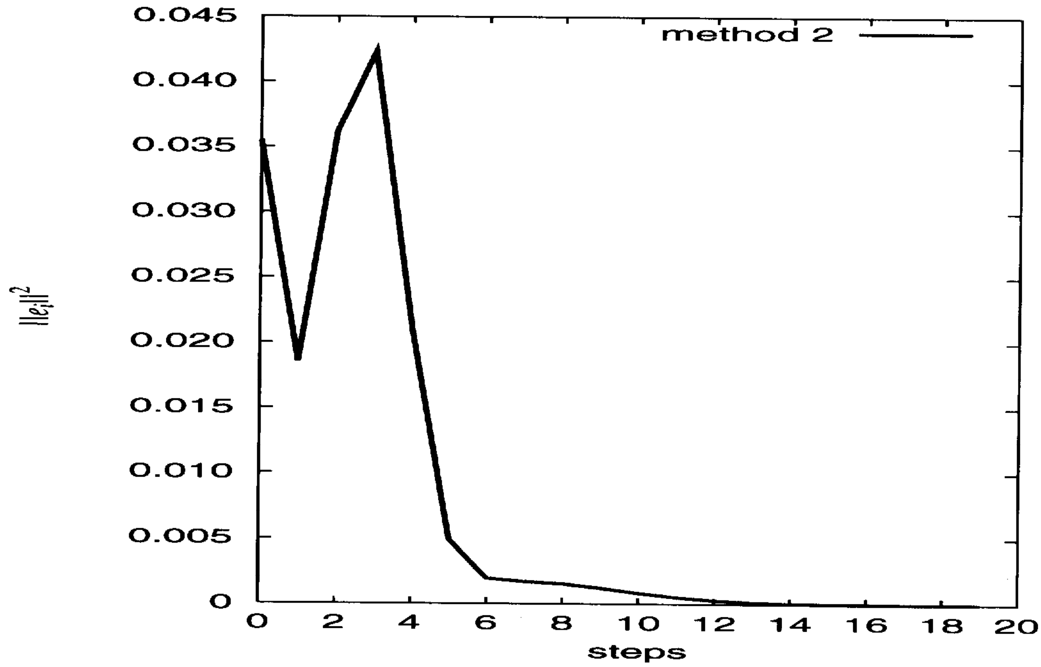


Figure 3.9: Tracking error : $||e_i||^2$

3.5 Application to Vibration Isolation Systems

We applied the proposed method to the control of an active vibration isolation system. Figure 3.10 and 3.11 show the configuration of this system and its coordinates. The stage is levitated by 4 air springs. The places indicated 0 - 7 have the air actuators, position and acceleration sensors.

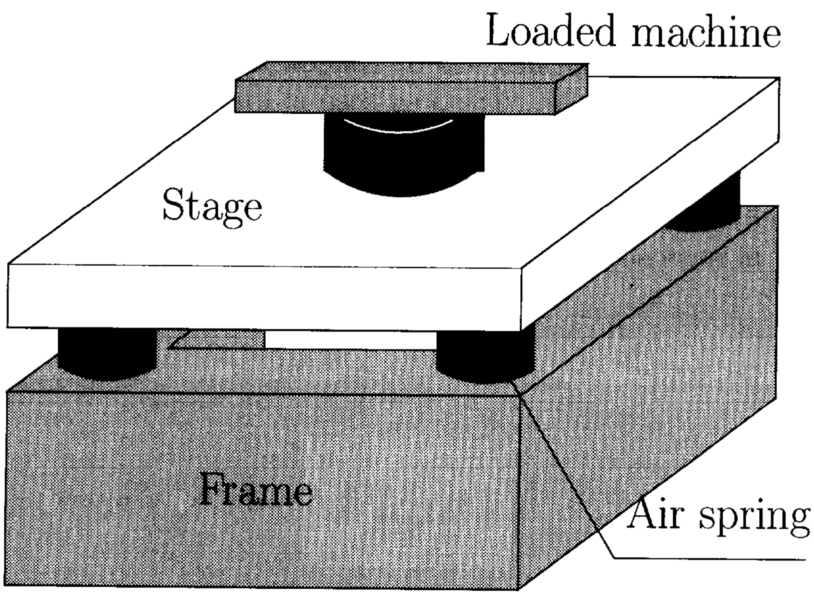


Figure 3.10: Active vibration isolation system

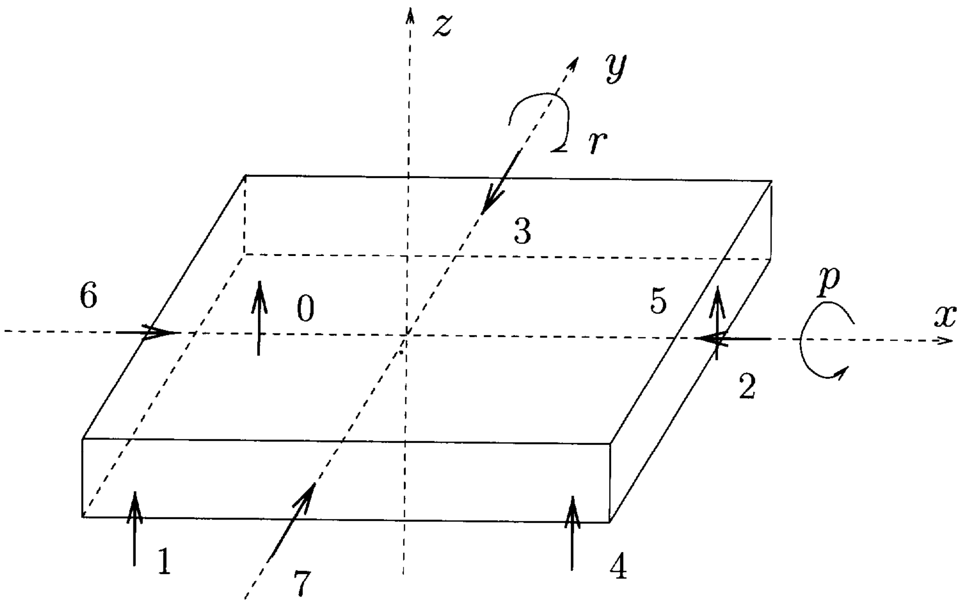


Figure 3.11: Actuating and sensing points of stage

The main purpose of this system is to realize environments that the loaded machine (e.g. X-Y stage to process LSI devices) can move precisely, and to decrease the vibration which is occurred by the machine. Many control methods have been designed in order to isolate the vibration from ground. Actually, because of the existence of the direct vibration from the machine, the isolation is disturbed by the motion.

Therefore, to overcome this problem we utilize the **method 2** and try to remove the effect from the machine. Figure 3.12 shows the block-diagram of this system which has a minor PI and H_∞ controller. The purpose of PI controller is to levitate the stage to a referenced level, and that of H_∞ controller is to isolate the vibration from the ground.

In this experiment, **Method2** is applied to the control of the horizontal direction (y in Figure 3.11) and only position data is used as output, so the system becomes SISO system. The criterion's weights Q and R were chosen as $Q = I,\ R = I$ through the experiment.

Figure 3.13 shows that $||z_i||$ is monotonously descending until 5-th iteration. Figure 3.14 indicates that $||e_i||$ decrease to the lowest value at 5-th step. Figure 3.15 shows the position of the system at step 0, 1, and 5. Using the proposed method, the error at step 5 is three times smaller than those at step 0.

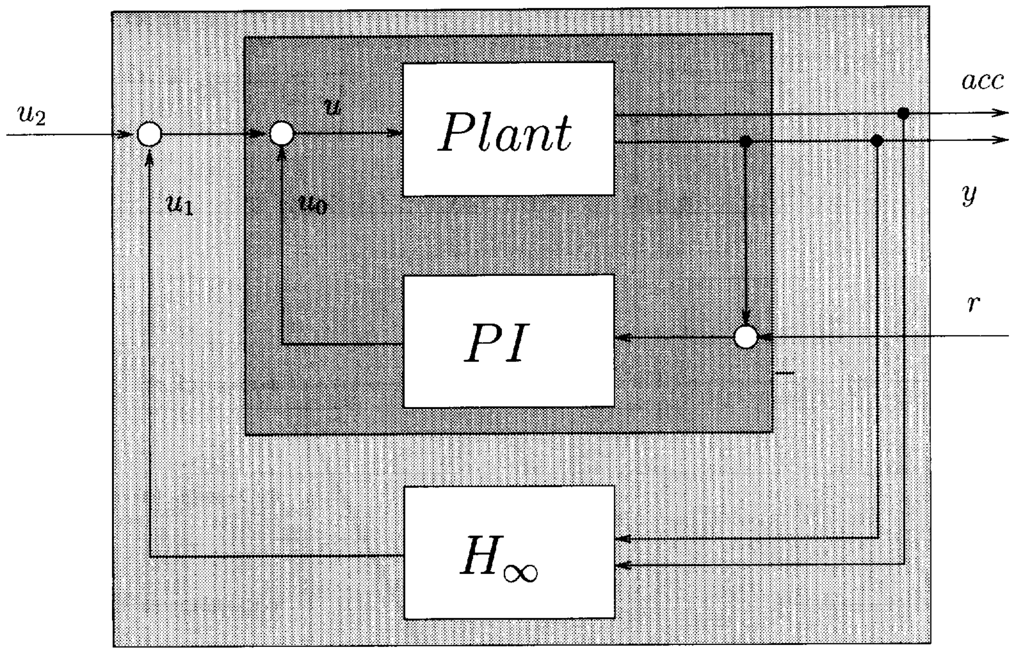


Figure 3.12: Minor-feedback loop

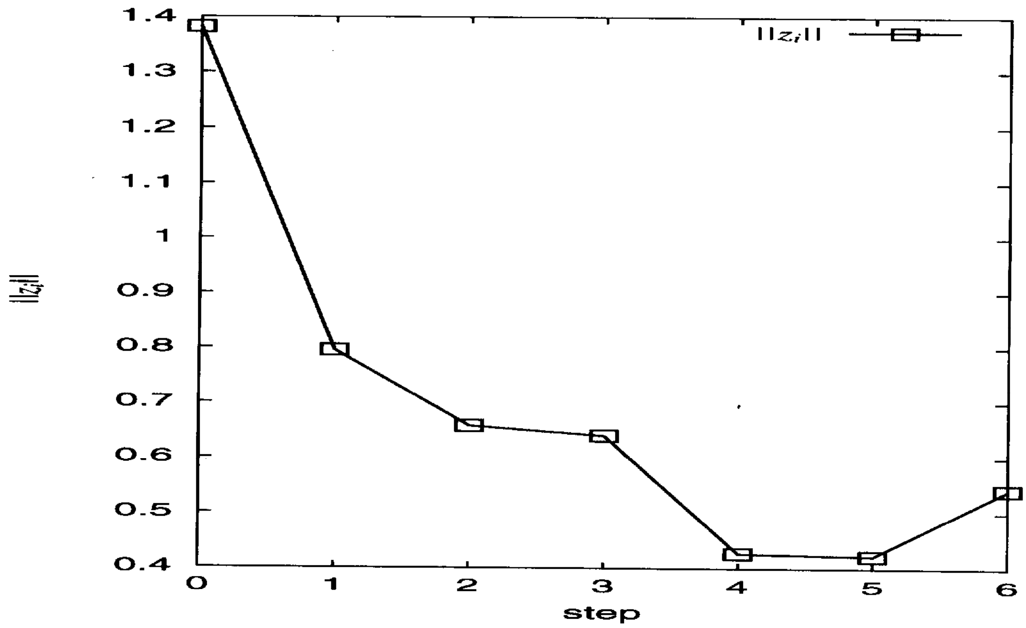


Figure 3.13: Cost function value : $\|z_i\|$

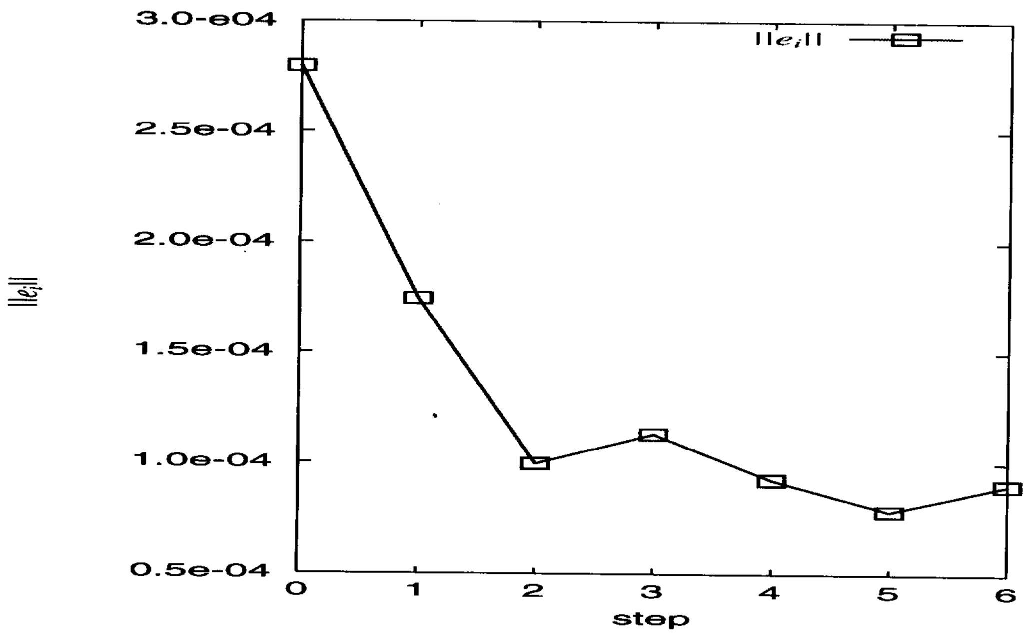


Figure 3.14: Tracking error : $\|e_i\|$

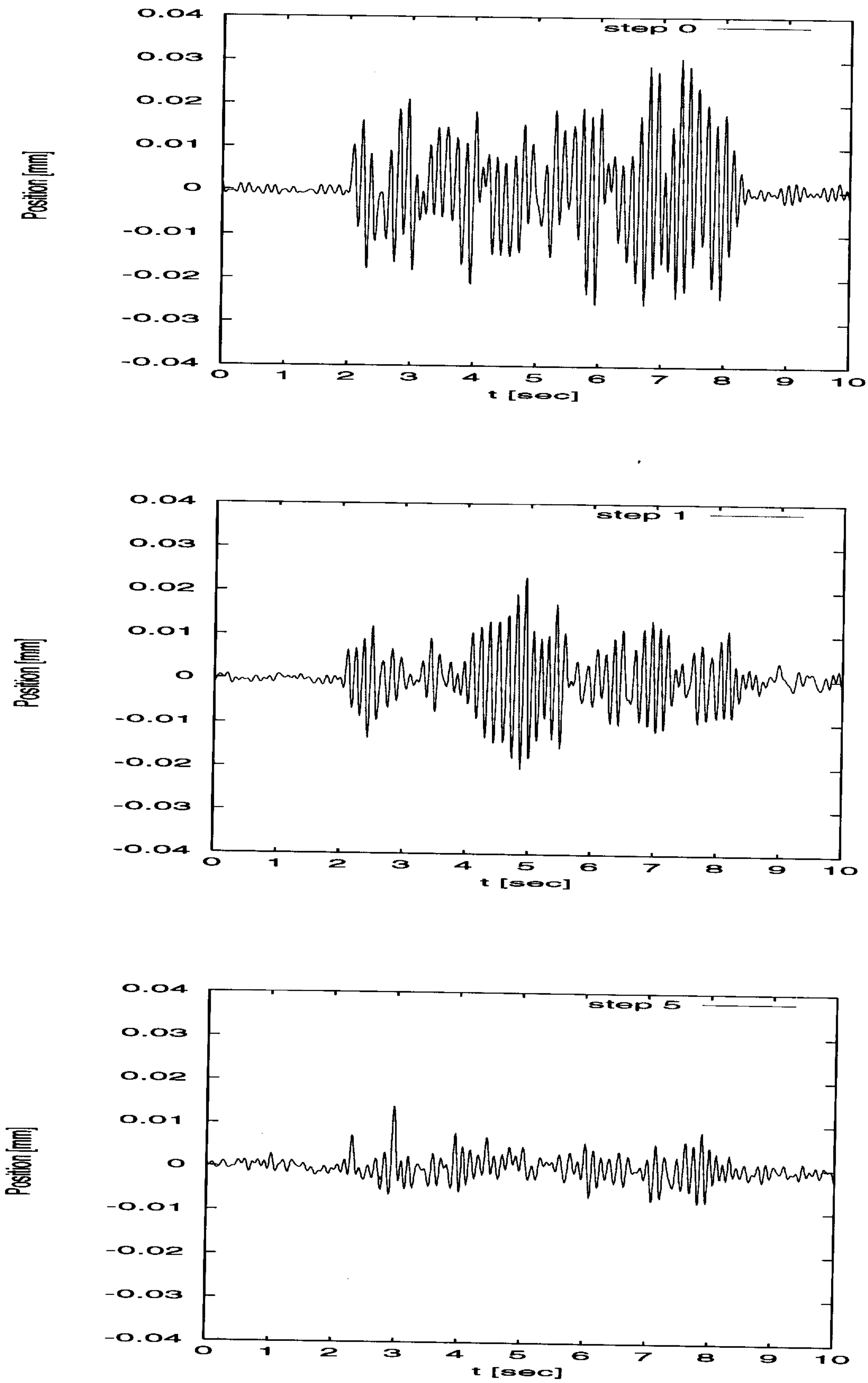


Figure 3.15: Position of the stage by tracking control (step 0, 1, 5)

3.6 Conclusions

We presented adaptive learning control methods using Markov parameter which provides input and parameter update simultaneously. Simulation results showed that these methods are effective on the systems which have modeling error and/or whose parameter are unknown, and the control performance is sufficiently good. Furthermore we illustrated the effectiveness of the proposed methods by applying them to a vibration isolation system.

Chapter 4

Time Optimal Control Design with Nonlinear Optimization

4.1 Introduction

Time optimal control under bounded input was studied and formulated by Pontryagin *et al.* [31] by his Maximum Principle giving the necessary condition of optimality, but it is not easy to obtain solutions for general nonlinear systems. It is reported that analytic solutions are impractical if the dimension of a system is higher than three [5]. Therefore, many numerical methods [9] [33] have been developed, and some new approaches such as the evolutionary approach [55] have been studied recently. Even for numerical solutions, difficulty arises when a system has uncontrollable states. The necessary conditions of optimality in the above case is studied in [1].

This chapter proposes a new computational approach using nonlinear optimization to attain a numerical solution for the time optimal control of a class of nonlinear systems such as the swing-up control of the rotating type pendulum.

While the time optimal control problem is generally defined as minimization of the terminal time to attain a desirable state under bounded input amplitude, the algorithm to solve is known complicated. Thus, we introduced an auxiliary problem of the time optimal control. It is to find the minimal input amplitude numerically so that the terminal state satisfies a specification at a given terminal time. We formulate the time optimal swing-up control according to the auxiliary problem, and solve it using the nonlinear optimization method. Furthermore, this approach leads to the next chapter with extending the targets to the linear time invariant discrete-time systems. There, it shall be shown that time optimal control and several similar optimal controls for linear system can be designed by the Linear Programming using only some system responses.

However, unfortunately, it is difficult to prove that the obtained solution is time optimal, because no sufficient condition of optimality is given in general. But, solutions satisfying the necessary condition are usually assumed to be optimal in practical sense [7] [29].

In this chapter, we consider the simplified model of a 2nd-order pendulum to verify our approach through numerical simulations, and ascertain that the simulation results of the time optimal control computed by the proposed method satisfy the necessary condition of optimality. Furthermore, this chapter presents an experimental result of a rotating type pendulum which is hinged to an arm connected to a direct-drive motor. This type of the rotating pendulum is called Furuta Pendulum developed in the Furuta Laboratory of Tokyo Institute of Technology

[2] [35].

Pendulums are known to have intrinsically interesting behaviors, and have been research objects for many researchers including Galileo Galilei. The swinging of a pendulum has been studied mathematically for a long time, and its control to keep upright state was paid attention to in early 60's in an analogy to the control of the launching rocket. When we consider the control of a pendulum connected to a moving cart by a hinge, the control problem is not only keeping it upright but also swinging up from the pendant to the upright position. Such control has been known practically difficult since the controlled system, whose control input is saturated, is nonlinear, underactuated, and has uncontrollable states.

The swing-up control of a pendulum was first studied by one of the authors [49]. This control law was feedforward and not robust to the change of system parameters. Furuta and others [21] studied the swing-up control of a rotating type pendulum by the bang-bang type feedback law paying attention on the state of the pendulum. Other control laws, for example based on energy, were also reported [2] [21] [26] [41] [50]. However, time optimal swing-up control of a pendulum has not been studied yet.

This chapter is organized as follows. In section 4.2, the algorithm to attain the time optimal control by nonlinear optimization is proposed, and optimality of the attained solution is verified. In section 4.3, the approach was applied to Furuta Pendulum by an experiment. In the last section, the robustness of the proposed approach to the change of system parameters and the comparison with an energy based approach are discussed, and concluding remarks are given.

4.2 Nonlinear Optimization Based on Time Optimal Control

Time optimal control formulated by Maximum Principle gives the necessary condition for optimal control, but an analytical and even a numerical solution is difficult to be determined for nonlinear systems.

On the other hand, it is reasonable to consider that the optimal time can be shorten as input bound is larger, and that the terminate time shall be longer as input bound can be smaller. Based on the consideration, an equivalent problem is proposed here. That is, we iteratively find the minimal input bound with which the state can be transferred from initial to terminal one for a given time interval. To find the minimal input bound, our approach uses nonlinear optimization. If the minimal input bound obtained is larger than a specification, we make the achievable time longer, otherwise make it shorter, and then compute again until the minimal input bound equals to the given one.

4.2.1 Time Optimal Control Problems

Consider a time optimal control problem of a class of nonlinear systems which are linear in input. The problem is to find the minimal achievable time t_f,

$$\min_u t_f \tag{4.1}$$

subject to

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ |u| &\leq u_{max} \\ x(t_0) &= x_0 \\ x(t_f) &= x_f \end{cases} \tag{4.2}$$

where u_{max} is the specification giving input bound, and

$x \in R^n$	:	the state vector	
$u \in R^m$	:	the control input	($m = 1$ here)
$f(\cdot),\ g(\cdot) \in R^n$	:	function vectors	
$x_0 \in R^n$	:	the initial state	
$x_f \in R^n$	:	the terminal state	

Obviously, as the input bound u_{max} is smaller, the minimal achievable time t_f becomes longer, otherwise, as u_{max} is bigger, t_f becomes shorter. Based on this observation, we consider an auxiliary problem of the time optimal control problem.

4.2.2 Auxiliary Problems of Time Optimal Control

An auxiliary problem is to search the minimal input bound so that the terminal state satisfies a specification at a given time. A numerical, iterative algorithm to solve this problem is proposed here. It is described as

1. Solve the following auxiliary problem:

$$\min \max_{t \in [t_0\ t_f]} |u(t)| \tag{4.3}$$

   subject to

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ x(t_0) &= x_0 \\ x(t_f) &= x_f \end{cases} \tag{4.4}$$

   where t_f is given.
2. If the minimal input bound obtained satisfies the specification, terminate this algorithm.
3. If the obtained bound is larger than the specification, lengthen t_f, otherwise shorten t_f, and then repeat from 1.

In order to solve this problem, our approach uses the nonlinear optimization:

$$\min_{\tilde{x}} h(\tilde{x}) \tag{4.5}$$

subject to

$$\begin{cases} c(\tilde{x}) &\leq 0 \\ c_{eq}(\tilde{x}) &= 0 \\ \\ A\,\tilde{x} &\leq b \\ A_{eq}\,\tilde{x} &= b_{eq} \\ \\ l_b \leq & \tilde{x} \leq u_b \end{cases} \tag{4.6}$$

where $\tilde{x}$ is a vector of variables, $h(\cdot)$ is a scalar objective function which may be linear or nonlinear, $c(\cdot)$ and $c_{eq}(\cdot)$ are function vectors as the nonlinear constraints, A and A_{eq} are constant matrices, b and b_{eq} are constant vectors as the linear constraints, each l_b and u_b is the lower and the upper bound vector. The inequality of vectors means that each row of vectors must satisfy the inequality.
To describe the auxiliary problem as a nonlinear optimization one, we need discretize the problem as follows:

$$\begin{aligned} t_k &\triangleq k\frac{t_f}{N} = kh \quad k = 1,\ 2,\ ...,\ N \\ x_k &\triangleq x(t_k) \\ u_k &\triangleq \mu v_k \end{aligned} \tag{4.7}$$

and

$$\tilde{x} \triangleq \begin{bmatrix} \mu & x_0^T & x_1^T & v_1 & x_2^T & v_2 & ... & x_N^T & v_N \end{bmatrix}^T \in R^{(n+1)(N+1)} \tag{4.8}$$

where either N or h is given, and input u_k is separated into the amplitude μ and the normalized input v_k. In our case, v_k are scalar because only single input case is considered. Obviously, the solution is more accurate as h is smaller, however much more computing time is required. Although there are many solvers of nonlinear optimization, we used one named **fmincon** in Optimization Toolbox of Matlab. The constraints of the initial and terminal states can be represented by the linear equality, and the state transform on every time interval by the nonlinear equality, i.e.

$$\begin{aligned} c_{eq,k}(\tilde{x}) &\triangleq -x_k + x_{k-1} + \int_{t_{k-1}}^{t_k} f(x)dt + \mu v_k \int_{t_{k-1}}^{t_k} g(x)dt \\ &= 0 \qquad\qquad \text{for } k = 1, 2, ..., N \end{aligned} \tag{4.9}$$

No inequality constraint is used in the problem. Lower and upper bound constraints play an important role in the optimization. Specifically, following two algorithms can be considered.

Algorithm 1:

$$\min_{\tilde{x}} \max_{1\le k\le N} |v_k| = \min_{\tilde{x}} \max_{1\le k\le N} \left|\tilde{x}_{(n+1)k+1:(n+1)k+1}\right| \tag{4.10}$$

subject to

$$\begin{cases} c_{eq}(\tilde{x}) = \begin{bmatrix} c_{eq,1}^T(\tilde{x}) & c_{eq,2}^T(\tilde{x}) & ... & c_{eq,N}^T(\tilde{x}) \end{bmatrix}^T = 0 \\ \tilde{x}_{2:n+1} = x_0 \\ \tilde{x}_{(n+1)N:(n+1)N+n} = x_N = x_f \\ \mu = 1 \\ -\infty < x_k < +\infty \\ -\infty \le v_k \le +\infty \end{cases} \tag{4.11}$$

where $\tilde{x}$ and $c_{eq,k}(\tilde{x})$ are defined in (4.8) and (4.9), respectively. $\tilde{x}_{i:j}$ is a subvector consisting of elements between the i-th and the j-th row. If we consider the bang-bang form of time optimal control, the another algorithm can be described as follows:

Algorithm 2:

$$\min_{\tilde{x}} \mu = \min_{\tilde{x}} \tilde{x}_{1:1} \tag{4.12}$$

subject to

$$\begin{cases} c_{eq}(\tilde{x}) = \left[\; c^T_{eq,1}(\tilde{x}) \;\; c^T_{eq,2}(\tilde{x}) \;\; \ldots \;\; c^T_{eq,N}(\tilde{x}) \; \right]^T = 0 \\ \tilde{x}_{2:n+1} = x_0 \\ \tilde{x}_{(n+1)N:(n+1)N+n} = x_N = x_f \\ -\infty < \; x_k \; < +\infty \\ -1 \leq \; v_k \; \leq 1 \\ 0 < \; \mu \; < +\infty \end{cases} \tag{4.13}$$

where μ is the maximal input magnitude for any input sequence $\{u_k\}_{k=1,\cdots,N}$ from (4.7), i.e.

$$\mu \triangleq \max_{1 \leq k \leq N} |u_k| \tag{4.14}$$

Some practical computations reveal that the Algorithm 2 converges more quickly than the Algorithm 1, whereas, that the Algorithm 1 is less dependent on the initial condition. It is known that nonlinear optimization depends on the initial condition, and that giving a satisfactory initial condition is hard. Therefore, in our approach, the Algorithm 1 is used to attain a feasible control for the initial condition $\{u_k = 0\}_{k=1,\cdots,N}$, and then, the feasible control is used as an initial condition of the Algorithm 2 to attain the optimal solution quickly.

4.2.3 Verification of the Proposed Approach

Generally, the analytical proof that a obtained solution is time optimal is known difficult. Even optimality of a numerical solution is hard to be verified, because no sufficient condition is given in general. Practically, a control satisfying the necessary condition, Pontryagin's Maximum Principle, is usually considered time optimal [7][29].

To show that our approach really gives the time optimal solution, we check it through two numerical simulations. First, this approach is applied to a 2nd-order simplified model of a single pendulum, and the result is compared with the time optimal control computed using the Linear Programming approach. To verify whether the solution satisfies Pontryagin's Maximum Principle, we propose utilizing a switching function which determines the switching condition of the optimal input. Second, we check the optimal solution of the 4th-order real pendulum model similar to the simplified model.

Comparison Using Simplified Model of a Single Pendulum

Consider the simplified model of a single pendulum, which is a 2nd-order system:

$$\ddot{\theta} = \sin\theta + \cos\theta u \tag{4.15}$$

where θ is an angle of the pendulum to the vertical line, and u is equivalent torque. (4.15) is nonlinear, and is uncontrollable at $\theta = k\pi + \frac{1}{2}\pi$.

Now, let us apply the proposed approach to (4.15) in order to find the time optimal swing-up control from the pendant, $\left[\theta,\ \dot{\theta}\right] = [\pi,\ 0]$, to the upright position, $\left[\theta,\ \dot{\theta}\right] = [0,\ 0]$. The result of this simulation is shown in Fig. 4.1, where the solid line is the result of the proposed approach, and the dotted line is the one based on Linear Programming appoaches in [23] [47].

From Fig. 4.1, it is observed that these two solutions are almost the same. In fact, for the condition of the terminal time of 5.7 seconds, the minimal input bound of the nonlinear optimization and the linear programming approach was 0.9793 and 0.9966, respectively.

The bound given by the linear programming is slightly larger than the one of nonlinear optimization because of the fact that the linear programming case has the non-control regions. It is considered that the non-control regions are brought by the uncontrollable states. The optimal control should be the bang-bang form from Maximum Principle, and the proposed approach has no non-control regions, and is almost the bang-bang form, therefore, it seems to be the time optimal control.

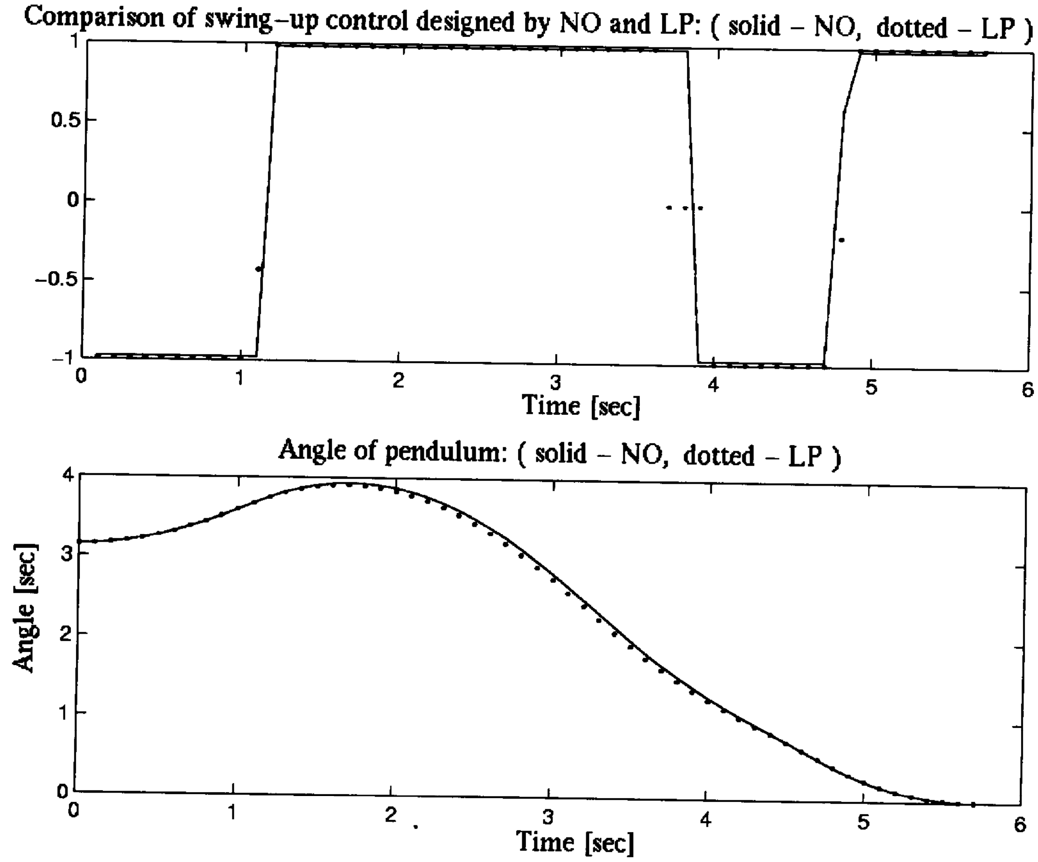


Figure 4.1: Comparison of time optimal control using nonlinear optimization (NO) and linear programming (LP): The upper shows the optimal input, and the lower shows the angle of the pendulum, where the solid line is of NO case and the dotted line is of LP case.

Verification of Necessary Condition

A solution not satisfying Pontryagin's Maximum Principle is not optimal, but all satisfying solutions may not be optimal, because the principle gives only the necessary condition. However, since the optimal control will be unique if exists, the solution satisfying the necessary condition may be considered optimal [29]. We examine that the solution given by our approach satisfies the necessary condition. In order to check the condition, the methodology in [29] is used, and is summarized as follows.

Criterion of (4.1) is equivalent to

$$J(t_0) = \int_{t_0}^{t_f} 1dt \tag{4.16}$$

and Hamiltonian for system (4.2) is

$$H(x, u, p) = 1 + p^T [f(x) + g(x)u] \tag{4.17}$$

where p is a co-state of system (4.2) and satisfies

$$\dot{p} = -\frac{\partial H}{\partial x} \tag{4.18}$$

Pontryagin's Minimum Principle states that the necessary condition is to minimize the Hamiltonian:

$$H(x^*, u^*, p^*) \leq H(x^*, u, p^*) \quad \text{for all admissible } u \tag{4.19}$$

where the superscript * denotes an optimal quantity. In addition, we have

$$H(x, u, p) = 0 \qquad t_0 \leq t \leq t_f \tag{4.20}$$

from [10]. From (4.2), (4.17), (4.19), the optimal control is represented as follows:

$$u^*(t) = \begin{cases} +u_{max} & \text{if } \; p^T g(\cdot) < 0 \\ (-u_{max} \;\; +u_{max}) & \text{if } \; p^T g(\cdot) = 0 \\ -u_{max} & \text{if } \; p^T g(\cdot) > 0 \end{cases} \tag{4.21}$$

where $p^T(t)g(x(t))$ is called a switching function. If the switching function is equal to zero over some time interval, (4.21) does not determine an unique solution, in which case the time optimal control is called singular. Conversely, if the switching function is zero at isolated times, (4.21) determines a bang-bang control, in which case it is called non-singular.

Here, we use (4.21) to check the necessary condition. To utilize (4.21), the initial co-state $p(t_0)$ is required. The solution of the linear adjoint differential equation (4.18) is given by

$$p(t) = \Phi(t, t_0)p(t_0) \tag{4.22}$$

where $\Phi(t, t_0)$ is the transition matrix associated with (4.18). Since both $x(t)$ and $u(t)$ are known, $\Phi(t, t_0)$ can be computed. If the input switches at time t_{si}, the corresponding switching function must satisfy

$$p^T(t_{si})g(x(t_{si})) = 0 \tag{4.23}$$

i.e.

$$p^T(t_0)\Phi^T(t_{si}, t_0)g(x(t_{si})) = 0 \tag{4.24}$$

at $t = t_{si}$. Furthermore, according to (4.17) and (4.20), the initial co-state must satisfy

$$p^T(t_0)\left[f(x_0) + g(x_0)u(t_0)\right] = -1 \tag{4.25}$$

Consider the case that the input u switches k times, then a non-homogeneous system consisting of $k+1$ linear equations is defined by both (4.24) and (4.25) with n unknown initial co-states $p(t_0)$. If $k+1 = n$, the system has a unique solution as long as these $k+1$ equations are linearly independent. If $k+1 > n$, the system has a solution as long as at least $k+1-n$ equations is linearly dependent, otherwise it has no solution. Finally, if $k+1 < n$, a set of solutions exists. Thus, it is claimed in [29] that the switching times is almost not more than $n-1$. However, in the previous example of the simplified single pendulum, the model is of 2nd-order dimension, while the input switches 3 times. It is considered that this is possibly due to the uncontrollable states of the system. The verification of the previous example is illustrated in Fig. 4.2. The initial co-state was computed from the state at the first switching time in reverse time. Obviously, we can confirm that left two switchings of input satisfy the condition derived by the switching function, therefore, the control is considered time optimal.

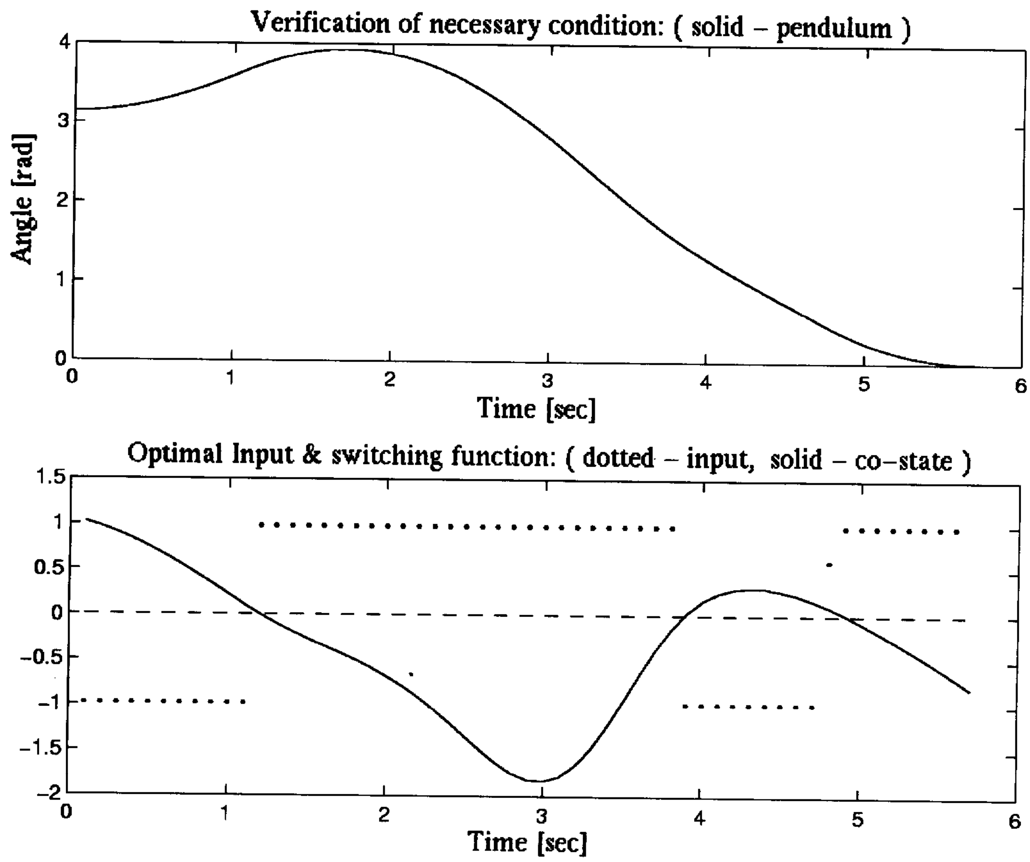


Figure 4.2: Verification of necessary condition using simplified pendulum model: the upper is the angle as a result of optimal swing-up control, and the lower shows the optimal input (dotted line) and the corresponding switching function (solid line).

Table 4.1: Identified parameters of Furuta Pendulum

Physical quantity	Symbol	Units
Arm's length	L_0	2.2343×10^{-2} [m]
Arm's inertia	I_0	2.5617×10^{-2} [kg m^2]
Viscous friction coefficient	C_{0v}	4.3281×10^{-2} [Nms]
Coulomb friction coefficient	C_{0c}	3.5686×10^{-1} [N]
Mass	m_1	8.3077×10^{-2} [kg]
Pendulum's length	l_1	1.0364×10^{-1} [m]
Pendulum's inertia	J_1	2.7684×10^{-4} [kg m^2]
Viscous friction coefficient	C_{1v}	2.4577×10^{-4} [Nms]
Threshold of Dead zone	v_d	0.68
Propotional constant	k_v	1.7391

In addition, the same verification is applied to the following real pendulum (4.26).

$$\begin{bmatrix} p_1 + p_2 \sin^2\theta_1 & p_3\cos\theta_1 \\ p_3\cos\theta_1 & p_4 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} C_{0v} + \frac{1}{2}p_2\sin(2\theta_1)\dot{\theta}_1 & -p_3\sin\theta_1\dot{\theta}_1 + \frac{1}{2}p_2\sin(2\theta_1)\dot{\theta}_0 \\ -\frac{1}{2}p_2\sin(2\theta_1)\dot{\theta}_0 & C_{1v} \end{bmatrix} \begin{bmatrix} \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} C_{0c}\,\mathrm{sign}(\dot{\theta}_0) \\ -p_5\sin\theta_1 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{4.26}$$

where p_i are defined as

$$\begin{aligned} p_1 &= I_0 + m_1 L_0^2 \\ p_2 &= m_1 l_1^2 \\ p_3 &= m_1 l_1 L_0 \\ p_4 &= J_1 + m_1 l_1^2 \\ p_5 &= m_1 l_1 g \end{aligned} \tag{4.27}$$

By linearizing (4.26) around the unstable equilibrium point $\left[\theta_1,\ \dot{\theta}_1,\ \dot{\theta}_0\right] = [0,\ 0,\ 0]$, the linear model is

$$\begin{bmatrix} p_1 & p_3 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} C_{ov} & 0 \\ 0 & C_{1v} \end{bmatrix} \begin{bmatrix} \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -p_5 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{4.28}$$

In this verification, for simplicity, no friction is considered, and model parameters were used in Table 4.1. Simulation arguments of nonlinear programming was chosen as $t_f = 0.95$, $h = 0.025$, $N = 38$.

Its solution and corresponding responses are depicted in Fig. 4.3, where the initial co-state $p(t_0)$ is computed according to the first three switching times as follow:

$$p(t_0) = \begin{bmatrix} -0.3040 & -0.0932 & 0.0225 & -0.0635 \end{bmatrix}^T$$

This system is of 4th-order while the solution has four switching times. It is observed that the fourth switching time coincides with the fourth zero crossing time of the switching function.

We can therefore conclude that Pontryagin's Maximum Principle is satisfied in this case, and can consider that our approach gives time optimal control.

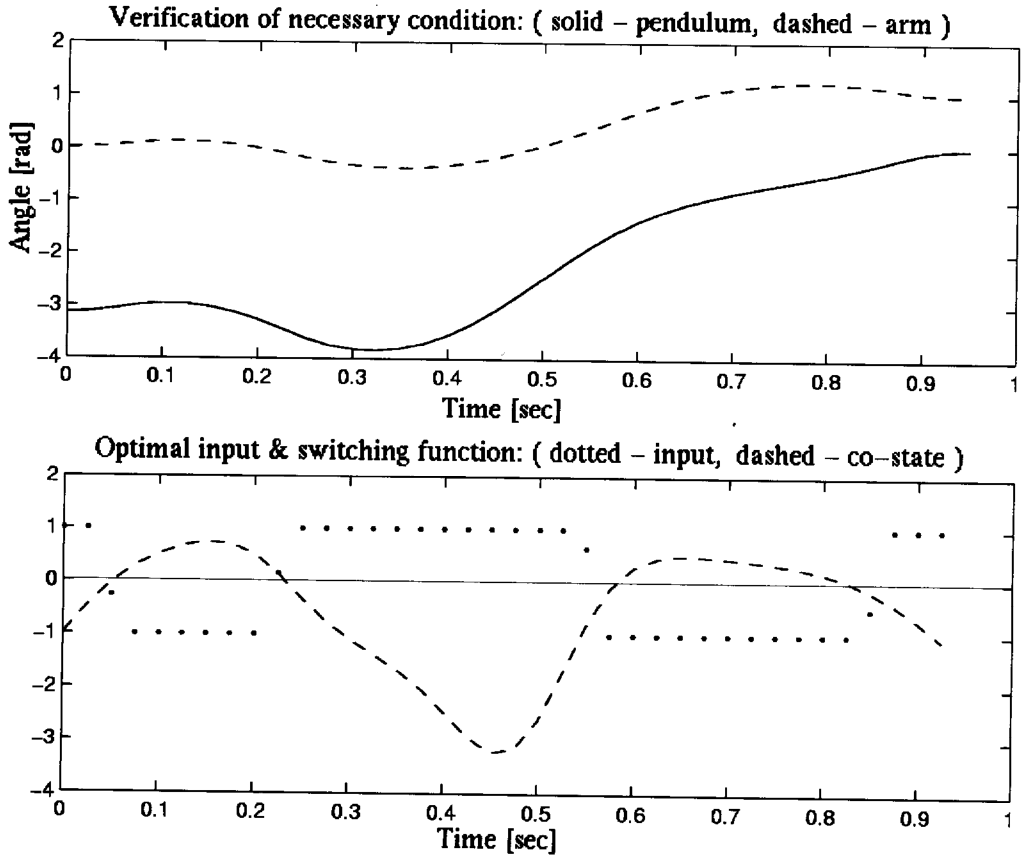


Figure 4.3: Verification of necessary condition using real pendulum model (4.26): the upper shows the angles of the pendulum (θ_1: solid line) and the arm (θ_0: dotted line). The lower shows the optimal input (dotted line) and the switching function (dashed line).

4.3 Experiment on Furuta Pendulum

This section gives experimental results of Furuta Pendulum based on the proposed approach. If a model of the pendulum is accurately described, the pendulum can swing up to the upright position where the velocity is zero by only the feedforward control. However, since modeling errors, parameter estimation errors and disturbances are unavoidable in practice, the pendulum cannot arrive at the upright position with exactly zero velocity, and it cannot stay at the unstable equilibrium point.

Therefore, to swing up to the upright position and keep the position, swing-up control is divided into 2 steps. The first step is to let the pendulum swing up to a neighborhood of the upright position by feedforward time optimal control designed by the proposed approach. The second step is to stabilize the pendulum in the neighborhood by LQ regulator where the linearized model (4.28) is used.

4.3.1 Feedforward Time Optimal Control

The time optimal control is attained by applying the proposed approach to Furuta Pendulum (4.26). The computed result is shown in Fig. 4.4, where the initial and final states are

$$\begin{aligned} \left[\ \theta_0(0)\ \ \theta_1(0)\ \ \dot\theta_0(0)\ \ \dot\theta_1(0)\ \right]^T &= \left[\ 0\ \ -\pi\ \ 0\ \ 0\ \right]^T \\ \left[\ \theta_0(t_f)\ \ \theta_1(t_f)\ \ \dot\theta_0(t_f)\ \ \dot\theta_1(t_f)\ \right]^T &= \left[\ 0\ \ 0\ \ 0\ \ 0\ \right]^T \end{aligned}$$

where the terminal time and the time interval were chosen as $t_f = 0.95$[s] and $h = 0.05$[s], respectively.

The attained minimal input bound was

$$\tau_m = 1.4355$$

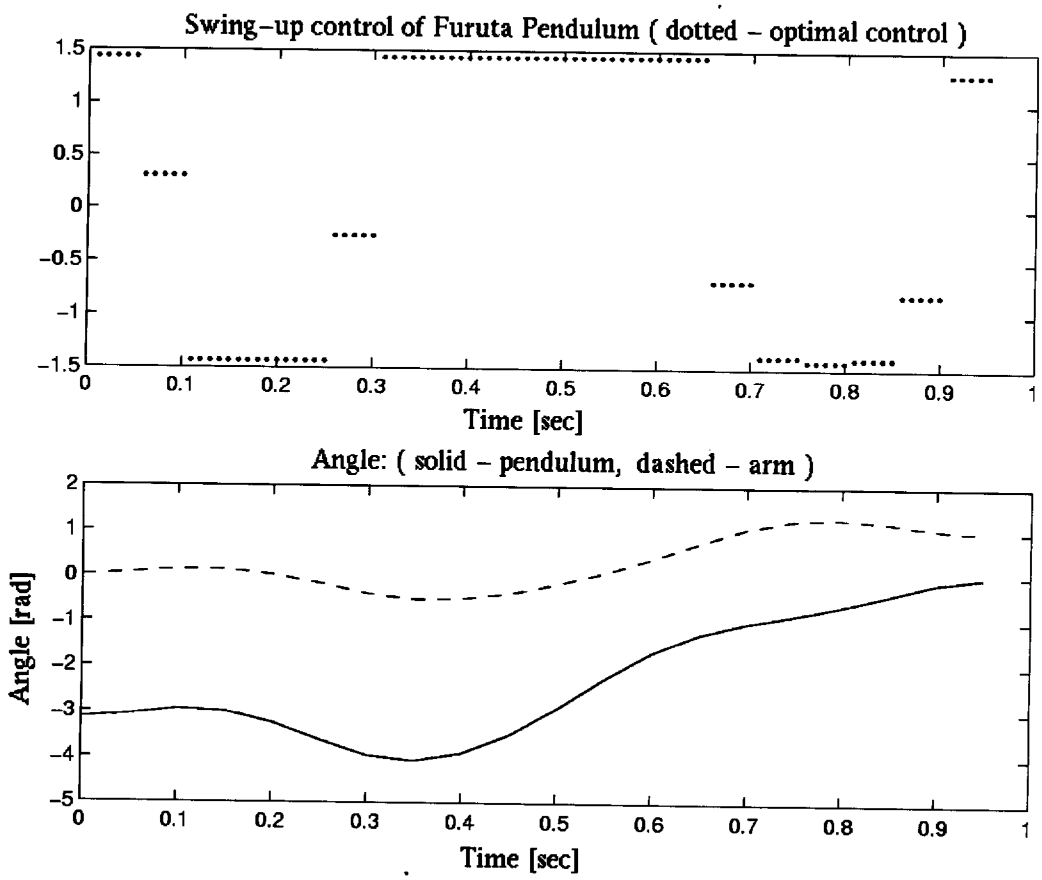


Figure 4.4: Design of time optimal swing-up control of Furuta Pendulum: the upper is the designed input for swing-up, and the lower shows angles of the pendulum (θ_1: solid line) and the arm (θ_0: dashed line) in swing-up motion.

4.3.2 Feedback Stabilization Control

LQ type state feedback control based on the linearized model (4.28) was chosen for stabilization. The quadratic criterion function J was

$$J = \int_0^\infty (x^T Q x + r u^2) dt \tag{4.29}$$

where x and u stand for $[\theta_0\ \theta_1\ \dot{\theta}_0\ \dot{\theta}_1]^T$ and τ, respectively. Q and r were specified as:

$$\begin{aligned} Q &= \mathrm{diag}\left(\ 30 \quad 2000 \quad 0.1 \quad 10\ \right)^T \\ r &= 1 \end{aligned} \tag{4.30}$$

The designed optimal stabilizing feedback was

$$\begin{aligned} u &= f^T x \\ f &= \left[\ 5.4772 \quad 61.5627 \quad 3.9337 \quad 5.8110\ \right]^T \end{aligned} \tag{4.31}$$

4.3.3 Results of the Experiment

If $|\theta_1| < 0.5$[rad] is satisfied, the control stage is changed from the swing-up control to the stabilizing control. The experimental and simulated results are shown in Fig. 4.5 – Fig. 4.7 corresponding to θ_1, θ_0 and input τ, respectively.

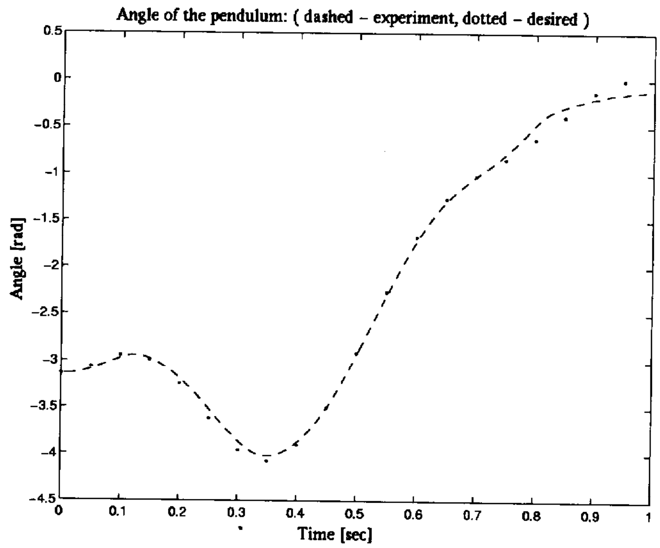


Figure 4.5: Result of experiment: the dotted line shows the computed trajectory of the pendulum angle θ_1. The dashed line is of the experimental result.

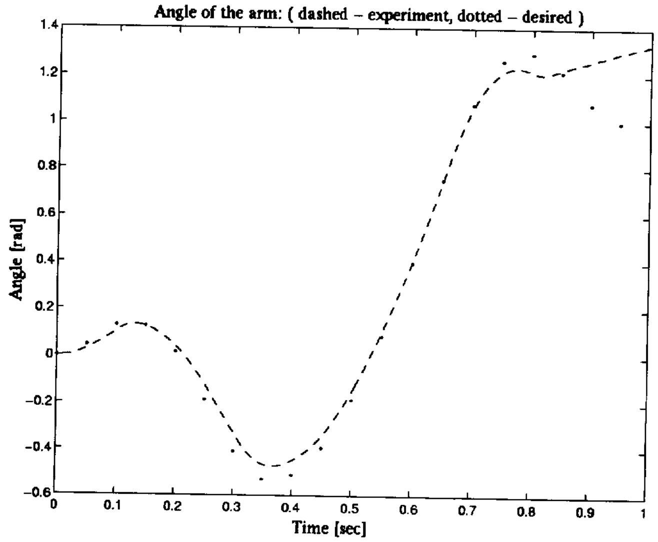


Figure 4.6: Result of experiment: the dotted line shows the computed trajectory of the arm rotation angle θ_0. The dashed line is of the experimental result.

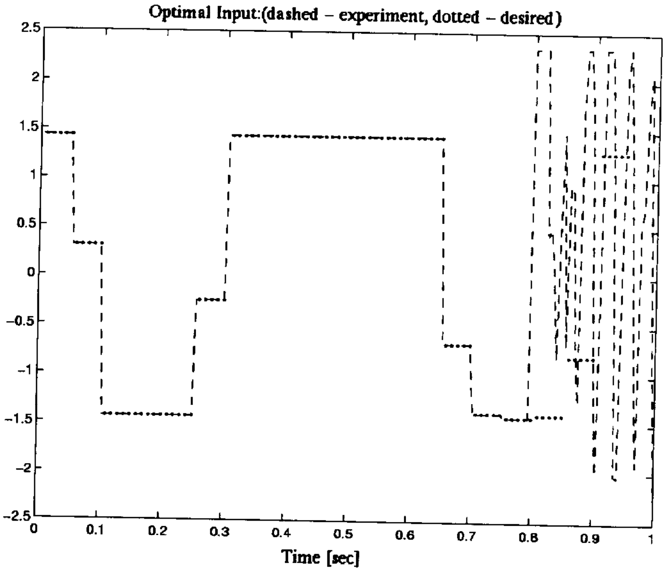
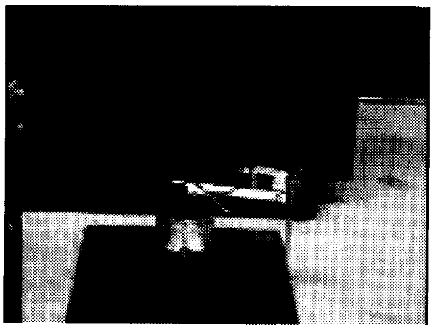


Figure 4.7: Optimal input used in experiment: the dotted line shows the designed input. The dashed line shows the input used in experiment.

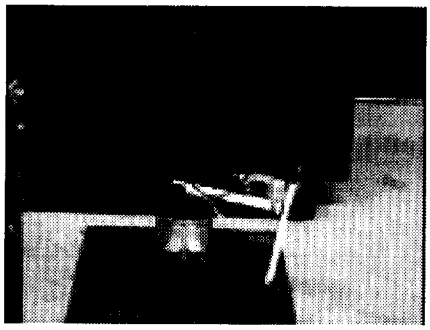
From the experimental results, it is observed that the angles θ_1 and θ_2 track the simulated trajectories, and it reveals that the assumptions of modeling are appropriate and the parameter estimation is effective.

Note that in the neighborhood of the upright position, the tracking error, especially θ_0, is big, because the control has been changed to the state feedback stabilizing control. In the weight matrix Q, the weight corresponding to θ_1 is bigger than the one of θ_0, and the tracking error of θ_1 would be smaller than the error of θ_0.

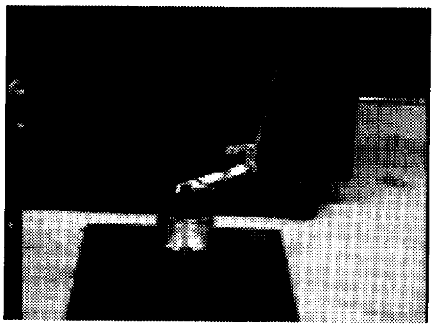
The input of the motor is limited within 2 volts, $|\tau| \leq 2.3476$. Actually, the required torque for the stabilizing control is larger than this limit, this causes the big tracking error around the upright position. The photos of swing-up control motion are shown in Fig. 4.8.



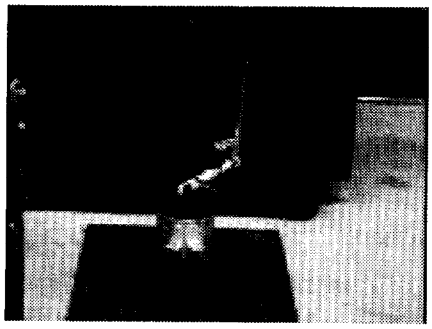
Phase 1: the position of pendulum is pendant



Phase 2: the pendulum begins to swing up



Phase 3: the control strategy is changed to the stabilizing of the pendulum at the neighborhood of upright position



Phase 4: the pendulum is stabilized by the LQ regulator.

Figure 4.8: Photos of swing-up control of Furuta Pendulum

4.4 Discussion and Conclusion

In this section, we summarize our approach, and discuss about merits and demerits. Comparison of our nonlinear optimization based approach with the energy-based is given. A robustness of the proposed feedforward control is also investigated.

4.4.1 Discussion about the Proposed Approach

The problem we focused is to solve time optimal control problem of nonlinear systems, for example, time optimal swing-up control of Furuta Pendulum. In general, solving this problem is known difficult, so we considered the auxiliary problem to find the minimal input bound satisfying the terminal condition of state at a given time. It depends implicitly on the following consideration: It is reasonable to. consider that the optimal time can become shorter as the input bound is larger, otherwise the optimal time shall be longer as the bound can be smaller. Accordingly, we minimize the input bound in stead of the achievable time, and the auxiliary problem can be formulated by the nonlinear optimization solved using many solvers, for example, Matlab.

It is known that solutions of nonlinear optimization depend on the initial conditions. We proposed two algorithms. Some practical computations revealed that the algorithm 1 is less dependent on the initial condition and that the algorithm 2 requires less computational time, and we consider the cause is that the algorithm 1 has no constraint of input v_k, while the algorithm 2 has the bounds of v_k. Therefore, we use the algorithm 1 to find a feasible solution, and then, the solution is used for the initial condition of the algorithm 2, so that we can quickly obtain the optimal solution which is less dependent on the initial condition.

By the way, it may be wonder that the solution of the algorithm 1 becomes like the bang-bang form since the algorithm 1 has no input constraints as mentioned above. Let us suppose that an optimal control obtained according to the algorithm 1 is not of the bang-bang form. In this case, values of the optimal control at some times are less than the value of the minimal input bound, but the values can increase to the value of the minimal input bound. Similarly to the consideration about the auxiliary problem, the achievable time should be shorter if more input can be used. Therefore, the supposed case is not the optimal one, and the values of the optimal control at each time will be equal to the minimal bound of the input, so that it is like the bang-bang form.

4.4.2 Comparison Against Energy-based Approach

Energy-based approaches are effective methods in swing-up control of both the single and the double pendulums. The comparison between our approach and the energy-based one for swing-up control of Furuta Pendulum is given, here. See the appendix A for the detail of the energy-based approach.

The input bound was set at $\tau_m = 1.4355$ as same as in the previous experiment. The simulation result is shown in Fig. 4.9. Obviously, the energy based approach requires more time for swing-up control than the proposed approach. This implies that the control based on our approach might be time optimal. Furthermore, since our approach can choose any terminal state as long as it is suitable, the approach has an advantage that the arm can be also controlled to any desired position, while the position control cannot be taken into consideration in the energy based approach.

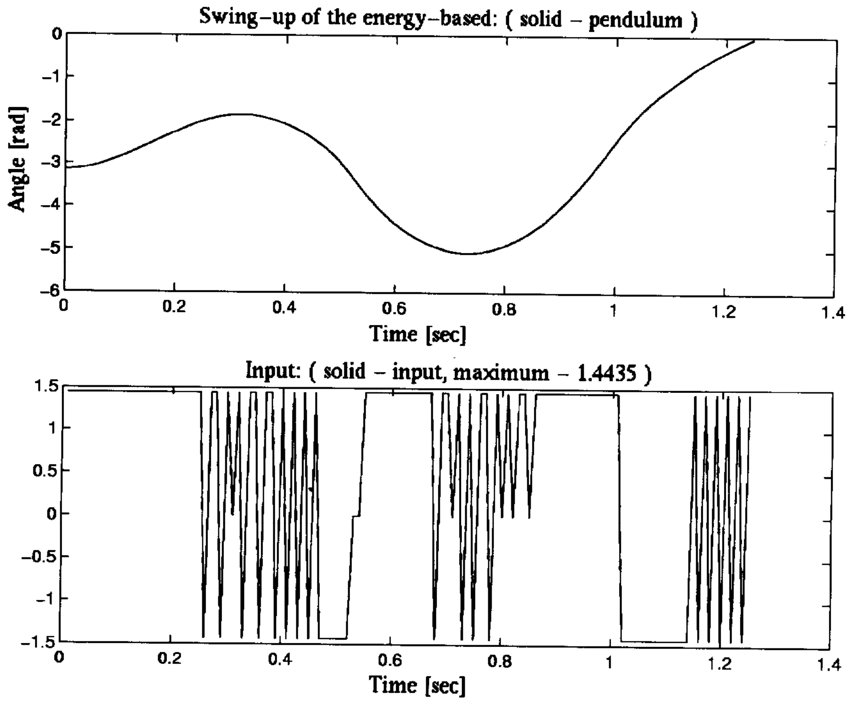


Figure 4.9: Simulation result of swing-up control based on energy-based approach: the upper shows the angle of pendulum, and the lower shows the input

4.4.3 Investigation of Robustness to Parameter Perturbations

It is known that feedforward control requires an accurate model and that it is not robust to parameter perturbations. As reported in [21], swing-up of the pendulum succeeds by a feedforward control if all parameters are correct, otherwise, it fails if an uncertainty is introduced, for example, by attaching a 15g weight to the end point of the pendulum in [21].

Here, robustness of our approach is investigated by introducing the same parametric uncertainty. The nominal and perturbed parameters are listed in Table 4.2.

Table 4.2: Nominal and perturbed parameters of Furuta Pendulum: the values of the perturbed parameters are as a result of attaching a 15g weight to the end point of the pendulum

Symbol	Nominal	Perturbed
p_1	2.9765×10^{-2}	3.0513×10^{-2}
p_2	8.9241×10^{-4}	1.0535×10^{-3}
p_3	1.9238×10^{-3}	2.2712×10^{-3}
p_4	1.1692×10^{-3}	1.3304×10^{-3}
p_5	8.4468×10^{-2}	9.9617×10^{-2}

The same input as shown in Fig. 4.4 is applied to the perturbed system, and the simulation result is shown in Fig. 4.10. It is observed that the pendulum still could swing up to the attraction region of stabilization control, while the swing-up failed in [21]. Of course, it could not swing up to the upright position where $\dot{\theta}_1 = 0$. Note that p_i has been perturbed more than 10%, it is suggested that time optimal control is not so sensitive to uncertainty as other feedforward control.

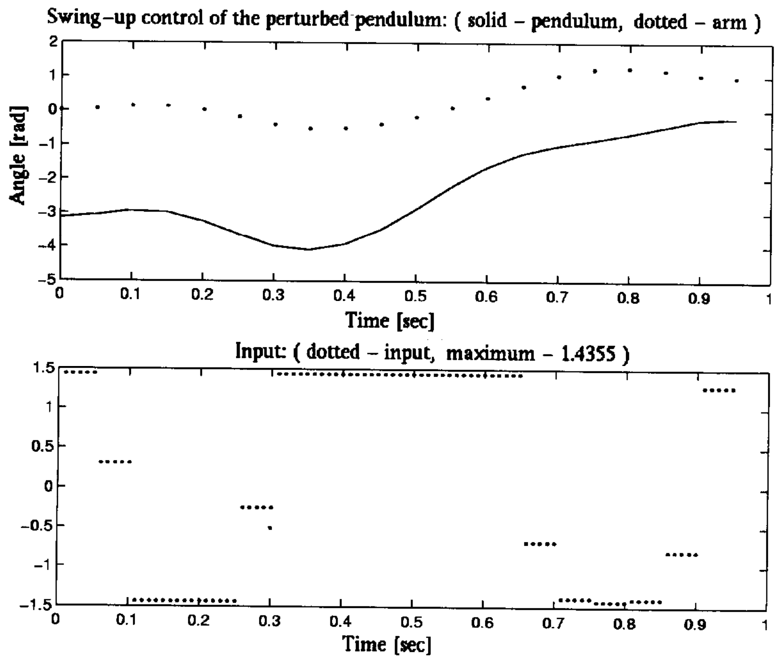


Figure 4.10: Result of swing-up control of the perturbed pendulum: the lower is the same control input as Fig. 4.4. The upper shows the angle of the pendulum (θ_1: solid line) and the arm (θ_0: dotted line)

4.4.4 Concluding Remarks

This chapter proposed a new computational approach to design time optimal control of general nonlinear systems with input bound. To solve the problem is known difficult in general, however, our approach can be computed by an auxiliary problem of finding the minimal input bound satisfying the terminal condition of state at the given time, and it could be solved by usual nonlinear optimization. The approach was applied to the time optimal swing-up control of the rotating type pendulum numerically and experimentally. The numerical solution obtained was verified using a simplified model of a single pendulum, and shown to satisfy the necessary condition of optimality. It was compared with the energy-based approach, and it was observed that our approach could make the pendulum swing up in shorter time and could control not only the angle of the pendulum but also the position of the arm. In addition, this approach was practically experimented on using Furuta Pendulum. In experiment, feedforward control was used to swing up to the neighborhood of the upright position from the pendant, and a conventional LQ regulator was used to stabilize the pendulum in the neighborhood. And then, adding some weight to the end of the pendulum, same experiment was carried out again to investigate the robustness against the parametric uncertainty. From the experiment and the analysis, the proposed approach not only could provide good reference trajectory, but also was less sensitive to parametric uncertainty. Another good feature of this approach is that it can be used for the time optimal control of general nonlinear systems.

Chapter 5

Time and Fuel Optimal Control Design by Linear Programming Using Input-Output Data

5.1 Introduction

Recent competitive development of computers has led to remarkable improvement of its ability, i.e. calculation speed, memory capacity and etc. Consequently, we can easily get these high performance computers, which can carry out an operation needing large memory capacity and which can execute loop routines repeated many times in a short time.

Under this circumstance, the application field of control system design is extended. For example, we can easily and in a short time solve the design problem difficult to be solved analytically and the searching problem of optimal solution needing a lot of repeats by numerical algorithms. Therefore, we should consider control designs more depending on computers in a further step.

In practice, such works have reported in [24][60]. In these works, the time optimal control design for some nonlinear systems is formulated as a numerical optimization problem and it is solved by computational algorithm as shown in the previous chapter.

Meanwhile, if only input-output data is obtained as system information and a control system can be designed directly by a reasonable method closed in input-output data, it is better than the case where the control system is designed on the basis of models constructed by non-parametric identification, because non-parametric identification gives no useful information any longer except approximation of the behaviour of the control object. That is a suggestion considered in the fields of control system designs recently.

In this chapter, in order to show examples of the design methods closed in input-output data, we consider the time optimal control, the fuel optimal control and the mixed time-fuel optimal control formulated as the linear programming problems for finite dimensional, linear time invariant, discrete-time systems. As a result, we can design such optimal control laws easily because these linear programming problems are solved easily by any usual solvers for the linear programming. In the proposed formulation, the system model and the specified initial and terminal conditions are treated together as equality constraints, and the input sequence which we should design is chosen as a variable of the linear programming, so the optimal feedforward control sequence can be obtained directly as the variable optimized by the solver.

The object function of the linear programming should be chosen to reflect the control purpose corresponding to the time optimal control, the optimal fuel control or the mixed optimal time-fuel control. However, time optimal control cannot be formulated as the linear programming in a straightforward way. While time optimal control is chosen out of the control laws satisfying the given conditions so that its terminal time is most short, we cannot formulate the design problem in the linear programming if the terminal time is chosen as the object function. Hence, we need to consider an auxiliary problem formulated in the linear programming for its design problem, and need to solve the original problem by utilizing the auxiliary problem.

Especially about time optimal control, its auxiliary problem is practically set on the analogy of the auxiliary problem, which has been proposed and formulated as a nonlinear optimization problem for nonlinear systems case in the previous chapter. In the mixed time-fuel optimal control case, the weight value is introduced to realize the trade-off between time optimality and fuel optimality. Furthermore, it is shown that these approach can be extended to having some inequality condition about the output.

How to choose the object function is the just point that we shall highlight, and we claim that it is new and easy approach. In this approach, if we were confronted with the case that any model of the object isn't given for design explicitly, the optimal control could be designed from some system responses as long as the responses can be measured. This is the advantage of the proposed method.

To verify the proposed methods, we apply the methods to the design problem of a positioning control for track seek motion of a hard disk drive. On each evaluation of the time optimal, fuel optimal and mixed time-fuel optimal, the optimal input is designed under the input and output restrictions. Simulations results show that each optimal control to satisfy constraints and restrictions can be designed easily by proposed methods.

5.2 Time Optimal Control Design Using Linear Programming

In this section we shall propose an algorithm that time optimal control for linear discrete time systems is designed using the linear programming on the analogy of the case that time optimal control for nonlinear systems is designed by the nonlinear optimization. We shall also show that not only time optimal control but also fuel optimal control, mixed time-fuel optimal control and etc are formulated and solved as the linear programming in later. Furthermore, in this approach, if information of the system, i.e., model and its parameters, is not given explicitly, it is shown that optimal control is designed using only the system responses as long as the responses are measured.

Consider a finite dimensional, linear time invariant, discrete time system (5.1):

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_k \\ y_k &= C x_k + D u_k \end{aligned} \tag{5.1}$$

where $x_k \in R^{n\times 1}$ is a state vector, $u_k \in R^{m\times 1}$ is a input vector, $y_k \in R^{p\times 1}$ is a output vector, $\Phi \in R^{n\times n}$, $\Gamma \in R^{n\times m}$, $C \in R^{p\times n}$ and $D \in R^{p\times m}$.

The output equation of (5.1) is represented in the vector form as shown in (5.2).

$$Y_N = \Gamma_N x_0 + \Phi_N U_N \tag{5.2}$$

$$Y_N = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}, \; U_N = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}, \; \Gamma_N = \begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{N-1} \end{bmatrix}$$

$$\Phi_N = \begin{bmatrix} D & & & O \\ C\Gamma & D & & \\ \vdots & & \ddots & \\ C\Phi^{N-2}\Gamma & \cdots & C\Gamma & D \end{bmatrix}$$

where x_0 is an initial state vector, Γ_N is the observability matrix, Φ_N is a toeplitz matrix consisting of the Markov parameters.

We consider the multi output case, and describe upper and lower bounds of each channel of u_i as $\{\bar{\alpha}_j\}_{j=1,\cdots,m}$, respectively. Each channel of input u_i is normalized by dividing it by the maximum of bounds, i.e., $\max_j \bar{\alpha}_j$. As a result, u_i is represented with a constant number μ and a normalized vector v_i as follows.

$$u_i = \mu v_i \tag{5.3}$$

$$\begin{cases} \mu > 0 \\ |v_{ij}| \leq \alpha_j \quad \text{for } j = 1, \cdots, m \\ v_i = \begin{bmatrix} v_{i1} & v_{i2} & \cdots & v_{im} \end{bmatrix}^T \end{cases} \tag{5.4}$$

where $\alpha_j = \bar{\alpha}_j / \max_j \bar{\alpha}_j$ is the normalized bound of each channel of u_i, therefore, $\max_j \alpha_j = 1$ and $\mu = \max u_i$. With input representation in (5.4), the output equation (5.2) is rewritten as the following equality constraint.

$$\begin{aligned} O &= \left[\; -Y_N + \Gamma_N x_0 \;\middle|\; \Phi_N \; \right] \xi \\ \xi &= \left[\; \tfrac{1}{\mu} \;\middle|\; v_0^T \; \cdots \; v_{N-1}^T \; \right]^T \end{aligned} \tag{5.5}$$

The time optimal control design of linear discrete time systems (5.1) can be formulated as the linear programming by introducing its auxiliary problem similarly to the way of nonlinear system case. First, we shall summarize the linear programming briefly.

Linear programming

Find ξ maximizing the cost function (5.6) under the constraints (5.7).

$$\max_{\xi} \; f^T \xi \tag{5.6}$$

$$\text{subject to} \begin{cases} A\xi \leq b \\ A_{eq}\xi = b_{eq} \\ l_b \leq \xi \leq u_b \end{cases} \tag{5.7}$$

where f, ξ, b, b_{eq}, l_b and u_b are vectors, A and A_{eq} are matrices. Each pair of (A, b) and (A_{eq}, b_{eq}) means inequality and equality constraints, respectively. l_b is a lower bound vector of

ξ and u_b is an upper bound vector. Equality or inequality in the vector form means that each component of a vector should satisfy the equality or inequality condition. $\diamondsuit$

Finally, the auxiliary problems is described using the input representation (5.4) and equality constraint (5.5) on the linear programming problem.

An auxiliary problem for time optimal control design of linear discrete time systems

The terminal time N and terminal condition Y_N are given. Find a variable ξ maximizing the cost function (5.8) under the constraints (5.9).

$$\max_{\xi} f^T\xi, \quad f^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{5.8}$$

$$\begin{cases} A\xi = B, \quad A = \left[\, -Y_N + \Gamma_N x_0 \,\middle|\, \Phi_N \, \right], \quad B = O \\ \xi = \begin{bmatrix} \frac{1}{\mu} & v_0^T & \cdots & v_{N-1}^T \end{bmatrix}^T \\ \mu > 0 \\ |v_{ij}| \leq \alpha_j, \quad \max \alpha_j = 1 \quad \text{for } i, j \end{cases} . \tag{5.9}$$

$\diamondsuit$

Once the auxiliary problem is solved, the maximum input amplitude μ is obtained. According to the examination with respect to the auxiliary problem for the time optimal control design, if μ is more than the specified input saturation in the original problem, the terminal time N must be extended, while, in the contrary case, N must be shortened, and then the auxiliary problem should be solved again under the new terminal time. We can search the optimal time N by repeating this algorithm until an approapriate solution is obtained.

Remark

- Note that the optimal solution satisfying constraints might not exist if the time slice is rough, i.e., the sampling time is not short enough.
- There are many solves of the linear programming problem (5.6)-(5.7), for exmaple, the function **linprog** in Optmization Tool Box of MATLAB. In our case, we have developed and used a solver like **linprog** on MATX[36].
- The iteration algorithm, in our case, was realized by the bisection method practically.

5.3 Fuel Optimal Control Design Using Linear Programming

We consider an other control design using the linear programming, i.e., the fuel optimal and the mixed time-fuel optimal control designs, as well as the time optimal control design. Each cost function of the time optimal control and fuel optimal control is put together in a cost function by introducing weight variables as follows.

$$J = \int_{t_0}^{t_f} \alpha \cdot 1 + \beta \cdot |u(t)| \, dt \tag{5.10}$$

where α and β are weight variables. $\beta = 0$ means the time optimal control problem, $\alpha = 0$ means the fuel optimal case and other the other means the mixed case.

We first show that the fuel optimal control design problem is formulated in the linear programming. As shown in the previous case, the control period is divided into N intervals, and, in each interval, the input assume to be constant because of choosing the zero-order holder as the input generator, for instance. Then the cost function (5.10) with $\alpha = 0$ and $\beta = 1$ can be approximated as follows

$$\min_{u} \int_{t_0}^{t_f} |u(t)| \approx \min_{u_k} \sum_{k=0}^{N-1} |u_k| \tag{5.11}$$

It is known that the problem of minimizing an absolute value under linear constraints, $\min_\xi |\xi|$, can be formulated in the linear programming by an appropriate arrangement [38].

Problem of minimizing an absolute cost, $\min_\xi |\xi|$, in linear programming

$$\max_{\xi} c^T \tilde{\xi}, \quad c^T = \begin{bmatrix} -1 & -1 \end{bmatrix} \tag{5.12}$$

$$\text{subject to} \begin{cases} \tilde{\xi} = \begin{bmatrix} \xi^+ & \xi^- \end{bmatrix}^T \\ \xi = \xi^+ - \xi^- \\ \\ \xi^+ \geq 0 \\ \xi^- \geq 0 \end{cases} \tag{5.13}$$

$\diamondsuit$

Using the above formulation, the fuel optimal control can be described in the linear programming problem as follows.

Fuel optimal control design using linear programming

The terminal time N and the terminal condition Y_N are given. Find a variable ξ maximizing the cost function (5.14) under the constraints (5.15).

$$\max_{\xi} f^T \xi \quad f^T = \begin{bmatrix} -1 & \cdots & -1 \end{bmatrix} \tag{5.14}$$

$$\text{subject to} \begin{cases} A\xi = B \\ \quad A = \begin{bmatrix} \Phi_N \mid -\Phi_N \end{bmatrix} \\ \quad B = -Y_N + \Gamma_N x_0 \\ \\ \xi = \begin{bmatrix} (u_0^+)^T & \cdots & (u_{N-1}^+)^T \mid (u_0^-)^T & \cdots & (u_{N-1}^-)^T \end{bmatrix}^T \\ \quad u_i^+ > 0 \\ \quad u_i^- > 0 \end{cases} \tag{5.15}$$

$\diamondsuit$

Remark

- The cost function of the fuel optimal control design is represented by the sum of absolute values of input sequences. To formulate this problem in linear programming, the input is devided into u_i^+ and u_i^- accroding to (5.12)-(5.13). Since the input u_i must satisfy the system constraints (5.5), (5.5) is also divided as shown in (5.15) in line with the input division.

Finally we shall integrate the time and fuel optimal control problem. The mixed cost function (5.10) is approximated similarly to (5.11) introducing the weight value γ newly:

$$J = \int_{t_0}^{t_f} 1 + \gamma \cdot |u(t)|\, dt = (t_f - t_0) + \gamma \sum_{k=0}^{N-1} |u_k| \tag{5.16}$$

As to the auxiliary problem of the time optimal control design (5.8)-(5.9) and the fuel optimal control design (5.14)-(5.15), these constraints and the cost functions can be put together into a constraint and a cost function, respectively.

An auxiliary problem for mixed time-fuel optimal control design in linear programming

The terminal time N and the terminal condition Y_N are given. Find a variable ξ maximizing the cost function (5.17) under constraints (5.18).

$$\max_{\xi} f^T \xi, \quad f^T = \left[\ 1 \mid -\gamma \ \cdots \ -\gamma \ \right] \tag{5.17}$$

$$\text{subject to} \begin{cases} A\xi = B \\ \quad A = \left[\ -Y_N + \Gamma_N x_0 \mid \Phi_N \ \ -\Phi_N \ \right] \\ \quad B = O \\ \\ \xi = \left[\ \frac{1}{\mu} \mid (v_0^+)^T \ \cdots \ (v_{N-1}^+)^T \ \ (v_0^-)^T \ \cdots \ (v_{N-1}^-)^T \ \right]^T \\ \mu > 0 \\ \alpha_j \geq v_{ij}^+ > 0, \quad \alpha_j \geq v_{ij}^- > 0, \quad \max \alpha_j = 1, \quad \text{for } i, j \end{cases} \tag{5.18}$$

◇

Remark

- Because this mixed design problem integrates the auxiliary problem of time optimal control design and the fuel optimal control design problem, iteration is needed similarly to the time optimal case.

While only the input which we should design has the inequality constraints such as saturations in previous design problems, the proposed method can be extended to one with output inequality constraints. In real system, due to some physical constraints and the care of safety, the request to let the output leave in some region exists. Especially time optimal control is easy to yield the large output because the input is in the switching form of the maximum amplitude. We shall show the design method explicitly coping with these cases with the output constraints.

To prevent from confusing notations, we make Y_N denote the terminal condition and Y_N^v denote an output variable. Then, an extended time optimal control design using the linear programming is shown as follows.

An auxiliary problem for time optimal control design with output inequality constraints

The terminal time N and the terminal condition Y_N are given. Find a variable ξ maximizing the cost function (5.19) under the constraints (5.20).

$$\max_{\xi} f^T\xi, \quad f^T = [\ 1 \ \ 0 \ \cdots \ \ 0\ ] \tag{5.19}$$

$$\begin{cases} A\xi = B \\ \quad A = \begin{bmatrix} \Gamma_N x_0 & \Phi_N & -I \\ -Y_N & O & I \end{bmatrix} \\ \quad B = \begin{bmatrix} O \\ O \end{bmatrix} \\ \xi = [\ \frac{1}{\mu} \mid v_0^T \ \cdots \ v_{N-1}^T \mid (Y_N^v)^T\ ]^T \\ \mu > 0 \\ |v_{ij}| \le \alpha_j, \quad \max \alpha_j = 1, \quad \text{for } i, j \\ \eta \le Y_N^v \le \zeta \end{cases} \tag{5.20}$$

where each η and ζ is the lower and upper bound vector of the output, respectively. ◇

Remark

- If the element of η or ζ is ∞, the corresponding element of Y_N^v has no restriction.
- The same extension can be applied to the fuel optimal control design or the mixed time-fuel optimal control design.

5.4 Time and/or Fuel Optimal Control Design Using System Responses

The proposed control design problems formulated in the linear programming for the system (5.1) need only the no-input response $\Gamma_N x_0$ and the toeplitz matrix Φ_N consisting of the Markov parameters as to system constraints. If these responses are measurable, we can design proposed optimal control even if the system model and parameters are unknown.

- No-input response $\rightarrow$ $\Gamma_N x_0$
- Unit impulse response $\rightarrow$ $\Gamma_N x_0 + \phi_N$

where $\phi_N = \left[D^T,\ (C\Gamma)^T,\ \cdots,\ (C\Phi^{N-1})\Gamma^T\right]^T$.

Considering suppression of an effect of measurement disturbances and the Persistent Exciting condition of input, $\Gamma_N x_0$ and Φ_N can be estimated from the large data set of Y_p, Y_f, U_p and U_f in a sense of the least square [54].

$$\begin{array}{rcl} \Gamma_N x_0 & \approx & \mathrm{L}_{[:,1:N(p+m)]}\mathrm{W}_{p[:,1:1]} \\ \Phi_N & \approx & \mathrm{L}_{[:,N(p+m)+1:end]} \end{array} \tag{5.21}$$

$$\left\{ \begin{array}{rcl} \mathrm{W}_p & := & \begin{bmatrix} \mathrm{Y}_p \\ \mathrm{U}_p \end{bmatrix}, \quad \mathrm{L} := \mathrm{Y}_f \begin{bmatrix} \mathrm{W}_p \\ \mathrm{U}_f \end{bmatrix}^{\dagger} \\ \mathrm{Y}_p & = & \begin{bmatrix} y_{-N} & \cdots & y_{-N+j} \\ \vdots & & \vdots \\ y_{-1} & \cdots & y_{-1+j} \end{bmatrix}, \ \mathrm{U}_p = \begin{bmatrix} u_{-N} & \cdots & u_{-N+j} \\ \vdots & & \vdots \\ u_{-1} & \cdots & u_{-1+j} \end{bmatrix} \\ \mathrm{Y}_f & = & \begin{bmatrix} y_0 & \cdots & y_j \\ \vdots & & \vdots \\ y_{N-1} & \cdots & y_{N-1+j} \end{bmatrix}, \ \mathrm{U}_f = \begin{bmatrix} u_0 & \cdots & u_j \\ \vdots & & \vdots \\ u_{N-1} & \cdots & u_{N-1+j} \end{bmatrix} \end{array} \right.$$

where $A_{[i:j,m:n]}$ denotes the sub matrix from the i-th to j-th rows and from m-th to n-th columns of A.

5.5 Numerical Examples: Positioning Control of Hard Disk Drive

To illustrate effectiveness of the proposed methods, the positioning control for the track seek motion of a hard disk drive is designed with respect to time optimal control, fuel optimal control and mixed time-fuel optimal control, respectively. Since the speedy seek motion is wanted and both input and output are restricted to some regions due to the physical constraints and the care of safety, the control design problem for this system is suitable for an application of the proposed method. We first consider a model of the drive briefly and then show simulation results for each design.

5.5.1 A Hard Disk Drive System [16]

The Head of a hard disk drive is attached on the top of the carriage arm which is driven by a voice coil motor, VCM. VCM generates torque in proportion to current of the coil. Let v denote a command voltage, c a coil current, R_v a resistor and L_v an inductance, respectively. Then, the transfer function from the command voltage to the current is

$$C(s) = \frac{1}{L_v s + R_v} V(s) \tag{5.22}$$

. Although (5.22) shows that the frequency property of VCM is of the low-pass type, its property is usually adjusted to flat by equipping VCM with a compensator for the current amplifier. Therefore VCM with the compensator is considered as a proportional element G_a [A/V].

Using G_a and a conversion coefficient of current-to-torque, K_t [N·m/A], the transfer function from v to the angular acceleration of the carriage arm $\ddot{x}_1$ is

$$\ddot{x}_1 = \frac{1}{I_b} K_t G_a v \tag{5.23}$$

Table 5.1: HDD parameter values

Explanation	Symbol	Value
VCM amplifier gain	G_a	2.59×10^{-1} A/V
Current-torque gain	K_t	7.57×10^{-2} N·m/A
Moment of inertia	I_b	5.70×10^{-6} kg·m^2
Pivot-head distance	r_1	5.20×10^{-2} m
Equivalent back EMF	k_{eb}	$2.05 \times 10^{+2}$ sec^{-1}
Sampling time	dt	1.32×10^{-6} sec
Maximum input	u_{max}	$9.32 \times 10^{+2}$ m/sec^2
Maximum velocity	v_{max}	$2.40 \times 10^{+0}$ m/sec

where I_b is moment of inertia. Since, for the hard disk system, the displacement of the head is measured with a track as an unit and the displacement angle of the arm is less than 30° in general, the displacement along the radius, p, is approximated by the arm angle x_1 and the arm length r_1 as $p_1 = r_1 x_1$.

Furthermore, we shall define equivalent constants corresponding to the ones in (5.23) as

$$K_f = \frac{K_t}{r_1} \quad \text{: Current-to-torque constant } K_f \text{ [N/A]}$$
$$m_b = \frac{I_b}{r_1^2} \quad \text{: Equivalent mass } m_b \text{ [kg]}$$

, and transform (5.23) into

$$\ddot{p} = \frac{1}{m_b} K_f G_a v \tag{5.24}$$

.

The state representation of (5.24) is shown as

$$\frac{d}{dt}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_f G_a}{m_b} \end{bmatrix} v \tag{5.25}$$

. Since the velocity, acceleration and input are restricted in design, we shall transform the input in (5.25) into an equivalent acceleration and reconstruct the state space form including effect of back electromotive force.

$$\frac{d}{dt}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cdot 0 & -k_{eb} \end{bmatrix}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5.26}$$
$$u = \frac{K_f G_a}{m_b} v$$

Using parameters in Table 5.1 and discretizing (5.26) with the sampling time 132μ [sec] lead to the discrete time state space representation:

$$x_{k+1} = \begin{bmatrix} 1.00 & 1.30 \times 10^{-4} \\ 0.00 & 9.73 \times 10^{-1} \end{bmatrix} x_k + \begin{bmatrix} 8.63 \times 10^{-9} \\ 1.30 \times 10^{-4} \end{bmatrix} u \tag{5.27}$$

In the following numerical examples, we shall use (5.27) for each control design.

5.5.2 Simulation results of positioning control for track seek motion

We shall design the positioning control for long track seek motion (6.00×10^{-3} [m]) and show the results in Fig 5.1-5.12, here. In this simulation, the restrictions for the acceleration, velocity and input are shown as below.

- Acceleration restriction: $\max|\ddot{p}| = 9.32 \times 10^{+2}$ [m/sec^2]
- Velocity restriction: $\max|v| = 2.40 \times 10^{+0}$ [m/sec]
- Input restriction: $\max|u| = 5.00 \times 10^{+2}$ [m/sec^2]

Time optimal control : Fig. 5.1~5.4

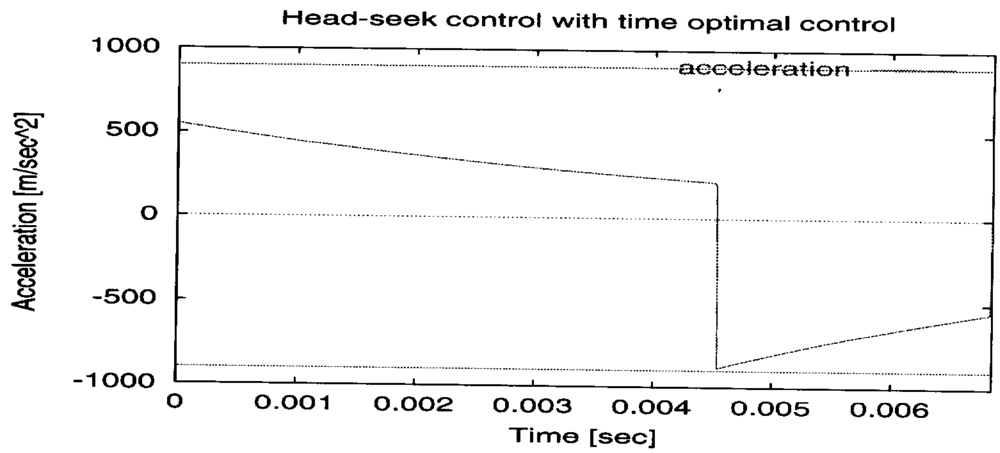


Figure 5.1: Acceleration in head-seek time optimal control

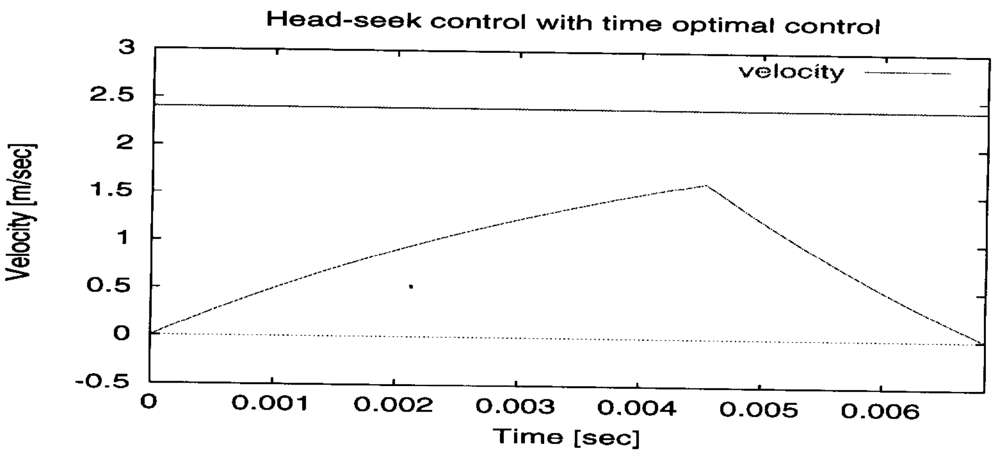
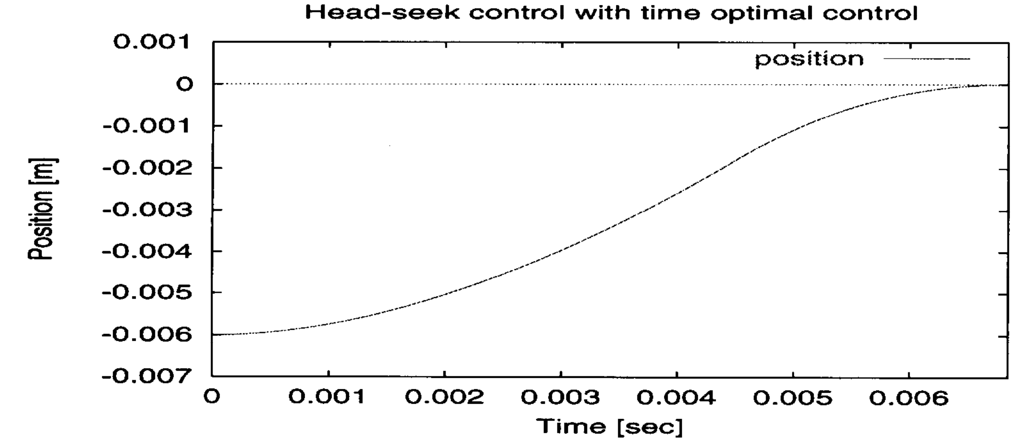


Figure 5.2: Velocity in head-seek time optimal control



n

Figure 5.3: Position in head-seek time optimal control

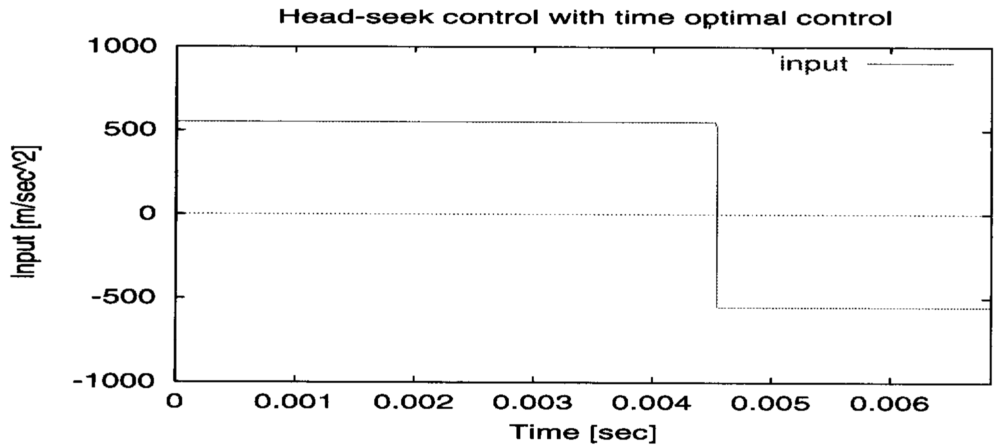


Figure 5.4: Input in head-seek time optimal control

Fuel optimal control : Fig. 5.5∼5.8

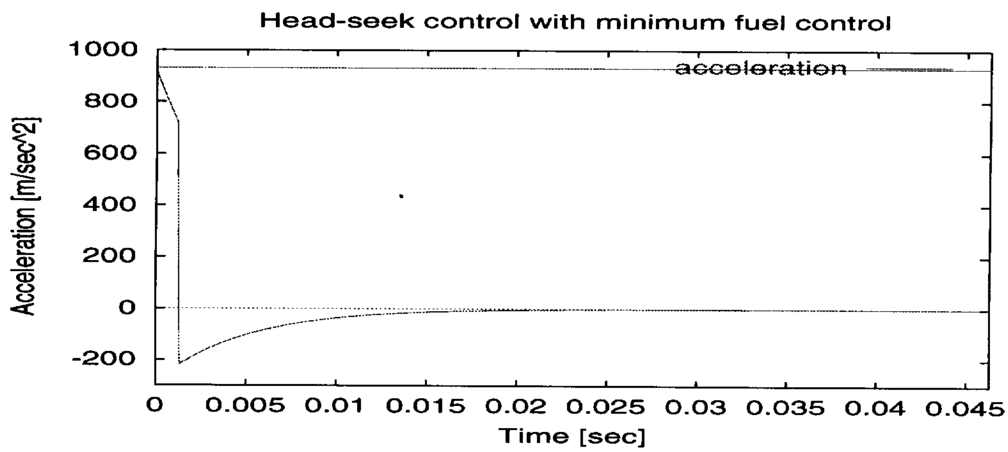


Figure 5.5: Acceleration in head-seek minimum fuel control

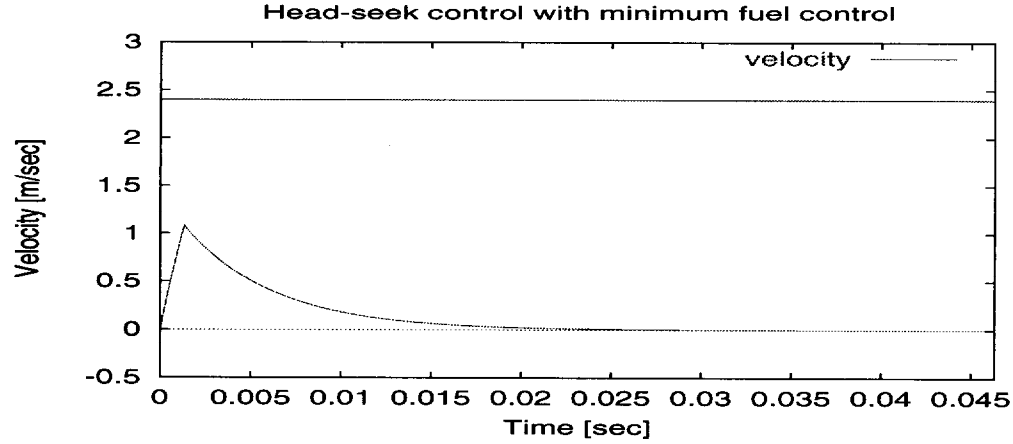


Figure 5.6: Velocity in head-seek minumum fuel control

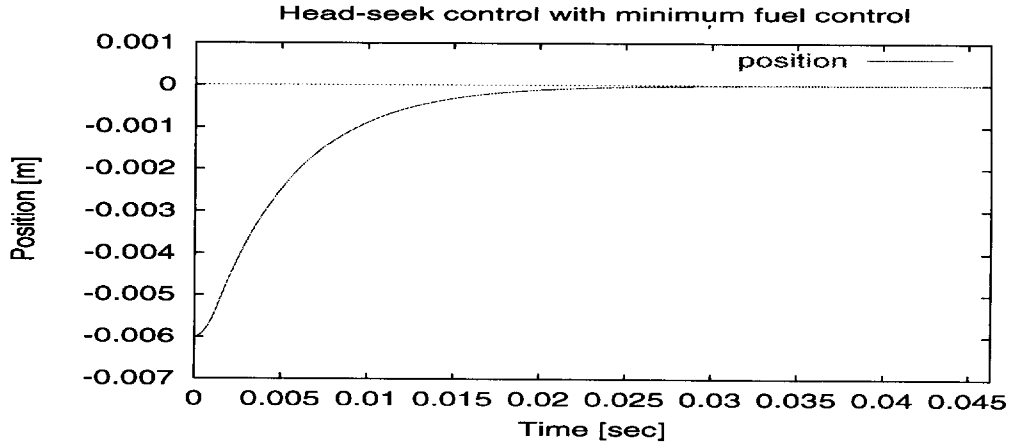


Figure 5.7: Position in head-seek minimum fuel control

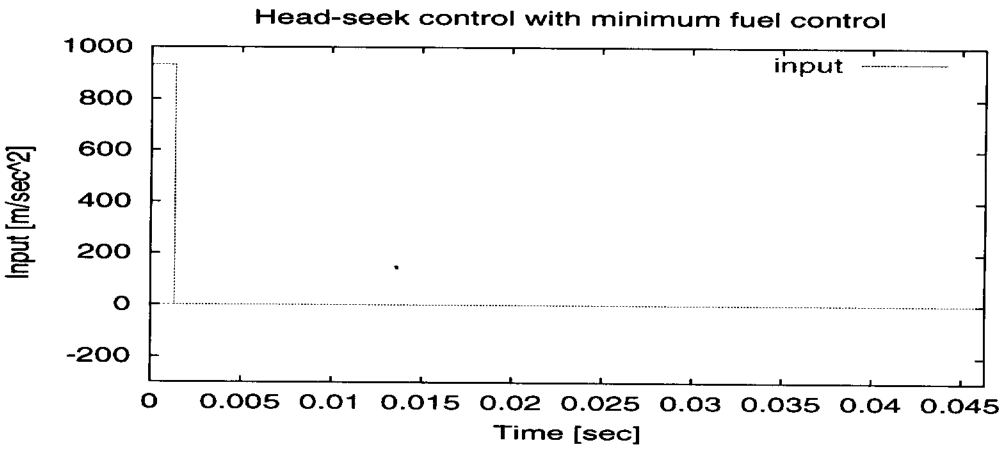


Figure 5.8: Input in head-seek minimum fuel control

Mixed time-fuel optimal control : Fig. 5.9∼5.12
(Weight value is set to $\gamma = 4.5 \times 10^{-6}$)

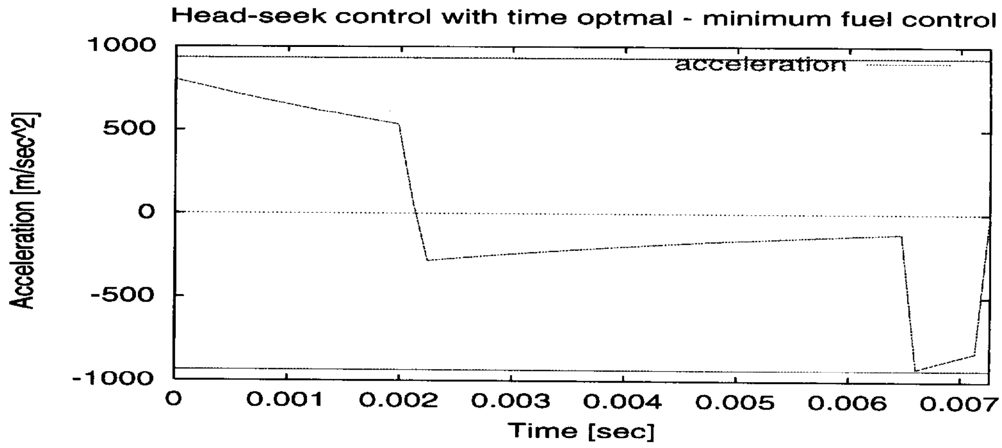


Figure 5.9: Acceleration in head-seek time optimal - minimum fuel control

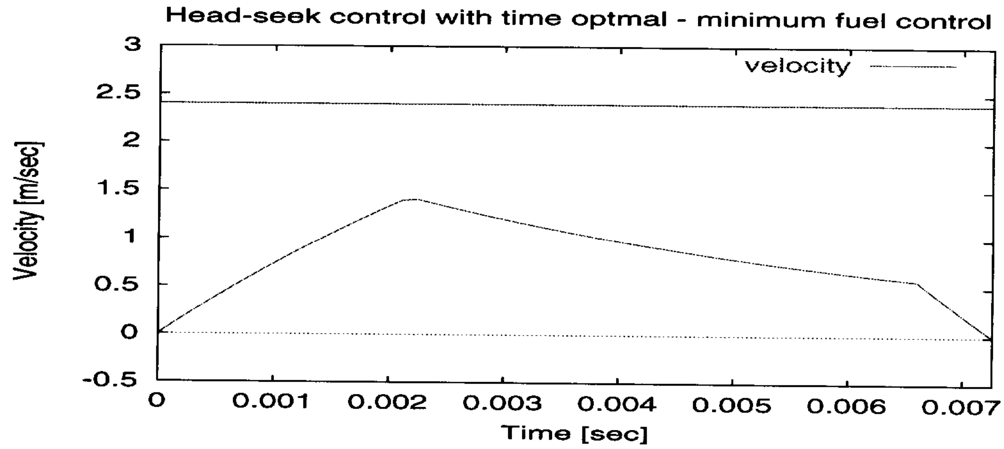


Figure 5.10: Velocity in head-seek time optimal - minumum fuel control

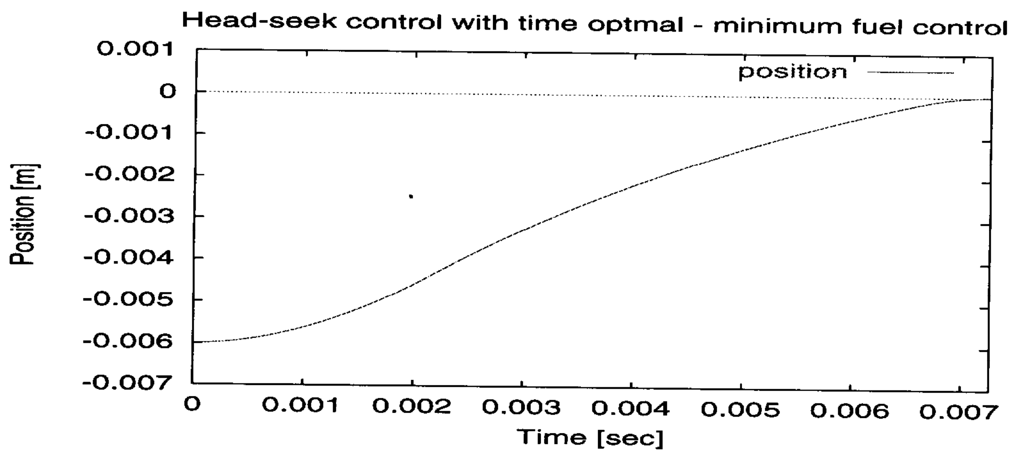


Figure 5.11: Position in head-seek time optimal - minimum fuel control

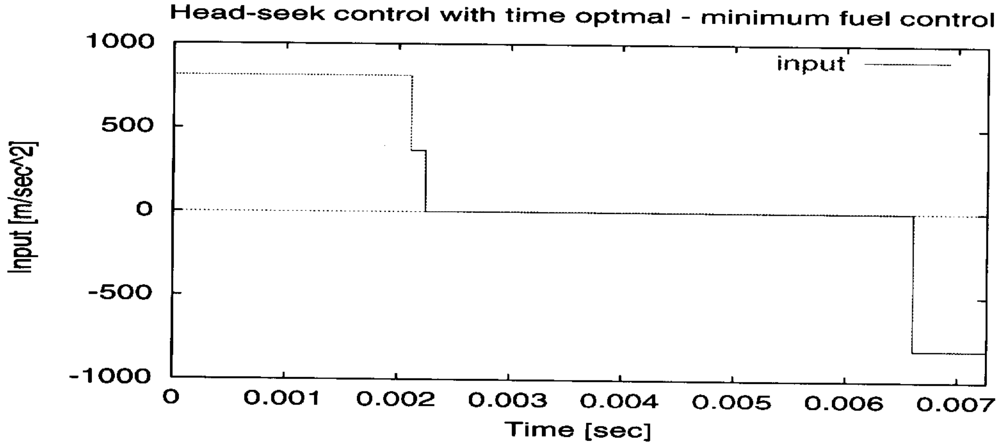


Figure 5.12: Input in head-seek time optimal - minimum fuel control

It is easily seen that the restrictions on the acceleration, velocity and input regions are satisfied in the time optimal, fuel optimal and mixed time-fuel optimal control design from simulation result Fig. 5.1-5.12. While the designed input should be of the rectangular form in theory, a part of the input gets out of rectangular shape as shown in Fig 5.4, 5.8 and 5.12. This fact is caused by discretization. Originally, in continuous time systems, the switching can be allowed to happen at any time in the control period. But, it is possible that the switching occurs in a sampling interval in discrete time system, and in this case the input cannot become the rectangular shape any more, that is, the shape should be broken a little in order to satisfy the given conditions. If the sampling interval becomes more rough, no optimal input satisfying the initial and terminal conditions with input saturation might exist. To avoid causing this situation, the sampling intervals should be chosen short enough.

5.6 Concluding Remarks

In this chapter, for the linear time invariant discrete time system, the control design methods which are formulated and solved as the linear programming problem with respect to (1) time optimal, (2) fuel optimal and (3) mixed time-fuel optimal case are proposed. These problems can be solved by existing solvers easily. Even if the parameters or model of system are not given explicitly, they can be also solved as long as some system responses, the no-input response and the unit pulse response, are measured or estimated. Furthermore, these problems have been extended to ones with output inequality constraints. To verify the effectiveness of proposed methods, they have been applied to the positioning control for track seek motion of a hard disk drive, and the control sequence has been designed in three cases. The simulation showed the correct results.

Chapter 6

Conclusions

6.1 Concluding remarks

This dissertation has proposed and has given new control system designs and applications based on input-output data. In the beginning, a new LQG controller design method has been proposed for linear time invariant continuous-time systems to extend the class with which data-based control can deal, because the majority of previous works studied for linear time invariant discrete-time systems. Next, in order to show that data-based control is effective in the case that obtained system information is only input-output data, that is, in the case that models, structures and dynamics are not given explicitly, a new adaptive learning control design using the Markov parameters has been proposed and has been applied to the microtremor vibration isolation system. At last, for an example that some control systems can be designed with operations closed in input-output data, it has been shown that time optimal and fuel optimal control design can be formulated in the form of a numerical optimization only with some system responses and can be solved easily by usual solvers.

The details of obtained results in each chapter are summarized as follows.

A LQG Controller Design for Linear Continuous Time Systems based on Laguerre Series Expansion

This chapter has proposed a new design method of a LQG dynamic controller for linear continuous-time systems. The Laguerre basis is introduced to discretize continuous signals and to expand the system. As a result a new system representation called "Laguerre system" has been given. Design methods of the optimal feedback gain and full order observer have been proposed with this Laguerre system representation, and it has been shown that by integrating them a new LQG controller is given. The designed controller has been described by expansion coefficients of a response, and the response is generated by injecting the time response of Laguerre basis to the original system. Finally this chapter has provided a realization method of the continuous-time state space model of the designed LQG controller represented with Laguerre expansion coefficients. To verify the proposed method, it has been applied to a 2nd order system and compared with a controller designed on the basis of the state space representation. Consequently, it has shown that the proposed method gives an almost same controller with the model-based controller as long as the coefficient's length is long enough.

An Adaptive Learning Control Using Markov Parameters

We presented adaptive learning control methods using Markov parameters. In these methods, input and parameter are updated simultaneously to minimize a quadratic cost function. Because the Method 1 cannot guarantee convergence if some disturbances are injected to the system, to overcome this problem a new algorithm utilizing an inverse system has been proposed in the Method 2. Simulation results showed that these methods are effective in the systems which have modeling error and/or whose parameters are unknown, and that the control performance is sufficiently good. Furthermore the proposed method has been applied to a vibration isolation system, and the effectiveness has been confirmed.

Time Optimal Control Design with Nonlinear Programming

This chapter proposed a new computational approach to design time optimal control of general nonlinear systems with input bound. This approach can be computed by an auxiliary problem of finding the minimal input bound satisfying the terminal state condition at the given time, and it could be solved by usual nonlinear optimization. To illustrate the efficiency of the proposed algorithm, the approach has been applied to time optimal swing-up control of a rotating type pendulum numerically and experimentally. The obtained numerical solution was verified using a simplified model of a single pendulum, and shown to satisfy the necessary condition of optimality. The algorithm has good feature that it is useful for the time optimal control of general nonlinear systems

Time and Fuel Optimal Control Design by Linear Programming Using Input-Output Data

In this chapter, it has been shown that problems of time optimal, fuel optimal and mixed time-fuel optimal control system design can be formulated and be solved using the linear programming for the linear time invariant discrete-time system. These problems can be solved by usual solvers. Even if parameters or models of systems are not given explicitly, the problems can be also solved as long as the non-input response and the unit pulse response are measured or estimated. To verify the effectiveness of proposed methods, they have been applied to the positioning control for track seek motion of a hard disk drive. The simulation has showed the correct results.

6.2 Future research directions

The following research direction can be expected in future.

- In Chapter 2, to discretize continuous signals the Laguerre function was introduced. However, other orthonormal functions, for example the Kautz function, can be considered as the function basis. It may be problem whether the continuous system can be represented with such functions in similar way to the Laguerre function.
- In Chapter 2, because only a LQG controller design method has been proposed, it may be extended to other controller designs, for example H_∞.
- Although the Laguerre function has been introduced to discretize continuous signals in

Chapter 2, that provides a good feature that a signal is approximated with functions. Application of this method to discrete-time system may solve the problem that many expansion coefficients, i.e. the Markov parameters, are needed in order to represent the system in the previous works, where the unit pulse is used as the basis of signal sequence.

- With the Laguerre system representation proposed in Chapter 2, a similar adaptive learning control of Chapter 3 may be considered for continuous-time systems. However, because it may be difficult to estimate the length of required expansion coefficients in the control period, the convergence of estimation of unknown system parameters may be taken as a problem.

- Although, in Chapter 4 and 5, time optimal and fuel optimal control has been designed in the form of sequences, it is convenient that the optimal control law can be given with the state conditions or functions. Key point is whether other auxiliary problem meeting this purpose can be considered and formulated.

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List of Papers

Published and Submitted papers

1. M. Iwase, M. Yamakita, Y. Maruta and H. Yamazoe, "An Adaptive Learning Control and Its Application to Vibration Isolation Systems", *accepted to Asian Journal of Control.*

2. Y. Xu, M. Iwase and K. Furuta, "Time Optimal Swing-up Control of Single Pendulum", *accepted to Trans. of ASME: Journal of Dynamics Systems, Measurement and Control.*

3. M. Iwase, M. Yamakita and K. Furuta, "A LQG Controller Design for Linear Continuous Time Systems based on Laguerre Series Expansion", *to be submitted*

Conference Papers

1. M. Iwase, T. Hoshino, K. Furuta, H. Yamazoe and H. Shigetoyo, "Vibration Suppressing Control of Microtremor Isolation System Using Markov Parameters", *Proc. of the 40th, Japan Joint Automatic Control Conference*, 1997.

2. M. Iwase, T. Hoshino and K. Furuta, "Vibration Suppressing Control of Microtremor Isolation System Using Input-Output Data", *Proc. of the SICE Annual Conf.*, 1998.

3. M. Iwase and K. Furuta, "Identification and State Feedback Control Design based on Series Expansion in Continuous-time Domain", *Proc. of the SICE Annual Conf.*, 1999.

4. M. Iwase, K. Furuta and M. Yamakita, "A Control System Design Based on Input-Output Data - An Application to Microtremor Isolation System -", *Prof. of the SICE Annual Conf.*, 2000.