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Some applications of flat connections over a surface

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Introduction

In low dimensional topology, the fundamental group of a manifold is an important object of the study of manifolds. Under the investigation of fundamental groups, more widely general groups, their representations to a finite group or a Lie group are powerful tools. In the theory of manifolds, representations of the fundamental group of a manifold are corresponding to flat connections over it. The object interpreted from different viewpoints is powerful as a tool and is an interesting one of study itself. For instance, if a representation of the fundamental group of a manifold is given, a local system over the manifold is constructed and then homology and cohomology groups with coefficients in it are obtained. These groups themselves are invariants of the manifold. For another example, Casson's invariants for homology three spheres are given. Roughly speaking it counts the number with signs of conjugacy classes of irreducible $SU(2)$ -representations of the fundamental group of a homology three sphere. Under the strict definition of it, the moduli space of $SU(2)$ -representations of the fundamental group of a surface appears. If it is regarded as the moduli space of $SU(2)$ -flat connections over a surface, it appears not only in the definition of the Casson's invariant but also in various fields. Moreover it turns out to be abundant in structure from recent researches. Therefore representations of fundamental groups or flat connections are powerful tools and are important objects of study. In this paper three applications of representations of the fundamental group of a surface are given.

In chapter 1, we give an application of a nontrivial representation of the fundamental group of a surface to $\mathbb{Z}_2$. From it, we obtain a local system over the surface, hence we have the first cohomology group of the surface with coefficients in it. A certain subgroup $\mathcal{M}_{g^*}^w$ of the mapping class group of the surface with a base point acts on this cohomology group. It turns out that this action induces a representation of $\mathcal{M}_{g^*}^w$ to a symplectic group over $\mathbb{Z}$. First we show that this representation is nontrivial on the Torelli group, more strongly, on the subgroup generated by all the Dehn twists along separating simple closed curves on the surface, which are of course contained in $\mathcal{M}_{g^*}^w$. Next, we consider only the case that the genus of a surface is 2. For each element of the subgroup $\mathcal{M}_{g^*}^w$, we construct a surface bundle over the circle and a representation of the fundamental group of it to $\mathbb{Z}_2$. Then we can obtain the Atiyah-Patodi-Singer ρ -invariant for them, which can be regarded as a function on $\mathcal{M}_{g^*}^w$. On the other hand, through the above representation of $\mathcal{M}_{g^*}^w$, we can define another function on $\mathcal{M}_{g^*}^w$ because of genus 2. This function is explicitly given by using the Dedekind sums. A new function on $\mathcal{M}_{g^*}^w$ is defined by the sum of two functions above. We show that the coboundary of this function coincides with the restriction to $\mathcal{M}_{g^*}^w$ of the signature 2-cocycle on the mapping class group of the surface of genus 2. Moreover we show that, on the Torelli group, the average of new functions obtained as above from all the nontrivial $\mathbb{Z}_2$ -representations of the fundamental group of the surface agrees with the Meyer function. Here the Meyer function is characterized as a unique function on the whole mapping class group of a surface of genus 2 whose coboundary coincides with the signature cocycle. This result means that the Meyer function is explicitly described on the Torelli group by the ρ -invariants and the Dedekind sums.

In chapter 2, we study an analogue to the construction of a symplectic manifold and its Lagrangian submanifolds from a Heegaard decomposition of a homology three sphere

by using $SU(2)$ -representations. We consider $SO(3)$ -representations of the fundamental group of a surface. More precisely, we consider the moduli space of $SU(2)$ -representations of the fundamental group of a nontrivial $SO(3)$ -bundle over a surface which satisfy a certain condition. It is known that this space is a symplectic manifold without singularities. Here we construct its Lagrangian submanifolds without singularities as follows. We consider the manifold with boundary which is the complement of a tubular neighborhood of pairwise disjoint embedded curves satisfying some conditions in a handlebody whose boundary is the surface considered. For an $SO(3)$ -bundle over this manifold which is nontrivial over each boundary component of it, we construct the moduli space of $SU(2)$ -representations with a certain condition as above. It is showed that this space is embedded as a smooth Lagrangian submanifold to the above symplectic manifold. Moreover we give some simple examples of Floer cohomologies for Lagrangian intersections.

In chapter 3, we consider a representation of the fundamental group of a closed oriented surface to a symplectic group or equivalently a flat symplectic vector bundle over it. For each symplectic vector bundle, a positive compatible complex structure with the symplectic one is uniquely determined up to homotopy. Since we handle the bundle over a closed oriented surface, we can obtain the first Chern number. Turaev showed the inequality for the number which is best possible. It is also proved by Domic-Toledo in a different form. In particular, in the case of a bundle of rank 2, this inequality is known as Milnor's one. In this chapter we give a simple proof of the inequality using Atiyah-Singer index theorem.

Chapter 1

On a function on the mapping class group of a surface of genus 2

ON A FUNCTION ON THE MAPPING CLASS GROUP OF A SURFACE OF GENUS 2

RYOJI KASAGAWA

ABSTRACT. We study representations of subgroups of the mapping class group $\mathcal{M}_g$ of a surface of genus $g \geq 2$ arising from the actions of them on the first cohomology groups of the surface with local coefficient systems which are defined by nontrivial homomorphisms $\pi_1(\Sigma_g, *) \rightarrow \mathbb{Z}_2 = \text{Aut}(\mathbb{Z})$. As an application, in the case of $g = 2$, we construct a function on $\mathcal{M}_2$ which agrees with the Meyer function $\phi: \mathcal{M}_2 \rightarrow \mathbb{Q}$ on the Torelli group $\mathcal{J}_2$.

1. INTRODUCTION

A well-known representation of the mapping class group of a surface of genus g to $Sp(2g, \mathbb{Z})$ is obtained from the action of it on the first cohomology group of the surface. In this case the coefficient group of the cohomology group is $\mathbb{Z}$, more precisely, the local coefficient system obtained from the trivial $\pi_1(\Sigma_g, *)$ -module $\mathbb{Z}$. In this paper we take local coefficient systems obtained from nontrivial $\pi_1(\Sigma_g, *)$ -module $\mathbb{Z}$'s and consider representations of subgroups of the mapping class group of the surface arising from the action of them on the first cohomology groups of the surface with the above local coefficient systems. Moreover as an application, we construct a function of the mapping class group of the surface of genus 2 and show some properties of it.

We state them more precisely as follows. Let Σ_g be a closed oriented surface of genus $g \geq 2$ with a base point $*$. For any nonzero class $w \in H^1(\Sigma_g; \mathbb{Z}_2) = \text{Hom}(\pi_1(\Sigma_g, *), \mathbb{Z}_2 = \{0, 1\})$, $\mathbb{Z}$ is regarded as the $\pi_1(\Sigma_g, *)$ -module with the action $(\alpha, m) \mapsto (-1)^{w(\alpha)}m$ for $(\alpha, m) \in \pi_1(\Sigma_g, *) \times \mathbb{Z}$, which is denoted by $\mathbb{Z}_w$ and the local coefficient system obtained from it is also denoted by the same letter. So we have the cohomology group $H^1(\Sigma_g; \mathbb{Z}_w)$. In this paper, it is identified with the first cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ of the surface group $\pi_1(\Sigma_g, *)$ with coefficients in $\mathbb{Z}_w$ since Σ_g is an Eilenberg-MacLane space. Let $\mathcal{M}_{g*}$ be the mapping class group of Σ_g relative to the base point and $\mathcal{M}_{g*}^w$ the subgroup of $\mathcal{M}_{g*}$ whose elements preserve the class w . Then the subgroup $\mathcal{M}_{g*}^w$ acts on the cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$ modulo torsion by the pull back of the inverse.

In order to state our results, we introduce subgroups of $\mathcal{M}_{g*}$ and $\mathcal{M}_g$. Let $\mathcal{J}_g$ be the subgroup of $\mathcal{M}_g$ acting on the first cohomology group $H^1(\Sigma_g, \mathbb{Z})$ trivially. Let $\mathcal{K}_g$ be the subgroup of $\mathcal{M}_g$ generated by all the Dehn twists along separating simple closed curves. Then we have $\mathcal{K}_g \subset \mathcal{J}_g \subset \mathcal{M}_{g*}^w \subset \mathcal{M}_g$. Similarly the subgroups $\mathcal{K}_{g*}$ and $\mathcal{J}_{g*}$ of $\mathcal{M}_{g*}$ are defined and we have $\mathcal{K}_{g*} \subset \mathcal{J}_{g*} \subset \mathcal{M}_{g*}^w \subset \mathcal{M}_{g*}$. The subgroups $\mathcal{J}_g$ and $\mathcal{J}_{g*}$ are called Torelli groups (see [4]).

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The following proposition is a collection of some results of section 2.

Proposition 1. *For each nonzero class $w \in H^1(\Sigma_g; \mathbb{Z}_2)$, the above action of $\mathcal{M}_{g*}^w$ on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$ gives a surjective homomorphism $\zeta_{g*}^w: \mathcal{M}_{g*}^w \rightarrow Sp(2(g-1), \mathbb{Z})$, whose restriction to $\mathcal{K}_{g*}$ is nontrivial. Moreover it descends to a surjective homomorphism $\zeta_g^w: \mathcal{M}_g^w \rightarrow PSp(2(g-1), \mathbb{Z})$ whose restriction to $\mathcal{K}_g$ is also nontrivial.*

Next we define a representation of the whole group $\mathcal{M}_{g*}$.

Let $Aut(\mathbb{H}, \Omega)$ be the group of automorphisms of $\mathbb{H}$ preserving Ω where $\mathbb{H}$ is the direct sum $\bigoplus_{w \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}} H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$ and Ω is the symplectic form defined by the cup product. For any element f of $Aut(\mathbb{H}, \Omega)$, we can put $f = (f_{uv})$, where f_{uv} is an isomorphism from $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_v)/\text{torsion}$ to $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_u)/\text{torsion}$ and u and v run over the nonzero classes of $H^1(\Sigma_g; \mathbb{Z}_2)$. Let $\mathcal{S}$ be the subgroup of $Aut(\mathbb{H}, \Omega)$ consisting of those elements $f = (f_{uv})$ for which there exists $\sigma \in Aut(H^1(\Sigma_g; \mathbb{Z}_2), \cup)$ such that f_{uv} is not 0 if and only if $v = \sigma(u)$. The action of $\mathcal{M}_{g*}$ on $\mathbb{H}$ induces a homomorphism $\zeta_{g*}: \mathcal{M}_{g*} \rightarrow \mathcal{S}$. The image of the kernel of the homomorphism $\mathcal{M}_{g*} \rightarrow \mathcal{M}_g$ agrees with a normal subgroup $N = \{((-1)^{u(\tau)} \delta_{uv}) \in \mathcal{S} \mid \tau \in \pi_1(\Sigma_g, *)\}$ of $\mathcal{S}$, where δ_{uv} is the identity map on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_u)$ if $u = v$, and is zero if not. Hence the homomorphism ζ_{g*} descends to a homomorphism $\zeta_g: \mathcal{M}_g \rightarrow \mathcal{S}/N$. With these notations, we have the following result which is stated as Corollary 9 in section 2.

Corollary 2. *The restrictions of ζ_{g*} and ζ_g to $\mathcal{K}_{g*}$ and $\mathcal{K}_g$ respectively are nontrivial.*

Here we state some relations between our representations and others. There is a representation of $\mathcal{M}_{g,1}$ constructed by Morita (see [12]) and Trapp (see [15]) which is nontrivial on $\mathcal{J}_{g,1}$, where $\mathcal{M}_{g,1}$ is the mapping class group of Σ_g relative to an embedded disk $(*)D^2 \subset \Sigma_g$ and $\mathcal{J}_{g,1}$ is the corresponding Torelli group. It was shown that it descends to a representation of $\mathcal{M}_{g*}$. Morita showed that this representation can be also obtained from his representation of $\mathcal{M}_{g,1}$ to the semi-direct product $\frac{1}{2} \bigwedge^3 H^1(\Sigma_g; \mathbb{Z}) \rtimes Sp(2g; \mathbb{Z})$. Morita's representation is an extension of Johnson's homomorphism from $\mathcal{J}_{g,1}$ to $\bigwedge^3 H^1(\Sigma_g; \mathbb{Z})$. It descends to a representation of $\mathcal{M}_{g*}$ to $\frac{1}{2} \bigwedge^3 H^1(\Sigma_g; \mathbb{Z}) \rtimes Sp(2g; \mathbb{Z})$. Moreover it descends to a representation of $\mathcal{M}_g$ to $\frac{1}{2} \bigwedge^3 H^1(\Sigma_g; \mathbb{Z}) / H^1(\Sigma_g; \mathbb{Z}) \rtimes Sp(2g; \mathbb{Z})$. The kernels of the Morita's representations of $\mathcal{M}_{g,1}$ and $\mathcal{M}_g$ are $\mathcal{K}_{g,1}$ and $\mathcal{K}_g$ respectively (see [5, 13]). This fact together with Proposition 1 and Corollary 2 imply that our representations ζ_{g*} and ζ_g are different from the above representations.

On the other hand there is representations of subgroups of the mapping class group $\mathcal{M}_g$ defined by Looijenga using finite abelian coverings. They are defined as follows (see [7]). Let $\pi: \Sigma_h \rightarrow \Sigma_g$ be a connected finite abelian covering with covering group G , then we obtain the homomorphism $c: H_1(\Sigma_g; \mathbb{Z}) \rightarrow G$ from it. Let $\mathcal{M}_g^c$ be the subgroup of $\mathcal{M}_g$ consisting of those elements which preserve the homomorphism c by pullback. Let Sp_G be the group of symplectic transformations of $H_1(\Sigma_h; \mathbb{Z})$ which commute with the action of the covering group G . The image of G is contained in the center of Sp_G . Any element of $\mathcal{M}_g^c$ lifts to a mapping class of Σ_h and the difference from any other lift is a covering transformation. It follows that we have a well-defined homomorphism $\mathcal{M}_g^c \rightarrow Sp_G/G$. We consider the case of $G = \mathbb{Z}_2$, hence $c = w$. If we consider the local coefficient system $\mathbb{Z}_w$ over Σ_g as one over Σ_h via the projection π , then it is trivial. This implies that

we have the induced homomorphism $\pi^*: H^1(\Sigma_g; \mathbb{Z}_w) \rightarrow H^1(\Sigma_h; \mathbb{Z})$. It turns out that its image is identified with $H^1(\Sigma_g; \mathbb{Z}_w)/\text{torsion}$ and is preserved by the image of Looijenga's representation. If we replace the range $Sp_{\mathbb{Z}_2}/\mathbb{Z}_2$ of it by its image, our representation ζ_g^w turns out to be factored by Looijenga's representation.

As an application, in the case of $g = 2$, we construct a function on the mapping class group $\mathcal{M}_2$ from our representations in the same way as [1]. This function is related to the signature of 4-manifolds which are Σ_2 -bundles over surfaces. Its restriction to the Torelli group $\mathcal{J}_2$ agrees with the restriction of the Meyer function ϕ to $\mathcal{J}_2$. We explain them more precisely as follows.

There is a unique conjugacy invariant $\mathbb{Q}$ -valued function ψ on $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$ satisfying $\sigma(A, B) = \psi(B) - \psi(AB) + \psi(A)$ for any $A, B \in SL(2, \mathbb{Z})$ where σ is the signature 2-cocycle of $SL(2, \mathbb{Z})$. This function ψ is described explicitly by using the Dedekind sums or the Rademacher ϕ function (see [1, 6, 9, 14]). Composing ψ with ζ_{2*}^w , we obtain the function $\Psi_*^w: \mathcal{M}_{2*}^w \rightarrow \mathbb{Q}$.

For any $f \in \mathcal{M}_{2*}^w$, let $M_f \rightarrow S^1$ be the Σ_2 -bundle over S^1 with the section s where M_f is given by $\Sigma_2 \times [0, 1]/(x, 1) \sim (f^{-1}(x), 0)$ and the section s is determined from the base point of Σ_2 . Let w_{M_f} be a unique class of $H^1(M_f; \mathbb{Z}_2)$ such that its restriction to the fiber Σ_2 at the base point $0 \in [0, 1]/\sim$ is w and its restriction to the image of the section s is 0. By considering w_{M_f} as a homomorphism $\pi_1(M_f, *) \rightarrow \mathbb{Z}_2 = S^0 \subset U(1)$, we obtain a flat complex line bundle over M_f . Then we have the Atiyah-Patodi-Singer ρ -invariant $\rho_{w_{M_f}}(M_f)$ for (M_f, w_{M_f}) (see [2]). In this case the value of $\rho_{w_{M_f}}(M_f)$ is in $\mathbb{Q}$. We define the function $\mu_*^w: \mathcal{M}_{2*}^w \rightarrow \mathbb{Q}$ by $\mu_*^w(f) = \rho_{w_{M_f}}(M_f) + \Psi_*^w(f)$. It turns out that this function μ_*^w descends to a function on $\mathcal{M}_2^w$, which is denoted by μ^w . Hence we can define the function $\mu: \mathcal{M}_2 \rightarrow \mathbb{Q}$ by $\mu(f) = \frac{1}{15} \sum_w \mu^w$, where w runs over the nonzero classes of $H^1(\Sigma_g; \mathbb{Z}_2)$ fixed by f^* .

It is known that there is a unique function, which is called the Meyer function, $\phi: \mathcal{M}_2 \rightarrow \mathbb{Q}$ such that the equality $\text{sign}(a, b) = \phi(b) - \phi(ab) + \phi(a)$ holds for any $a, b \in \mathcal{M}_2$ (see [8, 9]). Here the signature cocycle sign of $\mathcal{M}_2$ is defined as follows.

Let P denote the 2 sphere with 3 holes $S^2 \setminus \text{Int} D^2$, then $\pi_1(P, *)$ is a rank 2 free group whose generators are given by α and β . For any $a, b \in \mathcal{M}_2$, we define the homomorphism $h: \pi_1(P, *) \rightarrow \mathcal{M}_2$ by $\alpha \mapsto a, \beta \mapsto b$. Let Z_h be the Σ_2 -bundle over P (the inverse of) whose monodromy is given by h . Since Z_h is also a compact oriented 4-manifold, we have the signature $\text{sign}(Z_h)$ of it which depends only on a and b , so we can put $\text{sign}(a, b) = \text{sign}(Z_h)$.

Let $\mathcal{M}_2^H$ be the subgroup of $\mathcal{M}_2$ acting trivially on $H^1(\Sigma_2, \mathbb{Z}_2)$.

Theorem 3. *The function μ is conjugacy invariant on $\mathcal{M}_2$ and satisfies the equality $\text{sign}(a, b) = \mu(b) - \mu(ab) + \mu(a)$ for any $a, b \in \mathcal{M}_2^H$. Its restriction to the Torelli group $\mathcal{J}_2$ is a nontrivial homomorphism, and agrees with the restriction of Meyer function ϕ to $\mathcal{J}_2$.*

This theorem can be obtained by combining Corollary 18 and Proposition 19.

This paper is organized as follows. In section 2, we construct our representations and show some properties of them. In section 3, we review a certain function on $SL(2, \mathbb{Z})$ related to the signature 2-cocycle of it. In section 4, we define the function μ in Theorem

3 and prove this theorem. In section 5, we give an example, which is needed to show Theorem 3.

2. COHOMOLOGIES OF SURFACES WITH TWISTED COEFFICIENTS

In this section, for any nonzero class of the first cohomology group of a surface with $\mathbb{Z}_2$ -coefficient, we construct a representation of the subgroup of the mapping class group of the surface whose elements preserve the class.

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and $*$ $\in \Sigma_g$ a base point.

Let $w : \pi_1(\Sigma_g, *) \rightarrow \mathbb{Z}_2$ be a homomorphism, then it determines a cohomology class of $H^1(\Sigma_g; \mathbb{Z}_2)$ which is denoted by the same letter w . $\pi_1(\Sigma_g, *)$ acts on $\mathbb{Z}$ by

$$\begin{aligned} \pi_1(\Sigma_g, *) \times \mathbb{Z} &\rightarrow \mathbb{Z} \\ (\gamma, m) &\mapsto \gamma \cdot m := (-1)^{w(\gamma)} m, \end{aligned}$$

where the homomorphism w takes values in $\{0, 1\}$. So $\mathbb{Z}$ is regarded as a $\pi_1(\Sigma_g, *)$ -module, which is denoted by $\mathbb{Z}_w$.

Next we shall compute the first cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ (see [3]).

Let $Z^1 = Z^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ be the space of w -crossed homomorphisms u from $\pi_1(\Sigma_g, *)$ to $\mathbb{Z}$ which satisfy the equalities $u(\alpha\beta) = u(\alpha) + \alpha \cdot u(\beta)$ for $\forall \alpha, \beta \in \pi_1(\Sigma_g, *)$. Let $B^1 = B^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ be the space of principal w -crossed homomorphisms u , for which there exists $m \in \mathbb{Z}$ satisfying $u(\alpha) = \alpha \cdot m - m$ for $\forall \alpha \in \pi_1(\Sigma_g, *)$. Then $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ is given by Z^1/B^1 .

The fundamental group $\pi_1(\Sigma_g, *)$ is represented by

$$\langle \alpha_i, \beta_i (1 \leq i \leq g) \mid \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \rangle.$$

For any $u \in Z^1$, put $u(\alpha_i) = x_i$ and $u(\beta_i) = y_i$. Then we can regard Z^1 as a subset of $\mathbb{Z}^{2g}$ by the inclusion $u \mapsto (x_i, y_i)_{i=1}^g$.

A direct computation shows

$$\begin{aligned} Z^1 &= \{(x_i, y_i) \in \mathbb{Z}^{2g} \mid \sum_{i=1}^g \{w(\beta_i)x_i - w(\alpha_i)y_i\} = 0\}, \\ B^1 &= \{(x_i, y_i) \in \mathbb{Z}^{2g} \mid (x_i, y_i) = -2(w(\alpha_i)m, w(\beta_i)m), m \in \mathbb{Z}\}, \end{aligned}$$

where $w(\alpha_i)$ and $w(\beta_i)$ are in $\mathbb{Z}_2 = \{0, 1\}$. Thus we obtain the following lemma.

Lemma 4.

$$H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w) \cong \begin{cases} \mathbb{Z}^{2g} & w = 0 \\ \mathbb{Z}^{2(g-1)} \oplus \mathbb{Z}_2 & w \neq 0. \end{cases}$$

Remark. Since Σ_g is a $K(\pi, 1)$ -space, if we consider $\mathbb{Z}_w$ as a local coefficient system, then we have the identification $H^*(\Sigma_g; \mathbb{Z}_w) \cong H^*(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$. Hereafter we identify these groups by this identification.

Let $\mathcal{M}_{g*}$ be the mapping class group of orientation and base point preserving diffeomorphisms of Σ_g . For a nonzero class $w \in H^1(\Sigma_g; \mathbb{Z}_2)$, the subgroup of $\mathcal{M}_{g*}$ whose elements preserve the class w is denoted by $\mathcal{M}_{g*}^w$. This subgroup acts on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$.

In this paper, the action of $f \in \mathcal{M}_{g*}^w$ is defined by the pull-back $(f^{-1})^*$ of the inverse of f . Moreover this action preserves the symplectic form on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$ which is given by the cup-product and the identification $H^2(\pi_1(\Sigma_g, *), \mathbb{Z}) \cong \mathbb{Z}$.

Lemma 5. *The above symplectic lattice $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$ is isomorphic to $\mathbb{Z}^{2(g-1)}$ with the standard symplectic form.*

By Lemma 5, taking a symplectic basis for $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$, we obtain a homomorphism

$$\zeta_{g*}^w : \mathcal{M}_{g*}^w \rightarrow Sp(2(g-1), \mathbb{Z}).$$

Lemma 6. *The above homomorphism ζ_{g*}^w is surjective.*

Proof of Lemma 5 and 6. Since the action of the mapping class group of the surface on the first cohomology group $H^1(\Sigma_g; \mathbb{Z}_2)$ is transitive, it is sufficient to prove the case of $w = \alpha_g^*$ where α_g^* belongs to the dual basis of the one $\alpha_i, \beta_i (1 \leq i \leq g)$ of $H_1(\Sigma_g; \mathbb{Z}_2)$ which is given in Figure 1. Let $T \subset \Sigma_g$ be the submanifold of Σ_g depicted in Figure 1.

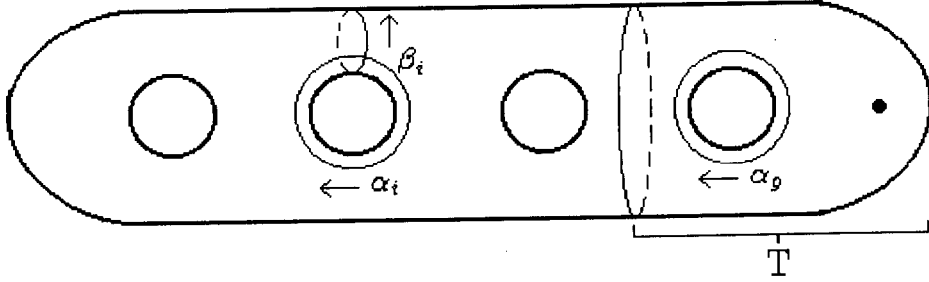


FIGURE 1. A basis α_i, β_i and the submanifold T .

Let $\Sigma_{g-1,1}$ be the closure of the complement of T in Σ_g . By the exact sequence of the pair (Σ_g, T) ;

$$\begin{aligned} 0 = H^0(T; \mathbb{Z}_w) &\rightarrow H^1(\Sigma_g, T; \mathbb{Z}_w) \rightarrow H^1(\Sigma_g; \mathbb{Z}_w) \rightarrow H^1(T; \mathbb{Z}_w) \\ &\rightarrow H^2(\Sigma_g, T; \mathbb{Z}_w) \rightarrow H^2(\Sigma_g; \mathbb{Z}_w) \rightarrow H^2(T; \mathbb{Z}_w) = 0, \end{aligned}$$

and isomorphisms $H^1(T; \mathbb{Z}_w) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $H^2(\Sigma_g, T; \mathbb{Z}_w) \cong \mathbb{Z}$ and $H^2(\Sigma_g; \mathbb{Z}_w) \cong \mathbb{Z}_2$, we have an isomorphism $H^1(\Sigma_g, T; \mathbb{Z}_w) \cong H^1(\Sigma_g; \mathbb{Z}_w)/\text{torsion}$. There is the following commutative diagram;

$$\begin{array}{ccc} H^1(\Sigma_g; \mathbb{Z}_w) \times H^1(\Sigma_g; \mathbb{Z}_w) & \xrightarrow{\cup} & H^2(\Sigma_g; \mathbb{Z}) \\ \uparrow & & \uparrow \cong \\ H^1(\Sigma_g, T; \mathbb{Z}_w) \times H^1(\Sigma_g, T; \mathbb{Z}_w) & \xrightarrow{\cup} & H^2(\Sigma_g, T; \mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ H^1(\Sigma_{g-1,1}, S^1; \mathbb{Z}) \times H^1(\Sigma_{g-1,1}, S^1; \mathbb{Z}) & \xrightarrow{\cup} & H^2(\Sigma_{g-1,1}, S^1; \mathbb{Z}), \end{array}$$

where S^1 is the boundary of $\Sigma_{g-1,1}$. From this we can deduce the following commutative diagram;

$$\begin{array}{ccc} H^1(\Sigma_g; \mathbb{Z}_w)/torsion \times H^1(\Sigma_g; \mathbb{Z}_w)/torsion & \xrightarrow{\cup} & H^2(\Sigma_g; \mathbb{Z}) \\ \parallel & & \parallel \\ H^1(\Sigma_{g-1,1}, S^1; \mathbb{Z}) \times H^1(\Sigma_{g-1,1}, S^1; \mathbb{Z}) & \xrightarrow{\cup} & H^2(\Sigma_{g-1,1}, S^1; \mathbb{Z}). \end{array}$$

Since the symplectic structure on $H^1(\Sigma_{g-1,1}, S^1; \mathbb{Z}) \cong H^1(\Sigma_{g-1}; \mathbb{Z})$ is the standard one, the proof of Lemma 5 is finished.

Let $\mathcal{M}_{g-1,1}$ denote the group of isotopy classes of diffeomorphisms of $\Sigma_{g-1,1}$ whose restriction to the boundary is the identity of it. For any element of $\mathcal{M}_{g-1,1}$, extending it to an isotopy class of diffeomorphism of Σ_g by the identity on T , we have a homomorphism $\mathcal{M}_{g-1,1} \rightarrow \mathcal{M}_{g*}$. By the choice of w , the image of the homomorphism is contained in $\mathcal{M}_{g*}^w$. Clearly the following diagram commutes;

$$\begin{array}{ccc} \mathcal{M}_{g-1,1} & \longrightarrow & Aut(H^1(\Sigma_{g-1,1}, S^1; \mathbb{Z}), \cup) \\ \downarrow & & \parallel \\ \mathcal{M}_{g*}^w & \longrightarrow & Aut(H^1(\Sigma_g; \mathbb{Z}_w)/torsion, \cup). \end{array}$$

The fact that $\mathcal{M}_{g-1} \rightarrow Aut(H^1(\Sigma_{g-1}; \mathbb{Z}), \cup)$ is surjective implies that the upper homomorphism in the above diagram is surjective, hence so is the lower one. The proof of Lemma 6 is completed. $\square$

Next we see that ζ_{g*}^w descends to the homomorphism from $\mathcal{M}_g^w$ to $PSp(2(g-1), \mathbb{Z})$.

There is the following exact sequence

$$1 \rightarrow \pi_1(\Sigma_g, *) \xrightarrow{\iota} \mathcal{M}_{g*} \rightarrow \mathcal{M}_g \rightarrow 1.$$

Since the image of ι is in $\mathcal{M}_{g*}^w$, we have the following exact sequence

$$1 \rightarrow \pi_1(\Sigma_g, *) \xrightarrow{\iota} \mathcal{M}_{g*}^w \rightarrow \mathcal{M}_g^w \rightarrow 1.$$

Lemma 7. *For any $\tau \in \pi_1(\Sigma_g, *)$, the equality $\zeta_{g*}^w \circ \iota(\tau) = (-1)^{w(\tau)} I$ holds. Hence the image of $\zeta_{g*}^w \circ \iota$ is $\{\pm I\} \subset Sp(2(g-1), \mathbb{Z})$.*

Proof. For each $\tau \in \pi_1(\Sigma_g, *)$, the action of $\iota(\tau) \in \mathcal{M}_{g*}$ on $\pi_1(\Sigma_g, *)$ is given by $\iota(\tau)_*(\gamma) = \tau\gamma\tau^{-1}$ for all $\gamma \in \pi_1(\Sigma_g, *)$ (see [11]).

For any $u \in Z^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$, we have

$$\begin{aligned} \iota(\tau^{-1})^* u(\gamma) &= u(\tau^{-1}\gamma\tau) = -\tau^{-1} \cdot u(\tau) + \tau^{-1} \cdot u(\gamma) + \tau^{-1}\gamma \cdot u(\tau) \\ &= \tau^{-1} \cdot u(\gamma) + (\gamma - 1)\tau^{-1} \cdot u(\tau). \end{aligned}$$

Since the map $\gamma \mapsto (\gamma - 1)\tau^{-1} \cdot u(\tau)$ belongs to $B^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$, we obtain $\iota(\tau^{-1})^*[u] = [\tau^{-1} \cdot u] = (-1)^{w(\tau^{-1})}[u] = (-1)^{w(\tau)}[u]$, where $[u] \in H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)$ is the class of u . The proof is completed. $\square$

By this lemma, the homomorphism ζ_{g*}^w descends to the homomorphism

$$\zeta_g^w: \mathcal{M}_g^w \rightarrow PSp(2(g-1), \mathbb{Z}).$$

Let $\mathcal{K}_{g^*}$ be the subgroup of $\mathcal{M}_{g^*}$ generated by all the Dehn twists along separating simple closed curves. Similarly the subgroup $\mathcal{K}_g \subset \mathcal{M}_g$ is defined. Clearly $\mathcal{K}_{g^*}$ and $\mathcal{K}_g$ are subgroups of $\mathcal{M}_{g^*}^w$ and $\mathcal{M}_g^w$ respectively for any $w \in H^1(\Sigma_g; \mathbb{Z}_2)$.

The example in section 5 shows the following proposition in the case of $g = 2$. For $g > 2$, similar examples can be given.

Proposition 8. *The restrictions of the homomorphisms $\zeta_{g^*}^w$ and ζ_g^w to the subgroups $\mathcal{K}_{g^*}$ and $\mathcal{K}_g$ respectively are nontrivial.*

We define a representation of the whole group $\mathcal{M}_{g^*}$ as follows. Let $\mathbb{H}$ be the direct sum $\bigoplus_{w \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}} H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$ and Ω be the symplectic form on it given by the direct sum of the cup products on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_w)/\text{torsion}$'s. By Lemma 5, $(\mathbb{H}, \Omega)$ is isomorphic to $\mathbb{Z}^{2(2^{2g}-1)(g-1)}$ with the standard symplectic form. Clearly $\mathcal{M}_{g^*}$ acts on $\mathbb{H}$ and preserves the symplectic form Ω . Let $\text{Aut}(\mathbb{H}, \Omega)$ be the group of automorphisms of $\mathbb{H}$ preserving Ω . For any $f \in \text{Aut}(\mathbb{H}, \Omega)$, we put $f = (f_{uv})$ where f_{uv} is an isomorphism from $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_v)/\text{torsion}$ to $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_u)/\text{torsion}$ and u and v run over the nonzero classes of $H^1(\Sigma; \mathbb{Z}_2)$. Let $\mathcal{S}$ be the subgroup of $\text{Aut}(\mathbb{H}, \Omega)$ consisting of those elements $f = (f_{uv})$ for which there exists $\sigma \in \text{Aut}(H^1(\Sigma_g; \mathbb{Z}_2), \cup)$ such that f_{uv} is not 0 if and only if $v = \sigma(u)$. Clearly the action of $\mathcal{M}_{g^*}$ on $\mathbb{H}$ induces a homomorphism

$$\zeta_{g^*} : \mathcal{M}_{g^*} \rightarrow \mathcal{S}.$$

By Lemma 7, the image of $\zeta_{g^*} \circ \iota$ agrees with $N = \{((-1)^{u(\tau)} \delta_{uv}) \in \mathcal{S} \mid \tau \in \pi_1(\Sigma_g, *)\} \subset \mathcal{S}$, where δ_{uv} is the identity map on $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_u)$ if $u = v$, and is zero if not. It is easy to see that N is a normal subgroup of $\mathcal{S}$. Hence the homomorphism ζ_{g^*} descends to a homomorphism

$$\zeta_g : \mathcal{M}_g \rightarrow \mathcal{S}/N.$$

Clearly the restriction of ζ_{g^*} to $\mathcal{M}_{g^*}^H$ is identified with $\prod_{w \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}} \zeta_{g^*}^w : \mathcal{M}_{g^*}^H \rightarrow \text{Sp}(2(g-1); \mathbb{Z})^{2^{2g}-1}$. For ζ_g , a similar result holds. Proposition 8 implies the following corollary.

Corollary 9. *The restrictions of ζ_{g^*} and ζ_g to $\mathcal{K}_{g^*}$ and $\mathcal{K}_g$ respectively are nontrivial.*

Remark. The above homomorphisms $\zeta_{g^*}^w$ are also obtained from the action of diffeomorphisms of Σ_g on the moduli spaces of flat $O(2)$ -connections on Σ_g . More precisely, for nonzero class $w \in H^1(\Sigma_g, \mathbb{Z}_2)$, we regard it as a homomorphism from $\pi_1(\Sigma_g, *)$ to $\mathbb{Z}_2 \subset O(2)$ whose image is not contained in the identity component of $O(2)$. Then it defines a nontrivial flat $O(2)$ bundle over Σ_g . We consider the moduli space of flat $O(2)$ connections on this bundle modulo automorphisms of the bundle which preserve an orientation of the fiber at the base point. For any diffeomorphism of Σ_g which preserves the base point, orientation and the class w , by taking a lift of it to an automorphism of the above $O(2)$ -bundle which preserves an orientation of the fiber at the base point, we obtain the action of the diffeomorphism on the moduli space. This action is independent of the choice of a lift and a representative of an isotopy class of a diffeomorphism. It is easy to see that the moduli space can be identified with a $2(g-1)$ -dim torus $T^{2(g-1)}$. The above action induces the action of $\mathcal{M}_{g^*}^w$ on the first homology group of the moduli space, so we obtain a homomorphism from $\mathcal{M}_{g^*}^w$ to $\text{Aut}(H_1(T^{2(g-1)}, \mathbb{Z}))$. It is easy to check that this homomorphism agrees with $\zeta_{g^*}^w$.

3. SOME FUNCTIONS ON $\mathcal{M}_{2*}^w$ AND $\mathcal{M}_{2*}$

In this section we consider the case of $g = 2$ and construct some functions on mapping class groups with values in $\mathbb{Q}$.

Let w be a nonzero class of $H^1(\Sigma_g; \mathbb{Z}_2)$. Take an oriented basis for $H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w)/\text{torsion}$ and fix it. Then by Lemma 4, $H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w)/\text{torsion}$ is identified with $\mathbb{Z}^2$. Hence the action of $\mathcal{M}_{2*}^w$ on the cohomology group induces the homomorphism

$$\zeta_{2*}^w: \mathcal{M}_{2*}^w \rightarrow SL(2, \mathbb{Z}).$$

Next we introduce some functions on $SL(2, \mathbb{Z})$ and $PSL(2, \mathbb{Z})$ (see [1, 6, 9, 14]).

Let

$$\phi: PSL(2, \mathbb{Z}) \rightarrow \mathbb{Z}$$

be the Rademacher ϕ function which is defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \phi(A) = \begin{cases} \frac{b}{d} & \text{if } c = 0 \\ \frac{a+d}{c} - 12\text{sign}(c)s(a, c) & \text{if } c \neq 0, \end{cases}$$

where $A \in SL(2, \mathbb{Z})$ is a lift of $[A] \in PSL(2, \mathbb{Z})$ and

$$s(a, c) := \sum_{k=1}^{|c|-1} \left(\left(\frac{k}{c} \right) \right) \left(\left(\frac{ka}{c} \right) \right)$$

is the Dedekind sums for coprime integers a and c . Here $((x)) = 0$ if x is an integer, and $= x - [x] - 1/2$ if not.

Moreover we define the $\mathbb{Z}$ -valued function

$$\nu: SL(2, \mathbb{Z}) \rightarrow \mathbb{Z}$$

by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \nu(A) = \begin{cases} \text{sign}(b) & \text{if } A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, k \in \mathbb{Z}, \\ \text{sign}(c(a+d-2)) & \text{if not,} \end{cases}$$

and the $\mathbb{Q}$ -valued function

$$\psi: SL(2, \mathbb{Z}) \rightarrow \mathbb{Q}$$

by

$$A \mapsto \psi(A) = \frac{1}{3}\phi(A) - \nu(A).$$

It is known that ψ is a unique function on $SL(2, \mathbb{Z})$ with values in $\mathbb{Q}$ satisfying $\sigma = \delta\psi$, where

$$\sigma: SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \rightarrow \mathbb{Z}$$

is the signature 2-cocycle and δ is the coboundary operator.

We define the function

$$\Psi_*^w: \mathcal{M}_{2*}^w \rightarrow \mathbb{Q}$$

by

$$\Psi_*^w(f) = \psi(\zeta_{2*}^w(f)).$$

Since ψ is conjugacy invariant on $SL(2, \mathbb{Z})$, Ψ_*^w is independent of the choice of a basis for $H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w)/\text{torsion}$.

We also define the function

$$\Psi_*: \mathcal{M}_{2*} \rightarrow \mathbb{Q}$$

by

$$\Psi_*(f) = \frac{1}{15} \sum_{\substack{w \in H^1(\Sigma_2; \mathbb{Z}_2) \setminus \{0\} \\ f^*w = w}} \Psi_*^w(f).$$

The following lemma is clear from the conjugacy invariance of ψ .

Lemma 10. Ψ_*^w is $\mathcal{M}_{2*}^w$ -conjugacy invariant and Ψ_* is $\mathcal{M}_{2*}$ -conjugacy invariant.

4. THE SIGNATURE OF SURFACE BUNDLES

In this section we define the function μ in Theorem 3 and show some properties of it.

Let X be a compact oriented surface with a base point x_0 and possibly with boundaries.

Let $\pi: Z \rightarrow X$ be a Σ_g -bundle with a section $s: X \rightarrow Z$ where $g \geq 2$. Considering (the inverse of) the monodromy of it, we have the homomorphism

$$h: \pi_1(X, x_0) \rightarrow \mathcal{M}_{g*},$$

where we identify Σ_g with the fiber Z_{x_0} at the base point x_0 of X . Conversely if a homomorphism from $\pi_1(X, x_0)$ to $\mathcal{M}_{g*}$ is given, then we can construct a surface bundle with a section since $B\text{Diff}^+(\Sigma_g, *) \simeq K(\mathcal{M}_{g*}, 1)$ holds for $g \geq 2$ (see [10]).

Lemma 11. Let w be a nonzero class of $H^1(\Sigma_g; \mathbb{Z}_2)$. Under the above setting, suppose $\text{Im } h \subset \mathcal{M}_{g*}^w$, then there exists a unique class $w_Z \in H^1(Z; \mathbb{Z}_2)$ satisfying $w_Z|_{\Sigma_g} = w$ in $H^1(\Sigma_g, \mathbb{Z}_2)$ and $s^*w_Z = 0$ in $H^1(X; \mathbb{Z}_2)$.

Proof. We consider a spectral sequence $(E_r^{p,q}, d_r)$ for the cohomology $H^*(Z; \mathbb{Z}_2)$ with $E_2^{p,q} = H^p(X; H^q(\Sigma_g; \mathbb{Z}_2))$ of the fibration $\pi: Z \rightarrow X$. In this case, we have isomorphisms $E_2^{0,1} \cong H^1(\Sigma_g; \mathbb{Z}_2)^{\pi_1(X, x_0)}$ and $E_2^{1,0} = H^1(X; \mathbb{Z}_2)$. Clearly we have $E_2^{1,0} = E_3^{1,0} = \dots = E_\infty^{1,0}$. It is easy to see that $d_2: E_2^{0,1} \rightarrow E_2^{2,0}$ is 0 because of the existence of a section s of the fibration by the assumption. So we get $E_2^{0,1} = E_3^{0,1} = \dots = E_\infty^{0,1}$. Hence we obtain isomorphisms $H^1(Z; \mathbb{Z}_2) \cong E_\infty^{1,0} \oplus E_\infty^{0,1} = E_2^{1,0} \oplus E_2^{0,1} \cong H^1(X; \mathbb{Z}_2) \oplus H^1(\Sigma_g; \mathbb{Z}_2)^{\pi_1(X, x_0)}$. For the above splitting of $H^1(Z; \mathbb{Z}_2)$, the projection to the second factor agrees with the restriction and one to the first factor may be given by pullback of s . By the hypothesis, the class w is in $H^1(\Sigma_g; \mathbb{Z}_2)^{\pi_1(X, x_0)}$. This completes the proof. $\square$

Hereafter we assume the hypothesis of Lemma 11, that is $w \neq 0$ and $\text{Im } h \subset \mathcal{M}_{g*}^w$.

Let $\mathbb{C}_{w_Z}$ be the $\pi_1(Z, *)$ -module $\mathbb{Z}_{w_Z} \otimes \mathbb{C}$ over $\mathbb{C}$. It defines the flat complex line bundle over Z , which is also denoted by $\mathbb{C}_{w_Z}$, so the cohomology groups $H^*(Z; \mathbb{C}_{w_Z})$ and $H^*(Z, \partial Z; \mathbb{C}_{w_Z})$ are defined. Put $\hat{H}^*(Z; \mathbb{C}_{w_Z}) := \text{Im}[H^*(Z, \partial Z; \mathbb{C}_{w_Z}) \rightarrow H^*(Z; \mathbb{C}_{w_Z})]$. There is the nondegenerate hermitian form on $\hat{H}^2(Z; \mathbb{C}_{w_Z})$ which is defined by the cup-product, the inner product on $\mathbb{C}$ and evaluation on the fundamental cycle $[Z, \partial Z]$. Its signature is denoted by $\text{sign}_{w_Z}(Z)$.

On the other hand, from the surface bundle $Z \rightarrow X$ with a section s , we construct the flat vector bundle $\mathcal{H}$ over X with the hermitian form as follows.

The fiber $\mathcal{H}_x$ at $x \in X$ is the cohomology group

$$H^1(\pi_1(\pi^{-1}(x), s(x)); \mathbb{C}_{w_Z}|_{\pi^{-1}(x)}) \cong H^1(\Sigma_g; \mathbb{C}_w) \cong \mathbb{C}^{2(g-1)}.$$

The hermitian form is given by i times the cup-product, the inner product on $\mathbb{C}$ and evaluation on the fundamental class of the surface.

By the definition of $\mathcal{H}$, (the inverse of) the holonomy homomorphism of the flat bundle $\mathcal{H}$ is given by

$$\tau: \pi_1(X, x_0) \rightarrow Sp(2(g-1), \mathbb{Z}) \hookrightarrow U(g-1, g-1),$$

which agrees with the homomorphism $\pi_1(X, *) \ni \alpha \mapsto (h(\alpha)^{-1})^* \in Aut(H^1(\Sigma_g; \mathbb{C}_w), i \times (\cdot \cup \cdot))$. By considering $\mathcal{H}$ as a local coefficient system, we obtain the first cohomology group $H^1(X; \mathcal{H})$ with the skew-hermitian form which is also defined by the cup-product and the hermitian form of the bundle. The skew-hermitian form multiplied by i is a hermitian form on $H^1(X; \mathcal{H})$, so we get its signature, which is denoted by $sign(X; \mathcal{H})$.

Lemma 12. *Under the above setting, the equality $sign_{w_Z}(Z) = sign(X; \mathcal{H})$ holds.*

Proof. There exists a spectral sequence $(E_r^{p,q}, d_r)$ for the cohomology group $H^*(Z, \partial Z; \mathbb{C}_{w_Z})$ with $E_2^{p,q} \cong H^p(X, \partial X; H^q(\Sigma_g; \mathbb{C}_w))$ of the fibration $Z \rightarrow X$. Note that the cohomology group $H^p(X, \partial X; H^q(\Sigma_g; \mathbb{C}_w))$ depends on the section s , hence the isomorphism from $E_2^{p,q}$ to it does so, and note that $E_2^{1,1} = H^1(X, \partial X; H^1(\Sigma_g; \mathbb{C}_w)) = H^1(X, \partial X; \mathcal{H})$. Since we have $H^0(\Sigma_g; \mathbb{C}_w) = H^2(\Sigma_g; \mathbb{C}_w) = 0$, we obtain $E_2^{2,0} = E_2^{0,2} = 0$. Hence we get the isomorphisms

$$H^2(Z, \partial Z; \mathbb{C}_{w_Z}) \cong \sum_{p+q=2} E_\infty^{p,q} \cong E_2^{1,1} \cong H^1(X, \partial X; \mathcal{H}).$$

Similarly we obtain isomorphisms $H^2(Z; \mathbb{C}_{w_Z}) \cong H^1(X; \mathcal{H})$ and $H^2(X, \partial X; H^2(\Sigma_g; \mathbb{C})) \cong H^4(Z, \partial Z; \mathbb{C})$. Moreover we have the following commutative diagrams;

$$\begin{array}{ccc} H^2(Z, \partial Z; \mathbb{C}_{w_Z}) & \xrightarrow{\quad} & H^2(Z; \mathbb{C}_{w_Z}) \\ \parallel & & \parallel \\ H^1(X, \partial X; \mathcal{H}) & \xrightarrow{\quad} & H^1(X; \mathcal{H}), \\ H^2(Z, \partial Z; \mathbb{C}_{w_Z}) \times H^2(Z; \mathbb{C}_{w_Z}) & \xrightarrow{\cup} & H^4(Z, \partial Z; \mathbb{C}) \\ \parallel & & \parallel \end{array}$$

$$H^1(X, \partial X; \mathcal{H}) \times H^1(X; \mathcal{H}) \xrightarrow{-(\cup)} H^2(X, \partial X; H^2(\Sigma_g; \mathbb{C})) \cong H^2(X, \partial X; \mathbb{C}),$$

where $-(\cup)$ on the lower arrow denotes minus the cup product on cohomology with local coefficient system. In this case $-(\cup)$ agrees with the cup product in the spectral sequence. Taking the two times multiple by i in the definition of $sign(X; \mathcal{H})$ into consideration, we obtain the equality $sign_{w_Z}(Z) = sign(X; \mathcal{H})$. $\square$

Put $P := S^2 \setminus \text{Int} D^2$, then $\pi_1(P, *)$ is a rank 2 free group whose generators are given by α and β .

Any homomorphism ξ from $\pi_1(P, *)$ to $U(p, q)$ defines a flat vector bundle E_ξ over P with a hermitian form. In the same way as above, we have the signature $sign(P, E_\xi)$ of the hermitian form on $\hat{H}^1(P; E_\xi) := Im[H^1(P, \partial P; E_\xi) \rightarrow H^1(P; E_\xi)]$. The flat bundle

E_ξ depends only on the homomorphism ξ , hence on the two element $(\xi(\alpha), \xi(\beta)) = (A, B)$ of $U(p, q)$. So we can put $\text{sign}(A, B) := \text{sign}(P, E_\xi)$ (see [1]).

Now we shall consider the case of $g = 2$.

Since the restriction of sign to $SL(2, \mathbb{Z}) \hookrightarrow U(1, 1)$ agrees with the signature 2-cocycle σ (see [1]), we have $\text{sign} = \sigma = \delta\psi$. Thus we obtain

$$\text{sign}(A, B) = \psi(B) - \psi(AB) + \psi(A).$$

Let X be a compact oriented surface with boundary $\partial X = \coprod_i S_i^1$. Let $\xi': \pi_1(X, x_0) \rightarrow SL(2, \mathbb{Z})$ be a homomorphism and ξ the composition of ξ' with $SL(2, \mathbb{Z}) \hookrightarrow U(1, 1)$, then by the signature additivity, we have

$$\text{sign}(X, E_\xi) = \sum_i \psi(\xi'(S_i^1)),$$

where S_i^1 also denotes a class in $\pi_1(X, x_0)$ corresponding to the boundary S_i^1 which is determined up to conjugation. Here we note that, since ψ is conjugacy invariant on $SL(2, \mathbb{Z})$, the right hand side of the above equality is well-defined.

Lemma 13.

$$\text{sign}_{w_Z}(Z) = \sum_i \Psi_*^w(h(S_i^1)).$$

Proof. The flat bundle $\mathcal{H}$ in Lemma 12 is constructed from the representation $\tau = \xi: \pi_1(X, x_0) \rightarrow SL(2, \mathbb{Z}) \hookrightarrow U(1, 1)$, so we get $\mathcal{H} = E_\xi$. Hence we have

$$\text{sign}_{w_Z}(Z) = \text{sign}(X, \mathcal{H}) = \text{sign}(X, E_\xi) = \sum_i \psi(\xi'(S_i^1)).$$

Since $\xi' = \zeta_{2*}^w h$ and $\Psi_*^w = \psi \circ \zeta_{2*}^w$ holds by definitions, we obtain

$$\psi(\xi'(S_i^1)) = \psi(\zeta_{2*}^w h(S_i^1)) = \Psi_*^w h(S_i^1),$$

hence $\text{sign}_{w_Z}(Z) = \sum_i \Psi_*^w(h(S_i^1))$. This completes the proof. $\square$

Now we recall the definition of the ρ -invariants (see [2]).

Let M be an oriented Riemannian manifold of dimension $2l - 1$ and $\alpha: \pi_1(M) \rightarrow U(n)$ a unitary representation. The self-adjoint operator on even forms on M ;

$$D: \Omega^{\text{even}}(M; \mathbb{C}) \rightarrow \Omega^{\text{even}}(M; \mathbb{C})$$

is defined by

$$D(\phi) = i^l (-1)^{p+1} (*d - d*)\phi,$$

where ϕ is a form of degree $2p$. Moreover the extension of D to the even forms with coefficients in the flat vector bundle of rank n defined by α is denoted by D_α , which is also a self-adjoint operator.

For the operator D_α , we define the function $\eta_\alpha(s)$ by

$$\eta_\alpha(s) = \sum_{\lambda \neq 0} (\text{sign} \lambda) |\lambda|^{-s},$$

where λ runs over the eigenvalues of D_α . The corresponding function to D is denoted by $\eta(s)$. These functions extend at $s = 0$ and have finite values here. Its values $\eta(0)$ and $\eta_\alpha(0)$ are called the η -invariants of a Riemannian manifold.

Put $\tilde{\eta}_\alpha(s) := \eta_\alpha(s) - n\eta(s)$.

Theorem 14 (Atiyah-Patodi-Singer [2]). *$\tilde{\eta}_\alpha(0)$ is independent of the metric. It is a diffeomorphism invariant of M and α which we shall denote by $\rho_\alpha(M)$. If $M = \partial N$ with α extending to a unitary representation of $\pi_1(N)$ then*

$$\rho_\alpha(M) = n \operatorname{sign}(N) - \operatorname{sign}_\alpha(N).$$

Now we return to our setting. Let $\partial Z = \coprod_i \partial_i Z$ be the decomposition of the boundary of Z by the connected components. Then $\partial_i Z$ is the Σ_2 bundle over S_i^1 with (the inverse of) a monodromy $h(S_i^1) \in \mathcal{M}_{2*}^w$ where S_i^1 is a connected component of the boundary $\partial X = \coprod_i S_i^1$. In Theorem 14, we put $N = Z$, $M = \partial Z$ and $\alpha = w_Z|_{\partial Z}$. Then using Lemma 13 we obtain

$$\begin{aligned} \operatorname{sign}(Z) &= \sum_i \rho_{w_{\partial_i Z}}(\partial_i Z) + \operatorname{sign}_{w_Z}(Z) \\ &= \sum_i \{\rho_{w_{\partial_i Z}}(\partial_i Z) + \Psi_*^w(h(S_i^1))\}. \end{aligned}$$

For an orientation and base point preserving diffeomorphism f of Σ_2 or its isotopy class, let M_f denote the Σ_2 bundle over S^1 with a monodromy f^{-1} .

For a nonzero class $w \in H^1(\Sigma_2; \mathbb{Z}_2)$, the function

$$\mu_*^w: \mathcal{M}_{2*}^w \rightarrow \mathbb{Q}$$

is defined by

$$\mu_*^w(f) = \rho_{w_{M_f}}(M_f) + \Psi_*^w(f)$$

for $f \in \mathcal{M}_{2*}^w$. Then we can rewrite the above equality as follows;

$$\operatorname{sign}(Z) = \sum_i \mu_*^w(h(S_i^1)).$$

Since the signature of a surface bundle Z_h over a compact surface X is determined by the action of the corresponding monodromy h^{-1} on the first cohomology group of the fiber Σ_2 , the equality $\operatorname{sign}(Z_h) = 0$ holds for any homomorphism $h: \pi_1(X, x_0) \rightarrow \mathcal{J}_{2*}$. This and the above equality imply that the restriction of μ_*^w to $\mathcal{J}_{2*}$ is a homomorphism.

Let $\mu_*: \mathcal{M}_{2*} \rightarrow \mathbb{Q}$ be the function defined by

$$\mu_*(f) = \frac{1}{15} \sum_{\substack{w \in H^1(\Sigma_2; \mathbb{Z}_2) \setminus \{0\} \\ f^* w = w}} \mu_*^w(f)$$

for $f \in \mathcal{M}_{2*}$. The above argument and the example in section 5 imply the following proposition.

Proposition 15. *The restrictions of μ_* and μ_*^w ($w \neq 0$) to $\mathcal{J}_{2*}$ are nontrivial homomorphisms.*

Next we shall prove that, for any nonzero class w in $H^1(\Sigma_2; \mathbb{Z}_w)$, the map $\mu_*^w: \mathcal{M}_{2*}^w \rightarrow \mathbb{Q}$ descends to the map

$$\mu^w: \mathcal{M}_2^w \rightarrow \mathbb{Q}.$$

For any $a, b \in \mathcal{M}_{2*}$, let $\text{sign}(a, b)$ be the signature $\text{sign}(Z_h)$ of the surface bundle Z_h over P constructed from the homomorphism $h: \pi_1(P, *) \rightarrow \mathcal{M}_{2*}$ defined by $\alpha \mapsto a$ and $\beta \mapsto b$. Then we have $\text{sign}(a, b) = \text{sign}(A, B)$, where $A, B \in U(2, 2)$ is the image of a, b by the obvious homomorphism $\mathcal{M}_{2*} \rightarrow \text{Aut}(H^1(\Sigma_2; \mathbb{Z}), \cup) \rightarrow \text{Sp}(4, \mathbb{Z}) \rightarrow U(2, 2)$, where we used a fixed basis of $H^1(\Sigma_2; \mathbb{Z})$.

Note that $\text{sign}(A, B) = 0$ if $A = 1$, $B = 1$ or $AB = 1$ (see [1]). Hence we have $\text{sign}(a, b) = 0$ if $a \in \pi_1(\Sigma_2, *)$, $b \in \pi_1(\Sigma_2, *)$ or $ab \in \pi_1(\Sigma_2, *)$.

The following lemma is easy or have already been proven.

Lemma 16. *For any nonzero class $w \in H^1(\Sigma_2; \mathbb{Z}_w)$, the map μ_*^w satisfies the following properties.*

1. $\mu_*^w(1) = 0$,
2. $\mu_*^w(a^{-1}) = -\mu_*^w(a)$,
3. $\mu_*^{(f^{-1})^*w}(faf^{-1}) = \mu_*^w(a)$,
4. $\text{sign}(a, b) = \mu_*^w(b) - \mu_*^w(ab) + \mu_*^w(a)$,

where $a, b \in \mathcal{M}_{2*}^w$ and $f \in \mathcal{M}_{2*}$.

Lemma 17. *For any $\alpha \in \pi_1(\Sigma_g, *)$ and $f \in \mathcal{M}_{2*}^w$, the followings hold.*

1. $\mu_*^w(\alpha) = 0$,
2. $\mu_*^w(\alpha f) = \mu_*^w(f\alpha) = \mu_*^w(f)$.

Proof. 1. Since we have $M_{\alpha^2} \cong \Sigma_2 \times S^1 = \partial(\Sigma_2 \times D^2)$, there exists $\tilde{w} \in H^1(\Sigma_2 \times D^2; \mathbb{Z}_2)$ satisfying $\tilde{w}|_{\partial(\Sigma_2 \times D^2)} = w_{M_{\alpha^2}}$. By Theorem 14 and $H^2(\Sigma_2 \times D^2; \mathbb{C}_{\tilde{w}}) = 0$, we have

$$\rho_{w_{M_{\alpha^2}}}(M_{\alpha^2}) = \text{sign}(\Sigma_2 \times D^2) - \text{sign}_{\tilde{w}}(\Sigma_2 \times D^2) = 0.$$

By Lemma 7, we have

$$\Psi_*^w(\alpha^2) = \psi(\zeta_{2*}^w(\alpha^2)) = \psi(1) = 0.$$

So we obtain $\mu_*^w(\alpha^2) = \rho_{w_{M_{\alpha^2}}}(M_{\alpha^2}) + \Psi_*^w(\alpha^2) = 0$. Since μ_*^w is a homomorphism on $\pi_1(\Sigma_2, *)$ by 4 of Lemma 16, we have $\mu_*^w(\alpha) = 0$.

2. $\text{sign}(f, \alpha) = 0$, 4 of Lemma 16 and 1 of this lemma imply 2. □

Corollary 18. *The functions μ_*^w and μ_* descend to the functions μ^w and μ on $\mathcal{M}_2^w$ and $\mathcal{M}_2$ respectively. For the functions μ^w and μ , the similar properties to Lemma 16 hold except for 4 for μ , but the restriction of μ to $\mathcal{M}_2^H$ satisfies the corresponding property to 4 of Lemma 16. Their restrictions to $\mathcal{J}_2$ are nontrivial homomorphisms.*

It is well-known that sign defines a 2-cocycle over $\mathbb{Z}$ on $\mathcal{M}_2$, which is called the signature cocycle, and that it is a coboundary over $\mathbb{Q}$. By the fact that $H^1(\mathcal{M}_2, \mathbb{Q}) = 0$, there exists a unique function, which is called Meyer function,

$$\phi: \mathcal{M}_2 \rightarrow \mathbb{Q}$$

such that $\text{sign}(a, b) = \phi(b) - \phi(ab) + \phi(a)$. It is known that ϕ satisfies the properties except for 4 in Lemma 16 on $\mathcal{M}_2$ and its image is in $\frac{1}{5}\mathbb{Z}$ (see [8, 9]).

Proposition 19. *On the Torelli group $\mathcal{J}_2$, the function μ agrees with the Meyer function ϕ .*

Proof. The Torelli group $\mathcal{J}_2$ is normally generated in $\mathcal{M}_2$ by the Dehn twist along a separating simple closed curve on Σ_2 (see [4]). Such a Dehn twist is given by the diffeomorphism f in the example in section 5. The functions μ and ϕ are conjugacy invariant on $\mathcal{M}_2$ and are homomorphisms on $\mathcal{J}_2$. Thus in order to prove this proposition, we have only to show the equality $\mu(f) = \phi(f)$, but it is true from Corollary 3.7 in [8] and the example in section 5. $\square$

5. AN EXAMPLE

In this section we shall give an example.

Let α_i, β_i ($i = 1, 2$) be generators for the fundamental group $\pi_1(\Sigma_2, *)$ of Σ_2 depicted in Figure 2.

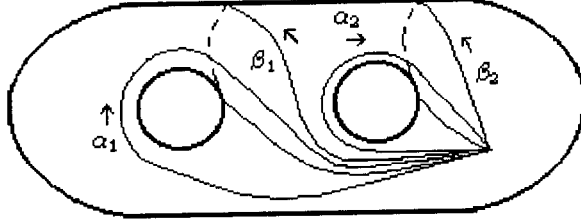


FIGURE 2. Generators of $\pi_1(\Sigma_2, *)$.

We use the same letters α_i, β_i for the corresponding basis for $H_1(\Sigma_2; \mathbb{Z}_2)$.

Take a class $w \in H^1(\Sigma_2; \mathbb{Z}_2)$ and fix it. We assume $w(\alpha_1) = 1$.

We can identify the space $Z^1 := Z^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w)$ of w -crossed homomorphisms from $\pi_1(\Sigma_2, *)$ to $\mathbb{Z}$ with the subset $\{(x_i, y_i)_{i=1}^2 \in \mathbb{Z}^4 \mid \sum_{i=1}^2 \{w(\beta_i)x_i - w(\alpha_i)y_i\} = 0\}$ of $\mathbb{Z}^4$ by the map $\iota: u \mapsto (x_1, y_1, x_2, y_2) := (u(\alpha_1), u(\beta_1), u(\alpha_2), u(\beta_2))$. We give elements E, F and W of Z^1 by $\iota E = (0, -w(\alpha_2), 0, 1)$, $\iota F = (0, w(\beta_2), 1, 0)$ and $\iota W = (w(\alpha_1), w(\beta_1), w(\alpha_2), w(\beta_2))$. It is easy to check that these elements form a basis for $Z^1 \cong \mathbb{Z}^3$ and $2W$ is a basis for the space $B^1 := B^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w) \cong 2\mathbb{Z}$ of the principal w -crossed homomorphisms. Thus we obtain

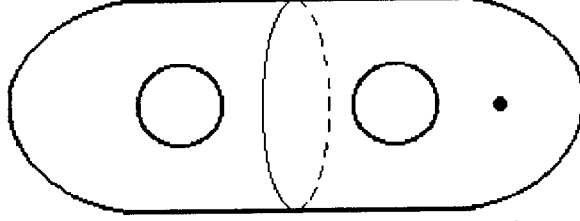
$$H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w) \cong \mathbb{Z}E \oplus \mathbb{Z}F \oplus \mathbb{Z}_2W.$$

Let $f: \Sigma_2 \rightarrow \Sigma_2$ be the base point preserving diffeomorphism of Σ_2 which is the positive Dehn twist along the loop in Figure 3, hence f belongs to $\mathcal{K}_{2+}$.

The action of f on the fundamental group of Σ_2 is given by

$$f_*: \pi_1(\Sigma_2, *) \rightarrow \pi_1(\Sigma_2, *)$$

$$\begin{cases} \alpha_1 \mapsto l\alpha_1l^{-1} \\ \beta_1 \mapsto l\beta_1l^{-1} \\ \alpha_2 \mapsto \alpha_2 \\ \beta_2 \mapsto \beta_2, \end{cases}$$

FIGURE 3. The loop defining f .

where $l = [\beta_1, \alpha_1]$.

A direct computation shows the following equalities.

$$\begin{aligned} f^* E &= (1 + 4w(\alpha_2)w(\beta_2))E + 4w(\alpha_2)^2 F - 4w(\alpha_2)W, \\ f^* F &= -4w(\beta_2)^2 E + (1 - 4w(\beta_2)w(\alpha_2))F + 4w(\beta_2)W, \\ f^* W &= W. \end{aligned}$$

Then the representation matrix of the homomorphism

$$f^* : H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w)/\text{torsion} \rightarrow H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_w)/\text{torsion}$$

with respect to the basis E, F is given by

$$\begin{pmatrix} 1 + 4w(\alpha_2)w(\beta_2) & -4w(\beta_2)^2 \\ 4w(\alpha_2)^2 & 1 - 4w(\beta_2)w(\alpha_2) \end{pmatrix}.$$

For $w \in H^1(\Sigma_2, \mathbb{Z}_2)$ such that $(w(\alpha_2), w(\beta_2)) = (0, 0), (0, 1), (1, 0), (1, 1)$, the representation matrices of $(f^{-1})^*$, which are $\zeta_{2*}^w(f)$ by the definition, are obtained as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix}$$

respectively. Here we note that, the basis E, F is oriented in the case of $(w(\alpha_2), w(\beta_2)) = (0, 1), (1, 0), (1, 1)$ and is not in the case of $(w(\alpha_2), w(\beta_2)) = (0, 0)$. But in the latter case, since the representation matrix of $(f^{-1})^*$ is identity, the one with respect to an oriented basis is also identity. Thus we have eight matrices in all in the case of $w(\alpha_1) = 1$.

Similarly in the case of $w(\alpha_1) = 0$, under appropriate choices of oriented bases, we obtain the same representation matrices of $(f^{-1})^*$ as above, but the number of the matrices are 4, 1, 1 and 1 respectively. Hence we obtain seven matrices in all.

Since we have

$$\psi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0, \quad \psi \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \psi \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} = \psi \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix} = \frac{1}{3},$$

we obtain

$$\Psi_*^w(f) = \begin{cases} 0 & \text{if } \zeta_{2*}^w(f) = id \\ \frac{1}{3} & \text{if not.} \end{cases}$$

Next we shall compute $\mu_*^w(f)$.

Let w be a class of $H^1(\Sigma_2; \mathbb{Z}_2)$ satisfying $(w(\alpha_1), w(\alpha_2), w(\beta_2)) = (1, 0, 1)$. Then we have the representation matrix

$$(f^{-n})^* = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix},$$

with respect to the above basis. Hence we obtain

$$\Psi_*^w(f^n) = \frac{1}{3}\phi \begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix} - \nu \begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix} = \frac{4}{3}n - \text{sgn}(n).$$

Lemma 20. *For any $w \in H^1(\Sigma_2; \mathbb{Z}_2)$, the set $\{\rho_{w(M_{f^n})}(M_{f^n}) | n \in \mathbb{Z}\}$ is bounded.*

Proof. We only prove the case of the class w given by $(w(\alpha_1), w(\beta_1), w(\alpha_2), w(\beta_2)) = (1, 0, 1, 0)$, since the other cases are shown in the same way.

Let H_2 be a handlebody of genus 2 with boundary Σ_2 such that the loop in Figure 3 is a boundary of an embedded disk. The diffeomorphisms f^n ($n \in \mathbb{Z}$) of $\Sigma_2 = \partial H_2$ extend to diffeomorphisms of H_2 . Let $H_{2,f^n} = H_2 \times [0, 1] / H_2 \times 1 \xrightarrow{f^{-n}} H_2 \times 0$ be the mapping torus, then it is a compact 4-manifold with a boundary $\partial H_{2,f^n} = M_{f^n}$. Clearly there exists a class $w(n) \in H^1(H_{2,f^n}; \mathbb{Z}_2)$ whose restriction to the boundary M_{f^n} is $w_{M_{f^n}}$. In order to prove this lemma, by Theorem 14 and the definition of signature, we have only to show that the set $\{\dim_{\mathbb{Q}} H^2(H_{2,f^n}; \mathbb{Q}), \dim_{\mathbb{C}} H^2(H_{2,f^n}; \mathbb{C}_{w(n)})\}$ is bounded. But it is true since the manifolds H_{2,f^n} are homotopic to each other, so the proof of this lemma is finished. $\square$

Since f^n belongs to $\mathcal{J}_{2*}$ and μ_*^w is a homomorphism on $\mathcal{J}_{2*}$, we have

$$\mu_*^w(f^n) = n\mu_*^w(f) = n(\rho_{w_{M_f}}(M_f) + \Psi_*^w(f)) = n(\rho_{w_{M_f}}(M_f) + \frac{1}{3}).$$

On the other hand, we have

$$\mu_*^w(f^n) = \rho_{w_{M_{f^n}}}(M_{f^n}) + \Psi_*^w(f^n) = \rho_{w_{M_{f^n}}}(M_{f^n}) + \frac{4}{3}n - \text{sgn}(n).$$

We consider $n \rightarrow \infty$. By Lemma 20, we obtain $\rho_{w_{M_f}}(M_f) = 1$, hence $\mu_*^w(f^n) = \frac{4}{3}n$ and $\rho_{w_{M_{f^n}}}(M_{f^n}) = \text{sgn}(n)$.

Similarly, for any $w \in H^1(\Sigma_2; \mathbb{Z}_2)$ such that the representation matrix of $(f^{-1})^*$ is not the identity, we obtain $\mu_*^w(f^n) = \frac{4}{3}n$. As a result, we have $\mu_*^w(f^n) = \frac{4}{3}n$ for the nine of the fifteen nonzero classes of $H^1(\Sigma_2; \mathbb{Z}_2)$ and $\mu_*^w(f^n) = 0$ for the other classes. Finally we obtain

$$\mu_*(f^n) = \frac{1}{15} \sum_{\substack{w \in H^1(\Sigma_2; \mathbb{Z}_2) \setminus \{0\} \\ (f^n)^* w = w}} \mu_*^w(f^n) = \frac{1}{15} \cdot 9 \cdot \frac{4}{3}n = \frac{4}{5}n.$$

By Corollary 18, the functions μ^w and μ take the same values at f^n as μ_*^w and μ_* do respectively.

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Chapter 2

Some examples of lagrangian submanifolds of the
moduli space of flat connections

SOME EXAMPLES OF LAGRANGIAN SUBMANIFOLDS OF THE MODULI SPACE OF FLAT CONNECTIONS

RYOJI KASAGAWA

ABSTRACT. We will construct monotone Lagrangian submanifolds of a well-known finite ramified covering of the moduli space of flat connections on the nontrivial $SO(3)$ -bundle over a closed oriented surface. Moreover we will give some examples of the Floer cohomology group of Lagrangian intersections for a pair of these Lagrangian submanifolds.

1. INTRODUCTION

Let Σ_g be a closed oriented surface of genus $g \geq 2$. The moduli space of representations of the fundamental group $\pi_1(\Sigma_g, *)$ of Σ_g to $SU(2)$:

$$\mathcal{R}(\Sigma_g) := \text{Hom}(\pi_1(\Sigma_g, *), SU(2)) / \text{Ad}SU(2)$$

is identified with the moduli space of flat $SU(2)$ -connections of the trivial $SU(2)$ -bundle over Σ_g . This is a $6g - 6$ dimensional symplectic manifold with singularities (see [6, 10]). Let H_g be an oriented handlebody of genus g whose boundary is identified with Σ_g . The inclusion $\Sigma_g = \partial H_g \hookrightarrow H_g$ induces an inclusion

$$\mathcal{R}(H_g) := \text{Hom}(\pi_1(H_g, *), SU(2)) / \text{Ad}SU(2) \hookrightarrow \mathcal{R}(\Sigma_g),$$

which is a Lagrangian embedding to the complement of the singular set, hence its image is a Lagrangian submanifold with singularities. If a Heegaard decomposition of a closed oriented 3-manifold is given, then we have a pair of Lagrangian submanifolds with singularities.

In this paper, we will replace the trivial $SU(2)$ bundle over Σ_g by the nontrivial $SO(3)$ -bundle over Σ_g , then we obtain a well-known smooth symplectic manifold, which is a finite ramified covering of the moduli space of flat $SO(3)$ connections on the nontrivial $SO(3)$ -bundle over Σ_g (see [1, 2, 4]). Moreover by replacing H_g by the complement of embedded curves in H_g , we can construct monotone Lagrangian submanifolds without singularities of the above symplectic manifold. Moreover it is found that, for some pairs of these Lagrangian submanifolds, Floer cohomology groups of Lagrangian intersections are well-defined (see [5, 8]). We will give some examples of these groups.

2. MONOTONE LAGRANGIAN SUBMANIFOLDS

In this section we recall the definition of monotone Lagrangian submanifolds (see [8]).

Let (M, ω) be a closed symplectic manifold. By choosing an almost complex structure J on (M, ω) such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric, we obtain the first Chern class $c_1(M)$ in $H^2(M; \mathbb{Z})$ which is independent of J . The evaluation of ω and $c_1(M)$ on the second homotopy group $\pi_2(M)$ of M defines two homomorphisms

$$I_\omega, I_c : \pi_2(M) \rightarrow \mathbb{R}, \mathbb{Z} (\subset \mathbb{R})$$

respectively.

Definition 1. A symplectic manifold (M, ω) is called monotone if two homomorphisms I_ω and I_c satisfy

$$I_c = \alpha I_\omega,$$

for some $\alpha > 0$.

For any Lagrangian submanifold L of M , we define two homomorphisms I_ω and $I_{\mu,L}$ on $\pi_2(M, L)$ as follows. Let $w : (D^2, \partial D^2) \rightarrow (M, L)$ be a smooth map of pairs, then the homomorphism I_ω is given by

$$I_\omega(w) = \int_{D^2} w^* \omega.$$

In order to define the homomorphism $I_{\mu,L}$, we prepare some notations. Let α be a map from S^1 to $L \subset M$. There is a trivialization $\Phi : \alpha^* TM \cong S^1 \times \mathbb{C}^n$ of pullback bundle $\alpha^* TM$ as a symplectic vector bundle. Using this trivialization Φ , we define a map α_Φ from S^1 to the space $\Lambda(\mathbb{C}^n)$ of Lagrangian subspaces of $\mathbb{C}^n$ by

$$\begin{aligned} \alpha_\Phi : S^1 &\rightarrow \Lambda(\mathbb{C}^n) \\ z &\mapsto \Phi(T_{\alpha(z)} L). \end{aligned}$$

We denote $\alpha_\Phi^* \mu[S^1] \in \mathbb{Z}$ by $\mu_\Phi(\alpha)$, where $\mu \in H^1(\Lambda(\mathbb{C}^n); \mathbb{Z})$ is the Maslov class.

Using these notations, we define $I_{\mu,L}$ as follows. There is a unique trivialization $\Phi : w^* TM \cong D^2 \times \mathbb{C}^n$ up to homotopy as a symplectic vector bundle. By restricting Φ to ∂D^2 , we obtain a trivialization $(\partial w)^* TM \cong \partial D^2 \times \mathbb{C}^n$ which is denoted by the same letter Φ . Here ∂w denotes the restricted map $w|_{\partial D^2} : \partial D^2 \rightarrow L \subset M$. Then $I_{\mu,L}$ is given by

$$I_{\mu,L}(w) = \mu_\Phi(\partial w).$$

$I_{\mu,L}$ defines a homomorphism on $\pi_2(M, L)$ (see [8]).

Definition 2. A Lagrangian submanifold L of M is called monotone if two homomorphisms $I_{\mu,L}$ and I_ω satisfy

$$I_{\mu,L} = \lambda I_\omega,$$

for some $\lambda > 0$.

Remark 3 (cf. [8]). 1. Let $a \in \pi_2(M, L)$ denote the image of $a' \in \pi_2(M)$ by the natural homomorphism $\pi_2(M) \rightarrow \pi_2(M, L)$. Then we have

$$\begin{aligned} I_\omega(a) &= I_\omega(a') \\ I_{\mu,L}(a) &= 2I_c(a'). \end{aligned}$$

Therefore we can deduce that if there is a monotone Lagrangian submanifold of M , then M is a monotone symplectic manifold.

2. Let (M, ω) be a monotone symplectic manifold. If M is simply connected, then any Lagrangian submanifold with a torsion fundamental group is monotone.

For a pair of monotone Lagrangian submanifolds, we can define the Floer cohomology group of Lagrangian intersections. More precisely, let $\Sigma(L)$ be the positive generator of the subgroup $Im I_{\mu,L}$ of $\mathbb{Z}$, where L is a monotone Lagrangian submanifold. Let L_0 and

L_1 be monotone Lagrangian submanifolds of a symplectic manifold M . Suppose that $\Sigma(L_i) \geq 3$ for $i = 0, 1$, then the Floer cohomology group of Lagrangian intersections $I^*(L_0, L_1; M)$ can be defined (see [8]). Moreover if $\Sigma(L_0)$ and $\Sigma(L_1)$ are even, then this group has even grading, hence the Euler number of it.

3. FLAT CONNECTIONS OVER A CLOSED ORIENTED SURFACE (cf.[4])

In this section we recall some facts about the moduli space of flat connections on the nontrivial $SO(3)$ -bundle over a closed oriented surface.

Let $p : P \rightarrow \Sigma_g$ be a nontrivial principal $SO(3)$ -bundle over a closed oriented surface of genus $g \geq 2$, which is unique up to isomorphisms. Let $\mathcal{A}_{flat}(P)$ be the space of flat connections on P and $\mathcal{G}_0(P)$ be the identity component of the gauge group $\mathcal{G}(P)$ of P . The moduli space

$$\mathcal{M}(P) = \mathcal{A}_{flat}(P) / \mathcal{G}_0(P)$$

is a closed, simply connected, symplectic manifold of dimension $6g - 6$ with symplectic structure

$$\omega_A(a, b) = - \int_{\Sigma_g} \text{tr}(a \wedge b),$$

for $a, b \in H^1(\Sigma_g, so(3)_{Ad_A}) = T_A \mathcal{M}(P)$. Moreover $(\mathcal{M}(P), \omega)$ is a monotone symplectic manifold. In fact the equality $I_c = \frac{1}{4\pi^2} I_\omega$ holds. Moreover we have $Im I_c = 2\mathbb{Z}$ (see [2, 3]). Any automorphism $\bar{f} : P \rightarrow P$ of the bundle P which descends to an orientation preserving diffeomorphism of Σ_g induces a symplectomorphism $\bar{f}^* : \mathcal{M}(P) \rightarrow \mathcal{M}(P)$ by the pullback of the connections. Via the holonomy, $\mathcal{M}(P)$ can be identified with the moduli space

$$\mathcal{R}(P) = Hom^{odd}(\pi_1(P), SU(2)) / Ad SU(2)$$

of odd representations to $SU(2)$ of the fundamental group $\pi_1(P)$ of P . Here a homomorphism $\rho : \pi_1(P) \rightarrow SU(2)$ is called odd if $\rho(\varepsilon) = -I$, where ε is the class of the non-trivial loop in the fiber. Moreover a tangent space $T_A \mathcal{M}(P)$ can be identified with $T_\rho \mathcal{R}(P) = H^1(\pi_1(P), su(2)_{Ad\rho}) \cong H^1(\pi_1(\Sigma_g), so(3)_{Ad\rho})$, where ρ ($\underline{\rho}$ resp.) is a representation of $\pi_1(P)$ ($\pi_1(\Sigma_g)$ resp.) to $SU(2)$ ($SO(3)$ resp.) corresponding to the connection A . Under this identification, the symplectic structure ω (at ρ) is defined by the cup-product, the trace on $so(3)$ and the evaluation on the fundamental class $[\Sigma_g]$ of Σ_g :

$$\omega_\rho : H^1(\pi_1(\Sigma_g), so(3)_{Ad\rho}) \times H^1(\pi_1(\Sigma_g), so(3)_{Ad\rho}) \xrightarrow{-\text{tr}(\cup)} H^2(\pi_1(\Sigma_g); \mathbb{R}) \xrightarrow{< \cdot, [\Sigma_g] >} \mathbb{R}.$$

4. FLAT CONNECTIONS OVER $H_g(\kappa)$

In this section we construct the manifolds which turn out to be Lagrangian submanifolds of $\mathcal{R}(P)$ in the next section.

Let H_g be an oriented handlebody of genus $g \geq 2$. Its boundary $\partial H_g = \Sigma_g$ is a closed oriented surface of genus g . Let κ be a union of mutually disjoint simple closed curves $\kappa_1, \dots, \kappa_n$ on Σ_g . Let $H_g(\kappa)$ be the complement of the union of open tubular neighborhoods of embedded curves in $int H_g$ which are parallel to $\kappa_1, \dots, \kappa_n$. We denote longitude and meridian curves in 2-dimensional torus T^2 by l and m respectively. Let T_i^2, l_i and m_i be copies of T^2, l and m respectively for $i = 1, \dots, n$.

Lemma 4. *There is a homotopy equivalence $h: H_g \bigcup_{\tilde{\kappa}} \prod_{i=1}^n T_i^2 \xrightarrow{\sim} H_g(\kappa)$, where $\tilde{\kappa}: \prod_{i=1}^n l_i \xrightarrow{\cong} \kappa \subset H_g$ is an attaching map which identifies l_i with κ_i for each i . Moreover $H_g(\kappa)$ is a $K(\pi, 1)$ -space if there exists a free basis of $\pi_1(H_g, *)$ which contains $\kappa_1, \dots, \kappa_n$.*

Remark 5. 1. We denote the class of κ_i in $\pi_1(H_g, *)$ by the same letter κ_i .
 2. When we consider κ_i as a class in $\pi_1(H_g, *)$, we move the curve κ_i by a homotopy in advance so that it passes through the base point $*$ of H_g .

Proof. It is obvious that $H_g \bigcup_{\tilde{\kappa}} \prod_{i=1}^n T_i^2$ is homotopy equivalent to $H_g(\kappa)$. We prove the latter. By the hypothesis, there is a free basis $\kappa_1, \dots, \kappa_g \in \pi_1(H_g, *)$ which contains $\kappa_1, \dots, \kappa_n$. We consider a map from $\bigvee_{i=1}^g S_i^1$ to H_g such that the class of the image of S_i^1 in $\pi_1(H_g)$ is κ_i for each i . By Whitehead's theorem, this map is homotopy equivalence one. Using it, we can construct a homotopy equivalence map from $H_g \bigcup_{\tilde{\kappa}} \prod_{i=1}^n T_i^2$ to $\bigvee_{i=1}^n T_i^2 \bigvee_{i=n+1}^g S_i^1$. It is easy to see that the universal covering space of $\bigvee_{i=1}^n T_i^2 \bigvee_{i=n+1}^g S_i^1$ is contractible. Hence $\bigvee_{i=1}^n T_i^2 \bigvee_{i=n+1}^g S_i^1$ is a $K(\pi, 1)$ -space, and so is $H_g(\kappa)$. $\square$

The following lemma is a well-known fact about the principal $SO(3)$ -bundles over a CW-complex of dimension ≤ 3 , so we omit the proof.

Lemma 6. *There is a one-to-one correspondence between the isomorphism classes of principal $SO(3)$ -bundles over a CW-complex C of dimension ≤ 3 and $H^2(C; \mathbb{Z}_2)$. This correspondence is given by taking the second Whitney class of an $SO(3)$ -bundle.*

We denote the principal $SO(3)$ -bundle over $H_g(\kappa)$ corresponding to $w \in H^2(H_g(\kappa); \mathbb{Z}_2)$ by $q: Q(w) \rightarrow H_g(\kappa)$. Put $w_i = w(h_*[T_i^2]) \in \mathbb{Z}_2 = \{0, 1\}$, where h and T_i^2 are in Lemma 4.

Lemma 7. *The fundamental groups of $H_g(\kappa)$ and $Q(w)$ has the following presentations:*

$$\pi_1(H_g(\kappa), *) = \langle a_1, \dots, a_g, m_1, \dots, m_n \mid [\kappa_i, m_i] = 1 (1 \leq i \leq n) \rangle$$

$\pi_1(Q(w), \tilde{*}) = \langle a_1, \dots, a_g, m_1, \dots, m_n, \varepsilon \mid [\kappa_i, m_i] = \varepsilon^{w_i} (1 \leq i \leq n), \varepsilon^2 = 1, \varepsilon : \text{center} \rangle$.
 Moreover $q_*: \pi_1(Q(w), \tilde{*}) \rightarrow \pi_1(H_g(\kappa), *)$ can be given by $a_i \mapsto a_i (1 \leq i \leq g), m_j \mapsto m_j (1 \leq j \leq n), \varepsilon \mapsto 1$.

Here, in $\pi_1(H_g(\kappa), *)$, the classes a_i, κ_i and m_i are the images by the map h in Lemma 4 of the free generators of $\pi_1(H_g, *) = \langle a_1, \dots, a_g \rangle$, the longitude and the meridian curves of T_i^2 's respectively. The class ε is the one of nontrivial loop of the fiber $SO(3)$. Base points $*$ and $\tilde{*}$ of $H_g(\kappa)$ and $Q(w)$ respectively are chosen so that $q(\tilde{*}) = *$.

Proof. We note that a presentation of the fundamental group $\pi_1(Q_{T^2}(w))$ of the bundle $Q_{T^2}(w)$ over T^2 corresponding to a class $w \in H^2(T^2; \mathbb{Z}_w) \cong \mathbb{Z}_2 = \{0, 1\}$ is given by

$$\langle l, m, \varepsilon \mid [l, m] = \varepsilon^w, \varepsilon^2 = 1, \varepsilon : \text{center} \rangle.$$

We apply van Kampen's theorem to the decompositions in Lemma 4. $\square$

Next we consider a representation space of the fundamental group of $Q(w)$ and its moduli space. We use the following notations:

$$\tilde{\mathcal{R}}(Q(w)) = \{\rho \in \text{Hom}(\pi_1(Q(w), \tilde{*}), SU(2)) \mid \rho(\varepsilon) = -I\},$$

$$\mathcal{R}(Q(w)) = \tilde{\mathcal{R}}(Q(w))/\text{Ad}SU(2).$$

Hereafter we use the same letter to present an element of $\tilde{\mathcal{R}}(Q(w))$ and its image by the projection to $\mathcal{R}(Q(w))$.

Proposition 8. *Suppose that there is a free basis of $\pi_1(H_g, *)$ which contains $\kappa_1, \dots, \kappa_n$, hence $n \leq g$. Let w be the class $\bigoplus_{i=1}^n 1 \in \bigoplus_{i=1}^n H^2(T_i^2; \mathbb{Z}_2) \cong H^2(H_g(\kappa); \mathbb{Z}_2)$, where $H^2(T_i^2; \mathbb{Z}_2)$ is considered as $\mathbb{Z}_2 = \{0, 1\}$. Then the moduli space $\mathcal{R}(Q(w))$ is diffeomorphic to $SO(3)^{n-1} \times SU(2)^{g-n}$.*

Proof. Let $\kappa_1 \dots \kappa_g \in \pi_1(H_g, *)$ be an extension of $\kappa_1 \dots \kappa_n$ as a free basis of $\pi_1(H_g, *)$. By the hypothesis $w = \bigoplus_{i=1}^n 1$ and Lemma 7, we obtain

$$\pi_1(Q(w), \tilde{*}) = \langle \kappa_1, \dots, \kappa_g, m_1, \dots, m_n, \varepsilon \mid [\kappa_i, m_i] = \varepsilon \ (1 \leq i \leq n), \varepsilon^2 = 1, \varepsilon : \text{center} \rangle.$$

We consider $\tilde{\mathcal{R}}(Q(w))$ as a subset of $SU(2)^{g+n+1}$ by the inclusion

$$\tilde{\mathcal{R}}(Q(w)) \ni \rho \mapsto (\rho(\kappa_i), \rho(m_i), \rho(\varepsilon)) =: (K_i, M_i, E) \in SU(2)^{g+n+1}.$$

We obtain

$$\begin{aligned} \tilde{\mathcal{R}}(Q(w)) &= \{(K_i, M_i, E) \in SU(2)^{g+n+1} \mid \\ &\quad [K_i, M_i] = E \ (1 \leq i \leq n), E^2 = 1, E : \text{center}, E = -1\} \\ &= \{(K_i, M_i, -I) \in SU(2)^{g+n+1} \mid [K_i, M_i] = -I \ (1 \leq i \leq n)\} \\ &\cong \{(K, M) \in SU(2)^2 \mid [K, M] = -I\}^n \times SU(2)^{g-n}. \end{aligned}$$

Claim 9. *For $K, M \in SU(2)$ satisfying $[K, M] = -I$, there exists $S \in SU(2)$ such that $S(K, M)S^{-1} = \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. Moreover such an S exists exactly two. If one is S_0 , the other $-S_0$.*

It is easy to prove this claim, so we omit the proof.

By this claim, we obtain

$$\{(K, M) \in SU(2)^2 \mid [K, M] = -I\} \cong \text{Ad}_{SU(2)}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cong SO(3),$$

hence $\tilde{\mathcal{R}}(Q(w)) \cong SO(3)^n \times SU(2)^{g-n}$. The adjoint action of $SU(2)$ on $\tilde{\mathcal{R}}(Q(w))$ induces a free action of $SO(3)$ on $\tilde{\mathcal{R}}(Q(w))$ by Claim 9, so we obtain $\mathcal{R}(Q(w)) \cong SO(3)^{n-1} \times SU(2)^{g-n}$. $\square$

Let $j: SU(2) \rightarrow SO(3) = SU(2)/\{\pm 1\}$ be the projection. For any homomorphism $\rho: \pi_1(Q(w)) \rightarrow SU(2)$, the homomorphism $\underline{\rho}: \pi_1(H_g(\kappa), *) \rightarrow SO(3)$ is defined by $\underline{\rho}(a) = j(\rho(q_*^{-1}(a)))$ for $a \in \pi_1(H_g(\kappa), *)$. It is easily checked that it is well-defined. Moreover the equality $q^* \underline{\rho} = j_* \rho$ holds, where $j_* \rho$ is a homomorphism from $\pi_1(Q(\kappa), \tilde{*})$ to $SO(3)$ defined by $j_* \rho(a) = j(\rho(a))$ for $a \in \pi_1(Q(\kappa), \tilde{*})$.

We need the next lemma in section 5.

Lemma 10. *Let w be as in Proposition 8.*

For any $\rho \in \tilde{\mathcal{R}}(Q(w))$, there are the following isomorphisms

$$\begin{aligned} T_\rho \mathcal{R}(Q(w)) &\cong H^1(\pi_1(Q(\kappa), \tilde{*}), su(2)_{Ad\rho}) \\ &\xrightarrow{j_*} H^1(\pi_1(Q(\kappa), \tilde{*}), so(3)_{Adj_*, \rho}) \\ &\xleftarrow{q^*} H^1(\pi_1(H_g(\kappa), *), so(3)_{Ad\rho}). \end{aligned}$$

Proof. The first identification is true since the space $\mathcal{R}(Q(w))$ is a smooth manifold by Proposition 8. The second one is induced by the isomorphism from $su(2)_{Ad\rho}$ to $so(3)_{Adj_*, \rho}$ as $\pi_1(Q(\kappa), \tilde{*})$ -module.

We shall prove the last. Let Z^1 and B^1 denote groups of crossed homomorphisms and principal crossed ones respectively. Let $\kappa_1, \dots, \kappa_g, m_1, \dots, m_n$ be the generators of $\pi_1(H_g(\kappa), *)$ in the proof of Proposition 8. For any $u \in Z^1(\pi_1(H_g(\kappa), *), so(3)_{Ad\rho})$, put $u(\kappa_i) = x_i$ and $u(m_i) = y_i$ in $so(3)$. Then we have

$$\begin{aligned} &Z^1(\pi_1(H_g(\kappa), *), so(3)_{Ad\rho}) \\ &\cong \{(x_1, \dots, x_g, y_1, \dots, y_n) \in so(3)^{g+n} \mid u([\kappa_i, m_i]) = 0 \ (1 \leq i \leq n)\}, \end{aligned}$$

where we consider $u([\kappa_i, m_i]) = 0$ as the equations for $x_1, \dots, x_g, y_1, \dots, y_n$ obtained by expanding the lefts with formulas of $Ad\rho$ -crossed homomorphisms. Similarly let $\kappa_1, \dots, \kappa_g, m_1, \dots, m_n, \varepsilon$ be the generators of $\pi_1(Q(w), \tilde{*})$ in the proof of Proposition 8 and put $\tilde{u}(\kappa_i) = \tilde{x}_i, \tilde{u}(m_i) = \tilde{y}_i$ and $\tilde{u}(\varepsilon) = \tilde{e}$ for $\tilde{u} \in Z^1(\pi_1(Q(w), \tilde{*}), so(3)_{Adj_*, \rho})$. Then we obtain

$$\begin{aligned} &Z^1(\pi_1(Q(w), \tilde{*}), so(3)_{Adj_*, \rho}) \cong \{(\tilde{x}_1, \dots, \tilde{x}_g, \tilde{y}_1, \dots, \tilde{y}_n, \tilde{e}) \in so(3)^{g+n+1} \mid \\ &\quad \tilde{u}([\kappa_i, m_i]) = \tilde{u}(\varepsilon) \ (1 \leq i \leq n), \tilde{u}(\varepsilon^2) = 0, \tilde{u}([s, \varepsilon]) = 0 \text{ for } s = \forall \kappa_i, m_i\} \\ &= \{(\tilde{x}_1, \dots, \tilde{x}_g, \tilde{y}_1, \dots, \tilde{y}_n, \tilde{e}) \in so(3)^{g+n+1} \mid \tilde{u}([\kappa_i, m_i]) = 0 \ (1 \leq i \leq n)\}. \end{aligned}$$

Since the defining equations of $Z^1(\pi_1(H_g(\kappa), *), so(3)_{Ad\rho})$ and $Z^1(\pi_1(Q(w), \tilde{*}), so(3)_{Adj_*, \rho})$ is the same via q^* , these groups are isomorphic by q^* . It is easy to see that q^* is an isomorphism between B^1 's, hence it induces an isomorphism from $H^1(\pi_1(H_g(\kappa), *), so(3)_{Ad\rho})$ to $H^1(\pi_1(Q(w), \tilde{*}), so(3)_{Adj_*, \rho})$. This completes the proof. $\square$

5. LAGRANGIAN EMBEDDINGS

In this section, we prove that the manifolds in Proposition 8 are Lagrangian submanifolds under some conditions.

We assume the hypotheses of Proposition 8. Then we have $w_2(\iota^*Q(w)) \equiv n \pmod{2}$, where ι is the inclusion $\iota: \Sigma_g \hookrightarrow H_g(\kappa)$ and $\iota^*Q(w)$ is the pullback bundle of $Q(w)$ by ι . The bundle $\iota^*Q(w)$ is isomorphic to the nontrivial bundle P if and only if n is odd.

We consider the case that n is odd. There is a bundle map $\bar{\iota}: P \rightarrow Q(w)$ over the inclusion ι which preserves base points.

Proposition 11. *Under the above setting, if the induced homomorphism $\iota_*: \pi_1(\Sigma_g, *) \rightarrow \pi_1(H_g(\kappa), *)$ is surjective, then the induced map $\bar{\iota}^*: \mathcal{R}(Q(w)) \rightarrow (\mathcal{R}(P), \omega)$ given by the pullback of the connections is a Lagrangian embedding.*

Proof. It is obvious that $\bar{\iota}_*: \pi_1(P, *) \rightarrow \pi_1(Q(w), \tilde{*})$ is surjective. This implies that $\bar{\iota}^*$ is an embedding. By Proposition 8, we have $\dim \mathcal{R}(Q(w)) = 3(g-1) = \frac{1}{2} \dim \mathcal{R}(P)$. Therefore we have only to prove the equality $\bar{\iota}^* \omega = 0$. We have the following commutative diagram:

$$\begin{array}{ccc}
 T_\rho \mathcal{R}(Q(w)) & \xrightarrow{d\bar{\iota}^*} & T_{\bar{\iota}^* \rho} \mathcal{R}(P) \\
 \parallel & & \parallel \\
 H^1(\pi_1(Q(w), \tilde{*}), su(2)_{Ad\rho}) & \xrightarrow{\bar{\iota}^*} & H^1(\pi_1(P, \tilde{*}), su(2)_{Ad\bar{\iota}^* \rho}) \\
 j_* \downarrow \cong & & j_* \downarrow \cong \\
 H^1(\pi_1(Q(w), \tilde{*}), so(3)_{Adj, \rho}) & \xrightarrow{\bar{\iota}^*} & H^1(\pi_1(P, \tilde{*}), so(3)_{Adj, \bar{\iota}^* \rho}) \\
 q^* \uparrow \cong & & p^* \uparrow \cong \\
 H^1(\pi_1(H_g(\kappa), *), so(3)_{Ad\rho}) & \xrightarrow{\iota^*} & H^1(\pi_1(\Sigma_g, *), so(3)_{Ad\iota^* \rho}).
 \end{array}$$

By this diagram and the definition of the symplectic form ω on $\mathcal{R}(P)$, we have only to show the equality $\omega_\rho(\iota^* a, \iota^* b) = 0$ for $a, b \in H^1(\pi_1(H_g(\kappa), *), so(3)_{Ad\rho})$ and $\rho \in \tilde{\mathcal{R}}(Q(w))$.

Since the boundary of $H_g(\kappa)$ is $\Sigma_g \amalg \coprod_{i=1}^n T_i^2$, we have $\iota_*[\Sigma_g] = -\sum_{i=1}^n \iota_{i*}[T_i^2]$ in $H^2(H_g(\kappa); \mathbb{Z})$, where $[\Sigma_g]$ and $[T_i^2]$'s are fundamental classes of Σ_g and T_i^2 's and $\iota_i: T_i^2 \hookrightarrow H_g(\kappa)$ are the inclusions. Since Σ_g , T_i^2 and H_g are $K(\pi, 1)$ -spaces, we may identify group (co)homology with singular (co)homology. We have

$$\begin{aligned}
 \omega_\rho(\iota^* a, \iota^* b) &= -tr(\iota^* a \cup \iota^* b)[\Sigma_g] = -tr(a \cup b)\iota_*[\Sigma_g] \\
 &= tr(a \cup b)\left(\sum_{i=1}^n \iota_{i*}[T_i^2]\right) = \sum_{i=1}^n tr(\iota_i^* a \cup \iota_i^* b)[T_i^2].
 \end{aligned}$$

The following claim is a well-known fact.

Claim 12. *For any homomorphism $\tau: \pi_1(T^2, *) \rightarrow SO(3)$ which defines a nontrivial flat $SO(3)$ -bundle $\mathbb{R}_\tau^3$ over T^2 , we have $H^1(T^2; \mathbb{R}_\tau^3) = 0$.*

This claim implies $\iota_i^* a = \iota_i^* b = 0$, hence $\omega_\rho(\iota^* a, \iota^* b) = 0$. □

Let $\mathcal{R}(Q(w))_{\bar{\iota}}$ be the image of $\mathcal{R}(Q(w))$ in $\mathcal{R}(P)$ by the pullback by $\bar{\iota}$, then we have a Lagrangian submanifold $\mathcal{R}(Q(w))_{\bar{\iota}}$ of $(\mathcal{R}(P), \omega)$. Moreover, by Remark 3, it is monotone.

We summarize the above results in the following theorem.

Theorem 13. *Let H_g be an oriented handlebody of genus $g \geq 2$. Let κ be a union of mutually disjoint simple closed curves $\kappa_1, \dots, \kappa_n$ on $\Sigma_g = \partial H_g$. Let $H_g(\kappa)$ be the complement of the union of open tubular neighborhoods of embedded curves in $\text{int} H_g$ which are parallel to $\kappa_1, \dots, \kappa_n$.*

Suppose that there is a free basis of $\pi_1(H_g, *)$ which contains the classes of $\kappa_1, \dots, \kappa_n$ hence $n \leq g$, and that the homomorphism $\iota_*: \pi_1(\Sigma_g, *) \rightarrow \pi_1(H_g(\kappa), *)$ is surjective. Moreover suppose that n is odd.

Then any base point preserving bundle map $\bar{\iota}^*: P \rightarrow Q(w)$ over the inclusion $\iota: \Sigma_g \hookrightarrow H_g(\kappa)$ induces a Lagrangian embedding $\bar{\iota}^*: \mathcal{R}(Q(w)) \rightarrow (\mathcal{R}(P), \omega)$, where $w \in H^2(H_g(\kappa); \mathbb{Z}_2)$ is nontrivial on any boundary component of $H_g(\kappa)$. Hence its image denoted by $\mathcal{R}(Q(w))_{\bar{\iota}}$ is a Lagrangian submanifold. Moreover $\mathcal{R}(Q(w))_{\bar{\iota}}$ is monotone and diffeomorphic to $SO(3)^{n-1} \times SU(2)^{g-n}$.

6. EXAMPLES

Let L be any simply connected Lagrangian submanifold of $(\mathcal{R}(P), \omega)$. Since $(\mathcal{R}(P), \omega)$ is monotone and has $Im I_c = 2\mathbb{Z}$ (see section 3), L is monotone and has $\Sigma(L) = 4$ by Remark 3. Hence, for a pair of such Lagrangian submanifolds, the Floer cohomology of Lagrangian intersections is well-defined. In particular, Theorem 13 with $n = 1$ gives examples of simply connected Lagrangian submanifolds.

Next we give some concrete examples in the case of $g = 2$.

Example 14.

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be a system of generators of $\pi_1(\Sigma_2, *)$ given by Figure 1.

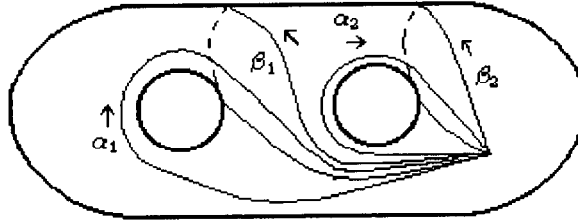


FIGURE 1. A system of generators of $\pi_1(\Sigma_2, *)$.

Then the fundamental group of the nontrivial $SO(3)$ -bundle P over Σ_2 is presented by

$$\pi_1(P, \tilde{*}) = \langle \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] = \varepsilon, \varepsilon^2 = 1, \varepsilon: \text{center} \rangle.$$

We can identify $\tilde{\mathcal{R}}(P)$ with the subset

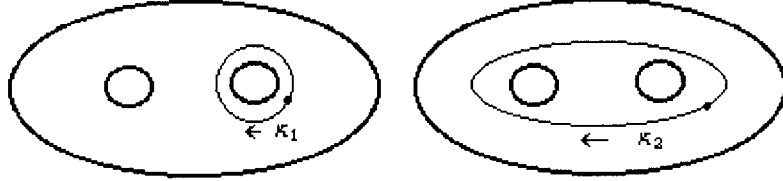
$$\{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid [A_1, B_1][A_2, B_2] = -I\}$$

of $SU(2)^4$ via the map $\rho \mapsto (A_i, B_i)_{i=1}^2 = (\rho(\alpha_i), \rho(\beta_i))_{i=1}^2$, where $\rho \in \tilde{\mathcal{R}}(P)$.

Let κ_1 and κ_2 be the simple closed curves on Σ_2 given by Figure 2.

Their classes of $\pi_1(H_2, *)$ are $\kappa_1 = a_2$ and $\kappa_2 = a_1 a_2$. Let Q_1 and Q_2 be the nontrivial $SO(3)$ -bundles over $H_2(\kappa_1)$ and $H_2(\kappa_2)$ respectively, then we have the following presentations of fundamental groups of them:

$$\begin{aligned} \pi_1(Q_1, \tilde{*}) &= \langle a_1, a_2, m, \varepsilon \mid [a_2, m] = \varepsilon, \varepsilon^2 = 1, \varepsilon: \text{center} \rangle, \\ \pi_1(Q_2, \tilde{*}) &= \langle a_1, a_2, m, \varepsilon \mid [a_1 a_2, m] = \varepsilon, \varepsilon^2 = 1, \varepsilon: \text{center} \rangle. \end{aligned}$$

FIGURE 2. Simple closed curves κ_1 and κ_2 .

Let $\bar{\iota}_i: P \rightarrow Q_i$ be a bundle map over $\iota_i: \Sigma_2 \hookrightarrow H_2(\kappa_i)$ for $i = 1, 2$ which preserves base points. The induced homomorphisms of them on the fundamental groups can be given by

$$\begin{aligned} \bar{\iota}_{1*}: \pi_1(P, \tilde{*}) &\rightarrow \pi_1(Q_1, \tilde{*}) & \bar{\iota}_{2*}: \pi_1(P, \tilde{*}) &\rightarrow \pi_1(Q_2, \tilde{*}) \\ \left\{ \begin{array}{l} \alpha_1 \mapsto \varepsilon^{\gamma(\alpha_1)} a_1 \\ \alpha_2 \mapsto \varepsilon^{\gamma(\alpha_2)} a_2 \\ \beta_1 \mapsto \varepsilon^{\gamma(\beta_1)} \\ \beta_2 \mapsto \varepsilon^{\gamma(\beta_2)} m \\ \varepsilon \mapsto \varepsilon \end{array} \right. & \left\{ \begin{array}{l} \alpha_1 \mapsto \varepsilon^{\delta(\alpha_1)} a_1 \\ \alpha_2 \mapsto \varepsilon^{\delta(\alpha_2)} a_2 \\ \beta_1 \mapsto \varepsilon^{\delta(\beta_1)} a_2 m a_2^{-1} \\ \beta_2 \mapsto \varepsilon^{\delta(\beta_2)} m \\ \varepsilon \mapsto \varepsilon, \end{array} \right. \end{aligned}$$

for certain $\gamma, \delta \in H^2(\Sigma_2; \mathbb{Z}_2) = \text{Hom}(\pi_1(\Sigma_2, *), \mathbb{Z}_2 = \{0, 1\})$. Let $\tilde{L}_i$ be the submanifold $\bar{\iota}_i^* \tilde{\mathcal{R}}(Q_i)$ of $\tilde{\mathcal{R}}(P)$ and $L_i = \tilde{L}_i / \text{Ad} SU(2)$ of $\mathcal{R}(P)$ for $i = 1, 2$. Then we have

$$\tilde{L}_1 = \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid B_1 = (-1)^{\gamma(\beta_1)} I, [A_2, B_2] = -I\}$$

$$\tilde{L}_2 = \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid B_1 = (-1)^{\delta(\beta_1) + \delta(\beta_2)} A_2 B_2 A_2^{-1}, [A_1 A_2, B_2] = -I\}$$

in $\tilde{\mathcal{R}}(P)$. A direct computation shows $\tilde{L}_1 \cap \tilde{L}_2 = \emptyset$, hence $L_1 \cap L_2 = \emptyset$. This implies $I^*(L_1, L_2; \mathcal{R}^\sim(P)) = \{0\}$.

Example 15.

Let $f: \Sigma_2 \rightarrow \Sigma_2$ be the orientation preserving diffeomorphism of Σ_2 whose induced homomorphism f_* on the fundamental group $\pi_1(\Sigma_2, *)$ of Σ_2 is given by $\alpha_i \rightarrow \beta_i^{-1}$ and $\beta_i \rightarrow \beta_i \alpha_i \beta_i^{-1}$ for $i = 1, 2$. For any bundle map $\bar{f}: P \rightarrow P$ over f which preserves the base point, the induced homomorphism $\bar{f}_*: \pi_1(P, \tilde{*}) \rightarrow \pi_1(P, \tilde{*})$ is given by

$$\left\{ \begin{array}{l} \alpha_i \mapsto \varepsilon^{\nu(\beta_i)} \beta_i^{-1} \\ \beta_i \mapsto \varepsilon^{\nu(\alpha_i)} \beta_i \alpha_i \beta_i^{-1} \\ \varepsilon \mapsto \varepsilon, \end{array} \right. \quad (i = 1, 2)$$

where ν is a certain class in $H^1(\Sigma_2; \mathbb{Z}_2) = \text{Hom}(\pi_1(\Sigma_2, *), \mathbb{Z}_2)$. Since it induces a symplectic diffeomorphism $\bar{f}^*: (\mathcal{R}(P), \omega) \rightarrow (\mathcal{R}(P), \omega)$ by pullback, the image $\bar{f}^* L_1$ of L_1 by $\bar{f}^*$ is also a Lagrangian submanifold of $(\mathcal{R}(P), \omega)$, where L_1 is the Lagrangian submanifold given in Example 14.

We calculate the Floer cohomology group of Lagrangian intersections of the pair $(L_1, \bar{f}^* L_1)$. The induced map $\bar{f}^*: \tilde{\mathcal{R}}(P) \rightarrow \tilde{\mathcal{R}}(P)$ is given by

$$(A_i, B_i)_{i=1}^2 \mapsto (A'_i, B'_i)_{i=1}^2 = ((-1)^{\nu(\beta_i)} B_i^{-1}, (-1)^{\nu(\alpha_i)} B_i A_i B_i^{-1})_{i=1}^2,$$

where $\tilde{\mathcal{R}}(P)$ is considered as in Example 14. The image of $\tilde{L}_1$ by $\bar{f}^*$ is given by

$$\bar{f}^* \tilde{L}_1 = \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid A_1 = (-1)^{\nu(\beta_1)} I, [A_2 B_2 A_2^{-1}, A_2^{-1}] = -I\},$$

and the intersection $\tilde{L}_1 \cap \bar{f}^* \tilde{L}_1$ of $\tilde{L}_1$ and $\bar{f}^* \tilde{L}_1$ is calculated as follows:

$$\begin{aligned} \tilde{L}_1 \cap \bar{f}^* \tilde{L}_1 &= \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid A_1 = (-1)^{\nu(\beta_1)} I, B_1 = I, \\ &\quad (A_2, B_2) = S \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) S^{-1}, S \in SU(2)\} \\ &\cong SO(3). \end{aligned}$$

This implies that $L_1 \cap \bar{f}^* L_1$ is one point. It is easy to see that L_1 and $\bar{f}^* L_1$ intersect transversely at this point, which shows $I^*(L_1, \bar{f}^* L_1; \mathcal{R}(P)) \cong \mathbb{Z}_2$.

Example 16.

Let p be any positive integer. Let $\kappa_{3,p}$ be the simple closed curve on Σ_2 given by Figure 3.

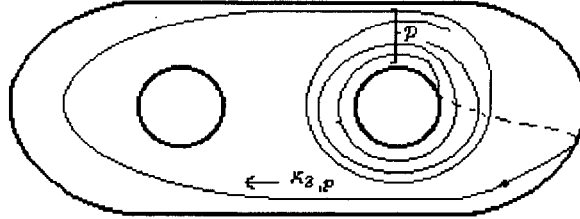


FIGURE 3. The simple closed curve $\kappa_{3,p}$.

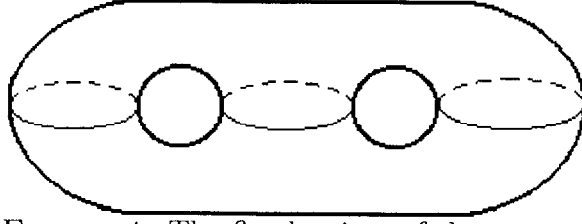
Since its class of $\pi_1(H_2, *)$ is $\kappa_{3,p} = a_1 a_2^p$, we have

$$\pi_1(H_2(\kappa_{3,p}), *) = \langle a_1, a_2, m \mid [a_1 a_2^p, m] = 1 \rangle.$$

The homomorphism $\iota_{3,p*}: \pi_1(\Sigma_2, *) \rightarrow \pi_1(H_2(\kappa_{3,p}), *)$ induced by the natural inclusion $\iota_{3,p}: \Sigma_2 \hookrightarrow H_2(\kappa_{3,p})$ is given by

$$\begin{cases} \alpha_1 \mapsto a_1 \\ \beta_1 \mapsto a_1^{-1} m a_1 \\ \alpha_2 \mapsto a_2^p m a_2^{-(p-1)} \\ \beta_2 \mapsto (m a_2)^p a_2^{-p}. \end{cases}$$

Let r be the reflection of Σ_2 whose fixed points are the union of the curves in Figure 4.

FIGURE 4. The fixed points of the map r

The induced homomorphism r_* on the fundamental group $\pi_1(\Sigma_2, *)$ of Σ_2 is given by

$$\begin{cases} \alpha_1 \mapsto \alpha_2^{-1} \alpha_1^{-1} \alpha_2 \\ \beta_1 \mapsto \alpha_2^{-1} \alpha_1 \beta_1 \alpha_1^{-1} \alpha_2 \\ \alpha_2 \mapsto \alpha_2^{-1} \\ \beta_2 \mapsto \beta_2. \end{cases}$$

Put $h = r \circ f$, then it is an orientation reversing diffeomorphism of Σ_2 . It is easy to see that the 3-manifold obtained by gluing two copies of H_2 by h is the three sphere. The induced homomorphism h^* on $\pi_1(\Sigma_2, *)$ is given by

$$\begin{cases} \alpha_1 \mapsto \alpha_2^{-1} \alpha_1 \beta_1^{-1} \alpha_1^{-1} \alpha_2 \\ \beta_1 \mapsto \alpha_2^{-1} \alpha_1 \beta_1 \alpha_1^{-1} \alpha_2 \alpha_2^{-1} \alpha_1 \beta_1^{-1} \alpha_1^{-1} \alpha_2 \\ \alpha_2 \mapsto \beta_2^{-1} \\ \beta_2 \mapsto \beta_2 \alpha_2^{-1} \beta_2^{-1}. \end{cases}$$

There is a bundle map $\bar{h}: P \rightarrow P$ over h such that the induced homomorphism $\bar{h}_*$ on $\pi_1(P, \tilde{*})$ is defined by

$$\begin{cases} \alpha_1 \mapsto \alpha_2^{-1} \alpha_1 \beta_1^{-1} \alpha_1^{-1} \alpha_2 \\ \beta_1 \mapsto \alpha_2^{-1} \alpha_1 \beta_1 \alpha_1^{-1} \alpha_2 \alpha_2^{-1} \alpha_1 \beta_1^{-1} \alpha_1^{-1} \alpha_2 \\ \alpha_2 \mapsto \beta_2^{-1} \\ \beta_2 \mapsto \beta_2 \alpha_2^{-1} \beta_2^{-1} \\ \varepsilon \mapsto \varepsilon. \end{cases}$$

Let $\bar{\iota}_1$ be the map in Example 6.1 with $\gamma \equiv 0$. Let $\bar{\iota}_{3,p}: P \rightarrow Q(\kappa_{3,p})$ be the lift of $\iota_{3,p}$ such that the induced homomorphism $\bar{\iota}_{3,p*}: \pi_1(P, \tilde{*}) \rightarrow \pi_1(Q(\kappa_{3,p}), \tilde{*})$ is given by

$$\begin{cases} \alpha_1 \mapsto a_1 \\ \beta_1 \mapsto a_1^{-1} m a_1 \\ \alpha_2 \mapsto a_2^p m a_2^{-(p-1)} \\ \beta_2 \mapsto (m a_2)^p a_2^{-p} \\ \varepsilon \mapsto \varepsilon. \end{cases}$$

Let $\tilde{L}_1$ and $\tilde{L}_{3,p}$ be $\tilde{\iota}_1^* \tilde{\mathcal{R}}(Q(\kappa_1))$ and $\tilde{\iota}_{3,p}^* \tilde{\mathcal{R}}(Q(\kappa_{3,p}))$ respectively. Put $L_1 = \tilde{L}_1 / \text{Ad}SU(2)$ and $L_{3,p} = \tilde{L}_{3,p} / \text{Ad}SU(2)$, then they are Lagrangian submanifolds of $\mathcal{R}(P)$. Since h is an orientation reversing diffeomorphism of Σ_2 , the induced map $\bar{h}^*: \mathcal{R}(P) \rightarrow \mathcal{R}(P)$ is an anti-symplectic (i.e. $\bar{h}^*\omega = -\omega$) diffeomorphism. Hence $\bar{h}^*L_1$ is also a Lagrangian submanifold of $\mathcal{R}(P)$.

Next we calculate the intersection points of the pair $(\bar{h}^*L_1, L_{3,p})$.

Since we have

$$\begin{aligned}\tilde{\mathcal{R}}(Q(\kappa_1)) &= \{(a'_1, a'_2, m') \in SU(2)^3 \mid [a'_2, m'] = -I\}, \\ \tilde{\mathcal{R}}(Q(\kappa_{3,p})) &= \{(a_1, a_2, m) \in SU(2)^3 \mid [a_1 a_2^p, m] = -I\},\end{aligned}$$

we obtain

$$\begin{aligned}\tilde{L}_1 &= \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid A_1 = a'_1, A_2 = a'_2, \\ &\quad B_1 = I, B_2 = m', (a'_1, a'_2, m') \in \tilde{\mathcal{R}}(Q(\kappa_1))\}, \\ \tilde{L}_{3,p} &= \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid A_1 = a_1, A_2 = a_2^p m a_2^{-(p-1)}, \\ &\quad B_1 = a_1^{-1} m a_1, B_2 = (m a_2)^p a_2^{-p}, (a_1, a_2, m) \in \tilde{\mathcal{R}}(Q(\kappa_{3,p}))\}\end{aligned}$$

and

$$\begin{aligned}\bar{h}^*L_1 &= \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid A_1 = I, A_2 = (m')^{-1}, \\ &\quad B_1 = a'_1, B_2 = m'(a'_2)^{-1}(m')^{-1}, (a'_1, a'_2, m') \in \tilde{\mathcal{R}}(Q(\kappa_1))\}.\end{aligned}$$

Then we have

$$\begin{aligned}\bar{h}^*L_1 \cap \tilde{L}_{3,p} &= \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid A_1 = I = a_1, A_2 = (m')^{-1} = a_2^p m a_2^{-(p-1)}, \\ &\quad B_1 = a'_1 = I, B_2 = m'(a'_2)^{-1}(m')^{-1} = (m a_2)^p a_2^{-p}, \\ &\quad [a'_2, m'] = -I, [a_1 a_2^p, m] = -I\} \\ &= \{(A_i, B_i)_{i=1}^2 \in SU(2)^4 \mid A_1 = I, A_2 = -m a_2, \\ &\quad B_1 = m, B_2 = (m a_2)^p a_2^{-p}, [a_2^p, m] = -I\} \\ &\cong \{(a_2, m) \in SU(2)^2 \mid [a_2^p, m] = -I\} \\ &= \{(a_2, m) \in SU(2)^2 \mid (a_2^p, m) = S \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} S^{-1}, S \in SU(2)\}.\end{aligned}$$

Since $\bar{h}^*L_1 \cap L_{3,p} = \bar{h}^*L_1 \cap \tilde{L}_{3,p} / \text{Ad}SU(2)$ holds, we have

$$\begin{aligned}\bar{h}^*L_1 \cap L_{3,p} &\cong \{(a_2, m) \in SU(2)^2 \mid (a_2^p, m) = \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)\} \\ &\cong \{a_2 \in SU(2) \mid a_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \lambda^p = i, \lambda \in \mathbb{C}\}.\end{aligned}$$

Hence we obtain $\# \bar{h}^*L_1 \cap L_{3,p} = p$. By direct computation, we can check that these Lagrangian submanifolds intersect transversely.

In general, if Floer cohomology has even-grading, then we can define its Euler number as the alternate sum of the ranks of its components up to sign. Moreover we can show that

this Euler number equals the intersection number of the two (Lagrangian) submanifolds up to sign.

In this setting, κ_1 and $\kappa_{3,p}$ form a link of 2-components in the three sphere $S^3 = H_2 \cup_h H_2$ and we can prove that the intersection number $\bar{h}^* L_1 \cdot L_{3,p}$ in $\mathcal{R}(P)$ with some orientations equals the linking number of κ_1 and $\kappa_{3,p}$ in S^3 up to sign (see [7]). In this case, we can show that this number is p . Since the cochain complex of this Floer cohomology group is generated by p points, we can deduce $I^*(\bar{h}^* L_1, L_{3,p}; \mathcal{R}(P)) \cong \mathbb{Z}_2^p$.

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Chapter 3

An extension of Milnor's inequality

AN EXTENSION OF MILNOR'S INEQUALITY

RYOJI KASAGAWA

ABSTRACT. We prove the inequality $|\langle c_1(V), [\Sigma_g] \rangle| \leq p(g-1)$ for flat symplectic vector bundles V of rank $2p$ over a closed oriented surface Σ_g of genus g via the Atiyah-Singer index theorem.

1. INTRODUCTION

Milnor showed the following theorem.

Theorem 1.1 (Milnor [4]). *Let $\rho : \pi_1(\Sigma_g) \rightarrow SL(2, \mathbb{R})$ be a representation of the surface group and let E be the corresponding flat vector bundle of rank 2 over Σ_g . Then the Euler number $\langle \chi(E), [\Sigma_g] \rangle$ satisfies the following inequality:*

$$|\langle \chi(E), [\Sigma_g] \rangle| \leq g - 1.$$

Conversely any integer χ such that $|\chi| \leq g - 1$ is realized as the Euler number of a flat $SL(2, \mathbb{R})$ -vector bundle.

There are various generalizations of this theorem, for which see [5].

In this paper we will prove an extension of this theorem to a representation of surface group to the symplectic group $Sp(2p, \mathbb{R})$. More precisely, let $\rho : \pi_1(\Sigma_g) \rightarrow Sp(2p, \mathbb{R})$ be a representation of the surface group and let V be the corresponding flat symplectic vector bundle of rank $2p$ over Σ_g . Since a maximal compact subgroup of $Sp(2p, \mathbb{R})$ is isomorphic to $U(p)$, V admits a positive compatible complex structure with the symplectic structure which always exists and is unique up to homotopy. Hence the first Chern class

$$c_1(V) \in H^2(\Sigma_g; \mathbb{Z})$$

is uniquely defined.

The main theorem of this paper is the following.

Theorem 1.2. *Let $g, p \geq 1$ be integers and Σ_g a closed oriented surface of genus g . For any representation $\rho : \pi_1(\Sigma_g) \rightarrow Sp(2p, \mathbb{R})$, let V be the corresponding flat symplectic vector bundle of rank $2p$ over Σ_g . Then the following inequality holds:*

$$|\langle c_1(V), [\Sigma_g] \rangle| \leq p(g-1).$$

Conversely any integer c_1 such that $|c_1| \leq p(g-1)$ is realized as the first Chern class of a flat symplectic vector bundle of rank $2p$.

2. THE SIGNATURE OF LOCAL COEFFICIENT SYSTEMS

In this section we shall review the result due to Lusztig and Atiyah ([1, 3]) which is needed to prove the main theorem.

Let Σ_g be a closed oriented surface of genus $g \geq 1$. Let $E \rightarrow \Sigma_g$ be a flat (possibly indefinite) hermitian vector bundle with the hermitian form h . It corresponds to a homomorphism

$$\rho: \pi_1(\Sigma_g) \rightarrow U(p, q),$$

where $U(p, q)$ is the unitary group of the hermitian form $\sum_{i=1}^p |z_i|^2 - \sum_{i=p+1}^{p+q} |z_i|^2$ on $(z_i) \in \mathbb{C}^{p+q}$.

Since a flat vector bundle is identified with a local coefficient system, we have the cohomology group $H^*(\Sigma_g; E)$. Moreover the first cohomology group $H^1(\Sigma_g; E)$ has a skew-hermitian form A induced by the cup product and the hermitian form h . Since $\sqrt{-1}A$ is a hermitian form, we can consider its signature. Let $\text{sign}(\Sigma_g, E)$ denote the signature of $\sqrt{-1}A$.

Take a splitting $E = E^+ \oplus E^-$ of E . Here a splitting of a hermitian vector bundle E is defined as a smooth decomposition $E = E^+ \oplus E^-$ of E which is orthogonal relative to the hermitian form and such that the form is positive on E^+ and negative on E^- . Such a splitting is identified with a reduction of the structure group $U(p, q)$ of E to its maximal compact subgroup $U(p) \times U(q)$. It always exists and is unique up to homotopy.

Theorem 2.1 (Lusztig [3], Atiyah [1]). *If E is a flat hermitian vector bundle over the closed surface Σ_g then*

$$\text{sign}(\Sigma_g, E) = 2(c_1(L^+) - c_1(L^-)),$$

where $L^\pm = \det E^\pm$ and $E = E^+ \oplus E^-$ is a splitting of E .

3. PROOF OF THE MAIN THEOREM

In this section we shall prove the main theorem.

Let $V \rightarrow \Sigma_g$ be a flat symplectic vector bundle of rank $2p$ over Σ_g and ω its symplectic form.

Put $E = V \otimes \mathbb{C}$ and extend the symplectic form ω on V to a complex symplectic form on E which is also denoted by ω . A hermitian form h on E is defined by

$$h_x(u, v) = -\sqrt{-1}\omega_x(u, \bar{v})$$

for any $x \in \Sigma_g$ and $u, v \in E_x$.

Let $J: V \rightarrow V$ be a positive compatible complex structure with the symplectic form ω , i.e. $J: V \rightarrow V$ is a bundle homomorphism which satisfies $J^2 = -1$ and such that $\omega(\cdot, J\cdot)$ is a Euclidean metric of V . Such a complex structure always exists and is unique up to homotopy. The complex vector bundle (V, J) of rank p over $\mathbb{C}$ is denoted by V_J . Extend J to a complex linear map on E , then by $J^2 = -1$, we have the decomposition $E = E' \oplus E''$ of E by the eigenspaces of J , where $J|_{E'} = \sqrt{-1}\text{id}_{E'}$ and $J|_{E''} = -\sqrt{-1}\text{id}_{E''}$. It is well known that the complex isomorphisms $E' \cong V_J$ and $E'' \cong \bar{E}' \cong \bar{V}_J$ hold, where $\bar{\cdot}$ denotes the complex conjugation. Thus we have $E = V_J \oplus \bar{V}_J$. Moreover it is easy to see that this decomposition gives a splitting of E where $E^+ \cong V_J$ and $E^- \cong \bar{V}_J$ (see [6]).

By Theorem 2.1, we have

$$\text{sign}(\Sigma_g, E) = 2(c_1(V_J) - c_1(\bar{V}_J)) = 4c_1(V_J).$$

Note that, for any complex vector bundle F , the equalities $c_1(F) = c_1(\det F)$ and $c_1(\bar{F}) = -c_1(F)$ hold.

On the other hand, since E is a flat vector bundle, we have the following de Rham complex with local coefficients in E :

$$d_E^\bullet: 0 \rightarrow \Omega^0(\Sigma_g; E) \xrightarrow{d_E} \Omega^1(\Sigma_g; E) \xrightarrow{d_E} \Omega^2(\Sigma_g; E) \rightarrow 0.$$

By the Atiyah-Singer index theorem ([2]), we obtain

$$\text{ind}(d_E^\bullet) = \text{rank}_{\mathbb{C}} E \cdot \chi(\Sigma_g) = 4p(1 - g).$$

We shall prove the inequality in the main theorem in the following two cases.

In the first case we suppose that $H^0(\Sigma_g; E) = 0$. We then have $H^2(\Sigma_g; E) = 0$ by the duality after identifications $E \cong \bar{E} \cong E^*$. Thus we have $\dim_{\mathbb{C}} H^1(\Sigma_g; E) = 4p(g - 1)$, and hence obtain

$$| \langle c_1(V_J), [\Sigma_g] \rangle | = \frac{1}{4} |\text{sign}(\Sigma_g, E)| \leq p(g - 1).$$

In the second case we suppose that $H^0(\Sigma_g; E) \neq 0$ and hence $H^0(\Sigma_g; V)$ is also non-trivial. Thus there exists a nonzero element s of $H^0(\Sigma_g; V)$ which does not vanish at any point of Σ_g . $W := \mathbb{R}s$ defines an $\mathbb{R}$ -line subbundle of V . Let $W^{\perp_\omega} \subset V$ be the subbundle of V orthogonal to W with respect to symplectic form ω , then W is also subbundle of $W^{\perp_\omega}$. The quotient bundle $W^{\perp_\omega}/W$ has the symplectic form $\underline{\omega}$ induced from ω . From the construction, it is clear that this bundle is a flat symplectic vector bundle of rank $2(p - 1)$.

Put $W_J = \mathbb{R}s \oplus J\mathbb{R}s$, then it is a trivial complex line subbundle of V_J . Let $W_J^{\perp_{h'}}$ be the subbundle of V_J orthogonal to W_J with respect to the hermitian form $h'(\cdot, \cdot) := \omega(\cdot, J\cdot) - \sqrt{-1}\omega(\cdot, \cdot)$ which is also a symplectic vector bundle with a positive compatible complex structure. It is easy to see that the inclusion $W_J^{\perp_{h'}} \hookrightarrow W^{\perp_\omega}$ induces the isomorphism

$$(W_J^{\perp_{h'}}, \omega|_{W_J^{\perp_{h'}}}) \cong (W^{\perp_\omega}/W, \underline{\omega})$$

as smooth symplectic bundles. So a positive compatible complex structure with $\underline{\omega}$ on $W^{\perp_\omega}/W$ is induced from the one on $W_J^{\perp_{h'}}$ via the isomorphism. Thus we have

$$c_1(V_J) = c_1(W_J^{\perp_{h'}} \oplus W_J) = c_1(W_J^{\perp_{h'}}) = c_1(W^{\perp_\omega}/W).$$

In particular if $p = 1$, then we obtain $c_1(V_J) = 0$. So we complete the proof of the inequality in Theorem 1.2 for $p = 1$.

For $p > 1$, we may suppose that $c_1(V) \neq 0$. If necessary, repeating the above argument, we can construct a flat symplectic vector bundle Z over Σ_g of rank $q < p$ such that $H^0(\Sigma_g; Z \otimes \mathbb{C}) = 0$ and $c_1(V) = c_1(Z)$. From the first case, we have

$$| \langle c_1(V), [\Sigma_g] \rangle | = | \langle c_1(Z), [\Sigma_g] \rangle | \leq q(g - 1) \leq p(g - 1).$$

This completes the proof of the inequality.

Conversely for any integer c_1 such that $|c_1| \leq p(g-1)$, since $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$, the existence of a flat symplectic vector bundle V over Σ_g of rank $2p$ with $\langle c_1(V), [\Sigma_g] \rangle = c_1$ is true for $p = 1$ by Theorem 1.1.

For $p > 1$, suppose that $c_1 = p(g-1)$ since the other cases can easily be obtained by the same method. Let V_1 be a flat symplectic vector bundle over Σ_g of rank 2 with $c_1(V_1) = g-1$. The direct sum $V = \bigoplus^p V_1$ of p copies of V_1 is a flat symplectic vector bundle of rank $2p$ with

$$\langle c_1(V), [\Sigma_g] \rangle = p \langle c_1(V_1), [\Sigma_g] \rangle = p(g-1).$$

This vector bundle V is a required one.

This finishes the proof of the main theorem.

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References

1. ρ -invariants, signature cocycles and the mapping class group of a surface
2. On intersection number of submanifolds of the moduli space of flat connections

ρ -INVARIANTS, SIGNATURE COCYCLES AND THE MAPPING CLASS GROUP OF A SURFACE

RYOJI KASAGAWA

ABSTRACT. If the monodromy of a surface bundle over the circle belongs to a certain subgroup of the mapping class group of a surface, we can give a certain representation of the fundamental group of this bundle. Then the ρ -invariant for the pair of the bundle and the representation defines a function on the subgroup. On the other hand, there are Meyer's signature 2-cocycle and a twisted version of it on the subgroup. In this paper we study some relations between them and some properties of the function.

1. INTRODUCTION

Atiyah-Patodi-Singer ρ -invariant is a real valued invariant for a pair of a closed oriented manifold of odd dimension and a unitary representation of its fundamental group. On the other hand, the signature is an integer valued invariant for a compact oriented manifold of even dimension possibly with boundary. More generally it is defined for a pair of such a manifold and a flat hermitian vector bundle over it which is equivalent to a unitary representation of its fundamental group. Atiyah, Patodi and Singer showed that, if the former pair is obtained by restricting the latter to its boundary, then the ρ -invariant for the former pair is the difference of a multiple of the usual signature of the even manifolds from the signature of the latter pair (see [3]).

On the other hand, Meyer defined a 2-cocycle on the mapping class group of a surface of genus greater than zero using the signature of a 4-manifold. It is called Meyer's signature 2-cocycle (see [2, 17]). In the case that the genus is 1 or 2, this cocycle is a coboundary of a function over the rationals. This function is unique and is known as Meyer's function (see [1, 13, 14, 16, 17]). In particular, in the case of genus 1, Atiyah showed that it relates to various geometric objects, e.g. Hirzebruch's signature defect, Dedekind η -function and so on (see [1]). From this, Meyer's signature cocycle is considered an important object.

In this paper, we construct twisted versions of Meyer's signature cocycle on subgroups of the mapping class group of a surface and functions on them as follows. We take a surface bundle over the circle. It is determined by its monodromy which is given by an element of the mapping class group of a surface. But we consider mainly the case that the monodromy belongs to the mapping class group with a base point, which corresponds to the case that a surface bundle has a cross section. To a surface bundle whose monodromy is in a certain subgroup of the mapping class group, we can give a certain unitary representation of the fundamental group of it. Thus the ρ -invariant can be considered as a function on this subgroup. On the other hand, a surface bundle over the two sphere with three holes is obtained from two elements of the mapping class group. Since this bundle is a four-manifold, we can consider the signature of a four-manifold for it. It defines Meyer's signature 2-cocycle on the mapping class group of a surface (see

[2, 17]). If the monodromies of this bundle are contained in the above subgroup, we can also give a certain flat hermitian vector bundle over the bundle. Then we can obtain the signature of the pair. It defines a twisted version of Meyer's signature 2-cocycle on the subgroup. Thus on this subgroup we have a function defined by the ρ -invariant and two 2-cocycles defined by the signatures. The purposes of this paper are to rewrite the above Atiyah, Patodi and Singer's result using the above function and signature cocycles and to show some properties of them.

We state them more precisely as follows. Let Σ_g be a closed oriented surface of genus $g \geq 1$ with a base point $*$. Let $\omega: \pi_1(\Sigma_g, *) \rightarrow U(n)$ be a unitary representation of the fundamental group of Σ_g . The mapping class group of orientation and base point preserving diffeomorphisms of Σ_g is denoted by $\mathcal{M}_{g*}$. Let $\mathcal{M}_{g*}^\omega$ be the subgroup of $\mathcal{M}_{g*}$ consisting of elements which preserve the representation ω by the pullback. For any $a \in \mathcal{M}_{g*}^\omega$ we define the surface bundle $M_a \rightarrow S^1$ over S^1 by $M_a = [0, 1] \times \Sigma_g / (1, x) \sim (0, a^{-1}(x))$. Moreover we have the section $s: S^1 \rightarrow M_a$ determined by the base point $* \in \Sigma_g$. We regard Σ_g as the fiber of M_a at the base point $* = 1$ of S^1 . We can define the homomorphism $\omega_a: \pi_1(M_a, s(1)) \rightarrow G$ by $\omega_a|_{\Sigma_g} = \omega$ and $s^*\omega = 1$. Thus we can define the real valued function

$$\rho_\omega: \mathcal{M}_{g*}^\omega \rightarrow \mathbb{R}$$

by $\rho_\omega(a) = \rho_{\omega_a}(M_a)$, where the right hand side is the Atiyah-Patodi-Singer ρ -invariant for the pair (M_a, ω_a) (see [3]).

Similarly for two elements $a, b \in \mathcal{M}_{g*}^\omega$, we define the surface bundle $Z_{a,b} \rightarrow P$ over the 2-sphere with 3 holes P and the homomorphism $\omega_{a,b}$ from the fundamental group of $Z_{a,b}$ to $U(n)$. Let $E_{a,b} \rightarrow Z_{a,b}$ be the flat hermitian vector bundle over $Z_{a,b}$ obtained from $\omega_{a,b}$. The signature cocycle $sign_\omega$ of $\mathcal{M}_{g*}^\omega$ is defined by $sign_\omega(a, b) = sign(Z_{a,b}, E_{a,b})$ for $a, b \in \mathcal{M}_{g*}^\omega$, where the right hand side denotes the signature of the pair $(Z_{a,b}, E_{a,b})$. The usual signature cocycle $sign$ of $\mathcal{M}_{g*}$ (which is also denoted by $sign_g$) is obtained in the case that ω is the trivial homomorphism to $U(1)$ (see [2, 17]).

Let V_ω be the $\pi_1(\Sigma_g, *)$ -module $\mathbb{C}^n$ obtained from the representation ω , then we have the group cohomology $H^1(\pi_1(\Sigma_g, *), V_\omega)$ with coefficients in it. This group has a natural hermitian form, which is preserved by the action of the subgroup $\mathcal{M}_{g*}^\omega$ by the pullback of inverse. It induces a homomorphism $\zeta_{g*}^\omega: \mathcal{M}_{g*}^\omega \rightarrow U(p, q)$ if we take an appropriate basis for the (possibly indefinite) hermitian vector space $H^1(\pi_1(\Sigma_g, *), V_\omega)$. In the case that ω is the trivial homomorphism to $U(1)$, we also use the symbol ζ_{g*} instead of ζ_{g*}^ω . Let $\mathcal{I}_{g*}$ and $\mathcal{I}_{g*}(\omega)$ be the kernels of the homomorphisms ζ_{g*} and ζ_{g*}^ω respectively. The former is called Torelli group. Let $\mathcal{K}_{g*}$ be the subgroup of $\mathcal{M}_{g*}$ generated by all the Dehn twists along separating simple closed curves on Σ_g which is contained in the Torelli group $\mathcal{I}_{g*}$ (see [12]).

Proposition 1. *The real valued function ρ_ω on $\mathcal{M}_{g*}^\omega$ is $\mathcal{M}_{g*}^\omega$ -conjugacy invariant and satisfies the equality $n \text{sign} - \text{sign}_\omega = \delta \rho_\omega$, where δ is the coboundary operator. Its restriction to $\mathcal{I}_{g*} \cap \mathcal{I}_{g*}(\omega)$ is a homomorphism.*

In general, we have both the cases that ρ_ω is trivial and nontrivial on $\mathcal{I}_{g*} \cap \mathcal{I}_{g*}(\omega)$. But we can show the nontriviality at least in the following cases.

Theorem 2. *Let $g \geq 2$. Assume that a nontrivial representation ω of $\pi_1(\Sigma_g, *)$ to $U(1)$ factors through $\mathbb{Z}_2 = \{\pm 1\}$. Then the function ρ_ω is an integer valued homomorphism on $\mathcal{I}_{g*} \cap \mathcal{I}_{g*}(\omega)$. Moreover its restriction to $\mathcal{K}_{g*} \cap \mathcal{I}_{g*}(\omega)$ is also nontrivial.*

These results can be regarded as an analogue of Morita's mapping $d: \mathcal{M}_{g,1} \rightarrow \mathbb{Z}$ (see [21] for details) and as an example of a secondary invariant.

Let M be a closed oriented 3-manifold. The direct product $2TM = TM \oplus TM$ has a natural $Spin(6)$ -structure and is trivial bundle over M . The homotopy class of a trivialization of this bundle as a $Spin(6)$ -bundle is called a 2-framing of M . There is a canonical choice α_M of a 2-framing for M which is well-known as Atiyah's canonical one (see [2]).

Let $\pi: \tilde{M} \rightarrow M$ be a connected finite covering of M , then we have two 2-framings $\alpha_{\tilde{M}}$ and $\pi^*\alpha_M$. The difference degree $d(\pi^*\alpha_M, \alpha_{\tilde{M}})$ between them is defined by the half of the relative Pontryagin number of the double $2T([0, 1] \times \tilde{M})$ of the tangent bundle of the product manifold $[0, 1] \times \tilde{M}$ trivialized on the boundaries $0 \times (-\tilde{M})$ and $1 \times \tilde{M}$ by the framings $-\pi^*\alpha_M$ and $\alpha_{\tilde{M}}$ respectively (see [2]).

Let $\omega: \pi_1(\Sigma_g, *) \rightarrow G$ be the associated homomorphism to a connected finite G -covering $\Sigma_h \rightarrow \Sigma_g$ which preserves base points.

Let $\mathcal{M}_{g*}^{ker\omega}$ be the subgroup of $\mathcal{M}_{g*}$ consisting of elements which induce homomorphisms on the fundamental group of Σ_g which map $ker\omega$ onto itself. For any $a \in \mathcal{M}_{g*}^{ker\omega}$, if we consider its lift to Σ_h which preserves the base point, then we obtain a homomorphism $\lambda: \mathcal{M}_{g*}^{ker\omega} \rightarrow \mathcal{M}_{h*}$. Moreover we obtain a natural covering $\pi_a: M_{\lambda(a)} \rightarrow M_a$ of degree $|G|$. Thus we can define the function d_ω on $\mathcal{M}_{g*}^{ker\omega}$ with values in $\mathbb{Q}$ by $d_\omega(a) = -\frac{1}{3}d(\pi_a^*\alpha_{M_a}, \alpha_{M_{\lambda(a)}})$ for $a \in \mathcal{M}_{g*}^{ker\omega}$.

Proposition 3. *Under the above situation, the equality $|G|sign_g - \lambda^*sign_h = \delta d_\omega$ holds on $\mathcal{M}_{g*}^{ker\omega}$.*

This paper is organized as follows. In section 2, we define the signature cocycles and the function ρ_ω more precisely and prove Proposition 1. In section 3, we show the similar result to Proposition 1 on the whole mapping class group. In section 4 and 5, we investigate the representation of a certain subgroup of the mapping class group arising from a homomorphism ω to $Aut(\mathbb{Z}) = \mathbb{Z}_2$. In section 6, we prove Theorem 2. In section 7, we show Proposition 3.

2. SIGNATURE COCYCLES OF SUBGROUPS OF THE MAPPING CLASS GROUPS

In this section we define the signature cocycles and the functions ρ_ω and prove Proposition 1. These are the extension of part of [13].

Let Y be a $2l$ -dimensional compact oriented manifold with boundary ∂Y . Let $E \rightarrow Y$ be a flat (possibly indefinite) hermitian vector bundle over Y . Then we have cohomology groups $H^*(Y; E)$ and $H^*(Y, \partial Y; E)$. There is a natural pairing $H^*(Y; E) \times H^*(Y, \partial Y; E) \rightarrow \mathbb{C}$ defined by the cup product, the hermitian product on E and the evaluation on the fundamental class $[Y, \partial Y]$. This pairing induces a nondegenerate form on $\hat{H}^*(Y; E) := Im[H^*(Y, \partial Y; E) \rightarrow H^*(Y; E)]$. On $\hat{H}^l(Y; E)$, it is a hermitian form for l even and a skew-hermitian for l odd, which is denoted by $(,)_E$. Let $<, >_E$ be the hermitian form on $\hat{H}^l(Y; E)$ given by $(,)_E$ for l even and by $i^l(,)_E$ for l odd. We define $sign(Y, E)$ by the signature of the hermitian form $<, >_E$ (see [3]).

Let Σ_g be a closed oriented surface of genus $g \geq 1$ and $*$ a base point of it. Let $\omega: \pi_1(\Sigma_g, *) \rightarrow U(n)$ be a unitary representation of the fundamental group of Σ_g .

The mapping class group of orientation and base point preserving diffeomorphisms of Σ_g is denoted by $\mathcal{M}_{g*}$. Let $\mathcal{M}_{g*}^\omega$ be the subgroup of $\mathcal{M}_{g*}$ consisting of elements which preserve the representation ω by the pullback.

Let X be a compact oriented surface possibly with boundary ∂X and with a base point x_0 . Let $\pi: Z \rightarrow X$ be a Σ_g -bundle over X with a section $s: X \rightarrow Z$ and with an identification ι_{x_0} from Σ_g to the fiber Z_{x_0} at the base point $x_0 \in X$. If we consider the holonomy of this surface bundle, we obtain the homomorphism $h: \pi_1(X, x_0) \rightarrow \mathcal{M}_{g*}$ defined by the inverse of holonomy. Conversely, if a homomorphism h is given, we can construct a surface bundle with above data from which we again obtain h since the classifying space of the group of orientation and base point preserving diffeomorphisms of Σ_g is an Eilenberg-MacLane space $K(\mathcal{M}_{g*}, 1)$ for $g \geq 1$ (see [7, 8, 20]).

Lemma 4. *Under the above setting, suppose that the image of h is contained in $\mathcal{M}_{g*}^\omega$. Then there exists a unique representation $\omega_Z: \pi_1(Z, z_0) \rightarrow U(n)$ such that $\iota_{x_0}^* \omega_Z = \omega$ and $s^* \omega_Z = 1$.*

Proof. In the case of $X = S^2$, this lemma is clear since any Σ_g -bundle over S^2 is trivial. Suppose that X is not S^2 , then, from the homotopy exact sequence for the Σ_g -bundle $\pi: Z \rightarrow X$, we obtain the following short exact one:

$$1 \rightarrow \pi_1(\Sigma_g, *) \xrightarrow{\iota_{x_0}^*} \pi_1(Z, *) \xrightarrow{\pi_*} \pi_1(X, *) \rightarrow 1.$$

Since the section s is given, the fundamental group $\pi_1(Z, *)$ is generated by the images of the induced homomorphisms ι_{x_0*} and s_* . Therefore if there exists a homomorphism satisfying the conditions, then it is unique.

Next we show the existence of ω_Z . The fundamental group $\pi_1(Z, *)$ of Z is given by

$$\langle \gamma \in \pi_1(\Sigma_g, *), t \in \pi_1(X, *) \mid t\gamma t^{-1} = h(t)_* \gamma \text{ for } \forall \gamma \in \pi_1(\Sigma_g, *) \text{ and } \forall t \in \pi_1(X, *) \rangle.$$

Here we consider the fundamental groups $\pi_1(\Sigma_g, *)$ and $\pi_1(X, x_0)$ as subgroups of $\pi_1(Z, *)$ by the injective homomorphisms ι_{x_0*} and s_* . We define the homomorphism $\omega_Z: \pi_1(Z, *) \rightarrow U(n)$ by $\omega_Z(\iota_{x_0*} \gamma) = \omega(\gamma)$ and $\omega_Z(s_* t) = I$ for $\gamma \in \pi_1(\Sigma_g, *)$ and $t \in \pi_1(X, *)$ where $I \in U(n)$ denotes the identity matrix. It is easy to see that it is well-defined because of the hypothesis $\text{Im } h \subset \mathcal{M}_{g*}^\omega$. $\square$

Let P be the 2-sphere with 3 holes. For $a, b \in \mathcal{M}_{g*}^\omega$, let $M_{a,b}$ be the Σ_g -bundle over P with monodromies a^{-1} and b^{-1} and s the section obtained from the base point of Σ_g . By Lemma 4, we have the representation $\omega_{a,b} := \omega_{M_{a,b}}: \pi_1(M_{a,b}, s(x_0)) \rightarrow U(n)$. Let $E_{\omega_{a,b}} \rightarrow M_{a,b}$ be the flat hermitian vector bundle over $M_{a,b}$ determined by $\omega_{a,b}$. Put $\text{sign}_\omega(a, b) := \text{sign}(M_{a,b}, E_{\omega_{a,b}})$, then the signature additivity implies that $\text{sign}_\omega: \mathcal{M}_{g*}^\omega \times \mathcal{M}_{g*}^\omega \rightarrow \mathbb{Z}$ is a 2-cocycle of $\mathcal{M}_{g*}^\omega$.

For $A, B \in U(p, q)$, let $E_{A,B} \rightarrow P$ be the flat hermitian vector bundle over P whose holonomies are given by A^{-1}, B^{-1} . Put $\text{Sign}(A, B) := \text{sign}(P, E_{A,B})$, then we also obtain a 2-cocycle $\text{Sign}: U(p, q) \times U(p, q) \rightarrow \mathbb{Z}$ of $U(p, q)$, which is called signature cocycle (see [1]).

Let V_ω be the $\pi_1(\Sigma_g, *)$ -module $\mathbb{C}^n$ defined by the representation ω . Hence we obtain the group cohomology $H^1(\pi_1(\Sigma_g, *), V_\omega)$ with coefficients in V_ω . This group has the hermitian

form which is given by i times the cup product, the hermitian product on V_ω and the evaluation on the fundamental class $[\Sigma_g]$. Note that this group cohomology is canonically identified with the singular cohomology group $H^1(\Sigma_g; V_\omega)$ with coefficients in V_ω , since Σ_g is an Eilenberg-MacLane space. Here V_ω also denotes the flat hermitian vector bundle or the local coefficient system obtained from ω . Hereafter we identify these groups.

The group $\mathcal{M}_{g^*}^\omega$ acts on the cohomology group $H^1(\pi_1(\Sigma_g, *), V_\omega)$ by the pullback of the inverse. This action preserves the above hermitian form. This implies that we obtain the homomorphism $\zeta_{g^*}^\omega$ from $\mathcal{M}_{g^*}^\omega$ to the automorphism group of the hermitian vector space $H^1(\pi_1(\Sigma_g, *), V_\omega)$. The following lemma is needed in later sections, but it is clear.

Lemma 5. *The equality $\zeta_{g^*}^\omega(a) = (f^{-1})^* \zeta_{g^*}^{f^* \omega} (f^{-1} a f) f^*$ holds for $f \in \mathcal{M}_{g^*}$ and $a \in \mathcal{M}_{g^*}^\omega$.*

If we take an appropriate basis with respect to the hermitian form, we obtain a homomorphism

$$\zeta_{g^*}^\omega: \mathcal{M}_{g^*}^\omega \rightarrow U(p, q),$$

where p and q are nonnegative integers determined by the hermitian form and we use the same symbol $\zeta_{g^*}^\omega$.

Proposition 6. $sign_\omega = (\zeta_{g^*}^\omega)^* Sign$.

Under the setting and the hypothesis of Lemma 4, let $E \rightarrow Z$ be the flat hermitian vector bundle over Z obtained from the representation ω_Z .

We define the flat hermitian vector bundle $\mathcal{H}$ over X as follows.

The fiber $\mathcal{H}_x$ at $x \in X$ is the cohomology group

$$H^1(\pi^{-1}(x); E|_{\pi^{-1}(x)}) \cong H^1(\pi_1(\pi^{-1}(x), s(x)), E|_{\pi^{-1}(x)}) \cong H^1(\pi_1(\Sigma_g, *), V_\omega).$$

The hermitian form is given by i times the cup-product, the hermitian product on E and the evaluation on the fundamental class of the fiber $\pi^{-1}(x)$ at $x \in X$. (The inverse of) the holonomy homomorphism of the flat bundle $\mathcal{H}$ is given by $\tau: \pi_1(X, x_0) \xrightarrow{h} \mathcal{M}_{g^*}^\omega \xrightarrow{\zeta_{g^*}^\omega} U(p, q)$.

Thus we have two signarures $sign(Z, E)$ and $sign(X, \mathcal{H})$.

Lemma 7. *Under the above setting, the equality $sign(Z, E) = sign(X, \mathcal{H})$ holds.*

For any $a, b \in \mathcal{M}_{g^*}^\omega$, we take the Σ_g -bundle $M_{a,b} \rightarrow P$ as the one $Z \rightarrow X$ in Lemma 7. Then this lemma implies Proposition 6.

Proof of Lemma 7. For the fiber bundle $\pi: Z \rightarrow X$ and the flat vector bundle (or the local system) $E \rightarrow Z$ over Z , there is a spectral sequence $(E_r^{p,q}(Z, \partial Z; E), d_r)$ for the cohomology group $H^*(Z, \partial Z; E)$ with $E_2^{p,q} \cong H^p(X, \partial X; H^q(\pi; E))$ of the fibration $\pi: Z \rightarrow X$. Here $H^q(\pi; E)$ denotes the flat vector bundle or the local system over X whose fiber $H^q(\pi; E)_x$ at $x \in X$ is given by $H^q(\pi^{-1}(x); E|_{\pi^{-1}(x)})$ (see [11, 18, 25]). Therefore we have $E_2^{p,q}(Z, \partial Z; E) = 0$ for $p \neq 0, 1, 2$ or $q \neq 0, 1, 2$. Thus we deduce that

$$H^2(Z, \partial Z; E) \cong \sum_{p+q=2} E_\infty^{p,q}(Z, \partial Z; E) \cong (E_3^{2,0} \oplus E_2^{1,1} \oplus E_3^{0,2})(Z, \partial Z; E).$$

Similarly we obtain isomorphisms $H^2(Z; E) \cong (E_3^{2,0} \oplus E_2^{1,1} \oplus E_3^{0,2})(Z; E)$ and $H^4(Z, \partial Z; \mathbb{C}) \cong E_2^{2,2}(Z, \partial Z; \mathbb{C}) \cong H^2(Z, \partial Z; H^2(\Sigma_g; \mathbb{C})) \cong \mathbb{Z}$. The product

$$\langle , \rangle_E: H^2(Z, \partial Z; E) \times H^2(Z; E) \rightarrow H^4(Z, \partial Z; \mathbb{C})$$

corresponds to the product

$$(E_3^{2,0} \oplus E_2^{1,1} \oplus E_3^{0,2})(Z, \partial Z; E) \times (E_3^{2,0} \oplus E_2^{1,1} \oplus E_3^{0,2})(Z; E) \rightarrow E_2^{2,2}(Z, \partial Z; \mathbb{C})$$

in the spectral sequence which is defined by the cup product and the hermitian form on E . Here we note $E_2^{2,2}(Z, \partial Z; \mathbb{C}) \cong E_3^{2,2}(Z, \partial Z; \mathbb{C})$. In this product, nonzero pairings are

$$E_r^{p,2-p}(Z, \partial Z; E) \times E_r^{2-p,p}(Z; E) \rightarrow E_r^{2,2}(Z, \partial Z)$$

for $p = 0, 1, 2$ and $r = 2, 3$ and the others are 0. In particular, under the identification $E_2^{1,1}(Z; E) \cong H^1(X; H^1(\pi; E))$ etc., the product on $E_2^{1,1}$ corresponds to $-(\ , \)_{H^1(\pi; E)}$ where $(\ , \)_{H^1(\pi; E)}$ is the one defined by the cup product and the skew-hermitian form on the local coefficient system $H^1(\pi; E)$ which is modelled on the skew-hermitian form on $H^1(\pi_1(\Sigma_g, *), V_\omega)$.

Let $\hat{E}_r^{p,q}$ be the image of the natural homomorphism $E_r^{p,q}(Z, \partial Z; E) \rightarrow E_r^{p,q}(Z; E)$. Then we have

$$\begin{aligned} \text{sign}(Z, E) &= \text{sign}(\hat{H}^2(Z; E)) = \text{sign}(\hat{E}_3^{2,0} \oplus \hat{E}_2^{1,1} \oplus \hat{E}_3^{0,2}) \\ &= \text{sign}(\hat{E}_2^{1,1}) + \text{sign}(\hat{E}_3^{2,0} \oplus \hat{E}_3^{0,2}) = \text{sign}(\hat{E}_2^{1,1}). \end{aligned}$$

Here $\text{sign}(\)$ except for the first in this equalities denotes the signature of the hermitian form on the vector space in it. The hermitian form on $\hat{E}_2^{1,1} = \hat{H}^1(X; H^1(\pi; E))$ is $-(\ , \)_{H^1(\pi; E)} = i^2(\ , \)_{H^1(\pi; E)}$. It is easy to see that the flat hermitian bundle $H^1(\pi; E)$ over X with the hermitian form $i(\ , \)_{H^1(\pi; E)}$ is isomorphic to $\mathcal{H}$ since the representation ω_Z is trivial on the image of the section $s: X \rightarrow Z$. By the definition of signature, we obtain $\text{sign}(Z, E) = \text{sign}(Z; \mathcal{H})$. $\square$

In the case that the representation ω factors through the orthogonal group $O(n)$, it turns out that the homomorphism $\zeta_{g^*}^\omega$ also factors through the symplectic group $Sp(2p, \mathbb{R})$. It is clear that the signature cocycle of $Sp(2p, \mathbb{R})$ agrees with the pullback of the one of $U(p, p)$ by the natural inclusion $Sp(2p, \mathbb{R}) \hookrightarrow U(p, p)$. This implies the same result which is obtained by replacing the targets $U(n)$, $U(p, p)$ of ω , $\zeta_{g^*}^\omega$ and the signature cocycle of $U(p, p)$ by $O(n)$, $Sp(2p, \mathbb{R})$ and the one of $Sp(2p, \mathbb{R})$ respectively.

Now we recall the definition of the ρ -invariants (see [3]).

Let M be an oriented Riemannian manifold of dimension $2l - 1$ and $\alpha: \pi_1(M) \rightarrow U(n)$ a unitary representation. The self-adjoint operator on even forms on M ;

$$D: \Omega^{\text{even}}(M; \mathbb{C}) \rightarrow \Omega^{\text{even}}(M; \mathbb{C})$$

is defined by

$$D(\phi) = i^l (-1)^{p+1} (*d - d*)\phi,$$

where ϕ is a form of degree $2p$. Moreover the extension of D to the even forms with coefficients in the flat vector bundle of rank n defined by α is denoted by D_α , which is also a self-adjoint operator.

For the operator D_α , we define the function $\eta_\alpha(s)$ by

$$\eta_\alpha(s) = \sum_{\lambda \neq 0} (\text{sign} \lambda) |\lambda|^{-s},$$

where λ runs over the eigenvalues of D_α . The corresponding function to D is denoted by $\eta(s)$. These functions extend at $s = 0$ and have finite values here. Its values $\eta(0)$ and $\eta_\alpha(0)$ are called the η -invariants of a Riemannian manifold.

Put $\tilde{\eta}_\alpha(s) := \eta_\alpha(s) - m\eta(s)$.

Theorem 8 (Atiyah-Patodi-Singer [3]). *$\tilde{\eta}_\alpha(0)$ is independent of the metric. It is a diffeomorphism invariant of M and α which we shall denote by $\rho_\alpha(M)$. If $M = \partial N$ with α extending to a unitary representation of $\pi_1(N)$ then*

$$\rho_\alpha(M) = n \operatorname{sign}(N) - \operatorname{sign}_\alpha(N).$$

For $a \in \mathcal{M}_{g^*}^\omega$, let M_a be the Σ_g -bundle over S^1 with monodromy a^{-1} and $s: S^1 \rightarrow M_a$ the section obtained from the base point $* \in \Sigma_g$. We may have an identification ι from Σ_g to the fiber of M_a at the base point 1 of S^1 . By Lemma 4, there is a unique homomorphism $\omega_a := \omega_{M_a}: \pi_1(M_a, s(1)) \rightarrow U(n)$ satisfying $\iota^* \omega_a = \omega$ and $s^* \omega_a = I$. Let $\rho_\omega: \mathcal{M}_{g^*}^\omega \rightarrow \mathbb{R}$ be the function defined by $a \mapsto \rho_{\omega_a}(M_a)$ for $a \in \mathcal{M}_{g^*}^\omega$. Clearly it is $\mathcal{M}_{g^*}^\omega$ -conjugacy invariant.

We apply Theorem 8 to the case of $N = M_{a,b}$ and $\alpha = \omega_{a,b}$ where $a, b \in \mathcal{M}_{g^*}^\omega$. Then we have Proposition 1.

3. SOME 1-COCHAINS ON THE WHOLE MAPPING CLASS GROUP

In this section we show a similar result to Proposition 1 on the whole mapping class group $\mathcal{M}_{g^*}$.

Let G be a finite group and $r: G \rightarrow U(n)$ a unitary representation. We consider an orbit by the action of $\mathcal{M}_{g^*}$ on the representation space $\operatorname{Hom}(\pi_1(\Sigma_g, *), G)$ by the pullback. The image of it by the map $r_*: \operatorname{Hom}(\pi_1(\Sigma_g, *), G) \rightarrow \operatorname{Hom}(\pi_1(\Sigma_g, *), U(n))$ is denoted by $\mathcal{O}$. Let $\mathcal{H}_\mathcal{O}$ denote the hermitian vector space $\bigoplus_{\omega \in \mathcal{O}} H^1(\pi_1(\Sigma_g, *), V_\omega)$. The action of $\mathcal{M}_{g^*}$ on $\mathcal{H}_\mathcal{O}$ defined by the pullback of the inverse induces the homomorphism

$$\zeta_{g^*}^\mathcal{O}: \mathcal{M}_{g^*} \rightarrow \dot{U}(\mathcal{H}_\mathcal{O}),$$

where $U(\mathcal{H}_\mathcal{O})$ is the unitary group of the hermitian vector space $\mathcal{H}_\mathcal{O}$. Let $\mathcal{M}_{g^*}^\mathcal{O}$ be the subgroup of $\mathcal{M}_{g^*}$ consisting of elements which induce the identity on $\mathcal{O}$. Then we have the following short exact sequence:

$$1 \rightarrow \mathcal{M}_{g^*}^\mathcal{O} \rightarrow \mathcal{M}_{g^*} \rightarrow F \rightarrow 1$$

where F is a subgroup of the symmetric group of $\mathcal{O}$.

Lemma 9. *The equality $[(\zeta_{g^*}^\mathcal{O})^* \operatorname{Sign}] = n\#\mathcal{O}[\operatorname{sign}]$ holds in $H^2(\mathcal{M}_{g^*}, \mathbb{Q})$.*

Proof. The restriction of $\zeta_{g^*}^\mathcal{O}$ to $\mathcal{M}_{g^*}^\mathcal{O}$ agrees with the homomorphism $\prod_{\omega \in \mathcal{O}} \zeta_{g^*}^\omega: \mathcal{M}_{g^*}^\mathcal{O} \rightarrow \prod_{\omega \in \mathcal{O}} U(p_\omega, q_\omega) \hookrightarrow U(\mathcal{H}_\mathcal{O})$, where p_ω and q_ω are some nonnegative integers. On the subgroup $\mathcal{M}_{g^*}^\mathcal{O}$, using Proposition 6, we have

$$\begin{aligned} (\zeta_{g^*}^\mathcal{O})^* \operatorname{Sign} &= \left(\prod_{\omega \in \mathcal{O}} \zeta_{g^*}^\omega \right)^* \operatorname{Sign} = \sum_{\omega \in \mathcal{O}} \{ (\zeta_{g^*}^\omega)^* \operatorname{Sign} \} \\ &= \sum_{\omega \in \mathcal{O}} \operatorname{sign}_\omega = n\#\mathcal{O} \operatorname{sign} - \delta \left(\sum_{\omega \in \mathcal{O}} \rho_\omega \right). \end{aligned}$$

Here we note that if a representation ω factors through a finite group, then the ρ -invariant obtained from it takes values in the rational numbers. Thus the equality $[(\zeta_{g^*}^{\mathcal{O}})^* \text{Sign}] = n\sharp[\text{sign}]$ holds in $H^2(\mathcal{M}_{g^*}^{\mathcal{O}}; \mathbb{Q})$.

Since the subgroup $\mathcal{M}_{g^*}^{\mathcal{O}}$ of $\mathcal{M}_{g^*}$ is of finite index, we can consider the transfer map. Then it turns out that the natural homomorphism $H^2(\mathcal{M}_{g^*}; \mathbb{Q}) \rightarrow H^2(\mathcal{M}_{g^*}^{\mathcal{O}}; \mathbb{Q})$ is injective (see [5]). Since the two cohomology classes are defined on $\mathcal{M}_{g^*}$, we have the equality in $H^2(\mathcal{M}_{g^*}; \mathbb{Q})$. $\square$

By this lemma and the fact of $H^1(\mathcal{M}_{g^*}, \mathbb{Q}) = 0$, there exists a unique 1-cochain $d^{\mathcal{O}}: \mathcal{M}_{g^*} \rightarrow \mathbb{Q}$ such that $n\sharp \mathcal{O} \text{sign} - (\zeta_{g^*}^{\mathcal{O}})^* \text{Sign} = \delta d^{\mathcal{O}}$. It is easy to see that it is conjugacy invariant. Using Lyndon-Hochschild-Serre spectral sequence for it (see [11]), we can deduce $H^1(\mathcal{M}_{g^*}^{\mathcal{O}}, \mathbb{Q})^F = 0$ from the fact of $H^1(\mathcal{M}_{g^*}, \mathbb{Q}) = 0$ and that F is a finite group. Here $H^1(\mathcal{M}_{g^*}^{\mathcal{O}}, \mathbb{Q})^F$ is the invariant subspace of the action of F on $H^1(\mathcal{M}_{g^*}^{\mathcal{O}}, \mathbb{Q})$ which is obtained from the short exact sequence $1 \rightarrow \mathcal{M}_{g^*}^{\mathcal{O}} \rightarrow \mathcal{M}_{g^*} \rightarrow F \rightarrow 1$ (see [5]).

On the other hand, the function

$$\sum_{\omega \in \mathcal{O}} \rho_{\omega}: \mathcal{M}_{g^*}^{\mathcal{O}} \rightarrow \mathbb{Q}$$

is $\mathcal{M}_{g^*}$ -conjugacy invariant, hence F -invariant. Moreover it satisfies the equality

$$n\sharp \mathcal{O} \text{sign} - (\zeta_{g^*}^{\mathcal{O}})^* \text{Sign} = \delta \left(\sum_{\omega \in \mathcal{O}} \rho_{\omega} \right)$$

on $\mathcal{M}_{g^*}^{\mathcal{O}}$ by Proposition 1. Therefore the difference $d^{\mathcal{O}} - \sum_{\omega \in \mathcal{O}} \rho_{\omega}$ defines an element of $H^1(\mathcal{M}_{g^*}^{\mathcal{O}}, \mathbb{Q})$. This implies $d^{\mathcal{O}} = \sum_{\omega \in \mathcal{O}} \rho_{\omega}$ on $\mathcal{M}_{g^*}^{\mathcal{O}}$.

Put $\mathcal{I}_{g^*}(\mathcal{O}) := \ker \zeta_{g^*}^{\mathcal{O}}$, then the restriction of $d^{\mathcal{O}}$ to $\mathcal{I}_{g^*} \cap \mathcal{I}_{g^*}(\mathcal{O})$ is a homomorphism. The above results are summed up to the following proposition.

Proposition 10. *The unique 1-cochain $d^{\mathcal{O}}: \mathcal{M}_{g^*} \rightarrow \mathbb{Q}$ such that $n\sharp \mathcal{O} \text{sign} - (\zeta_{g^*}^{\mathcal{O}})^* \text{Sign} = \delta d^{\mathcal{O}}$ agrees with $\sum_{\omega \in \mathcal{O}} \rho_{\omega}$ on $\mathcal{M}_{g^*}^{\mathcal{O}}$. Its restriction to $\mathcal{I}_{g^*} \cap \mathcal{I}_{g^*}(\mathcal{O})$ is a homomorphism.*

Next we give some applications of Proposition 1 and 10. For this, we recall the definition of Meyer's function ϕ_i on $\mathcal{M}_i$ for $i = 1, 2$. It is a unique $\mathbb{Q}$ -valued function which is characterized by the property that its coboundary is Meyer's signature 2-cocycle sign_i on $\mathcal{M}_i$ (see [16, 17]). Note that it is conjugacy invariant and that $\mathcal{M}_1$ is identified with $SL(2, \mathbb{Z})$. Moreover ϕ_1 is described explicitly by using the Dedekind sums or the Rademacher function (see [1, 14, 17, 23]).

We consider the case of $g = 1$. Let $\omega: \pi_1(\Sigma_1, *) \rightarrow U(1)$ be a nontrivial homomorphism. It is easy to see that the cohomology group $H^*(\pi_1(\Sigma_1, *), V_{\omega})$ vanishes. This and Proposition 6 imply $\text{sign}_{\omega} = 0$ on $\mathcal{M}_{1*}^{\omega}$. Then we have $\text{sign}_1 = \delta \rho_{\omega}$ on $\mathcal{M}_{1*}^{\omega}$ by Proposition 1. Considering the difference between the Meyer function ϕ_1 and ρ_{ω} on $\mathcal{M}_{1*}^{\omega}$, we have the following corollary.

Corollary 11. *The difference $\phi_1 - \rho_{\omega}$ is a homomorphism on $SL(2, \mathbb{Z})^{\omega}$. In other words, it defines an element of $H^1(SL(2, \mathbb{Z})^{\omega}, \mathbb{R})$.*

Here we remark that the groups $\mathcal{M}_1, \mathcal{M}_{1*}$ and $SL(2, \mathbb{Z})$ coincide.

Let $\omega: \pi_1(\Sigma_1, *) \rightarrow \mathbb{Z}_p \xrightarrow{\tau} U(1)$ be a nontrivial homomorphism factoring through $\mathbb{Z}_p$. Let $\mathcal{O}$ be the image of the orbit of ω . Then we have the following proposition.

Proposition 12. *On $SL(2, \mathbb{Z})^{\mathcal{O}}$, the equality $\phi_1 = \frac{1}{\#\mathcal{O}} \sum_{\omega \in \mathcal{O}} \rho_{\omega}$ holds.*

The same argument as above extends Proposition 19 in [13]. Now we recall the definition of the function μ on the mapping class group $\mathcal{M}_2$ of a surface of genus 2 in it. We consider a homomorphism $\omega: \pi_1(\Sigma_2, *) \rightarrow U(1)$ which factors through $r: \mathbb{Z}_2 = \{\pm 1\} \hookrightarrow U(1)$. In this case the action of $\mathcal{M}_2^{\omega}$ preserves the image $Im[H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_{\omega}) \rightarrow H^1(\pi_1(\Sigma_2, *), V_{\omega})]$ which is isomorphic to $\mathbb{Z}^2$. Here $\mathbb{Z}_{\omega}$ denotes the submodule of V_{ω} obtained from $\mathbb{Z} \subset \mathbb{C}$. Hence the target of the homomorphism ζ_{2*}^{ω} turns out to be $SL(2, \mathbb{Z})$. We define the function $\mu_{*}^{\omega}: \mathcal{M}_2^{\omega} \rightarrow \mathbb{Q}$ by $\mu_{*}^{\omega} = \rho_{\omega} + \phi_1 \circ \zeta_{2*}^{\omega}$. It turns out that this function μ_{*}^{ω} descends to a function on $\mathcal{M}_2^{\omega}$, which is denoted by μ^{ω} . Hence we can define the function $\mu: \mathcal{M}_2 \rightarrow \mathbb{Q}$ by $\mu(f) = \frac{1}{15} \sum_{\omega} \mu^{\omega}(f)$, where ω runs over the nonzero classes of $H^1(\Sigma_2; \mathbb{Z}_2)$ fixed by f^* , which is the one mentioned above. It is easy to see that the function μ is $\mathcal{M}_2$ -conjugacy invariant. Moreover, on $\mathcal{M}_2^{\mathcal{O}'}$ where $\mathcal{O}'$ denotes the image by r_* of $H^1(\Sigma_2, \mathbb{Z}_2) \setminus \{0\} = Hom(\pi_1(\Sigma_2, *), \mathbb{Z}_2) \setminus trivial$, its coboundary coincides with the signature cocycle of $\mathcal{M}_2$.

Theorem 13. *On $\mathcal{M}_2^{\mathcal{O}'}$, the function μ coincides with the Meyer function ϕ_2 .*

Since each ω is the pullback of the others by some mapping class in $\mathcal{M}_2$, we have half of the following corollary in the same way as above.

Corollary 14. *For any nontrivial homomorphism $\omega: \pi_1(\Sigma_2, *) \rightarrow U(1)$ factoring through $\mathbb{Z}_2$, the difference $\phi_2 - \mu^{\omega}$ is nontrivial homomorphism on $\mathcal{M}_2^{\omega}$.*

The nontriviality of the difference in this corollary is obtained by the example in [13].

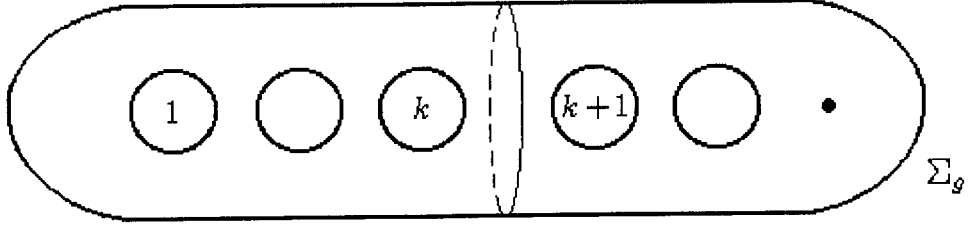
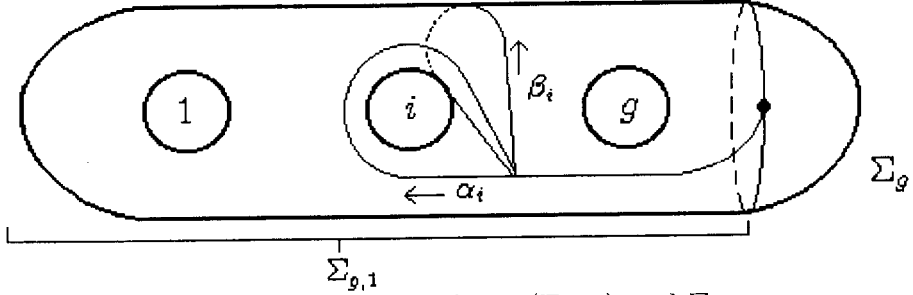
In the same way as this example in [13], we can give a nontrivial example of Corollary 11.

4. THE IMAGE OF $\mathcal{K}_{g*}$ BY ζ_{g*}^{ω}

In this section we determine the image of $\mathcal{K}_{g*}$ by ζ_{g*}^{ω} in the case of the homomorphism $\omega: \pi_1(\Sigma_g, *) \rightarrow U(1)$ factors through $\mathbb{Z}_2$.

In this case, the homomorphism ζ_{g*}^{ω} factors through the symplectic automorphism group of the cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_{\omega})/torsion$ modulo torsion, which is denoted by H_{ω} . Here $\mathbb{Z}_{\omega}$ is the $\pi_1(\Sigma_g, *)$ -module $\mathbb{Z}$ obtained from ω . Thus if we take a symplectic basis for H_{ω} , we have a decomposition $\zeta_{g*}^{\omega}: \mathcal{M}_{g*}^{\omega} \rightarrow Sp(2(g-1), \mathbb{Z}) \hookrightarrow U(g-1, g-1)$. In this section and the next, we use the same symbol ζ_{g*}^{ω} for the homomorphism obtained by replacing the target $U(g-1, g-1)$ of ζ_{g*}^{ω} by $Sp(2(g-1), \mathbb{Z})$. We note that it is an epimorphism to $Sp(2(g-1), \mathbb{Z})$ (see [13]).

Since nonzero elements of $H^1(\Sigma_g; \mathbb{Z}_2)$ move to each other by the action of $\mathcal{M}_{g*}$ by the pullback, the action on H_{ω} of the Dehn twist along a separating simple closed curve is conjugate to the one of $D_k \in \mathcal{K}_{g*}$ on H_{ω} by an element of $\mathcal{M}_{g*}$. Here D_k is the Dehn twist along the separating simple closed curve depicted in Figure 1 for $1 \leq k \leq g-1$ and $v \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus 0$ is one of α_1^*, α_g^* and $\alpha_1^* + \alpha_g^*$. Here a basis α_i^*, β_i^* for $H^1(\Sigma_g; \mathbb{Z}_2)$ is the dual to the homology basis depicted in Figure 2.

FIGURE 1. The separating simple closed curve defining D_k .FIGURE 2. A basis α_i, β_i for $\pi_1(\Sigma_g, *)$ and $\Sigma_{g,1}$.

The fundamental group $\pi_1(\Sigma_g, *)$ of the surface Σ_g is presented by $\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \mid \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \rangle$, where the basis α_i, β_i are depicted in Figure 2. The induced homomorphism D_k^* on $\pi_1(\Sigma_g, *)$ by the Dehn twist D_k for $1 \leq k \leq g-1$ is given by

$$\alpha_i \mapsto \begin{cases} l_k^{-1} \alpha_i l_k & 1 \leq i \leq k \\ \alpha_i & k+1 \leq i \leq g, \end{cases} \quad \beta_i \mapsto \begin{cases} l_k^{-1} \beta_i l_k & 1 \leq i \leq k \\ \beta_i & k+1 \leq i \leq g, \end{cases}$$

where $l_k = \prod_{i=1}^k [\alpha_i, \beta_i]$.

We consider the case of $v = \alpha_1^* + \alpha_g^*$. The space $Z^1(\pi_1(\Sigma_g, *), \mathbb{Z}_v)$ of v -crossed homomorphisms on $\pi_1(\Sigma_g, *)$ can be regarded as a subset of $Z^1(\pi_1(\Sigma_{g,1}, *), \mathbb{Z}_v)$ via the induced map by the inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_g$, where $\Sigma_{g,1}$ is the complement of an embedded open disk (see Figure 2). We note that the fundamental group of $\Sigma_{g,1}$ is a free group of rank $2g$. Let $E^1, F^1, \dots, E^g, F^g, W$ be the basis for $Z^1(\pi_1(\Sigma_{g,1}, *), \mathbb{Z}_v)$ determined by $E^i(\alpha_j) = \delta_j^i$, $F^i(\beta_j) = 0$, $F^i(\alpha_j) = 0$ and $F^i(\beta_j) = \delta_j^i$. Then as a basis for $Z^1(\pi_1(\Sigma_g, *), \mathbb{Z}_v)$, we can choose $U := -F^1 + F^g$, $V := E^g, E^2, F^2, \dots, E^{g-1}, F^{g-1}$, $W := E^1 + E^g$. The space $B^1(\pi_1(\Sigma_g, *), \mathbb{Z}_v)$ of principal crossed homomorphisms on $\pi_1(\Sigma_g, *)$ is given by $2\mathbb{Z}W$. Thus a basis for H_v can be given by $U, V, E^2, F^2, \dots, E^{g-1}, F^{g-1}$. It is easy to see that this is a symplectic basis (See [10] for calculation of symplectic forms).

Direct computation shows that the induced homomorphism D_k^* on H_v is given by $U \mapsto U + 4V - 4W$, $V \mapsto V$, $E^i \mapsto E^i$ and $F^i \mapsto F^i$ ($2 \leq i \leq g-1$). Hence we have

$$\zeta_{g^*}^v(D_k) = (D_k^{-1})^* = \begin{cases} I_{2(g-1)} & v = \alpha_1^* \text{ and } \alpha_g^* \\ \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \oplus I_{2(g-2)} & v = \alpha_1^* + \alpha_g^* \end{cases}$$

with respect to the above basis for H_v .

Similarly we have $\zeta_{g^*}^v(D_k) = I_{2(g-1)}$ for $1 \leq k \leq g-1$ in the cases of $v = \alpha_1^*$ and α_g^* .

This implies the following proposition since the subgroup $\mathcal{K}_{g^*}$ of $\mathcal{M}_{g^*}$ is generated by all the Dehn twists along separating simple closed curves on Σ_g and the homomorphism $\zeta_{g^*}^v$ is surjective (see [13]).

Proposition 15. *The image of $\mathcal{K}_{g^*}$ by the homomorphism $\zeta_{g^*}^\omega$ is the smallest normal subgroup of $Sp(2(g-1), \mathbb{Z})$ containing $\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \oplus I_{2(g-2)}$ for any nonzero $\omega \in H^1(\Sigma_g; \mathbb{Z}_2)$.*

5. THE IMAGE OF THE TORELLI GROUP $\mathcal{I}_{g^*}$ BY $\zeta_{g^*}^\omega$

In this section we determine the image of $\mathcal{I}_{g^*}$ by $\zeta_{g^*}^\omega$ for ω in section 4 and $g \geq 3$.

By the definition, a boundary pair (BP) map of genus k is the composition of the Dehn twists in opposite directions along a boundary pair of curves which are conjugate to the ones depicted in Figure 3. Let B_k be the BP map of genus k obtained from the curves in Figure 3 for $1 \leq k \leq g-2$. The action of any BP maps of genus k on H_ω is conjugate to the one of the BP map B_k on H_v where $v = a\alpha_1^* + b\alpha_{k+1}^* + c\beta_{k+1}^* + d\alpha_g^*$ ($a, b, c, d = 0, 1$).

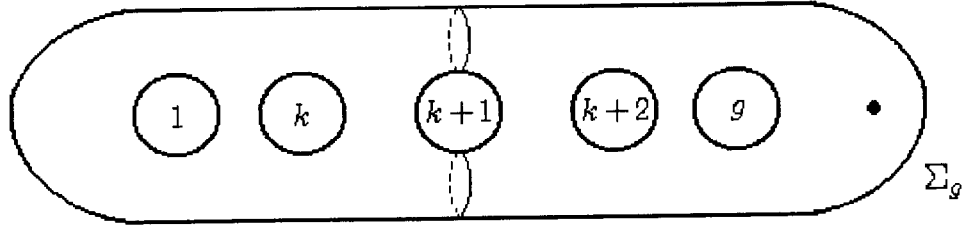


FIGURE 3. The bounding pair of curves defining B_k .

The induced homomorphism $B_{k*} : \pi_1(\Sigma_g, *) \rightarrow \pi_1(\Sigma_g, *)$ is given by

$$\alpha_i \mapsto \begin{cases} m_{k+1}^{-1} \alpha_i m_{k+1} & i \leq k \\ m_{k+1}^{-1} \alpha_{k+1} \beta_{k+1} & i = k+1 \\ \alpha_i & i \geq k+2 \end{cases} \quad \beta_i \mapsto \begin{cases} m_{k+1}^{-1} \beta_i m_{k+1} & i \leq k \\ \beta_i & i \geq k+1, \end{cases}$$

where $m_{k+1} = [\alpha_1, \beta_1] \dots [\alpha_k, \beta_k] \alpha_{k+1} \beta_{k+1} \alpha_{k+1}^{-1}$.

Next we compute the action of B_k on H_v and use the same notation in section 4.

In the case of $b = 1$, for the map B_k , we can choose a symplectic basis

$$\begin{aligned} & \langle E^1, (-1)^a (F^1 - aF^{k+1}), E^g, (-1)^d (F^g - dF^{k+1}) - 2adE^1, \\ & \quad E^2, F^2, \dots, \hat{E}^{k+1}, \hat{F}^{k+1}, \dots, E^{g-1}, F^{g-1} \rangle \end{aligned}$$

for H_v , where $\hat{E}^{k+1}, \hat{F}^{k+1}$ means that the bases E^{k+1}, F^{k+1} are removed from there. With respect to this basis the representation matrix of B_k^* is given by

$$\begin{pmatrix} (-1)^c & 2a(1-c) & 0 & 2acd \\ 0 & (-1)^c & 0 & 0 \\ 0 & -2acd & 1 & -2d(1-c) \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus (-1)^c I_{2(k-1)} \oplus I_{2(g-k-2)},$$

where $I_{2(k-1)}$ and $I_{2(g-k-2)}$ is the identity matrices of rank $2(k-1)$ and $2(g-k-2)$ respectively.

In the case of $b = 0$ and $c = 1$, a symplectic basis for H_v can be given by

$$\begin{aligned} < E^1, (-1)^a(F^1 + aE^{k+1}), E^g, (-1)^d(F^g + dE^{k+1}) - 2adE^1, \\ & E^2, F^2, \dots, \hat{E}^{k+1}, \hat{F}^{k+1}, \dots, E^{g-1}, F^{g-1} >. \end{aligned}$$

The representation matrix of B_k^* with respect to this basis is given by $-I_2 \oplus I_2 \oplus -I_{2(k-1)} \oplus I_{2(g-k-2)}$.

In the case of $b = c = 0$ and $d = 1$, a symplectic basis for H_v can be given by

$$< E^1, (-1)^a(F^1 - aF^g), E^{k+1}, F^{k+1}, E^2, F^2, \dots, \hat{E}^{k+1}, \hat{F}^{k+1}, \dots, E^{g-1}, F^{g-1} >.$$

The representation matrix of B_k^* with respect to this basis is given by

$$\begin{pmatrix} 1 & -4a & 0 & -2a \\ 0 & 1 & 0 & 0 \\ 0 & -2a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus I_{2(g-3)}.$$

In the case of $a = 1$ and $b = c = d = 0$, we obtain $B_k^* = I_{2(g-1)}$ for $1 \leq k \leq g-2$. The Torelli group $\mathcal{I}_{g^*}$ is generated by all the BP maps and the subgroup $\pi_1(\Sigma_g, *)$ of $\mathcal{M}_{g^*}$ (see [12]). The image of $\pi_1(\Sigma_g, *) \subset \mathcal{M}_{g^*}$ is $\{\pm I\}$ (see [13]). It is easy to see that minus identity can be obtained as a composition of above matrices and ones conjugate to them. The next proposition follows from the above computation.

Proposition 16. *For nonzero $\omega \in H^1(\Sigma_g; \mathbb{Z}_2)$, the image of $\mathcal{I}_{g^*}$ by $\zeta_{g^*}^\omega$ is the smallest normal subgroup of $Sp(2(g-1), \mathbb{Z})$ containing the above matrices.*

6. PROOF OF THEOREM 2

In this section we prove Theorem 2.

Let $\omega: \pi_1(\Sigma_g, *) \rightarrow U(1)$ be a nontrivial homomorphism factoring through $\mathbb{Z}_2$.

First we consider the case of $g = 2$. By the example in section 5 in [13], there exists $f \in \mathcal{K}_{2^*}$ such that $\zeta_{2^*}^\omega(f) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and $\mu_*^\omega(f) = \frac{4}{3}$. Here we consider $SL(2, \mathbb{Z})$ as the target of $\zeta_{2^*}^\omega$ by taking an oriented basis of $H^1(\pi_1(\Sigma_2, *), \mathbb{Z}_\omega)/torsion$. Since the homomorphism $\zeta_{2^*}^\omega$ is surjective, there exist s and t in $\mathcal{M}_{2^*}^\omega$ such that $\zeta_{2^*}^\omega(s) = S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\zeta_{2^*}^\omega(t) = T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We remark that $SL(2, \mathbb{Z})$ is presented by $\langle S, T | S^2 = (ST)^3, S^4 = I \rangle$ and that $\zeta_{2^*}^\omega(f)$ is T^4 .

For elements a, b of a group, put $Ad_a b := aba^{-1}$. Direct computation shows the following lemma.

Lemma 17. *The equality $Ad_S(T^4) \cdot T^4 \cdot Ad_A(T^4) \cdot Ad_{A^2}(T^4) \cdot Ad_{A^3 T^2 S}(T^4) \cdot Ad_{A^3}(T^4) = I$ holds in $SL(2, \mathbb{Z})$ where $A = STS^{-1}$.*

Put $l = Ad_s(f) \cdot f \cdot Ad_a(f) \cdot Ad_{a^2}(f) \cdot Ad_{a^3 t^2 s}(f) \cdot Ad_{a^3}(f)$, where $a = sts^{-1}$. Since $\mathcal{K}_{2*}$ is a normal subgroup of $\mathcal{M}_{2*}$, l belongs to it. Since μ_*^ω is conjugacy invariant on $\mathcal{M}_{2*}^\omega$ and is a homomorphism on $\mathcal{I}_{2*}$, we have $\mu_*^\omega(l) = 6\mu_*^\omega(f) = 6 \cdot \frac{4}{3} = 8$. On the other hand, Lemma 17 implies $l \in \mathcal{I}_{2*}(\omega)$, then we have $\rho_\omega(l) = \mu_*^\omega(l) = 8$. This shows that the function ρ_ω is nontrivial on $\mathcal{K}_{2*} \cap \mathcal{I}_{2*}(\omega)$.

Next we consider the case of $g \geq 3$. The above f , s and t can be chosen in $\mathcal{M}_{2,1}^\omega$. We regard $\Sigma_{2,1}$ as a subset of Σ_g in the obvious way. Let $\bar{\omega}: \pi_1(\Sigma_g, *) \rightarrow \mathbb{Z}_2 \rightarrow U(1)$ be the extension of ω whose restriction to $\Sigma_g \setminus \Sigma_{2,1}$ is trivial. Let $\bar{f}, \bar{s}$ and $\bar{t}$ be the elements of $\mathcal{M}_{g*}^\omega$ which are the extensions of f, s and t by the identity on $\Sigma_g \setminus \Sigma_{2,1}$ respectively. The element $\bar{l}$, which is similarly defined as l , belongs to $\mathcal{K}_{g*} \cap \mathcal{I}_{g*}(\bar{\omega})$.

Lemma 18. *Under the above situation, the equality $\rho_{\bar{\omega}}(\bar{l}) = \rho_\omega(l)$ holds.*

Proof. Let $D_1^2 \subset \Sigma_2$ and $D_2^2 \subset \Sigma_{g-2}$ be embedded disks. Put $\Sigma_{2,1} := \Sigma_2 \setminus \text{int} D_1^2$ and $\Sigma_{g-2,1} := \Sigma_{g-2} \setminus \text{int} D_2^2$. Since l is identity on D_1^2 , we can consider $D_1^2 \times S^1$ as a subset of M_l . Then we have the decomposition $M_l = (M_l \setminus D_1^2 \times S^1) \cup_{\partial D_1^2 \times S^1 = \partial D_2^2 \times S^1} (\Sigma_{g-2,1} \times S^1)$ where $\partial D_1^2 \times S^1$ is identified with $\partial D_2^2 \times S^1$ so that $\partial D_1^2 \times *$ is parallel to $\partial D_2^2 \times *$.

Let $\tilde{M}_l \rightarrow M_l$ be the $\mathbb{Z}_2$ -covering associated to the homomorphism $\omega_{M_l}: \pi_1(M_l, *) \rightarrow \mathbb{Z}_2$. There exists a compact oriented manifold Y such that its boundary is the disjoint union NM_l of N copies of M_l for some integer N and that the $\mathbb{Z}_2$ -covering $N\tilde{M}_l \rightarrow NM_l$ extends to a one $\tilde{Y} \rightarrow Y$ (see [4, 6]). This means that the pair $N(M_l, \omega_{M_l})$ extends to a one (Y, ω_Y) , where ω_Y is a homomorphism from $\pi_1(Y, *)$ to $\mathbb{Z}_2$. By Theorem 8, we have $N\rho_{\omega_{M_l}}(M_l) = \rho_{\omega_Y|_{NM_l}}(NM_l) = \text{sign}(Y) - \text{sign}_{\omega_Y}(Y)$. On the other hand, we note $\partial(H_{g-2} \times S^1) = \Sigma_{g-2} \times S^1$, where H_{g-2} is a handlebody of genus $g-2$. Let Z be the manifold $Y \cup_{N(D_1^2 \times S^1) = N(D_2^2 \times S^1)} (N(H_{g-2} \times S^1))$. It is easy to see that there exists a unique homomorphism $\omega_Z: \pi_1(Z, *) \rightarrow \mathbb{Z}_2$ satisfying $\omega_Z|_Y = \omega_Y$ and $\omega_Z|_{N(H_{g-2} \times S^1)} = 1$. Thus we have $\partial(Z, \omega_Z) = N(M_l, \bar{\omega})$. By Theorem 8, we have $N\rho_{\bar{\omega}}(M_l) = \rho_{N\bar{\omega}}(NM_l) = \text{sign}(Z) - \text{sign}_{\bar{\omega}}(Z)$. Applying the non-additivity formula for the signature by Wall (see [24]) to the above decomposition of Z , we have $\text{sign}(Z) = \text{sign}(Y) + N\text{sign}(H_{g-2} \times S^1) - N\sigma(V; A, B, C) = \text{sign}(Y)$, where $V = H_1(\partial(D_1^2 \times S^1); \mathbb{R})$, $A = \ker[V \rightarrow H_1(\Sigma_{g-2,1} \times S^1; \mathbb{R})]$, $B = \ker[V \rightarrow H_1(\Sigma_{2,1} \times S^1; \mathbb{R})]$ and $C = \ker[V \rightarrow H_1(D_1^2 \times S^1; \mathbb{R})]$. Here $\sigma(V; A, B, C)$ denote the signature of a certain symmetric bilinear form on the vector space $B \cap (C + A)/B \cap C + B \cap A$. In this case, since this vector space is $\{0\}$, we have $\sigma(V; A, B, C) = 0$. Similarly we can show $\text{sign}_{\omega_Z}(Z) = \text{sign}_{\omega_Y}(Y)$. Here we use Wall's nonadditivity formula for signature with local coefficient system (see [22]). Therefore we have $N\rho_{\bar{\omega}}(M_l) = \text{sign}(Z) - \text{sign}_{\omega_Z}(Z) = \text{sign}(Y) - \text{sign}_{\omega_Y}(Y) = N\rho_{\omega_{M_l}}(M_l)$, hence $\rho_\omega(M_l) = \rho_{\omega_{M_l}}(M_l)$. This completes the proof of the lemma. $\square$

This lemma implies that the function $\rho_{\bar{\omega}}$ is nontrivial on $\mathcal{K}_{g*} \cap \mathcal{I}_{g*}(\bar{\omega})$. Hence so is $\rho_{\omega'}$ on $\mathcal{K}_{g*} \cap \mathcal{I}_{g*}(\omega')$ for any nontrivial ω' since it is conjugate to $\rho_{\bar{\omega}}$ by an element of $\mathcal{M}_{g*}$.

In order to complete the proof of Theorem 2, we show that the homomorphism ρ_ω takes values in $\mathbb{Z}$ on $\mathcal{I}_{g*} \cap \mathcal{I}_{g*}(\omega)$. We need the fact that the ρ -invariants for $U(1)$ -representations on a closed oriented 3-manifold are constant as an $\mathbb{R}/\mathbb{Z}$ -valued function under deformation of $U(1)$ -representations (see [9])

For any $a \in \mathcal{I}_{g^*}$, the representation $\omega_a: \pi_1(M_a) \rightarrow U(1)$ can be deformed to the trivial representation through $U(1)$ -representations. Since the ρ -invariant for the trivial representation is zero, the above fact implies that the values of ρ_ω on $\mathcal{I}_{g^*} \cap \mathcal{I}_{g^*}(\omega)$ are in $\mathbb{Z}$. This completes the proof of Theorem 2.

7. σ_1 -INVARIANT AND DIFFERENCE DEGREE

Let M be a closed oriented three manifold. Since the double $2TM = TM \oplus TM$ of the tangent bundle of M is trivial and has a natural $Spin(6)$ -structure, we can consider a trivialization of $2TM$ as a $Spin(6)$ -bundle. Its homotopy class is called 2-framing of M . For two 2-framings β and γ of M , the difference degree $d(\beta, \gamma)$ between them is defined by the half of the first Pontrjagin class (or number) $\frac{1}{2}p_1(2T([0, 1] \times M), -\beta + \gamma)$ of the bundle $2T([0, 1] \times M)$ over $[0, 1] \times M$ trivialized on the boundaries $0 \times (-M)$ and $1 \times M$ by $-\beta$ and γ respectively in the obvious way. For any M , there is Atiyah's canonical 2-framing α_M which is characterized so that the equality $sign(Z) = \frac{1}{6}p_1(2TZ, \alpha_M)$ holds for any compact oriented four manifold Z with boundary $\partial Z = M$ (see [2]).

Let $\pi: \tilde{M} \rightarrow M$ be a finite covering over M . Then we have two 2-framing $\alpha_{\tilde{M}}$ and $\pi^*\alpha_M$ on $\tilde{M}$. The latter is the 2-framing obtained by pulling back of α_M by the covering map π . Thus we obtain the difference degree $d(\pi^*\alpha_M, \alpha_{\tilde{M}})$. Some relations of this degree to η -invariant are discussed by Morifuji (see [19]). In this section we also discuss the same in his paper, but from a different viewpoint.

Next we assume that the covering $\pi: \tilde{M} \rightarrow M$ is a G -covering, where G is a finite group. By G -signature theorem, we have invariants $\sigma_g(\tilde{M})$ for elements $g \neq 1$ of G . Put $\sigma_1(\tilde{M}) := -\sum_{g \in G \setminus 1} \sigma_g(\tilde{M})$. If there is a compact oriented four manifold Z with its boundary $\partial Z = M$ such that the G -covering $\pi: \tilde{M} \rightarrow M$ extends to a one $\pi: \tilde{Z} \rightarrow Z$, then the invariant $\sigma_g(\tilde{M})$ is given by $-sign(g, \tilde{Z})$ (see [4] for this symbol). Moreover we have $\sigma_1(\tilde{M}) = -sign(\tilde{Z}) + |G|sign(Z)$, where $|G|$ denote the order of G . In general, it is not possible to find such a manifold Z for any M . But it is possible for some multiple NM (see [4, 6]). Thus we define $\sigma_g(\tilde{M})$ by $\frac{1}{N}\sigma_g(N\tilde{M})$. It is known that any of the σ_g -invariants is calculated by using the ρ -invariants obtained from the homomorphism $\pi_1(M) \rightarrow G$ associated to the covering. The opposite direction is also true (see [3, 4]).

Lemma 19. *For any G -covering $\pi: \tilde{M} \rightarrow M$, the equality $\sigma_1(\tilde{M}) = -\frac{1}{3}d(\pi^*\alpha_M, \alpha_{\tilde{M}})$ holds.*

Proof(cf.[19]). For any G -covering $\pi: \tilde{M} \rightarrow M$, there exists a compact oriented four manifold Z such that the boundary of it is the disjoint union NM of N copies of M for some integer N and the G -covering $N\tilde{M} \rightarrow NM$ extends to a G -covering $\tilde{Z} \rightarrow Z$. Thus we have

$$\begin{aligned} & d(\pi^*\alpha_M, \alpha_{\tilde{M}}) \\ &= \frac{1}{2}p_1(2T([0, 1] \times M), -\pi^*\alpha_M + \alpha_{\tilde{M}}) = \frac{3}{N} \cdot \frac{1}{6}(p_1(T\tilde{Z}, N\alpha_{\tilde{M}}) - p_1(T\tilde{Z}, N\pi^*\alpha_M)) \\ &= \frac{3}{N}(sign(\tilde{Z}) - \frac{1}{6}|G|p_1(TZ, N\alpha_M)) = -\frac{3}{N}(-sign(\tilde{Z}) + |G|sign(Z)) \end{aligned}$$

$$= -\frac{3}{N}\sigma_1(NM).$$

By the definition of $\sigma_1(\tilde{M})$, we have $\sigma_1(\tilde{M}) = -\frac{1}{3}d(\pi^*\alpha_M, \alpha_{\tilde{M}})$. $\square$

As a corollary of the proof, we have the following which is essentially proved by Morifuji (see [19]).

Corollary 20. *Let $\pi: \tilde{M} \rightarrow M$ be a finite covering over a closed oriented 3-manifold M and Z a compact oriented 4-manifold with boundary M . If the covering π extends to a covering $\tilde{Z} \rightarrow Z$, then the equality $N\text{sign}(Z) - \text{sign}(\tilde{Z}) = -\frac{1}{3}d(\pi^*\alpha_M, \alpha_{\tilde{M}})$ holds where N denotes the degree of the covering.*

Next we rewrite the above result in terms of the mapping class group. Let Σ_g be a closed oriented surface of genus $g \geq 1$ and $*$ a base point of it. Let $p: \Sigma_h \rightarrow \Sigma_g$ be a connected G -covering, then the genus h of the surface Σ_h is given by $|G|(g-1)+1$. We give a base point $*$ of Σ_h so that the covering map p preserves the base points. Let $\omega: \pi_1(\Sigma_g, *) \rightarrow G$ be the homomorphism associated to the covering. Let $\mathcal{M}_{g*}^{ker\omega}$ be the subgroup of $\mathcal{M}_{g*}$ consisting of elements whose induced homomorphisms on the fundamental group of Σ_g map the subgroup $ker\omega$ onto itself. For any $a \in \mathcal{M}_{g*}^{ker\omega}$, there is a unique class $\lambda(a)$ of $\mathcal{M}_{h*}$ which is a lift of a preserving the base point of Σ_h . This define a homomorphism $\lambda: \mathcal{M}_{g*}^{ker\omega} \rightarrow \mathcal{M}_{h*}$ (cf.[15]). Moreover we have the following commutative diagram:

$$\begin{array}{ccc} M_{\lambda(a)} & \xrightarrow{\pi_a} & M_a \\ \downarrow & & \downarrow \\ S^1 & \xlongequal{\quad} & S^1, \end{array}$$

for any $a \in \mathcal{M}_{g*}^{ker\omega}$. Here the map $\pi_a: M_{\lambda(a)} \rightarrow M_a$ in the diagram is a covering of degree $|G|$. We define the function

$$d_\omega: \mathcal{M}_{g*}^{ker\omega} \rightarrow \mathbb{Z}$$

by $d_\omega(a) = d(\pi_a^*\alpha_{M_a}, \alpha_{M_{\lambda(a)}})$ for $a \in \mathcal{M}_{g*}^{ker\omega}$. If we restrict the domain to the subgroup $\mathcal{M}_{g*}^\omega$, we can also define the function

$$\sigma_\omega: \mathcal{M}_{g*}^\omega \rightarrow \mathbb{Q}$$

by $\sigma_\omega(a) = \sigma_1(M_{\lambda(a)})$ for $a \in \mathcal{M}_{g*}^\omega$. Then by Lemma 19, we have the following corollary.

Corollary 21. *The equality $\sigma_\omega = -\frac{1}{3}d_\omega$ holds on $\mathcal{M}_{g*}^\omega$.*

The above diagram is considered as a special case of the following. Let X be a compact oriented surface with a base point. Let $Z_\phi \rightarrow X$ be the Σ_g -bundle over X whose holonomies are determined by a homomorphism $\phi: \pi_1(X, *) \rightarrow \mathcal{M}_{g*}^{ker\omega}$ and $Z_{\lambda \circ \phi} \rightarrow X$ the Σ_h -bundle determined by the homomorphism $\lambda \circ \phi$. Then we have the following commutative diagram:

$$\begin{array}{ccc} Z_{\lambda \circ \phi} & \xrightarrow{\pi_\phi} & Z_\phi \\ \downarrow & & \downarrow \\ X & \xlongequal{\quad} & X, \end{array}$$

where $\pi_\phi: Z_{\lambda \circ \phi} \rightarrow Z_\phi$ is a covering of degree $|G|$. Applying Corollary 20 to the covering which is obtained by taking P as X , we can deduce Proposition 3.

In this setting, we consider the case of $g = 1$. Let $n \geq 2$ be an integer. We consider the covering $p: T^2 \rightarrow T^2$ induced by the map $n \times: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $\Sigma_1 = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The associated homomorphism is the natural one $\omega: \pi_1(T^2, *) \rightarrow H^1(T^2; \mathbb{Z}_n)$. In this case we take $g = h = 1$, $G = H^1(T^2; \mathbb{Z}_n)$, $|G| = n^2$ and $\lambda = \text{identity}$ in Proposition 3. Note that the identification $\mathcal{M}_1 = \mathcal{M}_{1*} = SL(2, \mathbb{Z})$ is well-known. Then we have the following corollary because of the uniqueness of the Meyer function ϕ_1 .

Corollary 22. *The equality $\phi_1 = -\frac{1}{3(n^2-1)}d_\omega$ holds on $SL(2, \mathbb{Z})$.*

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ON INTERSECTION NUMBER OF SUBMANIFOLDS OF THE MODULI SPACE OF FLAT CONNECTIONS

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ABSTRACT. We study an analogue of the construction of Casson's invariant. For a link of 2 components in a homology 3-sphere, by considering an appropriate Heegaard decomposition, we construct two submanifolds of the moduli space of flat connections on a nontrivial $PSU(N)$ -bundle over a closed surface. The above submanifolds and the moduli space are smooth manifolds without singularities. Therefore we can easily compute the intersection number of the submanifolds and the result is stated in terms of the linking number of the link.

1. INTRODUCTION

Casson's invariant is an integer valued invariant defined for oriented homology three spheres. Roughly speaking, it counts the number with signs of conjugacy classes of irreducible $SU(2)$ -representations of the fundamental group of an oriented homology three sphere. More precisely, we consider a Heegaard decomposition of the homology sphere. From this decomposition, we can construct two submanifolds in the moduli space of $SU(2)$ -representations of the fundamental group of the closed oriented surface which appears in the above decomposition. Each of the submanifolds consists of conjugacy classes of representations which are induced from ones of the fundamental group of a handlebody obtained by the decomposition. Casson's invariant is defined by the intersection number of the above submanifolds. The intersection points of them are nothing but the conjugacy classes of $SU(2)$ -representation of the fundamental group of the homology sphere. But the above submanifolds and the moduli space have singularities, so the situation is more complicated (see [1]).

In this paper we study an analogue of the construction of Casson's invariant. We notice that the associated $SU(2)$ -bundle to an $SU(2)$ -representation of the fundamental group of a homology three sphere is trivial. Thus we consider the exterior of a link of two components in a homology three sphere and $PSU(N)$ -representations of the fundamental group of it which define a nontrivial $PSU(N)$ -bundle instead of a homology three sphere and $SU(2)$ -representations respectively. By taking a Heegaard decomposition of the homology sphere satisfying certain conditions associated to the link, we construct two submanifolds of the moduli space of $PSU(N)$ -representations of the fundamental group of a surface which appears in the decomposition. Here the moduli space is identified with the one of stable bundles of a Riemann surface (see [2, 6]). In this case the above moduli space and two submanifolds are obtained as smooth manifolds without singularities. Thus we

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can easily compute the intersection number of them. The result is stated in terms of the linking number of a link.

This paper is organized as follows. In section 2, we state the main results. In section 3, we review the moduli space of $PSU(N)$ -representations of the surface group. In section 4, we construct submanifolds of the moduli space of representations. In section 5, we compute the linking number of a link in a homology three sphere in terms of Heegaard decomposition. In section 6, we prove the main theorem under additional conditions. In section 7, we complete the proof of the main theorem.

2. SETTING AND MAIN RESULT

Let H_g be an oriented handlebody of genus $g \geq 2$. Its boundary $\partial H_g = \Sigma_g$ is a closed oriented surface of genus g . A base point $*$ of H_g is chosen in its boundary Σ_g which is also a base point of Σ_g . Let $\kappa = \coprod_{i=1}^n \kappa_i$ be the union of pairwise disjoint embedded loops in the interior of H_g . Let $H_g(\kappa)$ be the complement of an open tubular neighborhood of κ in the interior of H_g . The inclusion of $\Sigma_g = \partial H_g$ into $H_g(\kappa)$ is denoted by j_g and its image is also denoted by Σ_g . Let T_i^2 be the connected component of the boundary $\partial H_g(\kappa)$ corresponding to κ_i for each i .

Since we want to consider the classes of κ_i 's in fundamental groups, we move them by homotopy so that they pass through the base point and fix them.

Definition 1. *A pair (H_g, κ) is called an admissible pair if it satisfies the following conditions.*

1. *There exists pairwise disjoint simple closed curves on Σ_g which are parallel to κ .*
2. *There exists a free basis of $\pi_1(H_g, *)$ which contains the classes of κ_i .*
3. *The induced homomorphism $j_{g*} : \pi_1(\Sigma_g, *) \rightarrow \pi_1(H_g(\kappa), *)$ is surjective.*

We remark that the inequality $n \leq g$ holds for an admissible pair.

Let $N \geq 2$ be an integer and e_N the complex number $e^{2\pi\sqrt{-1}/N}$. Put $G = SU(N)$ and $PG = PSU(N) = SU(N)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ is the center of $SU(N)$. The fundamental group of PG is given by

$$\pi_1(PG, 1) = \langle \varepsilon | \varepsilon^N = 1 \rangle,$$

where $1 \in PG$ is the identity and the generator ε is given as follows.

We take a path from the identity I to $e_N I$ in G which is unique up to homotopy since G is simply connected. Its image by the projection $G \rightarrow PG$ gives the class ε .

Let $Q \rightarrow H_g(\kappa)$ be a principal PG -bundle over $H_g(\kappa)$. The class $\varepsilon \in \pi_1(Q, \tilde{*})$ is defined by the image of the class $\varepsilon \in \pi_1(PG, 1)$ by the induced homomorphism from the map $PG \hookrightarrow Q$. This map is given by the action of PG on the principal PG -bundle Q at a basepoint $\tilde{*}$ of Q which is a lift of the one $*$ of $H_g(\kappa)$ for the bundle projection. Put

$$\begin{aligned} \tilde{\mathcal{R}}(Q) &= \{\rho \in \text{Hom}(\pi_1(Q, \tilde{*}), G) | \rho(\varepsilon) = e_N I\}, \\ \mathcal{R}(Q) &= \tilde{\mathcal{R}}(Q)/\text{Ad}G. \end{aligned}$$

Similarly we define $\tilde{\mathcal{R}}(P)$ and $\mathcal{R}(P)$ for any principal PG -bundle P . Note that the isomorphism classes of PG -bundles over a CW-complex of dimension ≤ 3 are classified by the primary obstruction of them, which is given by a class in the second cohomology

group with coefficient in $\mathbb{Z}_N = \pi_1(PG, 1)$ (see [5] or Lemma 6). We denote the primary obstruction of a PG -bundle P by $c(P)$ in this paper.

Put $c_0 := c(Q)([\Sigma_g])$ and $c_i := c(Q)([T_i^2])$ in $\mathbb{Z}_N$. Let $P := j_g^* Q \rightarrow \Sigma_g$ be the pullback bundle by the inclusion $j_g : \Sigma_g \rightarrow H_g(\kappa)$ and $\bar{j}_g : P \rightarrow Q$ the natural bundle map over j_g . It is well known that $\mathcal{R}(P)$ is a smooth manifold of dimension $2(\dim G)(g-1)$ if N and c are coprime (see [2, 6] or section 3). The following proposition is a part of Proposition 9 and an extension of a part of the main result of [4].

Proposition 2. *Suppose that a pair (H_g, κ) is admissible and that N and c_i are coprime for all $i = 0, 1, \dots, n$. Then the space $\mathcal{R}(Q)$ is diffeomorphic to $PG^{n-1} \times G^{g-n}$ and the induced map $\bar{j}_g^* : \mathcal{R}(Q) \rightarrow \mathcal{R}(P)$ by pullback is an embedding, hence its image is an orientable submanifold of $\mathcal{R}(P)$ of dimension $(\dim G)(g-1)$.*

Let M be an oriented homology 3-sphere and $\mathcal{L} = (\kappa, \kappa')$ a link of 2 components in it. Let (H'_g, Σ'_g) be a copy of (H_g, Σ_g) .

Definition 3. *A Heegaard decomposition $M = H_g \cup_f H'_g$ of M where $f : \Sigma_g = \partial H_g \rightarrow \Sigma'_g = \partial H'_g$ is an orientation reversing diffeomorphism is called admissible to a link $\mathcal{L}$ of 2 components if the following conditions are satisfied.*

1. $\kappa \subset \overset{\circ}{H}_g$ and $\kappa' \subset \overset{\circ}{H}'_g$.
2. (H_g, κ) and (H'_g, κ') are admissible.

Let $M(\mathcal{L}) = H_g(\kappa) \cup_f H'_g(\kappa')$ be the complement of an open tubular neighborhood of $\mathcal{L}$ in M , then we have $H^2(M(\mathcal{L}); \mathbb{Z}_N) \cong \mathbb{Z}_N$. Let $c \in H^2(M(\mathcal{L}), \mathbb{Z}_N)$ be a class with $(N, c) = 1$. The principal PG -bundle over $M(\mathcal{L})$ corresponding to the class c is denoted by $Q(c) \rightarrow M(\mathcal{L})$. Put $Q := Q(c)|_{H_g(\kappa)}$, $Q' := Q(c)|_{H'_g(\kappa')}$, $P := Q(c)|_{\Sigma_g}$ and $P' := Q(c)|_{\Sigma'_g}$, then they are also principal PG -bundles over $H_g(\kappa)$, $H'_g(\kappa')$, Σ_g and Σ'_g respectively. There are the induced bundle maps $\bar{j}_g, \bar{f}$ and $\bar{j}'_g$ over j_g, f and j'_g respectively in the following commutative diagram.

$$\begin{array}{ccccccc}
 Q & \xleftarrow{\bar{j}_g} & P & \xrightarrow{\bar{f}} & P' & \xrightarrow{\bar{j}'_g} & Q' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_g(\kappa) & \xleftarrow{j_g} & \Sigma_g & \xrightarrow{f} & \Sigma'_g & \xrightarrow{j'_g} & H'_g(\kappa').
 \end{array}$$

By Proposition 2, since N and c are coprime, we obtain two orientable submanifolds L and L' of $\mathcal{R}(P)$ where $L := \text{Im}[\bar{j}_g^* : \mathcal{R}(Q) \hookrightarrow \mathcal{R}(P)]$ and $L' := \text{Im}[(\bar{j}'_g \circ \bar{f})^* : \mathcal{R}(Q') \hookrightarrow \mathcal{R}(P)]$.

Theorem 4. *Let M be an oriented homology 3-sphere and $\mathcal{L}$ a link of 2 components in M . If N and c are coprime, then for any Heegaard decomposition of M admissible to $\mathcal{L}$, the equality $|L \cdot L'| = |lk_M(\kappa, \kappa')|^{N-1}$ holds, where $L \cdot L'$ denotes the intersection number of L and L' in $\mathcal{R}(P)$ and $lk_M(\kappa, \kappa')$ the linking number of κ and κ' in M with some orientations of L, L', κ and κ' .*

3. THE MODULI SPACE OF FLAT CONNECTIONS OVER A CLOSED SURFACE

In this section we shall recall some facts about the moduli space of flat PG -connections over a closed oriented surface (see [2, 6]).

Let $p : P \rightarrow \Sigma_g$ be the principal PG -bundle over a closed oriented surface Σ_g of genus $g \geq 2$ corresponding to a class $c \in H^2(\Sigma_g; \mathbb{Z}_N) \cong \mathbb{Z}_N$. Then the fundamental group of P is given by

$$\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \varepsilon \mid \prod_{i=1}^g [\alpha_i, \beta_i] - \varepsilon^c, \varepsilon^N = 1, \varepsilon : \text{center} \rangle.$$

The space $\mathcal{R}(P)$ can be identified with

$$\{\rho \in \text{Hom}(\pi_1(\Sigma_{g,1}), G) \mid \rho(\partial \Sigma_{g,1}) = e_N^c\} / \text{Ad}G,$$

where $\Sigma_{g,1}$ is the compact oriented surface of genus g with one boundary. If N and c are coprime, the space $\mathcal{R}(P)$ is a closed, simply connected, symplectic manifold of dimension $2(\dim G)(g-1)$ with symplectic structure ω . Note that $\dim G$ equals $N^2 - 1$.

The tangent space $T_{[\rho]} \mathcal{R}(P)$ at $[\rho] \in \mathcal{R}(P)$ can be identified with the first cohomology groups $H^1(\pi_1(P), su(N)_{\text{Ad}\rho}) \cong H^1(\pi_1(\Sigma_g), su(N)_{\text{Ad}\rho})$ with coefficients in the flat vector bundles $su(N)_{\text{Ad}\rho}$ and $su(N)_{\text{Ad}\underline{\rho}}$ defined by the adjoint representations of ρ and $\underline{\rho}$, where $\underline{\rho} : \pi_1(\Sigma_g, *) \rightarrow PG$ is the representation obtained from ρ in the obvious way. Under this identification, the symplectic structure ω is given by the cup-product, minus the trace on $su(N)$ and the evaluation on the fundamental class $[\Sigma_g]$:

$$\omega_{[\rho]} : H^1(\pi_1(\Sigma_g), su(N)_{\text{Ad}\rho}) \times H^1(\pi_1(\Sigma_g), su(N)_{\text{Ad}\rho}) \xrightarrow{-\text{tr}(\cup)} H^2(\pi_1(\Sigma_g); \mathbb{R}) \xrightarrow{\langle \cdot, [\Sigma_g] \rangle} \mathbb{R}.$$

Any bundle automorphism $\bar{f} : P \rightarrow P$ over an orientation preserving (resp. reversing) diffeomorphism $f : \Sigma_g \rightarrow \Sigma_g$ induces a symplectic (resp. anti-symplectic) diffeomorphism $f^* : \mathcal{R}(P) \rightarrow \mathcal{R}(P)$ by pullback. In this paper we need only the fact that the moduli space $\mathcal{R}(P)$ is a closed oriented smooth manifold of dimension $2(\dim G)(g-1)$.

4. FLAT CONNECTIONS OVER $H_g(\kappa)$

In this section we shall construct submanifolds of the moduli space in section 3. This is an extension of a part of the main result of [4].

Let H_g be an oriented handlebody of genus $g \geq 2$ with a base point $*$ in its boundary. The boundary of H_g is a closed oriented surface Σ_g of genus g . Let $\kappa = \coprod_{i=1}^n \kappa_i \subset H_g$ be the union of pairwise disjoint embedded loops in the interior of H_g . The closure of the complement of a tubular neighborhood of κ in H_g is denoted by $H_g(\kappa)$. Let $\partial H_g(\kappa) = \Sigma_g \coprod \coprod_{i=1}^n T_i^2$ be the decomposition of the boundary of $H_g(\kappa)$ by the connected components where T_i^2 is the torus boundary corresponding to κ_i . The fundamental group $\pi_1(H_g, *)$ of H_g is a free group of rank g . When we consider homotopy classes of κ_i , we move them by homotopy so that they pass through the base point. So we have classes $\kappa_i \in \pi_1(H_g, *)$ where we use the same letters as those for the loops. The next lemma implies that we can identify the singular cohomology of $H_g(\kappa)$ with the group cohomology of $\pi_1(H_g(\kappa), *)$ (see [3]).

Lemma 5. *If a pair (H_g, κ) satisfies the conditions 1 and 2 of Definition 1, then $H_g(\kappa)$ is an Eilenberg-MacLane space.*

This lemma is easily proven, so we omit the proof.

Let $N \geq 2$ be an integer. Put $G = SU(N)$ and $PG = PSU(N) = SU(N)/\mathbb{Z}_N$ where $\mathbb{Z}_N = \langle e_N^m I \mid m = 0, 1, \dots, N-1 \rangle$ is the center of $G = SU(N)$. From obstruction theory (see [5]), we can easily prove the following lemma.

Lemma 6. *Let C be a CW-complex of dimension ≤ 3 . Suppose that the homomorphism $H^2(C; \mathbb{Z}) \rightarrow H^2(C; \mathbb{Z}_N)$ induced by the projection $\mathbb{Z} \rightarrow \mathbb{Z}_N$ is surjective. Then the isomorphism classes of principal PG -bundles over C are identified with $H^2(C; \mathbb{Z}_N)$. The identification is obtained from taking the primary obstruction of a bundle.*

Let $q : Q(c) \rightarrow H_g(\kappa)$ be the principal PG -bundle over $H_g(\kappa)$ corresponding to a class $c \in H^2(H_g(\kappa); \mathbb{Z}_N)$. Put $c_0 := c([\Sigma_g])$ and $c_i := c([T_i^2])$ ($1 \leq i \leq n$) in $\mathbb{Z}_N$. The next lemma follows from van Kampen's theorem immediately.

Lemma 7. *Suppose the assumption of Lemma 5, then the fundamental groups of $H_g(\kappa)$ and $Q(c)$ are given by*

$$\pi_1(H_g(\kappa), *) = \langle a_1, \dots, a_g, m_1, \dots, m_n \mid [\kappa_i, m_i] = 1 \ (1 \leq i \leq n) \rangle,$$

$$\pi_1(Q(c), \tilde{*}) = \langle a_1, \dots, a_g, m_1, \dots, m_n, \varepsilon \mid [\kappa_i, m_i] = \varepsilon^{c_i} \ (1 \leq i \leq n), \varepsilon^N = 1, \varepsilon : \text{center} \rangle.$$

Moreover the induced homomorphism $q_* : \pi_1(Q(c), \tilde{*}) \rightarrow \pi_1(H_g(\kappa), *)$ is given by $a_i \mapsto a_i$ ($1 \leq i \leq g$), $m_j \mapsto m_j$ ($1 \leq j \leq n$), $\varepsilon \mapsto 1$.

Here the class κ_i is the word of $a_1, \dots, a_g$ which agrees with the one presenting the corresponding class κ_i in $\pi_1(H_g, *) = \langle a_1, \dots, a_g \rangle$. The class ε is defined as before. The class m_i in $\pi_1(H_g(\kappa), *)$ is represented by the meridian curve on torus boundary T_i^2 for each i .

Direct computation shows the following lemma.

Lemma 8. *Let $N > c \geq 1$ be coprime integers.*

Put

$$X_0 = \epsilon \begin{bmatrix} 1 & & & & \\ & \lambda & & & \\ & & \lambda^2 & & \\ & & & \ddots & \\ & & & & \lambda^{N-1} \end{bmatrix} \quad \text{and} \quad Y_0 = \begin{bmatrix} 0 & & & & (-1)^{N-1} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix},$$

where $\lambda = e^{2\pi\sqrt{-1}\frac{c}{N}}$ and $\epsilon = 1$ if N is odd and $\epsilon = e^{\frac{\pi\sqrt{-1}}{N}}$ if N is even. Then for any $A, B \in SU(N)$ satisfying $[A, B] = \lambda I$, there exists $C \in SU(N)$ such that the equalities $CAC^{-1} = X_0$ and $CBC^{-1} = Y_0$ hold. Moreover such a C exists exactly N and if one of them is C_0 then the others are given by $\lambda C_0, \lambda^2 C_0, \dots, \lambda^{N-1} C_0$.

This lemma implies that if N and c are coprime then the set $\{(A, B) \in G^2 \mid [A, B] = e_N^c I\}$ is diffeomorphic to PG and the adjoint action of G on it induces a free action of PG on it.

Proposition 2 is a part of the following proposition.

Proposition 9. *Suppose that a pair (H_g, κ) is admissible. Let c be a class in $H^2(H_g(\kappa), \mathbb{Z}_N)$ such that $(N, c_i) = 1$ for all $i = 0, 1, \dots, n$, then $\tilde{\mathcal{R}}(Q(c))$ and $\mathcal{R}(Q(c))$ are diffeomorphic to $PG^n \times G^{g-n}$ and $PG^{n-1} \times G^{g-n}$ respectively. Moreover the induced maps $\tilde{j}_g^* : \tilde{\mathcal{R}}(Q(c)) \rightarrow \tilde{\mathcal{R}}(P)$ and $\bar{j}_g^* : \mathcal{R}(Q(c)) \rightarrow \mathcal{R}(P)$ by pullback are embeddings hence their images are orientable submanifolds of $\tilde{\mathcal{R}}(P)$ and $\mathcal{R}(P)$ respectively.*

Proof. Since the classes κ_i ($1 \leq i \leq n$) extend to a free basis κ_i ($1 \leq i \leq g$) by the assumption, the fundamental group $\pi_1(Q(c), *)$ can be presented by

$$\langle \kappa_1, \dots, \kappa_g, m_1, \dots, m_n, \varepsilon | [\kappa_i, m_i] = \varepsilon^{c_i} \ (1 \leq i \leq n), \ \varepsilon^N = 1, \ \varepsilon : \text{center} \rangle.$$

Then we have

$$\begin{aligned} \tilde{\mathcal{R}}(Q(c)) &\cong \{(K_1, \dots, K_g, M_1, \dots, M_n) \in G^{g+n} | [K_i, M_i] = e_N^{c_i} I \ (1 \leq i \leq n)\} \\ &= \prod_{i=1}^n \{(K_i, M_i) \in G^2 | [K_i, M_i] = e_N^{c_i} I\} \times G^{g-n} \\ &= PG^n \times G^{g-n}. \end{aligned}$$

From the assumption that the induced homomorphism on the fundamental groups by $j_g : \Sigma_g \rightarrow H_g(\kappa)$ is surjective and an infinitesimal version of Lemma 8, it is easy to see that the map $\tilde{j}_g^* : \tilde{\mathcal{R}}(Q(c)) \rightarrow \tilde{\mathcal{R}}(P)$ is embedding.

Since the adjoint action of G on $\tilde{\mathcal{R}}(Q(c))$ descends to a free action of PG on it, the results for $\mathcal{R}(Q(c))$ follow from the ones for $\tilde{\mathcal{R}}(Q(c))$. $\square$

5. LINKING NUMBER

In this section, we shall describe the linking number in terms of Heegaard decomposition.

Let M be an oriented homology 3-sphere and $\mathcal{L} = (\kappa, \kappa')$ a link of 2 components in it. Let $M = H_g \cup_f H'_g$ be an admissible Heegaard decomposition to $\mathcal{L}$. Suppose that $\mathcal{L}$ is oriented in this section. Let $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ be a symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$ satisfying $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$ ($1 \leq i, j \leq g$), where $\cdot$ denotes the intersection pairing. Let $a_1, \dots, a_g$ denote the images of $\alpha_1, \dots, \alpha_g$ by the induced homomorphism $i_g : H_1(\Sigma_g; \mathbb{Z}) \rightarrow H_1(H_g; \mathbb{Z})$ by the inclusion $i_g : \Sigma_g = \partial H_g \hookrightarrow H_g$. $a_1, \dots, a_g$ form a basis of $H_1(H_g; \mathbb{Z})$. Put $\alpha = (\alpha_1 \dots \alpha_g)$, $\beta = (\beta_1 \dots \beta_g)$ and $a = (a_1 \dots a_g)$. Similarly we define a symplectic basis $\alpha'_1, \dots, \alpha'_g, \beta'_1, \dots, \beta'_g$ of $H_1(\Sigma'_g; \mathbb{Z})$, a basis $a'_1, \dots, a'_g$ of $H_1(H_g; \mathbb{Z})$ and α', β', a' .

Let $f_* : H_1(\Sigma_g; \mathbb{Z}) \rightarrow H_1(\Sigma'_g; \mathbb{Z})$ be the induced homomorphism of f given by

$$f_*(\alpha\beta) = (\alpha'\beta') \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Lemma 10. 1. B is invertible over $\mathbb{Z}$.

2. $-C + DB^{-1}A = -{}^t B^{-1}$.

Proof. The assumption that $M = H_g \cup_f H'_g$ is a homology 3-sphere implies 1.

Since f is an orientation reversing diffeomorphism from Σ_g to Σ'_g , f_* is an anti-symplectic isomorphism from $(H_1(\Sigma_g; \mathbb{Z}), \cdot)$ to $(H_1(\Sigma'_g; \mathbb{Z}), \cdot)$. This implies 2. $\square$

The first homology groups of $H'_g(\kappa')$ and $M(\kappa') := H_g \cup_f H'_g(\kappa')$ are given by

$$H_1(H'_g(\kappa'); \mathbb{Z}) = \bigoplus_{i=1}^g \mathbb{Z}a'_i \oplus \mathbb{Z}m' \cong \mathbb{Z}^{g+1},$$

$$H_1(M(\kappa'); \mathbb{Z}) = \mathbb{Z}m' \cong \mathbb{Z},$$

where a'_i 's are defined as before and m' is the class of the meridian curve on the torus boundary (we use the same letter).

Let $\iota: H_g \hookrightarrow M(\kappa')$ be the inclusion.

Lemma 11. *The induced homomorphism $\iota_*: H_1(H_g; \mathbb{Z}) \rightarrow H_1(M(\kappa'); \mathbb{Z})$ is given by*

$$\iota_* a = -m'(\tilde{\kappa}' \cdot \beta')^t B^{-1},$$

where $\tilde{\kappa}'$ is the class in $H_1(\Sigma'_g; \mathbb{Z})$ of a simple closed curve on $\Sigma'_g = \partial H'_g$ parallel to κ' and $\tilde{\kappa}' \cdot \beta' := (\tilde{\kappa}' \cdot \beta'_1, \dots, \tilde{\kappa}' \cdot \beta'_g)$.

Proof. The induced homomorphisms $i_{g*}: H_1(\Sigma_g; \mathbb{Z}) \rightarrow H_1(H_g; \mathbb{Z})$ and $j'_{g*}: H_1(\Sigma'_g; \mathbb{Z}) \rightarrow H_1(H'_g(\kappa'); \mathbb{Z})$ are given by $i_{g*}(\alpha\beta) = a(I_g \ 0)$ and $j'_{g*}(\alpha'\beta') = (a'm') \begin{pmatrix} I & 0 \\ 0 & \tilde{\kappa}' \cdot \beta' \end{pmatrix}$ respectively. We have the Mayer-Vietoris exact sequence for the decomposition $M(\kappa') = H_g \cup_f H'_g(\kappa')$;

$$H_1(\Sigma_g; \mathbb{Z}) \xrightarrow{i_{g*} \oplus -j'_{g*} f_*} H_1(H_g; \mathbb{Z}) \oplus H_1(H'_g(\kappa'); \mathbb{Z}) \rightarrow H_1(M(\kappa'); \mathbb{Z}) \rightarrow 0,$$

where $i_{g*} \oplus -j'_{g*} f_*$ is given by

$$(i_{g*} \oplus -j'_{g*} f_*)(\alpha\beta) = (aa'm') \begin{pmatrix} I_g & 0 \\ -A & -B \\ -(\tilde{\kappa}' \cdot \beta')C & -(\tilde{\kappa}' \cdot \beta')D \end{pmatrix}.$$

We multiply this equality by $\begin{pmatrix} I_g & 0 \\ -A & -B \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -B^{-1}A & -B \end{pmatrix}$ from the right. Using the equality in Lemma 10, we obtain

$$(i_{g*} \oplus -j'_{g*} f_*)(\alpha\beta) \begin{pmatrix} I_g & 0 \\ -A & -B \end{pmatrix}^{-1} = (a - m'(\tilde{\kappa}' \cdot \beta')^t B^{-1} \quad a' + m'(\tilde{\kappa}' \cdot \beta')DB^{-1}).$$

Then $\text{Im}(i_{g*} \oplus -j'_{g*} f_*)$ is generated by the components of $a - m'(\tilde{\kappa}' \cdot \beta')^t B^{-1}$ and $a' + m'(\tilde{\kappa}' \cdot \beta')DB^{-1}$. Since we can choose $\{m', a - m'(\tilde{\kappa}' \cdot \beta')^t B^{-1}, a' + m'(\tilde{\kappa}' \cdot \beta')DB^{-1}\}$ as a basis of $H_1(H_g; \mathbb{Z}) \oplus H_1(H'_g(\kappa'); \mathbb{Z})$, we obtain the isomorphisms $H_1(M(\kappa'); \mathbb{Z}) \xleftarrow{\cong} H_1(H_g; \mathbb{Z}) \oplus H_1(H'_g(\kappa'); \mathbb{Z}) / \text{Im}(i_{g*} \oplus -j'_{g*} f_*) \cong \mathbb{Z}m'$. This implies the assertion of this lemma. $\square$

Let $lk_M(\kappa', \kappa)$ be the linking number of κ' and κ in M . We define $\tilde{\kappa} \in H_1(\Sigma_g; \mathbb{Z})$ in the same way as $\tilde{\kappa}'$.

Corollary 12. $lk_M(\kappa', \kappa) = -(\tilde{\kappa}' \cdot \beta')^t B^{-1t} (\tilde{\kappa} \cdot \beta)$.

6. INTERSECTION NUMBER

In this section, we shall prove Theorem 4 under some additional conditions.

Let κ_0 be an embedded curve in $\overset{\circ}{H}_g$ which is parallel to the simple closed curve $\tilde{\kappa}_0$ on $\Sigma_g = \partial H_g$ depicted in Figure 1. Clearly the pair (H_g, κ_0) is admissible.

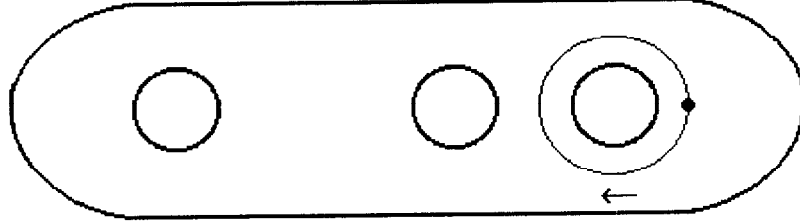


FIGURE 1. The simple closed curve $\tilde{\kappa}_0$.

Proposition 13. *Under the hypotheses of Theorem 4, suppose that the pairs (H_g, κ) and (H'_g, κ') are diffeomorphic to (H_g, κ_0) . Then the conclusion of Theorem 4 is true.*

Now we are going to prove this proposition, so we assume the hypotheses of it and use the notations defined before Theorem 4. By replacing the diffeomorphism f from Σ_g to Σ'_g if necessary, we may assume $(H_g, \kappa) = (H'_g, \kappa') = (H_g, \kappa_0)$.

The fundamental groups of Σ_g and $H_g(\kappa)$ are given by

$$\pi_1(\Sigma_g, *) = \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \mid \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \rangle,$$

$$\pi_1(H_g(\kappa), *) = \langle a_1, \dots, a_g, m \mid [a_g, m] = 1 \rangle.$$

Here the generators α_i and β_i of $\pi_1(\Sigma_g, *)$ are represented by the curves depicted in Figure 2. The generators a_i and m of $\pi_1(H_g(\kappa), *)$ are defined so that the induced homomorphism

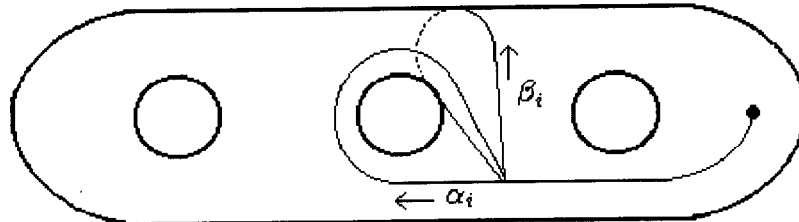


FIGURE 2. The generators of $\pi_1(\Sigma_g, *)$.

$j_{g!} : \pi_1(\Sigma_g, *) \rightarrow \pi_1(H_g(\kappa), *)$ of $j_g : \Sigma_g \rightarrow H_g(\kappa)$ is given by

$$\alpha_i \mapsto a_i \quad (1 \leq i \leq g), \quad \beta_i \mapsto \begin{cases} 1, & 1 \leq i \leq g-1, \\ m, & i = g. \end{cases}$$

The fundamental groups of the principal PG -bundles P and Q can be given by

$$\begin{aligned}\pi_1(P, \tilde{*}) &= \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \varepsilon \mid \prod_{i=1}^g [\alpha_i, \beta_i] = \varepsilon^c, \varepsilon^N = 1, \varepsilon: \text{center} \rangle, \\ \pi_1(Q, \tilde{*}) &= \langle a_1, \dots, a_g, m, \varepsilon \mid [a_g, m] = \varepsilon^c, \varepsilon^N = 1, \varepsilon: \text{center} \rangle,\end{aligned}$$

where the generators of $\pi_1(P, \tilde{*})$ are lifts of $\alpha_i, \beta_i \in \pi_1(\Sigma_g, *)$ for the bundle projection $P \rightarrow \Sigma_g$, which are denoted by the same letter. The generators of $\pi_1(Q, \tilde{*})$ are defined so that the induced homomorphism $\tilde{j}_g: \pi_1(P, \tilde{*}) \rightarrow \pi_1(Q, \tilde{*})$ by the map $\tilde{j}_g: P \rightarrow Q$ is given by

$$\alpha_i \mapsto a_i \ (1 \leq i \leq g), \quad \beta_i \mapsto \begin{cases} 1, & 1 \leq i \leq g-1, \\ m, & i = g, \end{cases}, \quad \varepsilon \mapsto \varepsilon.$$

In our setting, $\Sigma'_g, H'_g(\kappa), P'$ and Q' are copies of $\Sigma_g, H_g(\kappa), P$ and Q respectively. So we define generators of the fundamental groups of them in the same way, but we distinguish them by " ' ".

Let F_{g-1} and F'_{g-1} be free groups of rank $g-1$ with generators $\beta_1, \dots, \beta_{g-1}$ and $a'_1, \dots, a'_{g-1}$ respectively. Put $\tilde{\mathcal{R}}(F_{g-1}) = \text{Hom}(F_{g-1}, G)$. Let $\eta: \tilde{\mathcal{R}}(F_{g-1}) \rightarrow G^{g-1}$ be the evaluation map defined by $\eta(\rho) = (\rho(\beta_i))_{i=1}^{g-1} \in G^{g-1}$ for each $\rho \in \tilde{\mathcal{R}}(F_{g-1})$. We define the homomorphism $s: F_{g-1} \rightarrow \pi_1(P, \tilde{*})$ by $\beta_i \mapsto \beta_i$ ($1 \leq i \leq g-1$) and the homomorphism $q': \pi_1(Q', \tilde{*}) \rightarrow F'_{g-1}$ by

$$a'_i \mapsto \begin{cases} a'_i & 1 \leq i \leq g-1 \\ 1 & i = g \end{cases}, \quad m \mapsto 1, \quad \varepsilon \mapsto 1.$$

Let $\mathcal{H}: \tilde{\mathcal{R}}(P) \rightarrow G^{g-1}$ be the composition of the maps: $\tilde{\mathcal{R}}(P) \xrightarrow{s^*} \tilde{\mathcal{R}}(F_{g-1}) \xrightarrow{\eta} G^{g-1}$.

Lemma 14. 1. $I^{g-1} := (I, I, \dots, I) \in G^{g-1}$ is a regular value of $\mathcal{H}$.

2. $\mathcal{H}^{-1}(I^{g-1}) = \tilde{j}_g^* \tilde{\mathcal{R}}(Q)$.

This lemma is easily checked, so we omit the proof.

Put $\tilde{L} = \tilde{j}_g^* \tilde{\mathcal{R}}(Q)$, $\tilde{L}' = (\tilde{j}_g' \circ \tilde{f})^* \tilde{\mathcal{R}}(Q')$, $L = \tilde{j}_g^* \mathcal{R}(Q)$ and $L' = (\tilde{j}_g' \circ \tilde{f})^* \mathcal{R}(Q')$, then L and L' are submanifolds of $\mathcal{R}(P)$ by Proposition 9. $\tilde{\mathcal{R}}(Q')$ is identified with the subset

$$\{((A'_i)_{i=1}^g, M') \in G^{g+1} \mid [A'_g, M'] = e_N^c I\} = G^{g-1} \times \text{Ad}_G(X_0, Y_0)$$

of G^{g+1} by the map $\tilde{\mathcal{R}}(Q') \ni \rho \mapsto (\rho(\alpha'_1), \dots, \rho(\alpha'_g), \rho(m')) \in G^{g+1}$. Let $\sigma: G^{g-1} \rightarrow \tilde{\mathcal{R}}(Q')$ be the map defined by

$$(A'_i)_{i=1}^{g-1} \mapsto \sigma((A'_i)_{i=1}^{g-1}): \begin{cases} a'_i \mapsto A'_i & 1 \leq i \leq g-1 \\ a'_g \mapsto X_0 \\ m' \mapsto Y_0 \end{cases}$$

for $(A'_i)_{i=1}^{g-1} \in G^{g-1}$, where X_0 and Y_0 are the element of G defined in Lemma 8. The composition of σ and the projection $\tilde{\mathcal{R}}(Q') \rightarrow \mathcal{R}(Q')$ gives a diffeomorphism ζ from G^{g-1}

to $\mathcal{R}(Q')$. We define the section s' of the projection $\tilde{\mathcal{R}}(Q') \rightarrow \mathcal{R}(Q')$ by $\sigma = s' \circ \zeta : G^{g-1} \xrightarrow{\zeta} \mathcal{R}(Q') \xrightarrow{s'} \tilde{\mathcal{R}}(Q')$. Let $\phi : G^{g-1} \rightarrow G^{g-1}$ be the composition map:

$$\phi = \mathcal{H} \circ \bar{f}^* \circ \bar{j}_g'^* \circ s' \circ \zeta : G^{g-1} \xrightarrow{\zeta} \mathcal{R}(Q') \xrightarrow{s'} \tilde{\mathcal{R}}(Q') \xrightarrow{\bar{j}_g'^*} \tilde{\mathcal{R}}(P') \xrightarrow{\bar{f}^*} \tilde{\mathcal{R}}(P) \xrightarrow{\mathcal{H}} G^{g-1}.$$

We give some orientations to L and L' .

Lemma 15. *Under the above setting, the equality $|L \cdot L'| = |\deg \phi|$ holds.*

Proof. $\tilde{\mathcal{R}}(P)$ is a fiber bundle over $\mathcal{R}(P)$ with fiber PG and its restriction $\tilde{\mathcal{R}}(P)|_{L'}$ to L' is $\tilde{L}'$. This restricted bundle $\tilde{L}' \rightarrow L$ is identified with the bundle $\tilde{\mathcal{R}}(Q') \rightarrow \mathcal{R}(Q')$ via the map $(\bar{j}_g' \circ \bar{f})^*$. Let $s'_{L'} : L' \rightarrow \tilde{L}' \subset \tilde{\mathcal{R}}(P)$ be the section corresponding to the section $s' : \mathcal{R}(Q') \rightarrow \tilde{\mathcal{R}}(Q')$, then we have $|L \cdot L'| = |\tilde{L} \cdot s'_{L'}(L)|$. Since $\tilde{L} = \bar{j}_g^* \tilde{\mathcal{R}}(Q)$ is the inverse image of the regular value I^{g-1} by the map $\mathcal{H}$ by Lemma 14, we obtain

$$|\tilde{L} \cdot s'_{L'}(L)| = |\deg[L \xrightarrow{s'_{L'}} \tilde{L}' \hookrightarrow \tilde{\mathcal{R}}(P) \xrightarrow{\mathcal{H}} G^{g-1}]|,$$

hence

$$|L \cdot L'| = |\deg[\mathcal{R}(Q') \xrightarrow{s'} \tilde{\mathcal{R}}(Q') \xrightarrow{(\bar{j}_g' \circ \bar{f})^*} \tilde{\mathcal{R}}(P) \xrightarrow{\mathcal{H}} G^{g-1}]|.$$

Since ζ is a diffeomorphism from G^{g-1} to $\mathcal{R}(Q')$, the proof is completed. $\square$

Next we shall calculate $\deg \phi$. Let $\mathcal{F}$ be the composition of the following homomorphisms:

$$\mathcal{F} = j'_{g*} \circ \bar{f}_* \circ s : F_{g-1} \xrightarrow{s} \pi_1(P, \tilde{*}) \xrightarrow{\bar{f}_*} \pi_1(P', \tilde{*}) \xrightarrow{\bar{j}'_{g*}} \pi_1(Q', \tilde{*}).$$

Then we have $\mathcal{F}^* = s^* \circ \bar{f}^* \circ \bar{j}'^* : \tilde{\mathcal{R}}(Q') \rightarrow \tilde{\mathcal{R}}(F_{g-1})$.

Suppose that the homomorphism $\mathcal{F} : F_{g-1} \rightarrow \pi_1(Q', \tilde{*})$ is given by

$$\beta_i \mapsto (\varepsilon')^{s_i} \prod_{j=1}^{t_i} (a'_{u_{ij}})^{v_{ij}} \quad (1 \leq i \leq g-1),$$

where s_i, t_i, u_{ij} and v_{ij} are some integers with $t_i \geq 0$ and $1 \leq u_{ij} \leq g+1$, and where $a'_{g+1} = m'$. Then the map $\phi : G^{g-1} \rightarrow G^{g-1}$ is given by

$$(A'_i)_{i=1}^{g-1} \mapsto (e_N^{s_i} \prod_{j=1}^{t_i} (A'_{u_{ij}})^{v_{ij}})_{i=1}^{g-1},$$

where $A'_g = X_0$ and $A'_{g+1} = M' = Y_0$.

Let $V = (V_{ij})$ be the $(g-1) \times (g-1)$ -matrix defined by $V_{ij} = \sum_{u_{ik}=j} v_{ik}$. Let $\psi : G^{g-1} \rightarrow G^{g-1}$ be the map given by $(A_i)_{i=1}^{g-1} \mapsto (\prod_{k=1}^{g-1} A_k^{V_{ik}})_{i=1}^{g-1}$.

Lemma 16. $\deg \phi = \deg \psi = (\det V)^{N-1}$.

In order to prove this lemma, we recall the cohomology ring of $G = SU(N)$. The cohomology ring $H^*(SU(N); \mathbb{Q})$ of $SU(N)$ over $\mathbb{Q}$ is isomorphic to the exterior algebra over $\mathbb{Q}$ generated by the generators $x_3, x_5, \dots, x_{2N-1}$ with degree $\deg x_i = i$. Moreover we can choose the generators x_i so that the equalities $h^* x_i = 1 \otimes x_i + x_i \otimes 1$ hold for the multiplication $h: SU(N) \times SU(N) \ni (A, B) \mapsto AB \in SU(N)$. This property shows that the action of a map given by multiplication like ϕ on the cohomology group dose not change when order of multiplication defining it is changed. Moreover since G is connected, the multiplication by a constant matrix in G induces the identity on the cohomology group of G . These properties imply $\deg \phi = \deg \psi$.

Proof of Lemma 16 We have only to compute $\deg \psi$. Let $X = \bigotimes^{g-1} (x_3 \cup x_5 \cup \dots \cup x_{2N-1})$ be a generator of $H^{\dim SU(N)(g-1)}(SU(N)^{g-1}; \mathbb{Q})$. Here we identify $H^*(SU(N)^{g-1}; \mathbb{Q})$ with $\bigotimes^{g-1} H^*(SU(N); \mathbb{Q})$. Put $\psi_i(A) = \prod_{k=1}^{g-1} A_k^{V_{ik}}$ for $A = (A_i)_{i=1}^{g-1} \in G^{g-1}$, then we have $\psi(A) = (\psi_1(A), \dots, \psi_{g-1}(A))$. Using the equalities

$$\psi_i^* x_j = \sum_{k=1}^{g-1} V_{ik} 1 \otimes 1 \otimes \dots \otimes 1 \otimes x_j^k \otimes 1 \otimes \dots \otimes 1,$$

we have

$$\begin{aligned} \psi^*(x_j \otimes \dots \otimes x_j) &= \psi_1^* x_j \cup \psi_2^* x_j \cup \dots \cup \psi_{g-1}^* x_j \\ &= \left(\sum_{k_1=1}^{g-1} V_{1k_1} 1 \otimes \dots \otimes 1 \otimes x_j^{k_1} \otimes 1 \otimes \dots \otimes 1 \right) \\ &\quad \cup \left(\sum_{k_2=1}^{g-1} V_{2k_2} 1 \otimes \dots \otimes 1 \otimes x_j^{k_2} \otimes 1 \otimes \dots \otimes 1 \right) \cup \dots \\ &= \sum_{\sigma \in \mathfrak{S}_{g-1}} \text{sgn}(\sigma) V_{1\sigma(1)} V_{2\sigma(2)} \dots V_{g-1\sigma(g-1)} x_j \otimes \dots \otimes x_j \\ &= (\det V) x_j \otimes \dots \otimes x_j \end{aligned}$$

for each j , where $\mathfrak{S}_{g-1}$ is the symmetric group of degree $g-1$. Hence we have $\psi^* X = (\det V)^{N-1} X$. This completes the proof. $\square$

On the other hand, the induced homomorphism $(q' \circ \mathcal{F})_* : H_1(F_{g-1}; \mathbb{Z}) \rightarrow H_1(F'_{g-1}; \mathbb{Z})$ of $q' \circ \mathcal{F} : F_{g-1} \rightarrow F'_{g-1}$ is given by

$$\beta_i \mapsto \sum_{j=1}^{g-1} \left(\sum_{u_{ik}=j} v_{ik} \right) a'_j = \sum_{j=1}^{g-1} V_{ij} a'_j \quad (1 \leq i \leq g-1),$$

where the first homology groups of F_{g-1} and F'_{g-1} is the abelianizations of them. Therefore we obtain $\det(q'_* \circ \mathcal{F}_*) = \det V$ with respect to the bases $\{\beta_i\}_{1 \leq i \leq g-1}$ and $\{a'_i\}_{1 \leq i \leq g-1}$. Hence we have the following lemma by Lemma 16.

Lemma 17. $\deg(\phi) = (\det(q'_* \circ \mathcal{F}_*))^{N-1}$.

Now we shall prove Proposition 13. By Lemma 15 and 17, we have only to show the equality $|\det(q'_* \circ \mathcal{F}_*)| = |lk_M(\kappa', \kappa)|$.

The homomorphism $q'_* \circ \mathcal{F}_*$ agrees with the composition of the following homomorphisms

$$H_1(F_{g-1}; \mathbb{Z}) \xrightarrow{s_*} H_1(P; \mathbb{Z}) \xrightarrow{\bar{f}_*} H_1(P'; \mathbb{Z}) \xrightarrow{\bar{j}'_{g*}} H_1(Q'; \mathbb{Z}) \xrightarrow{q'_*} H^1(F'_{g-1}; \mathbb{Z}).$$

Suppose that the induced homomorphism $f^*: H_1(\Sigma_g; \mathbb{Z}) \rightarrow H_1(\Sigma'_g; \mathbb{Z})$ is described as in section 5. Then each homomorphism is given by

$$s_* \beta_{F_{g-1}} = (\alpha \beta) \begin{pmatrix} 0 \\ J \end{pmatrix}, \quad \bar{f}_*(\alpha \beta) = (\alpha \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$\bar{j}'_{g*}(\alpha \beta) = (a' m') \begin{pmatrix} I_g & 0 \\ 0 & \tilde{\kappa}' \cdot \beta' \end{pmatrix}, \quad q'_*(a' m) = \begin{pmatrix} a'_{F'_{g-1}} \end{pmatrix} \begin{pmatrix} {}^t J & 0 \end{pmatrix},$$

where $J = \begin{pmatrix} I_{g-1} \\ 0 \end{pmatrix}$ is $g \times (g-1)$ matrix and $\beta_{F_{g-1}} = (\beta_1 \dots \beta_{g-1})$ and $a'_{F'_{g-1}} = (a'_1 \dots a'_{g-1})$ are row vectors consisting of the free bases. $\{\alpha \beta\}$ is the basis of $H_1(P; \mathbb{Z})$ corresponding to the one of $H_1(\Sigma_g; \mathbb{Z})$ in section 5 via the bundle projection of $P \rightarrow \Sigma_g$. Using this description, we have

$$\begin{aligned} q'_* \circ \mathcal{F}_*(\beta_{F_{g-1}}) &= a'_{F'_{g-1}} \begin{pmatrix} {}^t J & 0 \end{pmatrix} \begin{pmatrix} I_g & 0 \\ 0 & \tilde{\kappa}' \cdot \beta' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 \\ J \end{pmatrix} \\ &= a'_{F'_{g-1}} {}^t J B J, \end{aligned}$$

hence $\det(q'_* \circ \mathcal{F}_*) = \det({}^t J B J) =$ the (g, g) -entry of B^{-1} .

Now we apply Corollary 12 to this setting. Since we have $(\tilde{\kappa}' \cdot \beta) = (\tilde{\kappa} \cdot \beta) = (0, \dots, 0, 1)$, we obtain $lk_M(\kappa', \kappa) = -$ the (g, g) -entry of B^{-1} , hence $|lk_M(\kappa', \kappa)| = |\det(q'_* \circ \mathcal{F}_*)|$. This completes the proof of Proposition 13.

7. PROOF OF THEOREM 4

In this section, we shall prove Theorem 4.

Let M be an oriented homology 3-sphere and $\mathcal{L} = (\kappa, \kappa')$ a link of two components in M . Let $M = H_g \cup_f H'_g$ be an admissible Heegaard decomposition of M to $\mathcal{L}$. By taking an appropriate orientation of κ , there exists a one handle h in H_g , which is a submanifold $D^2 \times [0, 1]$ of H_g satisfying $D^2 \times \{0, 1\} = (D^2 \times [0, 1]) \cap \partial H_g$, such that the followings are satisfied.

Let $M = H_{g+1} \cup_{f_+} H'_{g+1}$ be a new Heegaard decomposition of M where $H_{g+1} = \overline{H_g \setminus h}$, $H'_{g+1} = H'_g \cup h$ and f_+ is a diffeomorphism from ∂H_{g+1} to $\partial H'_{g+1}$ obviously constructed from f . Let (ν, ν') be a copy of (κ, κ') in (H_{g+1}, H'_{g+1}) . The above Heegaard decomposition is admissible to (ν, ν') and the pair (H_{g+1}, ν) is diffeomorphic to (H_{g+1}, κ_0) in Proposition 12 for $g+1$. Moreover we can present the fundamental groups as follows:

$$\begin{aligned} \pi_1(H_g(\kappa), *) &= \langle a_1, \dots, a_g, m \mid [\kappa, m] = 1 \rangle, \\ \pi_1(H'_g(\kappa'), *) &= \langle a'_1, \dots, a'_g, m' \mid [\kappa', m'] = 1 \rangle, \\ \pi_1(H_{g+1}(\nu), *) &= \langle a_1, \dots, a_g, a_{g+1}, m \mid [\nu, m] = 1, \nu = \kappa a_{g+1} \rangle, \\ \pi_1(H'_{g+1}(\nu'), *) &= \langle a'_1, \dots, a'_g, a'_{g+1}, m' \mid [\nu', m'] = 1, \nu' = \kappa' \rangle. \end{aligned}$$

Here κ (resp. κ') are a product of a_i 's (resp. a'_i 's) as in Lemma 7. Furthermore the induced homomorphisms on the fundamental groups from the inclusions $j_{g+1} : \Sigma_{g+1} = \partial H_{g+1} \hookrightarrow H_{g+1}(\nu)$, $j'_{g+1} : \Sigma'_g = \partial H'_{g+1} \hookrightarrow H'_{g+1}(\nu')$ and $f_+ : \Sigma_g \rightarrow \Sigma'_g$ are described as follows:

$$j_{g+1*} : \pi_1(\Sigma_{g+1}, *) \rightarrow \pi_1(H_{g+1}(\nu), *),$$

$$\alpha_i \mapsto \begin{cases} j_{g*} \alpha_i & 1 \leq i \leq g \\ a_{g+1} & i = g+1 \end{cases}, \quad \beta_i \mapsto \begin{cases} j_{g*} \beta_i & 1 \leq i \leq g \\ m & i = g+1, \end{cases}$$

$$j'_{g+1*} : \pi_1(\Sigma'_{g+1}, *) \rightarrow \pi_1(H'_{g+1}(\nu'), *),$$

$$\alpha'_i \mapsto \begin{cases} j'_{g*} \alpha'_i & 1 \leq i \leq g \\ a'_{g+1} & i = g+1 \end{cases}, \quad \beta'_i \mapsto \begin{cases} j'_{g*} \beta'_i & 1 \leq i \leq g \\ 1 & i = g+1, \end{cases}$$

$$f_{+*} : \pi_1(\Sigma_{g+1}, *) \rightarrow \pi_1(\Sigma'_{g+1}, *)$$

$$\alpha_i \mapsto \begin{cases} f_* \alpha_i & 1 \leq i \leq g \\ \alpha'_{g+1} \beta'_{g+1}{}^{-1} \alpha'_{g+1}{}^{-1} & i = g+1 \end{cases}, \quad \beta_i \mapsto \begin{cases} f_* \beta_i & 1 \leq i \leq g \\ \alpha'_{g+1}{}^{-1} & i = g+1. \end{cases}$$

Here $j_{g*} \alpha_i$ is the word corresponding to the word $j_{g*} \alpha_i \in \pi_1(H_g(\kappa), *)$ where $j_g : \Sigma_g = \partial H_g \hookrightarrow H_g(\kappa)$ is the inclusion. Similarly $j'_{g*} \alpha_i$, $f_* \alpha_i$, etc. are defined.

There exists a one-handle h satisfying the above conditions. In fact we may assume that the map f is given by Figure 3 near the base point.

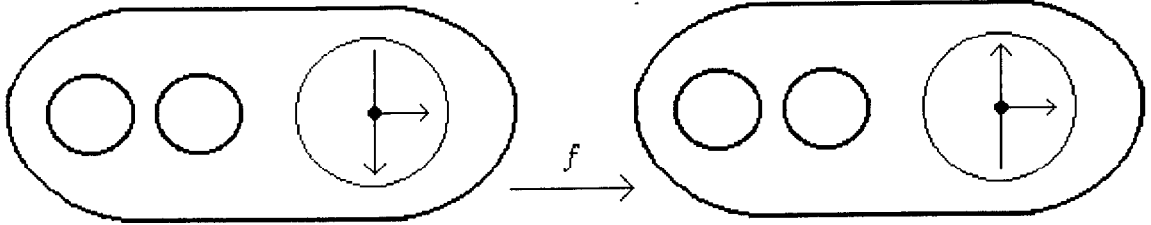
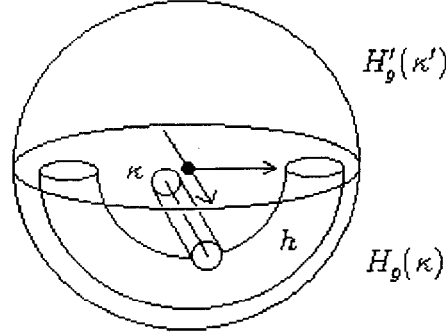
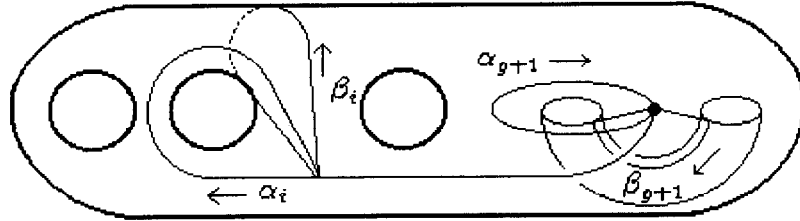
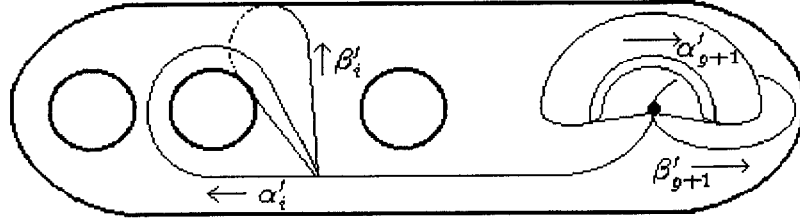


FIGURE 3. f near the base point.

Moreover we may assume that κ passes near the base point as in Figure 4. So we can choose h in $H_g(\kappa)$ as in Figure 4.

Generators α_i, β_i and α'_i, β'_i of the fundamental groups of $H_{g+1}(\nu)$ and $H'_{g+1}(\nu')$ are defined as in Figure 5 and 6 respectively.

Let $Q(\mathcal{L}) \rightarrow M(\mathcal{L})$ be a principal PG -bundle over $M(\mathcal{L})$ with $c = c(Q(\mathcal{L}))$. Then we have the restricted bundles Q_g, Q'_g, P_g, P'_g and $Q_{g+1}, Q'_{g+1}, P_{g+1}, P'_{g+1}$ associated with the decompositions $M = H_g(\kappa) \cup_f H'_g(\kappa')$ and $M = H_{g+1}(\nu) \cup_{f_+} H'_{g+1}(\nu')$ respectively as before. Let $\bar{j}_g : P_g \rightarrow Q_g$ be the induced bundle map over $j_g : \Sigma_g \rightarrow H_g(\kappa)$, and $\bar{j}'_g : P'_g \rightarrow Q'_g$ over $j'_g : \Sigma'_g \rightarrow H'_g(\kappa')$. Similarly we have the bundle maps $\bar{j}_{g+1}, \bar{j}'_{g+1}$ and f_+ over j_{g+1}, j'_{g+1} and f_+ respectively. The induced homomorphisms on the fundamental

FIGURE 4. One handle h .FIGURE 5. Generators of $\pi_1(H_{g+1}(\nu), *)$.FIGURE 6. Generators of $\pi_1(H'_{g+1}(\nu'), *)$.

groups by the bundle maps $\bar{j}_{g+1}$, $\bar{j}'_{g+1}$ and $\bar{f}_+$ can be described as follows:

$$\bar{j}_{g+1*}: \pi_1(P_{g+1}, \tilde{*}) \rightarrow \pi_1(Q_{g+1}, \tilde{*}),$$

$$\alpha_i \mapsto \begin{cases} \bar{j}_{g*} \alpha_i & 1 \leq i \leq g \\ a_{g+1} & i = g+1 \end{cases}, \quad \beta_i \mapsto \begin{cases} \bar{j}_{g*} \beta_i & 1 \leq i \leq g \\ m & i = g+1 \end{cases}, \quad \varepsilon \mapsto \varepsilon,$$

$$\bar{j}'_{g+1*}: \pi_1(P'_{g+1}, \tilde{*}) \rightarrow \pi_1(Q'_{g+1}, \tilde{*}),$$

$$\alpha'_i \mapsto \begin{cases} \bar{j}'_{g*} \alpha'_i & 1 \leq i \leq g \\ a'_{g+1} & i = g+1 \end{cases}, \quad \beta'_i \mapsto \begin{cases} \bar{j}'_{g*} \beta'_i & 1 \leq i \leq g \\ 1 & i = g+1 \end{cases}, \quad \varepsilon' \mapsto \varepsilon',$$

$$\bar{f}_{+*}: \pi_1(P_{g+1}, \tilde{*}) \rightarrow \pi_1(P'_{g+1}, \tilde{*}),$$

$$\alpha_i \mapsto \begin{cases} \bar{f}_* \alpha_i & 1 \leq i \leq g \\ \alpha'_{g+1} \beta'_{g+1}{}^{-1} \alpha'_{g+1}{}^{-1} & i = g+1 \end{cases}, \quad \beta_i \mapsto \begin{cases} \bar{f}_* \beta_i & 1 \leq i \leq g \\ \alpha'_{g+1}{}^{-1} & i = g+1 \end{cases}, \quad \varepsilon \mapsto \varepsilon',$$

where $\bar{j}_g \alpha_i, \bar{f}_* \alpha_i$, etc. are similarly defined as before.

Let $\tilde{L}_g, \tilde{L}'_g, \tilde{L}_{g+1}$ and $\tilde{L}'_{g+1}$ be the submanifolds $\bar{j}_g^* \tilde{\mathcal{R}}(Q_g), \bar{f}^* \bar{j}'_g{}^* \tilde{\mathcal{R}}(Q'_g), \bar{j}_{g+1}^* \tilde{\mathcal{R}}(Q_{g+1})$ and $\bar{f}_+^* \bar{j}'_{g+1}{}^* \tilde{\mathcal{R}}(Q'_{g+1})$ of $\tilde{\mathcal{R}}(P_g), \tilde{\mathcal{R}}(P_g), \tilde{\mathcal{R}}(P_{g+1})$ and $\tilde{\mathcal{R}}(P_{g+1})$ respectively. Let L_g, L'_g, L_{g+1} and L'_{g+1} be the submanifolds $\bar{j}_g^* \mathcal{R}(Q_g), \bar{f}^* \bar{j}'_g{}^* \mathcal{R}(Q'_g), \bar{j}_{g+1}^* \mathcal{R}(Q_{g+1})$ and $\bar{f}_+^* \bar{j}'_{g+1}{}^* \mathcal{R}(Q'_{g+1})$ of $\mathcal{R}(P_g), \mathcal{R}(P_g), \mathcal{R}(P_{g+1})$ and $\mathcal{R}(P_{g+1})$ respectively.

Lemma 18. *Under the above setting, the equality $|L_g \cdot_{\mathcal{R}(P_g)} L'_g| = |L_{g+1} \cdot_{\mathcal{R}(P_{g+1})} L'_{g+1}|$ holds, where $\cdot_{\mathcal{R}(P_g)}$ and $\cdot_{\mathcal{R}(P_{g+1})}$ denote the intersection pairings in $\mathcal{R}(P_g)$ and $\mathcal{R}(P_{g+1})$ respectively.*

Proof. We can identify $\tilde{\mathcal{R}}(P_{g+1})$ with the subset $\{(A_i, B_i)_{i=1}^{g+1} \in G^{2(g+1)} \mid \prod_{i=1}^{g+1} [A_i, B_i] = e_N^c I\}$ of $G^{2(g+1)}$ by the map $\tilde{\mathcal{R}}(P_{g+1}) \ni \rho \mapsto (A_i, B_i)_{i=1}^{g+1} := (\rho(\alpha_i), \rho(\beta_i))_{i=1}^{g+1} \in G^{2(g+1)}$. Let $\Phi : \pi_1(P_{g+1}, *) \rightarrow \pi_1(P_g, *)$ be the homomorphism defined by

$$\alpha_i \mapsto \begin{cases} \alpha_i & 1 \leq i \leq g \\ 1 & i = g+1 \end{cases}, \beta_i \mapsto \begin{cases} \beta_i & 1 \leq i \leq g \\ 1 & i = g+1 \end{cases}, \varepsilon \mapsto \varepsilon.$$

Since it is surjective, it is easy to see that the induced map Φ^* from $\tilde{\mathcal{R}}(P_g)$ (resp. $\mathcal{R}(P_g)$) to $\tilde{\mathcal{R}}(P_{g+1})$ (resp. $\mathcal{R}(P_{g+1})$) is an embedding. Through this embedding, $\tilde{\mathcal{R}}(P_g)$ (resp. $\mathcal{R}(P_g)$) is regarded as a subset of $\tilde{\mathcal{R}}(P_{g+1})$ (resp. $\mathcal{R}(P_{g+1})$). Hence $\tilde{L}_g, \tilde{L}'_g, \tilde{L}_{g+1}$ and $\tilde{L}'_{g+1}$ (resp. L_g, L'_g, L_{g+1} and L'_{g+1}) can be regarded as submanifolds of $\tilde{\mathcal{R}}(P_{g+1})$ (resp. $\mathcal{R}(P_{g+1})$). As subsets of $\tilde{\mathcal{R}}(P_{g+1}) \subset G^{2(g+1)}$, the submanifolds $\tilde{L}_g, \tilde{L}'_g, \tilde{L}_{g+1}$ and $\tilde{L}'_{g+1}$ are given as follows:

$$\begin{aligned} \tilde{L}_g &= \{(A_i, B_i)_{i=1}^{g+1} \in G^{2(g+1)} \mid \\ &\quad A_i = \begin{cases} \bar{J}_{\alpha_i} & 1 \leq i \leq g \\ I & i = g+1 \end{cases}, B_i = \begin{cases} \bar{J}_{\beta_i} & 1 \leq i \leq g \\ I & i = g+1 \end{cases}, \\ &\quad [\kappa, m] = e_N^c I, \quad a_i(1 \leq i \leq g), m \in G\}, \end{aligned}$$

$$\begin{aligned} \tilde{L}_{g+1} &= \{(A_i, B_i)_{i=1}^{g+1} \in G^{2(g+1)} \mid \\ &\quad A_i = \begin{cases} \bar{J}_{\alpha_i} & 1 \leq i \leq g \\ a_{g+1} & i = g+1 \end{cases}, B_i = \begin{cases} \bar{J}_{\beta_i} & 1 \leq i \leq g \\ m & i = g+1 \end{cases}, \\ &\quad [\kappa a_{g+1}, m] = e_N^c I, \quad a_i(1 \leq i \leq g+1), m \in G\}, \end{aligned}$$

$$\begin{aligned} \tilde{L}'_g &= \{(A_i, B_i)_{i=1}^{g+1} \in G^{2(g+1)} \mid \\ &\quad A_i = \begin{cases} (\bar{J}' \circ \bar{F})_{\alpha'_i} & 1 \leq i \leq g \\ I & i = g+1 \end{cases}, B_i = \begin{cases} (\bar{J}' \circ \bar{F})_{\beta'_i} & 1 \leq i \leq g \\ I & i = g+1 \end{cases}, \\ &\quad [\kappa', m'] = e_N^c I, \quad a'_i(1 \leq i \leq g), m' \in G\}, \end{aligned}$$

$$\begin{aligned} \tilde{L}'_{g+1} &= \{(A_i, B_i)_{i=1}^{g+1} \in G^{2(g+1)} | \\ A_i &= \begin{cases} (\bar{J}' \circ \bar{F})_{\alpha'_i} & 1 \leq i \leq g \\ I & i = g+1 \end{cases}, B_i = \begin{cases} (\bar{J}' \circ \bar{F})_{\beta'_i} & 1 \leq i \leq g \\ a'_{g+1}{}^{-1} & i = g+1 \end{cases}, \\ &[\kappa', m'] = e_N^c I, \quad a'_i (1 \leq i \leq g+1), m' \in G\}, \end{aligned}$$

where $\bar{J}_{\alpha_i}$ is the product of a_i ($1 \leq i \leq g$), m and $e_N I$ corresponding to the class $\bar{j}_g \alpha_i$ in $\pi_1(Q_g, \tilde{*})$ and $(\bar{J}' \circ \bar{F})_{\alpha'_i}$ is the product of a'_i ($1 \leq i \leq g$), m' and $e_N I$ corresponding to the class $(\bar{j}'_g \bar{f})_{\alpha'_i}$ in $\pi_1(Q'_g, *)$, where $e_N I$ is substituted for $\varepsilon \in \pi_1(Q'_g, *)$. Similarly $\bar{J}_{\beta_i}$ and $(\bar{J}' \circ \bar{F})_{\beta'_i}$ are defined.

Let $\tilde{\mathcal{T}}_g$ be the subset of $\tilde{\mathcal{R}}(P_{g+1}) \subset G^{2(g+1)}$ defined by

$$\{(A_i, B_i)_{i=1}^{g+1} \in G^{2(g+1)} | ((A_i, B_i)_{i=1}^g, I, I) \in \tilde{\mathcal{R}}(P_g), A_{g+1} = I, B_{g+1} \in G\}.$$

It is diffeomorphic to $\tilde{\mathcal{R}}(P_g) \times G$. Put $\mathcal{T}_g = \tilde{\mathcal{T}}_g / \text{Ad}G \subset \mathcal{R}(P_{g+1})$. It is a fiber bundle over $\mathcal{R}(P_g)$ with fiber G . Clearly we have $\tilde{L}'_{g+1} \subset \tilde{\mathcal{T}}_g$ and $L'_{g+1} \subset \mathcal{T}_g$. It is easy to see that the restricted bundle $\mathcal{T}_g|_{L'_g \subset \mathcal{R}(P_g)}$ is L'_{g+1} and it is trivial fiber bundle. Here if necessary, we move L'_{g+1} in $\mathcal{T}_g$ by an isotopy of $\mathcal{T}_g$ which preserves the fibers over $\mathcal{R}(P_g)$ such that the induced isotopy of $\mathcal{R}(P_g)$ moves L'_g so that it intersects L_g transversely. So we may assume that L_g and L'_g intersect transversely from the first, since the intersection number $L_g \cdot_{\mathcal{R}(P_g)} L'_g$ is not changed.

On the other hand, using Lemma 8, we have

$$\begin{aligned} \tilde{L}_{g+1} \cap \tilde{\mathcal{T}}_g &= \{(A_i, B_i)_{i=1}^{g+1} \in G^{2(g+1)} | \\ A_i &= \begin{cases} J_{\alpha_i} & 1 \leq i \leq g \\ I & i = g+1 \end{cases}, B_i = \begin{cases} J_{\beta_i} & 1 \leq i \leq g \\ m & i = g+1 \end{cases}, \\ &[\kappa, m] = e_N^c I, \quad a_i (1 \leq i \leq g), m \in G\} \\ &= \{((A_i, B_i)_{i=1}^g, I, \tilde{s}((A_i, B_i)_{i=1}^{g+1})) \in \tilde{\mathcal{T}}_g | (A_i, B_i)_{i=1}^{g+1} \in \tilde{L}_g\}, \end{aligned}$$

where $\tilde{s}: \tilde{L}_g \rightarrow G$ is an Ad_G -equivariant map. In other words, there exists a section $s: L_g \rightarrow \mathcal{T}_g|_{L_g}$ such that the image of s is $L_{g+1} \cap \mathcal{T}_g$. So we have

$$|(L_{g+1} \cap \mathcal{T}_g) \cdot_{\mathcal{T}_g} L'_{g+1}| = |L_g \cdot_{\mathcal{R}(P_g)} L'_g|.$$

For any $\rho \in L_{g+1} \cap \mathcal{T}_g$, the composition of the differential of the inclusion and the natural projection at ρ ;

$$T_\rho L_{g+1} \rightarrow T_\rho \mathcal{R}(P_{g+1}) \xrightarrow{\text{proj}} T_\rho \mathcal{R}(P_{g+1}) / T_\rho \mathcal{T}_g$$

is surjective. Since $L_{g+1} \cap \mathcal{T}_g$ is connected, we can deduce

$$|(L_{g+1} \cap \mathcal{T}_g) \cdot_{\mathcal{T}_g} L'_{g+1}| = |L_{g+1} \cdot_{\mathcal{R}(P_{g+1})} L'_{g+1}|.$$

As a result, we obtain $|L_{g+1} \cdot_{\mathcal{R}(P_{g+1})} L'_{g+1}| = |L_g \cdot_{\mathcal{R}(P_g)} L'_g|$. This completes the proof of

Lemma 18. □

If necessary, we use Lemma 18 twice, we obtain Theorem 4 by Proposition 13.

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