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H(2)-unknotting Number and Ozsváth-Szabó's d-invariant

Yuanyuan Bao

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Abstract

We study the H(2)-unknotting number of a link. This invariant is defined by adding twisted bands to link diagrams. Besides this definition, it also bears alternative 3-dimensional and 4-dimensional interpretations. This invariant is of great interest since it measures the complexity of a link, and the corresponding H(2)-Gordian distance provides a meaningful distance to the set of links. However it is usually hard to determine the value of this invariant because of its geometric definition. In this thesis, we study it by using Ozsváth-Szabó's d-invariant and some techniques in Dehn surgery.

First we provide a necessary condition for a knot to have H(2)-unknotting number one. In order to show that this condition is stronger in some cases than other known criteria, we consider the pretzel knot $P(13, 4, 11)$ as an example, proving that its H(2)-unknotting number is two, while this calculation cannot be achieved by any other methods as far as we know.

Then we study the H(2)-unknotting number of links as related to 2-bridge links, which is an interesting class of links parametrized by rational numbers. This study consists of three parts. In the first part, we consider a necessary and sufficient condition for a 2-bridge link to have H(2)-unknotting number one. In particular, this condition determines the parameters for such 2-bridge links. The second part concerns an explicit form of composite links with H(2)-unknotting number one, which we conjecture to be the only possible form. In the last part, we develop a method to study the H(2)-unknotting number of some tangle unknotting number one knots via 2-bridge knots.

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Chapter 1

Introduction

In knot theory, a knot is a smooth embedding of a circle S^1 into a closed oriented 3-manifold M, while a link is an embedding of a disjoint union of circles into M. In the following, we restrict ourselves to the case where M is the 3-sphere S^3.

The reason that knots and links are interesting study objects, is at least two-fold. On one hand, any closed oriented 3-manifold can be obtained from links via Dehn surgery (see Chapter 2). Therefore the study of 3-manifolds, in some aspects, is equivalent to the study of knots and links. Besides Dehn surgery, we can also obtain numerous interesting 3-manifolds from links by taking the complement of a link in S^3, or by taking the branched cover spaces of S^3 along a link. Knot theory can be regarded as a "representative" in 3-dimensional topology. On the other hand, knots and links themselves, are 1-dimensional manifolds in S^3, which are easy to imagine and can be studied via combinatorial methods. Therefore, problems on 3-manifolds can be sometimes transformed into problems on knots and links with easier statements.

We can ask many questions surrounding knots and links. One basic question is how to distinguish two links up to ambient isotopy. To answer this question, many link invariants have been constructed. Numerical invariants include signature, determinant etc., polynomial invariants contain the Alexander polynomial, the Jones polynomial and so on, and in recent years some homology invariants, such as Khonvanov homology and knot Floer homology, are defined. A link invariant, in theory, can be any object in Algebra.

There are questions concerning only partial information about knots or links, so some equivalence relations are introduced, such as link concordance and link homotopy. Regarding S^3 as the boundary of the 4-ball B^4, then we see every knot bounds a disk in B^4. If we require the disk to be smoothly embedded, we get an interesting class of knots, which are called slice knots. This concept was first made by Fox around 1962. Furthermore, the set of knots modulo the set of slice knots can be endowed with a group structure, the study of which has prospered since its appearance.

In order to know how different two given knots are, knot theorists have introduced some measures to connect them. For example, any two knots can be changed into each other by a local move called crossing change (see Chapter 3.3). So one can define the *X-Gordian distance* of two knots as the minimal number of crossing changes required to transform one into the other. Similar Gordian distances include Δ-Gordian distance, $\sharp$-Gordian distance, etc. Another way to measure the difference between two knots is by counting the complexity of surfaces bounded by them. The *3-genus* of a knot is the minimal genus of all orientable smooth surfaces in S^3 bounded by the given knot. If we replace the word "orientable" by "non-orientable", we get the definition of the crosscap number of a knot. Remember that a slice knot bounds a smooth disk in the 4-ball B^4. The *4-ball genus* of a knot is the minimal genus of all orientable smooth surfaces in B^4 bounded by the given knot in S^3. Similarly, if we replace the word "orientable" by "non-orientable", we get the definition of the 4-dimensional crosscap number of a knot. 3-genus and crosscap number measure the difference between a knot and the trivial knot, while 4-ball genus and 4-dimensional crosscap number measure the difference between a knot and slice knots.

Our topic in this thesis is H(2)-unknotting number. An H(2)-move is a local move defined by adding a twisted band to a knot (see Chapter 4 for details). The *H(2)-Gordian distance* of two knots is the minimal number of H(2)-moves required to transform one knot into the other. The *H(2)-unknotting number* of a knot is the H(2)-Gordian distance of the knot with the trivial knot. We can also define H(2)-Gordian distance from another viewpoint (see Theorem 4.1.1). Consider all smooth surfaces (orientable or non-orientable) in S^3 whose boundary is the union of the two given knots. Then the H(2)-Gordian distance of this two knots equals the minimal first Betti number of such surfaces minus one.

Though H(2)-unknotting number has good geometric meanings, its calculation is usually hard. In Chapter 4.2, we review some known criteria for detecting this value. In Chapter 5, we provide a necessary condition for a knot to have H(2)-unknotting number equal to one. The condition is described via the d-invariant associated with the double branched cover of S^3 along this knot. We further take the pretzel knot $P(13, 4, 11)$ an example of calculation, showing that its H(2)-unknotting number is two. This calculation cannot be fulfilled by other methods as far as we know.

In Chapter 6, we study the H(2)-unknotting number of three classes of links which are closely related to 2-bridge links. Here "bridge" indicates the bridge number of a link. 1-bridge links only include the trivial knot. 3 or higher-bridge links are complicated to handle. 2-bridge links, however, are parametrized by rational numbers, and are relatively convenient to study. Many link invariants, which are hard to study in general, have been formulated in the case of 2-bridge links.

This study consists of three parts. In the first part, we introduce a necessary and sufficient condition for a 2-bridge link to have H(2)-unknotting number equal to one. This condition determines the parameters for such 2-bridge links. Using this condition, we show that the 2-bridge knots 9_{21}, 9_{23}, 9_{26} and 9_{31} all have H(2)-unknotting number equal to one, which were unknown before. Here the names of knots come from Rolfson's knot table. The second part concerns an explicit form of composite links with H(2)-unknotting number one. Our purpose is to give an complete characterization of the forms for composite links with H(2)-unknotting number one. At present this is still an unsolved question, though, we conjecture that these composite links all have the explicit form we said above. In the last part, we develop a method to study the H(2)-unknotting number of some tangle unknotting number one knots via 2-bridge knots. This method is still described via the d-invariant.

Chapter 2

Preliminaries

In this chapter, we first recall some basic concepts about knots and links, hoping that for those who are not familiar with knot theory can have an overall impression about what knot theory concerns. Then we give the definition of Dehn surgery, which will be used in Chapter 6.

2.1 Knots and links

Knot theory considers knots and links as study objects. The purpose is to understand the topological properties of knots and links, and study low-dimensional manifolds via the study of knots and links. We take [28] as a reference. All the maps considered below are piecewise linear.

Definition 2.1.1. *An l-component oriented link* in S^3 is an embedding $\varphi : \coprod_{i=1}^{l} S^1 \to S^3$, where $\coprod_{i=1}^{l} S^1$ is a disjoint union of l copies of oriented 1-sphere S^1. Equivalently, we also regard the image $L = \varphi(\coprod_{i=1}^{l} S^1) \subset S^3$ as an oriented link in S^3. A 1-component link is called a knot, denoted $K \subset S^3$.

When a knot K bounds a piecewise linear 2-disk in S^3, K is called the trivial knot or the unknot.

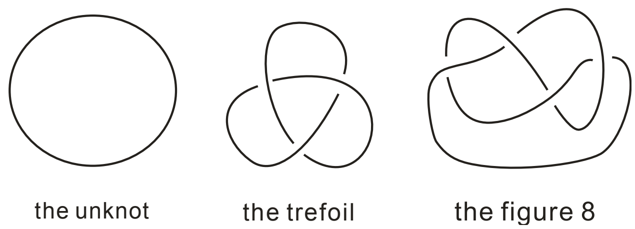

Figure 2.1: Examples of knots.

Definition 2.1.2. *An n-string tangle* is a pair (B^3, S) where B^3 is the 3-ball, and S is a disjoint union of n arcs properly embedded in B^3.

The concept of tangle will be used in Chapter 6. Now we come back to the discussion of links.

Definition 2.1.3. Two oriented links L_1 and L_2 are said to be *equivalent* if there exists a homeomorphism $h : S^3 \to S^3$ so that $h(L_1) = L_2$. Moreover, if h and $h|_{L_1}$ preserve the orientations of S^3 and L_1 respectively, L_1 and L_2 are said to be of *the same link-type*, denoted $L_1 \cong L_2$.

Given an oriented link $L \subset S^3$, a regular projection of L is a transverse map $\mathbf{P} : (S^3, L) \to (S^2, \mathbf{P}(L))$ which satisfies that $\sharp\mathbf{P}^{-1}(\mathbf{P}(a)) \leq 2$ for any point $a \in L$ and that the set $C_{\mathbf{P}} := \{b \in \mathbf{P}(L) | \sharp(\mathbf{P}^{-1}(b)) = 2\}$ is finite. We call the points in $C_{\mathbf{P}}$ the crossings of $\mathbf{P}$. The transversality of $\mathbf{P}$ guarantees that the local neighborhood of a crossing looks like . If the over and under positions of the two arcs near each crossing is remembered, the local neighborhood of a crossing will look like or , and the crossing is called a positive crossing or negative crossing of $\mathbf{P}(L)$.

An alternating link is a link which has a projection with alternating underpass and overpass near the crossings when travelling along it. The trefoil and figure-eight knot in Figure 2.1 are all alternating knots.

Whether two links have the same link type can be checked via their projections, as stated in the following theorem.

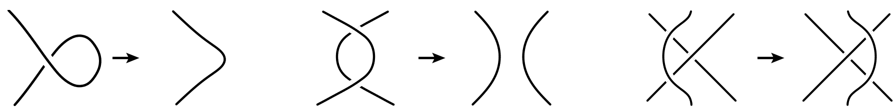

Figure 2.2: Three types of Reidemeister moves.

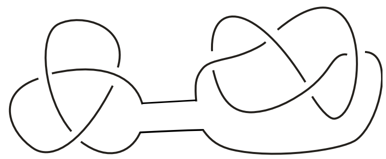

Figure 2.3: Connected sum of the trefoil and the figure-eight knot.

Theorem 2.1.4. *Given two knot $L_1, L_2 \subset S^3$ and their projections $\boldsymbol{P}_1, \boldsymbol{P}_2$, $L_1 \cong L_2$ if and only if $\boldsymbol{P}_1(L_1)$ can be converted to $\boldsymbol{P}_2(L_2)$ by a finite sequence of Reidemeister moves.*

See Figure 2.2 for the definition of Reidemeister moves. From now on, we do not distinguish a link from its projections.

We define the *connected sum* of two links in Figure 2.3. The connected sum of two links is not unique, but if we restrict this operation to the set of knots, it is uniquely defined.

Definition 2.1.5. *A link invariant* is a function from the set of all links to any other set, such as numbers, polynomials, groups and so on, such that the function preserves the link-type. In other words, a link invariant always assigns the same value to links of the same type (although different link types may have the same link invariant). When the link is a knot, the invariant is called a knot invariant.

The *linking number* of two knots K_1 and K_2 is an integer defined as

$$\mathrm{lk}(K_1, K_2) := \frac{1}{2} \sum_{p \in \mathbf{P}(K_1) \cap \mathbf{P}(K_2)} \mathrm{sign}(p),$$

where $\mathbf{P}$ is a projection of $K_1 \coprod K_2$. The sign of a crossing p is defined as

$$\operatorname{sign}(p) = \begin{cases} 1 & p \text{ is a positive crossing,} \\ \\ -1 & p \text{ is a negative crossing.} \end{cases}$$

The linking number of K_1 and K_2 does not depend on the choice of the projection.

2.2 Dehn surgery

Dehn surgery, introduced by Dehn, is a basic way to construct 3-manifolds from knots. Lickorish and Wallance showed that all orientable 3-manifolds can be obtained using Dehn surgery. In this section, we provide basic concepts around this construction.

Given an oriented knot $K \subset S^3$, and its tubular neighborhood $N(K) \subset S^3$, consider an orientation-preserving homeomorphism $f : (K \times D^2, K \times 0) \to (N(K), K)$, where D^2 is the unit disk in $\mathbb{R}^2$. The 2-disk $f(\{z\} \times D^2) \subset N(K)$ for some point $z \in K$ is called a meridian disk of K, and its boundary $\partial(f(\{z\} \times D^2)) = f(\{z\} \times \partial D^2)$ is called a *meridian* of K, denoted μ_K. The curve $f(K \times \{s\})$ for some point $s \in \partial D^2$ is called a longitude of K. The linking number $\operatorname{lk}(f(K \times \{s\}), K)$ is called the *framing* of the longitude. We let λ_K denote the one of framing zero. We call the pair (μ_K, λ_K) the *preferred meridian-longitude system* of K.

Let p and q be two relatively prime integers (if one integer is 0, the other is required to be 1). Then $p\mu_K + q\lambda_K$ is a simple closed curve in the boundary of $N(K)$. Attaching a solid torus $D^2 \times S^1$ to $\overline{S^3 - N(K)}$ by identifying $\partial D^2 \times \{t\}$ with $p\mu_K + q\lambda_K$ for a point $t \in S^1$, we get a closed 3-manifold. This procedure is called the *Dehn surgery* of slope p/q along the knot K, and the resulting manifold is denoted $S^3_{p/q}(K)$. The manifold $S^3_{p/q}(K)$ only depends on p/q and K. It is easy to see that $S^3_{1/0}(K)$ gives back S^3 for any $K \subset S^3$.

Example 2.2.1. *The complement of the unknot in* S^3 *is just a solid torus. Therefore* $S^3_{p/q}(O)$ *where* O *is the unknot is the result of gluing two solid torus. Precisely* $S^3_{p/q}(O)$ *is the lens space* $L(p,q)$.

Chapter 3

Heegaard Floer homology and the d-invariant

This chapter includes necessary backgrounds about the d-invariant. We start from the introduction of Heegaard Floer homology, from which the d-invariant is derived. Then we give the definition, basic properties and some applications of this invariant. Almost all constructions and results in this chapter can be found in Ozsváth and Szabó's original papers [36, 35, 34].

3.1 Heegaard Floer homology

Let Y be a closed oriented 3-manifold and $s \in \mathrm{Spin}^c(Y)$ be a spin^c-structure over Y. Ozsváth and Szabó [36] defined the Heegaard Floer homology associated with the pair (Y, s). In this section, we briefly recall the definition of Heegaard Floer homology. We assume that all the 3-manifolds mentioned below in this section are rational homology 3-spheres. This assumption will simplify the definition and is proper for our purpose.

As pointed by Ozsváth and Szabó, the construction of Heegaard Floer homology used Lagrangian intersection Floer homology as a reference. The later one is defined for a symplectic manifold and a pair of Lagrangian submanifolds, whose generators correspond to intersection points of the Lagrangian submanifolds, and whose boundary maps count pseudo-holomorphic disks with appropriate boundary conditions.

3.1.1 Heegaard digrams

Heegaard Floer complex is defined via a Heegaard diagram of the closed 3-manifold. The ambient manifold and its two submanifolds which play the role of the Lagrangians are derived from the Heegaard diagram.

Definition 3.1.1. A *Heegaard diagram* is a triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, which consists of a closed oriented surface Σ of genus g and two g-tuples of simple closed curves $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_g)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_g)$ embedded in Σ, satisfying the condition that the α-curves (respectively the β-curves) are disjoint pairwisely and linearly independent as homology classes in $H_1(\Sigma, \mathbb{Z})$.

One can constructed a closed oriented 3-manifold from a Heegaard diagram in the following way. Attaching 2-handles along the α-curves (respectively the β-curves) to Σ and capping off the resulting 2-sphere boundary by a 3-ball, we obtain a genus g-handlebody, denoted by U_α (respectively U_β). The orientations of U_α and U_β are fixed by requiring that the normal vectors of Σ point out of U_α and U_β. A closed oriented 3-manifold is obtained by attaching U_β to U_α by identifying $-\partial(U_\beta) = -\Sigma$ with $\partial(U_\alpha) = \Sigma$.

3-manifolds obtained in this way are not special. In fact, it is proved in [45] that any closed oriented 3-manifold can be constructed from a Heegaard diagram. Moreover, any two Heegaard diagrams providing the same 3-manifold can be transferred into each other by a sequence of well-studied moves (see Theorem 3.1.5).

Example 3.1.2. *The lens space $L(p, 1)$ for $p \geq 1$ has a genus one Heegaard digram. Figure 3.1 is a Heegaard diagram for $L(3, 1)$.*

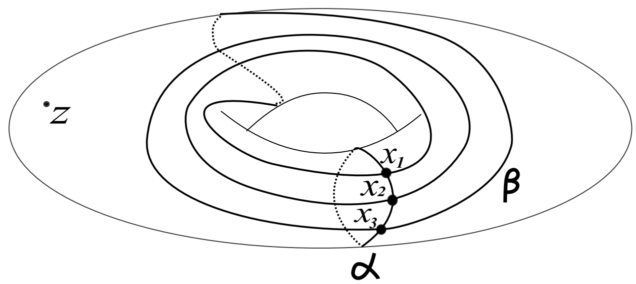

Figure 3.1: A Heegaard diagram for the lens space $L(3,1)$.

Definition 3.1.3. Two Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ are said to be *diffeomorphic* if there is an orientation-preserving diffeomorphism of Σ to Σ' which carries $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}$ to $\boldsymbol{\beta}'$.

Definition 3.1.4. Let $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram of genus g. Define the following three types of moves on H, which are called *Heegaard moves*.

1. An *isotopy move* of the Heegaard diagram is an isotopy move of certain α-curve (or β-curve) in Σ in such a way that the α-curves (or β-curves) keep disjoint.

2. A *handleslide* of α_i across α_j for $i \neq j$ is a replacement of α_i in $\boldsymbol{\alpha}$ by $\tilde{\alpha_i}$, which is a simple closed curve being disjoint from curves $\alpha_1, \alpha_2, \cdots, \alpha_g$ and having the property that α_i, $\tilde{\alpha_i}$ and α_j bound an embedded pair of pants in $\Sigma - \bigcup_{k \neq i,j} \alpha_k$. A handleslide of β-curves is defined in the same vein. A handleslide of an α-curve (respectively a β-curve) across a β-curve (respectively an α-curve) is not defined.

3. A *stablization* of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a replacement of this Heegaard diagram by a new Heegaard diagram $(\Sigma \sharp T, \boldsymbol{\alpha} \cup \{\alpha_{g+1}\}, \boldsymbol{\beta} \cup \{\beta_{g+1}\})$ where T is an oriented torus, and α_{g+1} and β_{g+1} are a pair of simple closed curves in T which meet transversally in a single point. A *destabilization* is the reverse of the procedure.

Theorem 3.1.5 ([43, 45, 44])**.** *Let Y be a closed oriented 3-manifold. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ be two Heegaard diagrams for Y. Then by applying sequences of isotopies, handlesides and (de)stabilizations we can change the above diagrams so that the new diagrams are diffeomorphic to each other.*

In the definition of Heegaard Floer homology, a basepoint $z \in \Sigma - \boldsymbol{\alpha} \cup \boldsymbol{\beta}$ is needed. Considering the existence of such a basepoint, Ozsváth and Szabó modified the above theorem in the following way.

Theorem 3.1.6. *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$ be two Heegaard diagrams for Y with basepoints z and z'. Then they can be connected (up to diffeomorphism) by a sequence of isotopies, handleslides and (de)stabilizations which occur in the complements of the basepoints.*

The proof can be completed by two observations. First given a sequence of Heegaard moves, one can introduce isotopies as needed to arrange that no handleslides and (de)stabilizations, only isotopies cross the basepoints z and z'. Second, any isotopy can be replaced by a series of handleslides and isotopies occurring in the complement of basepoints.

3.1.2 Generators of the complex

For a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ of genus g, the ambient space used in the definition of Heegaard Floer homology is

$$\mathrm{Sym}^g(\Sigma) := \Sigma \times \cdots \times \Sigma / S_g,$$

where S_g is the symmetry group on g letters. We can check by definition that $\mathrm{Sym}^g(\Sigma)$ is a smooth manifold. Moreover, a complex structure on Σ induces a complex structure on $\mathrm{Sym}^g(\Sigma)$.

The α-curves and β-curves induce a pair of smoothly embedded g-dimensional tori

$$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g \text{ and } \mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_g.$$

These tori are compatible with any complex structure on $\mathrm{Sym}^g(\Sigma)$ induced from Σ in the following sense.

Let L be a submanifold in a complex manifold Z with complex structure J. Then L is called *totally real* if for each point $p \in L$ its tangent space T_pL satisfies $T_pL \cap JT_pL = \{0\}$.

Lemma 3.1.7. *The tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are totally real submanifolds of $\mathrm{Sym}^g(\Sigma)$ for any complex structure induced from Σ.*

In Lagrangian intersection Floer homology, people consider a symplectic manifold and a pair of Lagrangian submanifolds in the symplectic manifold. In Heegaard Floer homology, we consider $\mathrm{Sym}^g(\Sigma)$ and its totally real tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. The group of the Heegaard Floer complex can be defined as follows.

Definition 3.1.8. The Heegaard Floer complex $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ associated with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is generated as a vector space over $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ by elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Example 3.1.9 (Continuation of Example 3.1.2)**.** *Consider the genus one Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for the lens space $L(p, 1)$. Since the genus is one, the symmetric manifold is just Σ, and $\mathbb{T}_\alpha = \alpha$, $\mathbb{T}_\beta = \beta$. The α-curve and β-curve intersect at p points $x_1, x_2, \cdots, x_p$. Therefore the Heegaard Floer complex $\widehat{\mathrm{CF}}(\Sigma, \alpha, \beta, z)$ is the vector space $\mathbb{F}^p$.*

Remark 3.1.10. *After Ozsváth and Szabó's construction of Heegaard Floer homology, Perutz [40] proved that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangian submanifolds for a well-chosen symplectic form on $\mathrm{Sym}^g(\Sigma)$. This implies that Heegaard Floer homology coincides with the Lagrangian Floer homology for the pair $(\mathbb{T}_\alpha, \mathbb{T}_\beta)$.*

3.1.3 Differential

Let $\mathbb{D}$ be a unit disk in the complex plane $\mathbb{C}$. Let e_1 and e_2 be the arcs in the boundary of $\mathbb{D}$ with $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(z) \leq 0$ respectively.

Definition 3.1.11. Given two elements $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a *Whitney disk* connecting $\mathbf{x}$ and $\mathbf{y}$ is a continuous map

$$u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma)$$

with the property that $u(-i) = \mathbf{x}$, $u(i) = \mathbf{y}$, $u(e_1) \subset \mathbb{T}_\alpha$ and $u(e_2) \subset \mathbb{T}_\beta$. Let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ and $\mathbf{y}$.

Endow $\mathrm{Sym}^g(\Sigma)$ a complex structure induced from Σ. For a given homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, let $\mathcal{M}(\phi)$ denote the moduli space of holomorphic representatives of ϕ. The moduli space $\mathcal{M}(\phi)$ admits an $\mathbb{R}$ action, which is induced by the automorphisms of the unit disk $\mathbb{D}$ that preserve i and $-i$. Then define the quotient space

$$\widehat{\mathcal{M}}(\phi) := \mathcal{M}(\phi)/\mathbb{R},$$

which is called the the unparametrized moduli space. The Maslov index [10] for the moduli space $\mathcal{M}(\phi)$ is denoted by $\mu(\phi)$.

For a generic complex structure on Σ, Ozsváth and Szabó showed that $\mu(\phi) > 0$ for non-constant holomorphic disk ϕ, and that if $\mu(\phi) = 1$ then $\widehat{\mathcal{M}}(\phi)$ is a compact oriented zero dimensional manifold. We always take the choice of complex structure into account in the definition of Heegaard Floer complex.

Definition 3.1.12. Let $\mathbf{x}, \mathbf{y}$ be two elements in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. For any point $p \in \Sigma$ which is in the complement of the α and β curves, let $n_p : \pi_2(\mathbf{x}, \mathbf{y}) \longrightarrow \mathbb{Z}$ be the algebraic intersection number, which is defined by

$$n_p(\phi) = \sharp\phi^{-1}(\{p\} \times \mathrm{Sym}^{g-1}(\Sigma)),$$

for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

Then the differential of the Heegaard Floer complex is defined in the following way.

Definition 3.1.13. Given an element $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, define the map $\widehat{\partial} : \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ by

$$\widehat{\partial}(x) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \left(\sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) | \mu(\phi) = 1, n_z(\phi) = 0\}} c(\phi) \cdot \mathbf{y} \right),$$

where $c(\phi)$ counts the number of points in $\mathcal{M}(\phi)$.

Ozsváth and Szabó proved that $(\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \widehat{\partial})$ is a chain complex, i.e. $\widehat{\partial}^2 = 0$. The proof follows from the compactifications of the one-dimensional moduli spaces with Maslov index two. They observed that the only boundary components in the compactifications consist of broken flow-lines, which "cancel out" with each other. Moreover, they proved the following theorem.

Theorem 3.1.14 (Ozsváth and Szabó)**.** *The homology* $\widehat{\mathrm{HF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ *does not depend on the choice of the complex structure on* Σ*, and the choice of the pointed Heegaard diagram. It is a topological invariant of the closed oriented 3-manifold* Y*, and therefore we can denote it by* $\widehat{\mathrm{HF}}(Y)$*.*

The definition of $(\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \widehat{\partial})$ depends on a Heegaard diagram and a complex structure J on Σ (and some perturbations of J which will not be considered here). The proof of Theorem 3.1.14 was fulfilled by many steps. Fixing a Heegaard diagram and varying J, the authors showed that there exists a chain map between two complexes which induces an isomorphism in homology. The isotopy of an α or β-curve induces an exact Hamiltonian isotopy on the Heegaard surface Σ, which further induces an isotopy of $\mathbb{T}_\alpha$ or $\mathbb{T}_\beta$. The chain map between the chain complexes before and after the isotopy move is defined by counting holomorphic disks connecting generators from the two complexes. The invariance under a handleslide move is as follows. Performing a handleslide move of β_1 cross β_2, we get a curve $\beta_1' \subset \Sigma$. Define $\boldsymbol{\beta'} := \{\beta_1', \beta_2, \cdots, \beta_g\}$. Then $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta'})$ is a Heegaard diagram for $\sharp^g(S^1 \times S^2)$. We call $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta'}, z)$ a Heegaard triple-diagram of $(Y, Y, \sharp^g(S^1 \times S^2))$. A chain map between the complexes before and after the handleslide move is defined by counting holomorphic triangles over the Heegaard triple. The definition of a holomorphic triangle is similar to that of a holomorphic disk, but the boundary of a triangle consists of three arcs lying in $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ and $\mathbb{T}_{\beta'}$. As for the invariance under (de)stabilization move, we see that the generators of the complexes before and after a (de)stabilization move has a natural identification, and that this identification extends to the differential maps. For detail exposition of the proof, nowhere is better than Ozsváth and Szabó's original paper [36].

The definition of $\widehat{\mathrm{HF}}(Y)$ uses the basepoint z in the way that the differential only counts holomorphic disks that "avoid" the point z. If we take all the holomorphic disks into account, different complexes can be defined.

Let $\mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be the vector space over $\mathbb{F}$ generated by pairs $[\mathbf{x}, i] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}$. The differential ∂^∞ is defined in the following way.

$$\partial^\infty([\mathbf{x}, i]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \left(\sum_{\{\phi \in \pi_2(\mathrm{x}, \mathrm{y}) \mid \mu(\phi) = 1\}} c(\phi) \cdot [\mathbf{y}, i - n_z(\phi)] \right).$$

Define $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to be the subspace of $\mathrm{CF}^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ generated by pairs $[\mathbf{x}, i]$ with $i < 0$. It is known that a holomorphic representative of a Whitney disk must have non-negative intersection number with the basepoint z. Then by definition we see that $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is a subcomplex under the induced differential ∂^-. Then let $(\mathrm{CF}^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \partial^+)$ be the corresponding quotient complex.

The homology spaces of the chain complexes above are all topological invariants of Y, which are denoted by $\mathrm{HF}^\infty(Y)$, $\mathrm{HF}^-(Y)$ and $\mathrm{HF}^+(Y)$.

There exist natural long exact sequences connecting these homology spaces. Let ι : $\mathrm{HF}^-(Y) \to \mathrm{HF}^\infty(Y)$ be the map induced by the inclusion map, and the map $\pi : \mathrm{HF}^\infty(Y) \to \mathrm{HF}^+(Y)$ be induced from the quotient map. Define $\hat{\iota} : \widehat{\mathrm{HF}}(Y) \to \mathrm{HF}^+(Y)$ by sending a generator $\mathbf{x}$ to $[\mathbf{x}, 0]$, and the map $U : \mathrm{HF}^+(Y) \to \mathrm{HF}^+(Y)$ by sending a generator $[\mathbf{x}, i]$ to $[\mathbf{x}, i-1]$. Then we have the long exact sequences

$$\cdots \longrightarrow \mathrm{HF}^-(Y) \xrightarrow{\iota} \mathrm{HF}^\infty(Y) \xrightarrow{\pi} \mathrm{HF}^+(Y) \longrightarrow \cdots,$$

and

$$\cdots \longrightarrow \widehat{HF}(Y) \xrightarrow{\hat{\iota}} \mathrm{HF}^+(Y) \xrightarrow{U} \mathrm{HF}^+(Y) \longrightarrow \cdots.$$

Heegaard Floer homology fits into a TQFT in the following sense. When W is a smooth cobordism from the 3-manifolds Y_1 to Y_2, there are induced maps between the long exact sequences as follows. Here F_W^∞, F_W^-, F_W^+, $\widehat{F_W}$ are unique homomorphisms induced from W (see [39] for the definition).

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{HF}^-(Y_1) \arrow[r, "\iota"] \arrow[d, "F_W^-"] & \mathrm{HF}^\infty(Y_1) \arrow[r, "\pi"] \arrow[d, "F_W^\infty"] & \mathrm{HF}^+(Y_1) \arrow[r] \arrow[d, "F_W^+"] & \cdots \\
\cdots \arrow[r] & \mathrm{HF}^-(Y_2) \arrow[r, "\iota"] & \mathrm{HF}^\infty(Y_2) \arrow[r, "\pi"] & \mathrm{HF}^+(Y_2) \arrow[r] & \cdots
\end{tikzcd},$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & \widehat{\mathrm{HF}}(Y_1) \arrow[r, "\hat{\iota}"] \arrow[d, "\widehat{F}_W"] & \mathrm{HF}^+(Y_1) \arrow[r, "U"] \arrow[d, "F_W^+"] & \mathrm{HF}^+(Y_1) \arrow[r] \arrow[d, "F_W^+"] & \cdots \\
\cdots \arrow[r] & \widehat{\mathrm{HF}}(Y_2) \arrow[r, "\hat{\iota}"] & \mathrm{HF}^+(Y_2) \arrow[r, "U"] & \mathrm{HF}^+(Y_2) \arrow[r] & \cdots
\end{tikzcd}.$$

3.1.4 Spin^c-structures

The chain complexes and maps introduced in Chapter 3.1.3 all split as a direct sum along Spin^c-structures. For instance, we have the splitting

$$\mathrm{CF}(Y) = \bigoplus_{s \in \mathrm{Spin}^c(Y)} \mathrm{CF}(Y, s),$$

where CF can be any of $\widehat{\mathrm{CF}}$, CF^∞, CF^- and CF^+, and $\mathrm{Spin}^c(Y)$ denotes the set of Spin^c-structures over Y.

In this section, we discuss the definition of Spin^c-structure and the splitting of Heegaard Floer complexes.

Our exposition of Spin^c-structure largely follows from Chapter 11 of Turaev's book [47]. To begin with, let $\mathrm{Spin}(n)$ be the universal cover space of $SO(n)$, and define $\mathrm{Spin}^c(n)$ as

$$\mathrm{Spin}^c(n) := (\mathrm{Spin}(n) \times U(1))/\langle(-1, -1)\rangle.$$

Recall that $SO(3) \cong SU(2)/\{\pm 1\} \cong U(2)/U(1)$ where $U(1)$ lies in $U(2)$ as the diagonal subgroup. Then $\mathrm{Spin}(3) \cong SU(2)$, and

$$\mathrm{Spin}^c(3) = (\mathrm{Spin}(3) \times U(1))/\langle(-1,-1)\rangle \cong (SU(2) \times U(1))/\langle(-1,-1)\rangle \cong U(2).$$

For the case $n = 4$, we have $\mathrm{Spin}(4) = SU(2) \times SU(2)$ and

$$\mathrm{Spin}^c(4) = \{(A, B) \in U(2) \times U(2) | \det(A) = \det(B)\},$$

and the identity

$$SO(4) = (SU(2) \times SU(2))/\{\pm 1\} = (U(2) \times U(2))/U(1).$$

Remember that the isomorphism classes of the principal $U(1)$ bundles over a CW-complex M are numbered by elements in $H^2(M, \mathbb{Z})$. If we think of $\mathrm{Spin}^c(n) \to SO(n)$ as a $U(1)$ bundle in the case of $n = 3, 4$, the first Chern class of the bundle corresponds to the non-trivial element of $H^2(SO(n), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Namely $\mathrm{Spin}^c(n) \to SO(n)$ is the unique non-trivial $U(1)$ bundle over $SO(n)$ for $n = 3, 4$.

In the following, we suppose Y is a 3-manifold and W is a 4-manifold, both are smooth compact oriented manifolds and possibly have boundary. Endow both manifolds with Riemannian metrics. Consider the bundle of oriented orthonormal frames over each manifold, and denote it by Fr_Y or Fr_X. It is a principal $SO(3)$ or $SO(4)$ bundle by definition.

A Spin^c-structure s over Y is a pair (P, α) where P is a principal $\mathrm{Spin}^c(3)$ bundle $P \to Y$ and $\alpha : P \to Fr_Y$ is a bundle map which fiberwise restricts to give the unique non-trivial $U(1)$ bundle $\mathrm{Spin}^c(n) \to SO(n)$. Note that $\alpha : P \to Fr_Y$ is a $U(1)$-bundle over Fr_Y. Two pairs (P, α) and (P', α') represent the same Spin^c-structure if there is a $\mathrm{Spin}^c(3)$ bundle isomorphism $P \to P'$ such that α is the composition of α' and the bundle isomorphism

$P \to P'$. A Spin^c-structure is essentially independent of the Riemannian metric. The set of Spin^c-structures over Y is denoted Spin$^c(Y)$.

Now we explain that there exists a free and transitive action of $H^2(Y, \mathbb{Z})$ on Spin$^c(Y)$.

The definition of Spin^c-structure equivalently says that a Spin^c-structure s is a $U(1)$ bundle $P \to Fr_Y$ which restricts over each fiber $SO(3) \hookrightarrow Fr_Y$ to give the unique non-trivial $U(1)$ bundle Spin$^c(3) \to SO(3)$. Recall the one-to-one correspondence between $U(1)$ bundles and elements of the second cohomology group. We can regard a Spin^c-structure $s \in$ Spin$^c(Y)$ as an element from $H^2(Fr_Y, \mathbb{Z})$ which on each fiber $SO(3) \hookrightarrow Fr_Y$ restricts to the unique non-trivial element $H^2(SO(3), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Fixing an orthogonal trivialization of TY, we obtain the diffeomorphism $Fr_Y \cong Y \times SO(3)$, and therefore

$$H^2(Fr_Y, \mathbb{Z}) \cong H^2(Y, \mathbb{Z}) \oplus H^2(SO(3), \mathbb{Z}) \cong H^2(Y, \mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

A Spin^c-structure on Y is therefore an element of $H^2(Fr_Y, \mathbb{Z})$ with non-trivial second coordinate in this decomposition. There is an obvious action of $H^2(Y, \mathbb{Z})$ on $H^2(Fr_Y, \mathbb{Z})$ given by the pullback map on second cohomology induced by $Fr_Y \to Y$. This action, viewed as an action on Spin$^c(Y)$, is obviously free and transitive.

Similar arguments apply to the 4-manifold W as well.

Ozsváth and Szabó revealed the splitting of the Heegaard Floer complex of Y along Spin$^c(Y)$ by assigning a Spin^c-structure to each generator of the complex. Precisely, consider the complex $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ defined via the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ of genus g. Then Ozsváth and Szabó constructed a map $\mathfrak{s}_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to$ Spin$^c(Y)$.

We do not give the definition of $\mathfrak{s}_z$. Instead, we recall a necessary and sufficient condition for two generators to have the same Spin^c-structure under $\mathfrak{s}_z$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be a pair of intersection points. Choose a pair of paths $a : [0,1] \to \mathbb{T}_\alpha$ and $b : [0,1] \to \mathbb{T}_\beta$ from $\mathbf{x}$ to $\mathbf{y}$

in $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ respectively. The difference $a-b$ gives a loop in $\mathrm{Sym}^g(\Sigma)$. Then we have the following property.

Proposition 3.1.15. *For the generators $\mathbf{x}, \mathbf{y}$, $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y})$ if and only if the homology class of $a-b$ is zero in $\dfrac{H_1(\mathrm{Sym}^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong H_1(Y, \mathbb{Z})$.*

It is obvious from the definition that if $a-b$ is homologically non-zero, then $\pi_2(\mathbf{x}, \mathbf{y}) = \emptyset$. This implies that the Heegaard Floer complex $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ splits along $\mathrm{Spin}^c(Y)$.

Example 3.1.16 (Continuation of Example 3.1.9)**.** *For the lens space $L(p,1)$, the set $\mathrm{Spin}^c(L(p,1))$ contains p elements. Recall the genus one Heegaard diagram $(\Sigma, \alpha, \beta, z)$. The Heegaard Floer complex $\mathrm{CF}^\infty(\Sigma, \alpha, \beta, z)$ has generators $x_1, x_2, \cdots, x_p$. Choose a path $a_{i,j} \subset \alpha$ and a path $b_{i,j} \subset \beta$ connecting the generators x_i and x_j for $i \neq j$. Then $(a_{i,j} - b_{i,j})$ as a homology class in $H_1(L(p,1), \mathbb{Z})$, is non-trivial. Therefore x_i and x_j for $i \neq j$ belong to different Spin^c-structures in $\mathrm{Spin}^c(L(p,1))$. Therefore the differential of $\mathrm{CF}^\infty(\Sigma, \alpha, \beta, z)$ is trivial, and the homology*

$$\mathrm{HF}^\infty(L(p,1)) = \bigoplus_{s \in \mathrm{Spin}^c(L(p,1))} \mathrm{HF}^\infty(L(p,1), s) = \bigoplus_{s \in \mathrm{Spin}^c(L(p,1))} \mathbb{F}[U, U^{-1}],$$

where $\mathbb{F}[U, U^{-1}]$ is the ring of Laurent polynomials with variable U. We identify U^i with $[x, -i]$ for any $i \in Z$, where x is the generator corresponding to $s \in \mathrm{Spin}^c(L(p,1))$.

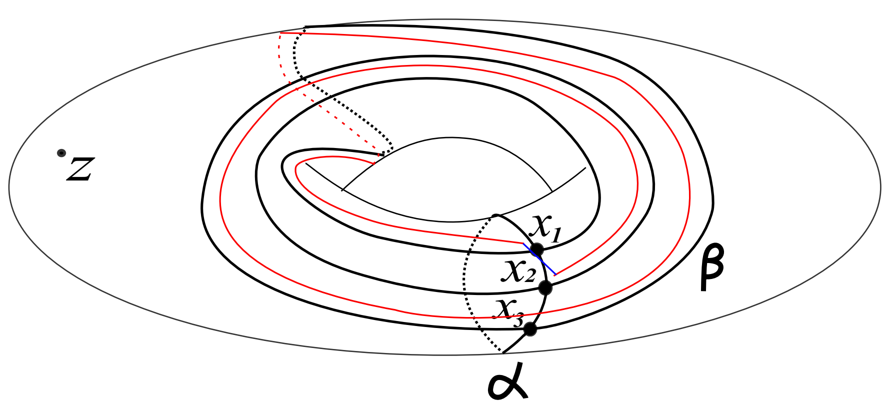

Figure 3.2: The blue arc is $a_{1,2}$, and the red arc is $b_{1,2}$.

Suppose W is a cobordism from Y_1 to Y_2. We also have the splitting

$$F_W = \bigoplus_{t \in \mathrm{Spin}^c(W)} F_{W,t},$$

where $F_{W,t}$ is a homomorphism from $\mathrm{HF}(Y_1, t\,\big|_{Y_1})$ to $\mathrm{HF}(Y_2, t\,\big|_{Y_2})$. Here $t\,\big|_{Y_1}$ is the restriction of t onto Y_i, and "HF" can be any version of Heegaard Floer homology.

3.1.5 Maslov grading

Besides the splitting along Spin^c-structures, Heegaard Floer homology is also a graded homology. There exists an absolute $\mathbb{Q}$-grading on it, called Maslov grading, which in fact works as a homological dimension. Namely, the differential decreases the Maslov grading by one.

Use $\mathrm{gr}(*)$ to denote the Maslov grading of an element. For generators $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, their relative Maslov grading difference is fixed by the formula

$$\mathrm{gr}(x) - \mathrm{gr}(y) = \mu(\phi) - 2n_z(\phi),$$

where ϕ is an element in $\pi_2(x, y)$. It is easy to check that the differential $\widehat{\partial}$ decreases the Maslov grading by one. Extend this formula to the complex $\mathrm{CF}^\infty(Y)$ by requiring that

$$\mathrm{gr}([x, i]) - \mathrm{gr}([y, j]) = (\mathrm{gr}(x) - \mathrm{gr}(y)) + (2i - 2j),$$

for any $i, j \in \mathbb{Z}$.

The Maslov grading can be now uniquely characterized by the following properties.

- The maps $\hat{\iota}$, ι and π all preserve the Maslov grading (see Chapter 3.1.3).
- The space $\widehat{\mathrm{HF}}(S^3) \cong \mathbb{F}$ is supported in Maslov grading zero.

- If (W,t) is a smooth cobordism from (Y_1,s_1) to (Y_2,s_2), and $\xi \in \mathrm{HF}^\infty(Y_1,s_1)$, then

$$\mathrm{gr}(F^\infty_{W,t}(\xi)) - \mathrm{gr}(\xi) = \frac{c_1^2(t) - 2\chi(W) - 3\sigma(W)}{4},$$

  where $c_1(t) \in H^2(W;\mathbb{Z})$ is the first Chern class of t and $c_1^2(t) \in \mathbb{Z}$ is the image of $(c_1(t), c_1(t))$ under the cup product, and $\chi(W)$ and $\sigma(W)$ are the Euler characteristic and signature of W respectively.

3.2 The definition and some properties of the d-invariant

Now we are ready to give the definition of Ozsváth-Szabó's d-invariant.

Definition 3.2.1. Let Y be a rational homology 3-sphere. The *d-invariant* of Y is a map $d : \mathrm{Spin}^c(Y) \to \mathbb{Q}$ which sends each $s \in \mathrm{Spin}^c(Y)$ to

$$d(Y,s) := \min\{\mathrm{gr}(\xi) | 0 \neq \xi \in \mathrm{Im}(\mathrm{HF}^\infty(Y,s) \xrightarrow{\pi} \mathrm{HF}^+(Y,s))\}$$

Since in general Heegaard Floer homology is not an easy invariant to calculate, the d-invariant therefore is not easy to calculate as well. The d-invariants for some manifolds with simple Heegaard Floer homology have been well studied. For instance, an inductive formula for the d-invariants of lens spaces is described in [34, Proposition 4.8].

Example 3.2.2 (Continuation of Example 3.2.2)**.** *The d-invariants for the lens space $L(p,1)$ where $p \geq 1$ has the following explicit form*

$$d(L(p,1),s_i) = \frac{(2i-2-p)^2 - p}{4p},$$

where s_i is the Spin^c-structure corresponding to x_i for $1 \leq i \leq p$.

The d-invariants for the 3-manifolds Y and $-Y$, where " $-$ " means the reverse of the orientation, are related by the formula

$$d(-Y,s) = -d(Y,s),$$

under the natural identification $\mathrm{Spin}^c(Y) \cong \mathrm{Spin}^c(-Y)$.

It is known that the d-invariant is additive under the connected sum of 3-manifolds. Given two homology 3-spheres Y_1 and Y_2, and Spin^c-structures $s_i \in \mathrm{Spin}^c(Y_i)$ for $i = 1, 2$, we have

$$d(Y_1 \sharp Y_2, s_1 \sharp s_2) = d(Y_1, s_1) + d(Y_2, s_2),$$

where $s_1 \sharp s_2$ means the sum of Spin^c-structures.

Suppose that Y is an oriented rational homology 3-sphere, that W is an oriented negative-definite smooth 4-manifold with $\partial W = Y$ and that $t \in \mathrm{Spin}^c(W)$. Then Ozsvath and Szabo [34] proved the following theorem.

Theorem 3.2.3. *With the assumptions above, we have*

$$\begin{aligned} d(Y, t\,\big|_Y) &\geq \frac{c_1^2(t) + b_2(W)}{4}, \\ d(Y, t\,\big|_Y) &= \frac{c_1^2(t) + b_2(W)}{4} \pmod 2, \end{aligned}$$

where $b_2(X)$ is the second Betti number of W.

When W is a rational smooth 4-cobordism from the rational homology 3-sphere Y_1 to the rational homology 3-sphere Y_2, we have the following corollary from Theorem 3.2.3.

Corollary 3.2.4. *With the assumption above, we have*

$$d(Y_1, t\,\big|_{Y_1}) = d(Y_2, t\,\big|_{Y_2}) \tag{3.1}$$

for any $t \in \mathrm{Spin}^c(W)$. In particular if Y_2 is the empty set, then $d(Y_1, t\,\big|_{Y_1}) = 0$.

Donaldson's celebrated diagonalization theorem can be regarded as a corollary of Theorem 3.2.3.

Theorem 3.2.5 (Donaldson)**.** *If W is a smooth closed oriented 4-manifold with definite intersection form, then the form is diagonalizable over $\mathbb{Z}$.*

3.3 Some known topological applications of the d-invariant

The d-invariant has many topological applications. In this section, we pick up two topics among them.

3.3.1 An application to X-unknotting number

The local transformation in Figure 3.3 occurring in a local neighbourhood of a knot projection is called a crossing change. It is easy to see that any knot projection can be changed into a projection of the unknot by performing crossing changes. Given a knot $K \subset S^3$, the *X-unknotting number* $u(K)$ of K is defined to be the minimal number of crossing changes needed to change K into the unknot.

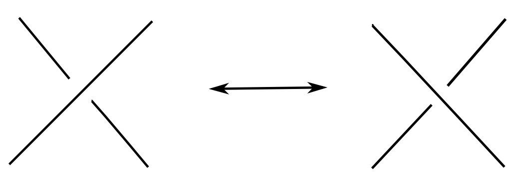

Figure 3.3: Crossing change.

Using the d-invariant, Ozsváth and Szabó [37] studied those knots which X-unknotting number one. Before stating their theorem, we introduce some necessary notations.

Given a knot $K \subset S^3$, let $\Sigma(K)$ denote the double branched cover of S^3 along K. If $u(K) = 1$, then by Montesinos's trick [31] we have $\Sigma(K) = \pm S^3_{p/2}(C)$ for some knot C in S^3, where p is a positive odd number. Suppose $p = 2n - 1$, and define a matrix

$A = \begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}$. Then A is a negative definite integral matrix. Considering A as a linear map from $\mathbb{Z}^2$ to $\mathbb{Z}^2$, then we see it determines a group $G_A := \mathbb{Z}^2/\mathrm{Im}A$. Define the map

$$\begin{aligned}\gamma_A : \mathbb{Z}/p\mathbb{Z} &\to \mathbb{Q} \\ [i] &\mapsto \max\left\{ \frac{\xi A^{-1}\xi^t + 2}{4} \middle| \xi = (2k-n, 2l), \xi - (i,0) \in \mathrm{Im}A, \forall k,l \in \mathbb{Z} \right\},\end{aligned}$$

where $0 \leq i \leq p-1$. Ozsváth and Szabó proved the following theorem for knots with X-unknotting number one.

Theorem 3.3.1. *If K is a knot with X-unknotting number one, then there is an isomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Spin}^c(\Sigma(K))$ and a number $\varepsilon = \{1,-1\}$ with the property that for all $i \in \mathbb{Z}/p\mathbb{Z}$, the following condition holds:*

$$T(i) := \varepsilon \cdot d(\Sigma(K), \phi(i)) - \gamma_A(i) \equiv 0 \pmod 2 \text{ and } T(i) \geq 0.$$

Furthermore, if K is an alternating knot and $|d(\Sigma(K), s_0)| \leq 1/2$ for the unique Spin^c structure s_0 with $c_1(s_0) = 0$, we have the following additional symmetry

$$T(i) = T(2t - i)$$

for $1 \leq i < k$ when $p = 4k-1$ and for $0 \leq i < k$ when $p = 4k+1$.

Using this theorem, Ozsváth and Szabó determined the X-unknotting number for many knots whose X-unknotting number was unknown before. We name the knots according to Rolfson's knot table.

Corollary 3.3.2. *The knots $8_{10}, 9_{29}, 9_{32}$, and other thirty-three ten crossing knots have X-unknotting number equal to two.*

Owens [33] further generalized Ozsváth and Szabó's idea to knots with higher X-unknotting number.

3.3.2 An application to knot concordance

An oriented knot K in the 3-sphere S^3 is said to be *smoothly slice* if there is a smoothly embedded 2-disk Δ in the 4-ball B^4 satisfying $\partial(B^4, \Delta) = (S^3, K)$. Here Δ is called a *slice disk* of K. A pair of knots K_1 and K_2 are said to be *smoothly concordant* (which is denoted by $K_1 \sim K_2$) if $K_1 \sharp (-K_2)$ is slice where $-K_2$ is the mirror image of K_2 with reversed orientation. Smooth concordance is an equivalence relation among knots and the set of equivalence classes becomes an abelian group under the operation of connected sum. The group is called the knot concordance group and denoted by C. Slice knots represent the zero element in C.

There have been many methods to judge whether a knot is smoothly slice or not, such as the signature of a knot, the twisted Alexander polynomials of a knot, the Casson-Gordon invariant, Rasmussen invariant and so on and so forth. The d-invariant also provides criteria for judging whether a knot is slice or not. Research on this direction has flourished very well in recent years. In the following paragraphs, we introduce an important criterion provided by the d-invariant.

Suppose p is a positive power of a positive prime number, and let $\Sigma^p(K)$ denote the p-fold branched cover of S^3 along a given knot K. The following lemma is included in [13].

Lemma 3.3.3. *When K is a smoothly slice knot, the manifold $\Sigma^p(K)$ is a rational homology 3-sphere.*

A slice knot K bounds a slice disk Δ in the 4-ball B^4. Let $W^p(\Delta)$ denote the p-fold branched cover of B^4 along Δ. Lemma 3.3.3 and Corollary 3.2.4 immediately imply the following fact.

Theorem 3.3.4. *When K is a smoothly slice knot, we have*

$$d(\Sigma^p(K), s) = 0$$

for any $s \in \mathrm{Spin}^c(K)$ which can be extended to a Spin^c-structure over $W^p(\Delta)$.

This fact has been contributed to a lot of important research, such as the confirmation of slice-ribbon conjecture for the case of 2-bridge knots [29] and pretzel knots of odd parameters [16]. Other papers, such as [17, 30, 24, 2], are also partially based on this fact.

If we replace every "smooth" by "locally flat" in the definition of slice knot, we obtain the definition of topologically slice knot. Distinguishing topological sliceness and smoothly sliceness has been a long-term problem since the appearance of knot concordance theory. Many invariants, such as signature, twisted Alexander polynomial and Casson-Gordon invariant, are defined in topological category and never tell the difference between topological sliceness and smooth sliceness. The d-invariant, however, is defined in smooth category. Theorem 3.3.4, for example, does not hold for topologically slice knots.

Using the d-invariant, recent work, such as [18, 6], constructed many topologically slice knots or links which are "far away" from smoothly slice knots or links. We expect further applications of the d-invariant in this direction.

Chapter 4

H(2)-unknotting number

Hoste, Nakanishi and Taniyama in [21] defined H(n)-unknotting operation. In this chapter, we focus on the case when n equals two. One can define the H(2)-unknotting number and H(2)-Gordian distance by using H(2)-unknotting operation. These geometric invariants carry interesting information about knots. They have alternative interpretations in different contexts and bear various relations with some other invariants of a knot. In this chapter, we review some of these basic facts about them.

4.1 Alternative interpretations of H(2)-unknotting number

See Figure 4.1 for the definition of an H(2)-move. The *H(2)-Gordian distance* $u_2(K_1, K_2)$ of two knots $K_1, K_2 \subset S^3$ is the minimal number of H(2)-moves needed to change the knot K_1 into the knot K_2. The *H(2)-unknotting number* $u_2(K)$ of a knot $K \subset S^3$ is the H(2)-Gordian distance of K and the unknot. Note that unlike crossing change, an H(2)-move does not preserve the orientation of a link. Therefore H(2)-unknotting number can be regarded as a "non-orientable" invariant. The definition involved in this thesis is slightly different from the one given in [21]. Here we allow a H(2)-move to change the number of components, while it is not allowed in [21].

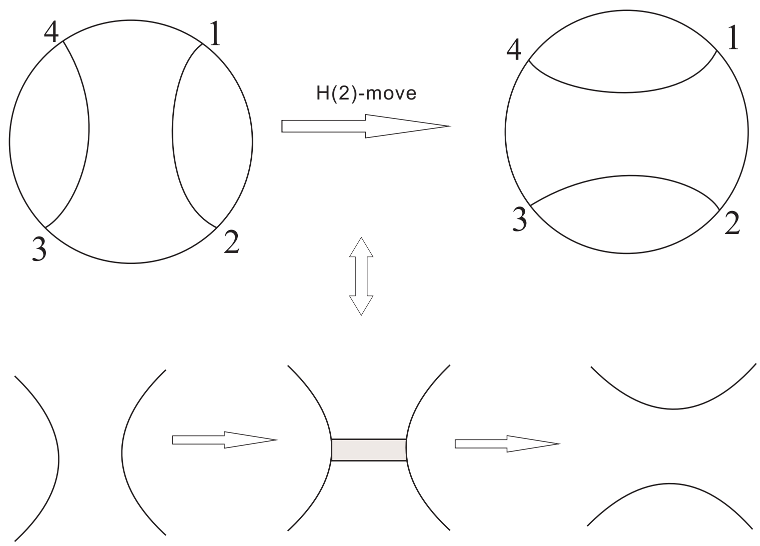

Figure 4.1: An H(2)-move.

This section deals with alternative interpretations of H(2)-Gordian distance. We largely follow Taniyama and Yasuhara's work [46].

Two knots K_1 and K_2 are said to be *g-bordant* if there is a compact connected surface (possibly non-orientable) $F \subset S^3$ with the first Betti number $\beta_1(F) = g + 1$ such that $\partial F = K_1 \cup K_2$. Then define

$$d_b(K_1, K_2) := \min\{g | K_1 \text{ is } g\text{-bordant to } K_2\}.$$

Consider a locally flat (possibly non-orientable) surface F properly embedded in $S^3 \times [0,1]$ with $\partial F = K_1 \times \{0\} \cup K_2 \times \{1\}$. The surface F is called *regular* if the natural projection $F \hookrightarrow S^3 \times [0,1] \to [0,1]$ is a Morse function on F. Then we define

$$c(K_1, K_2) := \min\{c(F) | F \text{ is a regular surface connecting } K_1 \times \{0\} \text{ to } K_2 \times \{1\} \},$$

where $c(F)$ is the number of critical points of the corresponding Morse function.

Taniyama and Yasuhara [46] proved the following relations. Please refer there for the proof.

Theorem 4.1.1 (Theorem 5.1 in [46]). *For any knots K_1 and K_2 in S^3, we have*

$$d_b(K_1, K_2) = u_2(K_1, K_2) = c(K_1, K_2).$$

There are two non-orientable invariants of knots which are closely related to H(2)-unknotting number. They are the crosscap number and the 4-dimensional crosscap number of a knot. Given a knot $K \subset S^3$, the *crosscap number* of K [7] is defined as follows:

$$\gamma(K) = \min \left\{ \beta_1(F) \,\middle|\, F \text{ is a non-orientable connected surface in } S^3 \text{ and } \partial F = K \right\}.$$

The *4-dimensional crosscap number* of K [32], which we denote $\gamma^*(K)$ here, is by name defined as follows:

$$\gamma^*(K) = \min \left\{ \beta_1(F) \,\middle|\, \begin{array}{l} F \text{ is a non-orientable connected smooth surface in } B^4 \text{ and} \\ \partial F = K \subset \partial B^4 = S^3 \end{array} \right\}.$$

Their relation with H(2)-unknotting number is as follows.

Lemma 4.1.2. *Given a knot $K \subset S^3$, we have $\gamma^*(K) - 1 \leq u_2(K) \leq \gamma(K)$.*

Proof. The knot K can be reconstructed from the unknot by adding $u_2(K)$ twisted bands successively. Let D be a disk bounded by the unknot and $b_1, b_2, \ldots, b_{u_2(K)}$ be the bands added to the boundary of D. Then we can push $F := D \cup \bigcup_{i=1}^{u_2(K)} b_i$ into B^4 to get a embedded surface with $\partial F = K$. If F is orientable, we add a trivial half-twisted band to F to make it non-orientable. Therefore we have $\gamma^*(K) \leq \beta_1(F) + 1 = u_2(K) + 1$. The second inequality is proved as follows. Suppose S is a non-orientable surface in S^3 which realizes the crosscap number of K. Namely we have $\beta_1(S) = \gamma(K)$ and $\partial S = K$. Then there are $\gamma(K)$ disjoint essential arcs in S, say $\tau_1, \tau_2, \cdots, \tau_{\gamma(K)}$, such that $S - \tau_i$ has one boundary component for

$i = 1, 2, \cdots, \gamma(K)$ and $S - \bigcup_{i=1}^{\gamma(K)} \tau_i$ is a disk. If we add twisted bands to K along τ_i for $i = 1, 2, \cdots, \gamma(K)$, the resulting knot is the unknot. Therefore we have $u_2(K) \leq \gamma(K)$. $\square$

4.2 Criteria for detecting H(2)-unknotting number

4.2.1 A criterion from linking form

This section comes from Lickorish's work [27]. Before reviewing his result, we first recall the definition of the bilinear form introduced in [15]. The definition is a generalization of both Seifert paring and Goeritz matrix. Bilinear forms associated to a surface have also been studied earlier by Blanchfield, Fox, Murasugi and some others.

Let $F_K \subset S^3$ be a spanning surface of the knot $K \subset S^3$, possibly orientable or non-orientable. Suppose α is a simple closed curve in F_K. Then let $\tau\alpha$ be the 1-cycle obtained by pushing off 2α into $S^3 - F$. Define

$$\begin{aligned} \theta : H_1(F_K, \mathbb{Z}) \times H_1(F_K, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ (\alpha, \beta) &\mapsto \mathrm{lk}(\alpha, \tau\beta). \end{aligned}$$

This is a non-degenerate bilinear form. Suppose A is a matrix representing the form. Then A is a non-singular matrix and $|\det(A)|$ is an invariant of K, which is called the *determinant* of K, denoted by $\det(K)$. Here is Lickorish's result and proof.

Theorem 4.2.1. *Let $\Sigma(K)$ be the double branched cover of S^3 along K. If $u_2(K) = 1$, then $H_1(\Sigma(K), \mathbb{Z})$ is cyclic of order $\det(K)$ and the linking form*

$$\lambda : H_1(\Sigma(K), \mathbb{Z}) \times H_1(\Sigma(K), \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

has the property that there is a generator $g \in H_1(\Sigma(K), \mathbb{Z})$ such that $\lambda(g, g) = \pm 1/\det(K)$.

Proof. We can choose knot diagrams for K and the unknot O so that they are identical except in a neighbourhood of a single point as in Figure 4.2. We can further assume that there is an orientable surface F_O for O meeting the neighbourhood only in the two shadow regions, and that there is a basis B of $H_1(F_O, \mathbb{Z})$ which makes the form $\theta : H_1(F_O, \mathbb{Z}) \times H_1(F_O, \mathbb{Z}) \to \mathbb{Z}$ a direct sum of copies of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

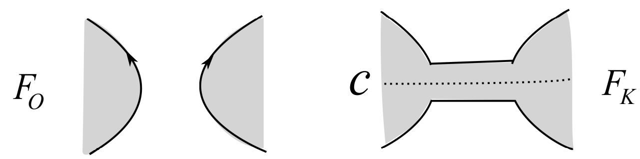

Figure 4.2: Surfaces F_O and F_K.

Adding a band to F_O produces a non-orientable surface F_K spanning K. Let c be a closed curve that travels once across the band. Then $B \cup \{c\}$ is a basis for $H_1(F_K, \mathbb{Z})$. Suppose that a neighbourhood of c in F_K contains n half-twists. The bilinear form θ : $H_1(F_K, \mathbb{Z}) \times H_1(F_K, \mathbb{Z}) \to \mathbb{Z}$ associated with $B \cup \{c\}$ is a matrix of the form

$$\left(\begin{array}{c|c} n & \rho \\ \hline \rho^T & \bigoplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right).$$

We can replace the curve c by a curve c' so that the new matrix associated with the basis $B \cup \{c'\}$ has the form

$$A = \left(\begin{array}{c|c} d & 0 \\ \hline 0 & \bigoplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}\right).$$

By [15] the matrix A is a presentation matrix for $H_1(\Sigma(K), \mathbb{Z})$ and with respect to the same basis the linking form λ on $H_1(\Sigma(K), \mathbb{Z})$ is represented by the matrix $\pm A^{-1}$. Hence $H_1(\Sigma(K), \mathbb{Z})$ has a generator g of order d for which $\lambda(g, g) = \pm 1/d$, while it is easy to see that $\det(K) = |\det(A)| = d$. $\square$

Here is the example included in Lickorish's paper [27].

Example 4.2.2. *For the figure-eight knot 4_1, the linking form has the form $\lambda(g, g) = 2/5$ for some generator $g \in H_1(\Sigma(4_1)) \cong \mathbb{Z}/5\mathbb{Z}$. If there is another generator $g' = xg$ such that $\lambda(g', g') = \pm 1/5$, we have $2x^2 \equiv \pm 1 \pmod 5$, while there is no such an integer x satisfying the condition. As a result $u_2(4_1) \geq 2$. It is easy to see that $u_2(4_1) = 2$ since 4_1 can be unknotted by performing two H(2)-moves.*

4.2.2 Criteria from other invariants

Th following theorem combines some results in [21, 48, 26], the proof of which will be omitted.

Theorem 4.2.3. *Given a knot $K \subset S^3$, we have the following facts about the H(2)-unknottting number of K.*

1. *For an integer $p \geq 2$, we have $u_2(K) \geq mg(\Sigma^p(K))/(p-1)$, where $\Sigma^p(K)$ is the p-fold branched cover of S^3 along K, and $mg(\Sigma^p(K))$ is the minimal number of generators of $H_1(\Sigma^p(K), \mathbb{Z})$.*

2. *Suppose $u_2(K) = 1$. Then*

   - *if $\sigma(K) \equiv 0 \pmod 8$, then $\mathrm{Arf}(K) = 0$;*
   - *if $\sigma(K) \equiv 4 \pmod 8$, then $\mathrm{Arf}(K) = 1$.*

   Here $\sigma(K)$ is the signature of K and $\mathrm{Arf}(K)$ is the Arf invariant of K.

3. *Suppose $u_2(K) = 1$. Assume that $\det(K) \equiv 0 \pmod 3$, and $\sigma(K) \equiv 2\epsilon \pmod 8$ for $\epsilon = \pm 1$. Then*

   - *if $\mathrm{Arf}(K) = 0$, then $J'_K(-1) \equiv 8\epsilon \pmod{24}$;*
   - *if $\mathrm{Arf}(K) = 1$, then $J'_K(-1) \equiv -8\epsilon \pmod{24}$.*

   Here $J'_K(t)$ is the first derivative of the Jones polynomial of K.

Chapter 5

On knots with H(2)-unknotting number one

In the previous chapter, we have reviewed some known criteria for detecting the H(2)-unknotting number of a knot. In this chapter, we give a necessary condition for a knot K to have $u_2(K) = 1$ (see [1]), by using a method introduced by Ozsváth and Szabó [37].

5.1 Preliminaries

Consider a negative-definite symmetric $n \times n$ matrix Q over $\mathbb{Z}$, and suppose $|\det(Q)|$ is p. Then define the group

$$G_Q := \mathbb{Z}^n/\mathrm{Im}(Q).$$

A *characteristic vector* for Q is an element in

$$\begin{aligned}\mathrm{char}(Q) &= \left\{\xi = (\xi_1, \xi_2, \cdots, \xi_n)^t \in \mathbb{Z}^n | \xi^t v \equiv v^t Q v \pmod 2 \text{ for any } v \in \mathbb{Z}^n\right\} \\ &= \{\xi \in \mathbb{Z}^n | \xi_i \equiv Q_{ii} \pmod 2 \text{ for } 1 \le i \le n\}.\end{aligned}$$

Suppose p is odd, and consider the map (cf. [33, 37])

$$M_Q : G_Q \longrightarrow \mathbb{Q}$$

defined by

$$M_Q(\alpha) = \max \left\{ \frac{\xi^t Q^{-1} \xi + n}{4} \middle| \xi \in \operatorname{char}(Q), [\xi] = \alpha \in G_Q \right\}.$$

In this section, we apply Theorem 3.2.3 in a simply setting. Let us adjust the theorem to fit the situation considered here. Suppose that X is simply-connected negative-definite 4-manifold whose boundary $\partial X = Y$ is a rational homology 3-sphere. Assume further that the order of $H^2(Y, \mathbb{Z})$ is odd. Then there exists a group structure on the space $\operatorname{Spin}^c(Y)$ by identifying $s \in \operatorname{Spin}^c(Y)$ with $c_1(s) \in H^2(Y, \mathbb{Z})$. In the following, we denote the d-invariant $d(Y, s)$ by $d(Y, c_1(s))$ if necessary. Let r denote the second Betti number of X. Then we have the following exact sequence:

$$0 \to H_2(X) = \mathbb{Z}^r \xrightarrow{\tau} H^2(X) = \mathbb{Z}^r \xrightarrow{j^*} H^2(Y) \to H_1(X) = 0.$$

Fix a basis for $H_2(X)$ and let B be the matrix of the intersection form of X. Then B is a symmetric negative-definite $r \times r$ integer matrix with $|\det B| = |H^2(Y, \mathbb{Z})|$. Any Spin^c-structure $s \in \operatorname{Spin}^c(Y)$ is the restriction of a Spin^c-structure $t \in \operatorname{Spin}^c(X)$ on Y if and only if $j^*(c_1(t)) = c_1(s)$.

In fact, the map τ under the given basis of $H_2(X)$ is presented by the matrix B. We define φ as the map $\operatorname{Coker}(\tau) = G_B \xrightarrow{j_1^*} H^2(Y)$, where j_1^* is the map induced from j^* on the cokernel of τ. It is obvious from the exact sequence that φ is an isomorphism. Under φ the set of characteristic vectors $\operatorname{char}(B)$ is equal to the set $\{c_1(t) \,|\, t \in \operatorname{Spin}^c(X)\} \subset H^2(X, \mathbb{Z})$. If we suppose the first Chern class $c_1(t)$ corresponds to the characteristic vector ξ, then $c_1^2(t)$ is equal to $\xi^t B^{-1} \xi$.

Under these identifications, Theorem 3.2.3 can be written as follows. For any $s \in \mathrm{Spin}^c(Y)$ and any $\xi \in \mathrm{char}(B)$ with $c_1(s) = \varphi([\xi])$, there are

$$d(Y, c_1(s)) \geq \frac{\xi^t B^{-1}\xi + r}{4} \text{ and } d(Y, c_1(s)) = \frac{\xi^t B^{-1}\xi + r}{4} \pmod 2.$$

This is equivalent to say under the isomorphism $\varphi : G_B \to H^2(Y, \mathbb{Z})$ the following hold for any $\alpha \in G_B$:

$$\begin{aligned} &d(Y, \varphi(\alpha)) \geq M_B(\alpha), \\ &d(Y, \varphi(\alpha)) = M_B(\alpha) \pmod 2. \end{aligned} \tag{5.1}$$

5.2 Result

For knots with H(2)-unknotting number one, we have the following lemma. It is a special case of Lemma 6.2.7 in next chapter, since the knot K in Lemma 5.2.1 is a special tangle unknotting number one knot (see Definition 6.2.5). Please refer there for a proof.

Lemma 5.2.1 (Montesinos's trick [31]). *If the H(2)-unknotting number of a knot K is one, then $\Sigma(K) = \epsilon \cdot S^3_{-p}(C)$ for some knot $C \subset S^3$ and $\epsilon \in \{+1, -1\}$. Here $p = |\det(K)|$ and $S^3_{-p}(C)$ denotes the $-p$-surgery of S^3 along the knot C.*

Our result about H(2)-unknotting number is as follows:

Theorem 5.2.2. *Let K be a knot with $|\det K| = p$. Since K is a knot, we see that p is an odd number. If $u_2(K) = 1$, then there is an isomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \longrightarrow H_1(\Sigma(K), \mathbb{Z})$ and a sign $\epsilon \in \{+1, -1\}$ with the properties that for all $i \in \mathbb{Z}/p\mathbb{Z}$:*

$$I_{\phi,\epsilon}(i) := \epsilon \cdot d(\Sigma(K), \phi(i)) + \frac{1}{4}\left(\frac{1}{p}\left(\frac{p + (-1)^i p}{2} - i\right)^2 - 1\right) = 0 \pmod 2,$$

$$\textit{and} \quad I_{\phi,\epsilon}(i) \geq 0.$$

Here we abuse i to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0,1,2,\cdots,p-1\}$.

If one is familar with the work in [37], the proof is immediate.

Proof. If the H(2)-unknotting number of K is one, then by Lemma 5.2.1 $\Sigma(K)=\epsilon\cdot S^3_{-p}(C)$ for some knot $C\subset S^3$ and $\epsilon\in\{+1,-1\}$ and $p=|\det(K)|$. Therefore $\epsilon\cdot\Sigma(K)=S^3_{-p}(C)$ bounds a 4-manifold X, which is obtained by attaching a 2-handle to the 4-ball along C with framing $-p$. The intersection form of X is $B=(-p)$. In this case $G_B=\mathbb{Z}/p\mathbb{Z}$, and X is a simply-connected negative-definite 4-manifold.

By (5.1), there is a group isomorphism $\varphi: G_B=\mathbb{Z}/p\mathbb{Z}\to H^2(S^3_{-p}(C),\mathbb{Z})\cong H^2(\Sigma(K),\mathbb{Z})$ with

$$d(S^3_{-p}(C),\varphi(i))=d(\epsilon\cdot\Sigma(K),\varphi(i))=\epsilon\cdot d(\Sigma(K),\varphi(i))\geq M_B(i)$$
$$\text{and}\quad \epsilon\cdot d(\Sigma(K),\varphi(i))\equiv M_B(i)\pmod 2.$$

In the following calculation, we abuse i to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0,1,2,\cdots,p-1\}$. By definition we see that for any $i\in\mathbb{Z}/p\mathbb{Z}$,

$$\begin{aligned} M_B(i) &= \max\left\{\frac{u^tB^{-1}u+1}{4}\,\middle|\, u \text{ is odd}, [u]=i\right\}\\ &= \max\left\{\frac{-u^2+p}{4p}\,\middle|\, u \text{ is odd}, [u]=i\right\}\\ &= \begin{cases} [-(p-i)^2+p]/4p & \text{if } i \text{ is even},\\ [-(i)^2+p]/4p & \text{if } i \text{ is odd}.\end{cases}\end{aligned}$$

Writing these two cases in one form we have $M_B(i)=-\frac{1}{4}(\frac{1}{p}(\frac{p+(-1)^ip}{2}-i)^2-1)$. This completes the proof of Theorem 5.2.2. □

In the case when K is an alternating knot, Theorem 5.2.2 can be described combinatorially. First we recall the definition of Goeritz matrix. Given a knot diagram, color

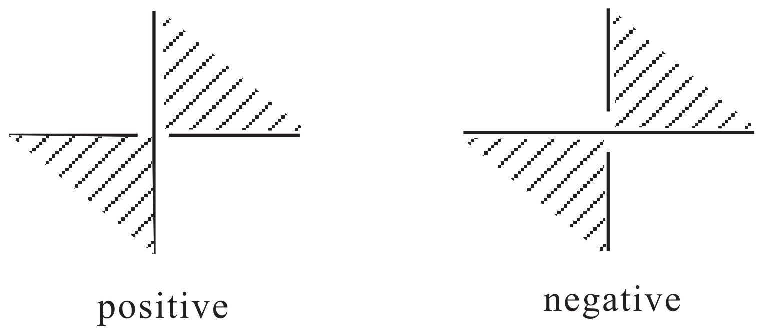

Figure 5.1: The sign convention of a crossing.

this diagram in checkerboard fashion such that the unbounded region has black color. Let $f_0, f_1, \ldots, f_k$ denote the black regions and f_0 correspond to the unbounded one. Define the sign of a crossing as in Figure 5.1. Then the Goeritz matrix $A = (a_{ij})$ is the $k \times k$ symmetric matrix defined as follows,

$$a_{ij} = \begin{cases} \text{the signed count of crossings adjacent to } f_i & \text{if } i = j, \\ \text{minus the signed count of crossings joining } f_i \text{ and } f_j & \text{if } i \neq j \end{cases}$$

for $i, j = 1, 2, \ldots, k$.

When K is an alternating knot in S^3, the d-invariants for $\Sigma(K)$ have an extremely easy combinatorial description as follows.

Theorem 5.2.3 (Ozsváth-Szabó[37, 38]). *If K is an alternating knot and A denotes a Goeritz matrix associated to a reduced alternating projection of K, and G_A is the group presented by A, then there is an isomorphism $\psi : H^2(\Sigma(K), \mathbb{Z}) \to G_A$, with the property that*

$$d(\Sigma(K), \beta) = M_A(\psi(\beta))$$

for all $\beta \in H^2(\Sigma(K), \mathbb{Z})$.

Then as a corollary of Theorem 5.2.2, we have the following corollary.

Corollary 5.2.4. *Let K be an alternating knot with $|\det K| = p$, and let A be the negative-definite Goeritz matrix corresponding to a reduced alternating diagram of K or its mirror image. Since K is a knot, we see that p is an odd number. Suppose G_A is the group presented by A. If $u_2(K) = 1$, then there is an isomorphism $\phi : \mathbb{Z}/p\mathbb{Z} \longrightarrow G_A$ and a sign $\epsilon \in \{+1, -1\}$ with the properties that for all $i \in \mathbb{Z}/p\mathbb{Z}$:*

$$I_{\phi,\epsilon}(i) := \epsilon \cdot M_A(\phi(i)) + \frac{1}{4}(\frac{1}{p}(\frac{p + (-1)^i p}{2} - i)^2 - 1) = 0 \pmod 2,$$

$$\textit{and} \quad I_{\phi,\epsilon}(i) \geq 0.$$

Here we abuse i to denote both the element in $\mathbb{Z}/p\mathbb{Z}$ and its representative in the set $\{0, 1, 2, \cdots, p-1\}$.

5.3 An example of calculation

In order to check that Theorem 5.2.2 works better in some cases than the existing criteria, we post the knot $P(13, 4, 11)$ as an example. We determine that it has H(2)-unknotting number 2, which cannot seem to be detected by the other methods that we know.

Corollary 5.3.1. *The pretzel knot $P(13, 4, 11)$ has H(2)-unknotting number* 2.

5.3.1 Proof of Corollary 5.3.1

The pretzel knot $K = P(13, 4, 11)$ is an alternating knot as shown in Figure 5.2. A negative-definite Goeritz matrix associated with the mirror image of this diagram is

$$A = \begin{pmatrix} -17 & 4 \\ 4 & -15 \end{pmatrix},$$

and the determinant is $\det(A) = \det(K) = 239$. Suppose G_A is the group presented by A. In fact, the group G_A is isomorphic to $\mathbb{Z}/239\mathbb{Z}$. In the following calculation, we take the

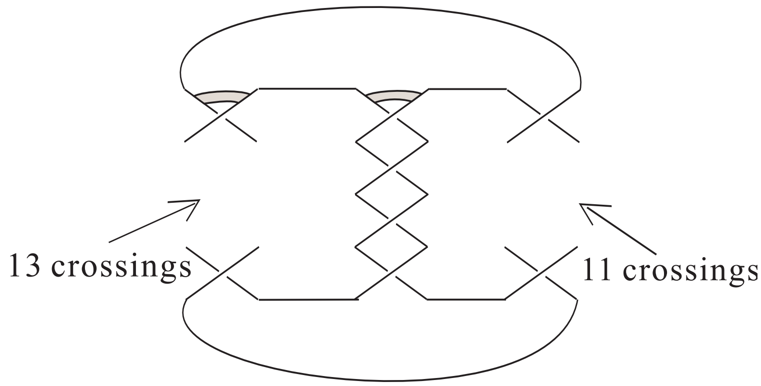

Figure 5.2: The pretzel knot $P(13, 4, 11)$.

vector $(0, 1)^t$ as a generator of G_A. The inverse of the matrix A is

$$A^{-1} = \frac{1}{239} \begin{pmatrix} -15 & -4 \\ -4 & -17 \end{pmatrix}.$$

Then by definition for any $0 \leq r \leq 238$ it holds that

$$\begin{aligned} & M_A((0,r)^t) \\ = \quad & \max \left\{ \frac{(u,v)^t A^{-1}(u,v) + 2}{4} \,\middle|\, (u,v)^t \in \mathrm{char}(A), [(u,v)^t] = (0,r)^t \in G_A \right\} \\ = \quad & \max \left\{ \frac{478 - (15u^2 + 8uv + 17v^2)}{956} \,\middle|\, u \text{ and } v \text{ are odd}, [(u,v)^t] = (0,r)^t \in G_A \right\}. \end{aligned}$$

From this expression, we see that in order to obtain the maximum we only need to focus on those representatives $(u, v)^t$ satisfying $|u| \leq 17$ and $|v| \leq 15$.

By calculation, it is easy to see that for any isomorphism $\phi : \mathbb{Z}/239\mathbb{Z} \to \mathbb{Z}/239\mathbb{Z}$ there is

$$I_{\phi,\epsilon}(0) = \epsilon \cdot M_A(\phi(0)) + 119/2 = \epsilon \cdot M_A((0,0)^t) + 119/2 = (\epsilon \cdot (-11) + 119)/2.$$

The vector which realizes the value of $M_A((0,0)^t)$ is $(u,v)^t = (13,11)^t$ or $(-13,-11)^t$.

We assume that $u_2(K) = 1$. Then by Theorem 6.2.1 the value $I_{\phi,\epsilon}(0)$ has to be an even number, and therefore $\epsilon = 1$. Next by calculation we have $I_{\phi,1}(1) = M_A(\phi(1)) - 119/478$.

Since 239 is a prime number, any $\phi_j =$ "multiplication by j" is an automorphism of $\mathbb{Z}/239\mathbb{Z}$. To guarantee that $I_{\phi_j,1}(1)$ is an even number, the isomorphism ϕ_j has to be either ϕ_{15} or ϕ_{224}. By calculation, we see that

$$I_{\phi_{15},1}(1) = M_A((0,15)^t) - 119/478 = -4.$$

The vector which realizes the value of $M_A((0,15)^t)$ is $(u,v)^t = (-9,-11)^t$. Same calculation tells us that $I_{\phi_{224},1}(1) = -4$ as well. Now we see -4 is a negative number, which conflicts with the necessary condition stated in Theorem 6.2.1. Therefore the H(2)-unknotting number of $P(13,4,11)$ has to be at least two. On the other hand, the knot $P(13,4,11)$ can be changed into the unknot by adding two twisted bands as shown in Figure 5.2. Hence the H(2)-unknotting number of $P(13,4,11)$ is two. This completes the proof of Corollary 5.3.1.

5.3.2 Comparisons with other criterions

There have been many criterions and properties which can be used to bound the H(2)-unknotting number of a knot. We want to apply them to the knot $P(13,4,11)$ and compare the results with Corollary 5.3.1.

The first one is Lickorish's obstruction that we recalled in Chapter 4.2.1. It does not work for the pretzel knot $K = P(13,4,11)$ because of the following reason. It is known that the Goeritz matrix A is a presentation matrix of $H_1(\Sigma(K),\mathbb{Z})$, and A^{-1} represents the linking form λ. It is not hard to see that $H_1(\Sigma(K))$ is cyclic of order 239, and that the generator $g = (0,1)^t$ satisfies $\lambda(g,g) = -17/239$. Then we see $\lambda(15g,15g) = (225 \times (-17))/239 = -3825/239 = -1/239$ over $\mathbb{Q}/\mathbb{Z}$. Since 239 is a prime number, the vector $g' = (0,15)^t$ can work as a generator of $H_1(\Sigma(K),\mathbb{Z})$.

Ichihara and Mizushima [22] calculated the crosscap numbers of pretzel knots. According to their calculation, the crosscap number of $P(13,4,11)$ is two. Gilmer and Livingston [11] studied the 4-dimensional crosscap number of a knot by using Heegaard Floer homology.

Their method and our result in this chapter are both in spirit derived from Theorem 9.6 in [34]. The author does not know whether their method can verify that the 4-dimensional crosscap number of $P(13, 4, 11)$ is 2 or not.

Yasuhara [48], and Kanenobu and Miyazawa [26] introduced some methods for detecting the H(2)-unknotting number of a knot, part of which is introduced in Chapter 4.2.2, but simple calculation shows that their methods cannot be applied to the knot $P(13, 4, 11)$. Alternative interpretations of H(2)-unknotting number in Chapter 4.1 do not seem to be helpful when considering the calculation of this invariant.

Chapter 6

H(2)-unknotting number as related to 2-bridge links

This chapter concerns the H(2)-unknotting numbers of links related to 2-bridge links (see [3]). It consists of three parts. In the first part, we consider a necessary and sufficient condition for a 2-bridge link to have H(2)-unknotting number one. The second part concerns an explicit form of composite links with H(2)-unknotting number one. In the last part, we develop a method to study the H(2)-unknotting numbers of some tangle unknotting number one knots via 2-bridge knots.

6.1 2-bridge links with H(2)-unknotting number one

In this section, we make some observations about 2-bridge links with H(2)-unknotting number one. Our purpose is to find out all the 2-bridge links whose H(2)-unknotting number is one. Here is our main observation.

Theorem 6.1.1. *The 2-bridge link $S(p, q)$ has H(2)-unknotting number one if and only if the lens space $L(p, q)$ can be obtained as an integral surgery along a Berge knot in S^3.*

We remark that this observation is just a corollary of some known results about integral Dehn surgery. Since our interest is H(2)-unknotting number, we think it is worth to write it down. In [26], an incomplete table of H(2)-unknotting numbers of knots is provided. Among knots with nine crossings, there are six knots whose H(2)-unknotting numbers are unknown. We confirm that among them the 2-bridge knots $9_{21}, 9_{23}, 9_{26}$ and 9_{31} are knots with H(2)-unknotting number one. We refer to Rolfson's table for the notations of knots.

Corollary 6.1.2. *The 2-bridge knots* $9_{21}, 9_{23}, 9_{26}$ *and* 9_{31} *have H(2)-unknotting number one.*

Now we explain how Theorem 6.1.1 is obtained.

The 2-bridge links have been widely studied in knot theory. The 2-bridge link $S(p, q)$ to be used here is the link illustrated in Figure 6.1. Here p/q is the continued fraction $[a_1, a_2, \cdots, a_n]$. Precisely

$$\frac{p}{q} = [a_1, a_2, \cdots, a_n] = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}.$$

Let $C(a_1, a_2, \ldots, a_n)$ denote the link diagram in Figure 6.1. Two unoriented links K_1 and K_2 are *equivalent* if there exists an orientation-preserving auto-homeomorphism of S^3 sending K_1 to K_2. The following fact is well-known: Two unoriented 2-bridge links $S(p_1, q_2)$ and $S(p_2, q_2)$ with $p_1, p_2 \geq 1$ are equivalent if and only if $p_1 = p_2$ and $q_1^{\pm} \equiv q_2 \pmod{p_1}$.

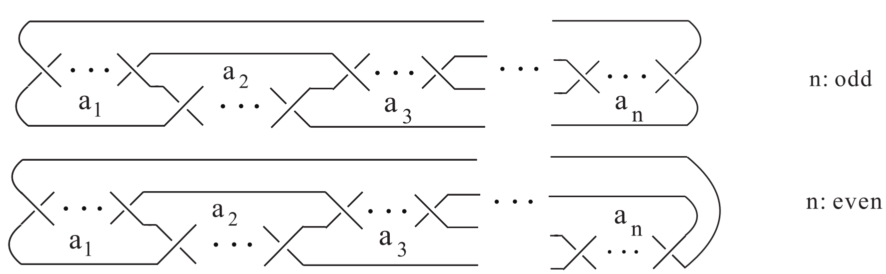

Figure 6.1: The link diagram $C(a_1, a_2, \ldots, a_n)$ for 2-bridge links.

We use $S^3_r(K)$ to denote the 3-manifold obtained by doing Dehn surgery to the 3-sphere S^3 along the knot K with coefficient r. Our orientation convention is the p/q-surgery along the trivial knot gives the lens space $L(p,q)$. An oriented knot C is called *strongly-invertible* if there is an orientation preserving homeomorphism, which is also an involution, of S^3 which takes the knot to itself but reverses the orientation along the knot. Given a link K, let $\Sigma(K)$ denote the double branched cover of S^3 along K. It is well-known that the double branched cover of S^3 along the 2-bridge link $S(p,q)$ is the lens space $L(p,q)$. We have the following lemma.

Lemma 6.1.3. *The 2-bridge link $S(p,q)$ has H(2)-unknotting number one if and only if the lens space $L(p,q)$ is $S^3_{\pm p}(C)$ for some strongly-invertible knot C.*

Proof. The proof is in fact a practice of Montesinos's trick [31]. In general, if a link K has H(2)-unknotting number one, then $\Sigma(K)$ equals $S^3_p(C)$ for some strongly-invertible knot C and $|p|$ equals the absolute value of the determinant of K. For 2-bridge links, the converse is true as well. The reason is as follows. For a integer p and a strongly-invertible knot C, one can always construct a link with H(2)-unknotting number one for which the double branched cover is $S^3_p(C)$. The double branched cover of S^3 along the 2-bridge link $S(p,q)$ is the lens space $L(p,q)$, and it is known [20] that there is no other links sharing the same double branched cover with $S(p,q)$. Therefore, if the lens space $L(p,q)$ equals $S^3_{\pm p}(C)$, the H(2)-unknotting number of the 2-bridge link $S(p,q)$ must be one. ☐

Since the set of strongly-invertible knots is too large, Lemma 6.1.3 does not help us simplify the task of finding out 2-bridge links with H(2)-unknotting number one. On the other hand, there have been many studies about integral surgeries which produce lens spaces. In the following paragraphs, we assemble some of these studies and come up with a practical criterion, which is Theorem 6.1.1, for distinguishing 2-bridge links with H(2)-unknotting number one.

If some integral surgery of S^3 along a knot gives rise to a lens space, we say this knot *admits integral lens space surgery*. It is known that doubly-primitive knots admit integral lens space surgeries. Here is the definition. Given a loop in the boundary of a genus two handlebody, it is called *primitive* if attaching a 2-handle produces a solid torus. A knot in S^3 is called *doubly-primitive* if it sits on a genus two Heegaard surface of S^3, and is primitive in handlebodies on both sides. Berge [4] found twelve classes of doubly-primitive knots, which are now called Berge knots. The lens spaces which arise as integral surgeries along Berge knots are listed as follows in [42]:

Theorem 6.1.4 (Berge)**.** *The lens space $L(\alpha, \beta)$ arises as an integral surgery along a Berge knot if there exists an integer k such that $\beta \equiv \pm k^2 \pmod{\alpha}$, and α, β and k satisfy one of the following conditions:*

1. $\alpha \equiv ik \pm 1 \pmod{k^2}$ *and* $\gcd(i, k) = 1, 2$ *for some i;*

2. $\alpha \equiv \pm(2k + \epsilon)d \pmod{k^2}$*, $d | k - \epsilon$ and $\frac{k-\epsilon}{d}$ is odd, for $\epsilon = \pm 1$;*

3. $\alpha \equiv \pm(k + \epsilon)d \pmod{k^2}$ *and $d | 2k - \epsilon$, for $\epsilon = \pm 1$;*

4. $\alpha \equiv \pm(k + \epsilon)d \pmod{k^2}$*, $d | k + \epsilon$ and d is odd, for $\epsilon = \pm 1$;*

5. $k^2 \pm k \pm 1 \equiv 0 \pmod{\alpha}$*;*

6. $\alpha = 22j^2 + 9j + 1$ *and $k = 11j + 2$ for some j;*

7. $\alpha = 22j^2 + 13j + 2$ *and $k = 11j + 3$ for some j.*

The following fact is implicated in [23], though the statement is not directly given.

Theorem 6.1.5. *The 2-bridge link $S(p, q)$ has H(2)-unknotting number one when the lens space $L(p, q)$ can be obtained as an integral surgery along a doubly-primitive knot in S^3.*

Greene [25] proved the following result.

Theorem 6.1.6 ([25])**.** *If a lens space is realized as an integral surgery along a knot in S^3, then it can be realized as an integral surgery along some Berge knot.*

The proof of Theorem 6.1.1 now easily follows from Theorems 6.1.5 and 6.1.6. Lens spaces which arise as integral surgeries along Berge knots have been completely listed in Theorem 6.1.4. Therefore, the corresponding 2-bridge links are those 2-bridge links whose H(2)-unknotting numbers are one. To prove Corollary 6.1.2, we only need to show that these four 2-bridge knots belong to the list.

Proof of Corollary 6.1.2. Within this proof, we do not distinguish a knot from its mirror image. If the corresponding lens spaces belong to the list in Theorem 6.1.4, then Theorem 6.1.1 tells us that the 2-bridge knots have H(2)-unknotting number one. For $9_{21} = S(43, 25)$, we have $43 = d(2k-1) \pmod{k^2}$ and $25 = k^2$ for $k = 5$ and $d = 2$. It belongs to Berge type (ii). For $9_{23} = S(45, 64)$, we have $45 = d(2k-1) \pmod{k^2}$ and $64 = k^2$ for $k = 8$ and $d = 3$. It belongs to Berge type (ii). For $9_{26} = S(47, 81)$, we have $47 = -d(2k-1) \pmod{k^2}$ and $81 = k^2$ for $k = 9$ and $d = 2$. It belongs to Berge type (ii). For $9_{31} = S(55, 144)$, we have $55 = d(k-1) \pmod{k^2}$ and $144 = k^2$ for $k = 12$ and $d = 5$. It belongs to Berge type (iii). □

6.2 Composite links with H(2)-unknotting number one

In this section, we study composite links with H(2)-unknotting number one. We mainly focus on the proofs of the following Propositions.

Theorem 6.2.1. *If K_1 is the 2-bridge link $S(q, p)$, and K_2 is a (p, q)-tangle unknotting number one link, then the H(2)-unknotting number of the composite $K_1 \sharp K_2$ is one.*

Bleiler [5, Theorem 1] and Eudave Muñoz [9, Corollary 2] proved that if a composite link has H(2)-unknotting number one, then it is a composition of a 2-bridge link and a prime link. We want to investigate the explicit forms of the two summands, and Theorem 6.2.1 is a partial answer to this desire. We further conjecture that the converse holds as well.

Conjecture 6.2.2. *A composite link with H(2)-unknotting number one always has the form described in Theorem 6.2.1.*

Conjecture 6.2.2 has close relation to the following question. It is obvious that any cable knot of a strongly invertible knot is still strongly invertible.

Question 6.2.3. *Is it true that a knot K is strongly invertible if and only if certain cable of K is strongly invertible?*

When we restrict the two summands to 2-bridge links, we have the following complete description.

Theorem 6.2.4. *Suppose $S(p,q)$ and $S(r,s)$ are two non-trivial 2-bridge links. Then the composite $S(p,q)\sharp S(r,s)$ has H(2)-unknotting number one if and only if either $S(r,s) = S(q,p)$, or $S(p,q) = S(v,\epsilon)$ and $S(r,s) = S(vab+\epsilon, va^2)$ for $\epsilon = \pm 1$ and some integers v, a and b satisfying $(a,b) = 1$.*

Now we give the definition of tangle unknotting number one knot.

Definition 6.2.5. A link K is a (p,q)-tangle unknotting number one link if there is a tangle decomposition $K = T_1 \cup T_2$ such that T_1 is the rational tangle as shown in Figure 6.2 and that $T_0 \cup T_2$ is the unknot. Here p/q is the continued fraction $[a_1, a_2, \cdots, a_n]$. We call T_1 a rational tangle with Conway notation $(a_1, a_2, \cdots, a_n)$.

Note that the definition depends on not only (p,q), but also the sequence of numbers $(a_1, a_2, \cdots, a_n)$. Our convention for Conway notation may be different from those in some references.

Remark 6.2.6. *The notation "tangle unknotting number one link" comes from the preprint [19], but the definition here is slightly modified.*

Lemma 6.2.7. *If a link K is a (p,q)-tangle unknotting number one link, then this link is also a $(p+aq, q)$-tangle unknotting number one link for any integer a. Furthermore, in this case there is an integer a_0 such that $\Sigma(K) = S_{(p+a_0q)/q}(C)$ for some strongly invertible knot C.*

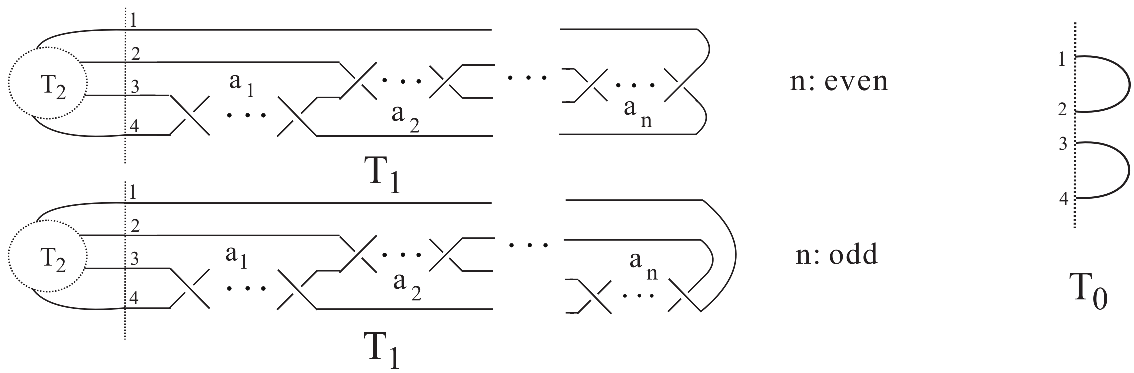

Figure 6.2: The tangle decomposition for a (p, q)-tangle unknotting number one knot.

Proof. We first explain the former statement. Let $K = T_2 \cup T_1$ be the tangle decomposition as in Definition 6.2.5 and T_1 be the rational tangle with notation $(a_1, a_2, \cdots, a_n)$. Given an integer a, let T_1^a be the rational tangle with notation $(a_1 + a, a_2, \cdots, a_n)$ and T_2^a be the tangle obtained from T_2 by making $(-a)$ half twists along the endpoints 3 and 4. It is easy to see that $K = T_1^a \cup T_2^a$ and that $T_2^a \cup T_0$ is the unknot. Therefore K is a $(p + aq, q)$-tangle unknotting number one link as well.

Now we prove the latter statement, which is again a practise of Montesinos's trick. Consider the tangle decomposition

$$(S^3, T_0 \cup T_2) = (D^3, T_0) \cup (D^3, T_2).$$

Since $T_0 \cup T_2$ is the unknot, the double branched cover of S^3 along $T_0 \cup T_2$ is still S^3. The double branched cover of D^3 along T_0 is a solid torus, which we denote S_0. Therefore, the manifold obtained by attaching S_0 to the double branched cover of D^3 along T_2 along their common torus boundary, is S^3. This implies that the double branched cover of D^3 along T_2 is the complement of a knot, say C, in S^3. In order to see that C is strongly invertible, we notice that the image of C in the base space S^3 is the dotted arc as shown in Figure 6.3. It is easy to see that the preimage of this arc, which is C, is strongly invertible.

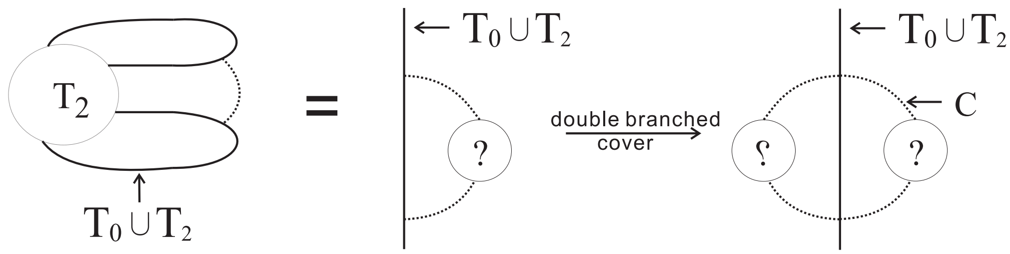

Figure 6.3: The knot C.

Next, we consider the tangle decomposition of K: $(S^3, K) = (D^3, T_1) \cup (D^3, T_2)$. Taking the double branched cover, we get

$$\Sigma(K) = S_1 \cup (S^3 \setminus \nu(C)),$$

where S_1 denotes the double branched cover of D^3 along T_1, which is a solid torus. Let (m_0, l_0), (m_1, l_1) and (m, l) be the preferred meridian-longitude systems of S_0, S_1 and C respectively. Here we choose the same orientation for S_0 and S_1. We can see that $m_1 = pm_0 + ql_0$, while we already know that $m_0 = m$ and $l_0 = l + a_0 m$ for some integer a_0, so as a conclusion we have $m_1 = (p + a_0 q)m + ql$. That is to say $\Sigma(K) = S^3_{(p+a_0q)/q}(C)$. □

Proof of Theorem 6.2.1. In fact, if $p/q = [a_1, a_2, \cdots, a_n]$, then $S(q, p)$ is equivalent to $S(q, p-a_1q)$. Note that $q/(p - a_1 q) = [a_2, \cdots, a_n]$. Then $C(a_2, \cdots, a_n)$ is a link diagram for $S(q, p)$. The composite link $K(p, q)\sharp S(q, p)$ can be unknotted by adding a band, as shown in Figure 6.4. This completes the proof. □

We have the following corollary since the 2-bridge link $S(p, q)$ is a (p, q)-tangle unknotting number one link. It is in fact Theorem 9.1 in [26].

Corollary 6.2.8. *The H(2)-unknotting number of the link $S(p, q)\sharp S(q, p)$ is one.*

Before giving the proof of Theorem 6.2.4, we introduce some facts. Given a knot K and two coprime integers p and q, we use $K_{p,q}$ to denote the (p, q)-cable knot of K. It is conjectured that (the cabling conjecture [12]) that if an integral surgery of S^3 along a knot

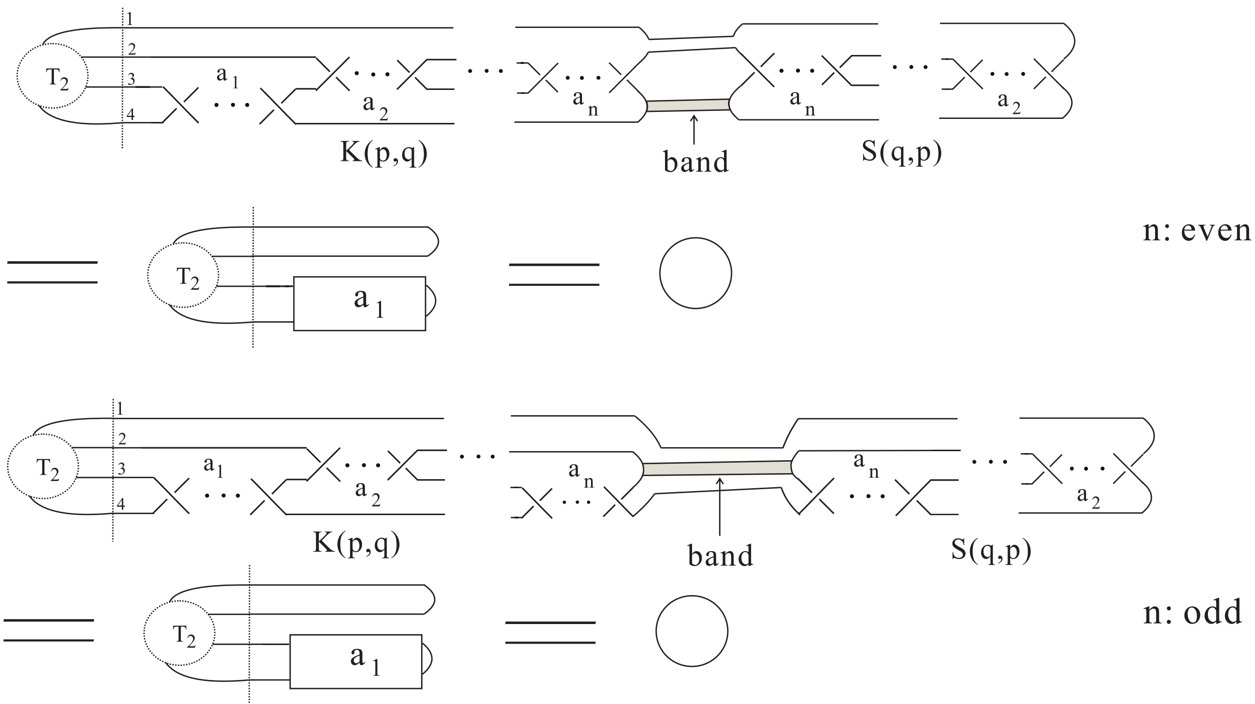

Figure 6.4: The H(2)-unknotting number of the composite link is one.

L produces a reducible manifold, then L is a cable knot and the slope of the surgery is pq. This conjecture holds when L is a strongly invertible knot [9].

Proof of Theorem 6.2.4. If $S(p,q)\sharp S(r,s)$ has H(2)-unknotting number one, then there exists a strongly invertible knot C and an integer l such that $\Sigma(S(p,q)\sharp S(r,s)) = L(p,q)\sharp L(r,s) = S^3_l(C)$. Since the cabling conjecture holds for strongly invertible knots, the knot C must be a cable knot, say $C = K_{u,v}$ for some knot K and coprime integers u and v. Then we have $l = uv$ and $S^3_l(C) = S^3_{u/v}(K)\sharp L(v,u)$ ([14, Corollary 7.3]), which in turn equals $L(p,q)\sharp L(r,s)$. By the prime decomposition theorem for 3-manifolds, we can suppose $L(v,u) = L(p,q)$. Then $S^3_{u/v}(K)$ has to be $L(r,s)$. The cyclic surgery theorem [8] implies that if a non-integral Dehn surgery of S^3 along a knot produces a lens space, then the knot is a torus knot. Since $|v| = |p| > 1$, the fact $S^3_{u/v}(K) = L(r,s)$ implies that K must be a torus knot.

If K is the unknot, then $S^3_{u/v}(K) = L(u,v) = L(r,s)$. Therefore $S(r,s)$ is equivalent to $S(q,p)$. (In fact, $S(r,s)$ may be $S(q+jp,p)$ for some integer j, but in this case, we can write $S(p,q)$ as $S(p,q+jp)$.) If K is non-trivial, suppose K is the (a,b)-torus knot. Then

$S^3_{u/v}(K)$ is a lens space only if $u = vab \pm 1$, and then $S^3_{u/v}(K) = L(vab \pm 1, va^2)$. In this case, $S(p,q) = S(v, vab \pm 1) = S(v, \pm 1)$ and $S(r,s) = S(vab \pm 1, va^2)$.

The converse can be proved easily. □

From Theorems 6.2.1 and 6.2.4, we have the following corollary:

Corollary 6.2.9. *1. The H(2)-unknotting number of a (p,q)-tangle unknotting number one link is less than or equal to $u_2(S(q,p)) + 1$.*

2. $u_2(C(a_1, a_2, \cdots, a_n)) \leq u_2(C(a_i, a_{i+1}, \cdots, a_n)) + i - 1$.

Remark 6.2.10. *The 2-bridge link $S(vab + \epsilon, va^2)$ in Theorem 6.2.4 is an (ϵ, v)-tangle unknotting number one link.*

6.3 Tangle unknotting number one knots

6.3.1 An inequality

The purpose of this subsection is to prove Relation (6.1), which will be applied in next subsection to calculate the H(2)-unknotting numbers of some tangle unknotting number one knots.

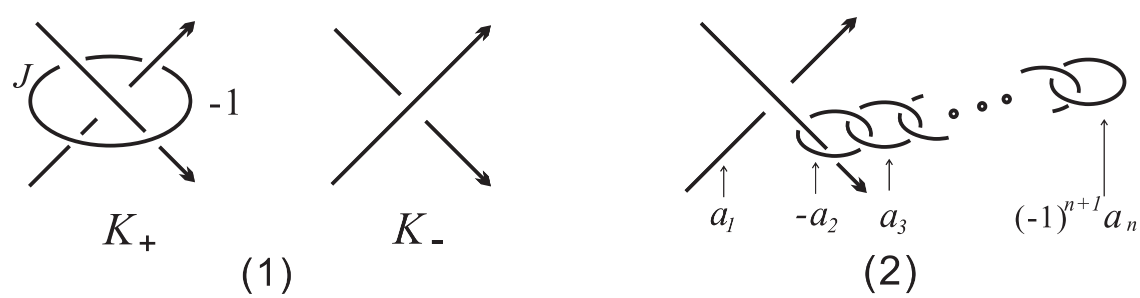

Figure 6.5: Kirby diagrams.

Suppose K_+ and K_- are two knots which differ only in a local neighborhood of a crossing, as shown in Figure 6.5-(1) (ignore the circle J around the crossing). Consider the two manifolds $Y_+ = S^3_{p/q}(K_+)$ and $Y_- = S^3_{p/q}(K_-)$, where p is odd and $(p,q) = 1$. There is a cobordism from Y_+ to Y_- given by attaching a 2-handle to $Y_+ \times [0,1]$ along the circle J with framing -1, and we denote the cobordism by $W : Y_+ \to Y_-$. Then we have the following property:

Proposition 6.3.1. *The cobordism W is a negative-definite cobordism from Y_+ to Y_-, and therefore:*

$$\begin{aligned} d(Y_-, t_-) - d(Y_+, t_+) &\geq \frac{c_1^2(s) - 2\chi(W) - 3\sigma(W)}{4} \quad \textit{and} \\ d(Y_-, t_-) - d(Y_+, t_+) &= \frac{c_1^2(s) - 2\chi(W) - 3\sigma(W)}{4} \pmod 2, \end{aligned}$$

where s is a Spin^c-structure over W with $s\big|_{Y_*} = t_*$ for $* = +, -$.

Proof. First we prove $H_2(W;\mathbb{Z}) = \mathbb{Z}$ by the following argument. Let W_+ be the plumbing manifold obtained by attaching 2-handles to a four-ball along the framed link in Figure 6.5-(2). The framings come from that p/q equals the continued fraction $[a_1, a_2, \cdots, a_n]$. By the slam-dunk move in Figure 6.6, we see the boundary of W_+ is indeed Y_+. Then we let $X = W_+ \cup W$, and clearly we have $H_2(X;\mathbb{Z}) = \mathbb{Z}^{n+1}$ and $H_2(W_+;\mathbb{Z}) = \mathbb{Z}^n$. Consider the Mayer-Vietoris sequence applied to the decomposition $X = W_+ \cup W$. In this case $W_+ \cap W = S^3_{p/q}(K_+) = Y_+$, so we have the following sequence for integral homology groups:

$$\cdots \to H_2(Y_+) \to H_2(W_+) \oplus H_2(W) \to H_2(X) \to H_1(Y_+) \to \cdots.$$

Since $H_2(Y_+) = 0$, $H_2(W_+) = \mathbb{Z}^n$, $H_2(X) = \mathbb{Z}^{n+1}$ and $H_1(Y_+) = \mathbb{Z}/p\mathbb{Z}$, we have $H_2(W) = \mathbb{Z}$.

It is easy to see that there exists a Seifert surface of J in S^3 which is disjoint with K_+. Let $\alpha \in H_2(W;\mathbb{Z})$ be the homology class of this Seifert surface of J capped off by the core disk of the two-handle attached along J. Then α is a generator of $H_2(W;\mathbb{Z})$, and we have $\alpha^2 = -1$, which implies that the cobordism W is negative-definite. By Theorem 3.2.3, we complete the proof of the proposition. ∎

Remark 6.3.2. *The idea of the proof comes from the proof of Theorem 1.3 in [41].*

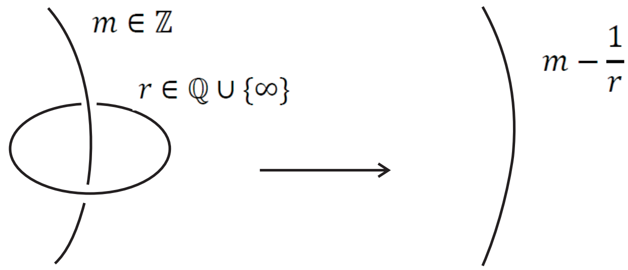

Figure 6.6: The slam-dunk move.

In the following two paragraphs, we figure out when the pair of Spin^c-structures $(t_+, t_-) \in \mathrm{Spin}^c(Y_+) \times \mathrm{Spin}^c(Y_-)$ can be realized as the restriction of a Spin^c-structure over W.

Note that W is constructed by attaching a 2-handle $D^4 = D^2 \times D^2$ to $Y_+ \times [0,1]$ along a solid torus $S^1 \times D^2 = (\partial D^2) \times D^2$. By considering the Mayer-Vietoris Sequence associated with the triple $(S^1 \times D^2, Y_+ \times [0,1] \coprod D^4, W)$, we have

$$0 \to H_2(W) \to H_1(S^1 \times D^2) = \mathbb{Z} \xrightarrow{f} H_1(Y_+ \times [0,1]) \oplus H_1(D^4) = \mathbb{Z}/p\mathbb{Z} \to H_1(W) \to 0.$$

It is easy to see that $H_1(S^1 \times D^2)$ is generated by a longitude of J in Figure 6.5, whose homology class in $H_1(Y_+ \times [0,1])$ is zero. Therefore the map f, which is induced by the inclusion map, is trivial. Therefore we have $H_1(W) = \mathbb{Z}/p\mathbb{Z}$. By the universal coefficient theorem, we have $H^2(W) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Noting that $\partial W = (-Y_+) \cup Y_-$, we have the following exact sequence with respect to the pair $(W, \partial W)$:

$$\begin{aligned} 0 \to H_2(W) = \mathbb{Z} \xrightarrow{\tau} H^2(W) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \xrightarrow{\alpha} H^2(-Y_+) \oplus H^2(Y_-) = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \\ \xrightarrow{\beta} H_1(W) = \mathbb{Z}/p\mathbb{Z} \to 0. \end{aligned}$$

Let $m_+ \subset Y_+$ be a meridian of K_+ and $m_- \subset Y_-$ be the image of m_+ after the Dehn surgery along J. Then $[m_+]$ and $[m_-]$ are generators of $H_1(Y_+)$ and $H_1(Y_-)$ respectively. We identify $H^2(Y_+)$ with $H_1(Y_+)$ and $H^2(Y_-)$ with $H_1(Y_-)$ by Poincaré duality. Then we have $\beta(-[m_+], [m_-]) = 0 \in H_1(W)$.

The set Spin^c(W) is an affine space over $H^2(W;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Given a pair of Spin^c-structures $t_+ \in \mathrm{Spin}^c(Y_+)$ and $t_- \in \mathrm{Spin}^c(Y_-)$, a Spin^c-structure $s \in \mathrm{Spin}^c(W)$ satisfies $s\,\big|_{Y_*} = t_*$ for $* = +, -$ if and only if $\alpha(c_1(s)) = (-c_1(t_+), c_1(t_-))$. From the exactness of the sequence, the element $(-i, i) := (-i[m_+], i[m_-]) \in H^2(-Y_+) \oplus H^2(Y_-)$ stays in the image of the map α for $0 \leq i \leq p-1$. Let $t_+ \in \mathrm{Spin}^c(Y_+)$ and $t_- \in \mathrm{Spin}^c(Y_-)$ be the Spin^c-structures for which $(-c_1(t_+), c_1(t_-)) = (-i, i)$. Then (t_+, t_-) is the restriction of some Spin^c-structures over W to (Y_+, Y_-).

We use $d(Y_-, i)$ (resp. $d(Y_+, i)$) to denote $d(Y_-, t_-)$ (resp. $d(Y_+, t_+)$). In the following, we want to show that

$$\begin{aligned} &d(Y_-, i) - d(Y_+, i) \geq 0 \quad \text{and} \\ &d(Y_-, i) - d(Y_+, i) = 0 \pmod 2, \end{aligned} \tag{6.1}$$

for any $i \in \mathbb{Z}/p\mathbb{Z}$. By Proposition 6.3.1, we have the following relations:

$$\begin{aligned} d(Y_-, i) - d(Y_+, i) &\geq \tfrac{c_1^2(s) - 2\chi(W) - 3\sigma(W)}{4} \quad \text{and} \\ d(Y_-, i) - d(Y_+, i) &= \tfrac{c_1^2(s) - 2\chi(W) - 3\sigma(W)}{4} \pmod 2, \end{aligned}$$

for any Spin^c-structure s over W whose restriction to (Y_+, Y_-) is (t_+, t_-). Note that $\chi(W) = 1$ and $\sigma(W) = -1$. Then we can prove (6.1) if we can prove that

$$\max\left\{ c_1^2(s) \,\middle|\, s \in \mathrm{Spin}^c(W), s\,\big|_{Y_*} = t_* \text{ for } * = +, - \right\} = -1. \tag{6.2}$$

We say $\xi \in \mathbb{Z}$ is a *characteristic element* of the matrix (-1) if ξ is odd. Let $t_{+,0}$ and $t_{-,0}$ denote the Spin^c-structures whose first Chern classes are trivial, over Y_+ and Y_- respectively. We define a set $H := \left\{ s \in \mathrm{Spin}^c(W) \,\middle|\, s\,\big|_{Y_*} = t_{*,0} \text{ for } * = +, - \right\}$. Then the first Chern classes of elements in H belong to the free part of $H^2(W;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Conversely any Spin^c-structure over W whose first Chern class belongs to the free part of $H^2(W;\mathbb{Z})$, belongs to H. The set $\{c_1(s)\,|s \in H\}$ is equal to the set of characteristic elements of the matrix (-1).

If $c_1(s)$ corresponds to the characteristic element ξ, then we have $c_1^2(s) = -\xi^2$. It is easy to calculate that $\max\{c_1^2(s)\,|s \in H\}$ is -1.

Note that $H^2(W;\mathbb{Z})$ acts transtively and freely on the set $\mathrm{Spin}^c(W)$. Any Spin^c-structure over W can be transformed into a Spin^c-structure in H by a torsion element of $H^2(W;\mathbb{Z})$. We know that for a Spin^c-structure s and a torsion element $a \in H^2(W;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}$ there is $c_1(s+a)^2 = c_1(s)^2$. Therefore Relation (6.2) is true, which in turn implies (6.1).

6.3.2 Application

We show that (6.1) can be used to study the H(2)-unknotting numbers of some tangle unknotting number one knots. Let us consider some (23,3)-tangle unknotting number one knots as examples. For the 2-bridge knot $S(23,3)$, the ordered set of the d-invariants of its double-branched cover, the lens space $L(23,3)$, is given as follows (for the calculation, refer to [37, Proposition 3.2]). Here we identify Spin^c(L(23,3)) with $\mathbb{Z}/23\mathbb{Z}$.

$$\{d(L(23,3),i)\}_{i=0}^{22} = \left\{\frac{3}{2}, \frac{85}{46}, \frac{41}{46}, \frac{29}{46}, \frac{49}{46}, \frac{9}{46}, \frac{1}{46}, \frac{25}{46}, \frac{-11}{46}, \frac{-15}{46}, \frac{13}{46}, \frac{-19}{46}, \frac{-19}{46},\right.$$
$$\left.\frac{13}{46}, \frac{-15}{46}, \frac{-11}{46}, \frac{25}{46}, \frac{1}{46}, \frac{9}{46}, \frac{49}{46}, \frac{29}{46}, \frac{41}{46}, \frac{85}{46}\right\}.$$

On the other hand, define $f(i) = \frac{1}{4}(\frac{1}{p}(\frac{p+(-1)^i p}{2} - i)^2 - 1)$ for $p = 23$ and any $0 \le i \le 22$. Then we have the following ordered set:

$$\{f(i)\}_{i=0}^{22} = \left\{\frac{11}{2}, \frac{-11}{46}, \frac{209}{46}, \frac{-7}{46}, \frac{169}{46}, \frac{1}{46}, \frac{133}{46}, \frac{13}{46}, \frac{101}{46}, \frac{29}{46}, \frac{73}{46}, \frac{49}{46}, \frac{49}{46},\right.$$
$$\left.\frac{73}{46}, \frac{29}{46}, \frac{101}{46}, \frac{13}{46}, \frac{133}{46}, \frac{1}{46}, \frac{169}{46}, \frac{-7}{46}, \frac{209}{46}, \frac{-11}{46}\right\}.$$

We will apply Theorem 5.2.2 to show that $u_2(S(23,3)) > 1$. Assume that $u_2(S(23,3)) = 1$. Then by Theorem 5.2.2, there exist an automorphism ϕ of $\mathbb{Z}/23\mathbb{Z}$ and a sign $\epsilon \in \{+1,-1\}$ such that $I_{\phi,\epsilon}(i)$ are even positive numbers for all $i \in \mathbb{Z}/23\mathbb{Z}$.

No matter which automorphism ϕ of $\mathbb{Z}/23\mathbb{Z}$ we choose, we have $I_{\phi,\epsilon}(0) = \epsilon \cdot d(L(23,3),0) + f(0) = (\epsilon \cdot 3 + 11)/2$. In order to make it be an even number, we need ϵ to be -1. By calculation, we see there exist two possible automorphisms of $\mathbb{Z}/23\mathbb{Z}$, the map ϕ_1 given by multiplication by 8 and ϕ_2 given by multiplication by 15, for which $I_{\phi_j,-1}(i) = -d(L(23,3),\phi_j(i)) + f(i)$ are even integers for all $1 \leq i \leq 22$ and $j = 1, 2$. Precisely in both cases, we have:

$$\{I_{\phi,-1}(i)\}_{i=0}^{22} = \{4, 0, 4, -2, 4, 0, 2, 0, 2, 0, 2, 0, 0, 2, 0, 2, 0, 2, 0, 4, -2, 4, 0\},$$

where ϕ is either ϕ_1 or ϕ_2. Among these numbers -2 is negative.

As a conclusion, we see that there exists no automorphism ϕ of $\mathbb{Z}/23\mathbb{Z}$ and sign $\epsilon \in \{+1, -1\}$ to guarantee that $I_{\phi,\epsilon}(i)$ are all even positive numbers for $i \in \mathbb{Z}/23\mathbb{Z}$. For simplicity, we say there exist no even positive matchings for the knot $S(23,3)$. Therefore the assumption that $u_2(S(23,3)) = 1$ is false, while it is easy to see that $u_2(S(23,3)) \leq 2$, so finally we have $u_2(S(23,3)) = 2$.

Suppose that K is a $(23,3)$-tangle unknotting number one knot and that $|\det(K)| = 23$. Then by Lemma 6.2.7 there is $\Sigma(K) = S^3_{23/3}(C)$ for some strongly invertible knot C.

If we get the unknot by changing a negative crossing of C into a positive crossing, then by (6.1) we have:

$$\begin{aligned} d(\Sigma(K), i) - d(L(23,3), i) &\geq 0 \quad \text{and} \\ d(\Sigma(K), i) - d(L(23,3), i) &= 0 \pmod 2. \end{aligned}$$

Now it is easy to see that there exist no even, negative matchings for the knot K as well. So we conclude that $u_2(K) > 1$. By Corollary 6.2.9, we have $u_2(K) \leq u_2(S(3,23)) + 1 = 2$, and therefore $u_2(K) = 2$. For example, let C be the left-hand trefoil knot T. Then in this case K is the knot in Figure 6.7-(2), whose double-branched cover is $S^3_{23/3}(T)$.

If there is a positive and a negative crossing in C for which either of the crossing change gives the unknot, and if particularly C is an amphicheiral knot with unknotting number one

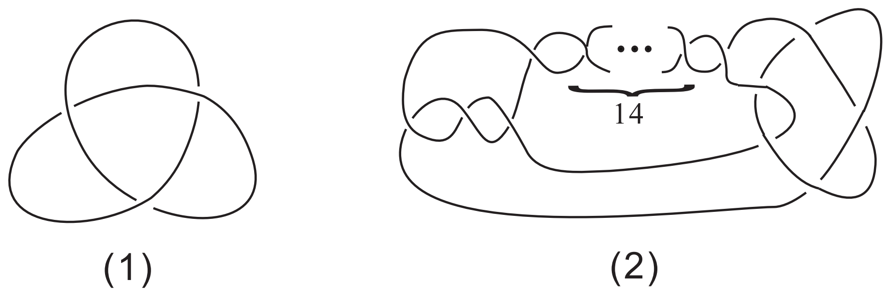

Figure 6.7: The left-hand trefoil knot T and the tangle unknotting number one knot K.

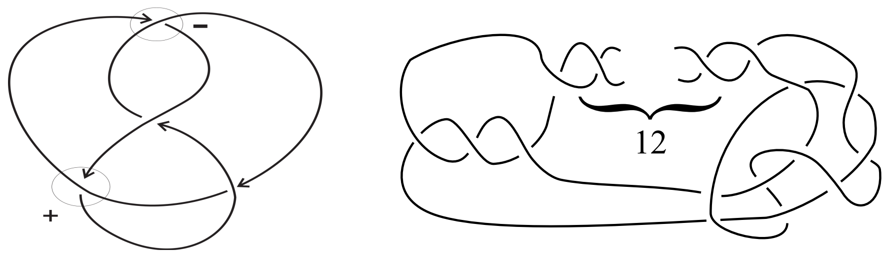

Figure 6.8: The figure eight knot and its corresponding tangle unknotting number one knot.

(For example the figure eight knot in Figure 6.8), then by (6.1) we have:

$$d(\Sigma(K), i) - d(L(23, 3), i) \quad = \quad 0.$$

By a similar argument as in the previous paragraph, we see in this case $u_2(K) = 2$ as well.

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