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著者(和文)	向田寿光
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THESIS

BOSE-FERMI TRANSMUTATION IN 3-DIMENSIONAL RELATIVISTIC QUANTUM THEORIES

D8009 Hisamitsu Mukaida

Department of Physics,

Tokyo Institute of Technology

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1 Introduction

Motivation In a system of many identical particles, several quantum states can be generated from a given state by interchanging particles. These states however describe the same physical state. When we interchange two particles, the state vector either does not alter or it acquires a factor of -1. The idea mentioned above leads us naturally to the concept of boson and fermion.

The procedure of second quantization reflects this property. In the path-integral formalism, we have to sum over all possible field configurations. Bose fields take values in the real or complex numbers. Fermi fields, however, take values in the grassmann numbers. Grassmann numbers play the role of producing the statistical factor of fermi particles.

In recent investigations of three dimensional field theories, for example the non-linear σ-model with the Hopf term or the Chern-Simons term, it is pointed out that the statistics of the particle is changed by interactions [1]. The $O(3)$ non-linear σ-model, or the CP^1 model in three dimensions describes a low-energy behaviour of antiferromagnetic magnons in two dimensions. For example, we can derive the CP^1 model in three dimensions as the effective theory of the Heisenberg antiferromagnets in two dimensions [7]. The CP^1 model consists of several charged scalar fields and a $U(1)$ gauge field. There are no kinetic terms for the $U(1)$ gauge field. In this case the magnons are described by these bosonic variables and obey bose statistics. However if the effective theory contains a $U(1)$ Chern-Simons term (or the Hopf term), the statistics of the magnons is transmuted.

One of the simplest examples of how particles acquire a statistical factor through interactions, is a system of two scalar particles in two spatial dimensions interacting with the Chern-Simons gauge field. The equations of motion inform us the fact that a magnetic flux is present only at positions of the particles: the field strength vanishes at all other points. If one particle goes around the other, a wave function of the two particles acquires a phase factor due to the Aharonov-Bohm effect. This phase is proportional to the strength of the magnetic flux which depends on the coefficient of the Chern-Simons action. Let us consider what happens if the phase were to take the value π when one particle winds a half times (which corresponds to interchanging of particles). In this case, the wave function acquires the factor of -1 by interchanging particles. This provides a clue as to how the statistics of scalar particles is transmuted into fermi statistics by the interaction. If the phase takes the value $\alpha\pi$ (where α is not an integer) when one particle turns a half times round the other, these particles interacting with the Chern-Simons gauge field obey 'fractional statistics' and are called 'anyons'.

It is an attractive problem to construct spin systems whose low-lying excitations are described by anyons. Answers are not apparent because the Chern-Simons action violates parity invariance. If the (microscopic) hamiltonian of a system is invariant under parity transformation, the parity invariance has to be spontaneously broken. Wen, Wilczek and Zee have constructed such a model of a two-dimensional antiferromagnetic spin system [5]. There are not only nearest neighbour interactions but also next-nearest ones in this model. They have explicitly shown in the mean-field approximation that magnons in this system obey fractional statistics. There are many investigations on the statistical properties of anyons. Readers are referred to ref.[8] for the review of this topic. We do not treat this subject in the text.

If statistics-changing-interactions of particles in an effective theory are generated by a topologically invariant term, there is a possibility that this theory has parity invariance. For example, suppose that interactions are generated by a Hopf term, which is a topological invariant. With the proper choice of coefficients, this term takes the value $(2n+1)\pi, n \in \mathbf{Z}$ for all field configurations. In this case the partition function of the theory remains unchanged even if the parity transformation changes the sign of the Hopf term because $e^{(2n+1)\pi i} = e^{-(2n+1)\pi i} = -1$. So far a spin system in two dimensions of which an effective theory contains the Hopf term has not yet been constructed . In one dimension however, there exists a spin system whose effective theory (which is described by a two-dimensional field theory) contains a topological term.

A topological term such as the Hopf term produces the interaction which changes the statistics of particles in three-dimensional field theories. On the other hand, in two dimensional field theories, a topological term can change the energy spectrum of particles. For example, consider the topological term in the $O(3)$ σ-model in two dimensions (called the 'θ-term'). We can obtain the $O(3)$ σ-model with the θ-term as an effective action of the Heisenberg antiferromagnets in one-dimension [6, 7]. The θ-term produces the difference between qualitative properties for integer spin and half-integer spin in the Heisenberg antiferromagnets. If the Heisenberg hamiltonian has integral spin, the θ-term takes the value $2\pi in, n \in \mathbf{Z}$ for all field configurations. This implies that the θ-term does not contribute to the partition function of the effective theory. It is well known that the energy spectrum of the $O(3)$ σ-model without the θ-term has a gap. In the other case, namely if the hamiltonian has half-integral spin, the θ-term takes the value $\pi in, n \in \mathbf{Z}$. We have to take account of an effect of the θ-term. The θ-term is expected to give important effects. In fact, the Heisenberg antiferromagnets with spin $\frac{1}{2}$ has a gapless spectrum. This implies that an excitation spectrum is changed by the θ-term. Note that the effective theory has parity invariance even if the sign of the θ-term changes under the parity transformation.

In relativistic quantum field theories the statistics of particles is closely related to their spin. The question arises in relativistic theories whether interactions changing the statistics of the particles also transmute their spin. Polyakov has dealt with the CP^1 model in 3 dimensions coupled with the Chern-Simons gauge field. This is regarded as an effective theory of a strongly correlated electron system in 2 dimensions [2]. He calculates the propagator of the charged scalar particle in the long-wave-length limit. The claim is that the self-energy of the particle generates spin degrees of freedom and that the bosonic propagator is transmuted into a fermionic one.

He first describes the propagator in terms of a functional integration over trajectories of a particle. The particle interacts with the Chern-Simons gauge field. Corrections coming from matter-loops are ignored in the long-wave-length limit. Next he takes the average over the Chern-Simons gauge field and the self-energy correction is estimated in a certain regularization. The resulting self-energy is equal to the torsion of the trajectory. Since the torsion term contains higher derivatives of the particle coordinates, the particle may acquire new degrees of freedom associated with the unit tangent vector along the path [9]. He rewrites the torsion as a spin factor which gives the commutation relation of spin to the unit tangent vector. Due to the spin factor, the propagator of the scalar particle, dressed with the Chern-Simons gauge field becomes the Dirac propagator.

In this argument, however, there are several unclear points. The measure in the path integral is not clearly defined. And the regularized self-energy seems to depend on the regularization procedure: the result may change if we take other regularization schemes [11]. For this reason, the relationship between the self-energy and the spin factor is not obvious. The results of Polyakov refer to the one particle sector, for this reason it has not been explicitly shown whether the statistics of the charged scalar particle are also transmuted by the Chern-Simons term.

The main purpose of this thesis is to elucidate these problems. We justfy Polyakov's argument. Further, we obtain an exact result on bose-fermi transmutation in second quantized field theories.

Organization of the thesis In the next section, we treat systems of free scalar and free Dirac fields. We discuss that the path-integral representations of partition functions and the free energies [3]. The partition functions are obtained by integration over all possible configurations of fields, and the free energies, by integration of all possible trajectories of particles. In the path-integral description of free energies, the action is determined in the bosonic case by the lengths of paths. In the fermionic case on the other hand, the spin factor is added to the bosonic action . We also comment on the path-integral representation of the propagators of both particles [2, 4, 17, 18, 19, 20, 22].

In section 3 we deal with a system of many scalar particles coupled with the Chern-Simons gauge field. Here we analyze the relative energy between different particles and the self-energy. The relative energy is found to be topologically invariant and its value is quantized. On the other hand, the self-energy is not completely topologically invariant. However its value is quantized if the trajectories lie on a certain plane. The relationship with the spin factor, which plays an important role in section 4, is clarified through this feature [14, 16, 24].

In section 4 the result obtained in section 2 is extended to the case where these systems are coupled to an external gauge field. Then a bosonic system coupled to the Chern-Simons field is studied. Here it is shown that the partition function of the system is identical with that of a free fermion theory. This is the result on the bose-fermi transmutation in relativistic quantum field theories [24, 25]. The last section contains our conclusion and discussions.

2 Path-integral representation of partition functions and free energies

In this section we show that the partition functions and free energies of second quantized systems can be represented in terms of functional integrations. Partition functions can be described by a summation over all configurations of a *field* while the free energies are described by summing over closed random paths of a *particle.*

2.1 A scalar field

Partition function In this paragraph we compute the following path integral (or the functional integral) over a complex scalar field

$$Z_B = \int \mathcal{D}\phi^* \mathcal{D}\phi \; e^{-S_B}, \tag{2.1}$$

where the action is defined by

$$S_B = \int_0^\beta dx^0 \int d^D\mathrm{x}\{|\partial_\mu \phi|^2 + m^2|\phi|^2\},\; \mu = 0, 1, \cdots D. \tag{2.2}$$

The field $\phi(x^0, \mathrm{x})$ is assumed to satisfy the periodic boundary condition $\phi(0, \mathrm{x}) = \phi(\beta, \mathrm{x})$. We wish to show that the above formula is identical with the partition function of the relativistic bose gas. The field $\phi(x^0, \mathrm{x})$ is expanded as follows:

$$\phi(x^0, \mathrm{x}) = \frac{1}{\beta}\sum_n \int \frac{d^D\mathbf{k}}{(2\pi)^D}\phi_{n,\mathbf{k}} e^{-i\frac{2\pi n}{\beta}x^0 + i\mathbf{k}\cdot\mathbf{x}} \tag{2.3}$$

The action S_B is written in terms of momentum representation:

$$S_B = \frac{1}{\beta}\sum_n \int \frac{d^D\mathbf{k}}{(2\pi)^D}\phi^*_{n,\mathbf{k}}(\omega_n^2 + E_{\mathbf{k}}^2)\phi_{n,\mathbf{k}}, \tag{2.4}$$

where $\omega_n \equiv 2\pi n/\beta$, and $E_{\mathrm{k}} \equiv \sqrt{\mathrm{k}^2 + m^2}$. The measure of the path integral eq.(2.1) is determined by the following norm of a functional space:

$$\begin{aligned} ||\delta\phi||^2 &= \int_0^\beta dx^0 \int d^D\mathrm{x}|\delta\phi(x^0, \mathrm{x})|^2 \\ &= \frac{1}{\beta}\sum_n \int \frac{d^D\mathbf{k}}{(2\pi)^D}\delta\phi^*_{n,\mathbf{k}}\delta\phi_{n,\mathbf{k}}. \end{aligned} \tag{2.5}$$

Therefore

$$\mathcal{D}\phi^*\mathcal{D}\phi = \prod_{n,\mathbf{k}} d(\frac{\phi^*_{n,\mathbf{k}}}{\sqrt{\beta}})d(\frac{\phi_{n,\mathbf{k}}}{\sqrt{\beta}}). \tag{2.6}$$

This shows that integration variables are not $\phi_{n,\mathbf{k}}$ but $\frac{\phi_{n,\mathbf{k}}}{\sqrt{\beta}}$. Substituting eq.(2.4) and eq.(2.6) into eq.(2.1), we can perform the following Gaussian integral:

$$\begin{aligned} Z_B &= \int \prod_{n,\mathbf{k}} d(\frac{\phi^*_{n,\mathbf{k}}}{\sqrt{\beta}}) d(\frac{\phi_{n,\mathbf{k}}}{\sqrt{\beta}}) \exp\{-\sum_n \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{\phi^*_{n,\mathbf{k}}}{\sqrt{\beta}} (\omega_n^2 + E_\mathbf{k}^2) \frac{\phi_{n,\mathbf{k}}}{\sqrt{\beta}}\} \\ &= \text{const.} \prod_{n,\mathbf{k}} (\omega_n^2 + E_\mathbf{k}^2)^{-1} \\ &= \text{const.} \exp\{-\sum_{n,\mathbf{k}} \ln(\omega_n^2 + E_\mathbf{k}^2)\}. \end{aligned} \tag{2.7}$$

Next we consider the summation over n in the above equation:

$$\sum_n \ln(\omega_n^2 + E^2). \tag{2.8}$$

We rewrite this in the following way.

$$\begin{aligned} (2.8) &= \sum_n \int^{E^2} dt^2 \frac{1}{\omega_n^2 + t^2} \\ &= \beta \int^{E^2} dt^2 \oint_C \frac{dz}{2\pi i} \frac{1}{e^{\beta z} - 1} \frac{1}{-z^2 + t^2} \\ &\equiv \beta \int^{E^2} dt^2 \oint_C \frac{dz}{2\pi i} g(z), \end{aligned} \tag{2.9}$$

where the contour C encircles the imaginary axis in z-plane. Since we have no interest in additive constants, the lower limit of the integrated interval is not specified. We see that a contribution to the contour integral comes from the residues at $z = \pm t$ after deforming the contour C. Then the last formula in the above equation becomes

$$\begin{aligned} & \beta \int^{E^2} dt^2 \oint_C \frac{dz}{2\pi i} g(z) \\ =\ & -\beta \int^{E^2} dt^2 \{\mathrm{Res}_{z=t}\, g(z) + \mathrm{Res}_{z=-t}\, g(z)\} \\ =\ & -\beta \int^{E^2} dt^2 \frac{1}{2t} (\frac{1}{e^{\beta t} - 1} - \frac{1}{e^{-\beta t} - 1}) \\ =\ & \beta \int^{E} dt (\frac{1}{e^{\beta t} - 1} - \frac{1}{e^{-\beta t} - 1}) \\ =\ & 2(\ln(1 - e^{-\beta E}) + \frac{\beta E}{2}). \end{aligned} \tag{2.10}$$

Inserting eq.(2.10) into eq.(2.7) we obtain

$$Z_B = \text{const.} \exp\{-2\beta V \int \frac{d^D\mathbf{k}}{(2\pi)^D} \{\frac{1}{\beta} \ln(1 - e^{-\beta E_\mathbf{k}}) + \frac{E_\mathbf{k}}{2}\}\}, \tag{2.11}$$

where V denotes the volume of the D-dimensional space. The exponent in the above equation is identical with the free energy of the ideal relativistic bose gas (up to the factor $-\beta$). The term $\frac{E_{\mathbf{k}}}{2}$ corresponds to the zero-point energy. Therefore Z_B is the partition function of this system. Eq.(2.11) indicates that Z_B gives a functional-integral form of the partition function.

The product in the second line of eq.(2.7) can be alternatively described by the product over eigen values of momentum operators:

$$Z_B \propto \prod_{n,\mathbf{k}} (\omega_n^2 + E_{\mathbf{k}}^2)^{-1} = \det{}^{-1}(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2), \tag{2.12}$$

or

$$Z_B \propto \prod_{n,\mathbf{k}} (\omega_n^2 + E_{\mathbf{k}}^2)^{-1} = \exp\{-\mathrm{Tr}\ln(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2)\}, \tag{2.13}$$

where $\hat{p}_0$ has the discrete spectrum $\frac{2\pi n}{\beta}$ and $\hat{\mathbf{p}}$ has the continuous one, $\mathbf{k}$. The trace in the above formula is taken using the following orthonormal basis:

$$\begin{aligned} \hat{p}_0|n> &= \frac{2\pi n}{\beta}|n>, \quad <n|m>= \delta_{nm}, \\ \hat{\mathbf{p}}|\mathbf{k}> &= \mathbf{k}|\mathbf{k}>, \quad <\mathbf{k}|\mathbf{k}'>= \delta(\mathbf{k}-\mathbf{k}'). \end{aligned} \tag{2.14}$$

Then

$$\begin{aligned} \mathrm{Tr}\ln(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2) &= \int d^D\mathbf{k} \sum_n <n|\otimes<\mathbf{k}|\ln(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2)|\mathbf{k}>\otimes|n> \\ &= \int d^D\mathbf{k} \sum_n \ln((\frac{2\pi n}{\beta})^2 + \mathbf{k}^2 + m^2) <\mathbf{k}|\mathbf{k}> \\ &= \mathrm{V}\int \frac{d^D\mathbf{k}}{(2\pi)^D} \sum_n \ln((\frac{2\pi n}{\beta})^2 + \mathbf{k}^2 + m^2). \end{aligned} \tag{2.15}$$

This is the same formula as in the last line of eq.(2.7). Thus we find

$$Z_B = \mathrm{const.}\exp\{-\mathrm{Tr}\ln(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2)\}. \tag{2.16}$$

If $\beta = \infty$, the summation over n appeared in eq.(2.15) are replaced by an integration over a continuous variable because p_0 has a continuous spectrum as well as $\hat{\mathbf{p}}$. In this case

$$\mathrm{Tr}\ln(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2) = V_{s.t.} \int \frac{dk_0}{2\pi}\frac{d^D\mathbf{k}}{(2\pi)^D} \ln(k_0^2 + \mathbf{k}^2 + m^2), \tag{2.17}$$

where $V_{s.t.} \equiv \beta \mathrm{V}$. The factor $\beta = \infty$ is absorbed by $V_{s.t.}$. Eq.(2.13) implies that the formula

$$\int \frac{dk_0}{2\pi} \frac{d^D \mathbf{k}}{(2\pi)^D} \ln(k_0^2 + \mathbf{k}^2 + m^2) \tag{2.18}$$

expresses the free energy per unit volume at $\beta = \infty$ of this system (see below). In the following we try to rewrite this quantity into a functional integral over random paths.

Free energy [1] The partition function Z_B and the free energy W_B are related with the following formula:

$$Z_B = e^{-\beta W_B}. \tag{2.19}$$

In this section we treat the free energy per unit volume $w_B \equiv W_B/\mathrm{V}$ rather than W_B. From eq.(2.16) and eq.(2.19), we have

$$\begin{aligned} Z_B &= \text{const.} \exp\{-\mathrm{Tr} \ln(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2)\} \\ &= \exp\{-\beta \mathrm{V} w_B\}. \end{aligned} \tag{2.20}$$

This equation shows

$$w_B = \frac{1}{\beta \mathrm{V}} \mathrm{Tr} \ln(\hat{p}_0^2 + \hat{\mathbf{p}}^2 + m^2). \tag{2.21}$$

From eq.(2.17), the above equation becomes

$$w_B = \int \frac{dk_0}{2\pi} \frac{d^D \mathbf{k}}{(2\pi)^D} \ln(k_0^2 + \mathbf{k}^2 + m^2) \tag{2.22}$$

when $\beta = \infty$.

The purpose of this paragraph is to show that w_B at $\beta = \infty$ can be written in terms of a functional integration over closed particle trajectories [2]:

$$w_B = \text{const.} \int \frac{\mathcal{D}X}{V_G V_{s.t.}} \exp\{-m_0 \int_0^1 d\tau \sqrt{\dot{X}^2}\}. \tag{2.23}$$

A closed trajectory of the particle is described by $X^\mu(\tau)(\mu = 0, \cdots, D)$ which satisfy $X(0) = X(1)$. The action $m_0 \int_0^1 d\tau \sqrt{\dot{X}^2}$ is invariant under the following reparametrizations (or local transformations):

$$\tau \to f(\tau), \quad f(0) = 0 \quad f(1) = 1 \quad \frac{df}{d\tau} > 0. \tag{2.24}$$

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[1] Our discussion in this paragraph is restricted to the case of $\beta = \infty$.

[2] From here we use the D+1-dimensional 'relativistic' notation $k^2 \equiv (k^0)^2 + (\mathbf{k})^2, \mathcal{D}X \equiv \prod_{\mu=0}^{D} \mathcal{D}X_\mu$, $a \cdot b = ab \equiv a^0 b^0 + \mathbf{a} \cdot \mathbf{b}$, etc.

In eq.(2.23), V_G (called the gauge volume) denotes the volume of the transformations. Since $X(\tau)$ and $X(f(\tau))$ represent the same path, summation over $X(f(\tau))$ varying $f(\tau)$ means infinitely counting contributions from one path. The gauge volume V_G in the denominator in eq.(2.23) eliminates this redundancy.

The above equation (2.23) is a formal expression since both sides of this equation have divergences. These have to be regularized in order to make sense. Following the asymptotic expansion

$$-\int_\epsilon^\infty \frac{dt}{t} e^{-tz} = \ln \epsilon z + \text{regular terms of } \epsilon, \tag{2.25}$$

we shall regularize the free energy w_B:

$$\begin{aligned} w_B &= \int \frac{dk_0}{2\pi} \frac{d^D\mathrm{k}}{(2\pi)^D} \ln(k_0^2 + \mathrm{k}^2 + m^2) \\ &\rightarrow -\int \frac{d^{D+1}k}{(2\pi)^{D+1}} \int_{\epsilon^2}^\infty \frac{dL}{L} \exp\{-L(k^2+m^2)\} \\ &\equiv w_B^\epsilon. \end{aligned} \tag{2.26}$$

We notice from eq.(2.11) that the free energy at $\beta = \infty$ is the summation of zero point energies of oscillators with eigen-modes E_{k}. As the cut-off parameter ϵ tends to zero, it diverges to $+\infty$.

We have to regularize the r.h.s. of eq.(2.23) as well. In this section dynamical variables in functional integrations are treated as those depending on a continuous parameter τ and high-frequency modes are cut off. In section 4, a different regularization will be adopted.

First we write eq. (2.23) in the following more convenient form:

$$\int \frac{\mathcal{D}X\mathcal{D}h}{V_G V_{s.t.}} \exp\{-\frac{1}{2}\int_0^1 d\tau(\dot{X}^2 h^{-1} + m_0^2 h)\}, \tag{2.27}$$

where $h(\tau)$ is an einbein which transforms under these local transformations (or the gauge transformations) (2.24) as follows: [3]

$$h(\tau) \rightarrow h'(f),\ h(\tau) = h'(f(\tau))\frac{df}{d\tau}. \tag{2.28}$$

We shall explain the equality between r.h.s. of eq.(2.23) and eq.(2.27). In classical theory, the equality is easily checked by using the equation of motion for $h(\tau)$. In

[3]See appendix A.

quantum theory, h can be integrated by using the gauge-invariant measure (see below) and by the formula

$$\int_0^\infty \frac{dx}{\sqrt{x}} e^{-a^2 x - b^2/x} = \frac{\sqrt{\pi}}{a} e^{-2ab}. \tag{2.29}$$

Then we turn back to the original formula eq.(2.23).

The measure in eq.(2.27) should be determined in a gauge-invariant way. This can be done on the basis of the following gauge invariant norms [3, 18].

$$\|\delta\phi\|^2 = \int_0^1 d\tau h(\tau)(\delta\phi)^2 \tag{2.30}$$

for a scalar $\delta\phi$,

$$\|\delta\eta\|^2 = \int_0^1 d\tau h(\tau)(h^{-1}(\tau)\delta\eta(\tau))^2 \tag{2.31}$$

for a covariant vector $\delta\eta$,

$$\|\delta\xi\|^2 = \int_0^1 d\tau h(\tau)(h(\tau)\delta\xi(\tau))^2 \tag{2.32}$$

for a contravariant vector $\delta\xi$. Note that a scalar or a vector is determined by its transformation law under the general coordinate transformation in one dimension[4].

Next we change the integration variable $\mathcal{D}h$. For any einbein $h(\tau)$ there exists some gauge transformation such that the einbein after the transformation is τ-independent. In fact, for a given einbein $h(\tau)$ we can do this by choosing the following local transformations

$$f(\tau) = \frac{1}{L}\int_0^\tau d\tau' h(\tau'), \quad L \equiv f(1). \tag{2.33}$$

Then the transformed einbein becomes

$$h'(f) = L. \tag{2.34}$$

This shows that the einbein can be parametrized by a pair $(L, f(\tau))$. In fact, from eq(2.28):

$$h(\tau) = L\frac{df}{d\tau}. \tag{2.35}$$

A small deformation δh is also parametrized by the new variables:

$$\begin{aligned} \delta h(\tau) &= \delta L \frac{df}{d\tau} + L\frac{d\delta f}{d\tau} \\ &= (\delta L + L\delta f)\dot{\tau}^{-1}, \end{aligned} \tag{2.36}$$

[4]The transformation properties of these quantities are summerized in appendix A.

or

$$h^{-1}(\tau)\delta h(\tau) = L^{-1}\delta L + \delta\dot{f}, \tag{2.37}$$

where the dot means d/df. The gauge invariant norm $||\delta h||$ follows eq.(2.31) [5]:

$$\begin{aligned} ||\delta h||^2 &= \int_0^1 Ldf(L^{-2}(\delta L)^2 + L^{-1}(\delta L)\delta\dot{f} + (\delta\dot{f})^2) \\ &= \int_0^1 df(L^{-1}(\delta L)^2 + L(\delta\dot{f})^2) \\ &= \frac{\delta L^2}{L} + \int_0^1 dfL(\delta\dot{f})^2, \end{aligned} \tag{2.38}$$

where the second term in the first line vanishes because $\delta f(0) = \delta f(1) = 0$. Since δf is a contravariant vector(see appendix A), the norm $||\delta f||^2$ should be determined by eq.(2.32):

$$\begin{aligned} ||\delta f||^2 &= \int_0^1 dfh'(h'\delta f)^2 \\ &= \int_0^1 df(L^{\frac{3}{2}}\delta f)^2. \end{aligned} \tag{2.39}$$

From eq.(2.38) and eq.(2.39), we get

$$\mathcal{D}h = \frac{dL}{\sqrt{L}}\mathcal{D}f \ \det{}'^{\frac{1}{2}}(-L^{-2}\frac{d^2}{df^2}), \tag{2.40}$$

where $\det'$ is the product of the eigenvalues except for the zero-eigenvalue. This determinant can be calculated under the regularization previously mentioned [3]. The result is (see appendix B)

$$\det{}'^{\frac{1}{2}}(-L^{-2}\frac{d^2}{df^2}) = N(\epsilon)\ e^{\frac{-L}{2\epsilon\sqrt{\pi}}}L, \tag{2.41}$$

where ϵ is an ultraviolet cut-off parameter. The constant factor N diverges as $\epsilon \to 0$.

It is easily seen that the action in eq.(2.27) does not contain $f(\tau)$ when we use the parameter $(L, f(\tau))$. The gauge volume in the denominator in eq.(2.27) means eliminating the integration over a redundant variable such as $\int \mathcal{D}f$. The gauge volume V_G contains not only $\int \mathcal{D}f$ but also the volume of the translation $\tau \to \tau + a$, $0 \le a \le 1$. In fact, $h(\tau)$ and $h(\tau + a)$ describes the same einbein because of the periodic boundary condition $h(0) = h(1)$. The gauge invariant norm $||\delta a||^2$ is subject to eq.(2.32). Therefore

$$V_G = \int \mathcal{D}f \int d(L^{\frac{3}{2}}a) = L^{\frac{3}{2}} \int \mathcal{D}f \tag{2.42}$$

[5]See appendix A

From eq.(2.40), eq.(2.41) and eq.(2.42) we get

$$\begin{aligned}
\frac{\int \mathcal{D}h}{V_G} &= \frac{\int \frac{dL}{\sqrt{L}}\mathcal{D}f \ \det'^{\frac{1}{2}}(-L^{-2}\frac{d^2}{df^2})}{L^{\frac{3}{2}}\int \mathcal{D}f} \\
&= N \int_{\epsilon^2}^{\infty} \frac{dL}{L} \ e^{\frac{-L}{2\sqrt{\pi\epsilon}}}.
\end{aligned} \tag{2.43}$$

Employing eq.(2.43) and the parametrization eq.(2.35), we now calculate the path integral eq.(2.27). The action is described as follows:

$$\int_0^1 df(\dot{X}^2 L^{-1} + m_0^2 L). \tag{2.44}$$

Note that the action does not contain the zero mode. This fact is checked as follows. According to the periodic boundary condition $X(0) = X(1)$, we can expand $X(f)$ as follows:

$$\begin{aligned}
X(f) &= X_c + \sum_{n\neq 0} a_n e^{2\pi i n f} \\
&\equiv X_c + Y(f),
\end{aligned} \tag{2.45}$$

where X_c is a constant which corresponds to the zero mode. Other modes are described by $Y(f)$. Thus

$$\int_0^1 df(\dot{X}^2 L^{-1} + m_0^2 L) = \int_0^1 df(\dot{Y}^2 L^{-1} + m_0^2 L). \tag{2.46}$$

Next we treat the measure $\mathcal{D}X$. The particle coordinates $X(f)$ transform as a scalar under reparametrizations. The measure is determined by the norm eq.(2.30):

$$\begin{aligned}
||\delta X||^2 &= \int h d\tau (\delta X)^2 \\
&= \int L df (\delta X)^2 \\
&= \int df (L^{\frac{1}{2}} \delta X)^2.
\end{aligned} \tag{2.47}$$

This implies that the integrated variables are $L^{\frac{1}{2}}X$ rather than X. Further we decompose the measure $\mathcal{D}X$ into the measure of the zero mode and that of other modes:

$$\begin{aligned}
\int \mathcal{D}X &= \int d^{D+1}(L^{\frac{1}{2}}X_c)\int \mathcal{D}Y \\
&= L^{\frac{D+1}{2}} V_{s.t.} \int \mathcal{D}Y,
\end{aligned} \tag{2.48}$$

where the integration over X_c gives the factor $V_{s.t.}$. We can perform the integration over Y as follows:

$$\begin{aligned}
\int \mathcal{D}Y e^{-\frac{1}{2}\int \dot{Y}^2 L^{-1}} &= \int \mathcal{D}Y e^{-\frac{1}{2}\int (\dot{Y}L^{\frac{1}{2}})^2 L^{-2}} \\
&= \int \mathcal{D}Y e^{-\frac{1}{2}\int (YL^{\frac{1}{2}})(-L^{-2}\frac{d^2}{df^2})(YL^{\frac{1}{2}})} \\
&= \text{const.}\det{}'^{-\frac{D+1}{2}}(-L^{-2}\frac{d^2}{df^2}).
\end{aligned} \tag{2.49}$$

Eqs.(2.46)-(2.49) give the following calculation:

$$\begin{aligned}
&\int \frac{\mathcal{D}X\mathcal{D}h}{V_G V_{s.t.}} \exp\{-\frac{1}{2}\int_0^1 d\tau(\dot{X}^2 h^{-1} + m_0^2 h)\} \\
&= N\int_{\epsilon^2}^{\infty} \frac{dL}{LV_{s.t.}}\, e^{\frac{-L}{2\sqrt{\pi}\epsilon}} \int d^{D+1}(L^{\frac{1}{2}}X_c) \int \mathcal{D}Y \exp\{-\frac{1}{2}\int_0^1 df(\dot{Y}^2 L^{-1} + m_0^2 L)\} \\
&= \text{const.} \int_{\epsilon^2}^{\infty} \frac{dL}{L} L^{\frac{D+1}{2}} \det{}'^{-\frac{D+1}{2}}(-L^{-2}\frac{d^2}{df^2}) \exp\{-\frac{1}{2}(\frac{1}{\sqrt{\pi}\epsilon} + m_0^2)L\} \\
&= \text{const.} \int_{\epsilon^2}^{\infty} \frac{dL}{L} L^{-\frac{D+1}{2}} \exp(-\frac{L}{2}m^2),
\end{aligned} \tag{2.50}$$

where a physical mass m has been defined in the last line:

$$m^2 \equiv m_0^2 - \frac{\text{const.}}{\epsilon}. \tag{2.51}$$

Another calculation gives

$$\begin{aligned}
w_B^\epsilon &\equiv -\int \frac{d^{D+1}k}{(2\pi)^{D+1}} \int_{\epsilon^2}^{\infty} \frac{dL}{L} \exp\{-L(k^2 + m^2)\} \\
&= -(\frac{1}{\sqrt{2\pi}})^{D+1} \int_{2\epsilon^2}^{\infty} \frac{dL}{L} L^{-\frac{D+1}{2}} \exp(-\frac{L}{2}m^2).
\end{aligned} \tag{2.52}$$

Comparing eq.(2.50) and eq.(2.52), we obtain eq.(2.23).

2.2 A Dirac field

Partition function Next we consider a functional integration over a Dirac field Z_F: [6]

$$\begin{aligned}
Z_F &\equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_F}, \\
S_F &\equiv \int_0^\beta d\tau \int d^D\mathrm{x}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi.
\end{aligned} \tag{2.53}$$

[6] Our convention for the γ matrices is $\gamma_\mu^\dagger = -\gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = -2\delta^{\mu\nu}$.

In this case the fields $\psi(x^0, \mathrm{x})$ and $\bar{\psi}(x^0, \mathrm{x})$ take values in grassmann numbers and satisfy the anti-periodic boundary condition, $\psi(0, \mathrm{x}) = -\psi(\beta, \mathrm{x})$. We expand the field $\psi(x^0, \mathrm{x})$ following the above boundary condition:

$$\psi(x^0, \mathrm{x}) = \frac{1}{\beta} \sum_n \int \frac{d^D \mathbf{k}}{(2\pi)^D} \psi_{n,\mathbf{k}} e^{-i\frac{2\pi(n+1)}{\beta}x^0 + i\mathbf{k}\cdot\mathbf{x}}. \tag{2.54}$$

The action S_F is described by $\psi_{n,\mathbf{k}}$ and $\bar{\psi}_{n,\mathbf{k}}$ in momentum space:

$$S_F = \frac{1}{\beta} \sum_n \int \frac{d^D \mathbf{k}}{(2\pi)^D} \bar{\psi}_{n,\mathbf{k}} (\gamma^0 \tilde{\omega}_n + \gamma \cdot \mathbf{k} + m) \psi_{n,\mathbf{k}}, \tag{2.55}$$

where $\tilde{\omega}_n \equiv \frac{2n+1}{\beta}$. After the same calculation as in the case of the scalar field, the integration measure in eq.(2.53) becomes

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_{n,\mathbf{k}} d(\frac{\bar{\psi}_{n,\mathbf{k}}}{\sqrt{\beta}}) d(\frac{\psi_{n,\mathbf{k}}}{\sqrt{\beta}}). \tag{2.56}$$

Before computation of the path integral eq.(2.53), we review integrations over grassmann variables.

Let us consider two grassmann numbers $\bar{c}$ and c. These satisfy the following relation:

$$\bar{c}\bar{c} = cc = \bar{c}c + c\bar{c} = 0. \tag{2.57}$$

Integration over $\bar{c}$ and c obeys the following rule:

$$\int dc = \int d\bar{c} = 0, \ \int dcc = \int d\bar{c}\bar{c} = 1. \tag{2.58}$$

Note that grassmann numbers and their differentials anticommute each other. For example we compute the following integral:

$$\int d\bar{c}\ dc\ e^{-\bar{c}ac}, \tag{2.59}$$

where a is a complex number. According to the anticommutation relation eq.(2.57), we notice

$$e^{-\bar{c}ac} = 1 - \bar{c}ac. \tag{2.60}$$

This leads to

$$\int d\bar{c}dc e^{-\bar{c}ac} = \int d\bar{c}dc(1 - \bar{c}ac) = +a \int d\bar{c}\bar{c}dcc = a. \tag{2.61}$$

Similarly one can show

$$\int \prod_i d\bar{c}_i dc_i e^{-\bar{c}_i A_{ij} c_j} = \det A, \tag{2.62}$$

where $\bar{c}_i$ and c_i are grassmann numbers $(i, j = 1, \cdots, n)$ while A is an $n \times n$ matrix whose entries take values in complex numbers.

Now we turn back to our problem. We can perform integration over grassmann fields $\psi_{n\mathrm{k}}$ and $\bar{\psi}_{n\mathrm{k}}$ following the same rule as eq.(2.58):

$$\begin{aligned} Z_F &= \int \prod_{n,\mathrm{k}} d(\frac{\bar{\psi}_{n,\mathrm{k}}}{\sqrt{\beta}}) d(\frac{\psi_{n,\mathrm{k}}}{\sqrt{\beta}}) \exp\{-\sum_n \int \frac{d^D\mathrm{k}}{(2\pi)^D} \frac{\bar{\psi}_{n,\mathrm{k}}}{\sqrt{\beta}} (\gamma^0 \tilde{\omega}_n + \gamma{\cdot}\mathrm{k} + m) \frac{\psi_{n,\mathrm{k}}}{\sqrt{\beta}}\} \\ &= \mathrm{const.} \prod_{n,\mathrm{k}} det(\gamma^0 \tilde{\omega}_n + \gamma{\cdot}\mathrm{k} + m). \end{aligned} \tag{2.63}$$

In the second line, det means taking the determinant of an $s \times s$ matrix, where s denotes the number of components of the Dirac field. The determinant in the above equation should be invariant under the change $n, \mathrm{k} \to -n, -\mathrm{k}$. Therefore

$$\begin{aligned} & det(\gamma^0 \tilde{\omega}_n + \gamma{\cdot}\mathrm{k} + m) \\ = \;& det^{\frac{1}{2}} \{(\gamma^0 \tilde{\omega}_n + \gamma{\cdot}\mathrm{k} + m)(-\gamma^0 \tilde{\omega}_n - \gamma{\cdot}\mathrm{k} + m)\} \\ = \;& det^{\frac{1}{2}} (\tilde{\omega}_n^2 + \mathrm{k}^2 + m^2) I \\ = \;& (\tilde{\omega}_n^2 + E_\mathrm{k}^2)^{\frac{s}{2}}, \end{aligned} \tag{2.64}$$

where we have used the relation $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$. In the second line I stands for the $s \times s$ unit matrix. Inserting this result into eq.(2.63) we have

$$\begin{aligned} Z_F &= \mathrm{const.} \prod_{n,\mathrm{k}} (\tilde{\omega}^2 + E_\mathrm{k}^2)^{\frac{s}{2}} \\ &= \mathrm{const.} \exp\{\frac{s}{2} \sum_{n,\mathrm{k}} \ln(\tilde{\omega}^2 + E_\mathrm{k}^2)\} \\ &= \mathrm{const.} \exp\{\frac{s}{2} \mathrm{V} \sum_n \int \frac{d^D\mathrm{k}}{(2\pi)^D} \ln(\tilde{\omega}^2 + E_\mathrm{k}^2)\}. \end{aligned} \tag{2.65}$$

Next we take the summation over n in the above formula as we have done in the case of the scalar field:

$$\begin{aligned} \sum_n \ln(\tilde{\omega}^2 + E^2) &= \beta \int^{E^2} dt^2 \oint_C \frac{dz}{2\pi i} \frac{1}{e^{\beta z} + 1} \frac{1}{-z^2 + t^2} \\ &= -\beta \int^E dt (\frac{1}{e^{\beta t} + 1} - \frac{1}{e^{-\beta t} + 1}) \\ &= 2(\ln(1 + e^{-\beta E}) + \frac{\beta E}{2}). \end{aligned} \tag{2.66}$$

Substituting this into eq.(2.65) leads to the final result:

$$Z_F = \text{const.}\exp\{s\beta \mathrm{V}\int \frac{d^D\mathbf{k}}{(2\pi)^D}\{\frac{1}{\beta}\ln(1+e^{-\beta E_{\mathbf{k}}})+\frac{E_{\mathbf{k}}}{2}\}\}. \tag{2.67}$$

Thus Z_F is the partition function of the free Dirac field.

The partition function Z_F is also calculated introducing the first-quantized operators $\hat{p}_0$ and $\hat{\mathbf{p}}$ as we have discussed in the previous case:

$$\begin{aligned} Z_F &= \text{const.}\det(\gamma^0\hat{p}_0+\boldsymbol{\gamma}\cdot\hat{\mathbf{p}}+m) \\ &= \text{const.}\exp\{\frac{s}{2}\mathrm{Tr}\ln(\gamma^0\hat{p}_0+\boldsymbol{\gamma}\cdot\hat{\mathbf{p}}+m)\}. \end{aligned} \tag{2.68}$$

The eigen-value of the operator $\hat{p}_0$, which is determined by the boundary condition for the fields, becomes $\tilde{\omega}_n = (2n+1)\pi/\beta$ instead of $\omega_n = 2n\pi/\beta$. These values lead to the change in sign of $e^{-\beta E_{\mathbf{k}}}$ in the integrand of eq.(2.67)(see eq.(2.11)). Note that the power of the determinant in the first line of eq. (2.68) has the opposite sign to that for the scalar field in eq.(2.12). This gives the statistical factor for a fermion loop.

As was shown in the scalar theory, the free energy of the Dirac field can also be described by a functional integral over closed paths of a particle. Paths of a spinning particle are usually described employing grassmann variables [3]. Here we give a description in terms of bosonic ones. Although a bosonic description is possible in arbitrary dimensions [12], here we only consider it for $D+1=3$. In this case, $s=2$ and we can choose $\gamma^\mu = -i\sigma^\mu$, where σ^μ are the Pauli matrices. The partition function Z_F at $\beta=\infty$ can be written as

$$\begin{aligned} Z_F &= \text{const.}\exp\{\mathrm{Tr}\ln(-i\sigma\cdot\hat{p}+m)\} \\ &= \text{const.}\exp\{V_{s.t.}\int\frac{d^3k}{(2\pi)^3}tr\ln(-i\sigma\cdot k+m)\}, \end{aligned} \tag{2.69}$$

where tr means taking the trace over spin indices. The main difference from the scalar particle is in the fact that we have to take the trace over the spin indices. For this purpose it is convenient to use $SU(2)$ coherent states [15].

$SU(2)$ coherent states In this paragraph, we shall explain $SU(2)$ coherent states. These are parametrized by the unit vector $e = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$:

$$|e> \equiv e^{-i\theta u\cdot S}|J,J>, \tag{2.70}$$

where $u \equiv (\sin\phi, -\cos\phi, 0)$. The spin generators S^μ satisfy $[S^\mu, S^\nu] = i\epsilon^{\mu\nu\lambda}S^\lambda$ and $|J, J>$ is the highest weight state in the spin-J representation. The $SU(2)$ coherent states have the following three properties (see appendix C):

$$\text{Partition of unity:} \qquad \int \frac{2J+1}{4\pi}\sin\theta d\theta d\phi |e><e| = 1. \tag{2.71}$$

$$\text{Inner product:} \qquad <e+\delta e|e> = e^{iJ\mathcal{A}[e+\delta e, e, e_0]} + O((\delta e)^2), \tag{2.72}$$

where $\mathcal{A}[e+\delta e, e, e_0]$ is the area of a spherical triangle with vertices $e, e+\delta e$ and $e_0 \equiv (0,0,1)$.

$$\text{Expectation value:} \qquad <e|S^\mu|e> = Je^\mu. \tag{2.73}$$

Employing these properties, the amplitude

$$<e_f|T\exp\left(\int_0^L dt S\cdot\mathcal{J}\right)|e_i> \tag{2.74}$$

can be written as a functional integral:

$$\begin{aligned}(2.74) &= \lim_{N\to\infty} <e_f|\prod_{i=0}^{N-1}(1+\Delta L S\cdot\mathcal{J}(t_i))|e_i> \\ &= \int \mathcal{D}e \exp\left(iJ\mathcal{A}[e_f, e_i, e_0] + \int_0^L dt e\cdot\mathcal{J}\right),\end{aligned} \tag{2.75}$$

where $\mathcal{J}$ is a c-number source. Since L is a gauge invariant quantity, ΔL plays a role of gauge-invariant cut-off. Therefore the measure

$$\mathcal{D}e \equiv \lim_{N\to\infty}\prod_{i=0}^{N-1}\frac{2J+1}{4\pi}\sin\theta(t_i)d\theta(t_i)d\phi(t_i) \tag{2.76}$$

is determined in a gauge-invariant way. In the last equality in eq.(2.75), $\mathcal{A}[e_f, e_i, e_0]$ is the area on the sphere encircled by the curve $e(t)$ and two geodesics. One geodesic connects e_0 and e_i, the other connects e_f and e_0. In particular, when $e_i = e_f$, $\mathcal{A}[e_f, e_i, e_0] \equiv \Phi[e]$ becomes the area encircled by the loop $e(t)$. We shall call $\Phi[e]$, the spin factor. An integral representation of the spin factor is given by [7]

$$\Phi[e] = \int_0^1 d\tau \int_0^1 d\tau' \epsilon^{\mu\nu\lambda}\bar{e}^\mu(\tau,\tau')\partial_\tau\bar{e}^\nu(\tau,\tau')\partial_{\tau'}\bar{e}^\lambda(\tau,\tau'), \tag{2.77}$$

where $\bar{e}(\tau,\tau')$ is an extension of $e(\tau)$ satisfying

$$\bar{e}(\tau,0) = e_0, \;\; \bar{e}(\tau,1) = e(\tau), \;\; \bar{e}(0,\tau') = \bar{e}(1,\tau'). \tag{2.78}$$

[7] Here we change the variable t to τ such that $\tau = t/L$

Different extensions lead to spin factors which differ by an integral multiple of 4π. This is explained as follows. Eq.(2.78) shows that the parameter space (τ, τ') can be regarded as a disc. Let us consider the two different extensions $\bar{e}(\tau, \tau')$ and $\bar{e}'(\tau, \tau')$. Parameter spaces of $\bar{e}$ and $\bar{e}'$ are denoted by D_+ and D_- respectively. Since $\bar{e}$ and $\bar{e}'$ take the same value on each boundary, we can make a sphere from the two discs identifying one boundary with the other ($D_+ \cup D_- \sim S^2$), and define the continuous map $\tilde{e} : S^2 \to S^2$:

$$\tilde{e}(p) = \begin{cases} \bar{e} & p \in D_+ \\ \bar{e}' & p \in D_- \end{cases} . \tag{2.79}$$

The difference of two spin factors can be rewritten as follows:

$$\int_{D_+} \epsilon^{\mu\nu\lambda} \bar{e}^\mu d\bar{e}^\nu \wedge d\bar{e}^\lambda - \int_{D_-} \epsilon^{\mu\nu\lambda} \bar{e}'^\mu d\bar{e}'^\nu \wedge d\bar{e}'^\lambda = \int_{S^2} \epsilon^{\mu\nu\lambda} \tilde{e}^\mu d\tilde{e}^\nu \wedge d\tilde{e}^\lambda \tag{2.80}$$

The r.h.s. of the above equation counts how many times the image $\tilde{e}$ wraps the sphere. Therefore

$$\int_{S^2} \epsilon^{\mu\nu\lambda} \tilde{e}^\mu d\tilde{e}^\nu \wedge d\tilde{e}^\lambda = 4\pi n, \; n \in \mathbf{Z}. \tag{2.81}$$

This result shows that $\exp\{iJ\Phi[e]\}$ is independent of the choice of extension if J is an integer or a half-integer. Note that the spin factor can also be represented in polar coordinates as follows (see appendix C):

$$\Phi[e] = \int_0^1 d\tau (1 - \cos\theta)\dot{\phi}. \tag{2.82}$$

Free energy Now we discuss the free energy of this system. From eq.(2.68) we notice that the free energy (per unit volume) is expressed as follows

$$w_F = -\frac{1}{V_{s.t.}} \frac{s}{2} \mathrm{Tr} \ln(\gamma^0 \hat{p}_0 + \boldsymbol{\gamma} \cdot \hat{\mathbf{p}} + m). \tag{2.83}$$

At $\beta = \infty$ the spectrum of $\hat{p}_0$ becomes continuous. We shall define the regularized free energy w_F^ϵ in the same way as w_B^ϵ (in the $D+1=3$ case): [8]

$$\begin{aligned} w_F^\epsilon &\equiv \frac{1}{V_{s.t.}} \frac{s}{2} \mathrm{Tr} \int_\epsilon^\infty \frac{dL}{L} \exp\{-L(\hat{p} \cdot \gamma + m)\} \\ &= tr \int_\epsilon^\infty \frac{dL}{L} \int \frac{d^3k}{(2\pi)^3} \exp\{-L(-ik \cdot \sigma + m)\} \end{aligned} \tag{2.84}$$

[8] Since ϵ has the dimension of length we have introduced the cut off ϵ instead of ϵ^2

Since the zero-point energy has the opposite sign to that of the scalar system (see eq.(2.11) and eq.(2.67)) the free energy at $\beta = \infty$ diverges to $-\infty$ as the cut-off ϵ tends to zero. Now we derive the expression corresponding to eq.(2.23) for the fermionic case,

$$w_F = \text{ const. } \int \frac{\mathcal{D}X}{V_G V_{s.t.}} \exp\{-\int_0^1 d\tau m_0 \sqrt{\dot{X}^2} + iJ\Phi[\dot{X}/\sqrt{\dot{X}^2}]\}, \tag{2.85}$$

where J is fixed at 1/2. The equivalent expression for the r.h.s. in eq.(2.85) is given by

$$\int \frac{\mathcal{D}\mu \mathcal{D}h}{V_G V_{s.t.}} e^{-S} \equiv K_f, \tag{2.86}$$

where

$$\mathcal{D}\mu \equiv \mathcal{D}X\mathcal{D}k\mathcal{D}e, \tag{2.87}$$

and

$$S \equiv \int_0^1 d\tau \{m_0 h + ik_\mu(\dot{X}^\mu - he^\mu)\} - iJ\Phi[e]. \tag{2.88}$$

We shall explain the equivalence between eq.(2.85) and eqs.(2.86)-(2.88). Since e^μ is a unit vector, it is integrated over a solid angle at each τ. The variable $k(\tau)$ plays a role of a Lagrange multiplier so that $\delta(\dot{X} - eh)$ is obtained after integrating out k. Then e and h are integrated, we obtain the r.h.s. of eq.(2.85). The measures $\mathcal{D}X, \mathcal{D}k$ and $\mathcal{D}e$ are determined by the gauge invariant norm eq.(2.30).

Now let us compute the path integral K_f. The action eq.(2.88) does not contain the zero mode of X. Then it becomes (after integrating by parts)

$$S = \int_0^1 df \{m_0 L - i\dot{k} \cdot Y - ik \cdot eL)\} - iJ\Phi[e], \tag{2.89}$$

where Y is defined by eq.(2.45). We have used the parametrization eq.(2.35) for the einbein h in the above equation. Next eq.(2.43) is inserted into eq.(2.86). Then

$$K_f = N \int_\epsilon^\infty \frac{dL}{L_V s.t.} e^{\frac{-L}{2\sqrt{\pi\epsilon}}} \int \mathcal{D}\mu e^{-S}. \tag{2.90}$$

Since the action does not contain the zero mode X_c, we separate the measure $\mathcal{D}X$ into $d(L^{\frac{1}{2}}X_c)$ and $\mathcal{D}Y$ as we have done in the case of the scalar particle. We have

$$\begin{aligned} K_f &= N \int_\epsilon^\infty \frac{dL}{LV_{s.t.}} e^{\frac{-L}{2\sqrt{\pi\epsilon}}} \int d^3(L^{\frac{1}{2}}X_c) \int \mathcal{D}Y\mathcal{D}k\mathcal{D}e e^{-S} \\ &= N \int_\epsilon^\infty \frac{dL}{L} e^{\frac{-L}{2\sqrt{\pi\epsilon}}} L^{\frac{3}{2}} \int \mathcal{D}Y\mathcal{D}k\mathcal{D}e \times \\ &\quad \exp\{-\int_0^1 df \{m_0 L - i\dot{k} \cdot Y - ik \cdot eL\} + iJ\Phi[e]\} \end{aligned} \tag{2.91}$$

Next we turn to the integration over Y. It should be noted that the integration variable is $YL^{\frac{1}{2}}$:

$$K_f = N\int_\epsilon^\infty \frac{dL}{L} e^{\frac{-L}{2\sqrt{\pi}\epsilon}} L^{\frac{3}{2}} \mathcal{D}k\mathcal{D}e\delta((\dot{k}L^{\frac{1}{2}})L^{-1}) \times \exp\{-\int_0^1 df\{m_0L - ik\cdot eL\} + iJ\Phi[e]\}. \tag{2.92}$$

We can easily perform the integration over k due to the delta functional. However the zero mode of k (denoted by k_c) still remains since the delta functional does not contain it. Therefore K_f becomes

$$K_f = N\int_\epsilon^\infty \frac{dL}{L} e^{\frac{-L}{2\sqrt{\pi}\epsilon}} L^{\frac{3}{2}} d^3(L^{\frac{1}{2}}k_c)\mathcal{D}e\det{}'^{-\frac{3}{2}}(-L^{-2}\frac{d^2}{df^2}) \times \exp\{-\int_0^1 df\{m_0L - ik_c\cdot eL\} + iJ\Phi[e]\} \tag{2.93}$$

Inserting eq.(2.41) we get

$$K_f = N^{-2}\int_\epsilon^\infty \frac{dL}{L}\int d^3k_c\mathcal{D}e\exp\{-Lm - \int_0^1 df\{-ik_c\cdot eL\} + iJ\Phi[e]\}, \tag{2.94}$$

where the physical mass m is defined by

$$m \equiv m_0 - \frac{\text{const.}}{\epsilon}. \tag{2.95}$$

Finally employing eq.(2.75), we complete the calculation:

$$K_f = N^{-2}tr\int_\epsilon^\infty \frac{dL}{L}\int d^3k_c\exp\{-(m - ik_c\cdot\sigma)L\}. \tag{2.96}$$

Comparing this result to eq.(2.84), K_f is proportional to the regularized free energy w_F^ϵ.

We have shown path-integral representations of both partition functions and free energies. In the case of partition functions, integration variables are fields. This means that partition functions contain many-particle contributions. On the other hand, free energies of free field theories contain single particle contributions since these are described by one-particle coordinates. A relationship between partition functions and free energies is given by the formula:

$$Z = e^{-\beta W} = 1 + (-\beta W) + \frac{1}{2}(-\beta W)^2 + \cdots. \tag{2.97}$$

This shows that contributions from the n-particle sector to the partition function Z are represented in terms of the product of the free energy W^n. In section 4, we analyzed the partition function of a scalar system coupled with the Chern-Simons gauge field using the particle-number expansion.

2.3 Propagators

We comment on the random-walk description of propagators. The propagators can be written in path-integral form, as in eq.(2.27) and eq.(2.86). The boundary condition however is different $X(0) = X_i, X(1) = X_f$, etc. The particle coordinates X can be expanded as

$$X(\tau) = (1-\tau)X_i + \tau X_f + Y(\tau), \tag{2.98}$$

which means the zero mode of $X(\tau)$ is fixed. Other modes $Y(\tau)$ are integrated. The different boundary condition changes the value of the determinant $\det(-L^2 d^2/df^2)$ [9]:

$$\det{}^{\frac{1}{2}}(-L^{-2}\frac{d^2}{df^2}) = N'(\epsilon)\ e^{\frac{-L}{2\epsilon\sqrt{\pi}}}\sqrt{L}. \tag{2.99}$$

Moreover, the gauge volume does not contain the volume of the translation. For these reasons eq.(2.43) is changed as follows:

$$\frac{\int \mathcal{D}h}{V_G} = N' \int_0^\infty dL\ e^{\frac{-L}{2\sqrt{\pi\epsilon}}}. \tag{2.100}$$

Following these facts, we can check that the path-integral eq.(2.27), with the boundary conditions $X(0) = X_i, X(1) = X_f$, gives the propagator of the scalar field [3, 18]:

$$\begin{aligned}
& N' \int_0^\infty dL\ e^{\frac{-L}{2\sqrt{\pi\epsilon}}} \int \mathcal{D}Y \exp\{-\frac{1}{2}\int_0^1 df(\dot{Y}^2 L^{-1} + (X_f - X_i)^2 + m_0^2 L)\} \\
= \ & N' \int_0^\infty dL \det{}^{-\frac{D+1}{2}}(-L^{-2}\frac{d^2}{df^2}) \exp\{-\frac{1}{2}(\frac{1}{\sqrt{\pi\epsilon}} + m_0^2)L - L^{-1}(X_i - X_f)^2\} \\
= \ & N'^{-D} \int_0^\infty dL L^{-\frac{D+1}{2}} \exp\{-\frac{1}{2}(Lm^2 + L^{-1}(X_f - X_i)^2)\} \\
= \ & \text{const.} \int \frac{d^{D+1}p}{(2\pi)^{D+1}} e^{-ip(X_f - X_i)} \frac{1}{p^2 + m^2}.
\end{aligned} \tag{2.101}$$

In the Dirac theory, the path integral eq.(2.86) (with fixed boundary conditions) reduces to the Dirac propagator [2, 17, 19, 20, 21, 22]:

$$\begin{aligned}
& N' \int_0^\infty dL e^{\frac{-L}{2\sqrt{\pi\epsilon}}} \int \mathcal{D}Y\mathcal{D}k\mathcal{D}e e^{-S[h=L]} \\
= \ & N' \int_0^\infty \frac{dL}{L} e^{\frac{-L}{2\sqrt{\pi\epsilon}}} \int \mathcal{D}k\mathcal{D}e\delta((\dot{kL^{\frac{1}{2}}})L^{-1}) \times \\
& \exp\{\int_0^1 df(ik(X_f - X_i) + ikeL - mL) + \frac{i}{2}\Phi[e]\}
\end{aligned}$$

[9] See appendix B.

$$
\begin{aligned}
&= N' \int_0^\infty dL e^{\frac{-L}{2\sqrt{\pi\epsilon}}} \det{}^{-\frac{3}{2}}(L^{-2}\frac{d^2}{df^2}) \int d^3(L^{\frac{3}{2}}k_c)\mathcal{D}e \times \\
&\quad \exp\{\int_0^1 df(ik(X_f - X_i) + ik_c eL - mL) + \frac{i}{2}\Phi[e]\} \\
&= \text{const.} \ < e_i| \int \frac{d^3k_c}{(2\pi)^3} \frac{1}{m - ik_c \cdot \sigma} e^{-ip(X_f - X_i)} |e_f> . \qquad (2.102)
\end{aligned}
$$

Finally we remark on the random-walk properties of both theories [3, 4]. The parameter L represents the length of a path. For the scalar theory, we can see from eq.(2.50) that the typical length of a path is $L \sim m^{-2}$. The correlation length R of the system is also determined by the physical mass $R \sim m^{-1}$. Therefore $L \sim R^2$. This indicates that the scalar particle mainly goes in a zigzag such that its path covers an area. On the other hand, it is found that $R \sim L$ for the Dirac particle. This implies the typical path of the Dirac particle is extremely smooth compared to that of the scalar particle. This fact is also confirmed by computing the expectation value of the velocity. We consider the periodic boundary condition. Let us consider a path with length L. For the scalar particle we obtain

$$
< \dot{X}^\mu(t)\dot{X}^\nu(t') > \equiv \int \mathcal{D}X \dot{X}^\mu(t)\dot{X}^\nu(t') e^{-\int_0^L ds \dot{X}^2} \sim \delta^{\mu\nu}\delta(t - t'). \qquad (2.103)
$$

This indicates $< \dot{X}(t)^2 >$ diverges for all t. On the other hand, for the Dirac particle, we get

$$
< \dot{X}^\mu(t)\dot{X}^\nu(t') > = < e^\mu(t)e^\nu(t') > = \sigma^\mu\sigma^\nu \sim \delta^{\mu\nu} \qquad (2.104)
$$

Comparing eq.(2.23) and eq.(2.85), we can understand that the difference arises from the presence of the spin factor. In section 4, we will examine more closely this fact.

3 Self-energy in Chern-Simons gauge field and the spin factor

In this section, we deal with a system of many particles interacting with the Chern-Simons gauge field [10]. In particular, the relative and the self energy of the particle are studied. It is found that these energies are related to links and knots of trajectories of particles. Consequently, the relationship between the self-energy and the spin factor is clarified. Once again we restrict ourselves to the case $D+1=3$.

3.1 Particles coupled to C.S. field

Here we discuss a relativistic n-particle system coupled to the Chern-Simons gauge field. The action of the Chern-Simons gauge field $S_{C.S.}$ is defined by

$$S_{C.S.} \equiv \frac{1}{8\pi J}\int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{3.1}$$

where J is some constant. This action is invariant under $U(1)$ gauge transformations $A_\mu \to A_\mu + \partial_\mu \Lambda$. Let us consider the following amplitude for closed paths $X_1, \cdots, X_n$:

$$K(X_1,\cdots,X_n) \equiv \mathrm{N}^{-1}\int \mathcal{D}A \exp\{i\sum_{p=1}^{n}\int_0^1 d\tau A(X_p)\cdot \dot{X}_p + iS_{C.S.}\}, \tag{3.2}$$

where $X_p(0)=X_p(1)$. The normalization factor N is determined by

$$\mathrm{N} = \int \mathcal{D}A e^{iS_{C.S.}}. \tag{3.3}$$

We take the average over the gauge field [11]

$$K(X_1,\cdots,X_n) = \exp\{-J\sum_{p<q}\Psi_{pq} - \frac{J}{2}\sum_{p}\Psi_{pp}\}, \tag{3.4}$$

and

$$J\Psi_{pq} \equiv \int_0^1 d\tau \int_0^1 d\tau' \dot{X}_p^\mu(\tau)\dot{X}_q^\nu(\tau') < A_\mu(X_p(\tau))A_\nu(X_q(\tau')) > . \tag{3.5}$$

[10] The argument in this section and in the next section is based on ref.[24]

[11] We have used the following formula which is derived from Wick's theorem:
$< \exp\varphi >= \exp(\frac{1}{2} < \varphi\varphi >)$, where $\varphi = \sum_{p=1}^{n}\int_0^1 d\tau A(X_p)\cdot X_p$.

These quantities are evaluated by using the propagator in the covariant gauge(see appendix D):

$$< A_\mu(x)A_\nu(y) >= iJ\epsilon^{\mu\nu\lambda}\frac{x^\lambda - y^\lambda}{|x-y|^3}. \tag{3.6}$$

The term $\Psi_{pq}(p \neq q)$ in eq.(3.4) represents the relative energy between the p-th and the q-th particle, while Ψ_{pp} represents the self-energy of the p-th particle. In the following, we investigate these energies respectively.

3.2 Relative energy

We first treat the relative energy. The propagator eq.(3.6) leads to

$$J\Psi_{pq} = J\int_0^1 d\tau \int_0^1 d\tau' \epsilon^{\mu\nu\lambda}\dot{X}_p^\mu(\tau)\dot{X}_q^\nu(\tau')\frac{X_p^\lambda(\tau) - X_q^\lambda(\tau')}{|X_p(\tau) - X_q(\tau')|^3}. \tag{3.7}$$

For convenience we introduce a unit vector $e^\mu(\tau,\tau')$ defined by

$$e^\mu(\tau,\tau') \equiv \frac{X_p^\mu(\tau) - X_q^\mu(\tau')}{|X_p(\tau) - X_q(\tau')|}. \tag{3.8}$$

We assume that two loops $X_p(\tau)$ and $X_q(\tau')$ have no intersecting points. This assumption guarantees that $e^\mu(\tau,\tau')$ is well-defined on the whole parameter space. (Due to the periodic boundary condition, the parameter space can be identified with torus, T^2.) The relative energy can be written in terms of the unit vector

$$\Psi_{pq} = \int_0^1 d\tau \int_0^1 d\tau' \epsilon^{\mu\nu\lambda} e^\mu \partial_\tau e^\nu \partial_{\tau'} e^\lambda. \tag{3.9}$$

The integrand is the surface element on S^2. It is known that a continuous map $T^2 \to S^2$ is classified by an integer which represents how many times the image $e(\tau,\tau')$ envelopes the sphere. Therefore Ψ_{pq} takes a value of $4\pi n, n \in \mathbf{Z}$ (where n is called the mapping degree). $(4\pi)^{-1}\Psi_{pq}$ is known as the Gauss linking number, which is a topological invariant of the link $(X_p(\tau), X_q(\tau'))$. Note that if J is an integer or a half-integer,

$$\exp\{iJ\Psi_{pq}[e]\} = 1. \tag{3.10}$$

This means that there are no interactions between two different particles!

3.3 Self-energy

Let us turn to the self-energy. Here we assume that no loops have self-intersecting points. The same expression as in the previous case is obtained as follows:

$$J\ \Psi_{pp} = J\int_0^1 d\tau \int_0^1 d\tau' \epsilon^{\mu\nu\lambda}\dot{X}_p^\mu(\tau)\dot{X}_p^\nu(\tau')\frac{X_p^\lambda(\tau)-X_p^\lambda(\tau')}{|X_p(\tau)-X_p(\tau')|^3}. \tag{3.11}$$

Both the denominator and the numerator vanish at $\tau=\tau'$ in the above expression, so that we have to check carefully if this integral is divergent or not. If $\tau'\sim\tau$, the numerator can be expanded as follows:

$$\begin{aligned} & \epsilon^{\mu\nu\lambda}\dot{X}_p^\mu(\tau)\times \\ & (\dot{X}_p^\nu(\tau)+(\tau'-\tau)\ddot{X}_p^\nu(\tau)+\frac{1}{2}(\tau'-\tau)^2\dddot{X}_p^\nu(\tau)+\cdots)\times \\ & (-(\tau'-\tau)\dot{X}_p^\lambda(\tau)-\frac{1}{2}(\tau'-\tau)^2\ddot{X}_p^\lambda(\tau)-\frac{1}{3!}(\tau'-\tau)^3\dddot{X}_p^\lambda(\tau)+\cdots) \\ \sim\ & \epsilon^{\mu\nu\lambda}(\tau'-\tau)^4\dot{X}_p^\mu(\tau)\ddot{X}_p^\nu(\tau)\dddot{X}_p^\lambda(\tau)+\cdots. \end{aligned} \tag{3.12}$$

This indicates that the numerator behaves like $(\tau-\tau')^4$ as $\tau'\to\tau$ while the denominator behaves like $(\tau-\tau')^3$. Therefore the integrand vanishes at $\tau=\tau'$. Thus we find that the self-energy takes a finite value for an arbitrary non-self-intersecting loop [13, 14, 24].

We introduce the variable $e(\tau,\tau')$ in the relative-energy case. In this case however, $e(\tau,\tau')$ is discontinuous at $\tau=\tau'$. That is, if we define a unit tangent vector $e(\tau)$ by

$$e(\tau)\equiv\lim_{\tau-\tau'\to+0}e(\tau,\tau'), \tag{3.13}$$

then

$$\lim_{\tau'-\tau\to+0}e(\tau,\tau')=-e(\tau). \tag{3.14}$$

In order to remove the discontinuity, we change the parameter space as follows (see fig.1):

$$(\sigma,\sigma')=\begin{cases}(\tau,\tau') & \text{if } \tau'\le\tau\\ (\tau+1,\tau') & \text{if } \tau'>\tau\end{cases} \tag{3.15}$$

and define

$$\tilde{e}(\sigma,\sigma')=\begin{cases}e(\sigma,\sigma') & \text{if } \sigma\le 1\\ e(\sigma-1,\sigma') & \text{if } \sigma>1\end{cases}. \tag{3.16}$$

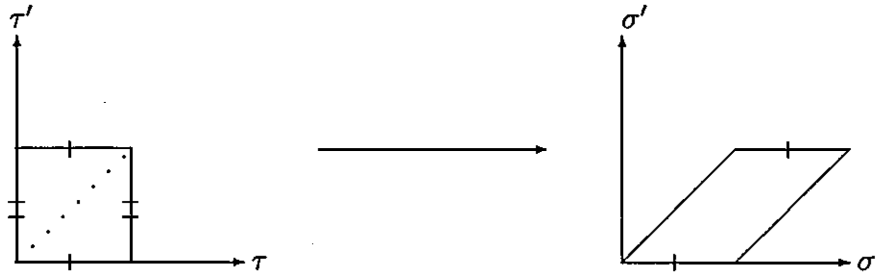


Figure 1: In (τ, τ') - plane, e is discontinuous along the dotted line. So we shift the left-upper-half part to τ - direction. The resulting space is parametrized by (σ, σ').

Hereafter we omit the $\tilde{}$ from $\tilde{e}(\sigma, \sigma')$. The unit vector $e(\sigma, \sigma')$ becomes continuous on the whole parameter space D, and satisfies the following boundary conditions:

$$e(\sigma, \sigma) = e(\sigma), \;\; e(\sigma, \sigma + 1) = -e(\sigma), \;\; e(\sigma, 0) = e(\sigma, 1). \tag{3.17}$$

The self-energy is written in terms of $e(\sigma, \sigma')$:

$$\Psi[e] = \int_D d\sigma d\sigma' \epsilon^{\mu\nu\lambda} e^\mu(\sigma, \sigma') \partial_\sigma e^\nu(\sigma, \sigma') \partial_{\sigma'} e^\lambda(\sigma, \sigma'), \tag{3.18}$$

where $\Psi_{pp}[e]$ is abbreviated to $\Psi[e]$ for simplicity. A small deformation δe of $e(\sigma, \sigma')$ gives

$$\begin{aligned}
\delta\Psi &= \delta \int_D \epsilon^{\mu\nu\lambda} e^\mu de^\nu \wedge de^\lambda \\
&= \int_D \epsilon^{\mu\nu\lambda} (\delta e^\mu de^\nu \wedge de^\lambda + 2e^\mu d\delta e^\nu \wedge de^\lambda) \\
&= 2 \int_D \epsilon^{\mu\nu\lambda} d(e^\mu \delta e^\nu de^\lambda) \\
&= 2 \int_{\partial D} dl \epsilon^{\mu\nu\lambda} e^\mu \delta e^\nu \partial_l e^\lambda,
\end{aligned} \tag{3.19}$$

where l parametrizes ∂D. In the second line of the above equation, we have used the fact that $\epsilon^{\mu\nu\lambda} \delta e^\mu de^\nu \wedge de^\lambda = 0$ since the three vectors $\delta e, \partial_\sigma e$ and $\partial_{\sigma'} e$ lies on a plane perpendicular to e. If $D \sim T^2$, its boundary ∂D is empty. Consequently, $\delta\Psi = 0$ and the value of Ψ is quantized. This is what happens in the case of the relative energy. However, the second equation in eq.(3.17) shows that the parameter space D cannot be regarded as torus. A small deformation of a boundary value changes the value of $\Psi[e]$. The self-energy is therefore *not quantized.*

3.4 The relationship between the spin factor and the self-energy

Comparing the self-energy $\Psi[e]$ to the spin factor $\Phi[e]$, we notice that the integrand has the same form but that the boundary conditions for the unit vector are different. See eq.(2.78) and eq.(3.17). The unit tangent vector of a trajectory lives on the boundary $(\sigma,\sigma)\cup(\sigma,\sigma+1)$ in the case of the self-energy, and on $(\tau,1)$ in the case of the spin factor. Consider the following quantity

$$\Psi[e]-2\Phi[e]. \tag{3.20}$$

A small deformation of a closed path changes the value of $\Psi[e]$ and $2\Phi[e]$ but these changes cancel each other out:

$$\begin{aligned}\delta\Psi[e]-2\delta\Phi[e] &= 2\int_{(\sigma,\sigma)\cup(\sigma,\sigma+1)}\epsilon^{\mu\nu\lambda}\delta e^{\mu}e^{\nu}de^{\lambda}-4\int_{(\tau,1)}\epsilon^{\mu\nu\lambda}\delta e^{\mu}e^{\nu}de^{\lambda}\\ &= 4\int_{(\sigma,\sigma)}\epsilon^{\mu\nu\lambda}\delta e^{\mu}e^{\nu}de^{\lambda}-4\int_{(\tau,1)}\epsilon^{\mu\nu\lambda}\delta e^{\mu}e^{\nu}de^{\lambda}\\ &= 0.\end{aligned} \tag{3.21}$$

We have used the result obtained in eq.(3.19) in the above equation. We can construct the knot diagram of a loop by deforming the loop on a plane without changing the value of eq.(3.20) provided that vector fields $\bar{e}(\tau,\tau')$ and $e(\sigma,\sigma')$ changes continuously. Note that at crossing points of the projected loop one line should be infinitesimally lifted from the other, as in fig.2.

The question of what values the quantity $\Psi[e]-2\Phi[e]$ takes for all loops in space becomes that of what values it takes for all possible knot diagrams on the plane. Let us consider the following two loops (fig.2), which are different from each other only at the crossing point. Non-zero contributions to the integrand arise in the case where one point $X(\tau)$ lies on the upper line near the crossing point. As the other point $X(\tau')$ goes along the loop, $e(\tau,\tau')$ the unit vector joining $X(\tau')$ to $X(\tau)$ describes a solid angle of 2π (fig.3). Since the same contribution is added when $X(\tau')$ lies on the upper line (fig.4), Ψ takes the value $+4\pi$ for loop A. On the other hand, for loop B, it takes the value -4π (figs.5, 6). Thus we can see that every crossing point of a loop contributes $\pm4\pi$ to the self-energy Ψ. The sign of its value depends on the crossing. As for the spin factor Φ, we can see from eq.(2.82) that its value depends on the number of times the tangent vector of a loop rotates.

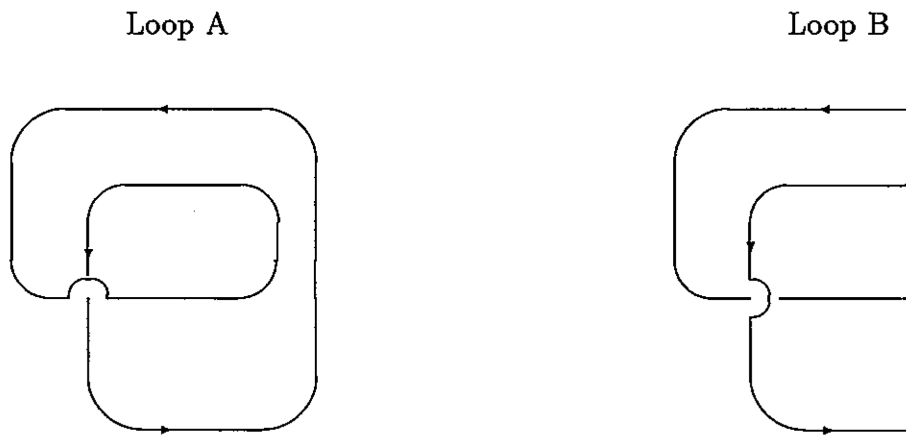


Figure 2: Two loops on a plane, whose tangent vectors are almost the same but whose linkings are different.

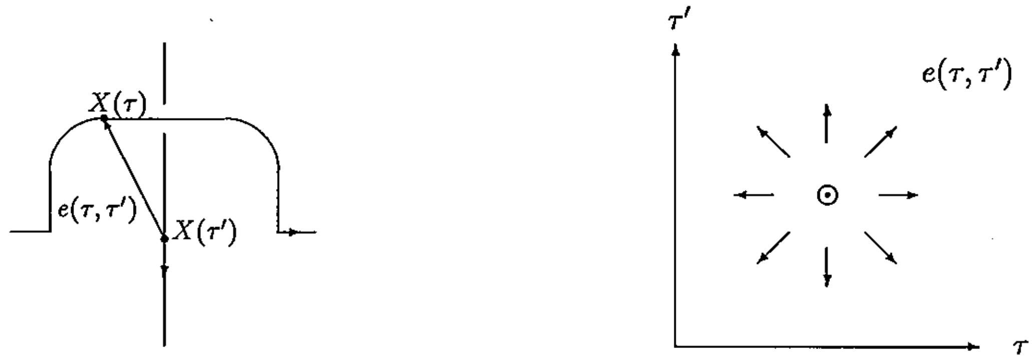


Figure 3: The crossing point for loop A in fig.2. A non-zero contribution to the integrand arises in the above situation. The figure on the right shows $e(\tau,\tau')$ covers the hemisphere around the crossing point.

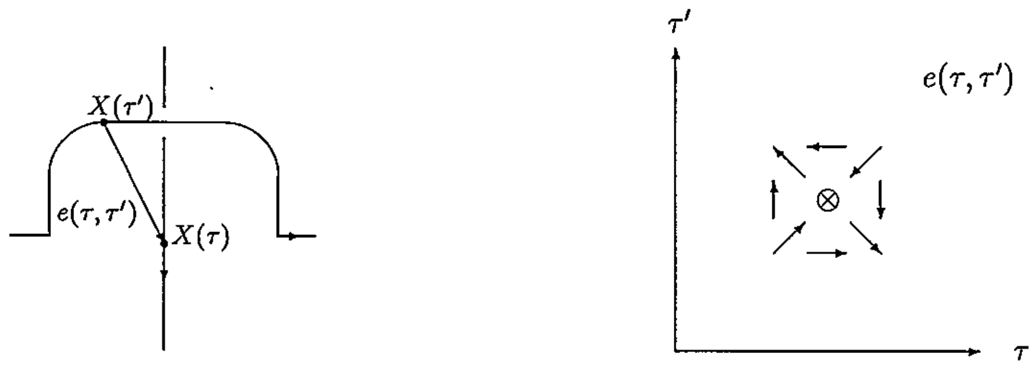
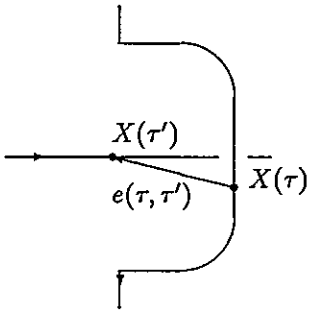


Figure 4: The second contribution at the crossing point of loop A. The figure of right shows that the contribution is the same as in fig.3.



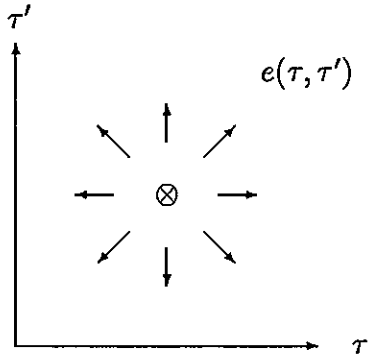
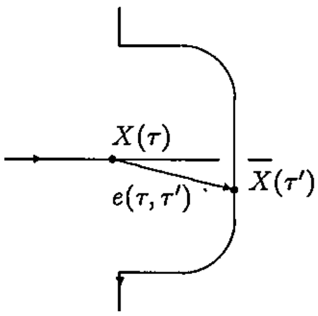


Figure 5: For loop B. Since the orientation of the covered hemisphere is the opposite of that for the loop A, the contribution to the self-energy is -2π.



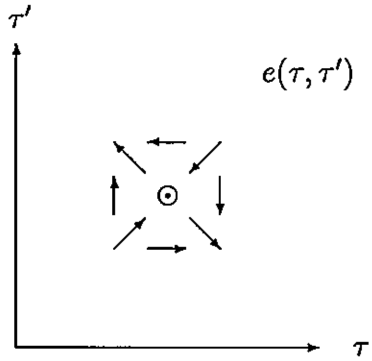


Figure 6: For loop B. Combining the contribution from this figure and fig.5, the self-energy takes the value -4π.

Figure 7: Simple examples of a loop on the plane.

To illustrate, we shall evaluate eq.(3.20) for several loops on the plane. First we consider a simple loop $\sim S^1$ (fig.7(a)). The self-energy vanishes since there are no crossing points. On the other hand, the spin factor takes the value of $\pm 2\pi$. Next we consider a nested loop (fig.2). As we have explained above, the self-energy takes the value $\pm 4\pi$. The spin factor takes the value $\pm 4\pi$, since the tangent vector rotates 2 times. At last we consider a figure of eight (fig.7(b)). Since there is one crossing point and the tangent vector does not rotate, the self-energy and the spin factor takes the value $\pm 4\pi$ and 0 respectively. To summarize we get the following table of results.

	Ψ	Φ	$\Psi - 2\Phi$
circle	0	$\pm 2\pi$	$\pm 4\pi$
nested loop	$\pm 4\pi$	$\pm 4\pi$	$\pm 12\pi, \pm 4\pi$
figure of eight	$\pm 4\pi$	0	$\pm 4\pi$

We can evaluate eq.(3.20) for an arbitrary loop in this way. First the loop is projected on some plane without changing the value of eq.(3.20). Suppose that the projected loop on a plane (called the knot diagram) has M crossing points and that the tangent vector rotates N times. Then the self-energy takes the value $4\pi M$ (mod 8π) and the spin factor $2\pi N$ (mod 4π). It can be shown that $N - M$ is an odd number for an arbitrary knot diagram. Therefore we get [14, 16, 24]

$$\Psi[e] - 2\Phi[e] = 4\pi \pmod{8\pi}. \tag{3.22}$$

Note that if J is a half-integer,

$$\exp\{i\frac{J}{2}\Psi[e]\} = -\exp\{iJ\Phi[e]\}. \tag{3.23}$$

The minus sign in the r.h.s. plays an important role in the next section.

The spin factor does not make sense if the coefficient J is neither an integer nor a half-integer since the value of $\exp\{iJ\Phi\}$ depends on an extension of the unit tangent vector. On the other hand, the self-energy does not have such an ambiguity. Therefore it is a well-defined problem to investigate a system interacting with the Chern-Simons field even if J takes another value. However, this is a more difficult problem because non-local interactions between different particles remain and because we have to analyze the self-energy without using the spin factor.

4 Bose-Fermi Transmutation

We have seen in section 2 that the free energies can be represented by path-integrals over trajectories. In this section we first show that similar representations are derived even if an external gauge field is turned on. Next we deal with the system of a scalar field interacting with both a Chern-Simons gauge field and an external field. The partition function of this system is shown to be equivalent to that of the Dirac system in the external field. This means that an arbitrary n-point correlation function of currents has the same form in both theories.

We shall comment on the regularization adopted in the following calculation. The average over the Chern-Simons gauge field is taken subject to the condition that all variables depend on the continuous parameter τ. There remain 'fields' in one dimension after the average is taken. In section 2, these fields have been treated without the need to descritize the interval. This regularization led to an evaluation of the determinant eq.(2.41). In the following calculation we regularize the einbein $h(\tau)$ in the same way, treating it as a continuous field. After gauge fixing, the interval $[0, L]$ is divided into equal pieces of length ϵ, and the remaining dynamical variables are treated as living at each breaking point.

4.1 Free energies in an external field

[12] We have obtained the path-integral description of the free energies in section 2. In this part we extend the result to the case where an external electro-magnetic field is present.

A scalar field Let us begin with the complex scalar system considered in section 2. Now an external gauge field C_μ is switched on. We wish to show that the following

[12] We shall comment on the term free energy in this section. Here we investigate a system in an external gauge field. This system generally does not have translational invariance. This implies that the free energy does not acquire the factor of the volume of the system. Further we deal with a covariant theory including the temperature direction. For this reason, we shall call W, which is defined in the following formula, the *free energy* of the system: $\exp\{-W\} = Z$, where Z is the partition function of this system.

path integral

$$\int \frac{\mathcal{D}X}{V_G} \exp\{\int_0^1 d\tau(-m_0\sqrt{\dot{X}^2} + iC\cdot\dot{X})\} \tag{4.1}$$

gives the free energy of the system. In the above equation particle coordinates $X(\tau)$ satisfy the periodic boundary condition $X(0) = X(1)$. We take the following more convenient expression instead of eq.(4.1):

$$\int \frac{\mathcal{D}\mu\mathcal{D}h}{V_G} \exp\{-S_b + i\int_0^1 d\tau C\cdot eh\},$$
$$S_b \equiv \int_0^1 d\tau(m_0 h + ik\cdot(\dot{X} - eh)) \tag{4.2}$$

where the measure $\mathcal{D}\mu$ is defined as in eq.(2.87). Note that the action S_b is the $J = 0$ case in eq.(2.88). We can make the replacement $\dot{X} \to eh$ in the interaction term because of the delta functional $\delta(\dot{X} - eh)$. After k, e and h being integrated out, we go back to eq.(4.1). We remove the redundant variables using eq.(2.43).

$$N\int_\epsilon^\infty \frac{dL}{L}\, e^{-L(\frac{1}{2\sqrt{\pi\epsilon}}+m_0)}\, \mathcal{K}_B[L],$$
$$\mathcal{K}_B[L] \equiv \int \mathcal{D}\mu \exp\{\int_0^L dt(ik(e-\dot{X}) + iCe)\}, \tag{4.3}$$

where t is the proper time defined by

$$t(\tau) \equiv \int_0^\tau d\tau' h(\tau'). \tag{4.4}$$

We shall regularize $\mathcal{K}_B[L]$ by splitting the interval $[0, L]$ into equal pieces ΔL. Since L is cut off at ϵ, it is reasonable to set $\Delta L = \epsilon$. Note that this regularization does not violate gauge invariance because $\Delta L = h\Delta\tau$ [3]. The measure $\mathcal{D}\mu$ is treated under the regularization as in finite dimensional integration:

$$\mathcal{D}\mu = \prod_{i=0}^{N-1} d^3X_i \frac{d^3k_i}{(2\pi)^3}\frac{1}{4\pi} d(cos\theta_i)d\phi_i, \tag{4.5}$$

where $N\epsilon \equiv L$. The discretized action is given by [13]

$$\begin{aligned} -S_b + i\int_0^1 d\tau Ceh &\to -\sum_{i=0}^{N-1}\{ik_i(X_{i+1} - X_i) - ik_ie_i\epsilon\} + i\sum_{i=0}^{N-1} C(X_i)e_i\epsilon \\ &\equiv -S_b^{disc} + i\sum_{i=0}^{N-1} C(X_i)e_i\epsilon, \end{aligned} \tag{4.6}$$

[13] Note that the mass term is absorbed by the exponent in front of $\mathcal{K}_B[L]$.

where all dynamical variables satisfy the periodic boundary condition such as $X_0 = X_N$. First we integrate e_i in $\mathcal{K}_B[L]$ (up to fourth order in ϵ).

$$\begin{aligned} &\prod_{i=0}^{N-1} \int de_i \exp\{i(k_i + C(X_i))e_i\epsilon\} \\ \sim &\prod_{i=0}^{N-1} \exp\{-\frac{1}{6}(k_i + C(X_i))^2\epsilon^2\}, \end{aligned} \tag{4.7}$$

where we have used the following formulae:

$$\begin{aligned} &\int de \equiv \int \frac{1}{4\pi} d(\cos\theta_i)d\phi_i = 1 \\ &\int dee^\mu = 0 \\ &\int dee^\mu e^\nu = \frac{1}{3}\delta^{\mu\nu}. \end{aligned} \tag{4.8}$$

This leads to

$$\begin{aligned} \mathcal{K}_B[L] &= \int \prod_{i=0}^{N-1} d^3X_i \frac{d^3k_i}{(2\pi)^3} \exp\{-\frac{1}{6}(k_i + C(X_i))^2\epsilon^2 - ik_i(X_{i+1} - X_i)\} \\ &= \int \prod_{i=0}^{N-1} d^3X_i < X_{i+1}|\exp\{-\frac{\epsilon^2}{6}(i\partial - \hat{C})^2\}|X_i > \\ &= \mathrm{Tr}\exp\{-\frac{\epsilon}{6}L(i\partial - \hat{C})^2\}. \end{aligned} \tag{4.9}$$

Substituting this into eq.(4.3), we complete the calculation of eq.(4.2):

$$\begin{aligned} \int \frac{\mathcal{D}\mu\mathcal{D}h}{V_G} e^{-S_b + i\int_0^1 d\tau C\cdot\dot{X}} &= N\mathrm{Tr}\int_{\frac{\epsilon^2}{6}}^{\infty} \frac{dL'}{L'} e^{-L'(-D^2+m^2)} \\ &= -NW_B^{\epsilon'}[C], \end{aligned} \tag{4.10}$$

where $D \equiv (\partial - iC)$ and $\epsilon' = \mathrm{const.}\epsilon$. The regularized free energy $W_B^{\epsilon'}[C]$ has been defined in the same way as eq.(2.26). The physical mass m is defined by

$$m^2 \equiv \frac{\mathrm{const.}}{\epsilon}(m_0 - \frac{\mathrm{const.}}{\epsilon}). \tag{4.11}$$

We have reduced the variable L with L' in eq.(4.10), where $L' = \epsilon L$. This implies that the length of typical paths L' has the same dimension as R^2 where R is the correlation length in this system. From the first equality in eq.(4.10) we find that the dominant length of paths is determined by $L'm^2 \sim 1$, which means

$$\epsilon L \sim R^2. \tag{4.12}$$

Let us regard ϵ as the unit length and put $L = \epsilon N$ and $R = \epsilon M$. The above relation tells us how many steps the particle needs to take to go back to the starting point: $N \sim M^2$. As a result, as we have mentioned in section 2, pathological paths dominate in the path integral eq.(4.1).

A Dirac field In this case, we have to add the spin factor to the previous action:

$$\int \frac{\mathcal{D}\mu \mathcal{D}h}{V_G} \exp\{-S_f + i\int_0^1 d\tau C \cdot \dot{X}\}, \qquad (4.13)$$
$$S_f \equiv S_b - \frac{i}{2}\Phi[e].$$

After dividing the gauge volume, we have

$$\begin{aligned} (4.14) &= N \int_\epsilon^\infty \frac{dL}{L} \, e^{-L(\frac{1}{\sqrt{\pi\epsilon}}+m_0)} \, \mathcal{K}_F[L], \\ \mathcal{K}_F[L] &\equiv \int \mathcal{D}\mu \exp\{-\int_0^1 d\tau\{ik(\dot{X}-eL) - iCeL\} + \frac{i}{2}\Phi[e]\}. \end{aligned} \qquad (4.14)$$

As we have done in the previous paragraph, $\mathcal{K}_F[L]$ is computed using the finite dimensional regularization. Introducing the proper time t, a discretized action S_f^{disc} for eq.(4.14) is given by

$$\begin{aligned} -S_f^{disc} &\equiv -S_b^{disc} + \frac{i}{2}\mathcal{A}[e_{i+1}, e_i, e_0], \\ -S_f^{disc} + i\sum_{i=0}^{N-1} C(X_i)e_i\epsilon &= \sum_{i=0}^{N-1}\{ik_i(X_{i+1}-X_i) - i\epsilon(k_i + C(X_i))\cdot e_i + \\ &\quad \frac{i}{2}\mathcal{A}[e_{i+1}, e_i, e_0]\}, \end{aligned} \qquad (4.15)$$

where all variables satisfy the periodic boundary condition as in the case of the scalar theory. Employing the formula

$$\begin{aligned} &< X_{i+1}|\otimes < e_{i+1}|\exp\{(i\hat{p}\cdot\sigma + i\hat{C}\cdot\sigma)\epsilon\}|e_i > \otimes|X_i > \\ = &\int \frac{dk_i}{2\pi}\exp\{-ik_i\cdot((X_{i+1}-X_i)) + i\epsilon(k_i + C(X_i))\cdot e_i \\ &+\frac{i}{2}\mathcal{A}[e_{i+1}, e_i, e_0]\}, \end{aligned} \qquad (4.16)$$

$\mathcal{K}_F$ becomes

$$\begin{aligned} &\int \prod_{i=0}^{N-1} dX_i de_i < X_{i+1}|\otimes < e_{i+1}|\exp\{\epsilon i(\hat{p}+\hat{C})\cdot\sigma\}|e_i > \otimes|X_i > \\ = \; &\mathrm{Tr}\exp\left(i(\hat{p}+\hat{C})\cdot\sigma L\right) \end{aligned} \qquad (4.17)$$

Substituting eq.(4.17) into eq.(4.14) we get

$$\begin{aligned}\int \frac{\mathcal{D}\mu\mathcal{D}h}{V_G} e^{-S} &= N \int_\epsilon^\infty \frac{dL}{L} \, \mathrm{Tr} e^{-L\{m+(\partial - iC)\cdot\sigma\}} \\ &= N W_F^\epsilon[C], \end{aligned} \tag{4.18}$$

where we have used the physical mass m defined by

$$m \equiv m_0 - \frac{\text{const.}}{\epsilon}. \tag{4.19}$$

We see from the above formula that the typical length of paths is $1/m$, which means $N \sim M$. Again, we have got the same result as we derived in section 2.

The difference between random-walk properties for a scalar particle and a Dirac particle originates from the following facts. When the unit vector e is integrated, the term of order ϵ vanishes in the case of the scalar particle:

$$\int de_i e^{ipe_i\epsilon} \sim 1 - \frac{1}{6} p^2 \epsilon^2 + \cdots. \tag{4.20}$$

Then the length of the path L is 'renormalized' by ϵ, and as a result, complicated paths dominate. On the other hand, the order of ϵ term remains due to the spin factor in the case of the Dirac particle

$$\int de_i e^{ipe_i\epsilon + \frac{i}{2}A[e_{i+1},e_i,e_0]} \sim < e_{i+1}|1 - ip\cdot\sigma\epsilon + \cdots |e_i > . \tag{4.21}$$

This shows that the spin factor suppresses the pathological paths [3], which dominate in the case of a scalar particle.

4.2 Equivalence of partition functions

Here we introduce the Chern-Simons gauge field A_μ to the system of the scalar field. The external field C_μ is still present. We investigate the partition function of this system. The second quantized action is given by

$$S = \int d^3x\{|D_\mu\phi|^2 + m^2|\phi|^2\} - iS_{C.S.}, \tag{4.22}$$

where $D_\mu \equiv \partial_\mu - iA_\mu - iC_\mu$. The Chern-Simons action $S_{C.S.}$ is defined by eq.(3.1). The value of J in $S_{C.S.}$ is fixed at $1/2$. Now the partition function of the system is written in the form of a particle-number series:

$$\begin{aligned}
Z &\equiv \int \mathcal{D}A\mathcal{D}\phi^*\mathcal{D}\phi e^{-S} \\
&= \int \mathcal{D}A \det{}^{-1}(-D_\mu D_\mu + m^2) e^{iS_{CS}} \\
&\equiv \, < e^{-W_B[C+A]} >_A \\
&= \, < \sum_n \frac{1}{n!}(-W_B[C+A])^n >_A . \qquad (4.23)
\end{aligned}$$

Namely, W_B^n contains n-charged-scalar particles. Let us consider the 1-particle sector. The free energy (the regularized free energy),

$$\begin{aligned}
& < NW_B[C+A] >_A \\
&= -< \int \frac{\mathcal{D}\mu\mathcal{D}h}{V_G} \exp\{-S_b + i\int_0^1 d\tau (C+A)\cdot \dot{X}\} >_A \quad \text{(eq.(4.10))} \\
&= -\int \frac{\mathcal{D}\mu\mathcal{D}h}{V_G} \exp\{-S_b + \frac{iJ}{2}\Psi + i\int_0^1 d\tau C\cdot \dot{X}\} \quad \text{(eq.(3.4))} \\
&= \int \frac{\mathcal{D}\mu\mathcal{D}h}{V_G} \exp\{-S_b + iJ\Phi + i\int_0^1 d\tau C\cdot \dot{X}\} \quad \text{(eq.(3.23))} \\
&= NW_F[C]. \quad \text{(eq.(4.18))} \qquad (4.24)
\end{aligned}$$

Note that the minus sign appeared in the third equality. This originates from the diffenence between the self-energy and the spin factor discussed in section 3. We have seen in section 2 that the zero-point energy of the scalar system is positive while in the Dirac system it is negative. We notice that the Chern-Simons gauge field changes the positive zero-point energy into the negative one. Moreover this interaction produces the spin factor. Thus the free energy of the scalar system coupled to the Chern-Simons gauge field is equal to that of the Dirac system. Next we consider many-particle sectors. As we have shown in section 3, the relative energies between the scalar particles vanish if $J = 1/2$. Therefore we get

$$\begin{aligned}
& < (NW_B[C+A])^n >_A \\
&= (-1)^n \int \prod_{p=1}^n \frac{\mathcal{D}\mu_p\mathcal{D}h}{V_G} \exp\{\sum_{p=1}^n(-S_b^p + i\int_0^1 d\tau C(X_p)\cdot \dot{X}_p) \\
&\quad + \frac{iJ}{2}\sum_{p=1}^n \Psi_{pp}[e] + iJ\sum_{p<q}\Psi_{pq}[e]\} \\
&= \int \prod_{p=1}^n \frac{\mathcal{D}\mu_p\mathcal{D}h}{V_G} \exp\{\sum_{p=1}^n(-S_b^p + i\int_0^1 d\tau C(X_p)\cdot \dot{X}_p) + iJ\sum_{p=1}^n \Phi_{pp}[e]\} \\
&= (NW_F[C])^n, \qquad (4.25)
\end{aligned}$$

where $\mathcal{D}\mu_p$ and S_b^p are the measure and the action for the p -th particle respectively. In particular,

$$< e^{-W_B[C+A]} >_A = e^{-W_F[C]}. \tag{4.26}$$

This is our main result for the bose-fermi transmutation of relativistic field theories [24, 25]. The result implies that arbitrary n-point functions of currents are equivalent:

$$< J^B_{\mu_1}(x_1) \cdots J^B_{\mu_n}(x_n) > = < J^F_{\mu_1}(x_1) \cdots J^F_{\mu_n}(x_n) >, \tag{4.27}$$

where $J^B_\mu(x) \equiv i\phi^*\partial_\mu\phi - i\partial_\mu\phi^*\phi$ and $J^F_\mu(x) \equiv i\bar{\psi}\gamma_\mu\psi$.

4.3 Extension to a non-abelian external field

We can extend the above result to the case where the external field is a non-abelian gauge field. The functional integration corresponding to eq.(4.2) is

$$\int \frac{\mathcal{D}\mu\mathcal{D}h\mathcal{D}g}{V_G} \exp\{-S_b + i\int_0^1 d\tau(\mathcal{I}^a C^a \cdot eh - < 0|g^{-1}\dot{g}|0 >)\}, \tag{4.28}$$

where $g \in SU(N)$ and $\mathcal{I}^a \equiv < g|T^a|g >$. Generators of the $su(N)$ algebra are represented by anti-hermitian matrices T^a. The state $|g >$, what is called an $SU(N)$ coherent state, is defined in the latter part of appendix C. The state $|0 >$ denotes the highest weight state of the representation to which a particle belongs. The term $g^{-1}\dot{g}$ is the '$SU(N)$ spin factor', which gives the Poisson bracket $\{\mathcal{I}^a, \mathcal{I}^b\} = if^{abc}\mathcal{I}^c$ [23]. After introducing the proper time t, we define the above formula by discretization as in the abelian case. The discretized action is given by

$$-S_b^{disc} + \sum_{i=0}^{N-1}\{i\mathcal{I}_i^a C^a e_i\epsilon - < 0|g_i^{-1}(g_{i+1} - g_i)|0 >\}. \tag{4.29}$$

In appendix C, the following formulae are proven:

$$\exp\{\mathcal{I}_i^a C^a e_i\epsilon - < 0|g_i^{-1}(g_{i+1} - g_i)|0 >\} = < g_{i+1}|\exp\{T^a C^a e_i\epsilon\}|g_i > + O(\epsilon^2), \tag{4.30}$$

and

$$\int dg_i |g_i >< g_i| = 1. \tag{4.31}$$

Therefore we can easily integrate g_i. The resulting action is given by

$$-S_b^{disc} + i\sum_{i=0}^{N-1} C^a(X_i)T^a e_i\epsilon. \tag{4.32}$$

This differs from the abelian case in that C^a is replaced by C^aT^a. However, the same calculation as eq.(4.7) still holds. Therefore we can conclude that the path integral eq.(4.28) gives the free energy. If we add the spin factor to the action in eq.(4.28), we obtain the free energy of the Dirac system. Arguing as we did in the previous case, we get the same result when the system lies in a non-abelian external gauge field, as in an abelian one. [14]

4.4 Remarks

Finally we make a couple of remarks on bose-fermi transmutation.

Transmutation into a spin 1 particle So far we have fixed the value of J at $\frac{1}{2}$. In this part, we shall discuss the $J = 1$ case [17]. In this case the Chern-Simons gauge field transmutes the scalar particle into a spin 1 particle. Since the epsilon tensor plays the role of $su(2)$ generator for $J = 1$, the corresponding field theory is a "massive Chern-Simons theory" (see eq.(2.102)):

$$\mathcal{L} = B^\mu(i\epsilon_{\mu\nu\lambda}\partial^\nu + m\delta_{\mu\lambda})B^\lambda. \tag{4.33}$$

The propagator for the spin-1 particle is

$$\frac{1}{p^2+m^2}(-\epsilon^{\mu\nu\lambda}p_\lambda + \frac{1}{m}p^\mu p^\nu + m\delta^{\mu\nu}).$$

In the canonical formalism the Hamiltonian is [15]

$$\mathcal{H} = m\{B_1^2 + B_2^2 + \frac{1}{m^2}(\partial_1 B_2 - \partial_2 B_1)^2\}, \tag{4.34}$$

and the commutation relation is

$$[B_i(x), B_j(y)]_{E.T.} = \frac{i}{2}\epsilon_{ij}\delta^{(2)}(x-y). \tag{4.35}$$

B_0 can be solved for B_i (i = 1, 2). The Hamiltonian being bilinear in B_i, it is easy to diagonalize and show that the spectrum is that of a free massive particle with mass m.

[14] The result is first obtained by Itoi.[10]

[15] Here we treat this problem in the Minkowski metric.

Transmutation of the propagator [16] When we discuss the transmutation of n-point functions of fields, we have to treat open paths of particles. In this case, the relationship between the self-energy and the spin factor is not so clear. We now consider eq.(3.20) in the case of an open path:

$$\Delta \equiv \frac{J}{2}\Psi[e] - J\Phi[e], \tag{4.36}$$

where $X(0) = X_i, X(1) = X_f$, in addition, the tangent vectors at these points are assumed to be fixed. If $X(\tau)$ is a straight path connecting $X(0)$ and $X(1)$, both $\Psi[e]$ and $\Phi[e]$ vanish and $\Delta = 0$. Under a continuous deformation of a path the variation of Δ does not vanish since the boundary of the parameter space is not empty. In eq.(2.86) the dominant contributions to the path integral comes from those paths whose lengths $L(P)$ satisfy $L(P) - L_0 < \frac{1}{m}$, where $L_0 = |X(1) - X(0)|$. For these paths, as L_0 becomes large $\delta\Delta$ decreases generally as L_0^{-2}. Therefore in the long distance limit of L_0 where $L_0 \gg \frac{1}{m}$, the self-energy $\frac{J}{2}\Psi[e]$ can be identified with the spin factor $J\Phi[e]$. For this reason we conclude that at $J = 1/2$ the dressed propagator of the charged scalar particle becomes that of a spinning particle in the long distance limit. On the other hand, if $L_0 \sim \frac{1}{m}$, the self-energy is far from the spin factor. In this case, the dressed propagator seems to be neither that of a free spinning particle nor that of a free scalar particle.

[16] This topic is investigated in ref.[25]

5 Conclusion

In this thesis we have shown that the partition function of a charged scalar system coupled with a $U(1)$ Chern-Simons gauge field is equivalent to that of the free Dirac system. The result also holds when the two systems couple to an external (abelian or non-abelian) gauge field. This indicates that arbitrary n-point functions of currents in these systems are identical.

In order to get the above result, we have used the path-integral forms of the free energies. These are described by a summation over all closed trajectories (loops). The action in the path-integral is determined by the length of paths in the case of scalar particles. On the other hand, the action for the Dirac particles contains the spin factor as well. We have shown that the spin factor changes the character of dominant paths: paths of the Dirac particle are much smoother than bosonic ones. The propagators of these particles are also written in the form of a path integral over random paths. The measures are determined in such a way that the reparametrization invariance is maintained. We have regularized the path integrals in two ways. One of these is to treat all dynamical variables as functions of a continuous parameter, and high-momentum modes are cut off. In the other way all dynamical variables except the einbein are regarded as variables defined on discrete points.

We have also discussed the interaction energy of the charged scalar particle coupled to the Chern-Simons gauge field. We have found that the relative energy of the particle is equal to the Gauss linking number. It vanishes when the coefficient of the Chern-Simons gauge field takes certain values. We have shown the relationship between the self-energy and the spin factor. This result has been related to knot diagrams. When a path lies on a plane, the value of the self-energy is determined by the number of crossing points, and the value of the spin factor, by the winding number of the unit tangent vector. This relationship shows that the self-energy generates not only the spin degrees of freedom but also the statistical factor for a fermion loop.

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Appendix A On the one-dimensional general coordinate transformation

We summarize a transformation law under the one-dimensional general coordinate transformation. A scalar or a vector (or a tensor) is determined by its transformation properties.

First we review these quantities in the case of four dimensions. Let us consider the following coordinate transformation:

$$x^\mu \to f^\mu(x^0, \cdots, x^3). \tag{A.1}$$

The differentials of the coordinates are related with the following formula:

$$dx^\mu \to df^\mu, \quad df^\mu = \frac{\partial f^\mu}{\partial x^\nu} dx^\nu. \tag{A.2}$$

A *contravariant vector* A^μ is defined as a quantity which transforms like the coordinate differentials:

$$A^\mu(x) \to A'^\mu(f), \ A'^\mu(f(x)) = \frac{\partial f^\mu}{\partial x^\nu} A^\nu(x). \tag{A.3}$$

The derivatives $\frac{\partial}{\partial x^\mu}$ transform under the coordinate transformation as follows:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial f^\mu}, \ \frac{\partial}{\partial x^\mu} = \frac{\partial f^\nu}{\partial x^\mu} \frac{\partial}{\partial f^\nu}. \tag{A.4}$$

A *covariant vector* B_μ is defined as a quantity which transforms like the derivatives:

$$B_\mu(x) \to B'_\mu(f), \ B_\mu(x) = \frac{\partial f^\nu}{\partial x^\mu} B'_\nu(f(x)). \tag{A.5}$$

A *scalar* $\phi(x)$ is defined as a invariant quantity under the coordinate transformation:

$$\phi(x) \to \phi'(f), \ \phi(x) = \phi'(f(x)). \tag{A.6}$$

A second or a higher rank tensor is defined as a quantity which transforms like products of vectors. For example a metric tensor, which is a second-rank covariant tensor, transforms under the coordinate transformation in the following rule:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(f), \quad g_{\mu\nu}(x) = \frac{\partial f^\rho}{\partial x^\mu} \frac{\partial f^\sigma}{\partial x^\mu} g'_{\rho\sigma}(f(x)). \tag{A.7}$$

A *vierbein* (or a *tetrad*) $e_\mu^{(a)}$, which is a covariant vector labeled by the index $a(a = 0,\cdots,3)$, is introduced as a quantity which represents a metric tensor $g_{\mu\nu}$ as follows:

$$e_\mu^{(a)} e_\nu^{(b)} \eta_{ab} = g_{\mu\nu}, \tag{A.8}$$

where $\eta_{ab} = \mathrm{diag}(+,-,-,-)$. Each $e_\mu^{(a)}$ transforms as follows:

$$e_\mu^{(a)}(x) \to e_\mu^{\prime(a)}(f),\ e_\mu^{(a)}(x) = \frac{\partial f^\nu}{\partial x^\mu} e_\nu^{\prime(a)}(f(x)). \tag{A.9}$$

According to the transformation laws eq.(A.2) and eq.(A.7), the invariant volume element under a coordinate transformation is not d^4x but $\sqrt{-\det g_{\mu\nu}}d^4x$. The invariant volume element is also written using the vierbein instead of the metric. Indeed,

$$\begin{aligned} -\det g_{\mu\nu} &= -\det(e_\mu^{(a)} e_\nu^{(b)} \eta_{ab}) \\ &= -\det e_\mu^{(a)} \det e_\nu^{(b)} \det \eta_{ab} \\ &= (\det e_\mu^{(a)})^2. \end{aligned} \tag{A.10}$$

Hence

$$\sqrt{-\det g_{\mu\nu}}d^4x = \det e_\mu^{(a)} d^4x \tag{A.11}$$

Let us turn to the one-dimensional case. Coordinate transformations $\tau \to f(\tau)$ are alternatively called reparametrizations or gauge transformations in the text. A scalar or a vector is defined in the same way as in the four-dimensional case.

- A contravariant vector $A(\tau)$

$$A(\tau) \to A'(f),\ A'(f(\tau)) = \frac{df}{d\tau} A(\tau). \tag{A.12}$$

- A covariant vector $B(\tau)$

$$B(\tau) \to B'(f),\ B(\tau) = \frac{df}{d\tau} B'(f(\tau)). \tag{A.13}$$

- A scalar $C(\tau)$

$$C(\tau) \to C'(f),\ C(\tau) = C'(f(\tau)). \tag{A.14}$$

- A metric tensor $g(\tau)$

$$g(\tau) \to g'(f),\ g(\tau) = (\frac{df}{d\tau})^2 g'(f(\tau)). \tag{A.15}$$

Now we introduce an *einbein* $e_0^{(0)}(\tau)$, which plays the same role as the vierbein in the four-dimensional case.

$$e_0^{(0)} e_0^{(0)} \eta_{00} = (e_0^{(0)})^2 = g. \tag{A.16}$$

From now on, we rewrite $e_0^{(0)}(\tau)$ into $h(\tau)$ which stands for an einbein in the text. Since $h(\tau)$ is a covariant vector, its transformation is determined as follows:

$$h(\tau) \to h'(f), \; h(\tau) = \frac{df}{d\tau} h'(f(\tau)). \tag{A.17}$$

An invariant volume element under reparametrizations is $hd\tau$ because $\det h = h$. We can make scalars using the einbein h. For example, $A(\tau)h(\tau), B(\tau)h^{-1}(\tau)$, etc. Since f is a coordinate, the infinitesimal difference δf transforms like contravariant vector. Combining the invariant volume element $h(\tau)$, we can obtain a gauge invariant norm $||\delta f||^2$ as in eq.(2.32):

$$\begin{aligned} ||\delta f||^2 &= \int_0^1 d\tau h(\tau)(h(\tau)\delta f(\tau))^2 \\ &= \int_0^1 df L(L\delta f(\tau))^2 \\ &= \int_0^1 df (L^{\frac{3}{2}} \delta f(\tau))^2, \end{aligned} \tag{A.18}$$

where $h(\tau)$ is parametrized by $(L, f(\tau))$ which is used in the text. Other gauge invariant norms defined in section 2 are also obtained in a similar way.

Appendix B Computation of the determinant

In this appendix, we derive the determinants eq.(2.41) and eq.(2.99) following reference [3].

Open paths First let us compute the determinant for open paths. Since the operator $-L^{-2}d^2/df^2$ acts on a function $\phi(f)$ satisfying $\phi(0) = \phi(1) = 0$, we take $\sin(n\pi f)$ as eigen functions. Then

$$\det(-L^{-2}\frac{d^2}{df^2}) = \prod_{n=1}^{\infty} \lambda_n = \exp \sum_{n=1}^{\infty} \ln \lambda_n, \tag{B.1}$$

where $\lambda_n = \frac{\pi n}{L}$. Following the asymptotic expansion eq.(2.25), we introduce the regularized expression as

$$\begin{aligned} \sum_{n=1}^{\infty} \ln \lambda_n &\rightarrow -\sum_{n=1}^{\infty} \int_{\epsilon^2}^{\infty} \frac{ds}{s} e^{-s\lambda_n} \\ &= -\sum_{n=1}^{\infty} \int_{(\epsilon/L)^2}^{\infty} \frac{dx}{x} e^{-x\pi^2 n^2} \end{aligned} \tag{B.2}$$

If we regard ϵ as the lattice spacing, wave lengths shorter than the lattice size are neglected. Poisson's formula

$$\sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \sqrt{\frac{\pi}{\alpha}} \sum_{m=-\infty}^{\infty} e^{-\frac{\pi^2 m^2}{\alpha}} \tag{B.3}$$

leads to the following estimate.

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-x\pi^2 n^2} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{-x\pi^2 n^2} - \frac{1}{2} \\ &= \frac{1}{2}\sqrt{\frac{1}{\pi x}} - \frac{1}{2} + O(e^{-c/x}) \end{aligned} \tag{B.4}$$

We rewrite eq.(B.2) as follows:

$$\begin{aligned} & -\sum_{n=1}^{\infty} \int_{(\epsilon/L)^2}^{1} \frac{dx}{x} e^{-x\pi^2 n^2} \\ = & -\int_{(\epsilon/L)^2}^{1} \frac{dx}{x} (\sum_{n=1}^{\infty} e^{-x\pi^2 n^2} - \frac{1}{2}\sqrt{\frac{1}{\pi x}} + \frac{1}{2}) \\ & -\int_{(\epsilon/L)^2}^{1} \frac{dx}{x} (\frac{1}{2}\sqrt{\frac{1}{\pi x}} - \frac{1}{2}) + \int_{1}^{\infty} \frac{dx}{x} \sum_{n=1}^{\infty} e^{-x\pi^2 n^2} \end{aligned} \tag{B.5}$$

The above estimate enables us to put $\epsilon = 0$ in the first term. Only the second term has an L-dependence:

$$-\int_{(\epsilon/L)^2}^{1} \frac{dx}{x}(\frac{1}{2}\sqrt{\frac{1}{\pi x}} - \frac{1}{2}) = \text{const.} - \sqrt{\frac{1}{\pi}}\frac{L}{\epsilon} - \ln\frac{\epsilon}{L}. \quad \text{(B.6)}$$

As a result, we obtain

$$\det(-L^{-2}\frac{d^2}{df^2}) = \text{const.} e^{\frac{-L}{\epsilon\sqrt{\pi}}} L \quad \text{(B.7)}$$

Closed paths Next we evaluate the determinant for closed paths. We take $e^{2\pi i n f}$ instead of $\sin n\pi f$ as eigen functions because the operator acts on a function $\phi(f)$ satisfying $\phi(0) = \phi(1)$. We have

$$\det{}'(-L^{-2}\frac{d^2}{df^2}) = \prod_{n\neq 0} \tilde{\lambda}_n = \exp\sum_{n\neq 0} \ln\tilde{\lambda}_n, \quad \text{(B.8)}$$

where $\tilde{\lambda}_n = \frac{2\pi n}{L}$. Comparing to the case of open paths, the number of modes doubles and each eigen-value occurs twice. This implies that the formula corresponding to eq.(B.6) is

$$2\int_{(2\epsilon/L)^2}^{1} \frac{dx}{x}(\frac{1}{2}\sqrt{\frac{1}{\pi x}} - \frac{1}{2}) = \text{const.} - \sqrt{\frac{1}{\pi}}\frac{L}{\epsilon} - 2\ln\frac{2\epsilon}{L}. \quad \text{(B.9)}$$

Therefore we get

$$\det{}'(-L^{-2}\frac{d^2}{df^2}) = \text{const.} e^{\frac{-L}{\epsilon\sqrt{\pi}}} L^2. \quad \text{(B.10)}$$

Appendix C On the Coherent states

C.1 The properties of $SU(2)$ Coherent states

The purpose of this part is to prove the three properties of the $SU(2)$ coherent state employed in section 2 [15].

Partition of unity The definition of the $SU(2)$ coherent states is given by eq.(2.70). We start with the $J = 1/2$ case. Explicit calculation gives

$$
\begin{aligned}
|e> &= \{\cos\frac{\theta}{2} + i(u\cdot\sigma)\sin\frac{\theta}{2}\}|+\frac{1}{2}> \\
&= \cos\frac{\theta}{2}\ |+\frac{1}{2}> - e^{i\phi}\sin\frac{\theta}{2}\ |-\frac{1}{2}>.
\end{aligned} \tag{C.1}
$$

Substituting this into the r.h.s. of eq.(2.71), we obtain

$$
\int de\ |e><e| = |+\frac{1}{2}><+\frac{1}{2}|\ +\ |-\frac{1}{2}><-\frac{1}{2}|\ = 1, \tag{C.2}
$$

where $de = \frac{2J+1}{4\pi} d(\cos\theta) d\phi$.

The higher spin representation can be made from the spin 1/2 representation. The coherent state can be expanded in terms of eigenstates of S^z:

$$
|e>= \sum_{m=-J}^{J} \sqrt{\frac{(2J)!}{(J+m)!(J-m)!}}(\cos\frac{\theta}{2})^{J+m}(-e^{i\phi}\sin\frac{\theta}{2})^{J-m}|J,m>. \tag{C.3}
$$

Using this we get

$$
\begin{aligned}
&\int de < J,m|e>< e|J,k> \\
&= 2\pi\frac{2J+1}{4\pi}\int d(cos\theta)\frac{(2J)!}{(J+m)!(J-m)!}(\cos\frac{\theta}{2})^{2J+2m}(\sin\frac{\theta}{2})^{2J-2m}\delta_{mk} \\
&= (2J+1)\frac{(2J)!}{(J+m)!(J-m)!}\frac{\Gamma(J+m+1)\Gamma(J-m+1)}{\Gamma(2J+2)}\delta_{mk} \\
&= \delta_{mk}.
\end{aligned} \tag{C.4}
$$

The above result expresses the partition of unity.

Inner product Next we prove the inner product eq.(2.72). We introduce the operator $U(t)$:

$$
U(t) \equiv e^{-t(H+\delta H)}e^{tH},\ U(0) = 1 \tag{C.5}
$$

This satisfies the following differential equation:

$$
\begin{aligned}
\dot{U}(t) &= -U(t)e^{-tH}\delta H e^{tH} \\
&= -U(t)\{\delta H + t[\delta H, H] + \cdots + \frac{t^k}{k!}[\cdots[\delta H, \underbrace{H]\cdots H]}_{k \text{ times}}\},
\end{aligned} \tag{C.6}
$$

where we have used Baker-Cambell-Hausdorff formula. The above equation can be solved by iteration. We get (up to the order of $O((\delta H)^2)$)

$$
U(1) = 1 - \int_0^1 dt\{\delta H + t[\delta H, H] + \cdots\}. \tag{C.7}
$$

Now we put $H = -i\theta u \cdot S$. The l.h.s. of eq.(2.72) becomes

$$
< e + \delta e | e > = < J, J | U(1) | J, J > . \tag{C.8}
$$

The operator H can be written in the following form.

$$
m_- S^+ + m_+ S^-, \ m_\pm \equiv \frac{\pm\theta}{2} e^{\pm i\phi} \tag{C.9}
$$

Since H does not contain S^z, terms containing an odd number of commutators are proportional to S^z. The others are written in linear combinations of $S^\pm$. For this reason, terms proportional to t^{2k} vanish in the r.h.s. of eq.(C.8). The result is

$$
\begin{aligned}
< J, J | U(1) | J, J > &= 1 + Ji\theta\delta\phi \int_0^1 dt \sin t\theta \\
&= 1 + iJ\delta\phi(1 - \cos\theta).
\end{aligned} \tag{C.10}
$$

The second term is equal to an infinitesimal area on the sphere. Thus we obtain eq.(2.72).

Expectation value Finally let us prove eq.(2.73). This can be done in a similar way to the previous case. For example,

$$
\begin{aligned}
< e | S^+ | e > &= < J, J | e^{-H} S^+ e^H | J, J > \\
&= < J, J | (S^+ + [S^+, H] + \cdots) | J, J > .
\end{aligned} \tag{C.11}
$$

In this case, the odd terms vanish. Then

$$
< e | S^+ | e > = J \sin\theta e^{i\phi}. \tag{C.12}
$$

Similarly one can show:

$$
< e | S^- | e > = J \sin\theta e^{-i\phi}, < e | S^z | e > = J \cos\theta. \tag{C.13}
$$

C.2 $SU(N)$ Coherent States

Next we explain the $SU(N)$ coherent states which appeared in section 4 [15, 23]. Let us consider the representation space of an $su(N)$ algebra. Generators are represented by anti-hermitian matrices T^a. These satisfy the commutation relation defined in the algebra: $[T^a, T^b] = f^{abc}T^c$. The $SU(N)$ coherent state is defined as follows:

$$|g> \equiv g|0>, \ <g| \equiv <0|g^{-1}, \ g = e^{\omega}, \ \omega \in su(N), \tag{C.14}$$

where $|0>$ stands for the highest weight state in the representation. The partition of unity is easily constructed if we use the invariant measure dg:

$$1 = \int dg |g><g|. \tag{C.15}$$

In fact we can check that the r.h.s. is commutative for arbitrary $h \in SU(N)$. A simple calculation shows that the infinitesimal kernel $< g+\delta g| \exp\{\mathcal{J}^a T^a \epsilon\}|g >$ can be expressed as follows:

$$\begin{aligned} & <g+\delta g| \exp\{\mathcal{J}^a T^a \epsilon\}|g> \\ = \ & 1- <0|g^{-1}\delta g|0> + <g|\mathcal{J}^a T^a \epsilon|g> + O(\epsilon^2) \\ = \ & \exp\{- <0|g^{-1}\delta g|0> + \mathcal{J}^a \mathcal{I}^a \epsilon\} + O(\epsilon^2), \end{aligned} \tag{C.16}$$

where $\mathcal{J}^a$ is a c-number source and $\mathcal{I}^a \equiv <g|T^a|g>$.

Appendix D The propagator of the Chern-Simons gauge field

In this appendix, we derive the propagator eq.(3.6). Let us consider the following action:

$$\int d^3x(\frac{i}{8\pi J}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + J_\mu A_\mu), \tag{D.1}$$

where the first term is the Chern-Simons action $S_{C.S.}$ having appeared in section 3 and J_μ is an external source. The above action leads to the following equation of motion:

$$\frac{i}{4\pi J}\epsilon^{\nu\beta\alpha}\partial_\beta A_\alpha + J^\nu = 0, \tag{D.2}$$

or

$$\frac{i}{4\pi J}(\partial_\mu\partial_\alpha - \Delta\delta_{\mu\alpha})A_\alpha + \epsilon^{\mu\lambda\nu}\partial_\lambda J^\nu = 0, \tag{D.3}$$

where Δ is the three-dimensional laplacian. Here we take the gauge condition $\partial_\mu A_\mu = 0$. Then

$$\Delta A_\mu = -4\pi i J\epsilon^{\mu\lambda\nu}\partial_\lambda J^\nu. \tag{D.4}$$

Solving the above equation, we have

$$\begin{aligned} A_\mu(x) &= iJ\int d^3y\epsilon^{\mu\lambda\nu}\partial_\lambda J^\nu(y)\frac{1}{|x-y|} \\ &= -iJ\int d^3y\epsilon^{\mu\lambda\nu}J^\nu(y)\frac{(x-y)_\lambda}{|x-y|^3}. \end{aligned} \tag{D.5}$$

This implies that the propagator, or the Green function is

$$\begin{aligned} < A_\mu(x)A_\nu(y) > &= -iJ\epsilon^{\mu\lambda\nu}\frac{(x-y)_\lambda}{|x-y|^3} \\ &= iJ\epsilon^{\mu\nu\lambda}\frac{(x-y)_\lambda}{|x-y|^3}. \end{aligned} \tag{D.6}$$

We can derive the same result starting from the following gauge fixed action S_{GF}:

$$S_{GF} \equiv iS_{C.S.} + \frac{1}{2\alpha}\int d^3x(\partial_\mu A^\mu)^2, \tag{D.7}$$

where α is a gauge-fixing parameter. The generating functional of connected Green functions $W[J]$ is defined as follows:

$$e^{W[J]} = \int \mathcal{D}Ae^{S_{GF}+\int d^3xJ_\mu A_\mu}. \tag{D.8}$$

After performing the integration of A_μ we have:

$$\exp\{W[J]\} = \exp\{-\frac{1}{2}\int d^3x d^3y J_\mu(x) K^{-1}_{\mu\nu}(x-y) J_\nu(y)\}, \tag{D.9}$$

where the operator $K^{-1}_{\mu\nu}$ is the inverse of $K_{\mu\nu}$:

$$\begin{aligned} K_{\mu\nu}(x-y) &\equiv (\frac{i}{4\pi J}\epsilon^{\mu\lambda\nu}\partial_\lambda - \frac{1}{\alpha}\partial_\mu\partial_\nu)\delta(x-y) \\ &= \int \frac{d^3p}{(2\pi)^3}(\frac{\epsilon^{\mu\lambda\nu}p_\lambda}{4\pi J} + \frac{1}{\alpha}p_\mu p_\nu)e^{-ip(x-y)} \end{aligned} \tag{D.10}$$

The propagator $< A_\mu(x)A_\nu(y) >$ is obtained using the generating functional:

$$< A_\mu(x)A_\nu(y) >= \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}W[J] = -K^{-1}_{\mu\nu}(x-y). \tag{D.11}$$

After a Fourier transformation, we wish to obtain the inverse operator in the following form:

$$K^{-1}_{\mu\nu}(p) = ap_\mu p_\nu + b\delta_{\mu\nu} + c\epsilon^{\mu\lambda\nu}p_\lambda. \tag{D.12}$$

We can find the unknown parameters a, b and c using the definition $K_{\mu\nu}(p)K^{-1}_{\nu\rho}(p) = \delta_{\mu\rho}$. The result is

$$a = -\frac{\alpha}{p^4},\ b = 0,\ c = -\frac{4\pi J}{p^2}. \tag{D.13}$$

Substituting this result into eq.(D.11), we get

$$\begin{aligned} < A_\mu(x)A_\nu(y) > &= -\int \frac{d^3p}{(2\pi)^3}K^{-1}_{\mu\nu}(p)e^{-ip(x-y)} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{\alpha p_\mu p_\nu + 4\pi J\epsilon^{\mu\lambda\nu}p_\lambda p^2}{p^4}e^{-ip(x-y)}. \end{aligned} \tag{D.14}$$

We can easily integrate over p in the $\alpha = 0$ gauge:

$$\begin{aligned} < A_\mu(x)A_\nu(y) > &= 4\pi J\int \frac{d^3p}{(2\pi)^3}\frac{\epsilon^{\mu\lambda\nu}p_\lambda}{p^2}e^{-ip(x-y)} \\ &= 4\pi iJ\epsilon^{\mu\lambda\nu}\partial_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2}e^{-ip(x-y)} \\ &= iJ\epsilon^{\mu\lambda\nu}\partial_\lambda\frac{1}{|x-y|} \\ &= iJ\epsilon^{\mu\nu\lambda}\frac{(x-y)_\lambda}{|x-y|^3}. \end{aligned} \tag{D.15}$$

Thus we have got the desired result.

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