

論文 / 著書情報
Article / Book Information

題目(和文)	平坦曲面のVeech群及びリーマン面のVeech正則族の研究
Title(English)	Veech groups of flat surfaces and Veech holomorphic families of riemann surfaces
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出典(和文)	学位:博士(理学), 学位授与機関:東京工業大学, 報告番号:甲第8879号, 授与年月日:2012年9月25日, 学位の種別:課程博士, 審査員:柳田 英二
Citation(English)	Degree:Doctor (Science), Conferring organization: Tokyo Institute of Technology, Report number:甲第8879号, Conferred date:2012/9/25, Degree Type:Course doctor, Examiner:
学位種別(和文)	博士論文
Type(English)	Doctoral Thesis

学位論文 平成24年度

Veech groups of flat surfaces and Veech
holomorphic families of Riemann
surfaces

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September 2012

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Chapter 1

Introduction

The classification of Riemann surfaces is one of the most important problem for complex analysis. A Riemann surface is called of type (g, n) if it is a closed Riemann surface of genus g with n points are removed. The set of all conformal equivalence classes of Riemann surfaces of type (g, n) is called the moduli space of Riemann surfaces of type (g, n) and we denote it by $M(g, n)$. The classification of Riemann surfaces is to analyze the moduli spaces. By the uniformization theorem which was proved by Poincaré and Koebe, if $(g, n) = (0, 0), (0, 1), (0, 2)$, and $(0, 3)$, there exists only one Riemann surface of type (g, n) and if $(g, n) = (1, 0)$, the moduli space coincides with $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$. Riemann saw that the complex structures have $3g - 3$ complex parameters if $g \geq 2$ and $n = 0$. Now, it is known that the moduli space $M(g, n)$ is a $(3g - 3 + n)$ -dimensional complex analytic space if $3g - 3 + n > 0$. The structure of $M(g, n)$ is induced by the Teichmüller space of a Riemann surface X of type (g, n) . Teichmüller spaces are introduced by Teichmüller in the 1930s. Fix a Riemann surface X of type (g, n) with $3g - 3 + n > 0$. Let us consider all pairs (Y, f) of a Riemann surface Y of type (g, n) and a quasiconformal map $f : X \rightarrow Y$. Two such pairs (Y_1, f_1) and (Y_2, f_2) are said to be Teichmüller equivalent if there exists a conformal map $h : Y_1 \rightarrow Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$. The Teichmüller space $T(X)$ of X is the set of all Teichmüller equivalence classes of such pairs. Teichmüller defined metrics on Teichmüller spaces which are called the Teichmüller metrics. With respect to this metric, $T(X)$ is a complete metric space and it is homeomorphic to $\mathbb{C}^{3g-3+n}$. Moreover, a Teichmüller space has an natural complex structure. In 1944, Teichmüller stated the existence of the natural complex structures on Teichmüller spaces. In 1960, Ahlfors [1] proved the existence in case that $n = 0$. In the same year, Bers [2] showed an easy way to induce the complex structures in case that $3g - 3 + n > 0$. With respect to the complex structure, the Teichmüller space $T(X)$ of a Riemann surface of type (g, n) is a simply connected $(3g - 3 + n)$ -dimensional complex manifold. Royden [18] proved in 1971 that the Kobayashi pseudo-metric on $T(X)$ coincides with the Teichmüller metric. Furthermore, the mapping class group $\mathrm{Mod}(X)$ of X which is the group of all homotopy classes of quasiconformal self-maps of X naturally acts on $T(X)$. It act on $T(X)$ isometrically and biholomorphically and the quotient space coincides with the moduli space $M(g, n)$. This means that the Teichmüller space $T(X)$ is the universal covering of the moduli space $M(g, n)$ of X . Therefore, we study Teichmüller spaces rather than moduli spaces.

In this paper, our main objects of interest are Veech groups of flat sur-

faces and related objects. A flat surface (X, u) is a pair of a Riemann surface X and a flat structure u on X . The Veech group $\Gamma(X, u)$ of a flat surface (X, u) is the group of all elements in $\mathrm{PSL}(2, \mathbb{R})$ which induce affine homeomorphisms on (X, u) . Until the 1970s, Veech groups were mainly studied in complex analysis. In the 1980s, they were also applied to ergodic theory. A flat surface (X, u) defines a holomorphic isometric embedding from a hyperbolic plane $\mathbb{H}$ into the Teichmüller space $T(X)$ and the image is called a Teichmüller disk. Veech [23] proved in 1989 that the Veech group $\Gamma(X, u)$ is regarded as the mirror of a representation of the action of the mapping class group $\mathrm{Mod}(X)$ of X into the Teichmüller disk. This also conclude that the Veech group is a Fuchsian group and the mirror of the orbifold $\mathbb{H}/\Gamma(X, u)$ is holomorphically and locally isometrically embedded into the moduli space $M(g, n)$ of X . If the Veech group $\Gamma(X, u)$ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$, such orbifold is called a Teichmüller curve. Teichmüller curves are important not only for complex analysis but also for algebraic geometry. Veech [23] also proved that Veech groups have information of geodesic flows on flat surfaces. The Veech dichotomy theorem states that if the Veech group $\Gamma(X, u)$ of a flat surface (X, u) is a lattice in $\mathrm{PSL}(2, \mathbb{R})$ then the geodesic flow in each direction on (X, u) is either periodic or uniquely ergodic. This theorem works essentially for studies of geodesics in Teichmüller spaces and measured foliations on Riemann surfaces.

In Chapter 2, we focus on flat surfaces which are finite coverings of the flat surface obtained by a regular $2n$ -gon ($n \geq 4$). In general, it is difficult to calculate the Veech groups of flat surfaces concretely. This prevented the study of Veech groups. In 2004, Schmithüsen [19] constructed an algorithm for calculating Veech groups of origamis. Origamis are unbranched finite coverings of the torus constructed from an unit square. She proved that the Veech group of the universal covering of origamis is equal to the modular group $\mathrm{PSL}(2, \mathbb{Z})$. This implies that all Veech groups of origamis are finite index subgroups of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ and hence, they are arithmetic Fuchsian groups.

To calculate Veech groups of finite coverings of the flat surface obtained by a regular $2n$ -gon, we extend Schmithüsen's method. To apply her method, we prove that the Veech group of the universal covering of the flat surfaces obtained by a regular $2n$ -gon coincides with the Veech group of the flat surface obtained by a regular $2n$ -gon. The claim also indicates that the Veech groups which we will obtain are non-arithmetic Fuchsian groups. According to her method, when we calculate the Veech group of a finite coverings of

the flat surface obtained by a regular $2n$ -gon, we observe the actions of affine homeomorphisms into the fundamental group of the flat surface and apply the Reidemeister-Schreier method. The Reidemeister-Schreier method is a way to obtain systems of generators and coset representatives of subgroups of a group. However, the calculations are very complicated in our case. We also prove that, it is enough to see the actions of the affine homeomorphisms into the one-dimensional homology group of the flat surface obtained by a regular $2n$ -gon when we calculate the Veech groups of certain finite Abelian coverings of the flat surface. It makes the calculation simple and gives a new algebraic point of view of Veech groups. At the end of this chapter, we give infinitely many new examples of Veech groups which are non-arithmetic Fuchsian groups by using the algebraic property.

In Chapter 3, we study holomorphic families of Riemann surfaces constructed from Veech groups which we call *Veech holomorphic families of Riemann surfaces*. A holomorphic family (M, π, B) of Riemann surfaces of type (g, n) over B is a triple of a two-dimensional complex manifold M , a Riemann surface B , and a holomorphic map $\pi : M \rightarrow B$ such that the fibers $X_t = \pi^{-1}(t)$ are Riemann surface of type (g, n) with $3g - 3 + n > 0$ for all $t \in B$ and the complex structures of X_t depend holomorphically on the parameter t , that is, the map $t \mapsto X_t$ from the Riemann surface B to the moduli space $M(g, n)$ of Riemann surfaces of type (g, n) is well-defined and holomorphic. Holomorphic families of Riemann surfaces are first studied in algebraic geometry as the Diophantine problems over function fields on compact Riemann surfaces. Take a three-variable irreducible homogeneous polynomial $f(X, Y, Z)$ whose coefficients are meromorphic functions on a compact Riemann surface. The equation $f(X, Y, Z) = 0$ is called a Diophantine equation over a function field. The Diophantine problems over function fields are to find all solutions of such Diophantine equations. It is known that a Diophantine equation has at most finitely many solutions. The statement is called *the Mordell conjecture* in function field case and proved by Manin [16], Grauert [7], and Miwa [17]. In 1992, Imayoshi and Shiga [12] also proved it analytically by using the Teichmüller theory. It is known that a Diophantine equation corresponds to a holomorphic family of Riemann surfaces over a Riemann surface. Every solution of a Diophantine equation is regarded as a holomorphic section of the corresponding holomorphic family of Riemann surfaces. They proved that holomorphic families of Riemann surfaces have at most finitely many holomorphic sections. Now, what is important for the study of holomorphic families of Riemann surfaces are to give upper bounds

of the numbers of holomorphic sections and to find all of them. It is also interesting to construct Diophantine equations from holomorphic families of Riemann surfaces and solve the equations by finding all holomorphic sections of the holomorphic families of Riemann surfaces. In 1997, Shiga [20] gave upper bounds of the numbers of holomorphic sections in case that the fibers are of type $(0, k)$ ($k \geq 4$), $(1, 2)$, or $(2, 0)$. He gave them by estimating the number of holomorphic families of Riemann surfaces. The upper bounds depend only on the topological types of the base Riemann surface B and the fibers. The first examples of holomorphic families of Riemann surfaces which are found all holomorphic sections are given by Imayoshi, Komori, and Nogi [11] in 2010.

We discuss about holomorphic sections of Veech holomorphic families of Riemann surfaces and Diophantine equations. This paper is the first study for Veech holomorphic families of Riemann surfaces. Veech holomorphic families of Riemann surfaces are constructed by using the holomorphic and local isometric embeddings from two-dimensional orbifolds into moduli spaces induced by Veech groups. In general, a holomorphic family (M, π, B) of Riemann surfaces also corresponds to a holomorphic map from B to the moduli space of the fibers. However, it is not local isometry or embedding. Thus, Veech holomorphic families of Riemann surfaces are special holomorphic families of Riemann surfaces. We characterize holomorphic sections of Veech holomorphic families of Riemann surfaces by points of corresponding flat surfaces with some property. Examples of Diophantine equations are constructed from some Veech holomorphic families of Riemann surfaces in this chapter. We solve the equations by applying the characterization of holomorphic sections of Veech holomorphic families of Riemann surfaces. These examples are the second examples of holomorphic families of Riemann surfaces which are found all holomorphic sections. They are the first examples of Diophantine equations constructed from holomorphic families of Riemann surfaces which are solved completely. A main purpose of this chapter is to give upper bounds of the numbers of holomorphic sections of certain Veech holomorphic families of Riemann surfaces. The upper bounds depend only on the topological types of the base Riemann surfaces and the fibers. To give the upper bounds, we prove a new property of Fuchsian groups which have parabolic elements. It is related to the Shimizu lemma which is an important lemma for hyperbolic geometry and Kleinian groups. The Shimizu lemma gives a lower bound of certain numbers for Fuchsian groups with parabolic elements in $\mathrm{PSL}(2, \mathbb{R})$. Our property gives upper bounds of the numbers.

The upper bound depends only on the genus, the number of punctures, and orders of cone points of the orbifold corresponding to each Fuchsian group. This property is also related to hyperbolic geodesics of the orbifold.

This work was supported by Global COE Program “Computationism as a Foundation for the Sciences”. The author would like to thank to Professor Hiroshige Shiga, Sudeb Mitra, Yunping Jiang, and Curtis T McMullen for their valuable suggestions.

Chapter 2

Veech groups of flat structures on Riemann surfaces

Veech groups of flat structures on Riemann surfaces

Yoshihiko Shinomiya

Abstract

In this chapter, we construct new examples of Veech groups by extending Schmithüsen's method for calculating Veech groups of origamis to Veech groups of unramified finite coverings of regular $2n$ -gons. We calculate the Veech groups of certain Abelian coverings of regular $2n$ -gons by using an algebraic method.

2.1 Introduction

The Teichmüller disk is a holomorphic isometric embedding of an upper-half plane $\mathbb{H}$ (or a unit disk) into a Teichmüller space. All such embeddings are constructed by flat structures on Riemann surfaces and $\mathrm{SL}(2, \mathbb{R})$ -orbit on flat structures. To study the image of a Teichmüller disk into the moduli space, we consider the stabilizer of the Teichmüller disk in the mapping class group. Veech [23] showed that this stabilizer is regarded as the group of all affine diffeomorphisms on a corresponding flat structure and its action can be represented by a Fuchsian group which acts on $\mathbb{H}$. The Fuchsian group is called a Veech group.

The first non-trivial examples of Veech groups were given by Veech [23] and [24]. His examples are constructed by gluing two congruent regular polygons along one side and identifying the parallel sides of the resulting polygons. However, not so many examples are known other than Veech's. Recently, Schmithüsen [19] showed an algorithm for finding Veech groups of "origami". An origami is an unramified finite covering of a once punctured torus constructed by a unit square. We apply her method to unramified finite coverings of regular $2n$ -gons instead of the unit square to obtain other examples of Veech groups. Veech groups of universal coverings play an important role in her method. We call these groups universal Veech groups.

In this chapter, we determine the universal Veech groups of $2n$ -gons and give an algorithm to calculate Veech groups of finite Abelian coverings of $2n$ -gons. In the case of origamis, Schmithüsen connected the Veech groups

of origamis with subgroups of $\mathrm{SL}(2, \mathbb{Z})$. She showed that the calculations of Veech groups stop in finitely many steps. In our case, for the Veech groups of Abelian coverings of $2n$ -gons whose degree is d , we connect them with subgroups of $\mathrm{SL}(n, \mathbb{Z}_d)$. We show that the calculations of Veech groups of certain Abelian coverings can be done by using the corresponding subgroups of $\mathrm{SL}(n, \mathbb{Z}_d)$.

2.2 Definitions

Let X be a Riemann surface of type (g, n) with $3g - 3 + n > 0$.

Definition 2.2.1 (Holomorphic quadratic differential). A tensor φ on X is called a holomorphic quadratic differential if its restriction to every coordinate neighborhood (U, z) is the form $f dz^2$, here f is a holomorphic function on U .

We define $|\varphi|$ to be the differential 2-form on X whose restriction to every coordinate neighborhood (U, z) has the form $|f| dx dy$ if φ equals $f dz^2$ in U . We say φ is integrable if its norm

$$\|\varphi\| = \iint_X |\varphi|$$

is finite.

We fix an integrable holomorphic quadratic differential φ . Denote by X' the Riemann surface constructed from X by removing zeros of φ .

Definition 2.2.2 (Flat structure). A flat structure u on X' is an atlas of X' which satisfies the following conditions.

- (1) Local coordinates of u are compatible with the orientation on X' induced by its Riemann surface structure.
- (2) For coordinate neighborhoods (U, z) and (V, w) of u with $U \cap V \neq \emptyset$, the transition function is the form

$$w = \pm z + c$$

in $z(U \cap V)$ for some $c \in \mathbb{C}$.

(3) u is maximal with respect to (1) and (2).

The holomorphic quadratic differential φ determines a flat structure u_φ on X' as follows.

For each $p_0 \in X'$, we can choose an open neighborhood U such that

$$z(p) = \int_{p_0}^p \sqrt{\varphi}$$

is a well-defined and injective function of U . This function is holomorphic in U since φ is a holomorphic quadratic differential. If (U, z) and (V, w) are pairs of such neighborhoods and functions with $U \cap V \neq \emptyset$, then we have $dw^2 = \varphi = dz^2$ in $U \cap V$. Hence $w = \pm z + c$ in $z(U \cap V)$ for some $c \in \mathbb{C}$. The flat structure u_φ is the maximal flat structure which contains such pairs.

Definition 2.2.3 (Affine group of φ). The affine group $\text{Aff}^+(X, \varphi)$ of the integrable holomorphic quadratic differential φ is the group of all quasiconformal mappings f of X onto itself which satisfy $f(X') = X'$ and are affine with respect to the flat structure u_φ . This means that for $(U, z), (V, w) \in u_\varphi$ with $f(U) \subseteq V$, the homeomorphism $w \circ f \circ z^{-1}$ is the form $z \mapsto Az + c$ for some $A \in \text{GL}(2, \mathbb{R})$ and $c \in \mathbb{C}$.

This A is uniquely determined up to the sign since u_φ is a flat structure. And A is always in $\text{SL}(2, \mathbb{R})$ since $\|\varphi\| = \int_X |\varphi| = \int_X f^*(|\varphi|) = \det(A) \|\varphi\|$. Thus we have a group homomorphism

$$D : \text{Aff}^+(X, \varphi) \rightarrow \text{PSL}(2, \mathbb{R}).$$

Definition 2.2.4 (Veech group of φ). We call $\Gamma(X, \varphi) = D(\text{Aff}^+(X, \varphi))$ the Veech group of φ .

Remark. Veech groups are discrete subgroups of $\text{PSL}(2, \mathbb{R})$ (see [4]).

2.3 Examples of Veech groups

In this section, we see two examples of Veech groups. The first example is a new example of Veech groups. The second one is the main target of this chapter. The purpose of this chapter is to determine Veech groups of some coverings of the second one. To do this, we need to determine the Veech group of the second one.

Example 2.3.1. Let X be a surface constructed as Figure 3.1. We induce an unique conformal structure on X such that the quadratic differential dz^2 on the interior of the rectangle of Figure 3.1 extends to a holomorphic quadratic differential φ on X . Then X is a Riemann surface of type $(2, 0)$ and vertices of four squares become two points on X . These points are zeros of φ of order 2. We can see that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ define elements in $\text{Aff}^+(X, \varphi)$ as Figure 3.2. Hence $\Gamma = \left\langle \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right] \right\rangle$ is a subgroup of the Veech group $\Gamma(X, \varphi)$. Since every element in $\text{Aff}^+(X, \varphi)$ must preserve the set of all lattice points, $\Gamma(X, \varphi)$ is a subgroup of $\text{PSL}(2, \mathbb{Z})$. It is known that $\left\langle \left[\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right] \right\rangle$ is the congruence subgroup of level 2 and has index 6 in $\text{PSL}(2, \mathbb{Z})$. Hence $\Gamma(X, \varphi)$ is either Γ or $\text{PSL}(2, \mathbb{Z})$. However, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ cannot be an element in $\Gamma(X, \varphi)$. Therefore $\Gamma(X, \varphi)$ must be Γ .

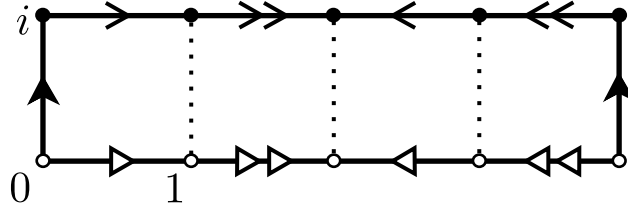


Figure 2.1: The flat surface (X, u)

The next example is given by Earle and Gardiner ([4]).

Example 2.3.2. Fix $n \geq 4$ and let Π_{2n} be a regular $2n$ -gon. We assume that Π_{2n} has two horizontal sides, lengths of the sides are 1 and its vertices are removed. We identify each side of Π_{2n} with the opposite parallel side by an Euclidean translation (see Figure 3.3) and denote the resulting surface by P_{2n} . We induce an unique conformal structure on P_{2n} such that the quadratic differential dz^2 on the interior of Π_{2n} extends to a holomorphic quadratic differential φ_{2n} on P_{2n} . If n is even, then P_{2n} is a Riemann surface of type $(\frac{n}{2}, 1)$ and if n is odd, then P_{2n} is a Riemann surface of type $(\frac{n-1}{2}, 2)$.

Now $R_{2n} = \begin{pmatrix} \cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\ \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}$ and $T_{2n} = \begin{pmatrix} 1 & 2 \cot \frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}$ induce elements

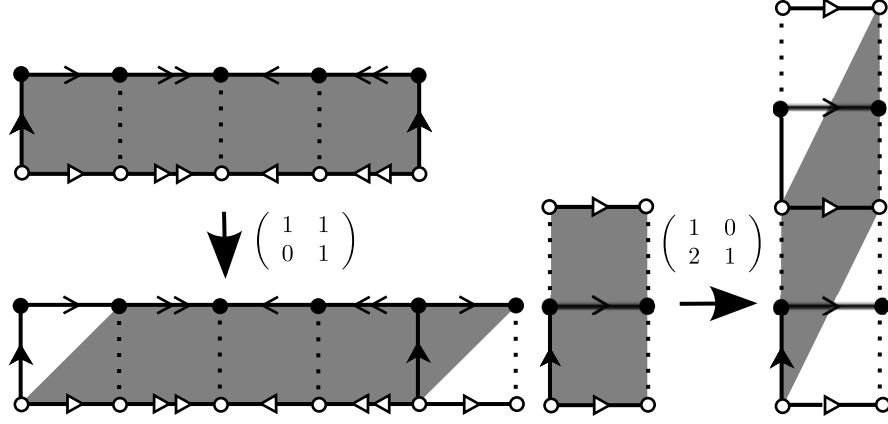


Figure 2.2: Actions of two matrices onto (X, u)

in $\text{Aff}^+(P_{2n}, \varphi_{2n})$. The action of R_{2n} on P_{2n} is the rotation about the center of Π_{2n} of angle $\frac{\pi}{n}$. To see the action of T_{2n} on P_{2n} , we cut P_{2n} along all horizontal segments which connect the vertices of Π_{2n} . If n is even, P_{2n} is decomposed into $\frac{n}{2}$ cylinders and the action of T_{2n} is the composition of the square of the right Dehn twist along a core curve of the cylinder which contains the center of Π_{2n} and the right Dehn twists along core curves of the other cylinders. If n is odd, P_{2n} is decomposed into $\frac{n-1}{2}$ cylinders and the action of T_{2n} is the composition of the right Dehn twists along core curves of all cylinders. Thus $\Gamma = \langle [R_{2n}], [T_{2n}] \rangle$ is a subgroup of the Veech group $\Gamma(P_{2n}, \varphi_{2n})$. It is easy to see that Γ is a (n, ∞, ∞) triangle group. Since only discrete group that contains Γ is a $(2, 2n, \infty)$ triangle group (see [4] and [22]) and this cannot be $\Gamma(P_{2n}, \varphi_{2n})$, we have $\Gamma(P_{2n}, \varphi_{2n}) = \langle [R_{2n}], [T_{2n}] \rangle$.

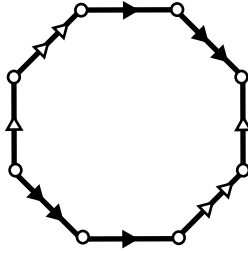


Figure 2.3: The flat surface (P_8, u_{q8})

2.4 Veech groups of coverings of P_{2n} and Universal Veech group of P_{2n}

Fix $n \geq 4$. Let P_{2n} be the same Riemann surface as in Example 2.3.2 and $p : X \rightarrow P_{2n}$ be an unramified finite covering mapping. Set $\varphi_X = p^* \varphi_{2n}$, here φ_{2n} is the holomorphic quadratic differential on P_{2n} defined in Example 2.3.2. Our purpose is to calculate the Veech group $\Gamma(X, \varphi_X)$. We denote $\Gamma(X, \varphi_X)$ by $\Gamma(X)$ hereafter. Schmithüsen constructed an algorithm for calculating Veech groups of origamis ([19]). We apply her method to our case.

Let $p_{2n} : \tilde{X}_{2n} \rightarrow P_{2n}$ be the universal covering mapping and set $\tilde{\varphi}_{2n} = p_{2n}^* \varphi_{2n}$. Note that $\|\tilde{\varphi}_{2n}\| = +\infty$. However, we can define the flat structure $u_{\tilde{\varphi}_{2n}}$ on $\tilde{X}_{2n}$ and the affine group $\text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ in the same manner as the case of integrable holomorphic quadratic differentials. Moreover, we have a homomorphism $D : \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \rightarrow \text{PGL}(2, \mathbb{R})$. Set $\Gamma(\tilde{X}_{2n}) = \text{Im}(D) \cap \text{PSL}(2, \mathbb{R})$.

Definition 2.4.1 (Universal Veech group of P_{2n}). We call $\Gamma(\tilde{X}_{2n})$ the universal Veech group of P_{2n} .

Remark. Let X be an unramified finite covering of P_{2n} . Then for each $f \in \text{Aff}^+(X, \varphi_X)$, there exists a lift $\tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ with $D(\tilde{f}) = D(f)$. Hence $\Gamma(X)$ is a subgroup of $\Gamma(\tilde{X}_{2n})$.

The following idea is due to Schmithüsen ([19]). For each finite covering X of P_{2n} , we take $\Gamma(X)$ as follows.

$$\begin{aligned} \Gamma(X) &= \left\{ [A] \in \Gamma(\tilde{X}_{2n}) \mid \exists \tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \text{ s.t. } D(\tilde{f}) = [A], \tilde{f} \text{ is a lift of a} \right. \\ &\quad \left. \text{homeomorphism of } X \text{ onto itself} \right\} \\ &= \left\{ [A] \in \Gamma(\tilde{X}_{2n}) \mid \exists \tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \text{ s.t. } D(\tilde{f}) = [A], \tilde{f}_*(\text{Gal}(\tilde{X}_{2n}/X)) = \right. \\ &\quad \left. \text{Gal}(\tilde{X}_{2n}/X) \right\}. \end{aligned}$$

To understand $\Gamma(X)$, we determine $\Gamma(\tilde{X}_{2n})$. The following theorem is a main theorem of this chapter.

Theorem 2.4.2. For all $n \geq 4$, $\Gamma(\tilde{X}_{2n}) = \langle [R_{2n}], [T_{2n}] \rangle = \Gamma(P_{2n})$.

For the proof of theorem, we represent $A \in \text{SL}(2, \mathbb{R})$ by

$$A = \begin{pmatrix} r \cos \alpha(A) & s \cos \beta(A) \\ r \sin \alpha(A) & s \sin \beta(A) \end{pmatrix}$$

for some $r, s > 0$ and $\alpha(A), \beta(A)$ with $0 \leq \alpha(A) < \beta(A) < 2\pi$. And set $\theta(A) = \beta(A) - \alpha(A)$. This $\theta(A)$ means the angle of $A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The following two lemmas give the proof of the theorem.

Lemma 2.4.3. *For $[A] \in \Gamma(\tilde{X}_{2n})$ with $|\cot \theta(A)| > \cot \frac{\pi}{2n}$, there exists $k, l \in \mathbb{Z}$ such that $|\cot \theta(A)| > |\cot \theta(T_{2n}^l R_{2n}^k A)|$.*

Lemma 2.4.4. *For $[A] \in \Gamma(\tilde{X}_{2n})$ with $|\cot \theta(A)| > \cot \frac{\pi}{2n}$, there exists $B \in \langle R_{2n}, T_{2n} \rangle$ such that $\cot \frac{\pi}{2n} \geq |\cot \theta(BA)|$.*

Proof of theorem 2.4.2. $\Gamma(P_{2n}) \subseteq \Gamma(\tilde{X}_{2n})$ is clear since $\Gamma(\tilde{X}_{2n})$ is the universal Veech group of P_{2n} . We show $\Gamma(\tilde{X}_{2n}) \subseteq \Gamma(P_{2n})$. By Lemma 2.4.4, for each $[A] \in \Gamma(\tilde{X}_{2n})$, there exists $[B] \in \langle [R_{2n}], [T_{2n}] \rangle$ such that

$$\cot \frac{\pi}{2n} \geq |\cot \theta(BA)|.$$

If we map Q_1 of Figure 2.4 by an affine transformation BA , the image is a parallelogram whose vertices correspond to vertices of $2n$ -gons and which has no such points in its interior. Moreover, it has the same area as Q_1 and each angle θ of its vertices satisfies $\pi/2n \leq \theta \leq \pi - \pi/2n$. We can see that such parallelograms are only Q_1, Q_2, Q_3 and Q_4 of Figure 2.4 up to the image of them by $[R_{2n}]$ and $[T_{2n}]$. Then BA is either

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cot \frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \cot \frac{\pi}{2n} \\ -\tan \frac{\pi}{2n} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \cot \frac{\pi}{2n} \\ -\tan \frac{\pi}{2n} & 0 \end{pmatrix}.$$

However, it does not happen except for the case that AB is the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since every vertex of $2n$ -gons must be mapped to a vertex. Hence $BA = I$ and so $[A] = [B^{-1}] \in \langle [R_{2n}], [T_{2n}] \rangle = \Gamma(P_{2n})$. $\square$

Proof of Lemma 2.4.3. We consider two cases : (a) $\cot \theta(A) > \cot \frac{\pi}{2n}$ and (b) $-\cot \theta(A) > \cot \frac{\pi}{2n}$

Case (a) : There exists $k \in \mathbb{Z}$ such that $B = R_{2n}^k A$ satisfies either $0 \leq \alpha(B) < \frac{\pi}{2n}$ or $\pi - \frac{\pi}{2n} \leq \alpha(B) < \pi$. We define the function

$$\begin{aligned} F_{\alpha(B)}^{\beta(B)}(x) = & 4 \cot^2 \frac{\pi}{2n} \cdot \frac{\sin \beta(B) \sin \alpha(B)}{\sin(\beta(B) - \alpha(B))} \cdot x^2 \\ & + 2 \cot \frac{\pi}{2n} \cdot \frac{\sin(\beta(B) + \alpha(B))}{\sin(\beta(B) - \alpha(B))} \cdot x + \cot \theta(B) \end{aligned}$$

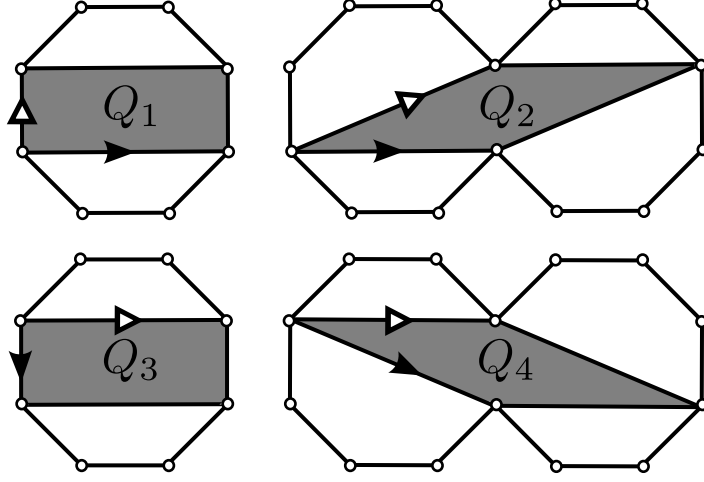


Figure 2.4: Parallelograms Q_1, Q_2, Q_3 and Q_4

of $x \in \mathbb{R}$. Note that $F_{\alpha(B)}^{\beta(B)}(l) = \cot(\beta(T_{2n}^l B) - \alpha(T_{2n}^l B)) = \cot \theta(T_{2n}^l B)$ for each $l \in \mathbb{Z}$.

(a)-1 : If $0 \leq \alpha(B) < \frac{\pi}{2n}$, there exists $\tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ with $D(\tilde{f}) = [B]$. And $\tilde{f}$ maps the rectangle Q_1 of Figure 2.4 to a parallelogram whose vertices correspond to vertices of $2n$ -gons and which has no such points in its interior. Hence we have $0 \leq \alpha(B) < \beta(B) \leq \frac{\pi}{2n}$. From this, if $\alpha(B) = 0$, then

$$F_0^{\beta(B)}(x) = 2 \cot \frac{\pi}{2n} \cdot x + \cot \theta(B)$$

and

$$F_0^{\beta(B)}(-\frac{1}{2}) = -\cot \frac{\pi}{2n} + \cot \theta(B) > 0.$$

So there exists a negative integer l such that

$$|F_0^{\beta(B)}(l)| < |F_0^{\beta(B)}(m)| \quad \text{for all } m \in \{0, -1, \dots, l+1, l-1\}.$$

Now we have

$$|\cot \theta(A)| = |F_0^{\beta(B)}(0)| > |F_0^{\beta(B)}(l)| = |\cot \theta(T_{2n}^l R_{2n}^k A)|.$$

If $0 < \alpha(B) < \beta(B) \leq \frac{\pi}{2n}$, then $F_{\alpha(B)}^{\beta(B)}(x)$ is a quadratic function of x and the axis of $F_{\alpha(B)}^{\beta(B)}$ is

$$x = -\frac{\cot \alpha(B) + \cot \beta(B)}{4 \cot \frac{\pi}{2n}} < -\frac{1}{2}$$

and

$$F_{\alpha(B)}^{\beta(B)}(-\frac{1}{2}) > 0.$$

Hence there exists a negative integer l such that

$$|F_{\alpha(B)}^{\beta(B)}(l)| < |F_{\alpha(B)}^{\beta(B)}(m)| \text{ for all } m \in \{0, -1, \dots, l+1, l-1\}.$$

And we have $|\cot \theta(A)| = |F_{\alpha(B)}^{\beta(B)}(0)| > |F_{\alpha(B)}^{\beta(B)}(l)| = |\cot \theta(T_{2n}^l R_{2n}^k A)|$.

(a)-2 : If $\pi - \frac{\pi}{2n} \leq \alpha(B) < \pi$, we have $\pi - \frac{\pi}{2n} \leq \alpha(B) < \beta(B) \leq \pi$ and

$$F_{\alpha(B)}^{\beta(B)}(x) = F_{(\pi-\beta(B))}^{(\pi-\alpha(B))}(-x).$$

By using the argument of (a)-1, we have $|\cot \theta(A)| > |\cot \theta(T_{2n}^l R_{2n}^k A)|$ for some $l \in \mathbb{Z}$.

Case (b) : We apply the same argument as in the Case (a) to the angle of two vectors $A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Then we have $|\cot \theta(A)| > |\cot \theta(T_{2n}^l R_{2n}^k A)|$ for some $k, l \in \mathbb{Z}$. $\square$

Proof of Lemma 2.4.4. Let $[A]$ be an element in $\Gamma(\tilde{X}_{2n})$ with $|\cot \theta(A)| > \cot \frac{\pi}{2n}$. From the proof of Lemma 2.4.3, we obtain $A_1 = T_{2n}^{l_1} R_{2n}^{k_1} A$ with $|\cot \theta(A_1)| < |\cot \theta(A)|$ for some $k_1 \in \mathbb{Z}$ and $l_1 \in \mathbb{Z} - \{0\}$. If $|\cot \theta(A_1)| > \cot \frac{\pi}{2n}$, then we obtain $A_2 = T_{2n}^{l_2} R_{2n}^{k_2} A_1$ with $|\cot \theta(A_2)| < |\cot \theta(A_1)|$ for some $k_2, l_2 \in \mathbb{Z} - \{0\}$ from the proof of Lemma 2.4.3 again. We repeat this operation. If there exists $m_0 \in \mathbb{N}$ such that $\cot \frac{\pi}{2n} \geq |\cot \theta(A_{m_0})|$ holds, then $B = A_{m_0} A^{-1}$ is what we want. Suppose that $|\cot \theta(A_m)| > \cot \frac{\pi}{2n}$ holds for every $m \in \mathbb{N}$. Then we have an infinite sequence $\{A_m\}$ in $\langle R_{2n}, T_{2n} \rangle \cdot A$ with $|\cot \theta(A_{m-1})| > |\cot \theta(A_m)| > \cot \frac{\pi}{2n}$ for all m . We represent A_m by

$$A_m = \begin{pmatrix} r_m \cos \alpha_m & s_m \cos \beta_m \\ r_m \sin \alpha_m & s_m \sin \beta_m \end{pmatrix}$$

for some $r_m, s_m > 0$ and $0 \leq \alpha_m < \beta_m < 2\pi$ with $r_m s_m \sin(\beta_m - \alpha_m) = 1$. For each m , there exists $\tilde{f}_m \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ such that $D(\tilde{f}_m) = [A_m]$ and $\tilde{f}_m$ maps Euclidean segment which connect vertices of $2n$ -gons to other segment. Thus we have

$$r_m = \left| A_m \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| \geq 1 \text{ and } s_m = \left| A_m \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \geq 1.$$

Moreover, we have

$$r_m s_m = \frac{1}{\sin(\beta_m - \alpha_m)} \leq \frac{1}{\sin(\beta_1 - \alpha_1)}.$$

Hence $\{\alpha_m\}, \{\beta_m\}, \{r_m\}$ and $\{s_m\}$ are bounded and there exists a subsequence $\{A_{m_i}\}$ of $\{A_m\}$ such that A_{m_i} converges to some $A_\infty \in \text{SL}(2, \mathbb{R})$. Since $\{A_{m_i}\}$ is in a discrete set $\langle R_{2n}, T_{2n} \rangle \cdot A$, there exists $i_0 \in \mathbb{N}$ such that $A_{m_i} = A_\infty$ for all $i \geq i_0$. However, this contradicts the construction of the sequence $\{A_m\}$. Hence there exists $m_0 \in \mathbb{N}$ such that $\cot \frac{\pi}{2n} \geq |\cot \theta(A_{m_0})|$. $\square$

2.5 Calculation of Veech groups

Let X be an unramified finite covering of P_{2n} . By theorem 2.4.2, we can write $\Gamma(X)$ as follows.

$$\Gamma(X) = \left\{ [A] \in \langle [R_{2n}], [T_{2n}] \rangle \mid \exists \tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \text{ s.t. } D(\tilde{f}) = [A], \right. \\ \left. \tilde{f}_*(\text{Gal}(\tilde{X}_{2n}/X)) = \text{Gal}(\tilde{X}_{2n}/X) \right\}.$$

Let z_0 be the point of P_{2n} which corresponds to the center of the $2n$ -gon Π_{2n} as in Example 2.3.2 and $\bar{z}_0$ be one of the preimages of z_0 in X . Let $\{x_1, x_2, \dots, x_n\}$ be the system of generators of $\pi_1(P_{2n}, z_0)$ as Figure 2.5. Then R_{2n} and T_{2n} define the following automorphisms $\gamma_{R_{2n}}$ and $\gamma_{T_{2n}}$ on $\pi_1(P_{2n}, z_0)$ (see Example 2.3.2).

$$\gamma_{R_{2n}} : \begin{cases} x_i \mapsto x_{i+1} & (i = 1, 2, \dots, n-1) \\ x_n \mapsto x_1^{-1} \end{cases}.$$

If n is even,

$$\gamma_{T_{2n}} : \begin{cases} x_1 \mapsto x_1 \\ x_{n+2-i}^{-1} x_i \mapsto x_{n+2-i}^{-1} x_i & (i = 2, 3, \dots, \frac{n}{2}) \\ x_i \mapsto (x_{n+2-i}^{-1} x_i) \cdots (x_{n-1}^{-1} x_3) (x_n^{-1} x_2) x_1^2 x_i & (i = 2, 3, \dots, \frac{n}{2}) \\ x_{\frac{n}{2}+1} \mapsto (x_{\frac{n}{2}+2}^{-1} x_{\frac{n}{2}}) \cdots (x_{n-1}^{-1} x_3) (x_n^{-1} x_2) x_1^2 x_{\frac{n}{2}+1} \end{cases}$$

and if n is odd,

$$\gamma_{T_{2n}} : \begin{cases} x_{n+1-i}^{-1} x_i \mapsto x_{n+1-i}^{-1} x_i & (i = 1, 2, \dots, \frac{n-1}{2}). \\ x_i \mapsto (x_{n+1-i}^{-1} x_i) \cdots (x_{n-1}^{-1} x_2) (x_n^{-1} x_1) x_i & (i = 1, 2, \dots, \frac{n-1}{2}) \\ x_{\frac{n+1}{2}} \mapsto (x_{\frac{n+3}{2}}^{-1} x_{\frac{n-1}{2}}) \cdots (x_{n-1}^{-1} x_2) (x_n^{-1} x_1) x_{\frac{n+1}{2}}. \end{cases}$$

Since $\text{Gal}(\tilde{X}_{2n}/P_{2n}) < \text{Ker}(D)$, $\text{Ker}(D)/\text{Gal}(\tilde{X}_{2n}/P_{2n}) = \{[id], [\tilde{h}^n]\}$ for some $\tilde{h} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ with $D(\tilde{h}) = [R_{2n}]$ and each element in $\text{Gal}(\tilde{X}_{2n}/P_{2n})$ defines an inner automorphism of $\text{Gal}(\tilde{X}_{2n}/P_{2n}) \cong \pi_1(P_{2n}, z_0)$, the action of each element of $\text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ on $\pi_1(P_{2n}, z_0)$ can be represented by a composition of $\gamma_{R_{2n}}$, $\gamma_{T_{2n}}$ and inner automorphisms of $\pi_1(P_{2n}, z_0)$.

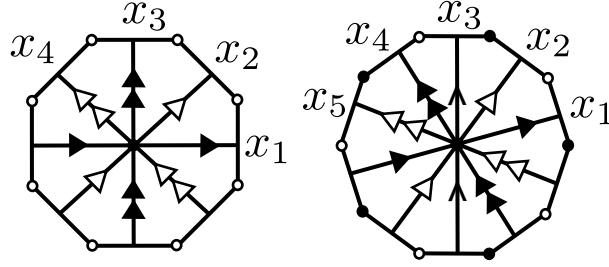


Figure 2.5: Systems of generators of $\pi_1(P_8, z_0)$ and $\pi_1(P_{10}, z_0)$

Hence we have the following.

Proposition 2.5.1. *For $\tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$, following two are equivalent. Here A is one of elements in $D(\tilde{f})$.*

- The mapping $\tilde{f}$ satisfies $\tilde{f}_*(\text{Gal}(\tilde{X}_{2n}/X)) = \text{Gal}(\tilde{X}_{2n}/X)$.
- There exists one of the preimages $\bar{z}_1 \in X$ of z_0 such that

$$\gamma_A(\pi_1(X, \bar{z}_0)) = \pi_1(X, \bar{z}_1) \text{ or } \gamma_{-A}(\pi_1(X, \bar{z}_0)) = \pi_1(X, \bar{z}_1).$$

By using this condition, we can determine whether $[A]$ is in $\Gamma(X)$ or not for each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$.

Now we can calculate the Veech group $\Gamma(X)$ of an unramified finite covering X of P_{2n} by using the following method. Schmithüsen([19]) also use this method to the calculations of Veech groups of origamis. The calculation is done on the following tree which we explain below.

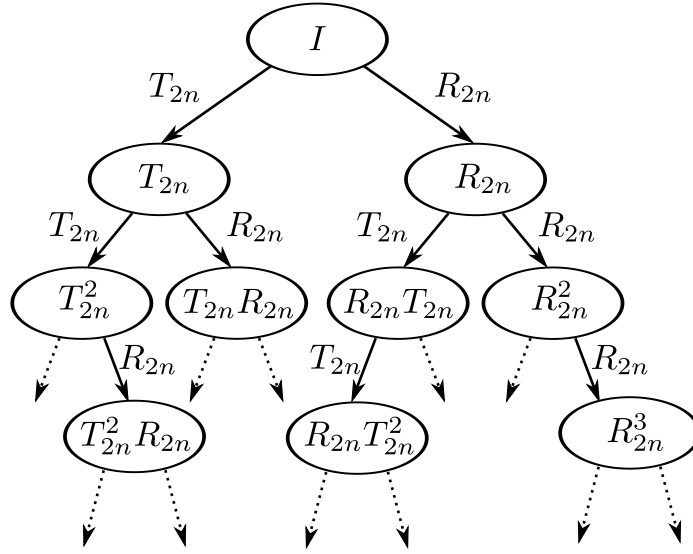


Figure 2.6: Tree of elements in $\langle [R_{2n}], [T_{2n}] \rangle$ which we consider in the following calculation

Calculation of $\Gamma(X)$ (Reidemeister-Schreier method).

Given an unramified finite covering X of P_{2n} .

Let **Rep** and **Gen** be empty sets.

Add $[I]$ to **Rep**. Set $A = I$.

Loop:

Set $B = A \cdot T_{2n}$, $C = A \cdot R_{2n}$.

Check whether B is already represented by **Rep**:

For each $[D]$ in **Rep**, check whether $[B] \cdot [D]^{-1}$ is in $\Gamma(X)$.

If so, add $[B] \cdot [D]^{-1}$ to **Gen**.

If none is found, add $[B]$ to **Rep**.

Do the same for C instead of B .

If there exists a successor of A in **Rep**,

let A be this successor and go to the beginning of the loop.

If not, finish the loop.

Result:

Gen : a list of generators of $\Gamma(X)$.

Rep : a list of coset representatives in $\langle [R_{2n}], [T_{2n}] \rangle$.

Proposition 2.5.2. *Let X be an unramified finite covering of P_{2n} . Then we have the following properties.*

- (1) *Any two elements in $\mathbf{Rep}$ belong to different cosets.*
- (2) *The calculation stops in finitely many steps.*
- (3) *In the end, each coset is represented by a member of $\mathbf{Rep}$.*
- (4) *In the end, $\Gamma(X)$ is generated by the elements in $\mathbf{Gen}$.*

Proof. (1) is clear and we can see a proof of (3), (4) in [19]. (2) is equivalent to what $\Gamma(X)$ is a finite index subgroup of $\langle [R_{2n}], [T_{2n}] \rangle$. By the next proposition, we conclude that $\Gamma(X)$ and $\Gamma(P_{2n}) = \langle [R_{2n}], [T_{2n}] \rangle$ are commensurable. Hence $\Gamma(X)$ is a finite index subgroup of $\langle [R_{2n}], [T_{2n}] \rangle$. Since all elements in $\mathbf{Rep}$ belong to different cosets of $\Gamma(X)$ in $\langle [R_{2n}], [T_{2n}] \rangle$, $\#\mathbf{Rep}$ cannot be greater than this index and hence the calculation of $\Gamma(X)$ stops in finitely many steps. $\square$

For a Riemann surface X and a holomorphic quadratic differential φ , denote by $C(X, \varphi)$ the set of all zeros of φ and punctures of X .

Proposition 2.5.3. ([8] and [9].) *Let $p : X \rightarrow Y$ be a covering mapping between Riemann surfaces. Let φ_X be a holomorphic quadratic differential on X and set $\varphi_Y = p_*\varphi_X$. Suppose that $p(C(Y, \varphi_Y)) = C(X, \varphi_X)$ and $p^{-1}(C(X, \varphi_X)) = C(Y, \varphi_Y)$. Then the Veech groups $\Gamma(X, \varphi_X)$ and $\Gamma(Y, \varphi_Y)$ are commensurable.*

Example 2.5.4. Let X be the covering of P_8 as Figure 2.7. We calculate the Veech group $\Gamma(X)$.

The fundamental group of X is

$$\pi_1(X, \bar{z}_0) = \langle x_1^2, x_2, x_4, x_1x_3, x_3x_1, x_1^{-1}x_2x_1, x_1^{-1}x_4x_1 \rangle.$$

Loop 1 : $\mathbf{Rep} = \{[I]\}$, $\mathbf{Gen} = \emptyset$, $A = I$, $B = T_8$, $C = R_8$.

We check $[B] \cdot [I]^{-1} = [T_8]$; the homomorphism γ_{T_8} maps the generators

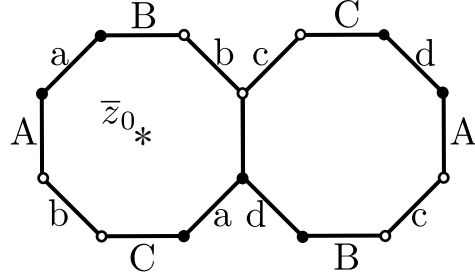


Figure 2.7: Double cover X of P_8

of $\pi_1(X, \bar{z}_0)$ as follows

$$\gamma_{T_8} : \begin{cases} x_1^2 \mapsto x_1^2 \\ x_2 \mapsto x_4^{-1} x_2 x_1^2 x_2 \\ x_4 \mapsto x_4^{-1} x_2 x_1^2 x_4 \\ x_1 x_3 \mapsto x_1 x_4^{-1} x_2 x_1^2 x_3 \\ x_3 x_1 \mapsto x_4^{-1} x_2 x_1^2 x_3 x_1 \\ x_1^{-1} x_2 x_1 \mapsto x_1^{-1} x_4^{-1} x_2 x_1^2 x_2 x_1 \\ x_1^{-1} x_4 x_1 \mapsto x_1^{-1} x_4^{-1} x_2 x_1^2 x_4 x_1 \end{cases} .$$

By taking $\bar{z}_0$ as a base point, all images represent closed curves. Hence $[T_8]$ is an element in $\Gamma(X)$ and add $[T_8]$ in **Gen**.

Now **Rep** = $\{[I]\}$, **Gen** = $\{[T_8]\}$.

We check $[C] \cdot [I]^{-1} = [R_8]$; there is no point of X such that $\gamma_{R_{2n}}(x_2) = x_3$ or $\gamma_{-R_{2n}}(x_2) = x_3^{-1}$ represent closed curves with the point as a base point. Hence $[R_8]$ is not in $\Gamma(X)$. We add $[R_8]$ in **Rep**.

Now **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8]\}$ and R_8 is a successor of $A = I$ and is in **Rep**. We set $A = R_8$.

Loop 2 : **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8]\}$, $A = R_8$, $B = R_8 T_8$, $C = R_8^2$.

We check $[B] \cdot [I]^{-1} = [R_8 T_8]$; it is not in $\Gamma(X)$.

We check $[B] \cdot [R_8]^{-1} = [R_8 T_8 R_8^{-1}]$; the homomorphism $\gamma_{R_8 T_8 R_8^{-1}}$ is the

form

$$\gamma_{R_8 T_8 R_8^{-1}} : \begin{cases} x_1 \mapsto x_1 x_2^{-2} x_3^{-1} x_1^{-1} \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_1 x_3 x_2^2 x_3 \\ x_4 \mapsto x_1 x_3 x_2^2 x_4 \end{cases}$$

and maps the generators of $\pi_1(X, \bar{z}_0)$ as follows

$$\gamma_{R_8 T_8 R_8^{-1}} : \begin{cases} x_1^2 \mapsto x_1 x_2^{-2} x_3^{-1} x_2^{-2} x_3^{-1} x_1^{-1} \\ x_2 \mapsto x_2 \\ x_4 \mapsto x_1 x_3 x_2^2 x_4 \\ x_1 x_3 \mapsto x_1 x_3 \\ x_3 x_1 \mapsto x_1 x_3 x_2^2 x_3 x_1 x_2^{-2} x_3^{-1} x_1^{-1} \\ x_1^{-1} x_2 x_1 \mapsto x_1 x_3 x_2^2 x_1^{-1} x_2 x_1 x_2^{-2} x_3^{-1} x_1^{-1} \\ x_1^{-1} x_4 x_1 \mapsto x_1 x_3 x_2^2 x_3 x_2^2 x_4 x_1 x_2^{-2} x_3^{-1} x_1^{-1} \end{cases}.$$

By taking $\bar{z}_0$ as a base point, all images represent closed curves. Hence $[R_8 T_8 R_8^{-1}]$ is an element in $\Gamma(X)$ and add $[R_8 T_8 R_8^{-1}]$ in **Gen**.

Now **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8], [R_8 T_8 R_8^{-1}]\}$.

We check $[C] \cdot [I]^{-1} = [R_8^2]$; the homomorphism γ_{R_8} maps the generators of $\pi_1(X, \bar{z}_0)$ as follows

$$\gamma_{R_8^2} : \begin{cases} x_1^2 \mapsto x_3^2 \\ x_2 \mapsto x_4 \\ x_4 \mapsto x_2^{-1} \\ x_1 x_3 \mapsto x_3 x_1^{-1} \\ x_3 x_1 \mapsto x_1^{-1} x_3 \\ x_1^{-1} x_2 x_1 \mapsto x_3^{-1} x_4 x_3 \\ x_1^{-1} x_4 x_1 \mapsto x_3^{-1} x_2^{-1} x_3 \end{cases}.$$

By taking $\bar{z}_0$ as a base point, all images represent closed curves. Hence $[R_8^2]$ is an element in $\Gamma(X)$ and add $[R_8^2]$ in **Gen**.

We check $[C] \cdot [R_8]^{-1} = [R_8]$; it is not in $\Gamma(X)$.

Now, **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8], [R_8 T_8 R_8^{-1}], [R_8^2]\}$ and there is no successor of $A = R_8$ in **Rep**. We finish the loop.

Result : **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8], [R_8 T_8 R_8^{-1}], [R_8^2]\}$.
As a result, $\Gamma(X) = \langle [T_8], [R_8 T_8 R_8^{-1}], [R_8^2] \rangle$ and coset representatives in $\langle [R_8], [T_8] \rangle$ is $\{[I], [R_8]\}$.

Remark. In the case of origamis, Schmithüsen showed that the calculations always stop by connecting the Veech groups of origami with subgroups of $SL(2, \mathbb{Z})$ (see [19]). In our case, for certain Abelian coverings of $2n$ -gons, we connect the Veech groups with subgroups of $SL(n, \mathbb{Z}_d)$ and calculate the Veech groups by using the corresponding matrices. It is seen in section 2.7.

2.6 Calculation of $\mathbb{H}/\Gamma(X)$

Let X be an unramified finite covering of P_{2n} . Assume that the calculation of $\Gamma(X)$ by the Reidemeister-Schreier method stopped. Then **Gen** is a list of generators of $\Gamma(X)$ and **Rep** is a list of coset representatives in $\langle [R_{2n}], [T_{2n}] \rangle$.

Let D be the fundamental domain of $\langle [R_{2n}], [T_{2n}] \rangle$ in $\mathbb{H}$ as Figure 2.8. Then

$$F = \text{Int} \left(\bigcup_{[A] \in \text{Rep}} [A](\overline{D}) \right)$$

is a fundamental domain of $\Gamma(X)$. Here $[A]$ means a Möbius transformation.

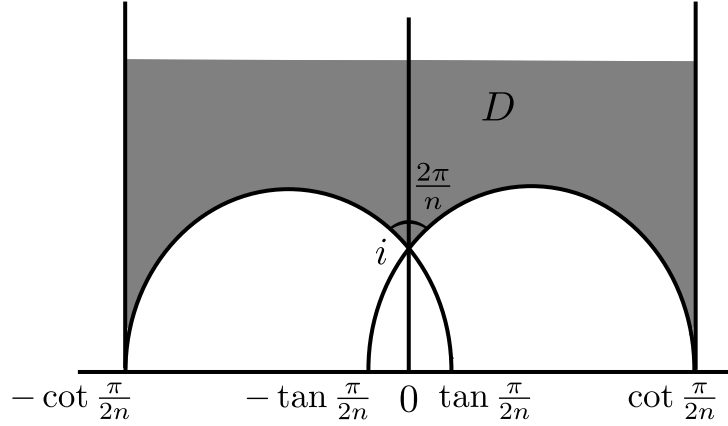


Figure 2.8: Fundamental domain of $\langle [R_{2n}], [T_{2n}] \rangle$

By reading **Gen**, we can know all pairs of sides of F which are identified by the action of $\Gamma(X)$. We call sides of $[A](D)$ which correspond to $(-\cot \frac{\pi}{2n}, i)$, $(\cot \frac{\pi}{2n}, i)$, $(-\cot \frac{\pi}{2n}, i\infty)$ and $(\cot \frac{\pi}{2n}, i\infty)$ the R^{-1} -side, the R -side, the T^{-1} -side and the T -side of $[A]$, respectively.

Proposition 2.6.1. *Assume that $\mathbf{Rep} = \{[A_1], [A_2], \dots, [A_k]\}$. Then for each $i, j \in \{1, 2, \dots, k\}$,*

- *The T -side of $[A_j]$ and the T^{-1} -side of $[A_i]$ are identified if and only if $[A_j T_{2n} A_i^{-1}] \in \Gamma(X)$.*
- *The R -side of $[A_j]$ and the R^{-1} -side of $[A_i]$ are identified if and only if $[A_j R_{2n} A_i^{-1}] \in \Gamma(X)$.*

We give a triangulation of $\mathbb{H}/\Gamma(X)$ by decomposing D as Figure 2.9. Then the number of triangles and sides are $2 \cdot \#\mathbf{Rep}$ and $3 \cdot \#\mathbf{Rep}$, respectively. Moreover, we can calculate the number v of vertices by using Proposition 2.6.1. When we calculate v , we decompose v as $v = v_\infty + v_{\cot} + v_{\text{cone}}$. Here v_∞ is the number of vertices corresponding to ∞ of D , $v_{\cot}$ is the number of vertices corresponding to $\pm \cot \frac{\pi}{2n}$ of D and v_{cone} is the number of vertices corresponding to i of D . Then $\mathbb{H}/\Gamma(X)$ has genus $(2 + \#\mathbf{Rep} - v)/2$ and $v_\infty + v_{\cot}$ punctures. We can also calculate the number of cone points and their orders in the calculation of v_{cone} .

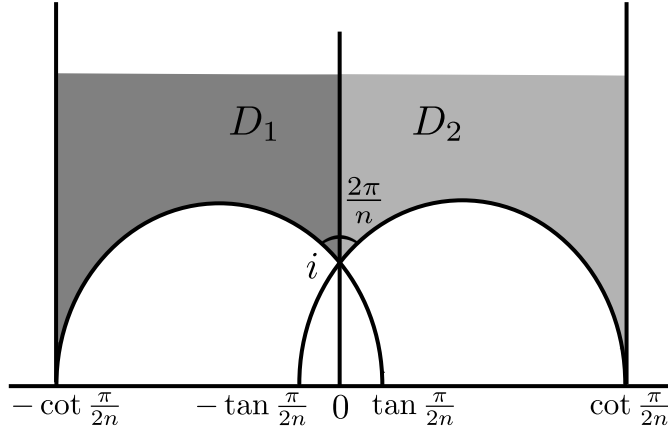


Figure 2.9: Decomposition of D into two triangles D_1 and D_2

Example 2.6.2. Let X be the Riemann surface as Figure 2.7. At the end of the calculation of $\Gamma(X)$, we have $\mathbf{Gen}=\{[T_8], [R_8 T_8 R_8^{-1}], [R_8^2]\}$ and $\mathbf{Rep}=\{[I], [R_8]\}$. Since $[T_8] = [I \cdot T_8 \cdot I^{-1}]$ is in $\Gamma(X)$, the T -side of $[I]$ and the T^{-1} -side of $[I]$ are identified by $\Gamma(X)$. In the same way the T -side of $[R_8]$ and the T^{-1} -side of $[R_8]$ are identified and the R -side of $[R_8]$ and the R^{-1} -side of $[I]$ are identified. Hence $\mathbb{H}/\Gamma(X)$ has no genus, three punctures and one cone point with order 2 (see Figure 2.10).

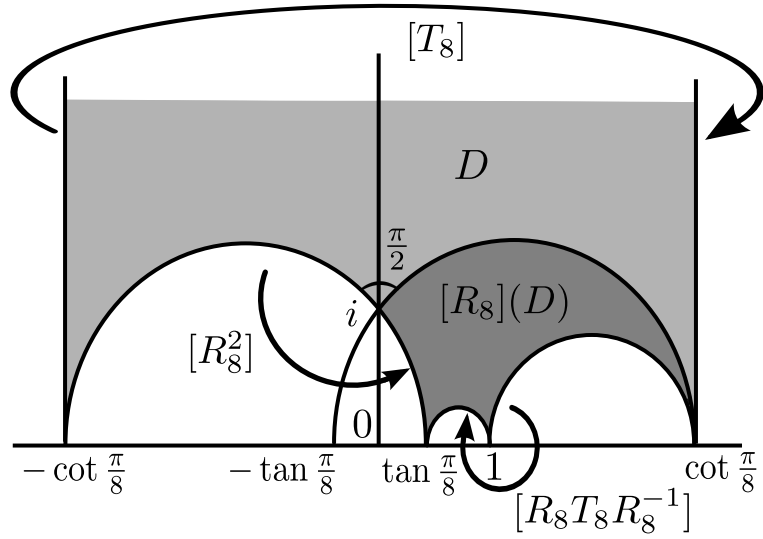


Figure 2.10: Fundamental domain of the Veech group $\Gamma(X)$ and the identification of the sides

2.7 Veech groups of Abelian coverings

In this section, we show that the calculation of Veech group $\Gamma(X)$ by the Reidemeister-Schreier method always stops if X is a finite Abelian covering of P_{2n} . And we show that the calculations of Veech groups of certain Abelian coverings can be done by using the corresponding subgroups of $\mathrm{SL}(n, \mathbb{Z}_d)$.

Recall that if $\Gamma(X)$ is a finite index subgroup of $\langle [R_{2n}], [T_{2n}] \rangle$, then the calculation of $\Gamma(X)$ stops by the proof of Proposition 2.5.2. We have a partial answer about the stop of calculations.

Theorem 2.7.1. *Let X be a finite Abelian covering of P_{2n} , that is, X is a finite Galois covering of P_{2n} and $\text{Gal}(X/P_{2n})$ is an Abelian group. Then the calculation of $\Gamma(X)$ stops.*

Proof. Recall that z_0 is the point of P_{2n} which corresponds to the center of the $2n$ -gon Π_{2n} as in Example 2.3.2 and $\bar{z}_0 \in X$ is one of the preimages of z_0 . Since X is a Galois covering, for each $w \in \text{Gal}(\tilde{X}_{2n}/P_{2n}) = \langle x_1, x_2, \dots, x_n \rangle$, w is in $\pi_1(X, \bar{z}_0)$ if and only if w is in $\pi_1(X, \bar{z}_1)$ for all $\bar{z}_1 \in X$. Hence $[A]$ is in $\Gamma(X)$ if and only if γ_A or γ_{-A} fix $\pi_1(X, \bar{z}_0)$ for each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$.

As $\text{Gal}(X/P_{2n}) \cong \pi_1(P_{2n}, z_0)/\pi_1(X, \bar{z}_0)$ is an Abelian group, $\overline{x_i x_j} = \overline{x_j x_i}$ and $x_i x_j = x_j x_i \cdot w$ for some $w \in \pi_1(X, \bar{z}_0)$. Moreover, set $d = \text{lcm}\{\text{ord}(\bar{x}_1), \text{ord}(\bar{x}_2), \dots, \text{ord}(\bar{x}_n)\}$, then $x_i^d \in \pi_1(X, \bar{z}_1)$ for all i and all $\bar{z}_1 \in X$.

We set $(e_1, e_2, \dots, e_{2n}) = I_{2n}$. Let us consider the homomorphism $\nu : \text{Gal}(\tilde{X}_{2n}/P_{2n}) \rightarrow \mathbb{Z}_d^n ; x_i \mapsto e_i$. Then there exists a homomorphism $\Phi_d : \langle \gamma_T, \gamma_R \rangle \rightarrow \text{SL}(n, \mathbb{Z}_d)$ such that the following diagram is commutative.

$$\begin{array}{ccc} \text{Gal}(\tilde{X}_{2n}/P_{2n}) & \xrightarrow{\gamma_A} & \text{Gal}(\tilde{X}_{2n}/P_{2n}) \\ \nu \downarrow & & \downarrow \nu \\ \mathbb{Z}_d^n & \xrightarrow{\Phi_d(A)} & \mathbb{Z}_d^n \end{array}$$

Set $V = \nu(\pi_1(X, \bar{z}_0))$. For each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$, if $[A]$ satisfies $\Phi_d(A)(V) = V$, then $[A] \in \Gamma(X)$.

Now we conclude that $\sharp \mathbf{Rep} \leq \sharp \text{SL}(n, \mathbb{Z}_d)$ at every step of the calculation. Suppose that $\sharp \mathbf{Rep} > \sharp \text{SL}(n, \mathbb{Z}_d)$ happens at some step of the calculation. Then there exists two distinct elements $[A]$ and $[B]$ in $\mathbf{Rep}$ such that $\Phi_d(A) = \Phi_d(B)$. Since $\Phi_d(AB^{-1}) = I_n$ stabilizes V , $[A] \cdot [B]^{-1}$ is in $\Gamma(X)$. However, since $[A]$ and $[B]$ are distinct elements in $\mathbf{Rep}$, $[A] \cdot [B]^{-1}$ is not in $\Gamma(X)$. This is a contradiction. $\square$

From the proof of theorem 2.7.1, we have the following.

Corollary 2.7.2. *Let X be a finite Abelian covering of P_{2n} . If there exists $d \in \mathbb{N}$ such that $\{\text{ord}(\bar{x}_1), \text{ord}(\bar{x}_2), \dots, \text{ord}(\bar{x}_n)\} = \{d\}$ or $\{1, d\}$, then $[A] \in \Gamma(X)$ if and only if $\Phi_d(A)(V) = V$ for each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$.*

Example 2.7.3. Let X be the covering of P_8 the same as Figure 2.7. Then X satisfies the assumption of Corollary 2.7.2. The fundamental group of X is

$$\pi_1(X, \bar{z}_0) = \langle x_1^2, x_2, x_4, x_1 x_3, x_3 x_1, x_1^{-1} x_2 x_1, x_1^{-1} x_4 x_1 \rangle$$

and

$$V = \langle e_2, e_4, e_1 + e_3 \rangle_{\mathbb{Z}_2}.$$

By Corollary 2.7.2, for $[A] \in \langle [R_8], [T_8] \rangle$, $[A]$ is in $\Gamma(X)$ if and only if $\Phi_2(A)$ satisfies the followings :

$$\begin{cases} \Phi_2(A)_{1,1} + \Phi_2(A)_{3,1} + \Phi_2(A)_{1,3} + \Phi_2(A)_{3,3} \equiv 0 \pmod{2} \\ \Phi_2(A)_{1,j} + \Phi_2(A)_{3,j} \equiv 0 \pmod{2} (j = 2, 4). \end{cases}$$

2.8 Examples.

Finally we show some examples of Veech groups that are calculated by the method of this chapter.

Example 2.8.1. Let X be the double covering of P_8 as Figure 2.11. Then X is a Riemann surface of type $(3, 2)$. Set $R = [R_8]$, $T = [T_8]$. Then

- For $[A] \in \langle R, T \rangle$, $[A]$ is in $\Gamma(X)$ if and only if $\Phi_2(A)_{1,j} \equiv 0 \pmod{2}$ ($j = 2, 3, 4$),
- $\Gamma(X)$
 $= \langle T, RT^2R^{-1}, RTRT^2(RTR)^{-1}, (RT)^3(RTRTR)^{-1},$
 $(RT)^2R^2T(RTR^2)^{-1}, (RT)^2R^3T(RTRTR^3)^{-1},$
 $RTR^2T(RTRTR^2)^{-1}, RTR^3T^2(RTR^3)^{-1},$
 $RTR^3TR, R^2TR^{-2}, R^3(RTR^3T)^{-1} \rangle,$
- $\Gamma(X) \setminus \langle R, T \rangle$
 $= \left\{ I, R, RT, R^2, RTR, RTRT, RTR^2, RTRTR, \right.$
 $\left. RTRTR^2, RTRTR^3, RTR^3, RTR^3T \right\}$ and
- $\mathbb{H}/\Gamma(X)$ is a Riemann surface of type $(0, 11)$.

Example 2.8.2. Let X be the covering of P_8 as Figure 2.12. Then X is a Riemann surface of type $(5, 4)$. Set $R = [R_8]$, $T = [T_8]$. Then

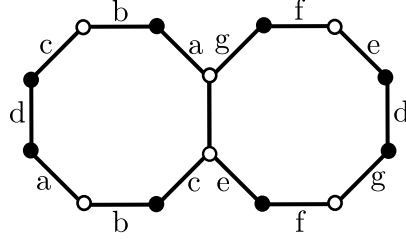


Figure 2.11: Double cover X of P_8

- For $[A] \in \langle R, T \rangle$, $[A]$ is in $\Gamma(X)$ if and only if $\Phi_4(A)$ satisfies the followings :

$$\left\{ \begin{array}{l} \sum_{i=1}^2 (\Phi_4(A)_{i,2} - \Phi_4(A)_{i,1}) \equiv \sum_{i=3}^4 (\Phi_4(A)_{i,2} - \Phi_4(A)_{i,1}) \pmod{4}, \\ \sum_{i=1}^2 (\Phi_4(A)_{i,1} + \Phi_4(A)_{i,j}) \equiv \sum_{i=3}^4 (\Phi_4(A)_{i,1} + \Phi_4(A)_{i,j}) \pmod{4} \end{array} \right. \quad (j = 3, 4),$$

- $\Gamma(X) = \langle T, R^2(RT)^{-1}, RT^2R^{-1}, RTRT(RTR)^{-1}, RTR^2 \rangle$,
- $\Gamma(X) \setminus \langle R, T \rangle = \{I, R, RT, RTR\}$ and
- $\mathbb{H}/\Gamma(X)$ is a Riemann surface of type $(0, 5)$.

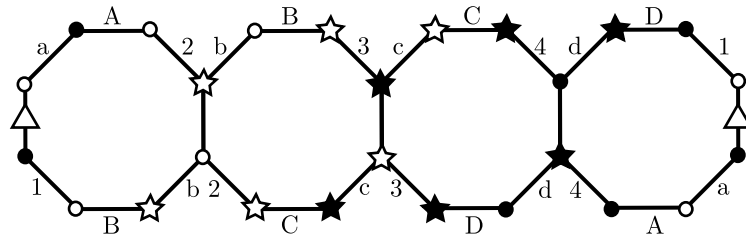


Figure 2.12: Covering X of P_8 of degree 4

Example 2.8.3. $n \geq 2$. Let X_{4n} be the double covering of P_{4n} as Figure 2.13. That is, X_{4n} is constructed by gluing two regular $4n$ -gons. Labels of small and capital letters appear in turn. The sides whose labels are capital letters are identified with the opposite sides of another polygon and others are identified with the opposite sides of the same polygon. Then X_{4n} is a Riemann surface of type $(2n - 1, 2)$.

- For $[A] \in \langle [R_{4n}], [T_{4n}] \rangle$, $[A]$ is in $\Gamma(X_{4n})$ if and only if $\Phi_2(A)$ satisfies the followings :

$$\begin{cases} \sum_{i=1}^n \Phi_2(A)_{2i-1,2j} \equiv 0 \pmod{2} (j = 1, \dots, n), \\ \sum_{i=1}^n (\Phi_2(A)_{2i-1,1} + \Phi_2(A)_{2i-1,2j-1}) \equiv 0 \pmod{2} (j = 2, \dots, n), \end{cases}$$

- $\Gamma(X_{4n}) = \langle [T_{4n}], [R_{4n}T_{4n}R_{4n}^{-1}], [R_{4n}^2] \rangle$,
- $\Gamma(X_{4n}) \setminus \langle [R_{4n}], [T_{4n}] \rangle = \{[I], [R_{4n}]\}$ and
- $\mathbb{H}/\Gamma(X_{4n})$ is an orbifold which has no genus, 3 punctures and one cone point whose order is n .

Example 2.8.4. For each $n \geq 2$, let X_{4n} be the double covering of P_{4n} as Figure 2.14. That is, horizontal and vertical sides of two polygons are identified with the opposite sides of another polygon and others are identified with the opposite sides of the same polygon. Then X_{4n} is a Riemann surface of type $(2n - 1, 2)$.

- For $[A] \in \langle [R_{4n}], [T_{4n}] \rangle$, $[A]$ is in $\Gamma(X_{4n})$ if and only if $\Phi_2(A)$ satisfies the followings :

$$\begin{cases} \Phi_2(A)_{1,1} + \Phi_2(A)_{n+1,1} + \Phi_2(A)_{n+1,1} + \Phi_2(A)_{n+1,n+1} \equiv 0 \pmod{2}, \\ \Phi_2(A)_{1,j} + \Phi_2(A)_{n+1,j} \equiv 0 \pmod{2} (j = 2, \dots, n, n+2, \dots, 2n), \end{cases}$$

- $\Gamma(X_{4n}) = \langle [R_{4n}^i T_{4n} R_{4n}^{-i}], [R_{4n}^n] \mid i = 0, 1, \dots, n-1 \rangle$,
- $\Gamma(X_{4n}) \setminus \langle [R_{4n}], [T_{4n}] \rangle = \{[I], [R_{4n}], [R_{4n}^2], \dots, [R_{4n}^{n-1}]\}$ and

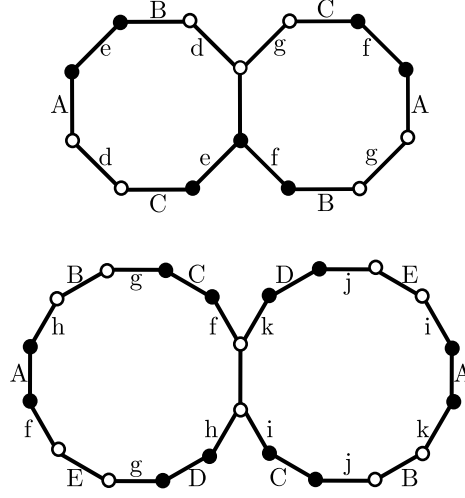


Figure 2.13: Double covers X_8 and X_{12} of P_8 and P_{12} , respectively.

- $\mathbb{H}/\Gamma(X_{4n})$ is an orbifold which has no genus, $2n + 1$ punctures and one cone point whose order is 2.

Example 2.8.5. Let X_d be the covering of P_8 with degree d as Figure 2.15. Then X_d is a Riemann surface of type $(d + 1, d)$. And , for $[A] \in \langle [R_8], [T_8] \rangle$, $[A]$ is in $\Gamma(X_d)$ if and only if $\Phi_d(A)_{1,j} \equiv 0 \pmod{d} (j = 2, 3, 4)$. The next is a chart about Veech groups $\Gamma(X_d)$. Here,

- $\# \text{ Rep}$ is the index of $\Gamma(X_d)$ in $\langle [R_8], [T_8] \rangle$,
- $\# \text{ Gen}$ is a number of generators of $\Gamma(X_d)$ by this calculation,
- “genus” is the genus of $\mathbb{H}/\Gamma(X_d)$,
- “puncture” is the number of punctures of $\mathbb{H}/\Gamma(X_d)$ and
- “cone point (order)” is the number of cone points of $\mathbb{H}/\Gamma(X_d)$ and their orders.

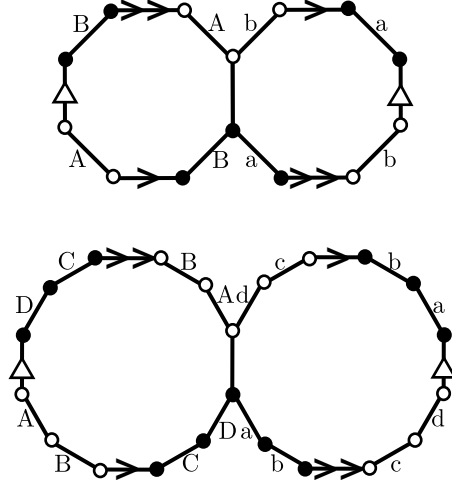


Figure 2.14: Double covers X_8 and X_{12} of P_8 and P_{12} , respectively.

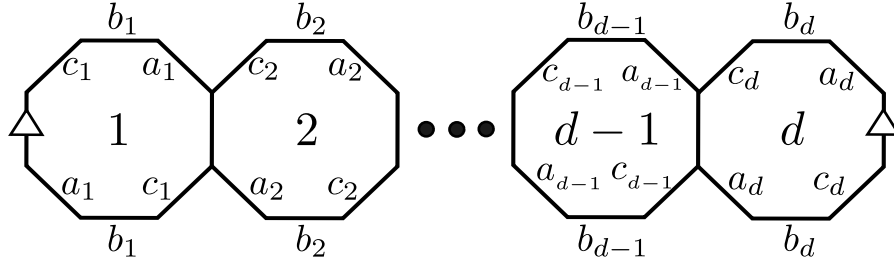


Figure 2.15: Covering X_d of P_8 of degree d

d	# Gen	# Rep	genus	puncture	cone point (order)
2	11	12	0	11	0
3	29	32	1	24	0
4	87	96	8	58	0
5	142	156	24	68	6 (2,2,2,2,2,2)
6	349	384	45	200	0
7	367	400	87	128	0
8	704	768	149	280	0
9	785	864	185	280	0
10	1704	1872	419	568	0
11	1353	1464	400	300	0

Example 2.8.6. Let X_d be the covering of P_8 with degree d as Figure 2.16. Then X_d is a Riemann surface of type $(d+1, d)$. And, for $[A] \in \langle [R_8], [T_8] \rangle$, $[A]$ is in $\Gamma(X_d)$ if and only if $\sum_{i=1}^4 (\Phi_d(A)_{i,j} - \Phi_d(A)_{i,1}) \equiv 0 \pmod{d} (j = 2, 3, 4)$. The next is a chart about Veech groups $\Gamma(X_d)$.

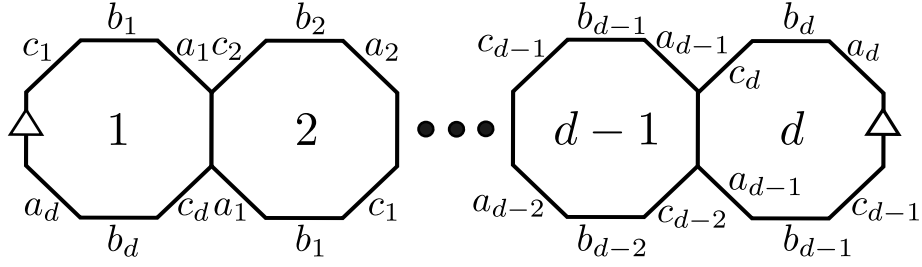


Figure 2.16: Covering X_d of P_8 of degree d

d	# Gen	# Rep	genus	puncture	cone point (order)
2	2	1	0	2	1 (4)
3	29	32	1	24	0
4	5	4	0	5	0
5	142	156	24	68	6 (2,2,2,2,2,2)
6	29	32	1	24	0
7	367	400	87	128	0
8	29	32	1	24	0
9	789	864	185	280	0
10	142	156	24	68	6 (2,2,2,2,2,2)
11	1353	1464	400	300	0
12	115	128	11	76	0
13	2220	2380	682	416	14 (2,2,2,2,2,2,2,2,2,2,2,2,2,2)
14	367	400	87	128	0

Chapter 3

Veech holomorphic families of
Riemann surfaces, holomorphic
sections, and Diophantine
problems

Veech holomorphic families of Riemann surfaces, holomorphic sections, and Diophantine problems

Yoshihiko Shinomiya

Abstract

In this chapter, we construct holomorphic families of Riemann surfaces from Veech groups and characterize their holomorphic sections by some points of corresponding flat surfaces. The construction gives us concrete solutions for some Diophantine equations over function fields. Moreover, we give upper bounds of the numbers of holomorphic sections of certain holomorphic families of Riemann surfaces.

3.1 Introduction

A holomorphic family (M, π, B) of Riemann surfaces of type (g, n) is a triple of a two-dimensional complex manifold M , a Riemann surface B and a holomorphic map $\pi : M \rightarrow B$ such that the fiber $X_t = \pi^{-1}(t)$ is a Riemann surface of type (g, n) for each $t \in B$ and the complex structure of X_t depends holomorphically on the parameter t . Holomorphic families of Riemann surfaces are first studied in algebraic geometry as the Diophantine problems over function fields on compact Riemann surfaces.

Let $M(B_0)$ be the field of all meromorphic functions on a compact Riemann surface B_0 . We take a three-variable irreducible homogeneous polynomial f whose coefficients are in $M(B_0)$. The Diophantine problem is to find all solutions $[X : Y : Z] \in \mathbb{P}^2(M(B_0))$ of the Diophantine equation $f(X, Y, Z) = 0$. It is known that a Diophantine equation defines holomorphic family of Riemann surfaces and all the solutions of a Diophantine equation correspond to all the holomorphic sections of the corresponding holomorphic family of Riemann surfaces. By using Teichmüller theory, Imayoshi and Shiga [12] proved that the number of holomorphic sections of a locally non-trivial holomorphic family of Riemann surfaces is finite. And Shiga [20] estimated the number of holomorphic families of Riemann surfaces of certain types. In this chapter, we consider holomorphic families of Riemann surfaces obtained by Veech groups.

Let X be a Riemann surface of type (g, n) . The Veech group $\Gamma(X, u)$ is the group of all elements in $\mathrm{PSL}(2, \mathbb{R})$ which induce affine homeomorphisms on a flat surface (X, u) . A Veech group induces a Teichmüller disk which is a holomorphic isometric embedding of a hyperbolic plane $\mathbb{H}$ into the Teichmüller space of the Riemann surface (X, u) . Veech [23] proved that the mirror Veech group $\bar{\Gamma}(X, u)$ is a representation of the action of mapping class group into the Teichmüller disk defined by (X, u) . And the action is regarded as the action of Fuchsian group $\bar{\Gamma}(X, u)$ into $\mathbb{H}$. Hence, $\mathbb{H}/\bar{\Gamma}(X, u)$ is holomorphically embedded into the moduli space $M(g, n)$. In this chapter, we construct holomorphic families of Riemann surfaces from such embeddings and observe their properties. We call such holomorphic families of Riemann surfaces *Veech holomorphic families of Riemann surfaces*. Especially, we study their holomorphic sections.

The chapter is organized as follows. In section 3.2, we define fundamental objects that we use in this chapter, and observe their properties. In section 3.3, we define Veech groups and show that orbifolds induced by Veech groups are embedded holomorphically and locally isometrically into moduli spaces. Moreover, a relation between geodesic flows on a flat surface and the Veech group is explained in this section. A construction of Veech holomorphic families of Riemann surfaces is discussed in section 3.4. Also, representations and monodromies of Veech holomorphic families of Riemann surfaces are observed in this section. Section 3.5 characterizes holomorphic sections of Veech holomorphic families of Riemann surfaces by some points of corresponding flat surfaces. Also, by applying this characterization, we find all the holomorphic sections of certain Veech holomorphic families of Riemann surfaces. In section 3.6, we construct examples of Diophantine equations from some Veech holomorphic families of Riemann surfaces. All solutions of the Diophantine equations are given by using the characterization of holomorphic sections. Section 3.7 gives an upper bound of the number of holomorphic sections for certain Veech holomorphic families of Riemann surfaces. The upper bound depends only on topological quantities of the base Riemann surface B and fiber Riemann surfaces $X_t = \pi^{-1}(t)$ of holomorphic families of Riemann surfaces (M, π, B) . To obtain the upper bound, we show a certain property of Fuchsian groups (Theorem 3.7.13), which may have its own interest.

3.2 Preliminaries

In this section, we see some properties of Teichmüller spaces, Kobayashi pseudo-metrics, Teichmüller modular groups, Bers fiber spaces and Fuchsian groups.

Let X be a Riemann surface of type (g, n) with $3g - 3 + n > 0$.

3.2.1 Teichmüller spaces

Let us consider all pairs (Y, f) of Riemann surfaces Y of type (g, n) and quasiconformal maps f from X onto Y . Two such pairs (Y_1, f_1) and (Y_2, f_2) are said to be equivalent if there exists a conformal map $h : Y_1 \rightarrow Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$. This equivalence relation is called the Teichmüller equivalence.

Definition 3.2.1 (Teichmüller space). The Teichmüller space $T(X)$ of X is the set of all equivalence classes of such pairs. We denote by $[Y, f]$ the equivalence class of a pair (Y, f) . The point $[X, id : X \rightarrow X]$ is called the base point of $T(X)$.

For any two points $[Y_1, f_1], [Y_2, f_2] \in T(X)$, we set

$$d_T([Y_1, f_1], [Y_2, f_2]) = \inf_h \log K(h).$$

Here, the infimum is taken over all quasiconformal maps $h : Y_1 \rightarrow Y_2$ which are homotopic to $f_2 \circ f_1^{-1}$ and $K(h)$ is the dilatation of h . That is, let $\mu_h = \bar{\partial}h/\partial h$ be the Beltrami coefficient of h . By the definition of quasiconformal maps, the essential supremum norm $\|\mu_h\|_\infty$ of h is less than 1. The dilatation $K(h)$ of h is defined by $K(h) = \frac{1+\|\mu_h\|_\infty}{1-\|\mu_h\|_\infty}$. Then d_T becomes a metric on $T(X)$.

Definition 3.2.2 (Teichmüller metric). The metric d_T is called the Teichmüller metric.

We consider geodesics of $T(X)$ with respect to the Teichmüller metric d_T .

Definition 3.2.3 (Holomorphic quadratic differential). A tensor φ on X is called a holomorphic quadratic differential if its restriction to every coordinate neighborhood (U, z) is the form $f dz^2$, here f is a holomorphic function on U .

We define $|q|$ to be the differential 2-form on X whose restriction to every

coordinate neighborhood (U, z) has the form $|f|dx dy$ if q equals $f dz^2$ in U . We say q is integrable if its norm

$$||q|| = \iint_X |q|$$

is finite. Let $\bar{X}$ be the compact Riemann surface of genus g obtained by filling all punctures of X . We construct a function $\text{ord} : \bar{X} \rightarrow \mathbb{Z}$. If $z \in \bar{X}$ is a zero of q of order n , we set $\text{ord}(z) = n$. If $z \in \bar{X}$ is a pole of q of order n , we set $\text{ord}(z) = -n$. And, we set $\text{ord}(z) = 0$ if $z \in \bar{X}$ is not a zero or pole of q .

Remark. Note that q is integrable if and only if $\text{ord}(z) \geq -1$ for all $z \in \bar{X}$. And, if $q \neq 0$, then by the Riemann-Roch theorem, we have

$$\sum_{z \in \bar{X}} \text{ord}(z) = 4g - 4.$$

Definition 3.2.4 (Teichmüller map). A quasiconformal map $f : X \rightarrow Y$ is said to be a Teichmüller map if its Beltrami coefficient μ_f is the form

$$\mu_f = k \frac{|q|}{q}$$

for some integrable holomorphic quadratic differential q on X and k ($0 \leq k < 1$).

The next two theorems claim that there is an unique Teichmüller map from X to Y realizing the distance between the base point $[X, id]$ and each point $[Y, f]$.

Theorem 3.2.5 (Teichmüller's uniqueness theorem). *Let $f_0 : X \rightarrow Y$ be a Teichmüller map and $\mu_{f_0} = k \frac{|q|}{q}$ for some integrable holomorphic quadratic differential q on X and k ($0 \leq k < 1$). Then, for every quasiconformal map $f : X \rightarrow Y$ which is homotopic to f_0 , the inequality*

$$||\mu_f||_\infty \geq ||\mu_{f_0}||_\infty = k$$

holds. Where, μ_f is the Beltrami coefficient of f . The equality holds if and only if $f = f_0$.

Theorem 3.2.6 (Teichmüller's existence theorem). *Let $f : X \rightarrow Y$ be a quasiconformal map. There exists a Teichmüller map $f_0 : X \rightarrow Y$ which is homotopic to f .*

Therefore, we conclude that the Teichmüller distance between the base point $[X, id]$ and each point $[Y, f]$ is

$$d_T([X, id], [Y, f]) = \log K(f_0) = \log \frac{1+k}{1-k}$$

if the Beltrami coefficient μ_{f_0} of the unique Teichmüller map $f_0 : X \rightarrow Y$ which is homotopic to f is the form $k \frac{|\bar{q}|}{q}$. For each geodesic ray $r : [0, \infty) \rightarrow T(X)$ with $r(0) = [X, id]$, there exists a unique holomorphic quadratic differential q on X such that $r(t) = [Y_t, f_t]$ if we set $f_t : X \rightarrow Y_t$ to be the Teichmüller map with the Beltrami coefficient $\mu_{f_t} = \frac{t-1}{t+1} \frac{|\bar{q}|}{q}$. Therefore, d_T is a complete metric. Since the complex dimension of the Banach space of all holomorphic quadratic differentials on X is $3g-3+n$, $T(X)$ is homeomorphic to $\mathbb{C}^{3g-3+n}$ and hence, $T(X)$ is proper. For the proof of them, see [13].

The Teichmüller space $T(X)$ of X has a complex structure induced by the Bers embedding which is an embedding of $T(X)$ into a certain Banach space of holomorphic quadratic differentials on X . The Teichmüller space is embedded into the Banach space as a bounded domain. See [13].

3.2.2 Kobayashi pseudo-metrics

On a complex manifold, we can define a pseudo-metric which is called the Kobayashi pseudo-metric ([15]). Let M be a connected complex manifold. Given two points $z, w \in M$, we choose points $z = z_0, z_1, \dots, z_{k-1}, z_k = w$ of M , points $a_1, b_1, \dots, a_k, b_k$ of a unit disk $\mathbb{D}$, and holomorphic maps $f_1, \dots, f_k$ from $\mathbb{D}$ to M with $f_i(a_i) = z_{i-1}$ and $f_i(b_i) = z_i$ for $i = 1, \dots, k$. For each choice of points and maps as above, we consider the number

$$d_{\mathbb{D}}(a_1, b_1) + \dots + d_{\mathbb{D}}(a_k, b_k),$$

where $d_{\mathbb{D}}$ is the hyperbolic metric on $\mathbb{D}$. Let $d_M^K(z, w)$ be the infimum of the numbers obtained as above. It is easy to see that $d_M^K : M \times M \rightarrow \mathbb{R}$ is continuous and satisfies the axioms of pseudo-metric:

$$d_M^K(z, w) \geq 0, d_M^K(z, w) = d_M^K(w, z) \text{ and } d_M^K(z, r) + d_M^K(r, w) \geq d_M^K(z, w).$$

Definition 3.2.7 (Kobayashi metric, Kobayashi hyperbolic manifold). The pseudo-metric d_M^K on M is called the Kobayashi pseudo-metric of M . If d_M^K satisfies the axioms of metric, we call d_M^K the Kobayashi metric of M , and M a Kobayashi hyperbolic manifold. If a complex manifold M is a Kobayashi hyperbolic manifold and the Kobayashi metric d_M^K is proper, we call M a proper Kobayashi hyperbolic manifold.

By definition, Kobayashi pseudo-metric has a distance decreasing property. That is, for complex manifolds M, N and a holomorphic map $f : M \rightarrow N$, we have

$$d_M^K(z, w) \geq d_N^K(f(z), f(w))$$

for all $z, w \in M$.

Royden [18] prove the following theorem.

Theorem 3.2.8. *On the Teichmüller space $T(X)$ of X , the Kobayashi pseudo-metric $d_{T(X)}^K$ coincides with the Teichmüller metric d_T .*

This implies that $T(X)$ is a proper Kobayashi hyperbolic manifold.

Remark. If M is a hyperbolic Riemann surface, d_M^K coincides with the complete hyperbolic metric which is compatible with its complex structure.

Proposition 3.2.9 ([15]). *Let M be a complex manifold and $\pi : \widetilde{M} \rightarrow M$ be a covering map. Choose any two points $z, w \in M$ and fix $\tilde{z} \in \pi^{-1}(z)$. Then*

$$d_M^K(z, w) = \inf_{\tilde{w} \in \pi^{-1}(w)} d_{\widetilde{M}}^K(\tilde{z}, \tilde{w}).$$

Proof. Since π is holomorphic, we have

$$d_M^K(z, w) \leq \inf_{\tilde{w} \in \pi^{-1}(w)} d_{\widetilde{M}}^K(\tilde{z}, \tilde{w}).$$

Assume that there exists $\epsilon > 0$ such that

$$d_M^K(z, w) + \epsilon < \inf_{\tilde{w} \in \pi^{-1}(w)} d_{\widetilde{M}}^K(\tilde{z}, \tilde{w}).$$

By the definition of Kobayashi pseudo-metric, there exist $a_1, b_1, \dots, a_k, b_k \in \mathbb{D}$ and holomorphic maps $f_1, \dots, f_k$ such that

$$z = f(a_1), f(b_i) = f(a_{i+1}) \ (i = 1, \dots, k-1), f_k(a_k) = w$$

and

$$d_M^K(z, w) + \epsilon > \sum_{i=1}^k d_{\mathbb{D}}(a_i, b_i).$$

We lift $f_1, \dots, f_k$ to holomorphic maps $\tilde{f}_1, \dots, \tilde{f}_k$ from $\mathbb{D}$ to $\widetilde{M}$ such that $\tilde{z} = \tilde{f}_1(a_1)$ and $\tilde{f}(b_i) = \tilde{f}(a_{i+1})$ ($i = 1, \dots, k-1$). If we set $\tilde{w} = \tilde{f}_k(b_k)$, then $\pi(\tilde{w}) = w$ and

$$d_M^K(\tilde{z}, \tilde{w}) \leq \sum_{i=1}^k d_{\mathbb{D}}(a_i, b_i).$$

Hence, $d_M^K(\tilde{z}, \tilde{w}) < d_M^K(z, w) + \epsilon$. This contradicts the assumption. $\square$

3.2.3 Teichmüller modular groups

Let $\text{Mod}(X)$ be the group of all homotopy classes of quasiconformal self-maps of X . We denote by $[h]$ the equivalence class of a quasiconformal self-map h of X .

Definition 3.2.10. The group $\text{Mod}(X)$ is called the the Teichmüller modular group of X .

The Teichmüller modular group $\text{Mod}(X)$ acts on $T(X)$ as follows. For each $[h] \in \text{Mod}(X)$, a map $[h]_* : T(X) \rightarrow T(X)$ is defined by

$$[h]_*[Y, f] = [Y, f \circ h^{-1}]$$

for all $[Y, f] \in T(X)$. Clearly, this action is isometric with respect to the Teichmüller metric d_T . Moreover, it is known that $[h]_*$ is a holomorphic self-map of $T(X)$ and the action of $\text{Mod}(X)$ is properly discontinuous. See [13]. Also, the quotient space $T(X)/\text{Mod}(X)$ coincides with the moduli space $M(g, n)$ of Riemann surfaces of type (g, n) .

3.2.4 Bers fiber spaces

Let $p : \mathbb{H} \rightarrow X$ be the universal covering map with the covering transformation group G . Let $f : X \rightarrow Y$ be a quasiconformal map with Beltrami coefficient μ_f . We define the Beltrami coefficient $\tilde{\mu}_f$ on $\hat{\mathbb{C}}$ which is the lift of μ_f on $\mathbb{H}$ and equals 0 on $\hat{\mathbb{C}} - \mathbb{H}$. Let $w^{\mu_f} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the unique $\tilde{\mu}_f$ -quasiconformal map which fixes $0, 1, \infty$. Then two domains $w^{\mu_{f_1}}(\mathbb{H})$ and $w^{\mu_{f_2}}(\mathbb{H})$ are equal if two quasiconformal maps $f_1 : X \rightarrow Y_1$ and $f_2 : X \rightarrow Y_2$ are Teichmüller equivalent. We denote by $\mathbb{H}^t$ the domain $w^{\mu_f}(\mathbb{H})$ for $t = [Y, f] \in T(X)$. Then

$$F(G) = \{(t, z) : t \in T(X), z \in \mathbb{H}^t\}$$

is called the Bers fiber space over $T(X)$. We induce the complex structure on $F(G)$ as a submanifold of $T(X) \times (\mathbb{C} - \{0, 1\})$. Bers prove the following.

Theorem 3.2.11 ([3]). *Let us fix a point $a \in X$ and set $\dot{X} = X - \{a\}$. The Bers fiber space $F(G)$ is biholomorphic to $T(\dot{X})$.*

3.2.5 Holomorphic families of Riemann surfaces

Definition 3.2.12 (Holomorphic family of Riemann surfaces). Let $\bar{M}$ be a two-dimensional complex manifold and A be a one-dimensional analytic subset of $\bar{M}$ or empty. Let B be a Riemann surface. Assume that there exists a proper holomorphic map $\bar{\pi} : \bar{M} \rightarrow B$ satisfying the following two conditions.

- (1) By setting $M = \bar{M} - A$ and $\pi = \bar{\pi}|_M$, the holomorphic map π is of maximal rank at every point of M .
- (2) The fiber $X_t = \pi^{-1}(t)$ over each $t \in B$ is a Riemann surface of fixed finite type (g, n) .

We call such triple (M, π, B) a holomorphic family of Riemann surfaces of type (g, n) over B . We assume that $3g - 3 + n > 0$.

Let (M, π, B) be a holomorphic family of Riemann surfaces of type (g, n) . Let $M(g, n)$ be the moduli space of Riemann surfaces of type (g, n) . Then we have a holomorphic map $\phi : B \ni t \mapsto X_t = \pi^{-1}(t) \in M(g, n)$. Let $p : \mathbb{H} \rightarrow B$ be the universal covering map with the covering transformation group $\Gamma < \text{PSL}(2, \mathbb{R})$. We fix $t_0 \in B$ and set $X = \pi^{-1}(t_0)$. Since $M(g, n) = T(X)/\text{Mod}(X)$, we have a lift $\Phi : \mathbb{H} \rightarrow T(X)$ of ϕ . The holomorphic map Φ induces a homomorphism $\chi : \Gamma \rightarrow \text{Mod}(X)$ such that $\Phi \circ A = \chi(A) \circ \Phi$ for every $A \in \Gamma$.

Definition 3.2.13 (Representation and monodromy). We call the holomorphic map Φ a representation of (M, π, B) into $T(X)$ and the homomorphism χ the monodromy with respect to Φ .

Definition 3.2.14 (Locally triviality and locally non-triviality). A holomorphic family of Riemann surfaces (M, π, B) is called locally trivial if the induced map $\phi : B \rightarrow M(g, n)$ is constant. And, (M, π, B) is called locally non-trivial if the induced map $\phi : B \rightarrow M(g, n)$ is non-constant.

3.2.6 Ford region and Shimizu's lemma

Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$ ($\nu_i \in \{2, 3, \dots, \infty\}$). That is, $\mathbb{H}/\Gamma$ has genus p and cone points whose orders are $\nu_1, \dots, \nu_k$. Now, a cone point of order ∞ means a puncture of $\mathbb{H}/\Gamma$.

Definition 3.2.15 (Fundamental domains). A domain D in $\mathbb{H}$ is called a fundamental domain for Γ if it satisfies the followings:

- (1) For all $A \in \Gamma$ ($A \neq id$), $A(D) \cap D = \emptyset$, and
- (2) For all $z \in \mathbb{H}$, there exists $A \in \Gamma$ such that $A(z) \in \text{cl}(D)$.

Here, $\text{cl}(D)$ is the closure of D .

Suppose that Γ contains $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and it is not a power of other elements of Γ . We set $\Gamma_0 = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \neq 0 \}$, and for all $A \in \Gamma_0$,

$$D_A = \{z \in \mathbb{H} : |z + \frac{d}{c}| > \frac{1}{|c|}\}.$$

Definition 3.2.16 (Ford regions). The region

$$D = \{z \in \mathbb{H} : |\text{Re}(z)| < \frac{1}{2}\} \cap \bigcap_{A \in \Gamma_0} D_A$$

is called the Ford region of Γ .

Proposition 3.2.17. *The Ford region D is a fundamental domain for Γ .*

For a proof of proposition 3.2.17, see [6].

Remark. It is clear that the hyperbolic area of $\mathbb{H}/\Gamma$ induced by the universal covering map $\mathbb{H} \rightarrow \mathbb{H}/\Gamma$ equals to that of the Ford region D in $\mathbb{H}$. And it is known that the Ford region D becomes a finite sided geodesic polygon in $\mathbb{H}$.

Lemma 3.2.18 (Shimizu's lemma). *For all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0$, the inequality $|c| \geq 1$ holds.*

For a proof, see [13].

3.3 Flat surface and Veech groups

In this section, we define Veech groups and see some applications of Veech groups. Let X be a Riemann surface of type (g, n) and C be a set of all punctures of X and finitely many points of X .

Definition 3.3.1 (Flat structure and flat surface). A flat structure u on X with critical points on C is an atlas of $X' = X - C$ which satisfies the following conditions.

- (1) Local coordinates of u are compatible with the orientation on X induced by its complex structure.
- (2) For coordinate neighborhoods (U, z) and (V, w) of u with $U \cap V \neq \emptyset$, the transition function is the form

$$w = \pm z + c$$

in $z(U \cap V)$ for some $c \in \mathbb{C}$.

- (3) The atlas u is maximal with respect to (1) and (2).

A pair (X, u) of a Riemann surface X and a flat structure u with critical points on C is called a flat surface with critical points on C . Each point in C is called a critical point of (X, u) . In this chapter, we assume that the area $\text{Area}(X, u)$ of X with respect to u is finite.

We construct a flat structure u_q on X which is compatible with the complex structure of X from a holomorphic quadratic differential q on X . We set C to be the set of all punctures of X and all zeros of q . For each $p_0 \in X' = X - C$, we can choose an open neighborhood U such that

$$z(p) = \int_{p_0}^p \sqrt{q}$$

is well-defined and an injective function in U . This function is holomorphic in U since q is a holomorphic quadratic differential on X . If (U, z) and (V, w) are pairs of such neighborhoods and functions with $U \cap V \neq \emptyset$, then we have $dw^2 = q = dz^2$ in $U \cap V$. Hence, $w = \pm z + c$ in $z(U \cap V)$ for some $c \in \mathbb{C}$. The flat structure u_q is the maximal flat structure which contains such pairs. The set C is the critical point set of u_q . And the area $\text{Area}(X, u_q)$ equals to the norm $\|q\|$ of q . Conversely, if $u = \{(U_\lambda, z_\lambda)\}$ is a flat structure on X which is compatible with the complex structure of X and the critical point set of u is C , then $q = \{dz_\lambda^2\}$ is a meromorphic quadratic differential on X with $\|q\| = \text{Area}(X, u)$. All zeros and poles of q are contained in C . If $p \in C$ is a zero of q with $\text{odr}(p) = n$, then the angle around p with respect to u is $(n + 2)\pi$. And the angle around a single pole of q with respect to u is π .

Fix an integrable holomorphic quadratic differential q on X . Let $u = u_q = \{(U_\lambda, z_\lambda)\}$ be the flat structure induced by q and C be the critical point set of u . For each $A \in \text{SL}(2, \mathbb{R})$, we denote by T_A the linear map

on $\mathbb{C} = \mathbb{R}^2$ corresponding to A . Then $u_A = \{(U_\lambda, T_A \circ z_\lambda)\}$ is also a flat structure on X with critical points on C . Note that (X, u_A) is not conformal equivalent to the original Riemann surface X in general. We obtain a map $\hat{i} : \mathrm{SL}(2, \mathbb{R}) \rightarrow T(X)$ such that $\hat{i}(A) = [(X, u_A), id : X \rightarrow (X, u_A)]$. Here, we consider u_A as a new complex structure on X . By the following lemma, $\hat{i}$ induces a map $\iota : \mathrm{SO}(2) \setminus \mathrm{SL}(2, \mathbb{R}) \rightarrow T(X)$.

Lemma 3.3.2. $\hat{i}(A) = \hat{i}(B)$ if $BA^{-1} \in \mathrm{SO}(2)$.

Proof. Let $A, B \in \mathrm{SL}(2, \mathbb{R})$ with $BA^{-1} \in \mathrm{SO}(2)$. For each $(U, z) \in u$, $(T_B \circ z) \circ (T_A \circ z)^{-1} = T_B \circ T_{A^{-1}} = T_{BA^{-1}}$ is a conformal map. Hence, $id : (X, u_A) \rightarrow (X, u_B)$ is a conformal map. $\square$

The identification of $\mathrm{SO}(2) \setminus \mathrm{SL}(2, \mathbb{R})$ with $\mathbb{H}$ by the bijective map

$$\mathrm{SO}(2) \setminus \mathrm{SL}(2, \mathbb{R}) \ni \mathrm{SO}(2) \cdot A \mapsto -\overline{A^{-1}(i)} \in \mathbb{H}$$

induces a map $\iota : \mathbb{H} \rightarrow T(X)$. Here, $A^{-1}(\cdot)$ is a Möbius transformation. It is known that this map ι is a holomorphic isometric embedding from a hyperbolic plane $\mathbb{H}$ into the Teichmüller space $T(X)$ with the Teichmüller distance. (See [10].) And ι satisfies $\iota(i) = [X, id]$. Moreover, it is easy to verify that the Beltrami coefficient of the Teichmüller map from $\iota(i)$ to $\iota(t)$ is $\frac{i-t}{i+t} \frac{|q|}{q}$. Conversely, we have the following.

Proposition 3.3.3. *Every holomorphic isometric embedding $\iota : \mathbb{H} \rightarrow T(X)$ with $\iota(i) = [X, id]$ is constructed from some flat structure as above.*

Proof. Let $\iota : \mathbb{H} \rightarrow T(X)$ be a holomorphic isometric embedding. Then the geodesic ray $r(t) = it$ ($t \geq 1$) in $\mathbb{H}$ maps to a geodesic ray in $T(X)$.

By theorem 3.2.5 and theorem 3.2.6, there exists a holomorphic quadratic differential q on X such that the Teichmüller map from $[X, id]$ to $\iota \circ r(t)$ has the Beltrami coefficient $\frac{1-t}{1+t} \frac{|q|}{q}$ for all $t \geq 1$. Note that these Teichmüller maps are deformations by $\begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix}$ of the flat structure u_q defined by q . Let $\iota_q : \mathbb{H} \rightarrow T(X)$ be the holomorphic isometric embedding constructed from u_q . Since $\iota(it) = \iota_q(it)$ for all $t \geq 1$, we conclude that $\iota = \iota_q$. $\square$

Definition 3.3.4 (Teichmüller disk). We call the image of a holomorphic isometric embedding of $\mathbb{H}$ into $T(g, n)$ a Teichmüller disk.

Let (X, u) be a flat surface with critical points on C . Let $\iota : \mathbb{H} \rightarrow T(X)$ be a holomorphic isometric embedding constructed from (X, u) . We consider the image of the Teichmüller disk $\Delta = \iota(\mathbb{H})$ into the moduli space $M(g, n)$ and hence, the subgroup $\text{Stab}(\Delta)$ of $\text{Mod}(X)$ which consists of all elements of $\text{Mod}(X)$ which fixing Δ . For this, we define the affine group $\text{Aff}^+(X, u)$ of (X, u) .

Definition 3.3.5 (Affine group of (X, u)). The affine group $\text{Aff}^+(X, u)$ of the flat surface (X, u) is the group of all quasiconformal maps h of X onto itself which satisfy $h(C) = C$ and are affine with respect to the flat structure u . This means that for $(U, z), (V, w) \in u$ with $h(U) \subseteq V$, the homeomorphism $w \circ h \circ z^{-1}$ is the form $z \mapsto Az + c$ for some $A \in \text{GL}(2, \mathbb{R})$ and $c \in \mathbb{C}$. Each element in $\text{Aff}^+(X, u)$ is called an affine homeomorphism on (X, u) .

For each $h \in \text{Aff}^+(X, u)$, the derivative $A \in \text{GL}(2, \mathbb{R})$ of the affine map $w \circ h \circ z^{-1}$ is uniquely determined up to the sign since u is a flat structure. And A is always in $\text{SL}(2, \mathbb{R})$ since $\text{Area}(X, u) = \int_X |q| = \int_X h^*(|q|) = \det(A) \cdot \text{Area}(X, u)$. Here, q is a holomorphic quadratic differential corresponding to (X, u) . Thus, we have a group homomorphism

$$D : \text{Aff}^+(X, u) \rightarrow \text{PSL}(2, \mathbb{R}).$$

We call this homomorphism the derivative map.

Definition 3.3.6 (Veech group of (X, u)). We call $\Gamma(X, u) = D(\text{Aff}^+(X, u))$ the Veech group of (X, u) .

Veech[23] prove the following.

Proposition 3.3.7. *All homomorphisms of $\text{Aff}^+(X, u)$ are not homotopic to each other. Therefore, $\text{Aff}^+(X, u)$ is regarded as a subgroup of $\text{Mod}(X)$.*

By setting $\iota(t) = [X_t, id]$ for each $t \in \mathbb{H}$, we have the following theorem.

Theorem 3.3.8 ([23], [4], [10]). *The group $\text{Stab}(\Delta)$ coincides with $\text{Aff}^+(X, u)$. For any $h \in \text{Aff}^+(X, u)$ and $[X_t, id] \in \Delta$,*

$$h_*[X_t, id] = [X_t, h^{-1}] = [X_{\bar{A}(t)}, id].$$

Here, $A = D(h)$, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\bar{A} = RAR^{-1}$ acts on $\mathbb{H}$ as a Möbius transformation.

Corollary 3.3.9 ([23], [4], [10]). *The Veech groups $\Gamma(X, u)$ is a Fuchsian group and we have the following commutative diagram. Here, $\bar{\Gamma}(X, u) = R\Gamma(X, u)R^{-1}$.*

$$\begin{array}{ccccc}
\mathbb{H} & \xrightarrow{\iota} & \Delta & \hookrightarrow & T(X) \\
\downarrow / \bar{\Gamma}(X, u) & & \downarrow / \text{Stab}(\Delta) & & \downarrow / \text{Mod}(X) \\
\mathbb{H} / \bar{\Gamma}(X, u) & \xrightarrow{\sim} & \Delta / \text{Stab}(\Delta) & \hookrightarrow & M(g, n)
\end{array}$$

We construct holomorphic families of Riemann surfaces from such holomorphic embeddings $\mathbb{H} / \bar{\Gamma}(X, u) \rightarrow M(g, n)$.

Next, we see an application of Veech groups for a dynamical behavior of the geodesic flows on flat surfaces (X, u) . We assume that the conformal structure defined by u is compatible with that of X . Let C be the critical point set of (X, u) . Since the transition functions of (X, u) are the from $z \mapsto \pm z + c$, geodesics on (X, u) and directions of geodesics of (X, u) are well-defined. That is, a geodesic $l : \mathbb{R} \rightarrow X$ on (X, u) is a curve in $X' = X - C$ such that for all $t \in \mathbb{R}$, sufficiently small $\epsilon > 0$, and $(U, z) \in u$ with $l((t - \epsilon, t + \epsilon)) \subset U$, $z \circ l|_{(t - \epsilon, t + \epsilon)}$ is an Euclidean line segment. The direction $\theta(l) \in [0, \pi)$ of a geodesic l is the direction of $z \circ l|_{(t - \epsilon, t + \epsilon)}$ on $\mathbb{C}$. A saddle connection is a path on X which connects two points in C and is locally an Euclidean line segment with respect to u .

Definition 3.3.10 (Jenkins-Strebel direction). A number $\theta \in [0, \pi)$ is called a Jenkins-Strebel direction of (X, u) if all points in X lie on θ -direction closed geodesics or θ -direction saddle connections of (X, u) .

If θ is a Jenkins-Strebel direction of (X, u) , the flat surface (X, u) is decomposed into finitely many Euclidean cylinders by cutting along the θ -direction saddle connections.

Definition 3.3.11 (Simple Jenkins-Strebel direction). If Jenkins-Strebel direction θ decomposes (X, u) into only one cylinder, we call the direction θ a simple Jenkins-Strebel direction of (X, u) .

Definition 3.3.12. For an Euclidean cylinder R whose length of circumference is b and height is a , the ratio $\frac{b}{a}$ is called the modulus of R and denote it by $\text{mod}(R)$.

Veech[23] proved the following theorem which is called the Veech dichotomy theorem. For the definition of the ergodicity, see [14].

Theorem 3.3.13. *Let (X, u) be a flat surface. Suppose that the Veech group $\Gamma(X, u)$ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$, that is, $\mathbb{H}/\Gamma(X, u)$ has finite hyperbolic area. Then for each direction θ , one of the following two possibilities occurs:*

- *The direction θ is a Jenkins-Strebel direction. Moreover, if θ decomposes X into k cylinders $R_1, \dots, R_k$, the ratio $\mathrm{mod}(R_i)/\mathrm{mod}(R_j)$ is a rational number for each $i, j \in \{1, \dots, k\}$.*
- *Every θ -direction geodesic is dense in X . And the θ -direction geodesic flow is uniquely ergodic.*

Finally, we see two examples of Veech groups.

Example 3.3.14 ([21]). Let X be a surface constructed as figure 3.1. We induce an unique conformal structure on X such that the quadratic differential dz^2 on the interior of the rectangle of figure 3.1 extends to a holomorphic quadratic differential q on X . Then X is a Riemann surface of type $(2, 0)$ and vertices of four squares become two points in X . These points are zeros of q of order 2. Let $u = u_q$ be the flat structure defined by q . Then $\theta = 0$ and $\frac{\pi}{2}$ are Jenkins-Strebel directions of (X, u) . The direction $\theta = \frac{\pi}{2}$ decomposes (X, u) into two cylinders. And the direction $\theta = 0$ decomposes (X, u) into only one cylinder. Therefore, $\theta = 0$ is a simple Jenkins-Strebel direction. The two Jenkins-Strebel directions $0, \frac{\pi}{2}$ of (X, u) imply that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ define elements in $\mathrm{Aff}^+(X, u)$ as figure 3.2. Hence, $\Gamma = \langle \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right] \rangle$ is a subgroup of $\Gamma(X, u)$.

Since every element in $\mathrm{Aff}^+(X, u)$ must preserve the set of all critical points, $\Gamma(X, u)$ is a subgroup of $\mathrm{PSL}(2, \mathbb{Z})$. It is known that the group $\langle \left[\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right] \rangle$ is the congruence subgroup of level 2 and has index 6 in $\mathrm{PSL}(2, \mathbb{Z})$. Hence, $\Gamma(X, u)$ is either Γ or $\mathrm{PSL}(2, \mathbb{Z})$. However, $\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right]$ cannot be an element in $\Gamma(X, u)$. Therefore $\Gamma(X, u)$ must be Γ . It is easy to see that $\mathbb{H}/\bar{\Gamma}(X, u)$ is an orbifold of genus 0 has 2 punctures and 1 cone point of order 2. Thus, $\Gamma(X, u)$ is a lattice in $\mathrm{PSL}(2, \mathbb{R})$ and the flat surface (X, u) satisfies the Veech dichotomy.

Example 3.3.15 ([4]). Fix $n \geq 2$ and let Π_{4n} be a regular $4n$ -gon. We assume that Π_{4n} has two horizontal sides, lengths of the sides are 1. We identify each side of Π_{4n} with the opposite parallel side by an Euclidean

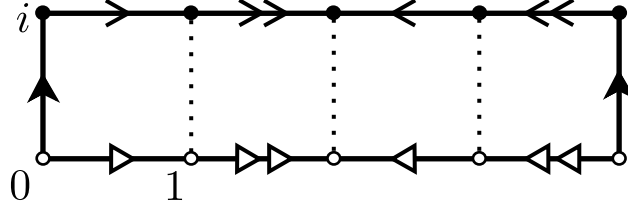


Figure 3.1: The flat surface (X, u)

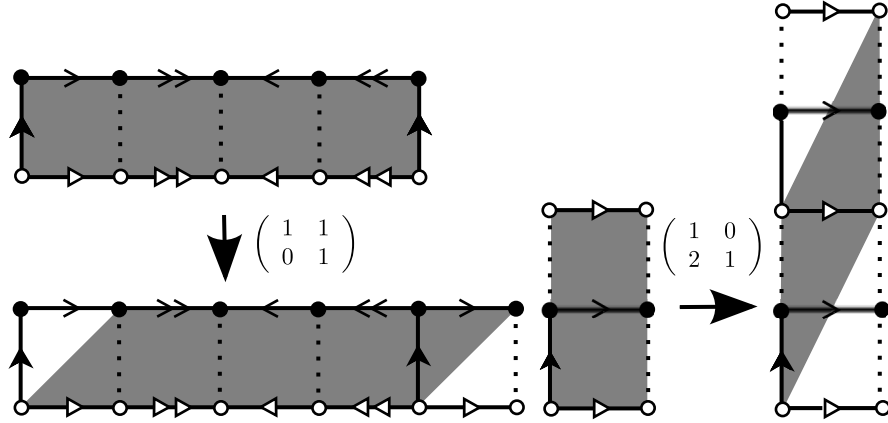


Figure 3.2: Actions of two matrices onto X

translation (see figure 3.3) and denote the resulting surface by P_{4n} . We induce an unique conformal structure on P_{4n} such that the quadratic differential dz^2 on the interior of Π_{4n} extends to a holomorphic quadratic differential q_n on P_{4n} . Then P_{4n} is a Riemann surface of type $(\frac{n}{2}, 0)$. And, the vertices of Π_{4n} become an unique zero of q_n of order $4n - 4$. Let $u_{q_{4n}}$ be the flat structure defined by q_n . Now, $R_{4n} = \begin{pmatrix} \cos \pi/2n & -\sin \pi/2n \\ \sin \pi/2n & \cos \pi/2n \end{pmatrix}$ and $T_{4n} = \begin{pmatrix} 1 & 2 \cot \pi/4n \\ 0 & 1 \end{pmatrix}$ induce elements in $\text{Aff}^+(P_{4n}, u_{q_{4n}})$. The action of R_{4n} on P_{4n} is the rotation about the center of Π_{4n} of angle $\frac{\pi}{2n}$. To see the action of T_{4n} on P_{4n} , we cut P_{4n} along all horizontal saddle connections. Then P_{4n} is decomposed into $\frac{n}{2}$ cylinders and the action of T_{4n} is the composition of the square of the right Dehn twist along a core curve of the cylinder which contains the center of Π_{4n} and the right Dehn twists along core curves of the other cylinders. Thus, $\Gamma = \langle [R_{4n}], [T_{4n}] \rangle$ is a subgroup of the Veech group $\Gamma(P_{4n}, u_{q_{4n}})$. It is

easy to see that Γ is a triangle group of type $(2n, \infty, \infty)$. Since only discrete group that contains Γ is a triangle group of type $(2, 4n, \infty)$ (see [4] and [22]) and this cannot be $\Gamma(P_{4n}, u_{q_{4n}})$, we have $\Gamma(P_{4n}, u_{q_{4n}}) = \langle [R_{4n}], [T_{4n}] \rangle$. And, $\Gamma(P_{4n}, u_{q_{4n}})$ is a lattice in $\text{PSL}(2, \mathbb{R})$ and the flat surface $(P_{4n}, u_{q_{4n}})$ satisfies the Veech dichotomy.

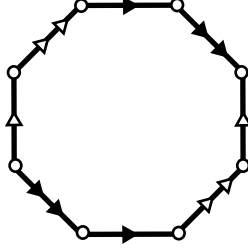


Figure 3.3: The flat surface (P_8, u_{q_8})

3.4 Construction of Veech holomorphic families of Riemann surfaces

In this section, we construct holomorphic families of Riemann surfaces from flat surfaces and the Veech groups. We call those holomorphic families *Veech holomorphic families of Riemann surfaces*. And we observe representations and monodromies of Veech holomorphic families of Riemann surfaces.

We assume that X is a Riemann surface of type (g, n) with $3g - 3 + n > 0$. Let (X, u) be a flat surface with critical points on C and assume that this flat structure is compatible with the complex structure of X . Let q be the corresponding holomorphic quadratic differential on X . In section 3.3, we constructed a holomorphic isometric embedding $\iota : \mathbb{H} \ni t \mapsto [X_t, id] \in T(X)$ from (X, u) . The Beltrami coefficient of the Teichmüller map from $\iota(i) = [X, id]$ to $\iota(t)$ is $\mu_t = \frac{i-t}{i+t} \frac{|q|}{q}$. Let $\rho : \mathbb{H} \rightarrow X$ be the universal covering map with the covering transformation group $G < \text{PSL}(2, \mathbb{R})$. For each $t \in \mathbb{H}$, we denote by $\tilde{\mu}_t$ the Beltrami coefficient on $\hat{\mathbb{C}}$ which is the lift of μ_t in $\mathbb{H}$ and equals to 0 in $\hat{\mathbb{C}} - \mathbb{H}$. Let $w^{\tilde{\mu}_t} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be an unique $\tilde{\mu}_t$ -quasiconformal map which fixes $0, 1, \infty$. We set

$$\widetilde{M} = \{(t, z) : t \in \mathbb{H}, z \in w^{\tilde{\mu}_t}(\mathbb{H})\}.$$

By identifying $\mathbb{H}$ with $\iota(\mathbb{H})$, $\widehat{M}$ is a two-dimensional complex manifold as a submanifold of the Bers fiber space $F(G)$. The covering transformation group G acts on $\widehat{M}$ holomorphically as $\sigma_*(t, z) = (t, w^{\tilde{\mu}_t} \sigma(w^{\tilde{\mu}_t})^{-1}(z))$ for all $\sigma \in G$ and $(t, z) \in \widehat{M}$. Set $\widetilde{M} = \widehat{M}/G$. Then we can write $\widehat{M}$ as

$$\widehat{M} = \{(t, z) : t \in \mathbb{H}, z \in X_t = f_t(X)\}.$$

Here, $f_t : X \rightarrow X_t = f_t(X)$ is the induced quasiconformal map of $w^{\tilde{\mu}_t}|_{\mathbb{H}}$. Of course, X_t coincides with $\iota(t)$ and f_t is a Teichmüller map with the Beltrami coefficient μ_t .

Recall that $\text{Aff}^+(X, u)$ is the group of all affine homeomorphisms from (X, u) onto itself which fixes C and the homomorphism $D : \text{Aff}^+(X, u) \rightarrow \text{PSL}(2, \mathbb{R})$ is the derivative map. The affine group $\text{Aff}^+(X, u)$ acts on $\widehat{M}$ holomorphically as

$$h_*(t, z) = (\bar{A}(t), f_{\bar{A}(t)} h(f_t)^{-1}(z))$$

for all $h \in \text{Aff}^+(X, u)$ and $(t, z) \in \widehat{M}$. Here, $A = D(h)$, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{A} = RAR^{-1}$ acts on $\mathbb{H}$ as a Möbius transformation. We set $N = \widehat{M}/\text{Aff}^+(X, u)$ and $\bar{B} = \mathbb{H}/\bar{\Gamma}(X, u)$. Let $\pi' : N \rightarrow \bar{B}$ be the projection, $\mathbb{H}^*$ be $\mathbb{H}$ with all elliptic fixed points of $\bar{\Gamma}(X, u)$ are removed, $B = \mathbb{H}^*/\bar{\Gamma}(X, u)$, and $N^* = (\pi')^{-1}(B)$. It is easy to see that N is homeomorphic to the product space $B \times (X/\text{Ker}(D))$. There exists a branched covering map $M \rightarrow N$ which corresponds to the branched covering map $B \times X \rightarrow B \times (X/\text{Ker}(D))$. Now, we obtain the holomorphic family (M, π, B) of Riemann surfaces of type (g, n) over B . Here, $\pi : M \rightarrow B$ is the projection.

Definition 3.4.1 (Veech holomorphic families). We call the holomorphic families of Riemann surfaces (M, π, B) which are constructed as above Veech holomorphic families of Riemann surfaces of type (g, n) over B .

Remark. The triple (N^*, π', B) is also a holomorphic family of Riemann surfaces. And, (M, π, B) and (N^*, π', B) are locally non-trivial holomorphic families of Riemann surfaces.

We observe a representation of a Veech holomorphic family (M, π, B) of Riemann surfaces and the monodromy. First, we have the holomorphic isometric embedding $\iota : \mathbb{H} \rightarrow T(g, n)$. Let $\bar{\rho} : \mathbb{H} \rightarrow \bar{B}$ and $\rho : \mathbb{H} \rightarrow B$ be the universal covering maps. Since $\mathbb{H}^*$ is a covering space of B , there exists a covering map $\rho_0 : \mathbb{H} \rightarrow \mathbb{H}^*$ which satisfies $\rho = \bar{\rho} \circ \rho_0$. Then, the map $\Phi = \iota \circ \rho_0$

is a representation of (M, π, B) . Since $B = \mathbb{H}^*/\bar{\Gamma}(X, u)$, we can identify $\bar{\Gamma}(X, u)$ with $\pi_1(B)/\pi_1(\mathbb{H}^*)$. And there exists a natural map $\bar{\Gamma}(X, u) \rightarrow \Gamma(X, u)$. The monodromy $\chi : \pi_1(B) \rightarrow \text{Aff}^+(X, u)$ of the representation Φ is a lift of this natural map by the maps $\pi_1(B) \rightarrow \pi_1(B)/\pi_1(\mathbb{H}^*)$ and $D : \text{Aff}^+(X, u) \rightarrow \Gamma(X, u)$.

$$\begin{array}{ccccc}
 \mathbb{H} & & & & \\
 \downarrow \rho & \searrow \rho_0 & \Phi & \searrow & \\
 & \mathbb{H}^* & \xrightarrow{\iota} & T(X) & \\
 & \searrow \bar{\rho}|_{\mathbb{H}^*} & & & \\
 & B & & & \\
 \\
 \pi_1(B) & \xrightarrow{\chi} & \text{Aff}^+(X, u) & & \\
 \downarrow & & \downarrow D & & \\
 \bar{\Gamma}(X, u) = \pi_1(B)/\pi_1(\mathbb{H}^*) & \longrightarrow & \Gamma(X, u) & &
 \end{array}$$

3.5 Holomorphic sections of Veech holomorphic families of Riemann surfaces.

In this section, we characterize holomorphic sections of holomorphic families of Riemann surfaces by certain points of a corresponding flat surface. We use the same notations as in section 3.4.

Definition 3.5.1 (Holomorphic sections). Let (M, π, B) be a holomorphic family of Riemann surfaces. A holomorphic map $s : B \rightarrow M$ is called a holomorphic section of (M, π, B) or a holomorphic section of π if it satisfies $\pi \circ s = id$.

Let (M, π, B) be a Veech holomorphic family of Riemann surfaces defined by a holomorphic quadratic differential q or the corresponding flat surface (X, u) . In section 3.4, we define the action of $\text{Aff}^+(X, u)$ onto $\widehat{M}$. We set $\widehat{N} = \widehat{M}/\text{Ker}(D)$ and let $\hat{\pi} : \widehat{N} \rightarrow \mathbb{H}$ be the projection. We denote each point

of $\widehat{N}$ by (t, w) . Here, $t \in \mathbb{H}$ is the image of the point by the map $\hat{\pi}$ into $\mathbb{H}$ and w is a point of the Riemann surface $\hat{\pi}^{-1}(t) = Y_t$. That is, we denote $\widehat{N}$ by

$$\widehat{N} = \{(t, w) : t \in \mathbb{H}, w \in Y_t\}.$$

Note that if we set q' to be a holomorphic quadratic differential on $Y = X/\text{Ker}(D)$ induced by q , then there exists the Teichmüller map $g_t : Y \rightarrow Y_t$ whose Beltrami coefficient is $\frac{i-t}{i+t} \frac{|q'|}{q'}$ for each $t \in \mathbb{H}$.

Let $s : B \rightarrow M$ be a holomorphic section of (M, π, B) . We project this holomorphic section to a holomorphic section $s' : B \rightarrow N^*$ of (N^*, π', B) . Where, (N^*, π', B) is a holomorphic family of Riemann surfaces constructed in section 3.4. We lift s' to a holomorphic section $\hat{s} : \mathbb{H}^* \rightarrow \widehat{N}^* = \hat{\pi}^{-1}(\mathbb{H}^*) \subset \widehat{N}$ of $\hat{\pi}|_{\widehat{N}^*}$.

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \uparrow \\ \widehat{N} \supset \widehat{N}^* & \xrightarrow{\quad} & N^* & & s \\ \downarrow \hat{\pi} & \downarrow \hat{s} & \downarrow \pi' & & \uparrow \\ \mathbb{H} \supset \mathbb{H}^* & \xrightarrow{\quad} & B & & \end{array}$$

Proposition 3.5.2. *The complex manifold $\widehat{N}$ is a proper Kobayashi hyperbolic manifold.*

Proof. Let $\widetilde{N}$ be the universal covering of $\widehat{N}$. First, we see that $\widetilde{N}$ is a proper Kobayashi hyperbolic manifold. Recall that q' is the holomorphic quadratic differential on $Y = X/\text{Ker}(D)$ which is the projection of the holomorphic quadratic differential q on X . By setting $\iota' : \mathbb{H} \rightarrow T(Y)$ to be the holomorphic isometric embedding defined by q' . Then we can write $\widetilde{N}$ as

$$\widetilde{N} = \{(\tilde{t}, \tilde{w}) : \tilde{t} \in \mathbb{H}, \tilde{w} \in \mathbb{H}^{\tilde{t}}\}$$

by identifying $\mathbb{H}$ as $\iota'(\mathbb{H})$. And $\mathbb{H}^{\tilde{t}}$ is defined in subsection 3.2.4. Therefore, $\widetilde{N}$ is a closed submanifold of the Bers fiber space $F(G')$. Here, G' is a Fuchsian group with $Y = \mathbb{H}/G'$. Since $F(G')$ is proper Kobayashi hyperbolic manifold by theorem 3.2.8 and theorem 3.2.11, $\widetilde{N}$ is also a proper Kobayashi hyperbolic manifold.

Next, we show that $\widehat{N}$ is a Kobayashi hyperbolic manifold. We suppose that $d_N^K((t_1, w_1), (t_2, w_2)) = 0$. Since $\hat{\pi}$ is holomorphic,

$$d_{\mathbb{H}}(t_1, t_2) \leq d_{\widehat{N}}^K((t_1, w_1), (t_2, w_2)) = 0.$$

Hence, we have $t_1 = t_2$. We set $t = t_1 = t_2$. Let $\rho : \widetilde{N} \rightarrow \widehat{N}$ be the universal covering map. Fix $(\tilde{t}, \tilde{w}_1) \in \rho^{-1}((t, w_1))$. By theorem 3.2.9, we have

$$0 = d_{\widehat{N}}^K((t, w_1), (t, w_2)) = \inf_{(\tilde{t}, \tilde{w}_2)} d_{\widetilde{N}}^K((\tilde{t}, \tilde{w}_1), (\tilde{t}, \tilde{w}_2)).$$

Here, infimum is taken over all points in $\rho^{-1}((t, w_2))$. For all $n \geq 2$, there exist $(\tilde{t}, \tilde{w}_n)$ such that $d_{\widetilde{N}}^K((\tilde{t}, \tilde{w}_1), (\tilde{t}, \tilde{w}_n)) < \frac{1}{n}$. Since $\widetilde{N}$ is holomorphically embedded in $\mathbb{H} \times (\mathbb{C} - \{0, 1\})$, we have

$$d_{\mathbb{C}-\{0,1\}}^K(\tilde{w}_1, \tilde{w}_n) \leq d_{\widetilde{N}}^K((\tilde{t}, \tilde{w}_1), (\tilde{t}, \tilde{w}_n)) < \frac{1}{n}.$$

Since the relative topology of $\mathbb{H}^t$ from $d_{\mathbb{C}-\{0,1\}}^K$ is same as the topology defined by $d_{\mathbb{H}^t}^K$ and $w_1 \in \mathbb{H}^t$, we have $d_{\mathbb{H}^t}^K(\tilde{w}_1, \tilde{w}_n) \rightarrow 0$ as $n \rightarrow \infty$. This implies that $\tilde{w}_n = \tilde{w}_{n+1}$ for sufficiently large n and, hence, $\tilde{w}_n = \tilde{w}_1$. Therefore, we have $(t_1, w_1) = (t_2, w_2)$.

Finally we prove that $\widehat{N}$ is a proper Kobayashi hyperbolic manifold. Let $U_r(t_0, w_0)$ be the closed ball of center $(t_0, w_0) \in \widehat{N}$ and radius r in $\widehat{N}$. Take any sequence $\{(t_n, w_n)\}_{n \in \mathbb{N}}$ in $U_r(t_0, w_0)$. We fix $(\tilde{t}_0, \tilde{w}_0) \in \rho^{-1}((t_0, w_0))$. By proposition 3.2.9, there exists $(\tilde{t}_n, \tilde{w}_n) \in \rho^{-1}((t_n, w_n))$ which satisfies $d_{\widetilde{N}}^K((\tilde{t}_0, \tilde{w}_0), (\tilde{t}_n, \tilde{w}_n)) < r$ for each $n \in \mathbb{N}$. Since $\widetilde{N}$ is proper, there exists a subsequence $\{(\tilde{t}_{n_l}, \tilde{w}_{n_l})\}$ which converges to some point $(\tilde{t}_\infty, \tilde{w}_\infty) \in \widetilde{N}$. Then,

$$d_{\widetilde{N}}^K((t_{n_l}, w_{n_l}), \rho(\tilde{t}_\infty, \tilde{w}_\infty)) \leq d_{\widetilde{N}}^K((\tilde{t}_{n_l}, \tilde{w}_{n_l}), (\tilde{t}_\infty, \tilde{w}_\infty)) \rightarrow 0 \ (l \rightarrow \infty).$$

Hence, $\widehat{N}$ is a proper Kobayashi hyperbolic manifold. □

By proposition 3.5.2, we can extend $\hat{s} : \mathbb{H}^* \rightarrow \widehat{N}^*$ to a holomorphic section $\hat{s} : \mathbb{H} \rightarrow \widehat{N}$ of $\hat{\pi}$. For holomorphic sections of $\hat{\pi}$, we have the following. Recall that q' is a holomorphic quadratic differential on $Y = X/\text{Ker}(D)$ induced by q and $g_t : Y \rightarrow Y_t$ is the Teichmüller map whose Beltrami coefficient is $\frac{i-t}{i+t} \frac{|q'|}{q'}$.

Theorem 3.5.3. *Let $\hat{s} : \mathbb{H} \rightarrow \widehat{N}$ be a holomorphic section of $\hat{\pi}$. Set $b = \hat{s}(i)$. Then*

$$\hat{s}(t) = (t, g_t(b))$$

for all $t \in \mathbb{H}$.

Proof. Set $\dot{Y} = Y - \{b\}$. Let $\iota' : \mathbb{H} \rightarrow T(Y)$ be the holomorphic isometric embedding defined by q' . Let $\tilde{s} : \mathbb{H} \rightarrow \tilde{N}$ be a lift of $\hat{s}$ to $\tilde{N}$. Since, $\tilde{N}$ is embedded holomorphically into the Bers fiber space $F(G')$, it is also embedded holomorphically into $T(\dot{Y})$. Here, G' is a Fuchsian group with $Y = \mathbb{H}/G'$. The composition of this embedding with $\tilde{s}$ is an injective holomorphic map $\kappa : \mathbb{H} \rightarrow T(\dot{Y})$ with $\kappa(i) = [\dot{Y}, id]$. Let $\tau : T(\dot{Y}) \rightarrow T(Y)$ be the forgetful map, that is, $\tau([f(\dot{Y}), f]) = [f(Y), f]$ for all $[f(\dot{Y}), f] \in T(\dot{Y})$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\quad} & T(\dot{Y}) \\ \downarrow \tilde{s} & \nearrow \kappa & \downarrow \tau \\ \mathbb{H} & \xrightarrow{\iota'} & T(Y) \end{array}$$

Since κ, τ is holomorphic and ι is holomorphic isometric embedding, by theorem 3.2.8, $\kappa : \mathbb{H} \rightarrow T(\dot{Y})$ is also an isometry. By proposition 3.3.3, there exists a holomorphic quadratic differential $\dot{q}$ on $\dot{Y}$ which defines κ . Take some $t \in \mathbb{H}$ ($t \neq i$) then the equation $\tau \circ \kappa(t) = \iota'(t)$ implies that $\frac{i-t}{i+t} \frac{|\dot{q}|}{\dot{q}} = \frac{i-t}{i+t} \frac{|q'|}{q'}$ on $\dot{Y}$. This means that $\hat{s}(t) = (t, g_t(b))$ for all $t \in \mathbb{H}$. $\square$

Now, we observe that N^* is a quotient manifold of $\hat{N}^*$ by the action of the Veech group $\Gamma(X, u)$. Let u' be the flat structure on $Y = X/\text{Ker}(D)$ defined by q' . Since $\Gamma(X, u) = \text{Aff}^+(X, u)/\text{Ker}(D)$, the Veech group $\Gamma(X, u)$ is considered as a subgroup of $\text{Aff}^+(Y, u')$ and acts on Y . We denote by h'_A each element A of $\Gamma(X, u)$ if we consider A as an element of $\text{Aff}^+(Y, u')$. We extend the action of $\Gamma(X, u)$ onto Y to an action onto $\hat{N}$. For all $(t, w) \in \hat{N}$ and $h'_A = A \in \Gamma(X, u) < \text{Aff}^+(Y, u')$, we set

$$(h'_A)_*(t, w) = (\bar{A}(t), g_{\bar{A}(t)} h'_A g_t^{-1}(w)).$$

Recall that $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{A} = RAR^{-1}$ acts on $\mathbb{H}$ as a Möbius transformation. This action is induced by the action of $\text{Aff}^+(X, u)$ on $\widehat{M}$ which we define in section 3.4. Then, it is clear that $N^* = \hat{N}^*/\text{Ker}(D)$. Now we have the following.

Corollary 3.5.4. *Let $\varphi : X \rightarrow Y$ be the branched covering map. For $b \in Y$ and any point $a \in \varphi^{-1}(b)$, the following are equivalent.*

- (1) *The map $\hat{s}_b : \mathbb{H} \ni t \mapsto (t, g_t(b)) \in \hat{N}$ induces a section of $\pi' : N^* \rightarrow B$,*

$$(2) \quad \Gamma(X, u)(\{b\}) = \{b\},$$

$$(3) \quad \text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\}).$$

Proof. (1) $\Rightarrow$ (2). If $\hat{s}_b$ induces a section of π' , then $(h'_A)_*(\hat{s}_b(t)) = \hat{s}_b(\bar{A}(t))$ for all $h'_A = A \in \Gamma(X, u)$. This implies $h'_A(b) = b$ for all $A \in \Gamma(X, u)$.

(2) $\Rightarrow$ (1). If $\Gamma(X, u)(\{b\}) = \{b\}$ holds, then $(h'_A)_*(s_b(t)) = s_b(\bar{A}(t))$ for all $h'_A = A \in \Gamma(X, u)$. Hence, we obtain (1).

Clearly, (2) and (3) are equivalent since $\Gamma(X, u) = \text{Aff}^+(X, u)/\text{Ker}(D)$. $\square$

Recall that $\bar{\rho}|_{\mathbb{H}^*} : \mathbb{H}^* \rightarrow B = \mathbb{H}^*/\bar{\Gamma}(X, u)$ is a covering map and $f_t : X \rightarrow X_t$ is the Teichmüller map with the Beltrami coefficient $\frac{i-t}{i+t} \frac{q}{q}$ for each $t \in \mathbb{H}^*$. Fix any $t \in \mathbb{H}^*$. We take an open neighborhood $\tilde{U}$ of $\tilde{t}$ in $\mathbb{H}^*$ such that $\bar{\rho}|_{\tilde{U}} : \tilde{U} \rightarrow \bar{\rho}(\tilde{U}) = U$ is a conformal map. Then $\pi^{-1}(U) \subset M$ is biholomorphic to $\widehat{M}(\tilde{U}) = \{(t, z) : t \in \tilde{U}, z \in X_t = f_t(X)\} \subset \widehat{M}$. By identifying U with $\tilde{U}$ and $\pi^{-1}(U)$ with $\widehat{M}(\tilde{U})$, we have the following.

Corollary 3.5.5. *Let $s : B \rightarrow M$ be a holomorphic section of π . Then there exists $a \in X$ with $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$ such that the induced holomorphic map $s|_{\tilde{U}} : \tilde{U} \rightarrow \widehat{M}(\tilde{U})$ satisfies*

$$s|_{\tilde{U}}(t) = (t, f_t(a))$$

for all $t \in U$.

Fix $\Gamma < \Gamma(X, u)$. Let $\rho' : \mathbb{H}^*/\Gamma \rightarrow B$ be a covering map. Set $M_\Gamma = \{(t', z') : t' \in \mathbb{H}^*/\Gamma, z' \in X_{\rho'(t')} = \pi^{-1}(\rho'(t'))\}$ and let $\pi_\Gamma : M_\Gamma \rightarrow \mathbb{H}^*/\Gamma$ be the projection. Then the triple $(M_\Gamma, \pi_\Gamma, \mathbb{H}^*/\Gamma)$ is a holomorphic family of Riemann surfaces, M_Γ is a covering space of M , and π_Γ is a lift of π . By identifying $V = (\rho')^{-1}(U)$ with $\tilde{U}$ and $\pi_\Gamma^{-1}(V)$ with $\widehat{M}(\tilde{U})$, we also have the following.

Corollary 3.5.6. *For a holomorphic section $s : \mathbb{H}^*/\Gamma \rightarrow M_\Gamma$, there exists to a point $a \in X$ with $D^{-1}(\Gamma)(\{a\}) = \text{Ker}(D)(\{a\})$ such that the induced holomorphic map $s|_{\tilde{U}} : \tilde{U} \rightarrow \widehat{M}(\tilde{U})$ satisfies*

$$s|_{\tilde{U}}(t') = (t', f_{t'}(a))$$

for all $t' \in \tilde{U}$.

Remark. It is not true that all points a of X with $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$ define holomorphic sections of $\pi : M \rightarrow B$. See example 3.6.1.

Example 3.5.7. Let (X, u) be the flat surface as in example 3.3.14. Recall that (X, u) is a Riemann surface of type $(2, 0)$ and the Veech group $\Gamma(X, u) = \langle [(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})], [(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix})] \rangle$. We can easily prove that $\text{Ker}(D) = \langle \alpha : z \mapsto z + 1, \beta : z \mapsto -z \rangle$. The candidates of holomorphic sections of the corresponding Veech holomorphic family of Riemann surfaces are the points a with $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$. The candidates are $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$, $(0, 1)$, $(\frac{1}{2}, 1)$, $(\frac{3}{2}, 1)$ in figure 3.1. They are the Weierstrass points of X and the hyperelliptic involution of X is α^2 .

Example 3.5.8. Let $(P_{4n}, u_{q_{4n}})$ be the flat surface as in example 3.3.15. Then $\Gamma(P_{4n}, u_{q_{4n}}) = \langle R_{4n}, T_{4n} \rangle$. It is easy to see that $\text{Ker}(D) = \langle z \mapsto -z \rangle$. Only the points of P_{4n} which correspond to the center and the vertices of the corresponding polygon satisfy $\text{Aff}^+(P_{4n}, u_{q_{4n}})(\{a\}) = \text{Ker}(D)(\{a\})$. Hence, the two points are the candidates of the holomorphic sections of the corresponding Veech holomorphic family of Riemann surfaces. Since, one of them is the critical point of $(P_{4n}, u_{q_{4n}})$ and the other is not a critical point of $(P_{4n}, u_{q_{4n}})$, if we construct sections locally from the points and extend them globally, the permutation of these two points does not happen. Hence, the two points define holomorphic sections and they are all the holomorphic sections of the Veech holomorphic family of Riemann surfaces.

3.6 Diophantine problems

In this section, we consider Diophantine problems over function fields. We associate the Diophantine problems with holomorphic families of Riemann surfaces and their holomorphic sections. At the end of this section, we construct some examples of Diophantine equations over function fields and give all solutions of them.

Let B_0 be a compact Riemann surface. Denote by $M(B_0)$ the field of all meromorphic functions on B_0 . We take an irreducible homogeneous polynomial

$$f(X, Y, Z) = \sum_{i+j+k=N} A_{i,j,k} X^i Y^j Z^k \quad (A_{i,j,k} \in M(B_0)).$$

Problem (Diophantine problem). Find all solutions of $f(X, Y, Z) = 0$ in $\mathbb{P}^2(M(B_0))$.

We see a relation between Diophantine problems and holomorphic families of Riemann surfaces. We set

$$f_t(x, y, z) = \sum_{i+j+k=N} A_{i,j,k}(t) x^i y^j z^k.$$

Since f is irreducible, there exist $t_1, \dots, t_n \in B_0$ such that $f_t(x, y, z) \in \mathbb{C}[x, y, z]$ is irreducible for each $t \in B = B_0 - \{t_1, \dots, t_n\}$. Then, for each $t \in B$,

$$X_t = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : f_t(x, y, z) = 0\}$$

is an algebraic curve, that is, a compact Riemann surface. Set

$$M = \{(t, [x : y : z]) \in B \times \mathbb{P}^2(\mathbb{C}) : f_t(x, y, z) = 0\}$$

and let $\pi : M \rightarrow B$ be the projection. Then the triple (M, π, B) is a holomorphic family of Riemann surfaces. Next, if $[X : Y : Z] \in \mathbb{P}^2(M(B_0))$ is a solution of $f(X, Y, Z) = 0$, then

$$f(X(t), Y(t), Z(t)) = \sum_{i+j+k=N} A_{i,j,k}(t) X(t)^i Y(t)^j Z(t)^k = 0$$

for all $t \in B_0$. This means that $[X(t) : Y(t) : Z(t)] \in X_t$ for all $t \in B$. Hence, the map $B \ni t \mapsto (t, [X(t) : Y(t) : Z(t)]) \in M$ is a holomorphic section of $\pi : M \rightarrow B$. Conversely, if $s : B \rightarrow M$ is a holomorphic section, then $s(t) = (t, [X(t) : Y(t) : Z(t)])$ for some $X, Y, Z \in M(B_0)$ and $[X : Y : Z]$ is a solution of $f(X, Y, Z) = 0$. In conclusion, a Diophantine problem is to find all holomorphic sections of the corresponding holomorphic families of Riemann surfaces.

Example 3.6.1. Let (X, u) be the flat surface as in example 3.3.14. Let (M, π, B) be the Veech holomorphic family of Riemann surfaces defined by (X, u) . Set $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Then X is a Riemann surface of type $(2, 0)$, $\Gamma(X, u) = \langle S, T \rangle$, $\text{Ker}(D) = \langle \alpha : z \mapsto z + 1, \beta : z \mapsto -z \rangle$, α^2 is the hyperelliptic involution of X , and the Weierstrass points of X are $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$, $(0, 1)$, $(\frac{1}{2}, 1)$ and $(\frac{3}{2}, 1)$. We denote the set of all Weierstrass points of X by WP . Since $\mathbb{H}/\Gamma(X, u)$ is genus 0 and has 2 punctures and

one cone point of order 2, $B = \mathbb{H}^*/\bar{\Gamma}(X, u)$ is a 3-punctured sphere. Recall that $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{\Gamma}(X, u) = R\Gamma(X, u)R^{-1}$.

We describe each fiber $X_t = \pi^{-1}(t)$ ($t \in B$) as an algebraic curve. By construction, $X_t = (X, A_t \circ u)$ for some $A_t \in \text{SL}(2, \mathbb{R})$. Let $T_{A_t} : (X, u) \rightarrow (X, A_t \circ u)$ be the identity map. Note that T_{A_t} is an affine map whose derivative is A_t with respect to the charts of the flat structures. Set $\alpha_t = T_{A_t} \circ \alpha \circ T_{A_t}^{-1}$, $\beta_t = T_{A_t} \circ \beta \circ T_{A_t}^{-1}$ and $WP_t = T_{A_t}(WP)$. Then $K_t = \langle \alpha_t, \beta_t \rangle$ is a group of conformal automorphisms of X_t . The automorphism α_t^2 is also the hyperelliptic involution of X_t and WP_t is the set of all Weierstrass points of X_t . Then $R_t = (X_t - WP_t)/\langle \alpha_t^2 \rangle$ is a Riemann surface of type $(0, 6)$ and has conformal automorphisms α'_t, β'_t induced by α_t, β_t . We may assume that $0, 1, \infty$ are punctures of Y_t , two critical points of X_t are mapped to $0, \infty$ and another Weierstrass point is mapped to 1. Then α'_t is a Möbius transformation of order 2 which fixes $0, \infty$ and β'_t is a Möbius transformation of order 2 which permutes 0 and ∞ . Therefore, $\alpha'_t(z) = -z$ and $\beta'_t(z) = \lambda_t/z$ for some $\lambda_t \in \mathbb{C} - \{0, \pm 1\}$. This implies that $R_t = \mathbb{C} - \{0, \pm 1, \pm \lambda_t\}$ and

$$X_t = \{(x, y) \in \hat{\mathbb{C}}^2 : y^2 = x(x^2 - 1)(x^2 - \lambda_t^2)\}.$$

Also, we can write X_t as

$$X_t = \{(x, y) \in \hat{\mathbb{C}}^2 : y^2 = x(x^2 - 1)(x^2 - 1/\lambda_t^2)\}.$$

Thus, we have the holomorphic map

$$\Psi : \mathbb{H}/\bar{\Gamma}(X, u) \ni t \mapsto \{\lambda_t^2, 1/\lambda_t^2\} \in (\mathbb{C} - \{0, 1\}) / \langle z \mapsto 1/z \rangle.$$

By construction, X_t and $X_{t'}$ are not conformal equivalent for two distinct points $t, t' \in \mathbb{H}/\bar{\Gamma}(X, u)$. This implies that Ψ is injective, and hence conformal map. The map $\phi(z) = \frac{1}{2}(z + 1/z)$ induces a conformal map from $(\mathbb{C} - \{0, 1\}) / \langle z \mapsto 1/z \rangle$ to $\mathbb{C} - \{1\}$. And, $-1 \in \mathbb{C} - \{1\}$ is a cone point of $(\mathbb{C} - \{0, 1\}) / \langle z \mapsto 1/z \rangle$. Hence, $B = \mathbb{H}^*/\bar{\Gamma}(X, u) = \mathbb{C} - \{\pm 1\}$. Since the parameter t is in B and $\lambda_t^2, 1/\lambda_t^2$ are in $\mathbb{C} - \{0, \pm 1\} = \phi^{-1}(B)$ which is a double cover of B , if the parameter t moves on B along a closed curve around 1 or -1 , then the descriptions of X_t as above appear reciprocally. (See figure 3.4.) We change the base space B to $\phi^{-1}(B) = \mathbb{C} - \{0, \pm 1\}$. By calculation, it is easily proved that $\Gamma = \langle S, T^2, TST^{-1} \rangle$ is the subgroup of $\Gamma(X, u)$ which corresponds to the covering map $\phi : \mathbb{C} - \{0, \pm 1\} \rightarrow B$. Let

$(M_\Gamma, \pi_\Gamma, \mathbb{C} - \{0, \pm 1\})$ be the holomorphic family of Riemann surfaces induced by (M, π, B) by replacing B to $B_\Gamma = \mathbb{H}^*/R\Gamma R^{-1} = \mathbb{C} - \{0, \pm 1\}$. Then

$$\begin{aligned} M_\Gamma &= \{(t, (x, y)) \in B_\Gamma \times \hat{\mathbb{C}}^2 : y^2 = x(x^2 - 1)(x^2 - t)\} \\ &= \{(t, [X : Y : Z]) \in B_\Gamma \times \mathbb{P}^2(\mathbb{C}) : Y^2 Z^3 = X(X^2 - Z^2)(X^2 - tZ^2)\}. \end{aligned}$$

From corollary 3.5.6, all holomorphic sections of π_Γ correspond to the points in X which satisfy $D^{-1}(\Gamma)(\{a\}) = \text{Ker}(D)(\{a\})$. By computation, it is easy to see that such points are exactly the Weierstrass points. Hence, the candidates of holomorphic sections are

$$\begin{aligned} (t, [X : Y : Z]) &= (t, [0 : 0 : 1]), (t, [1 : 0 : 1]), (t, [-1 : 0 : 1]), \\ &\quad (t, [0 : 1 : 0]), (t, [\sqrt{t} : 0 : 1]), (t, [-\sqrt{t} : 0 : 1]). \end{aligned}$$

They are well-defined except for the last two. Thus, we conclude that all the holomorphic sections of π_Γ are

$$(t, [X : Y : Z]) = (t, [0 : 0 : 1]), (t, [1 : 0 : 1]), (t, [-1 : 0 : 1]), (t, [0 : 1 : 0])$$

and all the solutions of the Diophantine equation $f(X, Y, Z) = X(X^2 - Z^2)(X^2 - tZ^2) - Y^2 Z^3$ in $\mathbb{P}^2(M(\hat{\mathbb{C}}))$ are

$$[X : Y : Z] = [0 : 0 : 1], [1 : 0 : 1], [-1 : 0 : 1], [0 : 1 : 0].$$

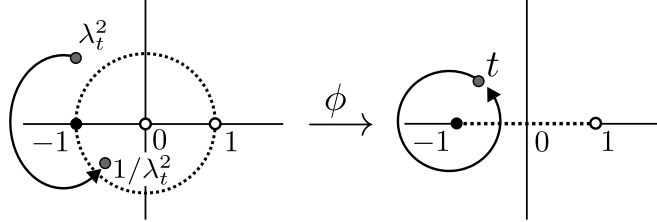


Figure 3.4: A closed curve with base point t and a lift of the curve

Remark. We can extend $(M_\Gamma, \pi_\Gamma, \mathbb{C} - \{0, \pm 1\})$ to a holomorphic family of Riemann surfaces over $\mathbb{C} - \{0, 1\}$. On the other hand, $t = 0, \infty$ correspond to the Riemann surface with one node obtained by contracting a horizontal closed curve on (X, u) . And $t = 1$ corresponds to a Riemann surface with two nodes obtained by contracting vertical closed curves on (X, u) which are the core curves of two vertical cylinders.

Example 3.6.2. By the same argument as example 3.6.1, we obtain the following examples. Let us consider the flat surface (X_m, u_m) as in figure 3.5 for $m \geq 2$. If m is even, X_m has genus m and two critical points. And if m is odd, X_m has genus $m - 1$ and four critical points. The Veech groups $\Gamma(X_m, u_m)$ are equal to $\Gamma(X, u) = \langle S, T \rangle$ as in example 3.6.1. Let $(M_m, \pi_m, \mathbb{C} - \{0, \pm 1\})$ be the holomorphic families of Riemann surfaces constructed in the same way as example 3.6.1. Then $(M_m, \pi_m, \mathbb{C} - \{0, \pm 1\})$ corresponds to a Diophantine equation

$$f_m(X, Y, Z) = X(X^m - Z^m)(X^m - tZ^m) - Y^2 Z^{2m-1} = 0.$$

And all solutions of $f_m(X, Y, Z) = 0$ are

$$[X : Y : Z] = [0 : 0 : 1], [0 : 1 : 0], [\exp(2\pi i \nu / m) : 0 : 1] \quad (\nu = 1, 2, \dots, m).$$

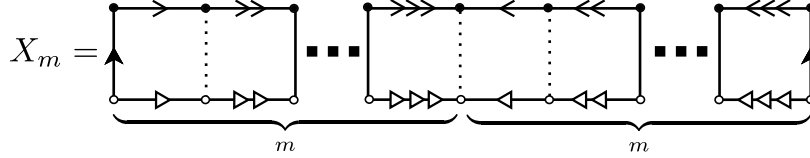


Figure 3.5: The flat surface X_m

3.7 Upper bound of the number of the holomorphic sections

In this section, we give an upper bound of the numbers of holomorphic sections of certain Veech holomorphic families of Riemann surfaces by topological quantities of B and the fiber Riemann surfaces.

Let X be a Riemann surface of type (g, n) and q be an integrable holomorphic quadratic differential on X . Denote by u the flat structure on X induced by q . Let (M, π, B) be the Veech holomorphic family of Riemann surfaces defined by (X, u) . We assume that $\Gamma(X, u)$ is a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$ ($\nu_i \in \{2, 3, \dots, \infty\}$). Then $B = \mathbb{H}^* / \Gamma(X, u)$ is a Riemann surface of type (p, k) . We prove the following theorem.

Theorem 3.7.1. *If q has a simple Jenkins-Strebel direction, then the number of holomorphic sections of (M, π, B) is at most*

$$32\pi(2p - 2 + k)(3g - 3 + n)^2(3g - 2 + n) - 2g + 2.$$

From corollary 3.5.5, the number of holomorphic sections is at most the cardinal of the set

$$S(X, u) = \{a \in X : \text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})\}.$$

We estimate the cardinal of $S(X, u)$. We may assume that $\theta = 0$ is a simple Jenkins-Strebel direction of q . Let $C(X, u)$ be the set of all critical points on X . Let $\varphi : X \rightarrow Y = X/\text{Ker}(D)$ be the branched covering map and $B(X)$ be the set of all branch points of φ . The Riemann surface $Y = X/\text{Ker}(D)$ has the flat structure u' induced by (X, u) . We denote the set of all critical points of (Y, u') by $C(Y, u')$ and the holomorphic quadratic differential on Y which corresponds to (Y, u') by q' . Note that $\varphi(C(X, u)) = C(Y, u')$ is not true. Recall that the Veech group $\Gamma(X, u)$ corresponds to a subgroup of the affine group $\text{Aff}^+(Y, u')$ of (Y, u') . If we consider $\Gamma(X, u)$ as a subgroup of $\text{Aff}^+(Y, u')$, we write $A \in \Gamma(X, u)$ by h'_A . Clearly, $\varphi(S(X, u)) = \{b \in Y : \Gamma(X, u)\{b\} = \{b\}\}$.

Since $\theta = 0$ is a simple Jenkins-Strebel direction, (X, u) is constructed from a rectangle whose vertical sides are glued by a translation and horizontal sides correspond to the union of all horizontal saddle connections of (X, u) . Let $H, W > 0$ be the height and the width of the rectangle, respectively. Let R be the domain in X which corresponds to the interior of the rectangle. Then there exists a chart $z : R \rightarrow (0, W) \times (-\frac{H}{2}, \frac{H}{2})$ in u . For every $h \in \text{Ker}(D)$, there exists an open set $U \subset R$ such that $h(U) \subset R$ and $z \circ h \circ z^{-1} : z(U) \rightarrow z(h(U))$ is the form $z \mapsto \pm z + c$ for some $c \in \mathbb{R}$. The sign of the representation of h is uniquely determined and c is also uniquely determined up to $\pm W$. We identify h with the representation. Then the following lemma is obvious.

Lemma 3.7.2. *The group $\text{Ker}(D)$ is one of the following form:*

$$\langle z \mapsto z + c \rangle, \langle z \mapsto -z + c \rangle, \langle z \mapsto z + c, z \mapsto -z + d \rangle.$$

Let $\text{sgn} : \text{Ker}(D) \rightarrow \{\pm 1\}$ be the homomorphism which maps $h \in \text{Ker}(D)$ to the derivative of $z \circ h \circ z^{-1}$. Denote by l_y the closed curve in X with $z(R \cap l_y) = (0, W) \times \{y\}$ for $y \in (-\frac{H}{2}, \frac{H}{2})$.

Lemma 3.7.3. *Every $h \in \text{Ker}(D)$ with $\text{sgn}(h) = -1$ has two fixed points on l_0 and $h^2 = \text{id}$.*

Proof. Since h fixes l_0 and reverse the orientation of l_0 , h has two fixed points in l_0 . This implies that $z \circ h^2 \circ z^{-1}$ is a translation with fixed points in l_0 . Hence, we have $h^2 = id$. $\square$

We set

$$\begin{aligned} b_0 &= \inf \left\{ |b| : \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma(X, u), b \neq 0 \right\}, \\ c_0 &= \inf \left\{ |c| : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(X, u), c \neq 0 \right\}. \end{aligned}$$

To prove theorem 3.7.1, we show the following.

Proposition 3.7.4. *If $\text{sgn}(\text{Ker}(D)) = \{1\}$, then*

$$\sharp S(X, u) \leq \text{mod}(X) c_0 \left\lceil \frac{b_0 \sharp \text{Ker}(D)}{\text{mod}(X)} + 1 \right\rceil - 2g + 2.$$

And if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, then

$$\sharp S(X, u) \leq 4 \text{mod}(X) c_0 \left\lceil \frac{b_0 \sharp \text{Ker}(D)}{4 \text{mod}(X)} + \frac{1}{2} \right\rceil - 2g + 2.$$

Here, $\text{mod}(X) = \frac{W}{H}$ and $\lceil x \rceil$ is the smallest integer which is greater than or equals to x .

By lemmas 3.7.2 and 3.7.3, (Y, u') is also constructed from some rectangle R' whose vertical sides are glued by translation and horizontal sides correspond to the union of all horizontal saddle connections of (Y, u') . Denote by W' and H' the width and height of R' , respectively. If $\text{sgn}(\text{Ker}(D)) = \{1\}$, we have $W' = \frac{W}{\sharp \text{Ker}(D)}$ and $H' = H$. If $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, we have $W' = \frac{2W}{\sharp \text{Ker}(D)}$ and $H' = \frac{H}{2}$. Note that, if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, the closed curve l_0 in X projects to one of horizontal sides of R' . The horizontal side has two critical points which are poles of q' of order 1. We fix a vertical open interval I in R' which connects two horizontal sides of R' . Let l'_w be the horizontal closed geodesic in Y passing through $w \in I$ and L' be the union of all horizontal saddle connections of Y . We set $h'_{A_0} = A_0 = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \in \Gamma(X, u)$, $I_0 = \{w \in I : l'_w \text{ is pointwise fixed by } h'_{A_0}\}$, and

$$\text{Cross}(A) = \left(\bigcup_{w \in I_0} (l'_w \cap h'_A(l'_w)) \right) \cup (L' \cap h'_A(L'))$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(X, u)$ with $c \neq 0$.

Lemma 3.7.5. *The sets $\varphi(C(X, u)), C(Y, u'), \varphi(B(X))$ and $\varphi(S(X, u))$ are contained in $\text{Cross}(A)$.*

Proof. Let $h_A \in \text{Aff}^+(X, u)$ such that $D(h_A) = A$. Since $h_A(C(X, u)) = C(X, u)$, we have $h'_A(\varphi(C(X, u))) = \varphi(C(X, u))$. And $h'_A(C(Y, u')) = C(Y, u')$ because $h'_A \in \text{Aff}^+(Y, u')$. Also, $h_A(B(X)) = B(X)$ implies $h'_A(\varphi(B(X))) = \varphi(B(X))$. As $\varphi(C(X, u)), C(Y, u')$ and $\varphi(B(X))$ are subsets of L' , they are contained in $L' \cap h'_A(L')$. Therefore, $\varphi(C(X, u)), C(Y, u'), \varphi(B(X)) \subset \text{Cross}(A)$. By the definition of $S(X, u)$, it is clear that $\varphi(S(X, u)) \subset \text{Cross}(A)$. $\square$

Lemma 3.7.6. *For all $w \in I$,*

$$\sharp(l'_w \cap h'_A(l'_w)) = \begin{cases} \frac{\text{mod}(X)}{\sharp \text{Ker}(D)} |c| & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\sharp \text{Ker}(D)} |c| & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

And,

$$\sharp(L' \cap h'_A(L')) = \begin{cases} \frac{\text{mod}(X)}{\sharp \text{Ker}(D)} |c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\sharp \text{Ker}(D)} |c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Here, g' is the genus of Y .

Proof. We identify l'_w with the vector $v = \begin{pmatrix} w' \\ 0 \end{pmatrix}$. Then $h_A(l'_w)$ is identified with $Av = \begin{pmatrix} w'_a \\ w'_c \end{pmatrix}$. The closed curve $h_A(l'_w)$ pass through l'_w whenever the height moves H' . Therefore,

$$\sharp(l'_w \cap h'_A(l'_w)) = W'c/H' = \begin{cases} \frac{\text{mod}(X)}{\sharp \text{Ker}(D)} |c| & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\sharp \text{Ker}(D)} |c| & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

For each $p' \in C(Y, u')$, let $\arg(p')$ be the angle around p' . Then $\arg(p') = n\pi$ for some $n \in \mathbb{N}$ and p' appears n times on the horizontal sides of the rectangle

R' . Hence,

$$\begin{aligned}
\#((L' \cap h'_A(L'))) &= \frac{1}{2} \left(2 \cdot W'c/H' - \sum_{p' \in C(Y, u')} \frac{\arg(p')}{\pi} \right) + \#C(Y, u') \\
&= W'c/H' - \frac{1}{2} \sum_{p' \in C(Y, u')} \left(\frac{\arg(p')}{\pi} - 2 \right) \\
&= W'c/H' - \frac{1}{2} \sum_{p' \in C(Y, u')} \text{ord}(p') \\
&= \begin{cases} \frac{\text{mod}(X)}{\#\text{Ker}(D)} |c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\#\text{Ker}(D)} |c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}
\end{aligned}$$

Recall that $\text{ord}(w)$ is the order of w with respect to q' . □

Lemma 3.7.7.

$$\#I_0 \leq \begin{cases} \lceil \frac{b_0 \# \text{Ker}(D)}{\text{mod}(X)} \rceil & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \lceil \frac{b_0 \# \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \rceil - 1 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Proof. On the rectangle R' , we can represent h'_{A_0} as the following form:

$$h_{A_0} \begin{pmatrix} x \\ y \end{pmatrix} = \pm \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} W'\xi \\ 0 \end{pmatrix}$$

for some $0 \leq \xi < 1$. If the representation holds by minus, $\#I_0 = 1$. If it holds by plus and $\text{sgn}(\text{Ker}(D)) = \{1\}$, by computation, we have

$$\#I_0 \leq \lceil \frac{b_0 \# \text{Ker}(D)}{\text{mod}(X)} \rceil.$$

If it holds by plus and $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, h'_{A_0} maps each horizontal sides of R' to itself. And one of the sides is the image of the horizontal closed geodesic l_0 and it has exactly two points of $\varphi(B(X))$. Moreover, the length of horizontal saddle connection on Y which connects the two points is $\frac{W'}{2}$. Hence, $\xi = 0$ or $\frac{1}{2}$. By computation, we have

$$\#I_0 \leq \lceil \frac{b_0 \# \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \rceil - 1.$$

□

Proof of proposition 3.7.4. By lemma 3.7.6 and lemma 3.7.7, we have

$$\sharp \text{Cross}(A) \leq \begin{cases} \frac{\text{mod}(X)}{\sharp \text{Ker}(D)} |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{\text{mod}(X)} + 1 \right\rceil \right) - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\sharp \text{Ker}(D)} |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \right\rceil \right) - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

By lemma 3.7.5,

$$\sharp S(X, u) \leq \sharp \varphi^{-1}(\text{Cross}(A)) = (\sharp \text{Cross}(A) - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X).$$

Therefore, if $\text{Ker}(D) = \{1\}$,

$$\begin{aligned} \sharp S(X, u) &\leq \text{mod}(X) |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{\text{mod}(X)} + 1 \right\rceil \right) \\ &\quad + (-2g' + 2 - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X). \end{aligned}$$

And, if $\text{Ker}(D) = \{\pm 1\}$,

$$\begin{aligned} \sharp S(X, u) &\leq 4\text{mod}(X) |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \right\rceil \right) \\ &\quad + (-2g' + 2 - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X). \end{aligned}$$

Moreover,

$$\begin{aligned} &(-2g' + 2 - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X) \\ &= (-2g' + 2) \cdot \sharp \text{Ker}(D) - \sum_{w \in \varphi(B(X))} (\sharp \text{Ker}(D) - \sharp \varphi^{-1}(w)) \\ &= (-2g' + 2) \cdot \sharp \text{Ker}(D) - \sum_{w \in \varphi(B(X))} \sum_{z \in \varphi^{-1}(w)} (e_z - 1) \\ &= (-2g' + 2) \cdot \sharp \text{Ker}(D) - \sum_{z \in B(X)} (e_z - 1) \\ &= -2g + 2. \end{aligned}$$

Here, e_z is of the ramification index of φ at z and the last equation is the Riemann-Hurwitz formula. Since we can take all $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(X, u)$ with $c \neq 0$, we obtain the claim. $\square$

Remark. Proposition 3.7.4 is useful to estimate the number of holomorphic sections if we construct an example of flat surface with a simple Jenkins-Strebel direction. To estimate the number of holomorphic sections, we only need estimating $\sharp \text{Ker}(D)$ and finding some element $A \in \Gamma(X, u)$ with $c \neq 0$. It is easy to see that $\lceil x + 1 \rceil \leq 4\lceil \frac{x}{4} + \frac{1}{2} \rceil$ for all $x \in \mathbb{R}$. Hence, the inequality

$$\sharp S(X, u) \leq 4\text{mod}(X)c_0 \lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \rceil - 2g + 2$$

always holds. Moreover, we have the following two lemmas.

Assume that $\theta = 0$ is a simple Jenkins-Strebel direction. We can estimate $\sharp \text{Ker}(D)$ by the following way. Let $\theta' (\neq 0)$ be a Jenkins-Strebel direction. Then the rectangle R is decomposed into finitely many parallelograms $R_1, R_2, \dots, R_k$ by the θ' -direction saddle connections. We label the parallelograms with respect to the rule that if R_i and R_j belong the same or congruent cylinders of Jenkins-Strebel direction θ' , then their labels are same. Assume that the labels which we use are $a_1, a_2, \dots, a_l$. Let a be the word in the free group $\langle a_1, a_2, \dots, a_l \rangle$ which is obtained by reading the labels from left to right on R . We set

$$n = \max\{n \in \mathbb{N} : a = b^n \text{ for some } b \in \langle a_1, a_2, \dots, a_l \rangle\}.$$

Lemma 3.7.8. $\sharp \text{sgn}^{-1}(\{1\}) \leq n$.

Proof. Let $h \in \text{sgn}^{-1}(\{1\})$ such that $\text{sgn}^{-1}(\{1\}) = \langle h \rangle$. Since h is a translation, if h sends R_1 to some R_m , then the labels of R_i and R_{i+m} are always same. This implies that $a = b^{\frac{k}{m}}$ for some $b \in \langle a_1, a_2, \dots, a_l \rangle$. Hence, $\sharp \text{sgn}^{-1}(\{1\}) = \frac{k}{m} \leq n$. $\square$

Lemma 3.7.9. *If a horizontal closed geodesic is separating curve of X , then*

$$\lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \rceil \leq 1.$$

Proof. By the assumption, identifications of two horizontal sides of R are independent. If $\text{sgn}(\text{Ker}(D)) = \{1\}$ then, since there exists the translation $z \mapsto z + \left(\begin{smallmatrix} W \\ \sharp \text{Ker}(D) \\ 0 \end{smallmatrix} \right)$ in $\text{Ker}(D)$, it is easy to see that $\left[\begin{pmatrix} 1 & \frac{\text{mod}(X)}{\sharp \text{Ker}(D)} \\ 0 & 1 \end{pmatrix} \right] \in \Gamma(X, u)$. By the definition of b_0 , there exists $m \in \mathbb{N}$ such that $\frac{\text{mod}(X)}{\sharp \text{Ker}(D)} = mb_0$. Hence, $\frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \leq \frac{1}{4m} + \frac{1}{2}$. By the same argument, if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, then $\left[\begin{pmatrix} 1 & \frac{2\text{mod}(X)}{\sharp \text{Ker}(D)} \\ 0 & 1 \end{pmatrix} \right] \in \Gamma(X, u)$ and hence, $\frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \leq \frac{1}{2m} + \frac{1}{2}$ for some $m \in \mathbb{N}$. $\square$

Now we come back to the proof of theorem 3.7.1. To prove this, we estimate $\frac{b_0}{\text{mod}(X)}$, $\frac{\text{mod}(X)}{b_0}$, $b_0 c_0$ and $\sharp \text{Ker}(D)$ by g, n, p , and k . As $\left[\begin{pmatrix} 1 & \text{mod}(X) \\ 0 & 1 \end{pmatrix} \right] \in \Gamma(X, u)$, we have $\frac{b_0}{\text{mod}(X)} \leq 1$ by the definition of b_0 . We set L_0 and L_1 to be the unions of all horizontal saddle connections corresponding to lower and

upper horizontal sides of the rectangle R , respectively. Let n_0 be the number of horizontal saddle connections contained in L_0 . And let n_1 be the number of horizontal saddle connections contained in L_1 . We may assume that $n_0 \geq n_1$.

Lemma 3.7.10. *The inequality*

$$\frac{n_0+n_1}{2} \leq 2(3g-3+n).$$

holds. The equality holds if and only if all punctures of X are poles of q of order 1 and all zeros of q is of order 1.

Proof. By considering the Euler characteristic of X , we have

$$\frac{n_0+n_1}{2} = 2g - 2 + \sharp C(X, u).$$

Clearly, $\sharp C(X, u) \leq 4g - 4 + 2n$. And, $\sharp C(X, u) = 4g - 4 + 2n$ if and only if all punctures of X are poles of q of order 1 and all zeros of q is of order 1. Hence, we obtain the claim. $\square$

Lemma 3.7.11.

$$\frac{\text{mod}(X)}{b_0} \leq \begin{cases} 8(3g-3+n)^2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ 4(3g-3+n)^2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Proof. Take $h_{A_0} \in \text{Aff}^+(X, u)$ such that $D(h_{A_0}) = A_0$. If $\text{sgn}(\text{Ker}(D)) = \{1\}$, then there exists $m_0 \leq n_0 + n_1$ such that $h_{A_0}^{m_0}$ fixes L_0 pointwise. And there exists $m_1 \leq n_1$ such that $h_{A_0}^{m_0 m_1}$ fixes L_1 pointwise. Then $m_0 m_1 b_0 = k \cdot \text{mod}(X)$ for some $k \in \mathbb{N}$. Hence, by lemma 3.7.10,

$$\frac{\text{mod}(X)}{b_0} = \frac{m_0 m_1}{k} \leq \frac{(n_0+n_1)^2}{2} \leq 8(3g-3+n)^2.$$

If $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, we may assume that $h_{A_0}(L_i) = L_i$ ($i = 0, 1$). There exists $m_0 \leq n_0$ such that $h_{A_0}^{m_0}$ fixes L_0 pointwise. And there exists $m_1 \leq n_1$ such that $h_{A_0}^{m_0 m_1}$ fixes L_1 pointwise. Then $m_0 m_1 b_0 = k \cdot \text{mod}(X)$ for some $k \in \mathbb{N}$. Hence,

$$\frac{\text{mod}(X)}{b_0} = \frac{m_0 m_1}{k} \leq n_0 n_1 \leq \frac{(n_0+n_1)^2}{4} \leq 4(3g-3+n)^2.$$

$\square$

Lemma 3.7.12.

$$\# \text{Ker}(D) \leq \begin{cases} 2(3g - 3 + n) & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ 4(3g - 3 + n) & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Proof. Let $h \in \text{sgn}^{-1}(\{1\})$. Then $h(L_i) = L_i$ ($i = 0, 1$) and there exists $m \leq n_1$ such that h^m fixes L_1 pointwise. Since h is a translation, $h^m = \text{id}$. Thus,

$$\# \text{sgn}^{-1}(\{1\}) \leq m \leq \frac{n_0 + n_1}{2} \leq 2(3g - 3 + n)$$

by lemma 3.7.10. □

Remark. For the flat surface X_3 as in example 3.6.2, the equations of lemma 3.7.10 and lemma 3.7.12 hold.

Finally, to estimate $b_0 c_0$, we prove a new property of finitely generated Fuchsian groups.

Theorem 3.7.13. *Let Γ be a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$. Here, $\nu_i \in \{2, \dots, \infty\}$. Assume that k_0 is the number of ν_i 's which are equal to ∞ . If Γ contains $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and it is not a power of other elements of Γ , then there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ such that*

$$1 \leq |c| < \text{Area}(\mathbb{H}/\Gamma) - k_0 + 1.$$

Here, $\text{Area}(\mathbb{H}/\Gamma)$ is the hyperbolic area of $\mathbb{H}/\Gamma$.

Remark. It is known that

$$\text{Area}(\mathbb{H}/\Gamma) = 2\pi \left(2p - 2 + \sum_{i=1}^k \left(1 - \frac{1}{\nu_i} \right) \right)$$

if Γ is a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$. See [5].

Proof. Set $A = \text{Area}(\mathbb{H}/\Gamma) - k_0 + 1$. Assume that $|c| \geq A$ or $c = 0$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. Set $\Gamma_0 = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \neq 0 \}$. We consider the Ford region D of Γ . Here, the Ford region D is defined by

$$D = \{ z \in \mathbb{H} : |\text{Re}(z)| < \frac{1}{2} \} \cap \bigcap_{B \in \Gamma_0} D_B$$

and

$$D_B = \{z \in \mathbb{H} : |z + \frac{d}{c}| > \frac{1}{|c|}\}.$$

By the assumption, D contains the region $D' = \{z \in \mathbb{H} : |\operatorname{Re}(z)| < \frac{1}{2}, \operatorname{Im}(z) > \frac{1}{A}\}$. Let $v_1, \dots, v_{k_0}$ be punctures of $\mathbb{H}/\Gamma$. Assume that $\infty \in \overline{\mathbb{H}}$ corresponds to the puncture v_{k_0} . There exists the horodisk U_{k_0} around v_{k_0} which is D' in D . And, by lemma 3.2.18, there exist horodisks U_i around v_i which are disjoint each other and their hyperbolic areas are 1 for all $i = 1, \dots, k_0 - 1$. Since every domain as figure 3.7 has area $\pi - 2$, we conclude that $U_i \cap U_{k_0} = \emptyset$ for all $i = 1, \dots, k_0 - 1$. Hence,

$$\begin{aligned} \operatorname{Area}(\mathbb{H}/\Gamma) &> \int_{D'} \frac{dx dy}{y^2} + (k_0 - 1) \cdot 1 \\ &= A + k_0 - 1 \\ &= \operatorname{Area}(\mathbb{H}/\Gamma). \end{aligned}$$

This is a contradiction. □

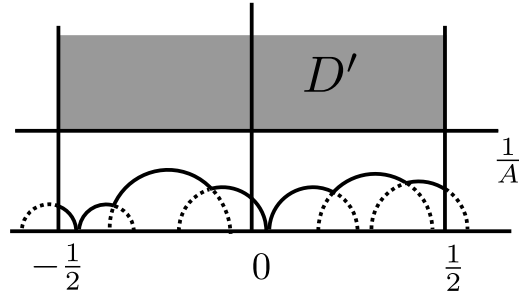


Figure 3.6: The region D' contained in the Ford region D of Γ

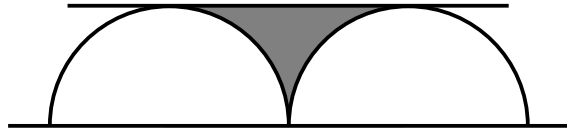


Figure 3.7: A region in $\mathbb{H}$ corresponding to a neighborhood of a puncture of $\mathbb{H}/\Gamma$ of area $\pi - 2$

Remark. Let Γ be a Fuchsian group of type $(p, k : \infty, \dots, \infty)$ with $k \geq 1$. We suppose that Γ satisfies the assumptions of theorem 3.7.13. Take the commutator Q of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. Since $\text{tr}(Q) = c^2 + 2 > 2$, theorem 3.7.13 implies that the Riemann surface $\mathbb{H}/\Gamma$ has a closed geodesic whose length is less than $2 \cosh^{-1} \left(\frac{(2\pi(2p-2+k)-k+1)^2}{2} + 1 \right)$. If this geodesic is a simple closed curve, it has a collar neighborhood whose width is greater than $2 \sinh^{-1} \left(\frac{2}{\gamma(p,k)\sqrt{\gamma(p,k)^2+4}} \right)$, where $\gamma(p, k) = 2\pi(2p - 2 + k) - k + 1$.

Proof of theorem 3.7.1. If $\text{sgn}(\text{Ker}(D)) = \{1\}$, by proposition 3.7.4, lemma 3.7.11, lemma 3.7.12, and theorem 3.7.13,

$$\begin{aligned} \sharp S(X, u) &\leq 8(3g - 3 + n)^2 \text{Area}(\mathbb{H}/\Gamma(X, u))(6g - 5 + 2n) - 2g + 2 \\ &\leq 32\pi(2p - 2 + k)(3g - 3 + n)^2(3g - \frac{5}{2} + n) - 2g + 2. \end{aligned}$$

And, if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$,

$$\begin{aligned} \sharp S(X, u) &\leq 16(3g - 3 + n)^2 \text{Area}(\mathbb{H}/\Gamma(X, u))(3g - 2 + n) - 2g + 2 \\ &\leq 32\pi(2p - 2 + k)(3g - 3 + n)^2(3g - 2 + n) - 2g + 2. \end{aligned}$$

□

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